# NORMAL ANALYTIC FUNCTIONS AND A QUESTION OF M. L. CARTWRIGHT

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In an address before the London Mathematical Society, M. L. Cartwright asked whether there exists a normal analytic function in $|z| < 1$ with an infinite radial limit at $z = 1$ but with a derivative that has no radial limit at $z = 1$. W. K. Hayman and D. A. Storvick [4] answered this in the affirmative, using a geometric construction to exhibit a univalent function with the required property. In this paper, we give three different explicit examples relevant to Cartwright's question.

## 1. *A logarithmic function with a Blaschke disturbance*

Let $\{z_n\}$ $(0 < z_1 < z_2 < \ldots < 1)$ be an interpolating sequence, and let $B(z) = \prod_{n=1}^{\infty} \left(\frac{z_n - z}{1 - z_n z}\right)$ be the Blaschke product with simple zeros at the points of $\{z_n\}$. Then $B$ is a real-valued function on the segment $[0, 1)$ of the real line, and it changes sign at each point $z_n$. Thus $\{B'(z_n)\}$ is a sequence of real numbers, negative when $n$ is odd, positive when $n$ is even. Since $\{z_n\}$ is an interpolating sequence, there exists a number $\delta > 0$ such that $(1-|z_n|^2)|B'(z_n)| \geqslant \delta$ $(n = 1, 2, \ldots)$ [3; p. 148]. Thus $B'(z_{2n}) \geqslant \delta/(1-z_{2n}^2)$ and $B'(z_{2n+1}) \leqslant -\delta/(1-z_{2n+1}^2)$. Consider the normal (Bloch) function

$$f(z) = \frac{\delta}{2}\log\frac{1+z}{1-z} - B(z).$$

Clearly, $\lim_{x \to 1} f(x) = \infty$. On the other hand, $f'(z_{2n}) \leqslant 0$ and $f'(z_{2n+1}) \geqslant 2\delta/(1-z_{2n+1}^2)$. By continuity, $f'$ vanishes between $z_{2n}$ and $z_{2n+1}$, while $f'(z_{2n+1}) \to \infty$. Thus $f'$ has no finite or infinite radial limit at $z = 1$.

## 2. *An example obtained by integration*

Our second example illustrates a connection between the non-Euclidean distance of consecutive points of an exponential interpolating sequence $\{z_n\}$ and the behaviour of the Blaschke product that vanishes at each point $z_n$. Let $\{z_n\}$ be an exponential interpolating sequence on the positive real line [3; p. 156]. For simplicity let us suppose that $z_n = 1 - c^n$ $(0 < c < 1)$. Let $B(z)$ be the Blaschke product with a simple zero at each $z_n$. Since the pseudo-non-Euclidean distance from $z_n$ to $z_{n+1}$ satisfies the condition

$$\frac{z_{n+1} - z_n}{1 - z_n z_{n+1}} = \frac{1-c}{1+c-c^{n+1}} > \frac{1-c}{1+c},$$

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we can find points $z'_n$ and $z''_{n+1}$ such that $z_n < z'_n < z''_{n+1} < z_{n+1}$,

$$\frac{z'_n - z_n}{1 - z'_n z_n} = \frac{1}{4}\frac{1-c}{1+c} = \frac{z_{n+1} - z''_{n+1}}{1 - z_{n+1} z''_{n+1}},$$

and the non-Euclidean distance $\rho(z'_n, z''_{n+1})$ from $z'_n$ to $z''_{n+1}$ is bounded below by a positive number $d$ depending on $c$ but not on $n$. Since $\{z_n\}$ is an interpolating sequence, there exists a number $\delta > 0$ such that

$$\prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| \frac{z_j - z_k}{1 - z_j z_k} \right| \geqslant \delta \qquad (k = 1, 2, \ldots)$$

