

Dimers, webs, and positroids

Thomas Lam

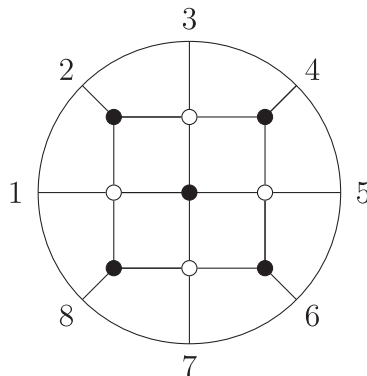
ABSTRACT

We study the dimer model for a planar bipartite graph N embedded in a disk, with boundary vertices on the boundary of the disk. Counting dimer configurations with specified boundary conditions gives a point in the totally non-negative Grassmannian. Considering pairing probabilities for the double-dimer model gives rise to Grassmann analogs of Rhoades and Skandera’s Temperley–Lieb immanants. The same problem for the (probably novel) triple-dimer model gives rise to the combinatorics of Kuperberg’s webs and Grassmann analogs of Pylyavskyy’s web immanants. This draws a connection between the square move of plabic graphs (or urban renewal of planar bipartite graphs), and Kuperberg’s square reduction of webs. Our results also suggest that canonical-like bases might be applied to the dimer model.

We furthermore show that these functions on the Grassmannian are compatible with restriction to positroid varieties. Namely, our construction gives bases for the degree 2 and degree 3 components of the homogeneous coordinate ring of a positroid variety that are compatible with the cyclic group action.

1. Introduction

Let N be a (weighted) planar bipartite graph embedded into the disk with n boundary vertices labeled $1, 2, \dots, n$ clockwise along the boundary of the disk. We study some algebraic aspects of the dimer configurations of N .



1.1. Dimers and the totally non-negative Grassmannian

A *dimer configuration*, or almost perfect matching Π of N is a collection of edges in N that uses each interior vertex exactly once, and some subset of the boundary vertices. The data of the subset of boundary vertices that are used in Π gives a boundary subset $I(\Pi) \subset [n]$. We

define a generating function

$$\Delta_I(N) = \sum_{\Pi: I(\Pi)=I} \text{wt}(\Pi),$$

where the weight $\text{wt}(\Pi)$ is the product of the weights of edges used in Π . These *boundary measurements* satisfy Plücker relations, and gives (see [9, 22, 26] and Theorem 2.1) a point $\tilde{M}(N)$ in the affine cone $\widetilde{\text{Gr}}(k, n)$ over the Grassmannian of k -planes in n -space (the value of k depends on N):

$$N \rightsquigarrow \tilde{M}(N) = (\Delta_I)_{I \in \binom{[n]}{k}} \in \widetilde{\text{Gr}}(k, n).$$

Indeed, the image $M(N) \in \text{Gr}(k, n)$ of $\tilde{M}(N)$ lies in Postnikov’s totally non-negative (TNN) Grassmannian $\text{Gr}(k, n)_{\geq 0}$ (see [21]), which is defined to be the set of points in the Grassmannian where all Plücker coordinates take non-negative values.

1.2. Double dimers and Temperley–Lieb immanants

A *double-dimer configuration* in N is an ordered pair (Π, Π') of two dimer configurations in N . Overlaying the two dimer configurations gives a collection of doubled edges, cycles of even length, and paths between boundary vertices (see the picture in Subsection 3.1), which we call a *Temperley–Lieb subgraph*. The paths between boundary vertices give a non-crossing pairing of some subset of the boundary vertices, studied, for example, by Kenyon and Wilson [6]. Let $\mathcal{A}_n = \{(\tau, T)\}$ (notation to be explained in Section 3) denote the set of partial non-crossing pairings (τ, T) on n -vertices. If in addition we fix the boundary subsets of Π and Π' , then this analysis gives the identity (Theorem 3.1)

$$\Delta_I(N)\Delta_J(N) = \sum_{(\tau, T)} F_{\tau, T}(N), \tag{1}$$

where $F_{\tau, T}(N)$ is a *Temperley–Lieb immanant* defined as the weight generating function of Temperley–Lieb subgraphs in N with specified partial non-crossing pairing (τ, T) , and the summation is over certain non-crossing pairings $(\tau, T) \in \mathcal{A}_n$ that are compatible with (I, J) . We show (Proposition 3.3) that $F_{\tau, T}$ are functions on the cone $\widetilde{\text{Gr}}(k, n)$ over the Grassmannian: that is, $F_{\tau, T}(N)$ only depends on $\tilde{M}(N)$. The functions $F_{\tau, T}$ are Grassmann analogs of the Temperley–Lieb immanants of Rhoades and Skandera [25].

Equation (1) leads to some inequalities between minors on $\text{Gr}(k, n)_{\geq 0}$. For example, we have (Proposition 3.12)

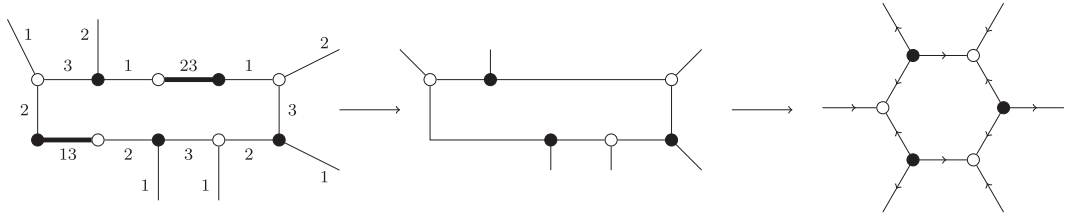
$$\Delta_{\text{sort}_1(I, J)}(X)\Delta_{\text{sort}_2(I, J)}(X) \geq \Delta_I(X)\Delta_J(X)$$

for $X \in \text{Gr}(k, n)_{\geq 0}$, where $\text{sort}_1, \text{sort}_2$ are defined in Subsection 3.5. This inequality was also independently discovered by Farber and Postnikov [2]. Indeed, there is an analogy between these inequalities and the Schur function inequalities of Lam, Postnikov and Pylyavskyy [15].

1.3. Triple dimers and web immanants

A *triple-dimer configuration* is an ordered triple (Π_1, Π_2, Π_3) of dimer configurations. Overlaying these dimer configurations on top of each other, we obtain a *weblike* subgraph $G \subset N$ (to be defined in the text) consisting of some tripled edges, some even length cycles that alternate between single and doubled edges, and some components illustrated below (thick edges are

present in two out of the three dimer configurations):



Informally, these components consist of trivalent vertices joined together by paths that alternate between single and doubled edges. Such a weblike graph gives rise to a web W (shown on the right) in the sense of Kuperberg [11]. Kuperberg’s webs have directed edges, and our bipartite webs should be interpreted with all edges directed toward white interior vertices. Kuperberg gave a reduction algorithm for such graphs, reducing any web to a linear combination of *non-elliptic webs*.

The set of non-elliptic webs on n boundary vertices, denoted by \mathcal{D}_n , should be thought of as the set of possible connections in a triple-dimer configuration. For each $D \in \mathcal{D}_n$, we define a generating function $F_D(N)$, counting weblike subgraphs $G \subset N$, called a *web immanant*, and we show that $F_D(N)$ only depends on $\tilde{M}(N)$. In particular, if N and N' are related by certain moves, such as the square move (also called urban renewal), then $F_D(N) = \alpha_{N,N'} F_D(N')$ for a constant $\alpha_{N,N'}$ not depending on D . We also obtain (Theorem 4.13) an identity

$$\Delta_I \Delta_J \Delta_K = \sum_D a(I, J, K, D) F_D,$$

where $a(I, J, K, D)$ counts the number of ways to ‘consistently label’ D with (I, J, K) . This is a Grassmann analog of a result of Pylyavskyy [24].

1.4. Boundary, pairing, and web ensembles in planar bipartite graphs

Given a planar bipartite graph N , we may define

$$\mathcal{M}(N) := \{I(\Pi)\} = \left\{ I \in \binom{[n]}{k} \mid \Delta_I(N) > 0 \right\}$$

to be the collection of boundary subsets $I = I(\Pi)$ that occur in dimer configurations Π in N . Similarly, one defines

$$\mathcal{A}(N) := \{(\tau, T) \in \mathcal{A}_n \mid F_{\tau, T}(N) > 0\}$$

to be the collection of partial non-crossing pairings (τ, T) that occur in double dimers in N , and

$$\mathcal{D}(N) := \{D \in \mathcal{D}_n \mid F_D(N) > 0\}$$

to be the collection of web connections D that occur in triple dimers in N . It is not obvious (but follows from our results) that knowing $\mathcal{M}(N)$ determines both $\mathcal{A}(N)$ and $\mathcal{D}(N)$. We propose to call $\mathcal{M}(N)$, $\mathcal{A}(N)$, and $\mathcal{D}(N)$ the *boundary ensemble*, *pairing ensemble*, and *web ensemble* of N , respectively.

1.5. Positroids and bases of homogeneous coordinate rings of positroid varieties

If $X \in \text{Gr}(k, n)$, then the matroid of X is the collection

$$\mathcal{M}_X := \left\{ I \in \binom{[n]}{k} \mid \Delta_I(X) \neq 0 \right\}$$

of k -element subsets labeling non-vanishing Plücker coordinates. A matroid \mathcal{M} is a *positroid* if $\mathcal{M} = \mathcal{M}_X$ for some $X \in \text{Gr}(k, n)_{\geq 0}$. Thus $\mathcal{M}(N) = \mathcal{M}_{M(N)}$ is always a positroid, and it follows from Postnikov’s work [21] that every positroid occurs in this way.

