EQUIVARIANT FORMAL GROUP LAWS

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Contents

1. Introduction 355
Part I. Complex oriented cohomology theories 358
2. The classifying space for line bundles 358
3. Complex stability and Euler classes 359
4. Orientations and the cohomology of $\mathbb{C}P^\infty$ 360
5. Orientations and Euler classes 362
Part II. Topological examples 363
6. Equivariant K-theory 363
7. The equivariant approach 364
8. Borel cohomology 365
9. Cohomology of the fixed point subspace 365
10. Bredon cohomology 367
Part III. The algebra of equivariant formal group laws 367
11. The definition 368
12. The definition relative to a flag 369
13. Comparison 371
14. The representing ring 373
15. Some relations in the representing ring: leading terms 375
16. Some relations in the representing ring: the general case 377
Appendix A. The additive and multiplicative group laws 380
Appendix B. The coinverse 381
Appendix C. The group of order 2 383
References 386

1. *Introduction*

The purpose of this article is to formulate and study a notion of an equivariant formal group law, at least for abelian compact Lie groups A of equivariance. Although the definition is algebraically natural, and may prove to be of interest elsewhere, it is obtained by abstracting the essential properties of orientable, complex stable, equivariant cohomology theories $E_A^*(\cdot)$. Accordingly, the justification for the definition, and the applications of results about it are at present principally topological. We have therefore taken trouble to provide topological background, motivation and examples before giving the definition. A reader uninterested in the topological applications can jump immediately to Part III, which begins with § 11.

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The aim is to understand cohomology theories which behave in a simple way on complex vector bundles, and hence give rise to a good theory of characteristic classes. In particular, we want to understand tom Dieck's homotopical equivariant complex bordism [**5**]. Bordism is universal amongst such complex orientable theories in a topological sense [**16**, **4**], and non-equivariantly Quillen showed that its coefficient ring is the Lazard ring, which is universal in the algebraic sense. This possibility is more seductive equivariantly, since our knowledge of the coefficient ring of equivariant bordism remains incomplete (although Kriz [**13**] and Sinha [**18**] provide useful information). In [**10**] an equivariant version of Quillen's theorem is proved, but still without making the structure of the equivariant Lazard ring explicit.

Since we work with an abelian group A, all simple complex representations are one dimensional, and it is enough to consider line bundles. Line bundles are classified by the A-space $\mathbb{C}P^\infty$ of lines in a complete A-universe, so we study $E_A^*(\mathbb{C}P^\infty)$, endowed with all the structure inherited from $\mathbb{C}P^\infty$. Since $\mathbb{C}P^\infty$ is a space, $E_A^*(\mathbb{C}P^\infty)$ is a ring, the tensor product of line bundles makes $\mathbb{C}P^\infty$ into an abelian group object in the homotopy category, so (when we have a Künneth isomorphism) $E_A^*(\mathbb{C}P^\infty)$ is a cogroup object, and finally, since one can tensor a line bundle with any one-dimensional representation, $\mathbb{C}P^\infty$ has an action of the dual group A^*, giving rise to a coaction of A^* on $E_A^*(\mathbb{C}P^\infty)$.

To ensure the additive structure is reasonable, we restrict attention to oriented complex stable theories. Cole has shown [**3**] that this condition ensures that $E_A^*(\mathbb{C}P(V))$ is well behaved. In the non-equivariant case, the ring structure is then necessarily that of a truncated polynomial ring, but equivariantly this is too much to ask. However, the Splitting Theorem [**3**] shows that for any complete flag $V^1 \subset V^2 \subset V^3 \subset \ldots \subset V^n = V$ in V the corresponding Schubert cell filtration of $E_A^*(\mathbb{C}P(V))$ splits additively, as a direct sum of copies of E_A^*, with basis elements $1, y(V^1), y(V^2), \ldots, y(V^{n-1})$ constructed from the orientation, using the A^*-action and the ring structure.

Thus (see Definition 11.1) an A-equivariant formal group law over a commutative ring k is a topological Hopf k-algebra R together with a coaction of A^* and special regular elements $y(\alpha)$ (related by the coaction as the notation suggests) so that $R/(y(\alpha)) = k$ for all α, and so that R is complete with respect to the ideal $(\prod_\alpha y(\alpha))$. This is a considerable refinement of the essentially non-equivariant, classical notion of an Okonek equivariant formal group law [**17**]. In effect our equivariant formal group law is a very special 'subcogroup of finite index' in an Okonek formal group law.

Returning to topology, we note that any complete flag F in the universe gives an additive topological basis for $E_A^*(\mathbb{C}P^\infty)$, and we may express the ring structure, the cogroup structure, and the A^*-action in terms of this basis using structure constants in E_A^*. Writing down the formal properties in terms of the basis, we obtain a list of conditions which must be satisfied by the structure constants, and this gives the notion of an (A, F)-equivariant formal group law, essentially that given in [**3**] for cyclic groups. This description shows that there is a representing ring $L_A(F)$ for such formal group laws. This process has an algebraic counterpart, so that an A-equivariant formal group law with a flag F gives an (A, F)-formal group law, and the latter gives a convenient way of calculating with A-equivariant formal group laws. Reversing the process, we conclude that the notion of an (A, F)-formal group law is essentially independent of the flag F, and that the ring $L_A(F)$ represents A-equivariant formal group laws. We begin the study of this ring by finding a more efficient set of generators, and further results are proved in [**10**], but the structure remains mysterious and calls for further investigation.

There are a number of other points of view possible. Firstly, one can give a coordinate-free description of the structure, which is done in geometric language in [**9**]. There is another coordinate-free description, which amounts to viewing an equivariant formal group as a certain type of deformation of a non-equivariant one. The structure studied by Hopkins, Kuhn and Ravenel [**12**] and Greenlees and Strickland [**11**] is equivalent to an equivariant formal group law over a suitably complete ring k.

