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Higher commutativity and nilpotency in finite groups

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Abstract

Given a finite group and an integer q, we consider the poset of nilpotent subgroups of class less than q and its corresponding coset poset. These posets give rise to a family of finite Dirichlet series parameterized by the nilpotency class of the subgroups, which in turn reflect probabilistic and topological invariants determined by these subgroups. Connections of these series to filtrations of the classifying space of a group are discussed.

1. *Introduction*

In 1936, Hall addressed to some extent the problem of computing the probability that a randomly chosen ordered s-tuple of elements in a finite group G generates the group. Hall proved that this probability, denoted by $P(G,s)$, can be expressed as a finite Dirichlet series. For instance (see [**3**]),

$$P(\mathrm{PSL}(2,7),s) = 1 - \frac{14}{7^s} - \frac{8}{8^s} + \frac{21}{21^s} + \frac{28}{28^s} + \frac{56}{56^s} - \frac{84}{84^s}.$$

Hall's results allow us to think of this as a function over the complex numbers. Several properties of this series and its connection to the structure of the group have been studied; see [**12**] and the references contained therein. In [**4**], this function was related to the coset poset of a finite group. This poset is the set of all proper cosets of a group ordered by inclusion. If we denote this poset by $\mathcal{C}(G)$, then Brown (following Bouc) proved that

$$P(G,-1) = 1 - \chi(\mathcal{C}(G)).$$

The reciprocal of $P(G,s)$ is called the probabilistic zeta function of G (see [**3**]).

We can twist the probability question to ask the following: given a finite group G, what is the probability that a randomly chosen ordered s-tuple generates an abelian subgroup of G? If we denote this probability by $P_2(G,s)$, then we have

$$P_2(G,s) = \frac{|\mathrm{Hom}(\mathbb{Z}^s,G)|}{|G|^s},$$

since the set of ordered commuting s-tuples in G can be identified with the set of group homomorphisms $\mathrm{Hom}(\mathbb{Z}^s,G)$. The number $P_2(G,s)$ was also studied in [**10**] under the name of multiple commutativity degree. The set (space if G has some topology) $\mathrm{Hom}(\mathbb{Z}^s,G)$ appears in different contexts of mathematics such as Differential Geometry, Group Cohomology and K-theory (see [**1**]). The sets $\mathrm{Hom}(\mathbb{Z}^s,G)$, $s \geqslant 0$, can be assembled to form a simplicial space. The realization of this simplicial space is denoted by $B(2,G)$ and it is the first layer of a filtration of the classifying space BG. The space $B(2,G)$ turns out to capture delicate information about the group G. For instance, when G is finite of odd order, the pull-back fibration $E(2,G) \to B(2,G)$, obtained from the universal G-bundle over BG, captures the celebrated Feit–Thompson Theorem; and from the point of view of group cohomology, it can be seen that there is a monomorphism $H^*(BG;\mathbb{F}_p) \to H^*(B(2,G);\mathbb{F}_p)$ modulo nilpotent elements.

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In this paper, we will show that the space $B(2,G)$ is closely related to the probability function $P_2(G,s)$.

The key is to show that the space $E(2,G)$ is homotopy equivalent to the poset $\mathcal{C}_2(G)$ of all cosets of proper, abelian subgroups of G. Moreover, we will show that $P_2(G,s)$ is a finite Dirichlet series determined by the poset of all abelian subgroups of G, and that $P_2(G,-1) = \chi(E(2,G))$.

The poset $\mathcal{C}_2(G)$ has some intriguing properties. Brown in his paper [4] asked: Can we characterize finite solvable groups in terms of the combinatorial topology of the coset poset? The answer is 'yes' for groups of odd order if instead we consider the poset $\mathcal{C}_2(G)$ since, as pointed out in [2], the following statements are equivalent.

(1) Every group of odd order is solvable.
(2) The map $H_1(\mathcal{C}_2(G)) \to H_1(B(2,G))$ is not onto when G is a group of odd order.

This probabilistic setting can be extended to higher classes of nilpotency. Consider the number

$$\frac{|\{(g_1,\ldots,g_s) \in G^s : \Gamma^q(\langle g_1,\ldots,g_s\rangle) = 1\}|}{|G|^s},$$

where Γ^q stands for the qth-stage of the lower central series. This latter ratio is precisely the probability that a randomly chosen s-tuple generates a nilpotent subgroup of class less than q. Likewise, for each $q \geqslant 2$ there is a space $E(q,G)$ that is the realization of the simplicial space $\mathrm{Hom}(F_n/\Gamma_q(F_n),G)$, $n \geqslant 0$. The main result of this paper is the following.

THEOREM 1.1. *Let $q \geqslant 2$ and G be a finite group such that $\Gamma^q(G) \neq 1$. Then,*

(i) *the space $E(q,G)$ is homotopy equivalent to the coset poset of all proper, nilpotent subgroups of class less than q;*
(ii) *there is a finite Dirichlet series $P_q(G,s)$ such that*

$$P_q(G,s) = \frac{|\mathrm{Hom}(F_s/\Gamma^q(F_s),G)|}{|G|^s},$$

where s is a non-negative integer and F_s is the free group on s generators;
(iii) $\chi(E(q,G)) = P_q(G,-1)$.

The paper is organized as follows. In Section 2, we cover some background. In Section 3, we relate the coset posets to probabilistic invariants via Möbius functions. In Section 4, we explore some properties of the finite Dirichlet series afforded by nilpotent subgroups. In Section 5, we provide an explicit formula to compute the number of commuting elements in a symmetric group and in Section 6, we present upper and lower bounds for the probabilistic invariants aforementioned. In this paper, a group will always be finite unless otherwise stated.