The positroid stratification [8, 21] is the stratification $\text{Gr}(k, n) = \bigcup_{\mathcal{M}} \mathring{\Pi}_{\mathcal{M}}$ obtained by intersecting n cyclically rotated Schubert stratifications. Each such stratum is labeled by a positroid \mathcal{M} . We denote the corresponding closed positroid variety by $\Pi_{\mathcal{M}}$ and the open stratum by $\mathring{\Pi}_{\mathcal{M}}$. For any $X \in (\Pi_{\mathcal{M}})_{>0} = \mathring{\Pi}_{\mathcal{M}} \cap \text{Gr}(k, n)_{\geq 0}$, we have $\mathcal{M}_X = \mathcal{M}$, so all TNN points in an open positroid stratum have the same matroid. Picking $X \in (\Pi_{\mathcal{M}})_{>0}$, one defines

$$\mathcal{A}(\mathcal{M}) := \{(\tau, T) \in \mathcal{A}_n \mid F_{\tau, T}(X) > 0\}$$

and

$$\mathcal{D}(\mathcal{M}) := \{D \in \mathcal{D}_n \mid F_D(X) > 0\}.$$

We show that $\mathcal{A}(\mathcal{M})$ and $\mathcal{D}(\mathcal{M})$ do not depend on the choice of X , but only \mathcal{M} . In particular, $\mathcal{A}(N) = \mathcal{A}(\mathcal{M}(N))$ and $\mathcal{D}(N) = \mathcal{D}(\mathcal{M}(N))$. In a future work, we will give a direct description of $\mathcal{A}(\mathcal{M})$ similar to Oh’s description [20] of \mathcal{M} as an intersection of cyclically rotated Schubert matroids. It would also be interesting to do so for $\mathcal{D}(\mathcal{M})$.

Let $\mathbb{C}[\Pi_{\mathcal{M}}]$ denote the homogeneous coordinate ring of a positroid variety. We prove the following statements.

- (1) For each positroid \mathcal{M} , the set

$$\{F_{\tau, T} \mid (\tau, T) \in \mathcal{A}(\mathcal{M})\}$$

forms a basis of the degree 2 part of $\mathbb{C}[\Pi_{\mathcal{M}}]$ (Theorem 3.10).

- (2) The set

$$\{F_D \mid D \in \mathcal{D}(\mathcal{M})\}$$

forms a basis of the degree 3 part of $\mathbb{C}[\Pi_{\mathcal{M}}]$ (Theorem 4.7).

Thus we have a combinatorially defined, cyclically invariant basis for these parts of the homogeneous coordinate rings. These bases are likely related to (but not identical to, see [7]) Lusztig’s dual canonical basis. We remark that Launiois and Lenagan [16] have studied the cyclic action on the quantized coordinate ring of the Grassmannian.

There is also a relation to cluster structures on Grassmannians and positroid varieties that for simplicity I have chosen to omit discussing in this work. We note that Fomin and Pylyavskyy [3] have constructed, using generalizations of Kuperberg’s webs, bases of certain rings of invariants, that include Grassmannians of 3-planes as special cases. Marsh and Scott [18] have investigated twists of Grassmannians in terms of dimer configurations. Recently, cluster structures related to the coordinate rings of positroid varieties have also been studied by Leclerc [17] and Muller and Speyer [19].

We hope to return to the connection with canonical and semicanonical bases, and cluster structures in the future.

2. The dimer model and the TNN Grassmannian

2.1. TNN Grassmannian

In this section, we fix integers k, n and consider the real Grassmannian $\text{Gr}(k, n)$ of (linear) k -planes in \mathbb{R}^n . Recall that each $X \in \text{Gr}(k, n)$ has Plücker coordinates $\Delta_I(X)$ labeled by k -element subsets $I \subset [n]$, defined up to a single common scalar. It will be convenient for us to talk about the Plücker coordinates Δ_I as genuine functions. We will thus often work with the affine cone $\widetilde{\text{Gr}}(k, n)$ over the Grassmannian. A point in $\widetilde{X} \in \widetilde{\text{Gr}}(k, n)$ is given by a collection

of Plücker coordinates $\Delta_I(\tilde{X})$, satisfying the Plücker relations [4] (without the equivalence relation where we scale all coordinates by a common scalar).

Suppose that $\tilde{X}, \tilde{X}' \in \widetilde{\text{Gr}}(k, n)$ represent the same point in $\text{Gr}(k, n)$. Then there exists a non-zero scalar $a \in \mathbb{R}$ such that $\Delta_I(\tilde{X}) = a\Delta_I(\tilde{X}')$ for all $I \in \binom{[n]}{k}$. As a shorthand, we then write $\tilde{X} = a\tilde{X}'$.

The TNN Grassmannian $\text{Gr}(k, n)_{\geq 0}$ is the subset of $\text{Gr}(k, n)$ consisting of points X represented by non-negative Plücker coordinates $\left\{ \Delta_I(X) \mid I \in \binom{[n]}{k} \right\}$. Similarly, one can define the TNN part $\widetilde{\text{Gr}}(k, n)_{\geq 0}$ of the cone over the Grassmannian.

The cyclic group acts on $\text{Gr}(k, n)_{\geq 0}$ (and on $\widetilde{\text{Gr}}(k, n)_{\geq 0}$) with generator χ acting by the map

$$\chi : (v_1, v_2, \dots, v_n) \longmapsto (v_2, \dots, v_n, (-1)^{k-1}v_1),$$

where v_i are columns of some $k \times n$ matrix representing X .

2.2. Dimer model for a bipartite graph with boundary vertices

Let N be a weighted bipartite network embedded in the disk with n boundary vertices, labeled $1, 2, \dots, n$ in clockwise order. Each vertex (including boundary vertices) is colored either black or white, and all edges join black vertices to white vertices. We let d be the number of interior white vertices minus the number of interior black vertices. Furthermore, we let $d' \in [n]$ be the number of white boundary vertices. Finally, we assume that all boundary vertices have degree 1, and that edges cannot join boundary vertices to boundary vertices. We shall also use the standard convention that in our diagrams unlabeled edges have weight 1.

Since the graph is bipartite, the condition that boundary vertices have degree 1 ensures that the coloring of the boundary vertices is determined by the interior part of the graph. So we will usually omit the color of boundary vertices from pictures.

A *dimer configuration* or almost perfect matching Π is a subset of edges of N such that

- (1) each interior vertex is used exactly once;
- (2) boundary vertices may or may not be used.

The boundary subset $I(\Pi) \subset \{1, 2, \dots, n\}$ is the set of black boundary vertices that are used by Π union the set of white boundary vertices that are not used. By our assumptions, we have $|I(\Pi)| = k := d' + d$.

Define the *boundary measurement* $\Delta_I(N)$ as follows. For $I \subset [n]$, a k -element subset,

$$\Delta_I(N) = \sum_{\Pi: I(\Pi)=I} \text{wt}(\Pi),$$

where $\text{wt}(\Pi)$ is the product of the weight of the edges in Π . The first part of the following result is essentially due to Kuo [9], and we will prove it using the language of Temperley–Lieb immanants in Section 3. The second part of the theorem is due to Postnikov [21] who counted paths instead of matchings; see also [13] for a proof in the spirit of the current work. The relation between Postnikov’s theory and the dimer model was suggested by the works of Talaska [26] and Postnikov, Speyer and Williams [22].

THEOREM 2.1. *Suppose that N has non-negative real weights. Then the coordinates $(\Delta_I(N))_{I \in \binom{[n]}{k}}$ defines a point $\tilde{M}(N)$ in the cone over the Grassmannian $\widetilde{\text{Gr}}(k, n)_{\geq 0}$. Furthermore, every $X \in \widetilde{\text{Gr}}(k, n)_{\geq 0}$ is realizable as $X = \tilde{M}(N)$ by a planar bipartite graph.*

We let $M(N)$ denote the equivalence class of $\tilde{M}(N)$ in $\text{Gr}(k, n)$. We will often implicitly assume that N does have dimer configurations, so that $M(N)$ is well defined.

2.3. Gauge equivalences and local moves

We now discuss operations on N that preserve $M(N)$.

Let N be a planar bipartite graph. If e_1, e_2, \dots, e_d are incident to an interior vertex v , then we can multiply all of their edge weights by the same constant $c \in \mathbb{R}_{>0}$ to get a new graph N' , and we have $M(N') = M(N)$. This is called a gauge equivalence.

We also have the following local moves, replacing a small local part of N by another specific graph to obtain N' .

(M1) Spider move [5], square move [21], or urban renewal [1, 23]: assuming that the leaf edges of the spider have been gauge fixed to 1, the transformation is

$$a' = \frac{a}{ac + bd}, \quad b' = \frac{b}{ac + bd}, \quad c' = \frac{c}{ac + bd}, \quad d' = \frac{d}{ac + bd}$$

(M2) Valent two-vertex removal. If v has degree 2, we can gauge fix both incident edges (v, u) and (v, u') to have weight 1, then contract both edges (that is, we remove both edges, and identify u with u'). Note that if v is a valent two-vertex adjacent to boundary vertex b , with edges (v, b) and (v, u) , then removing v produces an edge (b, u) , and the color of b flips.

(R1) Multiple edges with same endpoints are the same as one edge with sum of weights.

(R2) Leaf removal. Suppose that v is leaf, and (v, u) the unique edge incident to it. Then we can remove both v and u , and all edges incident to u . However, if there is a boundary edge (b, u) where b is a boundary vertex, then that edge is replaced by a boundary edge (b, w) , where w is a new vertex with the same color as v .

(R3) Dipoles (two degree 1 vertices joined by an edge) can be removed.

The following result is a case-by-case check.

PROPOSITION 2.2. Each of these relations preserves $M(N)$.

The following result is due to Postnikov [21] in the more general setting of plabic graphs.

THEOREM 2.3. Suppose that N and N' are planar bipartite graphs with $M(N) = M(N')$. Then N and N' are related by local moves and gauge equivalences.

2.4. Positroid stratification

Let $X \in \text{Gr}(k, n)$. The matroid \mathcal{M}_X of X is the collection

$$\mathcal{M}_X := \left\{ I \in \binom{[n]}{k} \mid \Delta_I(X) \neq 0 \right\}$$

of k -element subsets of $[n]$ labeling non-vanishing Plücker coordinates of X . If $X \in \text{Gr}(k, n)_{\geq 0}$, then \mathcal{M} is called a *positroid*. Unlike matroids in general, positroids have been completely classified and characterized [21]. Oh [20] shows that positroids are exactly the intersections of cyclically rotated Schubert matroids. In a forthcoming work, Lam and Postnikov show that positroids are exactly the matroids that are closed under sorting (see Subsection 3.5).