The rest of the paper is in three parts. In Part I (§§ 2 to 5), we summarize the basic topological definitions and the fundamental splitting theorem of [**3**]. In Part II (§§ 6 to 10), we discuss the cohomology of $\mathbb{C}P^\infty$ for a number of well-known cohomology theories, thereby providing a good supply of examples. Finally, in Part III (§§ 11 to 16), we come to the purely algebraic part of the paper. In § 11 we give the definition of an equivariant formal group law. To make calculations we need to introduce the framework of a complete flag F giving the definition of an (A, F)-equivariant formal group law in § 12. After showing that an A-equivariant formal group law together with a flag is equivalent to an (A, F)-formal group law, we find that the subsequent study of (A, F)-formal group laws provides tools for calculation with equivariant formal group laws. In particular, we show that there is a representing ring for equivariant formal group laws. In appendices we illustrate the theory by considering the special cases of additive and multiplicative formal group laws, and make formulae explicit for the group of order 2.

It is possible to read Part III first, as a piece of pure algebra, and then to return to Parts I and II as a source of examples. However we do not recommend this, since our experience shows that it is all too easy to underestimate the subtlety of the structure without examples to hand.

Here is a summary of our notational conventions (all are introduced in more detail in the text):

A is an abelian compact Lie group,

A^* is its dual group $\mathrm{Hom}(A, S^1)$,

$\alpha, \beta, \gamma, \ldots$ are typical one-dimensional complex representations,

ε is the trivial one-dimensional representation,

V is a complex representation,

$\mathbb{T}$ is the circle group,

z is the natural representation of $\mathbb{T}$,

$\mathcal{U}$ is a complete A universe,

F is a complete flag in $\mathcal{U}$,

$V^1, V^2, V^3, \ldots$ are the terms in F,

$\alpha_1, \alpha_2, \alpha_3, \ldots$ are the subquotients of F,

l_α is left multiplication by $\alpha \in A^*$,

$\lambda(V)$ is the Thom class giving a complex stable structure,
$\chi(V)$ is the pullback of $\lambda(V)$ along $S^0 \to S^V$,
$e(V)$ is the strict Euler class defined by a complex orientation.

Part I. Complex oriented cohomology theories

In Part I we introduce the basic topological ingredients: $\mathbb{C}P^\infty$, complex stable and complex oriented cohomology theories and Euler classes. Much of this is well known, but various pieces of notation and terminology introduced here are used elsewhere.

2. *The classifying space for line bundles*

For each complex representation V we may form the A-space $\mathbb{C}P(V)$ of complex lines in V. It is sometimes useful to consider the representation $V \otimes z$ of $A \times \mathbb{T}$, and then $\mathbb{C}P(V) = S(V \otimes z)/\mathbb{T}$.

Indeed $\mathbb{C}P$ defines a functor from the category of vector spaces and injective maps to the category of topological spaces and injective maps. Thus, in particular, if $W \subseteq V$, we have a pair $(\mathbb{C}P(V), \mathbb{C}P(W))$. For example, if W is one-dimensional, then $\mathbb{C}P(W)$ is a point, so that a one-dimensional subspace of V specifies a basepoint of $\mathbb{C}P(V)$: this is significant because basepoints may lie in different components of the fixed point set. At the other extreme, if α is one-dimensional, one may verify that there is a cofibre sequence

$$\mathbb{C}P(V) \longrightarrow \mathbb{C}P(V \oplus \alpha) \longrightarrow S^{V \otimes \alpha^{-1}}. \tag{2.1}$$

We remark that, as an A-space, V is isomorphic to $\overline{V}$, so that the cofibre may also be described as the Thom space of $\overline{V} \oplus \alpha$, as is often done when discussing Thom isomorphisms.

The A-invariant complex lines are exactly the subrepresentations of V, so it is easy to see that

$$\mathbb{C}P(V)^A = \coprod_\alpha \mathbb{C}P(V_\alpha) \tag{2.2}$$

where $V_\alpha = \mathrm{Hom}_A(\alpha, V)$ is the α-isotypical part of V. Note that if A is finite and $V = k\mathbb{C}A$ is a multiple of the regular representation, we have an isomorphism

$$A^* \times \mathbb{C}P(k\varepsilon) \xrightarrow{\cong} \mathbb{C}P(k\mathbb{C}A)^A$$

given by $(\alpha, W) \to \alpha \otimes W$.

For convenience we take $\mathscr{U} = \bigoplus_{k \geqslant 0} \bigoplus_{\alpha \in A^*} \alpha$ as our complete A-universe and we define $\mathbb{C}P^\infty = \mathbb{C}P(\mathscr{U})$, with its topology as a colimit of its subspaces $\mathbb{C}P(V)$ with V finite dimensional. It is also convenient to let $\mathbb{C}P^\infty_W = \bigcup_{k \geqslant 0} \mathbb{C}P(kW)$, so that $\mathbb{C}P^\infty_\varepsilon$ is an A-fixed infinite complex projective space.

The importance of projective spaces is the following standard fact.

Lemma 2.3. *The A-space $\mathbb{C}P^\infty$ classifies line bundles.*

The tensor product of line bundles is commutative and associative up to coherent isomorphism, and has ε as a unit, and we shall constantly use the represented counterpart.

COROLLARY 2.4. *The A-space $\mathbb{C}P^\infty$ is an abelian group object up to homotopy, and the inclusion of fixed points is a group homomorphism.*

We have seen that

$$(\mathbb{C}P^\infty)^A = \mathbb{C}P(\mathscr{U})^A = \coprod_\alpha \mathbb{C}P(\mathscr{U}_\alpha) \cong A^* \times \mathbb{C}P^\infty_\varepsilon.$$

It is useful to make explicit the map

$$i\colon A^* \times \mathbb{C}P^\infty_\varepsilon \longrightarrow \mathbb{C}P^\infty,$$
$$(\alpha, W) \longmapsto \alpha \otimes W.$$

This map of abelian groups is absolutely fundamental to our analysis.

