arcs of classes X, Y or Z. The order relation on a Jordan curve is the same as that on the circumference of a circle. Hence, when a Jordan curve of $J(r)$ is deformed continuously on F, the number of arcs of $\mathcal{L}_1$ resulting which are of class Z (that is, the number of arcs lying in S_2 which connect S_3, S_4) can only change by an even number. But curves of type (i) can be deformed to a point and so must give rise to an even number of arcs of class Z in $\mathcal{L}_1$. Also curves of type (ii) can be deformed to J_0 which has a single arc in S_2 joining S_3, S_4. Hence these must give rise to an odd number of arcs of class Z in $\mathcal{L}_1$.

However we have seen that there is an odd number of curves of type (ii) in $J(r)$. Hence there can only be an odd number of arcs of class Z in $\mathcal{L}_1$. By symmetry the same result is true for $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$; so that condition (ii) in the definition of a pseudo-square (§ 2) is satisfied by $Q(r)$.

Further $\qquad Q\text{-}\rho\left(\mathcal{L}_1,\ \mathcal{L}_3\right) \geqslant \mathcal{C}\text{-}\rho\left(\mathcal{L}_1,\ \mathcal{L}_3\right)$ by (1)

$$\geqslant \mathcal{C}\text{-}\rho\left(S_2,\ S_5\right) = b \text{ by (2)},$$

and $\qquad Q\text{-}\rho\left(\mathcal{L}_2,\ \mathcal{L}_4\right) \geqslant c.$

Thus $Q(r)$ is a pseudo-square of dimensions at least b, c; and therefore, by Lemma 2, the area of $Q(r)$ is at least bc.

A similar argument to that of Lemma 1 will now prove that

$$\Lambda^3\, G(a) \geqslant bc\, (a-\epsilon),$$

and the theorem is proved, since ϵ is arbitrary and $\mathcal{C} \supset G(a)$.

I am grateful to Prof. Besicovitch for giving me the manuscript of his paper before publication, and for pointing out to me some obscurities in my proof.

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ON MAHLER'S U-NUMBERS

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1. It is well known that a necessary and sufficient condition that the complex number ξ be transcendental is that to each $\omega > 0$ there correspond a positive integer $n = n(\omega)$ and infinitely many sets of rational integers $a_0, \ldots, a_n$, such that

$$0 < |a_0 + a_1\xi + \ldots + a_n\xi^n| < A^{-\omega},$$

where $A = \max_i (|a_i|)$. In general, n increases with ω; the numbers ξ for which n may remain bounded form an uncountable set of measure zero which K. Mahler [1] has termed the U-numbers. The Liouville numbers

are precisely the U-numbers for which n can always be chosen to be unity; while they have been studied more or less extensively, no general investigation has heretofore been made of the U-numbers of higher "degree". (See, however, [2], pp. 143-148.)

We shall say that the polynomials

$$f_n(x) = a_0^{(n)} + \ldots + a_m^{(n)} x^m \quad (n = 1, 2, \ldots) \tag{1}$$

of fixed degree m and height $H(f_n) = \max(|a_0^{(n)}|, \ldots, |a_m^{(n)}|)$ form a sequence of Liouville polynomials (more briefly, an L_m-sequence) for ξ if

$$H(f_1) \leqslant H(f_2) \leqslant \ldots$$

and $\qquad 0 < |f_n(\xi)| = H^{-\omega_n}(f_n) < 1 \quad (n = 1, 2, \ldots),$

where $\lim\limits_{n \to \infty} \omega_n = \infty$. We designate by U the set of all U-numbers, and by U_m the set of all ξ for which there exists an L_m-sequence but not an L_{m-1}-sequence. Thus U_1 is the set of Liouville numbers, and U is the union of all the (disjoint) sets U_m. An element of U_m does not, of course, have a unique L_m-sequence, since any infinite subsequence of an L_m-sequence is again an L_m-sequence, but it is shown in Theorem 4 that there is at most one sequence satisfying certain auxiliary conditions. This result is used to give a constructive proof that U_m is not empty for any $m \geqslant 1$, and an extension of it is used to show that numbers satisfying a transcendency condition due to T. Schneider are not U-numbers of too low degree.

