Two interesting examples of $\mathcal{D}$-modules in characteristic $p > 0$

Mordechai Katzman, Gennady Lyubeznik and Wenliang Zhang

Abstract

We provide two examples of $\mathcal{D}$-modules in prime characteristic p that answer two open questions in [G. Lyubeznik, 'A characteristic-free proof of a basic result on D-modules', *J. Pure Appl. Alg.* 215 (2011) 2019–2023] in the negative.

1. *Introduction*

Let K be a field, $R = K[x_1, \ldots, x_n]$ be the ring of polynomials in $x_1, \ldots, x_n$ over K and $\mathcal{D}$ be the ring of K-linear differential operators over K. In a remarkable paper [**1**], Bavula gave a characteristic-free definition of holonomic $\mathcal{D}$-modules. In characteristic zero, his definition coincides with the usual one. He proved, among other things, that his holonomic modules have one of the most important properties known from the characteristic zero case, namely, their length in the category of $\mathcal{D}$-modules is finite.

Using Bavula's ideas, Lyubeznik [**3**] gave a characteristic-free proof that R_f, for every non-zero element $f \in R$, is holonomic. This provided the first characteristic-free proof of the well-known fact that R_f has finite length in the category of $\mathcal{D}$-modules.

In view of these developments, it is interesting to see whether, in characteristic $p > 0$, holonomic modules, as defined by Bavula, have other properties known from the characteristic zero case.

Bavula proved that a submodule and a quotient module of a holonomic $\mathcal{D}$-module are holonomic. But in characteristic zero it is also true that an extension of two holonomic modules is holonomic. Does this property hold in characteristic $p > 0$ as well?

Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ be the Bernstein filtration on $\mathcal{D}$, let M be a holonomic $\mathcal{D}$-module generated by a finite set of elements $m_1 \ldots, m_s \in M$ and let $M_0 \subset M_1 \subset \ldots$ be the filtration on M defined by $M_i = \mathcal{F}_i m_1 + \ldots + \mathcal{F}_i m_s$. In characteristic zero, it is well-known that $\dim_k M_i$, for $i \gg 0$, is a polynomial in i of degree n; in particular, $\lim_{i \to \infty} (\dim M_i / i^n)$ exists and is finite. Does this property hold in characteristic $p > 0$ as well?

These two questions were raised in the last section of Lyubeznik [**3**]. In this paper, we give counter-examples to both of them. In Section 3, we produce a non-holonomic extension of two holonomic modules in characteristic $p > 0$ and in Section 4 we produce a holonomic $\mathcal{D}$-module in characteristic $p > 0$ such that the function $\dim_k M_i$ is very far from a polynomial and, in particular, $\lim_{i \to \infty} (\dim M_i / i^n)$ does not exist.

2. *Preliminaries*

As explained in [**3**, Section 2], a K-basis of $\mathcal{D}$ is the set of products $x_1^{i_1} \ldots x_n^{i_n} D_{t_1,1} \ldots D_{t_n,n}$ where $D_{t,i} = (1/t!)(\partial^t/\partial x_i^t) : R \to R$ is the $K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$-linear map that sends

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x_i^v to $\binom{v}{t} x_i^{v-t}$ ($D_{0,i}$ is the identity map) and $i_1, \ldots, i_n, t_1, \ldots, t_n$ range over all the $2n$-tuples of non-negative integers. The Bernstein filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ on $\mathcal{D}$ is defined by setting $\mathcal{F}_s$ to be the K-linear span of the products $x_1^{i_1} \ldots x_n^{i_n} D_{t_1,1} \ldots D_{t_n,n}$ with $i_1 + \ldots + i_n + t_1 + \ldots + t_n \leqslant s$. It is not hard to see that $\mathcal{F}_i \cdot \mathcal{F}_j \subset \mathcal{F}_{i+j}$.

By a $\mathcal{D}$-module, we always mean a left $\mathcal{D}$-module. By a K-filtration on a $\mathcal{D}$-module M, we mean an ascending chain of K-vector spaces $M_0 \subset M_1 \subset \ldots$ such that $\bigcup_i M_i = M$ and $\mathcal{F}_i M_j \subset M_{i+j}$. Bavula's definition of a holonomic $\mathcal{D}$-module [**1**, p. 185], as simplified by Lyubeznik [**3**, 3.4], is the following.

DEFINITION 2.1. A $\mathcal{D}$-module M is holonomic if it has a K-filtration $M_0 \subset M_1 \subset \ldots$ such that $\dim_k M_i \leqslant Ci^n$ for all i, where C is a constant independent of i.