2. *Preliminaries*

Recall that the lower central series of a group K is defined inductively by $\Gamma^1(K) = K$ and $\Gamma^q(K) = [\Gamma^{q-1}(K), K]$. We say that a group K is nilpotent of class c if $\Gamma^{c+1}(K) = 1$ and $\Gamma^q(K) \neq 1$ for $q \leqslant c$. By convention, we will write $\Gamma^\infty(K) = 1$. Let G be a group, and fix $q \geqslant 2$. We define

$$B_n(G,q) = \mathrm{Hom}(F_n/\Gamma^q_n, G)$$

and

$$E_n(G,q) = G \times \mathrm{Hom}(F_n/\Gamma^q_n, G),$$

where F_n is the free group on n generators and $\Gamma_n^q = \Gamma^q(F_n)$. Note that $B_n(G,q) \subseteq G^n$ and $E_n(G,q) \subseteq G^{n+1}$. The following functions provide the sets $B_*(G,q)$ and $E_*(G,q)$ with a simplicial structure: $d_i : E_n(q,G) \to E_{n-1}(q,G)$ for $0 \leqslant i \leqslant n$, and $s_j : E_n(q,G) \to E_{n+1}(q,G)$ for $0 \leqslant j \leqslant n$, are given by

$$d_i(g_0,\ldots,g_n) = \begin{cases} (g_0,\ldots,g_i \cdot g_{i+1},\ldots,g_n), & 0 \leqslant i < n, \\ (g_0,\ldots,g_{n-1}), & i = n \end{cases}$$

and

$$s_j(g_0,\ldots,g_n) = (g_0,\ldots,g_j,e,g_{j+1},\ldots,g_n).$$

Similarly, we have maps d_i and s_j for $B_*(q,G)$ defined in the same way, except that the first coordinate g_0 is omitted and the map d_0 takes the form $d_0(g_1,\ldots,g_n) = (g_2,\ldots,g_n)$. Note that G acts on $E_*(q,G)$ by multiplication on the first coordinate $g(g_0,g_1,\ldots,g_n) = (gg_0,g_1,\ldots,g_n)$, with orbit space homeomorphic to $B_*(q,G)$. We denote by $B(q,G)$ and $E(q,G)$ the geometric realizations of the aforementioned simplicial sets. Thus, we have a G-bundle $E(q,G) \to B(q,G)$ and a fibration $E(q,G) \to B(q,G) \to BG$ for each $q \geqslant 2$. Unlike the classical situation, the space $E(q,G)$ is not necessarily contractible. Note that if $E(q,G)$ is contractible, then $B(q,G) \simeq BG$. We can also consider the simplicial set of $BN_*(G)$ whose set of n-simplices is defined as

$$\{(g_1,\ldots,g_n) | \langle g_1,\ldots,g_n\rangle \neq G\}.$$

Likewise, we can define $NE_*(G)$ as the simplicial set whose n-simplices is defined as $G \times BN_n(G)$. The simplicial structure of these latter two spaces is the same as that of $B(q,G)$ and $E(q,G)$, respectively. We will denote their geometric realization by $BN(G)$ and $EN(G)$. Note that there is a fibration $EN(G) \to BN(G) \to BG$.

The inclusions $F_n/\Gamma_n^{q+1} \subseteq F_n/\Gamma_n^q$ induce the following filtrations:

$$E(2,G) \subseteq \cdots \subseteq E(q,G) \subseteq E(q+1,G) \subseteq \cdots \subseteq E(\infty,G) = EG$$

and

$$B(2,G) \subseteq \cdots \subseteq B(q,G) \subseteq B(q+1,G) \subseteq \cdots \subseteq B(\infty,G) = BG.$$

Moreover, if G is not nilpotent, then $B(q,G) \subseteq BN(G) \subset BG$, and $E(q,G) \subseteq EN(G) \subset EG$ for all $q \geqslant 2$.

Definition 1. Let $q \geqslant 2$. We define $\mathcal{N}_q(G)$ as the poset of all proper, nilpotent subgroups of G of class less than q, ordered by inclusion. When $q = \infty$, this poset becomes the poset of all subgroups of G, which we will denote by $\mathcal{L}(G)$.

Consider the functor $F_q : \mathcal{N}_q(G) \to G\text{-}Sets$ given by $H \mapsto G/H$. In [**2**], it was proved that if $\Gamma^q(G) \neq 1$, then $B(q,G) \simeq \operatorname{hocolim}(F_q)_{hG}$, that is,

$$B(q,G) \simeq \operatorname*{hocolim}_{H \in \mathcal{N}_q(G)} BH.$$

Likewise, we have

$$NB(G) \simeq \operatorname*{hocolim}_{H \in \mathcal{L}(G) - \{G\}} BH.$$

The isomorphism $(\operatorname{hocolim} F_q)_{hG} \cong \operatorname{hocolim}(F_q)_{hG}$ (see [**5**]) shows that

$$E(q,G) \simeq \operatorname*{hocolim}_{H \in \mathcal{N}_q(G)} G/H$$

and

$$EN(G) \simeq \operatorname*{hocolim}_{H \in \mathcal{L}(G) - \{G\}} G/H.$$

Suppose that H and K are subgroups of G, so one can see that $xH \subseteq yK$ if and only if $H \subseteq K$ and $xK = yK$. This allows us to see that an inclusion $H \subseteq K$ corresponds to a projection $G/H \to G/K$ so that the latter defines an inclusion in the poset of all cosets of G. The poset of all proper cosets will be denoted by $\mathcal{C}(G)$.

Definition 2. Let $q \geqslant 2$. We define $\mathcal{C}_q(G)$ as the poset of proper cosets of nilpotent subgroups of G of class less than q. That is, $\mathcal{C}_q(G) = \{xH \in \mathcal{C}(G) | \Gamma^q(H) = 1\}$. When $q = \infty$, this poset becomes the poset of all proper cosets of G, that is, $\mathcal{C}_\infty(G) = \mathcal{C}(G)$.