2. THEOREM 1. *Suppose that the sequence $\{f_n(x)\}$ of polynomials* (1) *is an L_m-sequence for each of the distinct numbers $\xi_1, \ldots, \xi_m$. Then there are numbers $\alpha_0, \ldots, \alpha_m$ such that*

$$\lim_{n \to \infty} \frac{a_k^{(n)}}{a_m^{(n)}} = \alpha_k \quad (k = 0, \ldots, m-1);$$

they are either Liouville numbers or rational, and one at least is Liouville. If α_k is rational, then $a_k^{(n)}/a_m^{(n)} = \alpha_k$ for large n, while if $\alpha_k \, \varepsilon \, U_1$, then $a_k^{(n)}/a_m^{(n)}$ (or $p_k^{(n)}/q_k^{(n)}$, in reduced form) is a convergent to α_k for large n. Finally the monotonic union sequence $\{q^{(n)}\}$ (consisting of all the integers t such that $t = q_k^{(n)}$ for some k, n, arranged by size) is a large-gap sequence; i.e.

$$\limsup_{n \to \infty} \frac{\log q^{(n+1)}}{\log q^{(n)}} = \infty.$$

For putting $\qquad f_n(x) = a_m^{(n)}(x - \xi_1^{(n)}) \ldots (x - \xi_m^{(n)})$

and using the fact that $f_n(\xi_i) \to 0$, we have $\xi_i^{(n)} \to \xi_i$ for $1 \leqslant i \leqslant m$ (after proper ordering), and so there exists

$$\lim_{n \to \infty} \frac{f_n(x)}{a_m^{(n)}} = \lim_{n \to \infty} \left(\frac{a_0^{(n)}}{a_m^{(n)}} + \ldots + \frac{a_{m-1}^{(n)}}{a_m^{(n)}} x^{m-1} + x^m \right) = \alpha_0 + \ldots + \alpha_{m-1} x^{m-1} + x^m.$$

We reserve the notation $\{f_n{}^*\}$ for the use to which it is put in this theorem, that is, to designate an L-sequence for ξ derived from a defining equation for ξ by replacing the coefficients by various of their convergents.

In the proofs of the next two theorems we shall need the following lemma:

LEMMA. *Suppose that $\phi(x)$, $\phi_1(x)$ are polynomials with integral coefficients, of degrees m, m_1 respectively, that $\phi(\xi_1) = \ldots = \phi(\xi_m) = 0$ and that $\phi_1 | \phi$. Then*

$$\max_k (|\xi_k|) \leqslant 2H(\phi), \tag{6}$$

and there is a c_7, depending only on m_1, such that[†]

$$H(\phi_1) < c_7 H^{m_1+1}(\phi). \tag{7}$$

We have, if $\phi(x) = ax^m + \ldots$ and $|x| \geqslant 2$,

$$\left| \frac{\phi(x)}{a} \right| \geqslant |x|^m - H(\phi)(1 + |x| + \ldots + |x|^{m-1}) \geqslant |x|^m - 2H(\phi)|x|^{m-1},$$

so that either $|\xi_k| < 2 \leqslant 2H(\phi)$ or $0 \geqslant |\xi_k| - 2H(\phi)$, and (6) holds. Since the leading coefficient of ϕ_1 divides that of ϕ, and since the coefficients of ϕ_1 are symmetric functions in certain of the zeros of ϕ, we have by (6),

$$H(\phi_1) \leqslant H(\phi) . \max (\Sigma|\xi_1|, \ldots, \Sigma|\xi_1 \ldots \xi_{m_1}|)$$

$$< c_7 2^{-m_1} H(\phi) . 2^{m_1} H^{m_1}(\phi),$$

and the proof is complete.