It is straightforward to see that every submodule and every quotient module of a holonomic module are holonomic. Some other properties are that the length of a holonomic module M in the category of D-modules is at most $n!C$ (see [**1**, 9.6; **3**, 3.5]) (in particular, the length is finite) and R_f, for every $0 \neq f \in R$, with its natural $\mathcal{D}$-module structure, is holonomic [**3**, 3.6].

For the rest of this paper, K denotes a perfect field of prime characteristic p. Let $\mathcal{D}_s$ be the (left) R-submodule of $\mathcal{D}$ generated by the products $D_{t_1,i_1} \ldots D_{t_n,i_n}$ such that $t_i < p^s$ for every i. It is not hard to see that $\mathcal{D}_s$ is a ring that (viewing $\mathcal{D}$ as a subring of $\mathrm{Hom}_k(R,R)$) is nothing but $\mathrm{Hom}_{R^{p^s}}(R,R)$. In particular, $\mathcal{D} = \bigcup_s \mathcal{D}_s$.

Our method of specifying a $\mathcal{D}$-module is as follows: we start with a sequence of $\{M^{(i)}\}_{i \geqslant 0}$ where each $M^{(i)}$ is an R^{p^i}-module and $R^{p^{i+1}}$-linear maps $\Theta_i : M^{(i+1)} \to M^{(i)}$ such that the R^{p^i}-module maps $\psi_i : R^{p^i} \otimes_{R^{p^{i+1}}} M^{(i+1)} \xrightarrow{r \otimes m \mapsto r\Theta(m)} M^{(i)}$ are bijective. This induces an R-module isomorphism $\phi_i : R \otimes_{R^{p^{i+1}}} M^{(i+1)} = R \otimes_{R^{p^i}} (R^{p^i} \otimes_{R^{p^{i+1}}} M^{(i+1)}) \xrightarrow{\mathrm{id} \otimes \psi_i} R \otimes_{R^{p^i}} M^{(i)}$. Clearly, the compositions $\varphi_i = \phi_0 \circ \phi_1 \circ \ldots \circ \phi_i : R \otimes_{R^{p^{i+1}}} M^{(i+1)} \to M^{(0)}$ are R-module isomorphisms. The natural action of $\mathcal{D}_s$ on R makes $R \otimes_{R^{p^s}} M^{(s)}$ a $\mathcal{D}_s$-module. This induces a structure of $\mathcal{D}_s$-module on $M^{(0)}$ via the isomorphism φ_i. It is not hard to check that if $s < s'$, then the $\mathcal{D}_{s'}$- and the $\mathcal{D}_s$-module structures thus defined are compatible with the natural inclusion $\mathcal{D}_s \subset \mathcal{D}_{s'}$, that is, the $\mathcal{D}_s$-module structure is obtained from the $\mathcal{D}_{s'}$-module structure via the restriction of scalars. Since $\mathcal{D} = \bigcup_s \mathcal{D}_s$, this gives $M^{(0)}$ a structure of the $\mathcal{D}$-module.

Both examples in this paper are special cases of the construction described in [**2**, Section 1]. Each $M^{(i)}$ is a free R^{p^i}-module with free generators $s_1^{(i)}$ and $s_2^{(i)}$, $\Theta_i(s_1^{(i+1)}) = s_1^{(i)}$ and $\Theta_i(s_2^{(i+1)}) = g_i s_1^{(i)} + s_2^{(i)}$ where, for all $i \geqslant 0$, g_i is an element of Rx^{p^i}. Since the elements $\Theta_i(s_1^{(i+1)})$ and $\Theta_i(s_2^{(i+1)})$ generate $M^{(i)}$ as an R^{p^i}-module, the associated map ψ_i (defined in the preceding paragraph) is surjective. Since ψ_i is a map between two free R^{p^i}-modules of rank 2, it is bijective. If we write $\sigma_n = -\sum_{r=0}^{n} g_r$, then the resulting $\mathcal{D}$-module structure on M is given by

$$\partial_{p^n}(f_1, f_2) = (\partial_{p^n} f_1 + (\partial_{p^n} \sigma_n) f_2, \partial_{p^n} f_2)$$

for all $n \geqslant 0$.

Note that we have a short exact sequence of $\mathcal{D}$-modules

$$0 \longrightarrow R \xrightarrow{\psi} M \xrightarrow{\phi} R \longrightarrow 0,$$

where $\psi(f) = (f, 0)$ and $\phi(f_1, f_2) = f_2$. Even though this exact sequence splits in the category of R-modules, it does not necessarily split in the category of $\mathcal{D}$-modules.