[3; p. 148]. Thus, for $z'_n < z < z''_{n+1}$,

$$\begin{aligned}
B^2(z) &= \prod_{j=1}^{\infty} \left( \frac{z_j - z}{1 - z_j z} \right)^2 \\
&\geqslant \left( \frac{z_n - z}{1 - z_n z} \right)^2 \left( \frac{z_{n+1} - z}{1 - z_{n+1} z} \right)^2 \prod_{j=1}^{n-1} \left( \frac{z_j - z_n}{1 - z_j z_n} \right)^2 \prod_{j=n+2}^{\infty} \left( \frac{z_j - z_{n+1}}{1 - z_{n+1} z_j} \right)^2 \\
&\geqslant \left( \frac{1}{4}\frac{1-c}{1+c} \right)^2 \left( \frac{1}{4}\frac{1-c}{1+c} \right)^2 \left( \prod_{\substack{j=1 \\ j \neq n}}^{\infty} \left| \frac{z_j - z_n}{1 - z_j z_n} \right| \right)^2 \left( \prod_{\substack{j=1 \\ n \neq n+1}}^{\infty} \left| \frac{z_j - z_{n+1}}{1 - z_j z_{n+1}} \right| \right)^2 \\
&\geqslant \delta^4 (1-c)^4 / 256 (1+c)^4 \\
&\equiv A > 0 .
\end{aligned}$$

Consider the normal (Bloch) function

$$F(z) = \int_0^z B^2(w)(1-w^2)^{-1} dw .$$

Clearly, $F'(z_n) = 0$ for each $n$. On the other hand

$$\begin{aligned}
F(z_{n+1}) &\geqslant \sum_{j=1}^{n} \int_{z'_j}^{z''_{j+1}} B^2(w)(1-w^2)^{-1} dw \\
&\geqslant \sum_{j=1}^{n} A \int_{z'_j}^{z''_{j+1}} (1-w^2)^{-1} dw \\
&\geqslant \sum_{j=1}^{n} A \, . \, d \\
&= nAd ,
\end{aligned}$$

and therefore $\lim_{x\to 1} F(x) = \infty$. Since $F(x) \to \infty$ as $x \to 1$, there exists a sequence of points $x_n$ on $[0, 1)$ such that $\lim_{n\to\infty} F'(x_n) = \infty$. Thus $\lim_{x\to 1} F'(x)$ fails to exist.

## 3. *A univalent example*

Our third example is inspired by the domain described by Hayman and Storvick; but instead of analyzing a univalent mapping defined by a precisely preassigned domain, we use a univalent function described by a simple formula.

Let $\{\theta_n\}$ denote a decreasing sequence of positive numbers such that $\theta_1 < 1$ and $\sum|\log\theta_n|^{-1/2} < \infty$. For $n = 1, 2, \ldots$, let $g_n(z) = e^{-i\theta_n}\log\frac{1}{1-ze^{i\theta_n}} + e^{i\theta_n}\log\frac{1}{1-ze^{-i\theta_n}}$, where the two logarithmic expressions represent principal values. Obviously the function

$$g(z) = z + \sum_{n=1}^{\infty} g_n(z)/|\log\theta_n|$$

is holomorphic in the unit disc $D$.

In the formula

$$g'(z) = 1 + \sum_{n=1}^{\infty}\left(1/(1-ze^{i\theta_n}) + 1/(1-ze^{-i\theta_n})\right)/|\log\theta_n|,$$

the real part of each term under the summation sign is positive; therefore $g$ is a univalent, close-to convex Bloch function in $D$.

It is easy to see that the domain $g(D)$ consists roughly of $D$ together with pairs of narrow fingers reaching to infinity in the directions $e^{i\theta_n}$ and $e^{-i\theta_n}$. The positivity of $g'$ on the segment $[0, 1)$ guarantees that $\lim_{r\to 1} g(r)$ exists. Since $g(r) \geqslant r + \sum_{n=1}^{m} g_n(r)/|\log\theta_n|$, we have the inequality

$$\lim_{r\to 1} g(r) \geqslant \lim_{m\to\infty}\left(1 + \sum_{n=1}^{m} g_n(1)/|\log\theta_n|\right),$$

which together with $g_n(1) \sim 2|\log\theta_n|$ proves that $\lim_{r\to 1} g(r) = \infty$. Since $g(r) \to \infty$ as $z \to 1$, the derivative $g'(r)$ is not bounded on $[0, 1)$.