THEOREM 3. *A necessary condition that ξ belong to U_m is that only finitely many elements of any L_m-sequence for ξ be reducible.*

Let $\{f_n\}$ be an L_m-sequence, with $|f_n(\xi)| = H^{-\omega_n}(f_n)$. Suppose that there is an infinite sequence $\{\phi_{n_k}\}$ of polynomials with integral coefficients such that $\phi_{n_k} | f_{n_k}$, $\deg \phi_{n_k} = m_1 < m$, and $|\phi_{n_k}(\xi)| \to 0$. Then by the lemma, $H(\phi_{n_k}) < c_7 H^{m_1+1}(f_{n_k})$, and so

$$|\phi_{n_k}(\xi)| < c_8 |f_{n_k}(\xi)|^{1/m} < c_8 \left(H(\phi_{n_k})/c_7 \right)^{-\omega_n/m(m_1+1)},$$

so that $\{\phi_{n_k}\}$ is an L_{m_1}-sequence for ξ.

THEOREM 4. *Suppose that ξ is a complex number to which corresponds a sequence of polynomials $F_n(x)$ with the following properties:*

1. $F_n(x) = a_0^{(n)} + \ldots + a_{m_n}^n) x^{m_n} = a_{m_n}^{(n)}(x - \xi^{(n)})(x - \xi_2^{(n)}) \ldots (x - \xi_{m_n}^{(n)}),$

where the numbers $a_k^{(n)}$ are integers, the zeros of F_n have been so ordered that $\xi_1^{(n)} = \xi^{(n)}$ is the zero nearest to ξ, and $H(F_1) \leqslant H(F_2) \leqslant \ldots$.

[†] C. L. Siegel, *Math. Zeitschrift*, 10 (1921), 716, has given the incomparably better bound $H(\phi_1) < \dfrac{m!}{m_1!} H(\phi)$, but the trivial result of the lemma suffices here.

2. $|F_n(\xi)| = H^{-\omega_n}(F_n)$, where

$$\lim_{n\to\infty} \omega_n = \infty. \tag{8}$$

3. All the F_n are irreducible.

4. For some m, $m_n \leqslant m$ for all n.

5. For some $\delta > 0$, $H^{\omega_n}(F_n) > H^\delta(F_{n+1})$.

Then this sequence is unique, in the sense that if $\{f_\nu\}$ is any L_r-sequence (r an arbitrary positive integer) for ξ, then for sufficiently large values of ν, f_ν is divisible by some F_n.

Clearly $|\xi - \xi^{(n)}| < c_9 H^{-\omega_n/m}(F_n)$,

so that we can put

$$\xi = \xi^{(n)} + t_n H^{-\omega_n/m}(F_n), \tag{9}$$

where $|t_n| < c_9$ for all n.

Now put

$$f_\nu(x) = d_r^{(\nu)} x^r + \ldots + d_0^{(\nu)} = d_r^{(\nu)}(x - \vartheta_1^{(\nu)}) \ldots (x - \vartheta_r^{(\nu)}).$$

Then from the relation

$$(a_{m_n}^{(n)})^r (d_r^{(\nu)})^{m_n} \prod_{\substack{1\leqslant i\leqslant r \\ 1\leqslant j\leqslant m_n}} (\vartheta_i^{(\nu)} - \xi_j^{(n)}) = (a_{m_n}^{(n)})^r \prod_{i=1}^{r} F_n(\vartheta_i^{(\nu)}) = (d_r^{(\nu)})^{m_n} \prod_{i=1}^{m_n} f_\nu(\xi_i^{(n)}),$$

we obtain the identity

$$|f_\nu(\xi^{(n)})| = \frac{\left| (d_r^{(\nu)})^{m_n} \prod\limits_{i=1}^{r} F_n(\vartheta_i^{(\nu)}) \right|}{\left| (a_{m_n}^{(n)})^r \prod\limits_{i=2}^{m_n} \dfrac{f_\nu(\xi_i^{(n)})}{H(f_\nu)} \right| . H^{m_n-1}(f_\nu)}. \tag{10}$$