We have a stratification

$$\text{Gr}(k, n) = \bigcup_{\mathcal{M}} \overset{\circ}{\Pi}_{\mathcal{M}}$$

of the Grassmannian by *open positroid varieties*, labeled by positroids \mathcal{M} . The strata $\overset{\circ}{\Pi}_{\mathcal{M}}$ are defined as the intersections of cyclically rotated Schubert cells (see [8]). The closure $\Pi_{\mathcal{M}}$ of $\overset{\circ}{\Pi}_{\mathcal{M}}$ is an irreducible subvariety of the Grassmannian called a (*closed*) *positroid variety*. Postnikov [21] showed the following theorem.

THEOREM 2.4. (1) *The intersection $(\Pi_{\mathcal{M}})_{>0} = \text{Gr}(k, n)_{\geq 0} \cap \overset{\circ}{\Pi}_{\mathcal{M}}$ is homeomorphic to $\mathbb{R}_{>0}^d$, where $d = \dim(\Pi_{\mathcal{M}})$.*

(2) *For each positroid \mathcal{M} , there exists a planar bipartite graph $N_{\mathcal{M}} = N_{\mathcal{M}}(t_1, t_2, \dots, t_d)$, where d of the edges have weights given by parameters t_1, \dots, t_d and all other weights are 1, such that*

$$(t_1, t_2, \dots, t_d) \mapsto M(N_{\mathcal{M}}(t_1, t_2, \dots, t_d))$$

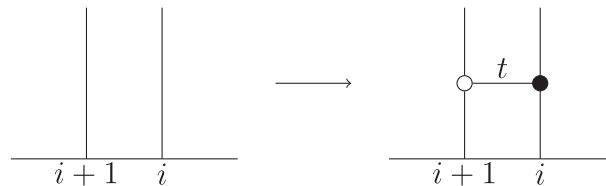
is a parameterization of $(\Pi_{\mathcal{M}})_{>0}$ as (t_1, t_2, \dots, t_d) vary over $\mathbb{R}_{>0}^d$.

In particular, positroids can be characterized completely in terms of planar bipartite graphs. Namely, \mathcal{M} is a positroid if and only if it is the matroid \mathcal{M}_X of a point $X = M(N)$, where N is a planar bipartite graph.

It follows from Theorem 2.4 that $(\Pi_{\mathcal{M}})_{>0}$ is Zariski-dense in $\Pi_{\mathcal{M}}$. We shall construct elements of the homogeneous coordinate ring $\mathbb{C}[\Pi_{\mathcal{M}}]$ using the combinatorics of planar bipartite graphs.

2.5. Bridge and lollipop recursion

We will require two additional operations on planar bipartite graphs that do not preserve $M(N)$. The first operation is *adding a bridge at i* , black at i , and white at $i + 1$. It modifies a bipartite graph near the boundary vertices i and $i + 1$:



The bridge edge is the edge labeled t in the above picture. Note that in general this modification might create a graph that is not bipartite, for example, if in the original graph i is connected to a black vertex. However, by adding valent two vertices using local move (M2), we can always assume we obtain a bipartite graph.

The second operation is *adding a lollipop* which can be either white or black. This inserts a new boundary vertex connected to an interior leaf. The new boundary vertices are then relabeled:



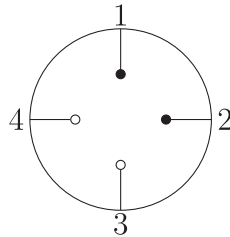
In the following, we will use $N_{\mathcal{M}}$ to denote any parameterized planar bipartite graph satisfying Theorem 2.4(2). The following result is proved in [13].

THEOREM 2.5. *Suppose that \mathcal{M} is not represented by the empty graph. Let $d = \dim(\Pi_{\mathcal{M}})$. Then there exists a positroid \mathcal{M}' such that either*

- (1) $\dim(\Pi_{\mathcal{M}'}) = d - 1$ and $N_{\mathcal{M}}(t_1, t_2, \dots, t)$ is obtained from $N_{\mathcal{M}'}(t_1, t_2, \dots, t_{d-1})$ by adding a bridge black at i and white at $i + 1$, such that the bridge edge has weight t , and all other added edges have weight 1; or
- (2) $\dim(\Pi_{\mathcal{M}'}) = d$ and $N_{\mathcal{M}}(t_1, t_2, \dots, t_d)$ is obtained from $N_{\mathcal{M}'}(t_1, t_2, \dots, t_d)$ by inserting a lollipop at some new boundary vertex i .

Note however that not all plabic graphs can be constructed recursively in this manner.

If $\dim(\Pi_{\mathcal{M}}) = 0$, then \mathcal{M} consists of a single subset I , and $\Pi_{\mathcal{M}}$ is the unique point in $\text{Gr}(k, n)$ where all Plücker variables are 0, except $\Delta_I \neq 0$. Such a point is represented by a lollipop graph N , with white lollipops at the locations specified by I . For example, the planar bipartite graph



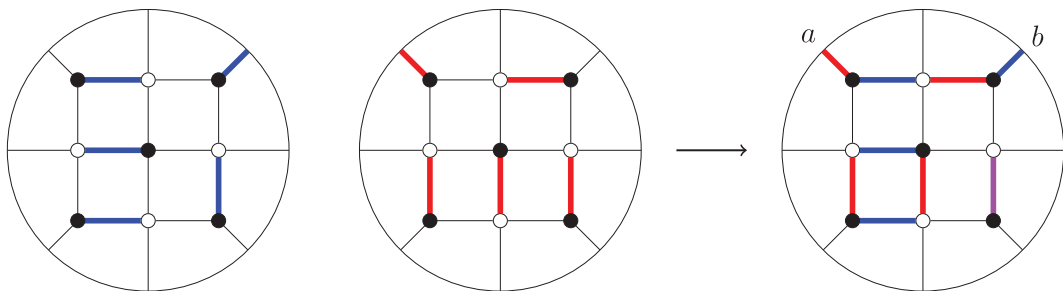
represents such a point with $I = \{3, 4\}$.

3. Temperley–Lieb immanants and the double-dimer model

3.1. Double dimers

A (k, n) -partial non-crossing pairing is a pair (τ, T) , where τ is a matching of a subset $S = S(\tau) \subset \{1, 2, \dots, n\}$ of even size, such that when the vertices are arranged in order on a circle, and the edges are drawn in the interior, then the edges do not intersect; and T is a subset of $[n] \setminus S$ satisfying $|S| + 2|T| = 2k$. Let $\mathcal{A}_{k,n}$ denote the set of (k, n) -partial non-crossing pairings.

A subgraph $\Sigma \subset N$ is a *Temperley–Lieb subgraph* if it is a union of connected components, each of which is: (a) a path between boundary vertices, or (b) an interior cycle, or (c) a single edge, such that every interior vertex is used. Let (Π, Π') be a double dimer (that is, a pair of dimer configurations) in N . Then the union $\Sigma = \Pi \cup \Pi'$ is a Temperley–Lieb subgraph:



The set S of vertices used by the paths on the Temperley–Lieb subgraph is given by $S = (I(\Pi) \setminus I(\Pi')) \cup (I(\Pi') \setminus I(\Pi))$. Thus each Temperley–Lieb subgraph Σ gives a partial non-crossing pairing on $S \subset \{1, 2, \dots, n\}$. For example, in the above picture we have that a is paired with b and $S = \{a, b\}$. Note that a Temperley–Lieb subgraph Σ can arise from a pair of matchings in many different ways: it does not remember which edge in a path came from which of the two original dimer configurations.

For each (k, n) -partial non-crossing pairing $(\tau, T) \in \mathcal{A}_{k,n}$, define the *Temperley–Lieb immanant*

$$F_{\tau,T}(N) := \sum_{\Sigma} \text{wt}(\Sigma)$$

to be the sum over Temperley–Lieb subgraphs Σ which give boundary path pairing τ , and T contains black boundary vertices used twice in Σ , together with white boundary vertices not used in Σ . Here $\text{wt}(\Sigma)$ is the product of all weights of edges in Σ times $2^{\#\text{cycles}}$; also, the weight of a connected component that is an edge in Σ is the square of the weight of that edge. The function $F_{\tau,T}$ is a Grassmann analog of Rhoades and Skandera’s Temperley–Lieb immanants [25]. It would also be reasonable to call these A_1 -web immanants.

Given $I, J \in \binom{[n]}{k}$, we say that a (k, n) -partial non-crossing pairing (τ, T) is compatible with I, J if:

- (1) $S(\tau) = (I \setminus J) \cup (J \setminus I)$, and each edge of τ matches a vertex in $(I \setminus J)$ with a vertex in $(J \setminus I)$, and
- (2) $T = I \cap J$.

THEOREM 3.1. For $I, J \in \binom{[n]}{k}$, we have

$$\Delta_I(N)\Delta_J(N) = \sum_{\tau,T} F_{\tau,T}(N),$$

where the summation is over all (k, n) -partial non-crossing pairings (τ, T) compatible with I, J .

Proof. The left-hand side is the generating function of double dimers (Π, Π') in N , where $I(\Pi) = I$ and $I(\Pi') = J$. By the above discussion, the union $\Pi \cup \Pi'$ is a Temperley–Lieb subgraph Σ . Given a Temperley–Lieb subgraph Σ of this form, there are exactly $2^{\#\text{cycles}}$ ways in which it arises from a double-dimer enumerated by the left-hand side. It remains to show that the Temperley–Lieb subgraphs that occur are exactly the ones with boundary pairing (τ, T) compatible with I, J .

Let Π, Π' be almost perfect matchings of N such that $I(\Pi) = I$ and $I(\Pi') = J$. Let p be one of the boundary paths in $\Pi \cup \Pi'$, with endpoints s and t . If s and t have the same color, then the path is even in length. If s and t have different colors, then the path is odd in length. In both cases, one of s and t belongs to $I \setminus J$ and the other belongs to $J \setminus I$. \square

3.2. Proof of first statement in Theorem 2.1

We shall use the following result.