Note also that since $\mathbb{C}P^\infty_\alpha$ is connected, there is a unique homotopy class $A^* \to (\mathbb{C}P^\infty)^A$ splitting the natural augmentation $(\mathbb{C}P^\infty)^A \to A^*$, and it is a group homomorphism. In particular, A^* acts on $\mathbb{C}P^\infty$ through A-maps, by $\alpha \cdot L = \alpha \otimes L$. To avoid confusion, we make it explicit. Any vector v in $\mathscr{U} = \bigoplus_\alpha \mathscr{U}_\alpha$ can be resolved into its components $v_\alpha \in \mathscr{U}_\alpha$, and under the isomorphism

$$\alpha \otimes : \mathscr{U}_\varepsilon \xrightarrow{\cong} \mathscr{U}_\alpha$$

it is best to view v as a function $v\colon A^* \to \mathscr{U}_\varepsilon$. The action of A^* on $\mathscr{U}$ is then given by $[\alpha \cdot v](\beta) = v(\beta\alpha^{-1})$, and the action of A by $[a \cdot v](\beta) = a(v(\beta))$. It is easy to verify that these commute. When A is finite, the action also restricts to an action on $k\mathbb{C}A$ for each k.

3. *Complex stability and Euler classes*

We have already seen that the fixed point spaces of interesting A-spaces are disconnected, so it is rare for there to be a preferred basepoint. This is one reason why it is convenient to work throughout in the *unbased* context.

A genuine equivariant cohomology theory $E^*_A(\cdot)$ is an exact contravariant functor on A-spaces, which admits an $RO(G)$-graded extension so that we have coherent suspension isomorphisms

$$\widetilde{E}^{V+n}_A(S^V \wedge X) \cong \widetilde{E}^n_A(X)$$

for all real representations V. Amongst these, the most familiar ones are those with a stronger stability property

$$\widetilde{E}^{|V|+n}_A(S^V \wedge X) \cong \widetilde{E}^n_A(X)$$

when V is a complex representation, where $|V|$ denotes the space V with trivial action. This is very convenient: for most purposes we only need to look at the theory in integer gradings. Following tom Dieck we call these theories *complex stable*. As examples, we have the cohomology theory of the Borel construction, defined in terms of a complex orientable non-equivariant cohomology theory by $X \mapsto E^*(EA \times_A X)$. A Serre spectral sequence argument shows that this is complex stable, since A acts trivially on $H^{|V|}(S^V)$ when V is complex. The other examples we discuss below include complex equivariant K-theory.

Now let us suppose given a multiplicative, complex stable equivariant cohomology theory $E^*_A(\cdot)$. For any complex representation V, complex stability provides an element

$$\lambda(V) \in \widetilde{E}_A^{|V|}(S^V)$$

corresponding to the unit in E_A^0, and the E_A^*-module $\widetilde{E}^*(S^V)$ is free of rank 1 on this generator. All complex stability isomorphisms are given by multiplication by $\lambda(V)$, and we have $\lambda(V \oplus W) = \lambda(V)\lambda(W)$. We then define the Euler class $\chi(V) = e_V^*(\lambda(V)) \in E_A^{|V|}$, where $e_V \colon S^0 \to S^V$ is the inclusion. Thus we have $\chi(V \oplus W) = \chi(V)\chi(W)$.

Note that for any based A-space X this gives

$$\widetilde{E}_*^A(X \wedge S^{\infty V}) = \widetilde{E}_*^A(X)[1/\chi(V)].$$

This leads to the localization theorem. In fact, if we take $S^{\infty\bar{\rho}} = \bigcup_{V^A=0} S^V$, the inclusion

$$X^A \wedge S^{\infty\bar{\rho}} \xrightarrow{\simeq} X \wedge S^{\infty\bar{\rho}}$$

is an equivalence by obstruction theory. It follows that $\widetilde{E}_*^A(X^A) \to \widetilde{E}_*^A(X)$ becomes an isomorphism if we invert all Euler classes $\chi(V)$ with $V^A = 0$. By duality we deduce the localization theorem we need [**6**]: the finiteness assumption in the statement is essential.

LEMMA 3.1. *If X is a based finite A-space then*

$$\widetilde{E}_A^*(X^A) \longleftarrow \widetilde{E}_A^*(X)$$

becomes an isomorphism if we invert all Euler classes $\chi(V)$ with $V^A = 0$.

4. *Orientations and the cohomology of $\mathbb{C}P^\infty$*

Notice that when $\varepsilon \subseteq \varepsilon \oplus \alpha \subseteq \mathscr{U}$ we have

$$*_\varepsilon = \mathbb{C}P(\varepsilon) \subseteq \mathbb{C}P(\varepsilon \oplus \alpha) = S^{\alpha^{-1}} \subseteq \mathbb{C}P(\mathscr{U}) = \mathbb{C}P^\infty.$$

DEFINITION 4.1 [**3**]. We say that $x(\varepsilon) \in E_A^*(\mathbb{C}P^\infty, \mathbb{C}P(\varepsilon))$ is an *orientation* if for all one-dimensional representations of $\alpha \in A^*$,

$$\mathrm{res}^{\mathscr{U}}_{\varepsilon\oplus\alpha}\, x(\varepsilon) \in E_A^*(\mathbb{C}P(\varepsilon \oplus \alpha), \mathbb{C}P(\varepsilon)) \cong \widetilde{E}_A^*(S^{\alpha^{-1}})$$

is a generator.

REMARK 4.2. We do not require that x restricts to the standard generator $\lambda(\alpha^{-1}) \in \widetilde{E}_A^*(S^{\alpha^{-1}})$. It is perhaps worth introducing the notation

$$\mathrm{res}^{\mathscr{U}}_{\varepsilon\oplus\alpha}\, x(\varepsilon) = u_{\alpha^{-1}}\lambda(\alpha^{-1})$$

for the unit concerned.

