HADAMARD GAP SERIES AND NORMAL FUNCTIONS

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In this note we prove the following theorem.

**Theorem 1.** Let \( f \) be an analytic function defined in the unit disc \( \{ z \mid |z| < 1 \} \) by
\[
f(z) = \sum_{k=0}^{\infty} c_k z^n \text{ where } n_{k+1}/n_k \geq q > 1.\]
Then \( f \) is normal if and only if \( f \) is Bloch if and only if \( \limsup |c_k| < \infty \).

J. S. Hwang [3] has proved the above theorem in the special case where \( c_k = n_k^m \) for some \( m > 1 \) and \( (n_{k+1}/n_k) \rightarrow \infty \) as \( k \rightarrow \infty \).

The proof of Theorem 1 is based on ideas of W. Fuchs and techniques which go back to Hardy and Littlewood. Throughout the remainder of the paper \( p_0 \) will denote the maximum of \( 2 \log 20/(q-1-\log q) \) and \( \log 10/(\log q-1+q^{-1}) \).

Consider the associated real valued function \( F \) defined by
\[
F(x) = \sum_{n=0}^{\infty} x^{n_{k+1}/n_k} q > 1.
\]

By accentuating the dominance of the largest term of this series by successive differentiations Fuchs [2] proved for all integers \( p \) with \( p \geq p_0 \) that \( F^{(p)}(x) \) behaves in certain intervals like a single term. To be precise, if \( p \geq p_0 \) and \( v \) is sufficiently large, then for any \( s \) in the interval
\[
\exp \left( \frac{-p}{n_c} \right) \leq s \leq \exp \left( \frac{-p}{2n_c} \left( 1 + \frac{\log q}{q-1} \right) \right),
\]
we have the inequality
\[
\sum_{k \leq r} n_k(n_k-1)...(n_k-p+1) s^n \leq \frac{1}{4 n_c(n_c-1)...(n_c-p+1)} s^n. \quad (1)
\]

This observation about the associated real valued function \( F \) lets us prove the following important result.

**Theorem 2.** Let \( f \) be an analytic function defined in \( \{ z \mid |z| < 1 \} \) by \( f(z) = \sum c_k z^n \), \( n_{k+1}/n_k \geq q > 1 \), where \( \limsup |c_k| = \infty \). Then for all integers \( p \) with \( p \geq p_0 \) and all \( M > 0 \), there is an \( s \) such that
\[
(1-|z|)^p |f^{(p)}(z)| \geq M
\]
for every \( z \) on the circle \( |z| = s \).

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Proof. Let \( \mu(r) = \sup_k |c_k| r_n^k \). The analyticity of \( f \) in \( |z| < 1 \) and \( \lim \sup |c_k| = \infty \) together imply that \( \mu(r) \) is a monotone increasing function which tends to infinity as \( r \to 1 \).

Let \( p \) be an arbitrary integer greater than \( p_0 \). Choose \( r_0 \) such that \( \mu(r_0) \geq 4M(2e)^p/p! \). Let \( v \) be the largest integer such that
\[
|c_v| r_0^{k_v} \geq \mu(r_0)/2.
\]

We note that \( r_0 \) can be chosen near enough to \( 1 \) so that \( n_v \geq p \), and we assume henceforth that \( r_0 \) is so chosen. Now set \( s = s_0 r_0 \) where \( s_0 = \exp(-p/n_v) \). Then
\[
|f^{(p)}(se^{i\theta})| \geq n_v(n_v - 1)(n_v - p + 1)|c_v| s_0^{n_v - p} - \sum_{k \neq v} n_k(n_k - 1)(n_k - p + 1)|c_k| s_0^{n_k - p} \tag{3}
\]

Using (1) and (2) we obtain
\[
\sum_{k \neq v} n_k(n_k - 1)(n_k - p + 1)|c_k| (s_0 r_0)^{n_k - p} \leq \sup_k (|c_k| r_0^{n_k}) s_0^{n_v - p} \sum_{k \neq v} n_k(n_k - 1)(n_k - p + 1)s_0^{n_k - p} \leq 1/2 n_v(n_v - 1)(n_v - p + 1)|c_v| s_0^{n_v - p} r_0^{n_v - p}. \tag{4}
\]

Thus from (3) and (4) we find
\[
|f^{(p)}(se^{i\theta})| \geq 1/2 n_v(n_v - 1)(n_v - p + 1)|c_v| M 2^p/p! \geq n_v(n_v - 1)(n_v - p + 1) M 2^p/p!.
\]

And therefore
\[
(1 - s)^p|f^{(p)}(se^{i\theta})| \geq (1 - s_0)^p n_v(n_v - 1)(n_v - p + 1) M 2^p/p!
\]
\[
\geq \left( \frac{p}{n_v} \right)^p n_v(n_v - 1)(n_v - p + 1) M/p! = \frac{M p^p}{p!} \left( 1 - \frac{1}{n_v} \right) \left( 1 - \frac{2}{n_v} \right) \left( 1 - \frac{p - 1}{n_v} \right)
\]
\[
\geq \frac{M p^p}{p!} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{2}{p} \right) \left( 1 - \frac{p - 1}{p} \right) = M,
\]

which concludes the proof of Theorem 2.