PROPOSITION 3.2. A non-zero vector $(\Delta_I)_{I \in \binom{[n]}{k}}$ lies in $\text{Gr}(k, n)$ if and only if the Plücker relation with one index swapped is satisfied:

$$\sum_{r=1}^k (-1)^r \Delta_{i_1, i_2, \dots, i_{k-1}, j_r} \Delta_{j_1, \dots, j_{r-1}, \hat{j}_r, j_{r+1}, \dots, j_{k+1}} = 0, \tag{2}$$

where \hat{j}_r denotes omission.

The convention is that Δ_I is antisymmetric in its indices, so, for example, $\Delta_{13} = -\Delta_{31}$.

Now use Theorem 3.1 to expand (2) with $\Delta_I = \Delta_I(N)$ as a sum of $F_{\tau,T}(N)$ over pairs (τ, T) (with multiplicity). We note that the set T is always the same in any term that comes up. We assume that $i_1 < i_2 < \dots < i_{k-1}$ and $j_1 < j_2 < \dots < j_{k+1}$.

So each term $F_{\tau,T}$ is labeled by (I, J, τ) , where I, J is compatible with τ , and I, J occur as a term in (2). We provide an involution on such terms. By the compatibility condition, all but one of the edges in τ uses a vertex in $\{i_1, i_2, \dots, i_{k-1}\}$. The last edge is of the form (j_a, j_b) , where $j_a \in I$ and $j_b \in J$. The involution swaps j_a and j_b in I, J , but keeps τ the same.

Finally, we show that this involution is sign-reversing. Let $I' = I \cup \{j_b\} - \{j_a\}$ and $J' = J \cup \{j_a\} - \{j_b\}$. Then the sign associated to the term labeled by (I, J, τ) is equal to (-1) to the power of $\#\{r \in [k-1] \mid i_r > j_a\} + a$. Note that by the non-crossingness of the edges in τ , there must be an even number of vertices belonging to $(I \setminus J) \cup (J \setminus I)$ strictly between j_a and j_b . Thus $(b - a) - 1 = \#\{r \in [k-1] \mid i_r > j_b\} - \#\{r \in [k-1] \mid i_r > j_a\}$ is even and $(a + \#\{r \in [k-1] \mid i_r > j_a\}) + (b + \#\{r \in [k-1] \mid i_r > j_b\})$ is odd. So the sign changes.

3.3. Transition formulae

So far $F_{\tau,T}$ has been defined as a function of a planar bipartite graph N .

PROPOSITION 3.3. *The function $F_{\tau,T}(N)$ depends only on $\tilde{M}(N)$, and thus gives a function $F_{\tau,T}$ on $\widetilde{\text{Gr}}(k, n)$.*

To prove this result, one could check the local moves and use Theorem 2.3. This is straightforward, and we will do a similar check later for web immanants (Proposition 4.3). Instead, here we will argue somewhat indirectly, by inverting the formula in Theorem 3.1.

We say that (I, J) is a *standard monomial* if $i_r \leq j_r$ for all r (in other words, I, J form the columns of a semistandard tableau).

PROPOSITION 3.4. *There is a bijection*

$$\theta : \left\{ \text{standard monomials in } \binom{[n]}{k} \right\} \longrightarrow \mathcal{A}_{k,n},$$

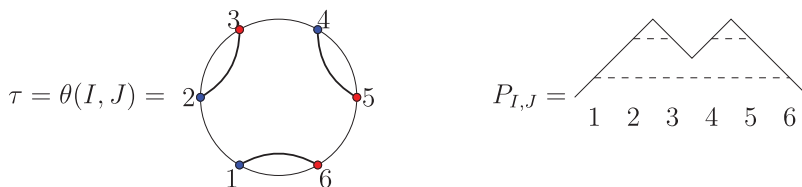
and a partial order \leq on standard monomials such that the transition matrix between $\{\Delta_I(N)\Delta_J(N) \mid (I, J) \text{ standard}\}$ and $\{F_{\tau,T} \mid (\tau, T) \in \mathcal{A}_{k,n}\}$ is unitriangular. More precisely,

$$\Delta_I(N)\Delta_J(N) = F_{\theta(I,J)}(N) + \sum_{(I',J')} a_{(I,J),(I',J')} F_{\theta(I',J')}(N),$$

where $a_{(I,J),(I',J')} \in \{0, 1\}$ and $a_{(I,J),(I',J')} = 1$ implies $(I', J') < (I, J)$.

Proof. Since the subset $T = I \cap J$ plays little role, we shall assume $T = \emptyset$, and for simplicity, $I \cup J = [n]$.

Then (I, J) is a two-column tableaux using the number $1, 2, \dots, 2k = n$. The bijection θ sends such I, J to the non-crossing pairing τ on $[2k]$ given by connecting i_r to $j_s > i_r$, where s is chosen minimal so that $\#(I \cap (j_s - i_r)) = \#(J \cap (j_s - i_r))$. This bijection can be described in terms of Dyck paths as follows: draw a Dyck path $P_{I,J}$ having a diagonally upward edge E_i at positions specified by $i \in I$, and a diagonally downward edge D_j at positions specified by $j \in J$. Then τ joins i to j if the horizontal rightwards ray starting at E_i intersects D_j before it intersects any other edge. For example, the bijection sends $(I, J) = (124, 356)$ to the following non-crossing pairing and Dyck path:



The partial order \leq is the following: $(I', J') \leq (I, J)$ if the Dyck path $P_{I,J}$ stays weakly below $P_{I',J'}$ the entirety of the path. To see this, suppose that $P_{I,J}$ goes above $P_{I',J'}$ somewhere. Let a be the first position this happens. Then a is an up step in $P_{I,J}$ (that is, $a \in I$) and a down step in $P_{I',J'}$ (that is, $a \in J'$). Suppose that a is paired with $a' < a$ in $\theta(I', J')$. Then the edges at positions a' and a are at the same height in $P_{I',J'}$. Since $P_{I,J}$ is weakly below $P_{I',J'}$ at position a' , it follows that the edges at positions a' and a are at different heights in $P_{I,J}$. So the cardinalities $|I \cap (a', a)|$ and $|J \cap (a', a)|$ differ, and thus pairing a' and a is not compatible with (I, J) . \square

Thus $F_{\tau,T}(N)$ can be expressed in terms of the Plücker coordinates $\Delta_I(N)$. It follows that $F_{\tau,T}$ are functions on $\text{Gr}(k, n)$ and Proposition 3.3 follows. Since $\{\Delta_I \Delta_J \mid (I, J) \text{ standard}\}$ form a basis for the degree 2 part of the homogeneous coordinate ring $\mathbb{C}[\text{Gr}(k, n)]$ in the Plücker embedding, we have the following corollary.

COROLLARY 3.5. *The set $\{F_{\tau,T}\}$ forms a basis for the degree 2 part of the homogeneous coordinate ring of the $\text{Gr}(k, n)$ in the Plücker embedding. In particular, the number $|\mathcal{A}_{k,n}|$ of (k, n) -non-crossing pairings is equal to the number of semistandard tableaux of shape 2^k filled with numbers $1, 2, \dots, n$.*

3.4. Restriction to positroid varieties

PROPOSITION 3.6. *Let N be obtained from N' by adding a bridge black at i to white at $i + 1$ with bridge edge having weight t . Then*

$$F_{\tau,T}(N) = \begin{cases} tF_{\tau-(i,i+1),T \cup \{i+1\}} + F_{\tau,T}, & (i, i + 1) \text{ is in } \tau, \\ F_{\tau,T}, & (i, a) \text{ and } (i + 1, b) \text{ are in } \tau, \\ F_{\tau,T}, & (i, a) \text{ is in } \tau \text{ and } i + 1 \in T, \\ F_{\tau,T} + tF_{\tau-(i,a) \cup (i+1,a),T}, & (i, a) \text{ is in } \tau, \text{ but } i + 1 \notin S \cup T, \\ F_{\tau,T} + tF_{\tau-(i+1,b) \cup (i,b),T}, & (i + 1, b) \text{ is in } \tau \text{ and } i \in T, \\ F_{\tau,T}, & (i + 1, b) \text{ is in } \tau, \text{ but } i \notin S \cup T, \\ F_{\tau,T}, & \text{neither } i \text{ nor } i + 1 \text{ is in } \tau, \text{ and } i \notin T \text{ or } i + 1 \in T \end{cases}$$

and

$$F_{\tau,T}(N) = t^2 F_{\tau,T-\{i\} \cup \{i+1\}} + t \sum_{(a,b) \in \tau} F_{\tau-(a,b) \cup (i,a) \cup (i+1,b),T-\{i\}} + 2t F_{\tau \cup (i,i+1),T-\{i\}} + F_{\tau,T}$$

if neither i nor $i + 1$ is in τ , and $i \in T$, but $i + 1 \notin T$. Here $F_{\tau,T} = F_{\tau,T}(N')$ and in the summation, (a, b) is ordered so that (i, a) and $(i + 1, b)$ are non-crossing.

Proof. The result is a case-by-case check. We explain only the last case.

Thus suppose that neither i nor $i + 1$ is in τ , and $i \in T$, but $i + 1 \notin T$. Let Σ be a Temperley–Lieb subgraph that contributes to $F_{\tau,T}(N)$. Then Σ does not use either of the two boundary edges incident to vertices i and $i + 1$. The possibilities for Σ are: (1) those that have the bridge edge as an isolated component; (2) those that have the bridge edge as part of a boundary path; (3) those that have the bridge edge as part of an interior cycle; and (4) those where the bridge edge is not used at all. These cases correspond to the four terms in the stated formula. \square

REMARK 3.7. The Lie group $\text{GL}(n)$ acts on $\text{Gr}(k, n)$. Since adding a bridge corresponds to acting by a one-parameter subgroup $x_i(t) = \exp(te_i)$ (see [13]), Proposition 3.6 determines the infinitesimal action of the Chevalley generators of $\mathfrak{gl}(n)$ on the functions $F_{\tau,T}$.