We may generate many other elements from an orientation. Firstly, pulling back along the action

$$\alpha^{-1} \colon (\mathbb{C}P^\infty, \mathbb{C}P(\alpha)) \longrightarrow (\mathbb{C}P^\infty, \mathbb{C}P(\varepsilon))$$

we have $x(\alpha) \in E_A^*(\mathbb{C}P^\infty, \mathbb{C}P(\alpha))$. To avoid confusion later, we write $l_\alpha = (\alpha^{-1})^*$; thus, in particular, $x(\alpha) = l_\alpha x(\varepsilon)$. Taking external direct products, if

$V = \alpha_1 \oplus \ldots \oplus \alpha_n$ then we obtain

$$x(V) = x(\alpha_1) * \ldots * x(\alpha_n) \in E_A^*(\mathbb{C}P^\infty, \mathbb{C}P(V)).$$

Here the product $*$ is defined by pulling back the external cup product along the map

$$\overline{\Delta}\colon (\mathbb{C}P(V \oplus W \oplus Z), \mathbb{C}P(V \oplus W)) \longrightarrow (\mathbb{C}P(V \oplus Z), \mathbb{C}P(V)) \times (\mathbb{C}P(W \oplus Z), \mathbb{C}P(W))$$

defined by $(v : w : z) \mapsto ((v : z), (w : z))$. Forgetting the subspace, note that $x(V)$ defines an element $y(V) \in E_A^*(\mathbb{C}P^\infty)$ which restricts to zero on $\mathbb{C}P(V)$. It turns out that the pair $(\mathbb{C}P(V \oplus W), \mathbb{C}P(V))$ defines a short exact sequence

$$0 \longleftarrow E_A^*(\mathbb{C}P(V)) \longleftarrow E_A^*(\mathbb{C}P(V \oplus W)) \longleftarrow E_A^*(\mathbb{C}P(V \oplus W), \mathbb{C}P(V)) \longleftarrow 0.$$

In particular, $y(\varepsilon)$ is the image of $x(\varepsilon)$, and, since the restriction turns out to be injective, either one determines the other. It is clear that $y(0) = 1$, that $y(V \oplus W) = y(V)y(W)$, and that $(\alpha^{-1})^* y(V) = y(V \otimes \alpha)$. Thus all the elements $y(V)$ can be obtained from $y(\varepsilon)$ using the action of A^* and the multiplication.

To obtain a topological additive base of $E_A^*(\mathbb{C}P^\infty)$ we choose a complete flag

$$F = (V^0 \subset V^1 \subset V^2 \subset \ldots),$$

so that $\dim_{\mathbb{C}}(V^i) = i$, and $\bigcup_{i \geqslant 0} V^i = \mathscr{U}$. Associated to any such complete flag F we have the sequence $\alpha_1, \alpha_2, \ldots$ of subquotients $\alpha_i = V^i / V^{i-1}$, so that $V^n \cong \alpha_1 \oplus \ldots \oplus \alpha_n$, and $y(V^n) = y(\alpha_1)y(\alpha_2)\ldots y(\alpha_n)$. The basis only depends on the isomorphism classes of the sequence V^n, so the important structure is represented by the path $\alpha_1, \alpha_2, \alpha_3, \ldots$ in the first orthant, $\mathbb{N}^{A^*}$. We also allow the use of flags in other complete universes, since none of the relevant structure depends on our identification of universes. We sometimes speak loosely as if the path were equivalent to the flag. The condition that F is a complete flag is simply that the path eventually enters the monoid-ideal generated by $V = (V_\alpha)_\alpha$ for each finite-dimensional representation V. We use the notation $k\{\{y_i \mid i \in I\}\}$ to denote the product $\prod_{i \in I} k$ where y_i is the characteristic function of the ith factor.

The Splitting Theorem of [**3**] is the appropriate substitute for the collapse of the Atiyah–Hirzebruch spectral sequence in the non-equivariant case.

THEOREM 4.3 (Cole [**3**]). *A complete flag* $F = (V^0 \subset V^1 \subset V^2 \subset \ldots)$ *specifies a basis of* $E_A^*(\mathbb{C}P^\infty)$ *as follows:*

$$E_A^*(\mathbb{C}P^\infty) = E_A^*\{\{y(V^0) = 1, y(V^1), y(V^2), \ldots\}\}.$$

Similar results hold for products of copies of $\mathbb{C}P^\infty$*, in the sense that the Künneth theorem holds with completed tensor products.*

Proof. We argue by induction that $E \wedge (\mathbb{C}P^\infty, \mathbb{C}P(0))$ splits as a wedge, and that $F((\mathbb{C}P^\infty, \mathbb{C}P(0)), E)$ is the corresponding product. The cofibre sequence

$$(S^{V^n \otimes \alpha_{n+1}^{-1}}, *) = (\mathbb{C}P(V^{n+1}), \mathbb{C}P(V^n)) \longrightarrow (\mathbb{C}P^\infty, \mathbb{C}P(V^{n+1})) \longrightarrow (\mathbb{C}P^\infty, \mathbb{C}P(V^n))$$

gives a split exact sequence in homology or cohomology: the splitting is given by $x(V^{n+1})$. This is equally true if $\mathbb{C}P^\infty$ is replaced by $\mathbb{C}P(U)$ whenever $V^{n+1} \subseteq U$. The result follows by passage to limits.

Evidently this theorem gives a means for expressing the cup product, the map induced by tensor product, and the action induced by the action of A^*, using collections of elements of the coefficient ring E_A^*. This lets us describe the coarsest features of the maps by identifying the leading terms. The framework of Part III, and especially the notion of an (A, F)-formal group law defined in § 12, allows us to discuss higher terms too.

5. *Orientations and Euler classes*

The main point is that suitable restrictions of the orientation class are unit multiples of the Euler classes.

