HADAMARD GAP SERIES AND NORMAL FUNCTIONS

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In this note we prove the following theorem.

THEOREM 1. *Let f be an analytic function defined in the unit disc $\{z \mid |z| < 1\}$ by $f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ where $n_{k+1}/n_k \geq q > 1$. Then f is normal if and only if f is Bloch if and only if* $\limsup |c_k| < \infty$.

J. S. Hwang [3] has proved the above theorem in the special case where $c_k = n_k^m$ for some $m > 1$ and $(n_{k+1}/n_k) \to \infty$ as $k \to \infty$.

The proof of Theorem 1 is based on ideas of W. Fuchs and techniques which go back to Hardy and Littlewood. Throughout the remainder of the paper p_0 will denote the maximum of $2 \log 20/(q-1-\log q)$ and $\log 10/(\log q - 1 + q^{-1})$.

Consider the associated real valued function F defined by

$$F(x) = \sum_{n=0}^{\infty} x^{n_k}, \quad n_{k+1}/n_k\, q > 1 .$$

By accentuating the dominance of the largest term of this series by successive differentiations Fuchs [2] proved for all integers p with $p \geq p_0$ that $F^{(p)}(x)$ behaves in certain intervals like a single term. To be precise, if $p \geq p_0$ and v is sufficiently large, then for any s in the interval

$$\exp\left(\frac{-p}{n_v}\right) \leq s \leq \exp\left(\frac{-p}{2n_v}\left(1 + \frac{\log q}{q-1}\right)\right),$$

we have the inequality

$$\sum_{k \neq v} n_k(n_k - 1) \dots (n_k - p + 1)\, s^{n_k} \leq 1/4\, n_v(n_v - 1) \dots (n_v - p + 1)\, s^{n_v} . \tag{1}$$

This observation about the associated real valued function F lets us prove the following important result.

THEOREM 2. *Let f be an analytic function defined in $\{z \mid |z| < 1\}$ by $f(z) = \sum c_k z^{n_k}$, $n_{k+1}/n_k \geq q > 1$, where* $\limsup |c_k| = \infty$. *Then for all integers p with $p \geq p_0$ and all $M > 0$, there is an s such that*

$$(1 - |z|)^p |f^{(p)}(z)| \geq M$$

for every z on the circle $|z| = s$.

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Proof. Let $\mu(r) = \sup_k |c_k| r^{n_k}$. The analyticity of f in $|z| < 1$ and $\limsup |c_k| = \infty$ together imply that $\mu(r)$ is a monotone increasing function which tends to infinity as $r \to 1$.

Let p be an arbitrary integer greater than p_0. Choose r_0 such that $\mu(r_0) \geq 4M(2e)^p/p!$. Let v be the largest integer such that

$$|c_v| r_0^{n_v} \geq \mu(r_0)/2\,. \tag{2}$$

We note that r_0 can be chosen near enough to 1 so that $n_v \geq p$, and we assume henceforth that r_0 is so chosen. Now set $s = s_0 r_0$ where $s_0 = \exp(-p/n_v)$. Then

$$|f^{(p)}(se^{i\theta})| \geq n_v(n_v-1)\dots(n_v-p+1)|c_v|s^{n_v-p} - \sum_{k \neq v} n_k(n_k-1)\dots(n_k-p+1)|c_k|s^{n_k-p} \tag{3}$$

Using (1) and (2) we obtain

$$\begin{aligned}
\sum_{k\neq v} & n_k(n_k-1)\dots(n_k-p+1)|c_k|(s_0r_0)^{n_k-p} \\
&\leq \sup_k (|c_k|r_0^{n_k}) r_0^{-p} s_0^{-p} \sum_{k\neq v} n_k(n_k-1)\dots(n_k-p+1)s_0^{n_k} \\
&\leq 1/2\, n_v(n_v-1)\dots(n_v-p+1)|c_v| s_0^{n_v-p} r_0^{n_v-p}\,.
\end{aligned} \tag{4}$$

Thus from (3) and (4) we find

$$\begin{aligned}
|f^{(p)}(se^{i\theta})| &\geq 1/2\, n_v(n_v-1)\dots(n_v-p+1)s_0^{n_v-p}\, r_0^{n_v-p}|c_v| \\
&\geq n_v(n_v-1)\dots(n_v-p+1)\, M2^p/p!
\end{aligned}$$

And therefore

$$\begin{aligned}
(1-s)^p|f^{(p)}(se^{i\theta})| &\geq (1-s_0)^p n_v(n_v-1)\dots(n_v-p+1)\, M2^p/p! \\
&\geq \left(\frac{p}{n_v}\right)^p n_v(n_v-1)\dots(n_v-p+1)M/p! \\
&= \frac{Mp^p}{p!}\left(1-\frac{1}{n_v}\right)\left(1-\frac{2}{n_v}\right)\dots\left(1-\frac{p-1}{n_v}\right) \\
&\geq \frac{Mp^p}{p!}\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\dots\left(1-\frac{p-1}{p}\right) \equiv M\,,
\end{aligned}$$

which concludes the proof of Theorem 2.