Define

$$\mathcal{A}(N) := \{(\tau, T) \in \mathcal{A}_{k,n} \mid F_{\tau,T}(N) \neq 0\}.$$

Let \mathcal{M} be a positroid of rank k on $[n]$. Let N be a planar bipartite graph representing \mathcal{M} . Then we define $\mathcal{A}(\mathcal{M}) := \mathcal{A}(N)$.

LEMMA 3.8. *The set $\mathcal{A}(\mathcal{M})$ does not depend on the choice of N .*

Proof. By Proposition 3.3, $F_{\tau,T}(N)$ depends (up to some global scalar) only on the point $M(N) \in \text{Gr}(k, n)_{\geq 0}$ representing N . Also, $\mathcal{A}(\mathcal{M})$ does not depend on the weights of N chosen, only the underlying unweighted bipartite graph. The result then follows from Theorem 2.4. \square

REMARK 3.9. The subset $\mathcal{A}(\mathcal{M}) \subset \mathcal{A}_{k,n}$ is a ‘degree 2’ analog of the positroid \mathcal{M} . It would be interesting to give a description of $\mathcal{A}(\mathcal{M})$ that does not depend on a choice of N , similar to Oh’s theorem [20] characterizing \mathcal{M} .

THEOREM 3.10. *The set*

$$\{F_{\tau,T} \mid (\tau, T) \in \mathcal{A}(\mathcal{M})\}$$

is a basis for the space for the degree 2 component of the homogeneous coordinate ring $\mathbb{C}[\Pi_{\mathcal{M}}]$.

In other words, the functions $F_{\tau,T}$ either restrict to 0 on $\Pi_{\mathcal{M}}$, or they form part of a basis.

Proof. It is known that every element of $\mathbb{C}[\Pi_{\mathcal{M}}]$ is obtained from restriction from $\mathbb{C}[\text{Gr}(k, n)]$ (see [8]). So certainly $\{F_{\tau,T} \mid (\tau, T) \in \mathcal{A}_{k,n}\}$ span the stated space. So it suffices to show that $\{F_{\tau,T} \mid (\tau, T) \in \mathcal{A}(\mathcal{M})\}$ is linearly independent.

We proceed by induction first on n and then on the dimension of $\Pi_{\mathcal{M}}$. The claim is trivially true when $\Pi_{\mathcal{M}}$ is a point.

Let \mathcal{M} be a positroid. By the bridge-lollipop recursion (Theorem 2.5), either

- (1) a plabic graph N for \mathcal{M} contains a lollipop, or
- (2) a plabic graph N for \mathcal{M} is obtained from a plabic graph N' for \mathcal{M}' by adding a bridge, where $\dim(\Pi_{\mathcal{M}'}) = \dim(\Pi_{\mathcal{M}}) - 1$.

In the first case, let N' be the plabic graph with $n - 1$ boundary vertices where a lollipop has been removed. The inductive hypothesis for $\mathcal{M}(N')$ immediately gives the claim for \mathcal{M} .

In the second case, let $d = \dim(\Pi_{\mathcal{M}'})$. Then a dense subset of $\Pi_{\mathcal{M}'}$ can be parameterized by assigning weights t_1, t_2, \dots, t_d to d of the edges of N' . By the inductive hypothesis, the functions $\{F_{\tau,T}(N') \mid (\tau, T) \in \mathcal{A}(\mathcal{M}')\}$ are then linearly independent polynomials in t_1, t_2, \dots, t_d . Let V denote the span of these polynomials. We may assume that N is obtained from N' by adding a bridge black at i to white at $i + 1$ with weight t , allowing us to use Proposition 3.6.

Note that

$$\{F_{\tau,T}(N) \mid (\tau, T) \in \mathcal{A}(\mathcal{M})\}$$

can then be thought of as a set of polynomials in t , with coefficients in V . We need to show that these polynomials $p_{\tau,T}(t) = F_{\tau,T}(N)$ are linearly independent. Suppose that there exists a linear relation

$$\sum_{(\tau,T) \in \mathcal{A}(\mathcal{M})} a_{\tau,T} p_{\tau,T}(t) = 0.$$

Then we will get a linear relation for each of the coefficients of $t^2, t, 1$. Consider first the linear relation for the constant coefficient. By Proposition 3.6, we get

$$0 = \sum_{(\tau, T) \in \mathcal{A}(\mathcal{M}')} a_{\tau, T} [t^0] p_{\tau, T}(t) = \sum_{(\tau, T) \in \mathcal{A}(\mathcal{M}')} a_{\tau, T} F_{\tau, T}(N').$$

By the inductive hypothesis, we see that $a_{\tau, T} = 0$ for $(\tau, T) \in \mathcal{A}(\mathcal{M}')$.

Now let us write

$$[t^1] \sum_{(\tau, T) \in \mathcal{A}(\mathcal{M})} a_{\tau, T} p_{\tau, T}(t) = \sum_{(\kappa, R) \in \mathcal{A}(\mathcal{M}')} b_{\kappa, R} F_{\kappa, R}(N').$$

By Proposition 3.6,

$$b_{\kappa, R} = \begin{cases} a_{\kappa \cup (i, i+1), R - \{i+1\}} & \text{if } i \text{ and } i+1 \text{ are not in } \kappa \text{ and } i+1 \in R, \\ a_{\kappa \cup (i, a) - (i+1, a), R} & \text{if } (i+1, a) \in \kappa \text{ and } i+1 \notin S(\kappa) \cup R, \\ a_{\kappa - (i, b) \cup (i+1, b), R} & \text{if } (i, b) \in \kappa \text{ and } i \in S(\kappa), \\ \frac{1}{2} a_{\kappa - (i, i+1), R \cup \{i\}} & \text{if } (i, i+1) \in \tau \text{ and } i, i+1 \notin R, \\ a_{\kappa \cup (a, b) - (i, a) - (i+1, b), R \cup \{i\}} & \text{if } (i, a), (i+1, b) \in \tau \text{ and } i, i+1 \notin R. \end{cases}$$

It follows from this that $a_{\tau, T}$ has to be 0 if the coefficient of t in $p_{\tau, T}(t)$ is non-zero. Similarly, $a_{\tau, T}$ is 0 if the coefficient of t^2 in $p_{\tau, T}(t)$ is non-zero. But by definition, one of the three coefficients of $p_{\tau, T}(t)$ is non-zero when $(\tau, T) \in \mathcal{A}(\mathcal{M})$. It follows that the stated polynomials are linearly independent. \square

A basis of standard monomials for $\mathbb{C}[\Pi_{\mathcal{M}}]$ follows from the methods of [8, 12]. Let $u \leq v$ be an interval in Bruhat order such that the Richardson subvariety X_u^v of the flag variety projects birationally to $\Pi_{\mathcal{M}}$ (see [8, Theorem 5.1]). Call an ordered pair (I, J) of k -element subsets \mathcal{M} -standard if $I = \pi_k(x)$ and $J = \pi_k(y)$ where $u \leq x \leq y \leq v$. Then by [8, Proposition 7.2], we have that

$$\{\Delta_I \Delta_J \mid (I, J) \text{ is } \mathcal{M}\text{-standard}\}$$

forms a basis for the degree 2 part of $\mathbb{C}[\Pi_{\mathcal{M}}]$. This suggests the following conjecture.

CONJECTURE 3.11. *The map θ of Proposition 3.4 restricts to a bijection between \mathcal{M} -standard pairs (I, J) and the set $\mathcal{A}(\mathcal{M})$.*

3.5. *Plücker coordinates for the TNN Grassmannian are ‘log-concave’*

Let $I = \{i_1 < i_2 < \dots, i_k\}, J = \{j_1 < \dots < j_k\} \in \binom{[n]}{k}$. Suppose that the multiset $I \cup J$, when sorted, is equal to $\{a_1 \leq b_1 \leq a_2 \leq \dots \leq a_k \leq b_k\}$. Then we define $\text{sort}_1(I, J) := \{a_1, \dots, a_k\}$ and $\text{sort}_2(I, J) := \{b_1, \dots, b_k\}$. Also, if $I \cap J = \emptyset$ define $\min(I, J) := \{\min(i_1, j_1), \dots, \min(i_k, j_k)\}$ and similarly $\max(I, J)$; for general I, J , we define $\min(I, J) := \min(I - J, J - I) \cup (I \cap J)$ and similarly for $\max(I, J) := \max(I - J, J - I) \cup (I \cap J)$.

The following result was independently obtained by Farber and Postnikov [2]. It says that the Plücker coordinates $\Delta_I(X)$ of $X \in \widetilde{\text{Gr}}(k, n)_{\geq 0}$ are log-concave. See [15] for a related situation in Schur positivity.

PROPOSITION 3.12. *Let $X \in \widetilde{\text{Gr}}(k, n)_{\geq 0}$. Then*

$$\Delta_I(X)\Delta_J(X) \leq \Delta_{\min(I,J)}(X)\Delta_{\max(I,J)}(X) \leq \Delta_{\text{sort}_1(I,J)}(X)\Delta_{\text{sort}_2(I,J)}(X).$$

Proof. Follows from Theorem 3.1 and an analysis of compatibility. □

A matroid \mathcal{M} is *sort-closed* if $I, J \in \mathcal{M}$ implies $\text{sort}_1(I, J), \text{sort}_2(I, J) \in \mathcal{M}$. We deduce the following result, first proved in a joint forthcoming work with Postnikov, in the context of alcoved polytopes [14].

COROLLARY 3.13. *Positroids are sort-closed.*

It will be shown in forthcoming work with Postnikov that the converse of Corollary 3.13 also holds: a sort-closed matroid is a positroid.

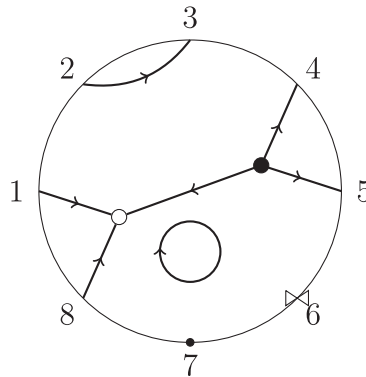
4. Webs and triple dimers

4.1. A_2 -webs and reductions

We review Kuperberg’s A_2 -webs, modified for our situation by allowing tagged boundary vertices.