LEMMA 5.1. *The restriction of $y(\varepsilon)$ to the point $*_\alpha = \mathbb{C}P(\alpha)$ is a unit multiple of the Euler class of α:*

$$\mathrm{res}_\alpha^{\mathscr{U}}(y(\varepsilon)) = u_{\alpha^{-1}}\chi(\alpha^{-1}),$$

where the unit $u_{\alpha^{-1}}$ is the one occurring in the definition of an orientation. In general,

$$\mathrm{res}_\alpha^{\mathscr{U}}(y(\beta)) = u_{\alpha^{-1}\beta}\chi(\alpha^{-1}\beta).$$

Proof. The result is immediate from the diagram

$$\begin{array}{ccc}
(\mathbb{C}P(\alpha\oplus\varepsilon), \mathbb{C}P(\varepsilon)) & \longrightarrow & (\mathbb{C}P^\infty, \mathbb{C}P(\varepsilon)) \\
\uparrow & & \uparrow \\
S^{\alpha^{-1}} \cong \mathbb{C}P(\alpha\oplus\varepsilon) & & \mathbb{C}P^\infty \\
\uparrow & & \uparrow \\
\mathbb{C}P(\alpha) & \xrightarrow{=} & \mathbb{C}P(\alpha)
\end{array}$$

It is convenient to introduce the notation $e(\beta) := u_\beta\chi(\beta)$ for the elements, and we refer to them as *strict* Euler classes associated to a complex oriented theory. For a general representation we define strict Euler classes so that $e(V\oplus W) = e(V)e(W)$.

We can now read off the leading term in the expression for $y(\beta)$ in any basis.

COROLLARY 5.2. *When working with a basis corresponding to a complete flag F with first term α, the coefficient of $y(V^0) = 1$ in the expression for $y(\beta)$ in the F-basis is $e(\alpha^{-1}\beta)$.*

Similarly, we may understand the coefficient of $y(V^j)$ in the expansion of $y(V^i)y(V^j)$.

LEMMA 5.3. *We have*

$$y(V)y(V^j) = e(\alpha_{j+1}^{-1}V)y(V^j) + \textit{ higher terms.}$$

Proof. By induction it suffices to prove the special case when $V = \beta$ is one dimensional. In this case $y(\beta) = e(\alpha_{j+1}^{-1}\beta)$ modulo $y(\alpha_{j+1})$, and the result follows.

Of course if V contains α_{j+1}, the displayed coefficient is zero. Later we consider the higher terms in some detail.

Our next task is to provide a good supply of examples (§§ 6 to 10). We will then give the formal definition and tools for calculation.

Part II. Topological examples

In Part II we consider four examples and one non-example. For each of the examples we describe $E_A^*(\mathbb{C}P^\infty)$, thereby providing an example for testing the arguments of Part III. The example of K-theory is important partly because it is so explicit and partly (see Proposition A.2 and [**8**]) because it provides a universal multiplicative formal group law. The examples of Borel cohomology and cohomology of the fixed point set are two extremes which must be covered by a common definition: the first gives a classical formal group law with Euler classes over a suitably complete ring (as in [**12**, **11**]) and the second gives an Okonek formal group law (as in [**17**]).

6. *Equivariant K-theory*

This is one of the few cases where calculations are easy. In fact, Bott periodicity shows that K-theory is complex stable, and that we may work entirely in degree 0. Thus the coefficient ring $K_A = R(A)$ is the complex representation ring. There is a severe danger of confusion here: the coefficient ring $R(A)$ acts on $K_A(X)$ for any X. On the other hand, when $X = \mathbb{C}P^\infty$ we have an action of A^* via ring homomorphisms: this is quite different from the action of $A^* \subseteq R(A)$. To minimize confusion, recall that we write $l_\alpha x$ for the image of the cohomology class under the action of $\alpha \in A^*$ so that $l_\alpha = (\alpha^{-1})^*$.

This is certainly a case where an equivariant approach is appropriate, since

$$K_{A\times\mathbb{T}}(S(V\otimes z)) = K_A(\mathbb{C}P(V)),$$

where z is the natural representation of $\mathbb{T}$. The point here is that the based cofibre sequence $S(V)_+ \to D(V)_+ \to S^V$ gives rise to the Gysin sequence. By Bott periodicity, it takes the form

$$0 \longleftarrow K_{A\times\mathbb{T}}(S(V\otimes z)) \longleftarrow R(A)[z, z^{-1}] \xleftarrow{\chi(V\,\otimes\,z)} R(A)[z, z^{-1}] \longleftarrow \dots .$$

Furthermore, if $V = \alpha_1 \oplus \dots \oplus \alpha_n$, then

$$\chi(V\otimes z) = (1-\alpha_1 z)(1-\alpha_2 z)\dots(1-\alpha_n z),$$

which is a regular element. Thus

$$K_A(\mathbb{C}P(V)) = K_{A\times\mathbb{T}}(S(V\otimes z)) = R(A)[z, z^{-1}]/\chi(V\otimes z).$$

Next, note that z is already invertible in $R(A)[z]/\chi(V\otimes z)$; indeed

$$1-\chi(V\otimes z) = z\cdot(V + \text{higher terms}).$$

Either by the completion theorem, or simply by passage to inverse limits, we see that

$$K_A(\mathbb{C}P_V^\infty) = R(A)[z]^\wedge_{\chi(V\otimes z)}.$$

Now observe that $y = 1-z$ is an orientation; K-theory is unusual in that this has finite degree in z. To verify that it is indeed an orientation, we note that $1-z$

makes sense as an element of $K_A(\mathbb{C}P(V))$ for any V, and that $1-z$ visibly generates the kernel of

$$K_A(\mathbb{C}P(\varepsilon \oplus \alpha)) = R(A)[z]/(1-z)(1-\alpha z) \longrightarrow R(A)[z]/(1-z) = K_A(\mathbb{C}P(\varepsilon)).$$

The element z, regarded as an element of $K_A(\mathbb{C}P(V))$, is the canonical line bundle over $\mathbb{C}P(V)$, so it is easy to identify the A^* action: $l_\alpha z = \alpha z$. Since the action is through ring homomorphisms, $y(\alpha) = l_\alpha(1-z) = 1-\alpha z$.

