REGULAR NEIGHBOURHOODS OF GRAPHS IN MANIFOLDS

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1. *Introduction*

It is the purpose of this note to sketch the proof of the following theorem.

THEOREM A. *Let M be a triangulated n-manifold. If G is a graph which is a subcomplex of M, and if N is a regular neighbourhood of G in M, then the double of N is homeomorphic to an n-sphere with handles.*

Here a graph is considered to be a connected 1-dimensional compact polyhedron. The term " regular neighbourhood " is used in the general simplicial complex setting as on p. 33 of **[12]**; in his earlier work **[9]** the author employed " barrel neighbourhood " to denote a canonic form of such a second derived mapping cylinder neighbourhood. If M is a connected n-manifold without boundary, the addition of a handle to M is defined as follows. Delete the interiors of two disjoint locally flat equivalently embedded n-cells from M and call the remaining subspace M_1. Adjoin the n-annulus $S^{n-1} \times I$ to M_1 via a homeomorphism of $\partial(S^{n-1} \times I)$ onto ∂M_1. In the terminology of **[12**; p. 74] this is attaching a 1-handle. We shall discuss this operation and its uniqueness in more detail in Section 3.

M. Brown and the author announced this theorem for oriented manifolds **[4]**. The proof has not appeared in print. Now that R. D. Edwards has shown that there exist non-combinatorial triangulations of S^n for each $n \geqslant 5$ **[5]**, there has been much work done recently on simplicial complexes which are manifolds. Accordingly, it seems appropriate at this time to place in the literature a proof of our early result. In the intervening years it has become possible to simplify drastically the proof, for example by eliminating several applications of the theory of stable manifolds **[3]**.

2. *Regular neighbourhoods of trees*

Our notation will mostly follow that of our earlier work **[8, 9, 10, 11]**. A manifold will be considered to have no boundary. The join of X and Y will be written as $X * Y$. If X and Y are homeomorphic, we write $X \approx Y$. The double of a generalized manifold with boundary X will be represented by $2X$.

First of all we consider the case in which the graph is a tree, i.e., an acyclic graph. In a research announcement we outlined a proof of a result which is a precursor of Theorem A.

THEOREM 2 OF **[9]**. *Let T be a tree which is a subcomplex in the triangulated n-manifold M. Then, if N is a regular neighbourhood of T in M, $2N \approx S^n$.*

It should be pointed out that for $n \geqslant 5$ this can now be deduced from Newman's

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work [7]. Of course, our proof holds for all n including dimension 4, about which little is known even now. Our construction will serve to simplify the exposition dealing with Theorem A.

Proof of Theorem 2. By the invariance of regular neighbourhoods [**12**; Theorem 3.8], we may use any convenient neighbourhood including second derived neighbourhoods.

Let $T = T_0 \cup e$ where T_0 is a subtree, e is a 1-simplex $|v_0 v_1|$ and $T_0 \cap e = \{v_0\}$. We proceed by induction on the number of edges in T. Let N_0 be a regular neighbourhood of T_0 in M.

For the basis of our induction, we have the case in which $T_0 = \{v_0\}$. Then the regular neighbourhood N_0 of T_0 is PL equivalent to $\mathrm{St}\,(M, v_0)$. By [**8**], $2N_0 \approx S^n$. Otherwise, in the induction step we assume that $2N_0 \approx S^n$. The proofs of both cases proceed in the same way from this point.

There exists $x_0 \in \mathrm{Int}\, e = (v_0 v_1)$ so that $N_0 \cap e = |v_0 x_0|$. Let N_2 be a regular neighbourhood of $|x_0 v_1|$ in M. We may assume that N_0 and N_2 have been chosen so that $N_0' = N_0 \cap \mathrm{St}\,(M, e)$ is a regular neighbourhood of $|v_0 x_0|$ in $L * |v_0 x_0|$, where $L = \mathrm{Lk}\,(M, e)$, and $(N_0 \cup N_2) \cap \mathrm{St}\,(M, e)$ is a regular neighbourhood of e in $\mathrm{St}\,(M, e)$. This may be done using barrel neighbourhoods as in Lemma 9 of [**9**]. More abstractly this can be carried out by using a relative form of Corollary 3.30 in [**12**] applied successively to maximal simplexes of $\mathrm{St}\,(M, e)$.