As usual, an integer n is fixed. A *web* is a planar bipartite graph embedded into a disk where all interior vertices are trivalent. Edges are always directed toward white interior vertices and away from black interior vertices. Furthermore, we allow some vertexless directed cycles in the interior, some directed edges from one boundary vertex to another, and some boundary vertices that are otherwise not used to be ‘tagged’.

Note that boundary vertices of a web are not colored. In the following picture, the boundary vertex 6 is tagged, but 7 is not.



The degree $d(W)$ of a web W is given by

$$\begin{aligned} d(W) = & 3\#\{\text{boundary tags}\} + 3\#\{\text{boundary paths}\} \\ & + \#\{\text{boundary vertices incident to a white interior vertex}\} \\ & + 2\#\{\text{boundary vertices incident to a black interior vertex}\}. \end{aligned}$$

A simple counting argument shows that $d(W)$ is always divisible by 3. In the above example, we get $d(W) = 12 = 3 \times 4$. Let $\mathcal{W}_{k,n}$ denote the (infinite) set of webs W on n boundary vertices, satisfying $d(W) = 3k$.

A web W is called *non-elliptic* if it has no contractible loops, then no pairs of edges enclosing a contractible disk, and no simple 4-cycles, all of whose vertices are internal and which enclose

a contractible disk. (Here contractible means that the enclosed region contains no other edges of the graph.) Let $\mathcal{D}_{k,n}$ denote the set of non-elliptic webs D on n boundary vertices, satisfying $d(D) = 3k$.

Any web W can be reduced to a formal (but finite) linear combination of non-elliptic webs, using the rules.

- (1) For either orientation,

$$\text{circle} = 3$$

- (2)

$$\text{circle with two vertices} = 2 \text{ arrow}$$

- (3)

$$\text{square with four vertices} = \text{two arcs} + \text{two arcs}$$

Note that our signs differ somewhat from Kuperberg’s, but agrees with those of Pylyavskyy [24]. Kuperberg [10, 11] shows that this reduction process is confluent: we get an expression

$$W = \sum_{D \in \mathcal{D}_{k,n}} W_D D$$

expressing a web W in terms of non-elliptic webs D , where the coefficients $W_D \in \mathbb{Z}$ do not depend on the choices of reduction moves performed.

4.2. Weblike subgraphs

Let $G \subset N$ be a subgraph consisting of

- (1) some connected components A_1, A_2, \dots, A_r where every vertex is either (a) internal trivalent, (b) internal bivalent, or (c) a boundary leaf, and such that if v and w are two trivalent vertices connected by a path consisting only of bivalent vertices, then v and w have different colors (or equivalently, the path between v and w has an odd number of edges), and
- (2) some (internal) simple cycles C_1, C_2, \dots, C_s (necessarily of even length), and
- (3) some isolated edges E_1, E_2, \dots, E_t (dipoles).

Furthermore, we require that any component A_i that has no trivalent vertices (and is thus a path between boundary vertices) is equipped with an orientation. We call such a subgraph G that uses all the internal vertices a *weblike* subgraph.

In the following, we shall abuse notation by using e to both denote an edge e , and the weight of that same edge. To each subgraph G , we associate the weight

$$\text{wt}(G) := \prod_{i=1}^r \text{wt}(A_i) \prod_{j=1}^s \text{wt}(C_j) \prod_{\ell=1}^t \text{wt}(E_\ell),$$

where the following conditions are satisfied.

(1) If A_i is a path between boundary vertices consisting of edges e_1, e_2, \dots, e_d (listed in order of the orientation), then

$$\text{wt}(A_i) = \begin{cases} e_1 e_2^2 e_3 e_4^2 \cdots & \text{if the first internal vertex along } A_i \text{ is white,} \\ e_1^2 e_2 e_3^2 e_4 \cdots & \text{if the first internal vertex along } A_i \text{ is black.} \end{cases}$$

(2) If A_i is not a path, then

$$\text{wt}(A_i) = \prod_{e \in A_i} e^{a_e}$$

and

$$a_e = \begin{cases} 1 & \text{if } e \text{ is incident to, or an even distance from, a trivalent vertex,} \\ 2 & \text{if } e \text{ is an odd distance from a trivalent vertex.} \end{cases}$$

(3) If C_i is a cycle with edges e_1, e_2, \dots, e_{2m} in cyclic order, then

$$\text{wt}(C_i) = e_1 e_2^2 \cdots e_{2m-1} e_{2m}^2 + e_1^2 e_2 \cdots e_{2m-1}^2 e_{2m}.$$

(4) If E_i is an isolated edge e , then

$$\text{wt}(E_i) = e^3.$$

To a weblike graph G , we associate a web $W = W(G)$ as follows.

(1) Each component A_i gives rise to a component W_i obtained by removing all bivalent vertices, and orienting all edges toward the white internal trivalent vertices. In the case that A_i has no internal trivalent vertex, we orient the edge using the orientation of the path in G .

(2) Each cycle C_i is replaced by a vertexless loop oriented arbitrarily.

(3) All internal edges E_i are removed; a black boundary vertex (that is, a boundary vertex that is adjacent to a white interior vertex) is ‘tagged’ if it belongs to an edge E_i ; a white boundary vertex (that is, a boundary vertex that is adjacent to a black interior vertex) is ‘tagged’ if it is not used in G .

We consider a boundary vertex i to be used in D if it belongs to a component that contains edges. Thus boundary vertices that are tagged are not considered used.

LEMMA 4.1. *Suppose that $G \subset N$ is a weblike subgraph with web $W = W(G)$. Suppose that W' is some other web that can be obtained from W by a series of reductions. Then there exists a weblike subgraph $G' \subset N$ such that $W(G') = W'$.*

Proof. A reduction $W \mapsto W'$ corresponds to removing some of the edges in W (and then removing bivalent vertices that result). This can be achieved on the level of weblike subgraphs by replacing a path of odd length by some isolated dipoles:



The same trick allows us to replace an even cycle by a number of isolated dipoles. □

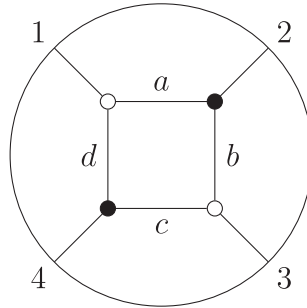
4.3. Web immanants

For each non-elliptic web $D \in \mathcal{D}_{k,n}$, we define a generating function, called the *web immanant*

$$F_D(N) := \sum_W W_D \sum_{W(G)=W} \text{wt}(G).$$

In other words, each subgraph G contributes a multiple of $\text{wt}(G)$ to F_D , where the multiple is equal to the coefficient of D in the web $W(G)$.

EXAMPLE 4.2. We compute $F_D(N)$ for the planar bipartite graph

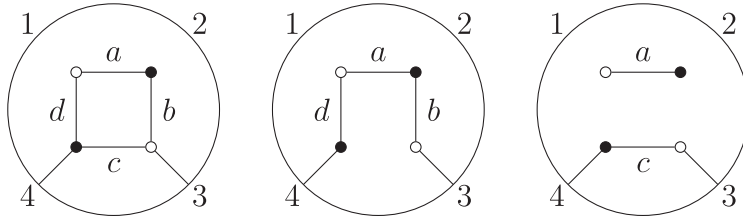


In the following table, we often list tagged boundary vertices as a subset. There are $|\mathcal{D}_{2,4}| = 50$ non-elliptic webs in this case.

D	$F_D(N)$
$\{1, 2\}$	b^3
$\{1, 3\}$	1
$\{1, 4\}$	c^3
$\{2, 3\}$	a^3
$\{2, 4\}$	$3(a^2bc^2d + ab^2cd^2) + a^3c^3 + b^3d^3$
$\{3, 4\}$	d^3
$(1 \rightarrow 2) \cup (3 \rightarrow 4)$	$a^2c^2 + abcd$
$(2 \rightarrow 1) \cup (3 \rightarrow 4)$	ac^2
$(1 \rightarrow 2) \cup (4 \rightarrow 3)$	a^2c
$(2 \rightarrow 1) \cup (4 \rightarrow 3)$	ac
$(1 \rightarrow 4) \cup (2 \rightarrow 3)$	bd^2
$(4 \rightarrow 1) \cup (2 \rightarrow 3)$	bd
$(1 \rightarrow 4) \cup (3 \rightarrow 2)$	$b^2d^2 + abcd$
$(4 \rightarrow 1) \cup (3 \rightarrow 2)$	b^2d
$\{1\} \cup (3 \rightarrow 4)$	c^2
$\{1\} \cup (4 \rightarrow 3)$	c
$\{1\} \cup (2 \rightarrow 3)$	b
$\{1\} \cup (3 \rightarrow 2)$	b^2
$\{1\} \cup (2 \rightarrow 4)$	bc^2
$\{1\} \cup (4 \rightarrow 2)$	b^2c
$\{2\} \cup (3 \rightarrow 4)$	$a^3c^2 + 2a^2bcd + ab^2d^2$
$\{2\} \cup (4 \rightarrow 3)$	$a^3c + a^2bd$
$\{2\} \cup (1 \rightarrow 4)$	$b^3d^2 + 2ab^2cd + a^2bc^2$
$\{2\} \cup (4 \rightarrow 1)$	$b^3d + ab^2c$
$\{2\} \cup (1 \rightarrow 3)$	a^2b
$\{2\} \cup (3 \rightarrow 1)$	ab^2
$\{3\} \cup (1 \rightarrow 2)$	a^2
$\{3\} \cup (2 \rightarrow 1)$	a
$\{3\} \cup (4 \rightarrow 1)$	d
$\{3\} \cup (1 \rightarrow 4)$	d^2
$\{3\} \cup (4 \rightarrow 2)$	a^2d
$\{3\} \cup (2 \rightarrow 4)$	ad^2
$\{4\} \cup (1 \rightarrow 2)$	$a^2c^3 + 2abc^2d + b^2cd^2$
$\{4\} \cup (2 \rightarrow 1)$	$ac^3 + bc^2d$
$\{4\} \cup (3 \rightarrow 2)$	$b^2d^3 + 2abcd^2 + a^2c^2d$
$\{4\} \cup (2 \rightarrow 3)$	$bd^3 + acd^2$
$\{4\} \cup (3 \rightarrow 1)$	c^2d
$\{4\} \cup (1 \rightarrow 3)$	cd^2

	$abd^2 + a^2cd$
	cd
	$b^2cd + abc^2$
	ab
	bc
	$ab^2d + a^2bc$
	ad
	$bcd^2 + ac^2d$
	bcd
	abc
	abd
	acd

Note that all the 50 polynomials are linearly independent. For example, to obtain the answer for $D = \{2\} \cup (3 \rightarrow 4)$ we need to consider the following weblike graphs, contributing $2a^2bcd$, ab^2d^2 , and a^3c^2 , respectively.