Next, we specialize to the case where A is finite and V is the regular representation, and we let $\Pi = \chi(\mathbb{C}A \otimes z) = \prod_\alpha (1-z\alpha)$. There is a straightforward and standard way to adapt the discussion to an arbitrary abelian compact Lie group A. The inclusion i: $A^* \times \mathbb{C}P^\infty_\varepsilon \to \mathbb{C}P^\infty$ induces a map

$$i^*\colon R(A)[z]^\wedge_\Pi \longrightarrow \prod_\alpha R(A)[z]^\wedge_{(1-\alpha z)}.$$

We have chosen coordinates so that the αth component is induced by completing the identity map of $R(A)[z]$ with respect to Π in the domain and $(1-\alpha z)$ in the codomain, as is legitimate since $(1-\alpha z)$ divides Π. Note in particular that i^* is injective, since the same primes contain the product and the intersection of the ideals $(1-\alpha z)$. It is also not hard to see that if we invert all the Euler classes $\chi(\alpha) = 1-\alpha$ then the ideals become coprime. Thus if we invert the Euler classes *before* completion, we obtain an isomorphism by the Chinese Remainder Theorem.

Working modulo all Euler classes, we see that $1-\alpha z = 1-z$, so that i^* is the diagonal inclusion. If we complete at the augmentation ideal, the components of i^* are each isomorphisms.

Finally, it will be shown in Proposition A.2 (see also [**8**]) that, as in the non-equivariant case, K-theory gives rise to the universal multiplicative equivariant formal group law.

7. *The equivariant approach*

This is a class of examples generalizing equivariant K-theory, and certainly including equivariant complex bordism and related theories.

It often happens that there is an $A \times \mathbb{T}$-equivariant form of the cohomology theory, so that $E^*_{A\times\mathbb{T}}(X) = E^*_A(X/\mathbb{T})$ when X is $\mathbb{T}$-free. In this case we have

$$E^*_A(\mathbb{C}P(V)) = E^*_{A\times\mathbb{T}}(S(V\otimes z)),$$

where z is the natural representation of $\mathbb{T}$, just as for K-theory. We also have a completion theorem in this context, stating

$$E^*_A(\mathbb{C}P^\infty_V) = (E_{A\times\mathbb{T}})^\wedge_{\chi(V\otimes z)}.$$

Because we are only considering the cohomology of the single infinite sphere $S(\infty(V\otimes z))$, the completion theorem only requires complex stability, and not highly structured ring and module technology. The most naive form of the statement would involve local homology, but the calculation of Theorem 4.3 shows that this reduces to the classical completion.

We still find that if $V = \alpha_1 \oplus \ldots \oplus \alpha_n$, then

$$\chi(V\otimes z) = \chi(\alpha_1\otimes z)\chi(\alpha_2\otimes z)\ldots\chi(\alpha_n\otimes z).$$

Thus, (with the same adaptions to the infinite case as for K-theory) the inclusion $i\colon A^* \times \mathbb{C}P^\infty_\varepsilon \to \mathbb{C}P^\infty$ induces a map

$$i^*\colon (E^*_{A\times\mathbb{T}})^\wedge_\Pi \longrightarrow \prod_\alpha (E^*_{A\times\mathbb{T}})^\wedge_{\chi(\alpha z)},$$

where $\Pi = \prod_\alpha \chi(\alpha z)$. The map i^* is again injective, and the analogue, in which Euler classes $\chi(V)$ with $V^A = 0$ are inverted before completion, is an isomorphism.

We have an action of A^* on $\mathbb{C}P^\infty$, and hence on the completion of $E^*_{A\times\mathbb{T}}$. It can be rather useful to know that the action exists before completion. In fact the group A^* acts on the group $A \times \mathbb{T}$ by the formula

$$\alpha \cdot (a, z) = (a, \alpha(a)z).$$

This induces an action of A^* on the cohomology theory $E^*_{A\times\mathbb{T}}(\,\cdot\,)$. Since composition with $\alpha\colon \mathscr{U} \otimes z \to \mathscr{U} \otimes z$ takes a subspace $V \otimes z$ of $\mathscr{U}$ to $V\alpha \otimes z$, we see that the action on $E^*_{A\times\mathbb{T}}(S(\mathscr{U} \otimes z))$ is the same as that on $E^*_A(\mathbb{C}P^\infty)$ discussed earlier.

8. *Borel cohomology*

If E is any non-equivariant complex oriented theory, we may consider the associated Borel cohomology:

$$b(E)^*_A(X) = E^*(EA \times_A X).$$

Since A acts trivially on $H^*(S^V)$ if V is a complex representation, a Serre spectral sequence argument shows that this is a complex stable theory, and a complex stable structure is provided by choosing Thom classes. Any class in $\widetilde{b(E)}^*_A(\mathbb{C}P^\infty)$ restricting to a generator of $\widetilde{b(E)}^*_A(S^2)$ is automatically a complex orientation.

These examples are special because $b(E)^*_A$ is already complete at the ideal I generated by the Euler classes. This means as expected that the associated formal group law is essentially the non-equivariant formal group law with its base extended. This is made precise in Example 11.3(iv) below.

We may consider ordinary Borel cohomology with integer coefficients. From the equivariant point of view of § 7 or otherwise, we see that $b(H)^*_A(\mathbb{C}P^\infty) = H^*(BA) \otimes H^*(BS^1)$. Choosing an orientation c we find that this is $H^*(BA)[c]$. Furthermore, $H^2(BA) \cong A^*$, and we have $e(\alpha) = \alpha$. It turns out that this gives an additive formal group law. We shall see in Appendix A that the fact that Euler classes are $\mathbb{Z}$-torsion for finite A is a necessary consequence, and that the associated formal group law is closely related to the universal additive formal group law (and equal to it when A is a torus).