From the join structure of $\mathrm{St}\,(M, e) = L * e$ it may be concluded that N_0' is PL equivalent to $X \times |v_0 x_0|$ and that $(N_0 \cup N_2) \cap \mathrm{St}\,(M, e)$ is PL equivalent to $X \times e$, where $X = C(L)$—the cone over L. If N_0' and $N_2' = N_2 \cap \mathrm{St}\,(M, e)$ have been nicely chosen as in Lemma 9 of [**9**] we may assume that in both mapping cylinder structures, if $p \in e$ then the fibre over p corresponding to $X \times \{p\}$ is $(p * L) \cap (N_0' \cup N_2')$. Let X_0 denote the fibre over x_0 in N_0'.

From the choice of N_0 and N_2 it now follows that $N_1 = \mathrm{Cl}\,(N_2 - N_0)$ is a regular neighbourhood of $|x_0 v_1|$ in $\mathrm{Cl}\,(M - N_0)$. Also $N_0 \cup N_1 = N$ is a regular neighbourhood of T in M and $N_0 \cap N_1 = X_0$.

Select a point $x_0' \in (v_0 x_0)$, and let X_0' be the fibre over x_0'. Then $2X_0'$ is a flat suspension $(n-1)$-sphere in $2N_0 \approx S^n$. That is, $2X_0'$ has the following properties: if $S(2X_0')$ stands for the suspension of $2X_0'$ then

$$S(2X_0') \approx S(2X) \approx S(S(L)) \approx S^1 * L \approx S^n$$

by [**8**]; since $2X_0'$ is bicollared in $2N_0$ it is flat [**9**, **11**], that is, each closed domain in $2N_0$ bounded by $2X_0'$ is a topological cone over $2X_0'$.

Symmetrically we choose $x_1' \in (x_0 v_1)$ so that if X_1' is the fibre over x_1' in $N_1' = N_1 \cap \mathrm{St}(M, e)$, it follows that $2X_1'$ is a flat suspension $(n-1)$-sphere in $2N_1 \approx 2\,\mathrm{St}\,(M, v_1) \approx S^n$.

We may now represent the double of N as $2N = W_0 \cup W_1$, so that $W_0 \cap W_1 = 2X_0$ and $W_i - 2X_0 \approx 2N_i - X_0$ for $i = 0, 1$. This is a canonic representation induced by $N = N_0 \cup N_1$ because in N_i we have $X_0 \subseteq \partial N_i$ so that X_0 is injected into $2N_i$, for $i = 0, 1$. In N, however, $X_0 \cap \partial N = \partial X_0$, so that $2X_0$ is injected into $2N$. It may be observed that W_0 is homeomorphic to the closed domain in $2N_0$ bounded by $2X_0'$ which is exterior to X_0. The latter closed domain is homeomorphic to $C(2X)$ as previously noted. This gives us the pair equivalence $(W_0, 2X_0) \approx (C(2X), 2X)$. Similarly, $(W_1, 2X_0) \approx (C(2X), 2X)$. Consequently, $2N \approx 2C(2X_0) \approx S(2X_0) \approx S^n$.

This completes the proof.

3. *Attaching handles to manifolds*

Let M be a connected n-manifold. Suppose that D_0 and D_1 are disjoint locally flat n-cells in M which are annularly equivalent [3; p. 19]. Set $M_1 = M - \operatorname{Int}(D_0 \cup D_1)$ and $A^n = S^{n-1} \times I$. If h_i is a homeomorphism of $S^{n-1} \times i$ onto ∂D_i, $i = 0, 1$, then form the adjunction space from the disjoint union $M_1 + A^n$ via h_0 and h_1. The adjunction space $M_1 \cup A^n$ is said to be formed by adding a handle to M. If h_0 and h_1 carry the same sense of orientation from the product orientation on A^n, then the handle is called orientable. If h_0 and h_1 carry opposite orientations, the handle is called non-orientable.

To discuss the conditions under which this operation is well defined and unique up to orientation, it is convenient to redescribe the attaching of handles using the method of Brown and Gluck [3; p. 46].

Consider the following propositions.

(SHC_n): Every orientation-preserving self-homeomorphism of S^n is stable.

(A_n): Two disjoint flatly embedded $(n-1)$-spheres in S^n bound an annulus.

(I_n): Every orientation-preserving self-homeomorphism of S^n is isotopic to the identity.