Note that the leftmost graph has an elliptic web, which we must first reduce.

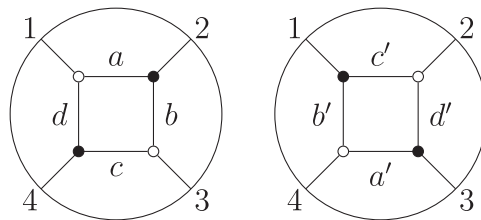
Write

$$\mathcal{D}(W) := \{D \in \mathcal{D}_{k,n} \mid W_D \neq 0\} \quad \text{and} \quad \mathcal{D}(G) := \{D \in \mathcal{D}_{k,n} \mid W(G)_D \neq 0\}.$$

PROPOSITION 4.3. *Suppose that N and N' are such that $M(N) = M(N')$. Then $F_D(N) = \alpha^3 F_D(N')$, where the scalar α is given by $\tilde{M}(N) = \alpha \tilde{M}(N')$.*

Proof. By Theorem 2.3, it suffices to consider the gauge equivalences and local moves (M1-2) and (R1-3). Suppose that N' is obtained from N by multiplying all edge weights incident to a vertex v by α . Then $\tilde{M}(N') = \alpha \tilde{M}(N)$ and $F_D(N') = \alpha^3 F_D(N)$ for any D . The moves (M2) and (R1-3) are similarly easy.

Suppose that N' is obtained from N by applying the spider/square move (M1) in Subsection 2.3. Due to the confluence of reduction, to find the relationship between $F_D(N')$ and $F_D(N)$, it suffices to compute F_D for N and N' being the two graphs



where a, b, c, d and a', b', c', d' are related as in the local move (M1), see Subsection 2.3. We compute directly that $\tilde{M}(N) = (ac + bd)\tilde{M}(N')$.

To check the statement in theorem, we use the following symmetry. Suppose that D' is obtained from D by 90° rotation, sending each boundary vertex i to $i + 1 \pmod 4$. Then $F_{D'}(N')(a', b', c', d') = F_D(N)(d', a', b', c')$. Thus it suffices to check that for every D we have

$$F_{D'}(N)(a, b, c, d) = (ac + bd)^3 F_D(N)(d', a', b', c').$$

This follows from the tables in Example 4.2. □

The non-trivial local move (M1) for planar bipartite graphs involves a square shape, as does the most interesting reduction move for webs. The calculation in Proposition 4.3 relates these two moves. As a corollary, we obtain the following.

COROLLARY 4.4. *For each $D \in \mathcal{D}_{k,n}$, the function $F_D(N)$ depends only on $\tilde{M}(N) \in \widehat{\text{Gr}}(k, n)$, and is a degree 3 element of the homogeneous coordinate ring $\mathbb{C}[\text{Gr}(k, n)]$.*

4.4. *Restriction to positroid varieties*

Let N be a planar bipartite graph. Define

$$\mathcal{D}(N) := \{D \in \mathcal{D}_{k,n} \mid F_D(N) \neq 0\}.$$

If \mathcal{M} is a positroid of rank k on $[n]$, then we define $\mathcal{D}(\mathcal{M}) = \mathcal{D}(N)$ where N represents \mathcal{M} .

LEMMA 4.5. *The set $\mathcal{D}(\mathcal{M})$ does not depend on the choice of N .*

Proof. Follows immediately from Proposition 4.3. □

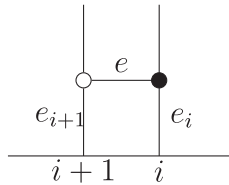
REMARK 4.6. It would be interesting to find the analog of Oh’s theorem (see Remark 3.9), and the analogs of Proposition 3.4 and Conjecture 3.11 for $\mathcal{D}_{k,n}$ and $\mathcal{D}(\mathcal{M})$.

THEOREM 4.7. *The set*

$$\{F_D \mid D \in \mathcal{D}(\mathcal{M})\}$$

is a basis for the space of functions on $\Pi_{\mathcal{M}}$ spanned by $\{\Delta_I \Delta_J \Delta_K\}$. Equivalently, this set forms a basis for the degree 3 component of the homogeneous coordinate ring $\mathbb{C}[\Pi_{\mathcal{M}}]$.

Let N be obtained from N' by adding a bridge e , black at i to white at $i + 1$. Denote the edges joining e to the boundary vertex i (respectively, $i + 1$) by e_i (respectively, e_{i+1}), as illustrated here:



The following is straightforward to check.

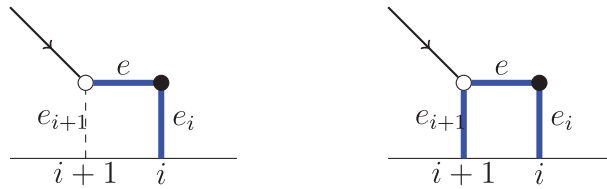
LEMMA 4.8. *Suppose that $D' \in \mathcal{D}(N')$ and G' represents D' . Let $G \subset N$ be weblike such that $G \cap N' = G'$ and $G \setminus G'$ contains e . Then*

$$G \setminus G' = \begin{cases} \{e\} \text{ or } \{e, e_i, e_{i+1}\}, & i \text{ is a source in } D' \text{ and } i + 1 \text{ is a sink in } D', \\ \{e\}, & i \text{ is a sink in } D' \text{ and } i + 1 \text{ is a source in } D', \\ \{e, e_i\}, & i \text{ and } i + 1 \text{ are sources in } D', \\ \{e, e_{i+1}\}, & i \text{ and } i + 1 \text{ are sinks in } D', \\ \{e, e_i\}, & i \text{ is not used and not tagged, but } i + 1 \text{ is a source in } D', \\ \{e, e_{i+1}\}, & i + 1 \text{ is not used and not tagged, but } i \text{ is a sink in } D', \\ \{e, e_i\} \text{ or } \{e, e_i, e_{i+1}\}, & i \text{ is not used and not tagged, but } i + 1 \text{ is a sink in } D', \\ \{e, e_{i+1}\} \text{ or } \{e, e_i, e_{i+1}\}, & i + 1 \text{ is not used and not tagged, but } i \text{ is a source in } D', \\ \{e\}, & i \text{ and } i + 1 \text{ are not used and not tagged in } D'. \end{cases}$$

Furthermore, other D' cannot occur in this way.

We illustrate the two possibilities where i is not used and not tagged (so the boundary vertex i in D' is not incident to any edges in G'), but $i + 1$ is a sink in D' . Here the blue edges are

the ones in $G \setminus G'$.



Note that the two cases are distinguished by the fact that the edge e contributes $\text{wt}(e)^2$ in the picture on the left, but contributes $\text{wt}(e)^1$ in the picture on the right.

Abusing notation, we say that D is obtained from D' by adding the edges e, e_i and/or e_{i+1} if for a weblike graph G' with $W(G') = D'$, we have $W(G) = D$, where G is obtained from G' by adding the same edges. Let $D' \in \mathcal{D}(N')$ be an irreducible web for N' . We say that D' is *stable* if either D' does not use i or $i + 1$, or if joining i and $i + 1$ does not cause D' to become non-elliptic.

LEMMA 4.9. *Suppose $D \in \mathcal{D}(N) \setminus \mathcal{D}(N')$. Then there exists a stable $D' \in \mathcal{D}(N')$ such that D is obtained from D' by adding the edge e , and some (possibly empty) subset of the edges $\{e_i, e_{i+1}\}$.*

Proof. We use Lemma 4.1 in the following. Suppose that $D \in \mathcal{D}(N)$ is represented by a weblike graph G . Then $G' = G \cap N'$ is a weblike subgraph of N' . If $G \setminus G'$ does not include the edge e , then we have $W(G') = W(G)$ and so $D \in \mathcal{D}(N')$ as well. For example, if e_{i+1} is an isolated dipole in G , then the boundary vertex $i + 1$ is tagged in both $W(G')$ and $W(G)$. Thus if $D \in \mathcal{D}(N) \setminus \mathcal{D}(N')$, then we must have $e \in G \setminus G'$. \square

Proof of Theorem 4.7. In Theorem 4.13, we will show that $\{F_D \mid D \in \mathcal{D}_{k,n}\}$ span the degree 3 component of $\mathbb{C}[\text{Gr}(k, n)]$. Since the restriction map $\mathbb{C}[\text{Gr}(k, n)] \rightarrow \mathbb{C}[\Pi_{\mathcal{M}}]$ is surjective, it remains to show that $\{F_D \mid D \in \mathcal{D}(\mathcal{M})\}$ is linearly independent in $\mathbb{C}[\Pi_{\mathcal{M}}]$.

The general strategy of the proof is the same as the proof of Theorem 3.10. We use the same setup and notation here. Suppose that there exists a linear relation

$$\sum_{D \in \mathcal{D}(\mathcal{M})} a_D p_D(t) = 0.$$

The same argument as in the proof of Theorem 3.10 gives $a_D = 0$ if $D \in \mathcal{D}(\mathcal{M}')$.