Thus, for finite q, we have the equivalences: $E(q,G) \simeq \mathcal{C}_q(G)$ if $\Gamma^q(G) \neq 1$, and $EN(G) \simeq \mathcal{C}(G)$. If $\Gamma^q(G) = 1$, then $E(q,G) \simeq EG$. But note that $EG = E(\infty, G)$ is not, in general, homotopy equivalent to $\mathcal{C}(G)$. Note that $B(q,G)$ is always connected since $B_0(q,G) = \{1\}$. This implies that $E(q,G)$ is always connected. On the other hand, it was proved in [4] that $EN(G)$ is connected if and only if G is not isomorphic to a cyclic group of order a prime power. So, G is not isomorphic to a cyclic group of prime power order if and only if $BN(G)$ is connected.

3. *The coset poset*

We can make our identification of $E(q,G)$ more efficient by considering only the maximal subgroups of G of nilpotency class less than q.

Definition 3. Let $\mathcal{M}_q(G)$ be the poset determined by the maximal nilpotent subgroups of G of class less than q; that is, any subgroup in $\mathcal{M}_q(G)$ is an intersection of maximal nilpotent subgroups of G of class less than q. The corresponding coset poset is defined as $\mathcal{MC}_q(G) = \{xH \in \mathcal{C}(G) \,|\, H \in \mathcal{M}_q(G)\}$.

Recall the following lemma.

Lemma 3.1 (Quillen, Webb–Thevenaz). *Let $f : P \to Q$ be a G-map of G-posets. If the subposet $f_{\geqslant y} = \{x \in P \,|\, f(x) \geqslant y\}$ is G_y-contractible for all $y \in Q$, then f is a homotopy equivalence.*

Remark 1. The group G acts on the coset poset in two ways: (1) by translation and (2) by conjugation $g(xH)g^{-1} = gxg^{-1}(gHg^{-1})$. We also have the action of $\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$ given by $(g\sigma) \cdot xH = g\sigma(x)\sigma(H)$. Moreover, the inclusion $\mathcal{N}_q(G) \subseteq \mathcal{N}_{q+1}(G)$ induces a G-equivariant inclusion $\mathcal{C}_q(G) \subseteq \mathcal{C}_{q+1}(G)$.

Note that if $f : \mathcal{MC}_q(G) \to \mathcal{C}_q(G)$ denotes the inclusion map, then, for $xH \in \mathcal{C}_q(G)$, the poset $f_{\geqslant xH}$ has an initial element, namely the coset xM_H, where M_H is the intersection of all the subgroups in $\mathcal{M}_q(G)$ containing H. Moreover, one can see that xM_H is fixed by G_{xH} (under both the translation and conjugation action). Thus, $\mathcal{C}_q(G) \simeq \mathcal{MC}_q(G)$.

Proposition 3.2. *For any finite group G such that $\Gamma^q(G) \neq 1$, there are homotopy equivalences*

$$E(q,G) \simeq \mathcal{MC}_q(G)$$

and

$$B(q,G) \simeq \operatorname*{hocolim}_{A \in \mathcal{M}_q(G)} BA.$$

EXAMPLE 1. Let $q = 2^n$ with $n \geqslant 2$. Every pair of maximal abelian subgroups of $\mathrm{SL}(2,\mathbb{F}_q)$ has as intersection the trivial subgroup (see Example 2). Thus,

$$\begin{aligned} B(2,\mathrm{SL}(2,\mathbb{F}_q)) \simeq & \left[\bigvee^{q+1} B(\mathbb{Z}/q)^n \right] \\ & \vee \left[\bigvee^{(1/2)q(q-1)} B\mathbb{Z}/(q+1) \right] \vee \left[\bigvee^{(1/2)q(q+1)} B\mathbb{Z}/(q-1) \right] \end{aligned}$$

and

$$E(2,\mathrm{SL}(2,\mathbb{F}_q)) \simeq \bigvee_{(q^2-1)^2(q+1)-q^2(q^2+1)+1} S^1.$$

We will now focus on computing the Euler characteristic of $E(q,G)$. For this purpose, we will use Möbius functions.

DEFINITION 4. If $\mathcal{X}$ is a finite poset, then the Möbius function of this poset is a function $\mu_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \to Z$ determined recursively by the following equations:

$$\mu_{\mathcal{X}}(a,a) = 1,$$

and, for $a < b$,

$$\mu_{\mathcal{X}}(a,b) = -\sum_{a \leqslant u < b} \mu_{\mathcal{X}}(a,u).$$

Let $\mathcal{X}$ be a subposet of the poset of all subgroups of G, and assume that G is in $\mathcal{X}$. Then Hall showed that $\mu_{\mathcal{X}}(H,G)$ is computed by a signed sum of the number of chains of subgroups in $\mathcal{X}$ from H to G. We say that $H = K_0 < K_1 < \cdots < K_n = G$ has length n and it is counted with the sign $(-1)^n$. Now (following [4]), note that a chain of the form $H = K_0 < K_1 < \cdots < K_n = G$ corresponds to $|G : H|$ chains of cosets $C_0 < C_1 < \cdots < C_{n-1} < G$, where $C_0 = xH$ and x is a representative of the $|G : H|$ cosets of H in G. As the chains $C_0 < C_1 < \cdots < C_{n-1}$ form the simplices of the simplicial complex afforded by $\mathcal{C}_{\mathcal{X}}$, we see that

$$1 - \chi(\mathcal{C}_{\mathcal{X}}) = \sum_{H \leqslant G} \mu_{\mathcal{X}}(H,G)|G : H|,$$

and thus

$$\chi(\mathcal{C}_{\mathcal{X}}) = -\sum_{H \in \mathcal{X} - \{G\}} \mu_{\mathcal{X}}(H,G)|G : H|.$$

DEFINITION 5. Let $q \geqslant 2$. We define $\mathcal{L}_q(G)$ as the poset $\mathcal{N}_q(G) \cup \{G\}$, and μ_q as the Möbius function of the poset $\mathcal{L}_q(G)$.