9. *Cohomology of the fixed point subspace*

The other extreme case is given by cohomology theories in which all Euler classes are invertible. Thus for this section we suppose $E^*_A(\,\cdot\,)$ is a complex stable theory in which all Euler classes $\chi(\alpha)$ (with $\alpha \neq \varepsilon$) are invertible. We shall see that such a theory is complex orientable provided it is orientable as a cohomology theory on A-fixed spaces. Theories of this type give Okonek formal groups [**17**].

Since Euler classes are invertible, we have an equivalence

$$E = E \wedge S^0 \xrightarrow{\ \simeq\ } E \wedge S^{\infty\bar{\rho}},$$

where $S^{\infty\bar{\rho}} = \bigcup_{V^A=0} S^V$ as before. It follows that for any A-space X, the inclusion $X^A \to X$ is a cohomology isomorphism (since the cohomology of any non-fixed cell is zero). This applies to $X = \mathbb{C}P^\infty$, with $(\mathbb{C}P^\infty)^A = A^* \times \mathbb{C}P^\infty_\varepsilon$, where the multiplication comes from the group structure in A^* and $\mathbb{C}P^\infty_\varepsilon$. Thus

$$E_A^*(\mathbb{C}P^\infty) \cong \prod_\alpha E_A^*(\mathbb{C}P^\infty_\varepsilon) = E_A^*(\mathbb{C}P^\infty_\varepsilon)^{A^*}.$$

We should explain the notation: if k is a cogroup in the category of rings, we use the notation k^{A^*} for the object which is a product $\prod_\alpha k$ as a ring, but where the coproduct combines that of k with the group operation on A^*. This should not be confused with the group ring $k[A]$. More explicitly, if we choose a coordinate y_ε then

$$E_A^*(\mathbb{C}P^\infty) = \prod_\alpha E_A^*[[y_\alpha]]$$

and the coproduct is determined by the group structure of A^* and its effect on the generator y_ε.

More generally, (2.2) shows that

$$E_A^*(\mathbb{C}P(V)) = \prod_\alpha E_A^*[y_\alpha]/(y_\alpha^{|V_\alpha|}),$$

so it is easy to understand orientations, $x = x(\varepsilon) = (x_\alpha)_\alpha$. An element $(f_\alpha(y_\alpha)) \in E_A^*(\mathbb{C}P^\infty)$ gives an element of $E_A^*(\mathbb{C}P^\infty, \mathbb{C}P(\varepsilon))$ if $f_\varepsilon(y_\varepsilon)$ has zero constant term. To give a generator of $E_A^*(\mathbb{C}P(\varepsilon \oplus \varepsilon), \mathbb{C}P(\varepsilon))$ the coefficient of y_ε must be a unit. Finally, to give a generator of $E_A^*(\mathbb{C}P(\varepsilon \oplus \alpha), \mathbb{C}P(\varepsilon))$ when $\alpha \neq \varepsilon$, the constant term in $f_\alpha(y_\alpha)$ must be a unit. Thus, if Euler classes are given, an orientation is specified by

(i) a classical orientation of the classical formal group law $E_A^*(\mathbb{C}P^\infty_\varepsilon)$ and

(ii) for each $\alpha \neq \varepsilon$, an arbitrary power series $g_\alpha(y_\alpha)$ so that $f_\alpha(y_\alpha) = e(\alpha) + y_\alpha g_\alpha(y_\alpha)$.

The simplest orientation is $x = (y_\varepsilon, (\chi(\alpha))_{\alpha \neq \varepsilon})$. We use this orientation unless otherwise stated, and thus we have

$$y(\beta) = (y_\beta, (\chi(\beta\alpha^{-1}))_{\alpha \neq \beta}),$$

(that is, the βth component of $y(\beta)$ is y_β, and the αth component is $\chi(\beta\alpha^{-1})$ if $\alpha \neq \beta$).

EXAMPLE 9.1. It is perhaps worth a short calculation with A of order 2 to kill certain preconceptions. Let η denote the non-trivial one-dimensional representation, and $\chi = \chi(\eta)$. We thus have

$$E_A^*(\mathbb{C}P^\infty) = E_A^*[[y_\varepsilon]] \times E_A^*[[y_\eta]],$$

with E_A^* acting diagonally on the factors, and A^* exchanging the factors. Note that

$$y(\varepsilon) = (y_\varepsilon, \chi) \quad \text{and} \quad y(\eta) = (\chi, y_\eta).$$

We can thus write $y(\eta)$ in terms of the $\varepsilon, \eta, \varepsilon, \eta, \varepsilon, \ldots$ basis

$$y(\eta) = \chi \cdot 1 + (-1) \cdot y(\varepsilon) + \chi^{-1} \cdot y(\varepsilon) y(\eta).$$

A purely algebraic version of this example is given as Example 11.3(v).

10. *Bredon cohomology*

Guided by the non-equivariant case, we might have expected to find additive formal groups associated to ordinary cohomology. We show here that this is misguided: readers without prior prejudice should omit this section.

First we show that ordinary cohomology is hardly ever complex stable. Recall that an ordinary cohomology is one satisfying the dimension axiom, and it is thus determined by its values $M(B) := E_A^0(A/B)$. From the suspension isomorphisms, the functor $A/B \mapsto M(B)$ is an additive functor on the stable orbit category: a Mackey functor. Bredon has shown how to construct the ordinary cohomology theory $H_A^*(\cdot\,; M)$ associated to M [**2**], and it extends to an $RO(G)$-graded theory if and only if M is a Mackey functor [**14**].