Now suppose that $D \in \mathcal{D}(\mathcal{M}) \setminus \mathcal{D}(\mathcal{M}')$. By Lemma 4.9, there exists some stable $D' \in \mathcal{D}(\mathcal{M}')$ such that D is obtained from D' by adding some of the edges e, e_i, e_{i+1} . Whether or not i (respectively, $i + 1$) are used in D tells us whether or not the edge e_i (respectively, e_{i+1}) is added.

We deduce that D' is uniquely determined by D except for one situation: when D uses neither i nor $i + 1$, in which case G could either (a) have e as an isolated dipole, or (b) have e belong to a component of G' that uses both i and $i + 1$.

Let $V' = \text{span}(F_{D'}(N') \mid D' \text{ is unstable})$. If the stable D' is uniquely determined, then we have

$$p_D(t) = t^a F_{D'} \pmod{V' \otimes \mathbb{C}[t]}, \tag{3}$$

where $a \in \{1, 2, 3\}$. In the case that the stable D' is not uniquely determined, we have

$$p_D(t) = t^3 F_{D'} + t^a F_{D''} \pmod{V' \otimes \mathbb{C}[t]}, \tag{4}$$

where $a \in \{1, 2\}$, for stable $D', D'' \in \mathcal{D}(N')$.

By Lemma 4.8, each stable D' either occurs once in (3) or (4), or it occurs twice, but with different powers of t . It follows that for a fixed $a \in \{1, 2, 3\}$, the coefficient of $F_{D'}(N')$ in

$[t^a] \sum_{D \in \mathcal{D}(\mathcal{M})} a_D p_D(t)$ is either 0 or a single a_D . This proves that $a_D = 0$ for all $D \in \mathcal{D}(\mathcal{M}) \setminus \mathcal{D}(\mathcal{M}')$, as required. \square

By standard results about the homogeneous coordinate ring $\mathbb{C}[\text{Gr}(k, n)]$, we have the following corollary.

COROLLARY 4.10. *The cardinality of $\mathcal{D}_{k,n}$ is equal to the number of semistandard tableaux of shape 3^k , filled with numbers $\{1, 2, \dots, n\}$.*

The reader is invited to check that this agrees with $|\mathcal{D}_{2,4}| = 50$.

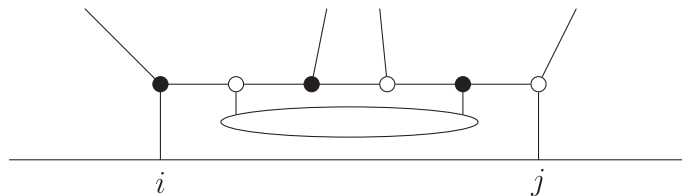
4.5. *Products of three minors*

Many ideas in this section are already present in Pylyavskyy [24]. Pylyavskyy works in the setting of an $n \times n$ matrix. We have modified the results for the Grassmannian situation, and we believe also simplified the presentation.

Let W be a web on $[n]$. A *labeling* (W, α) of W is an assignment of one of the three labels $\{1, 2, 3\}$ to each non-isolated edge in W with the property that the three edges incident to any trivalent vertex have distinct labels. Internal loops in W are labeled by a single label. (One can also think of isolated edges as labeled by all three labels.)

LEMMA 4.11. *Every web W has a labeling.*

Proof. This can be proved by induction on the number of vertices of the web W . Pick two adjacent boundary vertices of some component of W , and find a self-avoiding path between these two vertices. Then we get a situation that looks like:



By induction, we can first label all the webs hanging off this path. Then it is easy to see that there is a labeling of the edges in this path. \square

Let (I, J, K) be a triple of k -element subsets of $[n]$ and denote by $\bar{I} = [n] \setminus I$ (respectively, \bar{J}, \bar{K}) the complement subset. We say that a labeled web (W, α) is *consistently labeled* with (I, J, K) if for any boundary vertex $i \in [n]$,

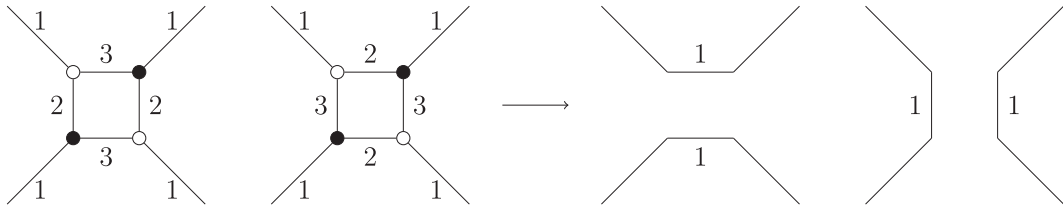
- (1) if i is a black sink or white source in W , and the edge incident to i is labeled by a 1 (respectively, 2, 3), then $i \in \bar{I} \cap J \cap K$ (respectively, $I \cap \bar{J} \cap K, I \cap J \cap \bar{K}$);
- (2) if i is a white sink or black source in W , and the edge incident to i is labeled by a 1 (respectively, 2, 3), then $i \in I \cap \bar{J} \cap \bar{K}$ (respectively, $\bar{I} \cap J \cap \bar{K}, \bar{I} \cap \bar{J} \cap K$);
- (3) if i is black and tagged or white and untagged in W then $i \in I \cap J \cap K$;
- (4) if i is white and tagged or black and untagged in W then $i \in \bar{I} \cap \bar{J} \cap \bar{K}$.

Let $a(I, J, K; W)$ denote the number of consistent labelings of W with (I, J, K) .

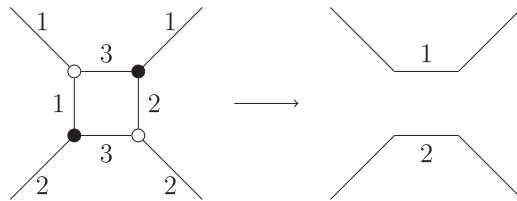
LEMMA 4.12. Suppose that $W = \sum_{D \in \mathcal{D}_{k,n}} W_D D$ is the decomposition of W into non-elliptic webs. Then

$$a(I, J, K; W) = \sum_{D \in \mathcal{D}_{k,n}} W_D a(I, J, K; D).$$

Proof. The identity is checked case-by-case for each of the elementary reduction moves of Subsection 2.3. For example, if we apply move (M1), then we have



when the boundary edges of the square all have the same label, and



when they do not. □

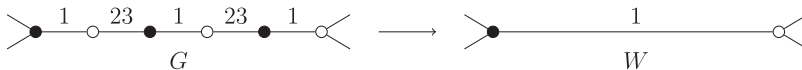
THEOREM 4.13. As functions on the cone over the Grassmannian, we have

$$\Delta_I \Delta_J \Delta_K = \sum_{D \in \mathcal{D}_{k,n}} a(I, J, K; D) F_D.$$

Proof. Let N be a planar bipartite graph with boundary vertices $[n]$. We have

$$\Delta_I(N) \Delta_J(N) \Delta_K(N) = \sum_{\Pi_1, \Pi_2, \Pi_3} \text{wt}(\Pi_1) \text{wt}(\Pi_2) \text{wt}(\Pi_3),$$

where the summation is over triples (Π_1, Π_2, Π_3) of dimer configurations with boundary configurations $I(\Pi_1) = I$, $I(\Pi_2) = J$, and $I(\Pi_3) = K$, respectively. Overlaying these dimer configurations on top of each other, we obtain a weblike subgraph $G \subset N$ and a labeling α of $W(G)$: isolated dipoles are edges occurring in all three dimer configurations, and for a path whose endpoints are trivalent (but other vertices are bivalent), we do the following to obtain α :



Here an edge labeled by 1 in G indicates an edge that is present in only Π_1 , while the edges labeled 23 are present in both Π_2 and Π_3 , but not Π_1 . Conversely, a weblike subgraph $G \subset N$ together with a consistent labeling (W, α) with (I, J, K) arises from a triple of dimer configurations. Comparing with the definition of weight of G , we have

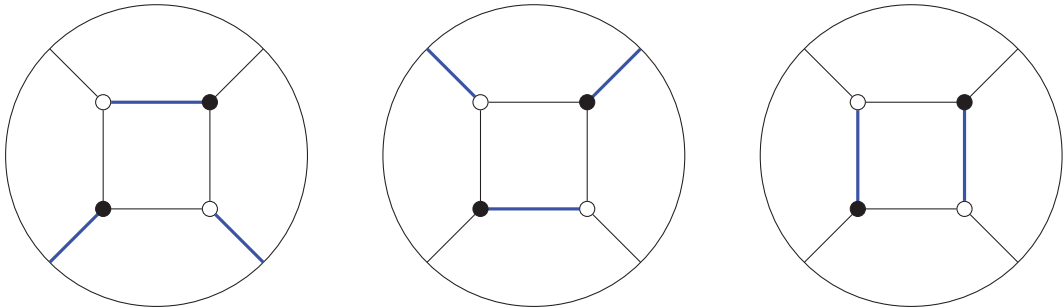
$$\Delta_I(N) \Delta_J(N) \Delta_K(N) = \sum_W a(I, J, K; W) \sum_{G: W(G)=W} \text{wt}(G).$$

Note that if a cycle C_i of even length in G is labeled by I in W , then it comes from two different triples of dimer configurations. Using Lemma 4.12, we have

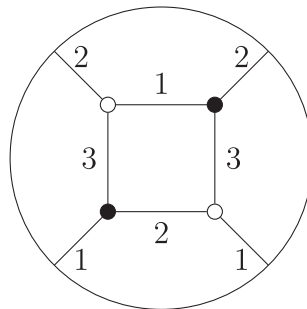
$$\Delta_I(N)\Delta_J(N)\Delta_K(N) = \sum_D a(I, J, K; D)F_D(N).$$

Now both sides depend only on the point $M(N) \in \text{Gr}(k, n)$, so we have an identity on the Grassmannian. □

For example, the three dimer configurations



gives the labeled (elliptic) web



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Thomas Lam
Department of Mathematics
University of Michigan
2074 East Hall
530 Church Street
Ann Arbor, MI 48109-1043
USA
tfylam@umich.edu