The numerator on the right is an integer, and we have the bounds

$$|a_{m_n}^{(n)}|^r \leqslant H^r(F_n),$$

$$\prod_{i=2}^{m_n} \left| \frac{f_\nu(\xi_i^{(n)})}{H(f_\nu)} \right| \leqslant \prod_{i=2}^{m_n} (1 + |\xi_i^{(n)}| + \ldots + |\xi_i^{(n)}|^r)$$

$$\leqslant \max\left(2^{(r+1)(m_n-1)}, \; \{2 \max (|\xi_i^{(n)}|^r)\}^{m_n-1} \right),$$

the last inequality being obtained by the argument used in proving (6). By the lemma this last quantity does not exceed $\left(2^{r+1} H^r(F_n)\right)^{m-1}$. Hence if $F_n(\vartheta_i^{(\nu)}) \neq 0$ for $i = 1, \ldots, r$, then

$$|f_\nu(\xi^{(n)})| \geqslant \frac{c_{10}}{H^{rm}(F_n) H^{m-1}(f_\nu)}. \tag{11}$$

Since $\{f_\nu\}$ is an L-sequence for ξ, we can write

$$|f_\nu(\xi)| = H^{-\gamma_\nu}(f_\nu), \tag{12}$$

where $\gamma_\nu \to \infty$ with ν. But from (9),

$$|f_\nu(\xi)| = \left| f_\nu\left(\xi^{(n)} + t_n H^{-\omega_n/m}(F_n)\right)\right|,$$

or $\qquad |f_\nu(\xi)| = |f_\nu(\xi^{(n)})| + t_{n,\nu} H(f_\nu) H^{-\omega_n/m}(F_n), \tag{13}$

where $|t_{n,\nu}| < c_{11}$ for all n and ν.

For each sufficiently large ν, there is an n such that

$$H(F_n) \leqslant H(f_\nu) < H(F_{n-1});$$

hereafter let n be so defined as a function of ν. Let ϵ be positive; we consider two cases, according as the inequality

$$H(f_\nu) < H^{\omega_n/\{m^2(r+1)+\epsilon\}}(F_n) \tag{14}$$

is true or false. Suppose first that (14) holds for infinitely many ν. Then for such ν,

$$\frac{c_{10}}{H^{rm}(F_n)H^{m-1}(f_\nu)} > \frac{c_{10}}{H^{rm+m-1}(f_\nu)} \tag{15}$$

and $\qquad \dfrac{c_{11}H(f_\nu)}{H^{\omega_n/m}(F_n)} < \dfrac{c_{11}}{H^{rm+m-1+(\epsilon/m)}(f_\nu)}. \tag{16}$

But (11), (12), (13), (15) and (16) are incompatible for large ν; hence for large ν for which (14) holds, $F_n(\vartheta_i^{(\nu)}) = 0$ for some i, which implies that $F_n | f_\nu$.

Now consider the case where (14) is false, so that

$$H^{\omega_n/\{m^2(r+1)+\epsilon\}}(F_n) \leqslant H(f_\nu) < H(F_{n+1}). \tag{17}$$

Then by hypothesis 5,

$$\frac{1}{H^{\gamma_\nu}(f_\nu)} \leqslant \frac{1}{H^{\gamma_\nu\omega_n/\{m^2(r+1)+\epsilon\}}(F_n)} < \frac{1}{H^{\delta\gamma_\nu/\{m^2(r+1)+\epsilon\}}(F_{n+1})} \tag{18}$$

and $\qquad \dfrac{c_{10}}{H^{rm}(F_{n+1})H^{m-1}(f_\nu)} > \dfrac{c_{10}}{H^{rm+m-1}(F_{n+1})}. \tag{19}$

Moreover, $\qquad \dfrac{c_{11}H(f_\nu)}{H^{\omega_{n+1}/m}(F_{n+1})} < \dfrac{c_{11}}{H^{(\omega_{n+1}/m)-1}(F_{n+1})}. \tag{20}$

But (11)-(13), with n replaced by $n+1$, and (18)-(20) are again incompatible for large ν, so that if (17) holds for large ν then $F_{n+1}(\vartheta_i^{(\nu)}) = 0$ for some i, and so $F_{n+1} | f_\nu$. The proof is complete.