Our examples below result from a judicious choice of the sequences $\{g_i\}_{i \geqslant 0}$.

3. *An example of a non-holonomic extension of holonomic modules*

The main result in this section is Theorem 3.2, which answers [**3**, Section 4, Question 1] in the negative. We do so by analysing the $\mathcal{D}$-module obtained by setting $g_r = x^{p^r + p^{2r}}$ for all $r \geqslant 0$ in the construction of $\mathcal{D}$-modules described in Section 1.

We start with the following calculation to which we shall refer repeatedly.

LEMMA 3.1.   *For any integers $0 \leqslant \alpha \leqslant \beta$ and $K \geqslant 0$, we have,*

$$\partial_{p^k}(x^{p^\alpha} x^{p^\beta}) = \begin{cases} x^{p^\alpha} + x^{p^\beta}, & \text{if } k = \alpha = \beta, \\ x^{p^\beta}, & \text{if } k = \alpha < \beta, \\ x^{p^\alpha}, & \text{if } \alpha < \beta = k, \\ x^{p^\alpha}, & \text{if } p = 2, \alpha = \beta = k - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof.   We first note that

$$\partial_j(x^{p^\alpha}) = \begin{cases} x^{p^\alpha}, & \text{if } j = 0, \\ 1, & \text{if } j = p^\alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that ∂_j is $K[x^{p^\gamma}]$-linear whenever $j < p^\gamma$, and so, if $K < \alpha$, we have

$$\partial_{p^k}(x^{p^\alpha} x^{p^\beta}) = x^{p^\alpha} \partial_{p^k} x^{p^\beta} = x^{p^\alpha} x^{p^\beta} \partial_{p^k} 1 = 0.$$

If $K = \alpha$, then we use [**3**, Proposition 2.1] to compute

$$\partial_{p^k}(x^{p^\alpha} x^{p^\beta}) = \sum_{j=0}^{p^k} \partial_j x^{p^\alpha} \partial_{p^k - j} x^{p^\beta} = \partial_0 x^{p^\alpha} \partial_{p^k} x^{p^\beta} + \partial_{p^k} x^{p^\alpha} \partial_0 x^{p^\beta}$$

$$= \begin{cases} x^{p^\alpha} + x^{p^\beta}, & \text{if } k = \alpha = \beta, \\ x^{p^\beta}, & \text{if } k = \alpha < \beta. \end{cases}$$

If $\alpha < \beta = k$, then we compute

$$\partial_{p^k}(x^{p^\alpha} x^{p^\beta}) = \sum_{j=0}^{p^k} \partial_j x^{p^\alpha} \partial_{p^k - j} x^{p^\beta} = \partial_0 x^{p^\alpha} \partial_{p^k} x^{p^\beta} + \partial_{p^\alpha} x^{p^\alpha} \partial_{p^k - p^\alpha} x^{p^\beta} = x^{p^\alpha}.$$

If $\alpha \leqslant \beta < k$, then we compute

$$\partial_{p^k}(x^{p^\alpha} x^{p^\beta}) = \sum_{j=0}^{p^k} \partial_j x^{p^\alpha} \partial_{p^k - j} x^{p^\beta}$$

$$= \partial_0 x^{p^\alpha} \partial_{p^k} x^{p^\beta} + \partial_{p^\alpha} x^{p^\alpha} \partial_{p^k - p^\alpha} x^{p^\beta}$$

$$= \begin{cases} x^{p^\alpha}, & \text{if } \alpha = \beta = k - 1 \text{ and } p = 2, \\ 0, & \text{if } \alpha < \beta \text{ or } \beta < k - 1 \text{ or } p > 2. \end{cases} \qquad \square$$

THEOREM 3.2.   *The $\mathcal{D}$-module M is not holonomic in the sense of Lyubeznik [**3**, Definition 3.4].*

Proof.   Let $s_1 = (1, 0)$ and $s_2 = (0, 1)$ be the free generators of M. Let $\{\mathcal{F}_i\}_{i \geqslant 0}$ denote the Bernstein filtration of $\mathcal{D}$. Let $\{M_i\}_{i \geqslant 0}$ be any K-filtration of M. Our aim is to show that

$\lim_{i\to\infty} \dim_K M_i/i = \infty$; we may shift the indices to ensure that $s_1, s_2 \in M_0$ and we henceforth assume that this holds.