(WI_n): Every orientation-preserving self-homeomorphism h of S^n is weakly isotopic to the identity, that is, there is a homeomorphism G of $S^n \times I$ onto itself such that for all $x \in S^n$ we have $G(x, 0) = (x, 0)$ and $G(x, 1) = (h(x), 1)$.

In [3; Theorem 9.3, p. 12] it was shown that $(\mathrm{A}_n) + (\mathrm{WI}_{n-1}) \Leftrightarrow (\mathrm{SHC}_n)$. Further, $(\mathrm{SHC}_n) \Rightarrow (\mathrm{I}_n)$ [3; p. 6]. Obviously $(I_n) \Rightarrow (\mathrm{WI}_n)$.

From classical results it had been known that (SHC_n) is true for $n \leqslant 3$. Kirby proved that (SHC_n) is true for $n \geqslant 5$ [**6**].

Thus, we make the following observation.

1. (WI_n) *is true for all n.*

If h is a self-homeomorphism of S^{n-1}, let X_h be the identification space derived from $S^{n-1} \times I$ by identifying $(x, 0)$ with $(h(x), 1)$. Note that $X_1 \approx S^{n-1} \times S^1$, which we denote by T_n. If h is orientation-preserving, it may be easily proved that (WI_{n-1}) implies that $X_h \approx T_n$.

Now let $r : S^{n-1} \to S^{n-1}$ be the standard equatorial reflection. We call X_r the n-dimensional Klein bottle and denote it by K_n. If h is orientation-reversing, it may be established that (WI_{n-1}) implies that h is weakly isotopic to r and consequently that $X_h \approx K_n$.

If M_1 and M_2 are connected n-manifolds, D_i is a locally flat n-cell in M_i, $i = 1, 2$, and $h : \partial D_1 \approx \partial D_2$, then the connected sum $M_1 \# M_2$ is the (possibly ambiguous) adjunction space formed from $(M_1 - \operatorname{Int} D_1) + (M_2 - \operatorname{Int} D_2)$ via identification by h.

A connected n-manifold M is called homogeneous if for any two locally flat embeddings $f_i : D^n \to M$, $i = 1, 2$, there exists a self-homeomorphism h of M so that $hf_1 = f_2$.

Now suppose that we have attached the handle A^n to M_1 along ∂D_0 and ∂D_1, D_0 and D_1 being disjoint annularly equivalent locally flat n-cells in M. Let D_2 be a locally flat n-cell in M which is annularly equivalent with each D_i, and such that $\operatorname{Int} D_2 \supseteq D_0 \cup D_1$. It may now be seen that because (WI_{n-1}) is true, $M_1 \cup A^n$ is homeomorphic to $M \# T_n$ if the handle is orientable, and homeomorphic to $M \# K_n$ if the handle is non-orientable. The connected sum is derived from M by deleting the interior of D_2.

In [3; pp. 55–56] it is demonstrated that T_n has exactly two global equivalence classes of locally flat embeddings of S^{n-1}; those which separate or do not separate T_n.

This has the following consequences.

2. *T_n is homogeneous.*

3. *By* [3; Theorem 11.1, p. 53], *$M \# T_n$ is uniquely defined independently of the adjunction map. Furthermore, if M is homogeneous, $M \# T_n$ is homogeneous as well.*

By slight modifications of the proofs of the above results, it is possible to establish similar properties for K_n.

For example, to adapt Lemma 14.1 of [3; p. 56] to K_n, we proceed in the following way. The space $S^{n-1} \times R$ is the universal covering space of K_n with projection map p satisfying the relation, $p(x, t) = p(r(x), t+1)$. A generating covering transformation τ of $S^{n-1} \times R$ is defined by $\tau(x, t) = (r(x), t+1)$. Now, as in the cited lemma, let $f : S^{n-1} \to K_n$ be a locally flat embedding which does not separate K_n and let $\tilde{f}$ cover f. Then $\tilde{f}$ separates the ends of $S^{n-1} \times R$.

It may be verified that $\tilde{f}$ and $\tau \tilde{f} r$ have the same sense of orientation, have disjoint images and are stably equivalent in $S^{n-1} \times R$. This allows us to produce an embedding $G : S^{n-1} \times I \to S^{n-1} \times R$ which agrees with $\tilde{f}$ on $S^{n-1} \times 0$, and with $\tau \tilde{f} r$ on $S^{n-1} \times 1$. This gives us a homeomorphism $h : K_n \to K_n$ satisfying $hp = pG$. Accordingly, $hp(x, 0) = f(x)$; that is, f is globally equivalent to the standard embedding $p|S^{n-1} \times 0$.