REMARK 2. Hereinafter, we will assume that $\Gamma^q(G) \neq 1$ (in particular, q has to be finite).

PROPOSITION 3.3. *Let G be a finite group. Then*

$$\chi(E(q,G)) = 1 - \sum_{H\in\mathcal{L}_q(G)} \mu_q(H,G)|G:H| = - \sum_{H\in\mathcal{M}_q(G)} \mu_q(H,G)|G:H|.$$

REMARK 3. In the formula above, the sum can run either over $\mathcal{M}_q(G)$ or $\mathcal{N}_q(G)$. There is a topological reason for this as we have seen, but there is a combinatorial one which follows from the fact that if $\mathcal{X}$ is a subposet of the poset of all subgroups of G (assume $G \in \mathcal{X}$), then $\mu_{\mathcal{X}}(H,G) = 0$ if H is not an intersection of maximal subgroups in $\mathcal{X}$ (see [**9**, Theorem 2.3]). Another important property that facilitates the computation of the Möbius function is that if $\mathcal{X}$ is invariant under $\mathrm{Aut}(G)$, then so is $\mu_{\mathcal{X}}$.

EXAMPLE 2. A finite group G is said to be *transitively commutative*, TC for short, if $[g,h] = 1 = [h,k]$ implies $[g,k] = 1$ for all non-central elements. The poset of maximal abelian subgroups of a TC-group G consists of the maximal abelian subgroups and the center of G which is the intersection of any pair of distinct maximal abelian subgroups of G. Then we obtain the formula

$$1 - \chi(E(2,G)) = 1 - |G:Z(G)| + \sum_{1\leqslant i\leqslant N} (|G:Z(G)| - |G:M_i|),$$

where $M_1,\dots,M_N$ are the maximal abelian subgroups of G (see [**2**]). The family of TC-groups includes groups such as any non-abelian group of order less than 24, dihedral groups, quaternion groups and $\mathrm{SL}(2,\mathbb{F}_{2^n})$ with $n \geqslant 2$; see [**14**, p. 519], for a classification of these groups. When G is a TC-group, it turns out that $E(2,G)$ has the homotopy type of a wedge of $1-\chi(E(2,G))$ circles (see [**2**]).

4. *Probability*

Hall defined $\phi(G,s)$ as the number of ordered s-tuples $(g_1,\dots,g_s) \in G^s$ such that $\langle g_1,\dots,g_s\rangle = G$. Thus, the probability that a randomly chosen s-tuple generates G is given by

$$P(G,s) = \frac{\phi(G,s)}{|G|^s}.$$

Analogously, we can define $P_q(G,s)$ as the probability that a randomly chosen ordered s-tuple generates a nilpotent subgroup of class less than q. Of course, we have

$$P_q(G,s) = \frac{|\mathrm{Hom}(F_s/\Gamma^q, G)|}{|G|^s},$$

since $\mathrm{Hom}(F_s/\Gamma^q, G) = \{(g_1,\dots,g_s) \in G^s | \Gamma^q(\langle g_1,\dots,g_s\rangle) = 1\}$.

LEMMA 4.1. *Let μ be the Möbius function of a poset $\mathcal{X}$, and define the function $\tilde{\mu}$ inductively by*

$$\tilde{\mu}(a,a) = 1,$$
$$\tilde{\mu}(a,b) = - \sum_{a<u\leqslant b} \mu(u,b).$$

Then $\mu = \tilde{\mu}$.

PROPOSITION 4.2. *Let μ_q be the Möbius function of the poset $\mathcal{L}_q(G) = \mathcal{N}_q(G) \cup \{G\}$. Then*

$$P_q(G, s) = -\sum_{H \in \mathcal{N}_q(G)} \frac{\mu_q(H, G)}{|G : H|^s}.$$

Proof. Note that

$$\sum_{H \in \mathcal{N}_q(G)} \mu_q(H, G)|H|^s = \sum_{H \in \mathcal{N}_q(G)} \sum_{(g_1, \ldots, g_s) \in H^s} \mu_q(H, G).$$

If we interchange the sums, then we get

$$\sum_{H \in \mathcal{N}_q(G)} \mu_q(H, G)|H|^s = \sum_{(g_1, \ldots, g_s) \in \mathrm{Hom}(F_s/\Gamma_s^q, G)} \sum_{\langle g_1, \ldots, g_s \rangle \subseteq H} \mu_q(H, G),$$

where the inner sum runs over all $H \in \mathcal{N}_q(G)$ containing $\langle g_1, \ldots, g_s \rangle$. Recall that $\mu_q(G, G) = 1$, thus

$$\sum_{H \in \mathcal{N}_q(G)} \mu_q(H, G)|H|^s = \sum_{(g_1, \ldots, g_s) \in \mathrm{Hom}(F_s/\Gamma_s^q, G)} \left(-1 + \sum_{\langle g_1, \ldots, g_s \rangle \subseteq H \subseteq G} \mu_q(H, G) \right).$$

By the previous lemma, the inner sum is zero. Therefore,

$$\sum_{H \in \mathcal{N}_q(G)} \mu_q(H, G)|H|^s = \sum_{(g_1, \ldots, g_s) \in \mathrm{Hom}(F_s/\Gamma_s^q, G)} (-1) = -|\mathrm{Hom}(F_s/\Gamma_s^q, G)|.$$

The result follows. □

REMARK 4. The previous result allows to regard $P_q(G, s)$ as a function defined over the complex numbers. The reciprocal of the function $P_q(G, s)$ may be called the probabilistic zeta function of G of class q.