Since $M_i \supseteq \mathcal{F}_i M_0$, it is enough to show that the function $d(i) = \dim_K \mathcal{F}_i M_0$ is such that $\lim_{i\to\infty} d(i)/i = \infty$.

For any pair of integers (j, k) with $j, k \geqslant 0$, we have $x^j \partial_{p^k} s_2 = (x^j \partial_{p^k} \sigma_k) s_1$. Now consider the set of elements

$$E = \{r_{jk} := x^j \partial_{p^k} \sigma_k \mid j + p^k \leqslant p^i\} \subseteq \mathcal{F}_{p^i} M_0 \subseteq M_{p^i}.$$

Lemma 3.1 gives $\partial_{p^k}(x^{p^k} x^{p^{2k}}) = x^{p^{2k}}$, hence $\deg \partial_{p^k} g_k = p^{2k}$ and $\deg r_{jk} = j + \deg \partial_{p^k} \sigma_k = j + \deg(\sigma_k) - p^k = j + p^{2k}$. Hence, for all different pairs (j, k) and (j', k') with $K, k' \geqslant i/2$, we have

$$\deg r_{jk} = j + p^{2k} \neq j' + p^{2k'} = \deg(r_{j'k'}),$$

otherwise $p^{2k} - p^{2k'} = j' - j$, which implies $p^i \mid (j' - j)$, contradicting the fact that $j, j' < p^i$. We deduce that elements r_{jk} of E with $K \geqslant i/2$ and $j + p^k \leqslant p^i$ have distinct degrees and hence are linearly independent over K. Let $\lceil i/2 \rceil$ denote the least integer greater than or equal to $i/2$. For each $K \geqslant i/2$, there are $p^i - p^k$ many r_{jk}; therefore,

$$\dim_K \mathcal{F}_{p^i} M_0 \geqslant \sum_{i \geqslant k \geqslant \lceil i/2 \rceil} (p^i - p^k)$$

$$= (i - \lceil i/2 \rceil + 1)p^i - p^{\lceil i/2 \rceil} \frac{p^{i - \lceil i/2 \rceil + 1} - 1}{p - 1},$$

which implies that

$$\lim_{i\to\infty} \frac{d(p^i)}{p^i} = \lim_{i\to\infty} \left((i - \lceil i/2 \rceil + 1) + \frac{p^{i - \lceil i/2 \rceil + 1} - 1}{p^{i - \lceil i/2 \rceil}(p - 1)} \right) = \infty. \qquad \square$$

4. An example of a holonomic module whose multiplicity does not exist

Let M be as in the previous section with g_r replaced by $g_r = x^{(p+1)p^r}$. In this section, we show that $\mathcal{D}s_2$, the $\mathcal{D}$-submodule of M generated by s_2, is a holonomic $\mathcal{D}$-module for which $\lim_{i\to\infty}(\mathcal{F}_i s_2/i)$ does not exist. This gives a negative answer to [**3**, Section 4, Question 2].

We start with the following calculation.

LEMMA 4.1. Let $K_1, \ldots, k_t, e_1, \ldots, e_t$ be non-negative integers.

(a)

$$\partial_{p^k} \sigma_k = \begin{cases} x^p, & \text{if } k = 0, \\ x^{p^{k+1}} + x^{p^{k-1}}, & \text{otherwise.} \end{cases}$$

(b)

$$(\partial_{p^{k_1}})^{e_1} \ldots (\partial_{p^{k_t}})^{e_t} s_2 = \begin{cases} s_2, & \text{if } e_1 = \ldots = e_t = 0, \\ x^p s_1, & \text{if } t = 1, e_1 = 1, k_1 = 0, \\ (x^{p^{k_t+1}} + x^{p^{k_t-1}})s_1, & \text{if } t = 1, e_1 = 1, k_t \geqslant 1, \\ s_1, & \text{if } t = 2, e_1 = e_2 = 1, \\ & k_1 = k_2 + 1 \text{ or } k_1 = k_2 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (a) Lemma 3.1 implies that, for $K \geqslant 0$, $\partial_{p^k} g_r$ vanishes when $r + 1 < k$, that $\partial_{p^k} g_{k-1} = x^{p^{k-1}}$ and that $\partial_{p^k} g_k = x^{p^{k-1}} = x^{p^{k+1}}$, hence

$$\partial_{p^k} \sigma_k = \begin{cases} x^p, & \text{if } k = 0, \\ x^{p^{k+1}} + x^{p^{k-1}}, & \text{otherwise.} \end{cases}$$