Proof of Theorem 1. Pommerenke [**8**] proved $\limsup |c_k| < \infty$ implies f is Bloch, and it is well known that Bloch functions are normal [**9**, p. 268]. It therefore suffices to prove that $\limsup |c_k| = \infty$ implies f is not normal.

Fix an integer p for which $p > p_0$. Lappan [**4**] proved that if f is normal, then there is a finite constant K such that

$$\frac{|f^{(p)}(z)|(1-|z|)^p}{1+|f(z)|^{p+1}} \leq K \tag{5}$$

for all z in the unit disc. According to Theorem 2 there is a sequence of radii s_n such that for $|z| = s_n$

$$(1-|z|)^p |f^{(p)}(z)| \geqslant n.$$

If min $|f(z)|$ on $|z| = s_n$ tends to ∞, then f has Koebe arcs and is therefore non-normal [9, p. 267]. If min $|f(z)|$ on $|z| = s_n$ does not tend to ∞, then by passing to a subsequence we can find an integer M and a sequence of points z_n, $|z_n| = s_n$, such that $|f(z_n)| \leqslant M < \infty$. For this sequence of points

$$\frac{(1-|z_n|)^p |f^{(p)}(z_n)|}{1+|f(z_n)|^{p+1}} \geqslant \frac{n}{1+M^{p+1}},$$

which proves that (6) cannot hold. Therefore f must be non-normal.

COROLLARY. *Let f be defined in the unit disc by $f(z) = \sum_{k=1}^{\infty} k^k z^{2^k}$. Then all of its derivatives and all of its integrals are non-normal functions which are analytic in $\{z \mid |z| < 1\}$.*

Remark 1. Motivated by a result of MacLane [5, p. 46], Bonar [1, p. 59] posed the following question. If f is a strongly annular function, can f be written as $f(z) = g(z) + h(z)$ where $g(z) = \sum a_k z^{\mu_k}$, $\liminf \mu_{k+1}/\mu_k > 3$, and $h(z)$ is bounded in the unit disc? The answer is no. To see this, let $F(z) = \sum 2^k z^{2^k}$. Since $n_{k+1}/n_k = 2$, and $|c_k| \to \infty$, it is an easy consequence of Theorem 2 that for an integer $p > p_0$, there is an increasing sequence of positive numbers s_n such that $(1-|z|)^p |F^{(p)}(z)| \geqslant n$ for all z on the circle $|z| = s_n$. Thus $F^{(p)}$ is strongly annular. Suppose $F^{(p)}(z)$ could be written as $g(z) + h(z)$ where $g(z) = \sum a_j z^{\mu_j}$, $\liminf \mu_{j+1}/\mu_j > 3$, and h is bounded in the unit disc. The coefficients in the power series expansion of h about zero must go to zero since h is bounded. Therefore for k sufficiently large the index set $\{\mu_j\}$ must contain all exponents of the form $\{2^j - p\}_{j=k}^{\infty}$. Consequently, $3 < \liminf \mu_{j+1}/\mu_j \leqslant 2$, which is absurd. An appropriate modification works for any $q > 1$.

Remark 2. Piranian [7] asked whether a bounded function of finite area must have a normal derivative. Theorem 1 lets us answer this in the negative. Let f be defined by

$$f(z) = \sum_{n=1}^{\infty} n(2^n+1)^{-1} z^{2^n+1} = \sum_{j=1}^{\infty} a_j z^j.$$

Then f is bounded by $\sum n2^{-n}$, and $\sum j|a_j|^2 \leqslant \sum n^2 2^{-n}$ shows that f has finite area. Theorem 1 guarantees that $f'(z) = \sum n z^{2^n}$ defines a non-normal function in the unit disc.

We close with two open questions.

Question 1. If $f(z) = \sum c_k z^{n_k}$, $n_{k+1}/n_k \geqslant q > 1$, $\limsup |c_k| = \infty$, must f be annular? If the maximum modulus of f grows rapidly enough, the answer is yes [c.f. 6, Thm. 4].

Question 2. What is the best value for p_0 for which Theorem 2 is true?

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