COROLLARY. *A sufficient condition that ξ belong to U_l is that there be a sequence $\{F_n\}$ as described in Theorem 4 for which* $\liminf m_n = l$.

This result can be used to demonstrate the following basic fact:

THEOREM 5. *U_m is not empty for any $m \geqslant 1$.*

Let P_k be a sequence of primes, with $\log P_{k+1}/\log P_k$ increasing monotonically and without bound, so that

$$\zeta = \sum_{k=1}^{\infty} 2^{-P_k}$$

is a Liouville number. Put

$$q_n = 2^{P_n}, \quad p_n = q_n \sum_{k=1}^{n} 2^{-P_k};$$

then $\{q_n x^m - p_n\}$ is an L_m-sequence for $\zeta^{1/m}$, and for this sequence the hypotheses of Theorem 4 are easily verified. Hence U_m contains $\zeta^{1/m}$, and so is not empty.

3. We shall say that the Liouville number ξ with convergents P_n/Q_n is strong if $\{Q_n x - P_n\}$ is an L_1-sequence for ξ, and weak if every L_1-sequence for ξ is a proper subsequence of $\{Q_n x - P_n\}$. Thus if $|Q_n \xi - P_n| = Q_n^{-\kappa_n}$, then ξ is strong or weak according as $\liminf \kappa_n$ is infinite or finite (of course $\limsup \kappa_n = \infty$, since ξ is Liouville). The number ζ considered in the proof of Theorem 5 is a strong Liouville number, while the numbers

$$\zeta' = \sum_{n=1}^{\infty} (10^{-(2n)!} + 10^{-3(2n)!} + \ldots + 10^{-3^{n'}(2n)!}),$$

and

$$\zeta'' = \sum_{n=1}^{\infty} (10^{-(2n+1)!} + 10^{-3(2n+1)!} + \ldots + 10^{-3^{n''}(2n+1)!}),$$

where $3^{n'} \leqslant 2n < 3^{n'+1}$ and $3^{n''} \leqslant 2n+1 < 3^{n''+1}$, are weak, since there are infinitely many n for which each of the inequalities

$$\left| \zeta' - \frac{P_n'}{Q_n'} \right| > \frac{1}{Q_n'^3}, \quad \left| \zeta'' - \frac{P_n''}{Q_n''} \right| > \frac{1}{Q_n''^3}$$

holds. It is clear that if $\alpha_0, \ldots, \alpha_{m-1}$ are strong Liouville numbers, then the monotonic union sequence formed from the denominators of their convergents is a large-gap sequence; that this is not necessarily true if the α's are not strong is seen by taking $m = 2$, $\alpha_0 = \zeta'$, $\alpha_1 = \zeta''$.

We say that $\alpha_0, \ldots, \alpha_{m-1}$ are comparable Liouville numbers if there is an increasing sequence $\{N_n'\}$ of positive integers such that

$$\limsup_{n \to \infty} \frac{\log \max_k (Q_k^{(h_k(N_n'))})}{\log \min_k (Q_k^{(h_k(N_n'))})} < \infty, \tag{21}$$

where $h_k(N)$ is defined as in the proof of Theorem 2. If $\alpha_0, \ldots, \alpha_{m-1}$ are both strong and comparable, it is easily seen that the sequence $\{N_n\}$ of Theorem 2 and the new sequence $\{N_n'\}$ may be taken to be identical. Suppose that $\{M_n\}$ is both an N- and an N'-sequence, and that $\{f_n^*\}$ is the

corresponding polynomial sequence, and suppose that no sequence $\{\bar{M}_n\}$ is both an N- and an N'-sequence, where $\{\bar{M}_n\}$ is the monotonic sequence formed by adjoining to $\{M_n\}$ an infinite number of elements to which correspond polynomials $\bar{f}^*$ distinct from all the f^*. Then clearly

$$h_k(M_{n+1}) = h_k(M_n) + 1$$

for n large and $0 \leqslant k \leqslant m-1$, and it is straightforward to show that both

$$\frac{\log H(f_n{}^*)}{\log \min_k (Q_k^{(h_k(M_n))})} \quad \text{and} \quad \frac{\log H^{\omega_n}(f_n{}^*)}{\log \min_k (Q_k^{(h_k(M_n)+1)})}$$

are bounded away from zero and infinity, and so, taking $F_n = f_n{}^*$, it is easily verified that all the hypotheses of Theorem 4, except possibly the third, are satisfied. The following result then follows from Theorems 3 and 4:

THEOREM 6. *Suppose that $\alpha_0, \ldots, \alpha_{m-1}$ are comparable strong Liouville numbers or rational numbers, one at least being irrational, and suppose that $\alpha_0 + \ldots + \alpha_{m-1} \xi^{m-1} + \xi^m = 0$. Then $\xi \varepsilon U_m$ if and only if every L_m-sequence $\{f_n{}^*\}$ for ξ has only finitely many reducible elements.*

The relation of comparability [defined by (21) for arbitrary increasing sequences $\{Q_k^{(n)}\}$] is rather difficult to work with, since it is not transitive. For example $\{10^{n!}\}$ and $\{10^{(2n)!}\}$ are comparable, as are $\{10^{n!}\}$ and $\{10^{(2n+1)!}\}$, but $\{10^{(2n)!}\}$ and $\{10^{(2n+1)!}\}$ are not. The concept was introduced by Maillet [3] who showed that if α, β are comparable Liouville numbers then each of $\alpha \pm \beta$, $\alpha\beta$ and α/β is either a Liouville number or rational, and those which are Liouville are comparable to each other and to α and β. (Unfortunately, he neglected the non-transitivity of comparability, and concluded that the set of all Liouville numbers comparable with a given one, together with the rationals, forms a field.)

That some restriction is necessary in Theorem 6, in addition to the requirement that $\alpha_0, \ldots, \alpha_{m-1}$ be Liouville numbers, is shown by a theorem recently proved by P. Erdös, to the effect that every real number is the root of an algebraic equation with Liouville or rational coefficients.

4. The following transcendence criterion is due to T. Schneider in the case where $q_n'' = 1$; Mahler pointed out that the proof applies in the more general theorem (*cf.* [4] for a detailed proof):

Suppose that ϑ is a real number, and that there is a constant κ and an infinite sequence of fractions p_n/q_n whose denominators increase monotonically, such that

$$\left| \vartheta - \frac{p_n}{q_n} \right| = q_n^{-\kappa_n} \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n} < \infty. \tag{22}$$

Then ϑ is transcendental if either of the following conditions is satisfied:

(i) $\kappa_n \geqslant \kappa > 2$;

(ii) $\kappa_n \geqslant \kappa > 1$ *and, for each n, q_n can be represented as a product $q_n' q_n''$, where q_n' is a power of the fixed integer q, and*

$$\lim_{n \to \infty} \frac{\log q_n''}{\log q_n} = 0.$$

On account of this theorem, it is of some interest to consider the numbers ϑ having approximations satisfying (22). If in Theorem 4, $F_n(x)$ is taken to be $q_n x - p_n$, and ξ is replaced by ϑ and ω_n by κ_n, then all hypotheses except (8) are fulfilled if κ_n remains bounded as $n \to \infty$; that this is the case is an easy consequence of (22). But an examination of the proof shows that the conclusion still holds if

$$\liminf_{n \to \infty} \kappa_n > m^2(r+1).$$

Since here the polynomials F_n are linear, $m = 1$ and we have the following theorem:

THEOREM 7. *If ϑ is a real number having a sequence of approximations p_n/q_n satisfying* (22), *where for all n*

$$r+1+\epsilon < \kappa_n \quad (r \text{ an integer, } \epsilon > 0),$$

then $\vartheta \notin U_1 \cup \ldots \cup U_r$.

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ON THE LATTICE DETERMINANTS OF TWO PARTICULAR POINT SETS

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It is well known that the star domain

$$K: \ |xy| \leqslant 1$$