(b) This follows immediately from (a). $\square$

THEOREM 4.2. *Let S denote $\mathcal{D}s_2$ and S_i denote $\mathcal{F}_i s_2$. Then*

$$\dim_k(S_i) = \begin{cases} 2i + p^{e+1} - p^e + 2, & \text{if } p^{e+1} - p^e + p^{e-1} \leqslant i < p^{e+1}, \\ 3i - p^{e-1} + 3, & \text{if } p^e \leqslant i < p^{e+1} - p^e + p^{e-1}, \end{cases}$$

where e is the unique integer such that $p^e \leqslant i < p^{e+1}$. Consequently, S is holonomic and $\lim_{i \to \infty}(\dim_k(S_i)/i)$ does not exist.

Proof. Consider a general element $x^j (\partial_{p^{k_1}})^{e_1} \ldots (\partial_{p^{k_t}})^{e_t} s_2$ with $j + \sum_{i=0}^{t} e_i p^{k_i} \leqslant i$ in S_i. Lemma 4.1 shows that $x^j (\partial_{p^{k_1}})^{e_1} \ldots (\partial_{p^{k_t}})^{e_t} s_2$ equals

$$\begin{cases} x^j s_2, & \text{with } 0 \leqslant j \leqslant i, & \text{if } e_1 = \ldots = e_t = 0, \\ x^j x^p s_1, & \text{with } 0 \leqslant j \leqslant i - 1, & \text{if } t = 1, e_1 = 1, k_1 = 0, \\ x^j (x^{p^{k_1+1}} + x^{p^{k_1-1}}) s_1, & \text{with } 0 \leqslant j \leqslant i - p^{k_1}, & \text{if } t = 1, e_1 = 1, k_1 \geqslant 1, \\ x^j s_1, & \text{with } 0 \leqslant j \leqslant i - p^{k_2} - p^{k_2+1}, & \text{if } t = 2, e_1 = e_2 = 1, k_1 = k_2 + 1, \\ x^j s_1, & \text{with } 0 \leqslant j \leqslant i - p^{k_2} - p^{k_2-1}, & \text{if } t = 2, e_1 = e_2 = 1, k_1 = k_2 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

From this we can see that, among $\{x^j (\partial_{p^{k_1}})^{e_1} \ldots (\partial_{p^{k_t}})^{e_t} s_2 \,|\, j + \sum_{i=0}^{t} e_i p^{k_i} \leqslant i\}$, there are three types of elements: elements obtained when $t = 0$ (that is, $e_1 = \ldots = e_t = 0$), elements obtained when $t = 1$ and elements obtained when $t = 2$. Let V_1 be the K-span of all elements of the first type, that is,

$$V_1 = K \langle x^j s_2 \,|\, 0 \leqslant j \leqslant i \rangle.$$

Let V_2 be the K-span of all elements of the second type, that is,

$$V_2 = K \langle x^j x^p s_1 \,|\, 0 \leqslant j \leqslant i - 1 \rangle + K \langle x^j (x^{p^{k+1}} + x^{p^{k-1}}) \,|\, 0 \leqslant j + p^k \leqslant i, \ k \geqslant 1 \rangle.$$

Let V_3 be the K-span of all elements of the third type, that is,

$$V_3 = K \langle x^j s_1 \,|\, 0 \leqslant j \leqslant i - 1 - p \rangle.$$

It should be pointed out that all elements of the first type are in the copy of R generated by s_2 and all elements of the other two types are in the copy of R generated by s_1; and hence $\dim_K(\mathcal{F}_i s_2) = \dim_K(V_1) + \dim_K(V_2 + V_3)$ since s_1 and s_2 are linearly independent. It is clear that V_1 is a $(i + 1)$-dimensional K-vector space; it remains to calculate $\dim_K(V_2 + V_3)$.