EXAMPLE 3. If G is a TC-group with maximal abelian subgroups $M_1, \ldots, M_N$, then we have

$$P_2(G, s) = \frac{1 - N}{|G : Z(G)|^s} + \sum_{i=1}^{N} \frac{1}{|G : M_i|^s}.$$

We have the following computations:

(i)
$$P_2(\Sigma_3, s) = \frac{1}{2^s} + \frac{3}{3^s} - \frac{3}{6^s}.$$

(ii) If D_{2n} is the dihedral group of order $2n$ and $n = 2k \geqslant 4$, then
$$P_2(D_{2n}, s) = \frac{1}{2^s} + \frac{k}{k^s} - \frac{k}{n^s}.$$

(iii)
$$P_2(A_4, s) = \frac{1}{3^s} + \frac{4}{4^s} - \frac{4}{12^s}.$$

(iv)
$$P_2(A_5, s) = \frac{6}{12^s} + \frac{5}{15^s} + \frac{10}{20^s} - \frac{20}{60^s},$$

whereas from [3], we have

$$P(A_5, s) = 1 - \frac{5}{5^s} - \frac{6}{6^s} - \frac{10}{10^s} + \frac{20}{20^s} + \frac{60}{30^s} - \frac{60}{60^s}.$$

(v)

$$P_2(\Sigma_4, s) = \frac{7}{6^s} + \frac{4}{8^s} - \frac{6}{12^s} - \frac{4}{24^s}.$$

(vi)

$$P_2(\mathrm{PSL}(2,7), s) = \frac{8}{24^s} + \frac{35}{42^s} + \frac{28}{56^s} - \frac{42}{84^s} - \frac{28}{168^s}.$$

(vii)

$$P_3(\Sigma_4, s) = \frac{3}{3^s} - \frac{2}{6^s} + \frac{4}{8^s} - \frac{4}{24^s}.$$

PROPOSITION 4.3. *If $q \geqslant 2$, then $\chi(E(q,G)) = P_q(G,-1)$.*

COROLLARY 4.4. *Let $m_q(G)$ be the greatest common divisor of the indices in G of the maximal subgroups in $\mathcal{N}_q(G)$. Then $m_q(G)$ divides $\chi(E(q,G))$.*

EXAMPLE 4. If G is a non-abelian p-group, then $\chi(E(2,G))$ is divisible by p^α, where p^α is the index of a maximal abelian subgroup of G.

Let us consider the simple group $\mathrm{PSL}(2,p)$, where p is a prime number greater than 3. The structure of this group is well known. We can write the order of $\mathrm{PSL}(2,p)$ as $2pqr$, where $q = \frac{1}{2}(p-1)$ and $r = \frac{1}{2}(p+1)$. The numbers p, q, r are relatively prime. The maximal abelian subgroups of $\mathrm{PSL}(2,p)$ are $\mathbb{Z}/p, \mathbb{Z}/q, \mathbb{Z}/r$ and D_4, where D_{2n} is the dihedral group of order $2n$. This information yields the following result.

PROPOSITION 4.5. *Let p be a prime number greater than 3. Then*

$$\begin{aligned} P_2(\mathrm{PSL}(2,p), s) = {} & \frac{p+1}{(2qr)^s} + \frac{pr}{(2pr)^s} + \frac{pq}{(2pq)^s} + \frac{(1/6)pqr}{((1/2)pqr)^s} \\ & - \frac{(1/2)pqr}{(pqr)^s} - \frac{p + pr + pq - pqr/3}{(2pqr)^s}. \end{aligned}$$

REMARK 5. Note that the isomorphism $\mathrm{Hom}(A, B \times C) \cong \mathrm{Hom}(A,B) \times \mathrm{Hom}(A,C)$ induces a homeomorphism $E(q, G \times H) \cong E(q,G) \times E(q,H)$, and produces the identity

$$P_q(G \times H, s) = P_q(G,s) P_q(H,s).$$

This tells us that

$$\chi(E(q, G \times H)) = \chi(E(q,G)) \chi(E(q,H)),$$

which is simpler than the formula obtained in [4] for the poset of proper cosets. What makes the ordinary coset poset more complex with respect to products is that we do not have, in general, an isomorphism between $BN_n(G \times H)$ and $BN_n(G) \times BN_n(H)$.

This latter formula allows one to compute the Euler characteristic of $\mathcal{C}_q(G)$ for any nilpotent group G in terms of that of its Sylow supgroups.

Example 5. Let G be the central product of Q_8 with itself. Here G is an extraspecial group of order 32. The maximal abelian subgroups of G have order 8 and the center of G is cyclic of order 2. It follows that $E(2,G)$ has the homotopy type of a two-dimensional complex. We can see that

$$\mu_2(H,G)=\begin{cases}-1 & \text{if } |H|=8,\\ 2 & \text{if } |H|=4,\\ -16 & \text{if } |H|=2,\end{cases}$$

and find that $\chi(E(2,G))=76$. We also have

$$P_2(G,s)=\frac{15}{4^s}-\frac{30}{8^s}+\frac{16}{16^s}.$$

We can also compute the homology of $E(2,G)$ by using MAGMA:

$$H_*(E(2,G);\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{if } *=0,\\ \mathbb{Z}/2 & \text{if } *=1,\\ \mathbb{Z}^{75} & \text{if } *=2,\\ 0 & \text{if } *\geqslant 3.\end{cases}$$

Note that this is the same as the homology of $\mathbb{R}P^2\vee\bigvee^{75}S^2$.

Remark 6. This latter example shows that $E(q,G)$ does not need to have the homotopy type of a bouquet of spheres when G is solvable.