Proof of Theorem 1. Pommerenke [8] proved \( \lim \sup |c_k| < \infty \) implies \( f \) is Bloch, and it is well known that Bloch functions are normal [9, p. 268]. It therefore suffices to prove that \( \lim \sup |c_k| = \infty \) implies \( f \) is not normal.

Fix an integer \( p \) for which \( p > p_0 \). Lappan [4] proved that if \( f \) is normal, then there is a finite constant \( K \) such that
\[
\frac{|f^{(p)}(z)(1 - |z|)^p}{1 + |f(z)|^{p + 1}} \leq K \tag{5}
\]
for all \( z \) in the unit disc. According to Theorem 2 there is a sequence of radii \( s_n \) such that for \( |z| = s_n \)

\[
(1 - |z|)^p |f^{(p)}(z)| \geq n.
\]

If \( \min |f(z)| \) on \( |z| = s_n \) tends to \( \infty \), then \( f \) has Koebe arcs and is therefore non-normal [9, p. 267]. If \( \min |f(z)| \) on \( |z| = s_n \) does not tend to \( \infty \), then by passing to a subsequence we can find an integer \( M \) and a sequence of points \( z_n, |z_n| = s_n \), such that \( |f(z_n)| \leq M < \infty \). For this sequence of points

\[
\frac{(1 - |z_n|)^p |f^{(p)}(z_n)|}{1 + |f(z_n)|^{p+1}} \geq \frac{n}{1 + M^{p+1}},
\]

which proves that (6) cannot hold. Therefore \( f \) must be non-normal.

**Corollary.** Let \( f \) be defined in the unit disc by \( f(z) = \sum_{k=1}^{\infty} k^k z^k \). Then all of its derivatives and all of its integrals are non-normal functions which are analytic in \( \{z | |z| < 1\} \).

**Remark 1.** Motivated by a result of MacLane [5, p. 46], Bonar [1, p. 59] posed the following question. If \( f \) is a strongly annular function, can \( f \) be written as \( f(z) = g(z) + h(z) \) where \( g(z) = \sum a_k z^{\mu_k}, \lim \inf \mu_{k+1}/\mu_k > 3 \), and \( h(z) \) is bounded in the unit disc? The answer is no. To see this, let \( F(z) = \sum z^k \). Since \( n_{k+1}/n_k \) is a growth function for \( g(z) + h(z) \) where \( g(z) = \sum a_j z^{\mu_j}, \lim \inf \mu_{j+1}/\mu_j > 3 \), and \( h \) is bounded in the unit disc. The coefficients in the power series expansion of \( h \) about zero must go to zero since \( h \) is bounded. Therefore for \( k \) sufficiently large the index set \( \{\mu_j\} \) must contain all exponents of the form \( \{2^{j-1}\} \). Consequently, \( 3 \leq \lim \inf \mu_{j+1}/\mu_j \leq 2 \), which is absurd. An appropriate modification works for any \( q > 1 \).

**Remark 2.** Piranian [7] asked whether a bounded function of finite area must have a normal derivative. Theorem 1 lets us answer this in the negative. Let \( f \) be defined by

\[
f(z) = \sum_{n=1}^{\infty} n(2^n + 1)^{-1} z^{2^n+1} = \sum_{j=1}^{\infty} a_j z^j.
\]

Then \( f \) is bounded by \( \sum n2^{-n} \), and \( \sum |a_j| \leq \sum n^2 2^{-n} \) shows that \( f \) has finite area. Theorem 1 guarantees that \( f'(z) = \sum n z^{2^n} \) defines a non-normal function in the unit disc.

We close with two open questions.

**Question 1.** If \( f(z) = \sum c_k z^k \), \( n_{k+1}/n_k \geq q > 1 \), \( \lim \sup |c_k| = \infty \), must \( f \) be annular? If the maximum modulus of \( f \) grows rapidly enough, the answer is yes [c.f. 6, Thm. 4].

**Question 2.** What is the best value for \( p_0 \) for which Theorem 2 is true?
References


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