To calculate the dimension of $V_2 + V_3$, we break V_2 into pieces as follows:

$$V_{2,0} = K\langle x^j x^p s_1 \,|\, 0 \leqslant j \leqslant i-1\rangle,$$
$$V_{2,1} = K\langle x^j (x^{p^2} + x)s_1 \,|\, 0 \leqslant j \leqslant i-p\rangle,$$
$$\vdots$$
$$V_{2,e-1} = K\langle x^j (x^{p^e} + x^{p^{e-2}})s_1 \,|\, 0 \leqslant j \leqslant i-p^{e-1}\rangle,$$
$$V_{2,e} = K\langle x^j (x^{p^{e+1}} + x^{p^{e-1}})s_1 \,|\, 0 \leqslant j \leqslant i-p^e\rangle.$$

If $i \geqslant 2p+1$, then $V_{2,0} + V_3 = K\langle x^j s_1 \,|\, 0 \leqslant j \leqslant i-1+p\rangle$. Since $V_{2,0} + V_3$ contains $x^j x s_1$ with $0 \leqslant j \leqslant i-p$, we see that $V_{2,1} + V_{2,0} + V_3$ contains $x^j x^{p^2} s_1$ with $0 \leqslant j \leqslant i-p$. Consequently,

$$V_{2,1} + V_{2,0} + V_3 = K\langle x^j s_1 \,|\, 0 \leqslant j \leqslant i-p+p^2\rangle.$$

Similarly, we have

$$V_{2,e-1} + \ldots + V_{2,0} + V_3 = K\langle x^j s_1 \,|\, 0 \leqslant j \leqslant i-p^{e-1}+p^e\rangle.$$

It remains to analyse $V_{2,e} = K\langle x^j (x^{p^{e+1}} + x^{p^{e-1}})s_1 \,|\, 0 \leqslant j \leqslant i-p^e\rangle$. There are two cases:

Case 1: $i \geqslant p^{e+1} - p^e + p^{e-1}$, that is, $[p^e, i-p^{e-1}+p^e] \cap [p^{e+1}, i-p^e+p^{e+1}] \neq \emptyset$. In this case, similar to the consideration of $V_{2,e-1} + \ldots + V_{2,0} + V_3$, we can see that $V_{2,e} + \ldots + V_{2,0} + V_3$ consists of all polynomials of degree less than or equal to $i - p^e + p^{e+1}$. Therefore,

$$\dim_K(V_2 + V_3) = i - p^e + p^{e+1} + 1,$$

and hence

$$\dim_K(\mathcal{F}_i s_2) = \dim_K(V_1) + \dim_K(V_2 + V_3) = 2i - p^e + p^{e+1} + 2.$$

Case 2: $i < p^{e+1} - p^e + p^{e-1}$, that is, $[p^e, i-p^{e-1}+p^e] \cap [p^{e+1}, i-p^e+p^{e+1}] = \emptyset$. In this case, the degrees of the basis elements of $V_{2,e}$ (which are all distinct) exceed the degree of any element in $V_{2,e-1} + \ldots + V_{2,0} + V_3$, thus $\dim_K V_{2,e} + V_{2,e-1} + \ldots + V_{2,0} + V_3 = \dim_K V_{2,e} + \dim_K V_{2,e-1} + \ldots + V_{2,0} + V_3 = (i - p^{e-1} + p^e + 1) + (i - p^e + 1)$ and

$$\dim_K(\mathcal{F}_i s_2) = (i+1) + (i - p^{e-1} + p^e + 1) + (i - p^e + 1) = 3i - p^{e-1} + 3.$$

We note that, for all i, we have $\dim_K(S_i) \leqslant 4i$, therefore $S = \mathcal{D}s_2$ is holonomic. For all i of the form p^e, we have

$$\lim_{e\to\infty} \frac{\dim_K(S_{p^e})}{p^e} = \lim_{e\to\infty} \frac{3p^e - p^{e-1} + 3}{p^e} = 3 - \frac{1}{p};$$

but, for all i of the form $p^{e+1} - p^e$, we have

$$\lim_{e\to\infty} \frac{\dim_K(S_{p^{e+1}-p^e})}{p^{e+1} - p^e} = \lim_{e\to\infty} \frac{3(p^{e+1} - p^e) - p^{e-1} + 3}{p^{e+1} - p^e} = 3 - \frac{1}{p^2 - p}.$$

Therefore, $\lim_{i\to\infty}(\dim_K(S_i)/i)$ does not exist. $\qquad\square$

REMARK 4.3. A reasonable theory of holonomic modules should include the polynomial ring itself as a holonomic module. However, our example in this section indicates that any such theory of holonomic modules cannot have both the extension property and the existence of multiplicity at the same time. If extensions of holonomic modules are holonomic, then our module $M = \mathcal{D}s_2$ in this section will be holonomic, but as we have seen the multiplicity of M does not exist.

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Mordechai Katzman
Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield
S3 7RH
United Kingdom

M.Katzman@sheffield.ac.uk

Gennady Lyubeznik
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA

gennady@math.umn.edu

Wenliang Zhang
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

wlzhang@umich.edu