Remark 7. Note that $P_q(G,1)=1$ for all G. Define

$$R_q(G,s)=1-P_q(G,s)=\sum_{\mathcal{N}_q(G)\cup\{G\}}\frac{\mu_q(H,G)}{|G:H|^s}.$$

Thus $-R_q(G,-1)=\tilde{\chi}(\mathcal{C}_qG))$, the reduced Euler characteristic of $\mathcal{C}_q(G)$. This function $R_q(G,s)$ has a probabilistic interpretation: $R_q(G,s)$ is the probability that a randomly chosen ordered s-tuple does not generate a nilpotent subgroup of class less than q. The inclusions $\mathrm{Hom}(F_n/\Gamma_n^q,G)\subseteq\mathrm{Hom}(F_n/\Gamma_n^{q+1},G)$ show that, for all integers $s\geqslant 1$, we have

$$R_2(G,s)\geqslant\cdots\geqslant R_q(G,s)\geqslant R_{q+1}(G,s)\geqslant\cdots.$$

Likewise, the function $P_q(G,s)-P_{q-1}(G,s)$ is the probability that a randomly chosen s-tuple generates a nilpotent subgroup of class $q-1$.

Remark 8. If G is finite, then the series $P_q(G,s)$ stabilize, that is, there is some q_G such that $P_{q_G}(G,s)=P_q(G,s)$ for all $q\geqslant q_G$. The integer q_G is the smallest integer so that $\Gamma^{q_G}(H)=1$ for all proper, nilpotent subgroups of G.

4.1. *Factorizations*

The ring of finite Dirichlet series with integer coefficients

$$\mathcal{R}=\left\{\sum_{n\geqslant 0}\frac{a_n}{n^s}\colon a_n\in\mathbb{Z},\ a_n\neq 0 \text{ for finitely many subindices } n\right\}$$

is a unique factorization domain as it is a polynomial ring over $\mathbb{Z}$ in $1/2^s, 1/3^s, 1/5^s, \ldots$. It is then natural to study the factorization of $P_q(G,s)$ and $R_q(G,s)$ in $\mathcal{R}$.

EXAMPLE 6. (i) Recall that D_{2n} is nilpotent if and only if n is a power of 2. Suppose that $n = 2^r$ and $r > 1$; so D_{2n} has class r. Then

$$R_{r-k+1}(D_{2n}, s) = 1 - \frac{1}{2^s} - \frac{2^k}{2^{ks}} + \frac{2^k}{(2^{k+1})^s} = -\left(1 - \frac{1}{2^s}\right)\prod_{d|k} \Phi_d\left(\frac{2}{2^s}\right),$$

where Φ_d is the dth cyclotomic polynomial, and $1 \leqslant r-k < r$. Moreover,

$$E(r-k+1, D_{2n}) \simeq \bigvee_{2^{2k}-1} S^1$$

for $1 \leqslant r-k < r$.

(ii) If G is the central product of Q_8 with itself, then

$$R_2(G,s) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{2}{2^s}\right)\left(1 + \frac{3}{2^s} - \frac{8}{4^s}\right).$$

(iii) If M_{11} is the Mathieu group of degree 11, then

$$\begin{aligned} R_2(M_{11}, s) = 1 &- \frac{144}{720^s} - \frac{55}{880^s} - \frac{495}{990^s} - \frac{660}{1320^s} - \frac{396}{1584^s} \\ &+ \frac{330}{1980^s} + \frac{660}{2640^s} + \frac{1980}{3960^s} - \frac{561}{7920^s} \end{aligned}$$

is irreducible.

(iv) If $q = 2^n$ and $n \geqslant 2$, then

$$R_2(\mathrm{SL}(2,\mathbb{F}_q), s) = 1 + \frac{q(q+1)}{[q(q-1)]^s} - \frac{q+1}{[q^2-1]^s} - \frac{(1/2)q(q-1)}{[q(q-1)]^s} - \frac{(1/2)q(q+1)}{[q(q+1)]^s}.$$

It has been checked, for several small values of n, that $R_2(\mathrm{SL}(2,\mathbb{F}_q), s)$ is irreducible.

CONJECTURE 1. If G is a finite simple group, then $R_2(G,s)$ is irreducible in $\mathcal{R}$.

Note that if G is nilpotent, then $P_q(G,s)$ is reducible for all finite $q \geqslant 2$. More precisely, if the Sylow subgroups of G are $P_1, \ldots, P_n$, then $P_q(G,s) = P_q(P_1,s) \cdots P_q(P_n,s)$.

5. *Commuting tuples in the symmetric group*

Note that $|\mathrm{Hom}(\mathbb{Z}^2, \Sigma_n)| = n!p(n)$, where $p(n)$ is the number of partitions of n. Thus,

$$P_2(\Sigma_n, 2) = \frac{p(n)}{n!}.$$

Computing the number of commuting s-tuples in Σ_n is more elaborate than the previous observation as we will see. Let $\mathcal{P}_n$ be the set of all partitions of n. A partition of n will be denoted by $1^{a_1}2^{a_2}\cdots n^{a_n}$, where a_k is the number of k-cycles in the partition. To compute $|\mathrm{Hom}(\mathbb{Z}^s, \Sigma_n)|$, one has to note that an element in $\mathrm{Hom}(\mathbb{Z}^s, \Sigma_n)$ can be identified with a $\mathbb{Z}^s$-set of degree n. A $\mathbb{Z}^s$-set of degree n decomposes into transitive $\mathbb{Z}^s$-sets of degree a_i so that $1^{a_1}2^{a_2}\cdots n^{a_n}$ is a partition of n. A transitive $\mathbb{Z}^s$-set of degree r can be identified with a subgroup of $\mathbb{Z}^s$ of index r. Let $j_r(\mathbb{Z}^s)$ be the number of subgroups of $\mathbb{Z}^s$ of index r. The number $j_r(\mathbb{Z}^s)$ turns out to be finite for all s and r, and can be computed by using the following two formulae (see [**11**, Section 15.2; **15**]):

$$j_r(\mathbb{Z}^s) = \sum_{r_1\cdots r_s = r} r_2 r_3^2 \cdots r_s^{s-1}$$

and

$$j_r(\mathbb{Z}^s) = \sum_{d|r} j_d(\mathbb{Z}^{s-1}).$$

Thus, we have the following result.