We subject the $\theta_n$ to an additional requirement. Assuming that $\theta_1, \theta_2, \ldots, \theta_j$ have been chosen, let $D_j$ denote the set of points in $D$ where $|g_j'(z)| \geqslant |\log\theta_j|^{1/2}$. Since $|g'_j(z)| < |z-e^{i\theta_j}|^{-1} + |z-e^{-i\theta_j}|^{-1}$, the set $D_j$ lies in the union of the two overlapping discs $|z-e^{\pm i\theta_j}| \leqslant 2|\log\theta_j|^{-1/2}$. However $g'_j(1) = 1$ which implies there is an open disc around $z = 1$ which is disjoint from $D_j$. We choose $\theta_{j+1}$ so that $D_{j+1}$ lies in this open disc and is therefore disjoint from $D_j$. Each neighborhood of $z = 1$ contains a point $z_m$ of $[0, 1) - \bigcup_{n=1}^{\infty} D_n$ satisfying $|g_n'(z_m)| < |\log \theta_n|^{1/2}$ for every $m = 1, 2, \ldots$. Therefore,

$$|g'(z_m)| \leqslant 1 + \sum_{n=1}^{\infty} |g_n'(z_m)|/|\log \theta_n|$$

$$< 1 + \sum_{n=1} |\log \theta_n|^{1/2}.$$

This inequality together with $g'$ being unbounded on $[0, 1)$ proves that $g'$ has no radial limit at $z = 1$ and completes the proof.

The function constructed by Hayman and Storvick in [4] has infinite planar area and maps the disc onto a Jordan domain (on the Riemann sphere). A slight modification of our example produces a function which has finite planar area and which also maps the disc onto a Jordan domain ( on the Riemann sphere). It suffices to move the logarithmic branch points slightly beyond the unit circle. To be precise let

$$G(z) = z + \sum_{n=1}^{\infty} \left\{ e^{-i\theta_n} \log \frac{1}{1+\theta_n^2 - ze^{i\theta_n}} + e^{i\theta_n} \log \frac{1}{1+\theta_n^2 - ze^{-i\theta_n}} \right\} \Big/ |\log \theta_n|.$$

Then $G(z) \to \infty$ as $z \to 1$. If $\{\theta_n\} \to 0$ fast enough, then the sequence $\{G'(1-\theta_n^2)\}$ is bounded, the stereographic image of $G(D)$ is a Jordan domain, and the domain $G(D)$ has finite planar area.

## 4. *Concluding remarks*

We note that the existence of a normal analytic function in $D$ for which $\lim_{z \to 1} f(z)$ is finite while $\lim_{x \to 1} f'(x)$ fails to exist is easily established. The bounded function $(1-z)^{1+i}$ tends to zero as $z \to 1$ radially while its derivative (which is also bounded) has no radial limit at 1. In fact, there exists a univalent function $f$ and a set $E$ of measure $2\pi$ such that for each $\theta$ in $E$ the radial limit $f(re^{i\theta})$ exists while $\lim_{r \to 1} f'(re^{i\theta})$ fails to exist. Simply let

$$f(z) = \int_0^z \exp\left(\frac{1}{8} \sum_{n=1}^{\infty} w^{2^n}\right) dw.$$

Since $(1-|z|^2)|f''(z)/f'(z)| < 1/2$, the function $f(z)$ is univalent [6; p. 172] and therefore has a radial limit almost everywhere [2; p. 56]. On the other hand, $\log f'(z) = \frac{1}{8} \sum_{n=1}^{\infty} z^{2^n}$ has no finite radial limits by the high-indices theorem of Hardy and Littlewood (see [1] for an elegant proof). Thus $f'(z)$ can have only 0 and $\infty$ as radial limits. Since $f'(z)$ is normal, each radial limit is an angular limit. Thus $f'$ can have only 0 and $\infty$ as angular limits. It follows from Privalov's uniqueness theorem for angular limits [2; p. 146] that the radial limits 0 and $\infty$ can occur only on a set of measure 0. Therefore there exists a set $E$ of measure $2\pi$ such that $\lim_{r \to 1} f(re^{i\theta})$ exists for all $\theta$ in $E$ while $\lim_{r \to 1} f'(re^{i\theta})$ exists for no $\theta$ in $E$.

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