Lemma 14.2 of [3; p. 57] may be used without change of proof to show that if $f : S^{n-1} \to K_n$ is a locally flat embedding which separates K_n, then f may be extended over D^n.

In the proof that T_n is homogeneous, only one additional change need be made to adapt the proof to K_n. In the last line of p. 57 of [3], a stable homeomorphism h_2 of T_n is described. More precisely, h_2 may be required to be the identity on a non-empty open set. This construction carries over without change to K_n. Since h_2 is somewhere the identity, the appropriately chosen covering $\tilde{h}_2$ is stable on $S^{n-1} \times R$. The other details go through unchanged.

Hence, we make the following observations.

4. *K_n is homogeneous.*

5. *$M \# K_n$ is uniquely defined. If M is homogeneous, then so is $M \# K_n$.*

6. *If M is a connected sum of S^n with p copies of T_n and q copies of K_n, then M is homogeneous and topologically unique depending only on the integers p and q.*

Applying (6) to spheres with handles, we see that such spaces are uniquely defined provided that at each stage the handle has been attached along annularly equivalent n-cells. This is automatic for $n \neq 4$ by (A_n). For $n = 4$, in our construction in the next section the cells will always be subcomplexes of some common triangulation of the manifold. Using Lemma 1 from the next section or the proof of Theorem 11.2 from [3; p. 33], we are assured that here, too, the cells will be annularly equivalent.

The author acknowledges a helpful discussion with M. Brown concerning the material in this section.

4. *A proof of Theorem A*

Let T be a spanning tree in G. Since the addition of handles to S^n is interchangeable in order and can, in fact, be carried out in our case so that the handles are " disjoint ", we may reduce the proof to consideration of the case in which we add in one additional edge.

So we assume that $G = T \cup e$, that $e = |v_0 v_1|$ is a 1-simplex of M and that $T \cap e = \partial e = \{v_0, v_1\}$. Select $x_0 \in (v_0 v_1)$ and $x_1 \in (x_0 v_1)$. Choose N_0 to be a regular neighbourhood of T in M and N_1 to be a regular neighbourhood of $|x_0 x_1|$ in $\mathrm{Cl}\big(\mathrm{St}(M, e) - N_0\big)$ so that $N = N_0 \cup N_1$ is a regular neighbourhood of G in M. Furthermore, we may suppose that if $L = \mathrm{Lk}(M, e)$ and $X = C(L)$, $N_1 \approx X \times |x_0 x_1|$ and $N_0 \cap N_1 = X_0 \cup X_1$ with X_i corresponding to $X \times \{x_i\}$, $i = 0, 1$.

Next select $x_0' \in (v_0 x_0)$ and $x_1' \in (x_1 v_1)$. Let $X_i'(\approx X)$ be the fibre over x_i', $i = 0, 1$, of the mapping cylinder $N_0 \cap \mathrm{St}(M, e)$. It follows that $2X_i'$ is flat in $2N_0 \approx S^n$. Again each closed domain in $N \cap \mathrm{St}(M, e)$ bounded by any distinct pair of X_0', X_0, X_1, X_1' is homeomorphic to $X \times I$. By Lemma 5 of [9] we may select flat $(n-1)$-spheres S_0 and S_1 in $2N_0$ so that S_0 separates $2X_0'$ and $2X_0$ in $2\big(N \cap (\mathrm{Int\,St}(M, e))\big)$ and S_1 separates $2X_1$ and $2X_1'$ in $2\big(N \cap (\mathrm{Int\,St}(M, e))\big)$.

If $n \geqslant 5$, then S_0 and S_1 bound an annulus $A_0(\approx S^{n-1} \times I)$ in $2N_0$ [6]. Similarly the closed domain A_1 bounded by S_0 and S_1 in $2\big(N \cap (\mathrm{Int\,St}(M, e))\big)$ is also an annulus. Thus $2N = A_0 \cup A_1$ and $A_0 \cap A_1 = \partial A_i = S_0 \cup S_1$. Therefore $2N$ is an n-sphere with handle.