Proposition 5.1. *The number of commuting s-tuples in Σ_n is given by*

$$|\mathrm{Hom}(\mathbb{Z}^s, \Sigma_n)| = \sum_{1^{a_1}2^{a_2}\cdots n^{a_n} \in \mathcal{P}_n} \frac{n!}{\prod_{i=1}^n i^{a_i} a_i!} \prod_{i=1}^n j_i(\mathbb{Z}^s)^{a_i}.$$

6. *Conjugacy classes, lower bounds and upper bounds*

Another important invariant of a group is the number of conjugacy classes of ordered commuting n-tuples. If $n \geqslant 1$, then we denote by $k_n(G)$ the number of conjugacy classes of commuting n-tuples in G. In this section, we will use the letter n instead of s to stress the fact that all the results concerning P_2 are for non-negative integers.

Remark 9. As we mentioned in Section 1, the number $P_2(G, n)$ is also called multiple commutativity degree. The idea of estimating the size of the set of commuting n-tuples of a finite group has been modified to include other ideas such as: counting the number of n-tuples whose higher commutator is trivial; counting pairs of commuting elements formed from different subgroups and extending these results to topological groups. To get a taste of these results, the reader may consult [**6**, **7**], and the references contained therein.

We begin with the following useful lemma.

Lemma 6.1. *Let K be any discrete group. Then*

$$|\mathrm{Hom}(K \times \mathbb{Z}, G)|/|G| = |\mathrm{Hom}(K, G)/G|.$$

Proof. Note that

$$\mathrm{Hom}(K \times \mathbb{Z}, G) = \bigsqcup_{\phi \in \mathrm{Hom}(K,G)} C_G(\phi).$$

So

$$\frac{1}{|G|}|\mathrm{Hom}(K \times \mathbb{Z}, G)| = \frac{1}{|G|}\sum_{\phi \in \mathrm{Hom}(K,G)} |C_G(\phi)| = \sum_{\phi \in \mathrm{Hom}(K,G)} \frac{1}{|G : C_G(\phi)|}$$
$$= \sum_{[\phi] \in \mathrm{Hom}(K,G)/G} 1 = |\mathrm{Hom}(K, G)/G|. \qquad \square$$

The previous result shows that

$$k_n(G) = |\mathrm{Hom}(\mathbb{Z}^n, G)/G| = |\mathrm{Hom}(\mathbb{Z}^{n+1}, G)|/|G| = P_2(G, n+1)|G|^n.$$

This latter in turn shows that

$$k_n(G) = -\frac{1}{|G|} \sum_{H \in \mathcal{N}_2(G)} \mu_2(H, G)|H|^{n+1}.$$

Proposition 6.2. *Let H be a subgroup of G. Then*

$$|G : H|^{-n} P_2(H, n) \leqslant P_2(G, n) \leqslant P_2(H, n),$$

and therefore

$$|G : H|^{-1} k_n(H) \leqslant k_n(G) \leqslant |G : H|^n k_n(H).$$

Proof. We proceed by induction on n to prove all the inequalities. The case $n = 1$ is well known (see [8]). Suppose that $n > 1$. For the right-hand side inequalities, we have

$$\begin{aligned} |\mathrm{Hom}(\mathbb{Z}^n, G)| &= \sum_{g \in G} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_G(g))| \\ &\leqslant \sum_{g \in G} |C_G(g) : C_H(g)|^{n-1} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_H(g))|. \end{aligned}$$

Since $|C_G(g) : C_H(g)| \leqslant |G : H|$, it follows that

$$|\mathrm{Hom}(\mathbb{Z}^n, G)| \leqslant \sum_{g \in G} |G : H|^{n-1} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_H(g))|.$$

On the other hand,

$$\begin{aligned} \sum_{g \in G} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_H(g))| &= \sum_{\phi \in \mathrm{Hom}(\mathbb{Z}^{n-1}, H)} |C_G(\phi)| \\ &= \sum_{[\phi] \in \mathrm{Hom}(\mathbb{Z}^{n-1}, H)/H} |C_G(\phi)||H : C_H(\phi)| \\ &= \sum_{[\phi] \in \mathrm{Hom}(\mathbb{Z}^{n-1}, H)/H} |H||C_G(\phi) : C_H(\phi)| \\ &\leqslant \sum_{[\phi] \in \mathrm{Hom}(\mathbb{Z}^{n-1}, H)/H} |H||G : H| \\ &= |G| k_{n-1}(H) \\ &= |G : H||\mathrm{Hom}(\mathbb{Z}^n, H)|. \end{aligned}$$

The right-hand side inequalities follow.

For the left-hand side inequalities, we only need to note that $|\mathrm{Hom}(\mathbb{Z}^n, H)| \leqslant |\mathrm{Hom}(\mathbb{Z}^n, G)|$ for all $n \geqslant 1$. □

Remark 10. Since $P_2(G \times H, n) = P_2(G, n)P_2(H, n)$, it follows that $k_n(G \times H) = k_n(G)k_n(H)$.

Proposition 6.3. *Let $N \lhd G$. Then $P_2(G, n) \leqslant P_2(G/N, n)P_2(N, n)$, and therefore*

$$k_n(G) \leqslant k_n(G/N)k_n(N).$$

Proof. We proceed by induction on n. The result, for $n = 1$, is well known. For $n > 1$, we have

$$\begin{aligned}
|\mathrm{Hom}(\mathbb{Z}^n, G)| &= \sum_{g\in G} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_G(g))| \\
&\leqslant \sum_{g\in G} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_G(g)N/N)||\mathrm{Hom}(\mathbb{Z}^{n-1}, C_N(g))| \\
&\leqslant \sum_{g\in G} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_{G/N}(gN))||\mathrm{Hom}(\mathbb{Z}^{n-1}, C_N(g))| \\
&= \sum_{xN\in G/N} \sum_{g\in xN} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_{G/N}(gN))||\mathrm{Hom}(\mathbb{Z}^{n-1}, C_N(g))| \\
&= \sum_{xN\in G/N} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_{G/N}(xN))| \sum_{g\in xN} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_N(g))|.
\end{aligned}$$

On the other hand,

$$\sum_{g\in xN} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_N(g))| = \sum_{\phi\in \mathrm{Hom}(\mathbb{Z}^{n-1},N)} |C_{xN}(\phi)|.$$

Note that if y is in $C_{xN}(\phi)$, then $C_{xN}(\phi) = C_G(\phi) \cap xN = yC_G(\phi) \cap yN = yC_N(\phi)$. So $C_{xN}(\phi)$ is either empty or a coset of $C_N(\phi)$. Thus,

$$\begin{aligned}
\sum_{g\in xN} |\mathrm{Hom}(\mathbb{Z}^{n-1}, C_N(g))| &\leqslant \sum_{\phi\in \mathrm{Hom}(\mathbb{Z}^{n-1},N)} |C_N(\phi)| \\
&= \sum_{[\phi]\in \mathrm{Hom}(\mathbb{Z}^{n-1},N)/N} |N| \\
&= |N|k_{n-1}(N) = |\mathrm{Hom}(\mathbb{Z}^n, N)|.
\end{aligned}$$

The result follows. □

The following result provides a slight improvement over that in [8, Lemma 2].

PROPOSITION 6.4. *Let x be an element of $G - Z(G)$ so that $|C_G(x)|$ is as large as possible, and let p be the smallest prime divisor of the index of $Z(G)$ in G. Let c denote $|C_G(x)|$. Then, for $n \geqslant 1$,*

$$P_2(G, n+1) \leqslant \frac{(p^n + \cdots + p + 1)c^n}{p^n|G|^n}.$$

Proof. If $\phi = (x_1, \ldots, x_n)$ is in $\mathrm{Hom}(\mathbb{Z}^n, G)$, then $C_G(\phi) \subseteq C_G(x_i)$. If at least one of the components of ϕ is not in $Z(G)$, then $|C_G(\phi)| \leqslant c$. Then, for $n \geqslant 1$, we have

$$|\mathrm{Hom}(\mathbb{Z}^n, G)| \geqslant |Z(G)|^n + (k_n(G) - |Z(G)|^n)\frac{|G|}{c}.$$

Note that $c/|Z(G)| \geqslant p$. So

$$\begin{aligned}
P_2(G, n+1) &\leqslant \frac{c}{|G|}P_2(G, n) + \left(1 - \frac{c}{|G|}\right)\frac{|Z(G)|^n}{|G|^n} \\
&\leqslant \frac{c}{|G|}P_2(G, n) + \frac{|Z(G)|^n}{|G|^n} \\
&\leqslant \frac{c}{|G|}P_2(G, n) + \frac{c^n}{p^n|G|^n}.
\end{aligned}$$

An inductive argument applied to the latter inequality completes the proof. □

Proposition 6.5. *Let p be the smallest prime divisor of $|G:Z(G)|$, and m be the index of $Z(G)$ in G. Then*

$$\frac{mn+m-n}{m^{n+1}} \leqslant P_2(G,n+1) \leqslant \frac{p^{n+1}+p^n-1}{p^{2n+1}}.$$

Proof. If p is the smallest prime divisor of $|G|$, then $|G:C_G(\phi)| \geqslant p$ for all $\phi \notin Z(G)^n$. So

$$|\mathrm{Hom}(\mathbb{Z}^n,G)| \geqslant |Z(G)|^n + p(k_n(G)-|Z(G)|^n),$$

and thus

$$k_n(G) \leqslant \frac{1}{p}(|\mathrm{Hom}(\mathbb{Z}^n,G)| + (p-1)|Z(G)|^n).$$

Dividing by $|G|^n$, we have

$$P_2(G,n+1) \leqslant \frac{1}{p}\left(P_2(G,n) + (p-1)\frac{1}{|G:Z(G)|^n}\right).$$

Note that $|G:Z(G)| \geqslant p^2$, else $G/Z(G)$ is a cyclic group of order p. So

$$P_2(G,n+1) \leqslant \frac{1}{p}\left(P_2(G,n) + \frac{p-1}{p^{2n}}\right).$$

An inductive argument applied to the latter inequality completes the proof for the right-hand side inequality. The left-hand side inequality is obtained in a similar way. □

In the last result p is of course at least 2. The same proof yields the following result proved in [**10**].

Corollary 6.6. *If G is a non-abelian group, then*

$$P_2(G,n+1) \leqslant \frac{3\cdot 2^n - 1}{2^{2n+1}}.$$

We close this section by providing a family of congruences for $k_n(G)$. The congruence for $k_1(G)$ was proved in [**13**].

Proposition 6.7. *Let $p_1,\ldots,p_l$ be the prime divisors of $|G|$, and D_n be the greatest common divisor of $p_1^n-1,\ldots,p_l^n-1$. Then*

$$k_{n-1}(G) \equiv |G|^{n-1} \pmod{D_n}.$$

Remark 11. The number $|G|^n - k_{n-1}(G)|G|$ is precisely the number of non-commuting n-tuples in G, that is, $|G^n - \mathrm{Hom}(\mathbb{Z}^n,G)|$.

The idea to obtain the congruence is as follows: first recall that

$$\sum_{H\in\mathcal{L}_2(G)} \mu_2(H,G) = 0.$$

So

$$-|G|k_{n-1}(G) = -|G|^n + \sum_{H\in\mathcal{L}_2(G)} \mu_2(H,G)(|H|^n + a)$$

for any number a. Let us restrict to the case $a = -1$. Thus, the last two equations show that if d divides $|H|^n - 1$ for all $H \subseteq G$, then $k_{n-1}|G| \equiv |G|^n \pmod{d}$. If D_n is as defined in the proposition, then one can see that D_n is relatively prime to $|G|$ and that it divides $|H|^n - 1$ for all $H \subseteq G$. This yields the desired congruence.

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