If $n = 4$, then $L \approx S^2$ because $S^1 * L \approx S^4$ [10; Section I]. In this case we have that X_0 and X_1 are 3-cells so that $2X_0$ and $2X_1$ are 3-spheres. The closed domain A_1 in $2\big(N \cap (\mathrm{Int\,St}(M, e))\big)$ bounded by $2X_0$ and $2X_1$ has already been observed to be homeomorphic to $2X \times I$. Since $2X \approx S^3$, A_1 is a *bona fide* annulus. We may subdivide $2N_0$ by starring from x_0 and x_1 so that $2X_0$ and $2X_1$ become subcomplexes of this triangulation of S^4. By the construction, $2X_0$ and $2X_1$ are flat in S^4. We may then conclude from [3; p. 13] that these 3-spheres bound an annulus A_0 in $2N_0$. Alternatively, this step can be verified as a separate lemma given below.

In any event, for $n = 4$ we again have a representation $2N = A_0 \cup A_1$ with $A_0 \cap A_1 = \partial A_i = 2X_0 \cup 2X_1$. Thus again $2N$ is a 4-sphere with handle.

If M is orientable, then clearly so is N and, therefore, $2N$ as well. In this case, $2N$ is an n-sphere with oriented handles. If M is non-orientable, $2N$ may or may not be orientable depending on how G is situated in M.

The result we have used for dimension 4 may be stated as follows.

LEMMA 1. *Let S_0 and S_1 be subcomplexes of a triangulation of S^n. Suppose that $S_0 \cap S_1 = \varnothing$, and S_i is a flat $(n-1)$-sphere, $i = 0, 1$. Then A, the closed domain in S^n bounded by $S_0 \cup S_1$, is an annulus.*

Remark. Substantially the same proof works if we assume that S_0 and S_1 are homeomorphic to the same suspension $(n-1)$-sphere.

Proof. Let K be the $(n-2)$-skeleton of S^n. The simplicial structure on S^n restricts to a PL structure on $S^n - K$ (the union of the interiors of all n-dimensional and $(n-1)$-dimensional simplexes).

Hence, we may run a PL flat arc J in $A - K$ from a point $x_0 \in S_0 - K$ to a point

$x_1 \in S_1 - K$. The arc J is chosen so that it is the core of a PL flat n-cell $C \subseteq A - K$, so that $C \cap S_i$ is an $(n-1)$-simplex F_i interior to an $(n-1)$-simplex of S_i, $i = 0, 1$.

It follows easily that $S = (S_0 \cup \partial C \cup S_1) - \text{Int}(F_0 \cup F_1)$ is a flat $(n-1)$-sphere and bounds an n-cell $D \subseteq A$ so that $\text{Cl}(A-D) = C$. There is a homeomorphism h of $\partial(I^{n-1} \times I)$ onto ∂D, carrying $I^{n-1} \times 0$ onto $S_0 - \text{Int } F_0$ and $I^{n-1} \times 1$ onto $S_1 - \text{Int } F_1$. Of course, h can be extended over $I^{n-1} \times I$ onto D giving a product structure under which $h(\partial I^{n-1} \times I) = \partial C - \text{Int}(F_0 \cup F_1)$. This structure may be extended over C to give a homeomorphism $(A, S_0, S_1) \approx (S^{n-1} \times I, S^{n-1} \times \{0\}, S^{n-1} \times \{1\})$.

5. *Concluding observations*

It is well known that a closed connected combinatorial n-manifold M can be expressed as the union of $n+1$ PL n-cells. In [4] an analogous result was described without assuming the triangulation of M to be combinatorial; in this situation, of course, the cells are only topologically embedded.

THEOREM B. *Let M be a connected triangulated n-manifold. Then M is the union of $n+1$ open n-cells. If M is compact, then each open cell may be chosen to be the interior of an n-cell.*

This can be established by use of the following elementary lemmas.

LEMMA 2. *Let $\Gamma = \{C_1, C_2, C_3, \ldots\}$ be a discrete collection of cellular subsets of a connected n-manifold M. Then $\bigcup \Gamma$ lies in an open n-cell U. If Γ is finite then U may be chosen to be the interior of a closed n-cell.*

The proof reduces to the fact that a point $x \in M - D$, where D is an n-cell, may be engulfed by Int D. See Lemmas 1 and 2 of [2].

LEMMA 3. *Let S be a simplex of a triangulation of the n-manifold M. Suppose that A is a compact starlike subset of* Int S. *Then A is cellular in M.*

This involves a straight-forward application of Theorem 2 of [8].

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