We shall see that Bredon cohomology is very rarely orientable. However, even when the theory is orientable, it is obvious that all Euler classes of non-zero representations are trivial, since they lie in the zero group.

Let us consider the special case when A is cyclic. First, consider the case $M(B) = \Lambda^B$ for an A-module Λ. If V is A-free then $H_A^*(S(V); M) = H^*(S(V)/A; \Lambda)$. Choosing V to be of large dimension we see that this is the group cohomology $H^*(A; \Lambda)$ in a range of degrees. Now consider the cofibre sequence $S(V)_+ \to S^0 \to S^V$, and deduce that $H_A^*(S^V; \Lambda) = \Sigma H^*(A; \Lambda)$ from degree 2 up to the dimension of V. Thus complex stability implies $H^*(A; \Lambda)$ is zero in positive degrees. The main examples occur when the order of A is invertible in Λ and when Λ is projective over $\mathbb{Z}A$. Both these cases are relatively dull.

Now suppose that A is an arbitrary finite abelian group. If V is a non-trivial one-dimensional complex representation of A, it has a kernel K so that A/K is cyclic, generated by c say. We then have two cofibre sequences

$$A/K_+ \xrightarrow{1-c} A/K_+ \longrightarrow S(V)_+ \quad \text{and} \quad S(V)_+ \longrightarrow S^0 \longrightarrow S^V.$$

The first gives an exact sequence

$$0 \longleftarrow H_A^1(S(V); M) \longleftarrow M(K) \xleftarrow{1-c} M(K) \longleftarrow H^0(S(V); M) \longleftarrow 0,$$

and the second shows that $H_A^1(S(V); M) \cong H_A^2(S^V; M)$ and that there is an exact sequence

$$0 \longleftarrow H_A^1(S^V; M) \longleftarrow H_A^0(S(V); M) \longleftarrow M(A) \longleftarrow \widetilde{H}_A^0(S^V; M) \longleftarrow 0.$$

Suppose that $H_A^*(\cdot\,; M)$ is complex stable, so that $H_A^*(S^V) = H_A^*(S^2)$. There is thus an exact sequence

$$0 \longleftarrow M(A) \longleftarrow M(K) \xleftarrow{1-c} M(K) \xleftarrow{\mathrm{res}_K^A} M(A) \longleftarrow 0,$$

showing that $M(A) = M(K)_{A/K}$ and that restriction gives an isomorphism $M(A) \cong M(K)^{A/K}$. Applying a similar argument to subgroups, we see that M is the Mackey functor associated to an A-module Λ by $M(B) = \Lambda^B$. Furthermore, the discussion of the case Λ above shows that it too is very restricted.

PART III. THE ALGEBRA OF EQUIVARIANT FORMAL GROUP LAWS

Part III is entirely algebraic, and may be of interest independent of our topological applications. Nonetheless, we encourage the reader to become acquainted with the examples of Part II so as to be aware of the variety and complexity of phenomena which arise.

Section 11 gives the definition of an equivariant formal group law, § 12 gives a second definition relative to a flag, which is more suitable for calculation. These are then shown to be essentially equivalent, and the definition relative to a flag used to prove the existence of a universal ring and to establish some basic properties of it.

11. *The definition*

In this section we give the definition of an A-equivariant formal group law. As usual, the word *law* refers to the fact that the definition is relative to a particular choice of coordinate specified by the orientation.

The definition is motivated by the formal properties of the classifying space $\mathbb{C}P^\infty$ of line bundles and the definition of a complex orientation of a complex orientable cohomology theory. Thus, because $\mathbb{C}P^\infty$ is an abelian group object, we see that $E_A^*(\mathbb{C}P^\infty)$ has a product and a coproduct, and because the inclusion of $A^* = \coprod_\alpha \mathbb{C}P(\alpha) \to \mathbb{C}P^\infty$ is a group homomorphism, it induces a map compatible with product and coproduct. Finally, there is an axiom encoding the definition of the orientation element.

Definition 11.1. If A is a finite abelian group, an *A-equivariant formal group law* over a commutative ring k is a complete topological k-algebra R together with

Afgl1: a continuous comultiplication

$$\Delta\colon R \longrightarrow R \widehat{\otimes} R$$

which is a map of k-algebras, cocommutative, coassociative and counital;

Afgl2: an augmentation

$$\theta\colon R \longrightarrow k^{A^*}$$

which is a map of k-algebras compatible with the coproduct, so that $\ker(\theta)$ defines the topology;

Afgl3: an orientation element $y(\varepsilon) \in R$ so that

(i) $y(\varepsilon)$ is regular,

(ii) $\theta(\varepsilon)$ induces an isomorphism $R/(y(\varepsilon)) \cong k$.

If A is a general compact abelian group, the definition is the same except that the topology is defined by the system of all finite products of ideals $\ker(\theta(\alpha)\colon R \to k)$. The space k^{A^*} is topologised as a product of copies of the discrete ring k.

Remark 11.2. (i) The coproduct allows us to define an action of A^* on R by

$$l_\alpha(r) = (\theta(\alpha^{-1}) \otimes \mathrm{id})\Delta(r).$$

We may then define elements $y(\alpha) := l_\alpha(y(\varepsilon))$. By Afgl3, $y(\alpha)$ is regular and $\theta(\alpha)$ induces an isomorphism $R/(y(\alpha)) \cong k$. Thus the topology on R is defined by $\Pi = (\prod_\alpha y(\alpha))$ if A is finite and generally by the system of all finite product ideals $(\prod_i y(\alpha_i))$.

(ii) This data allows us to define Euler classes by

$$e(\alpha) = \theta(\varepsilon)(y(\alpha))$$

(or equivalently, $e(\alpha) = \theta(\alpha^{-1})(y(\varepsilon))$). Indeed, θ may be viewed as a coordinate-free packaging of Euler classes.

(iii) In view of both the topological motivation and the terminology, the reader may have expected a coinverse as part of the structure. We shall show in Appendix B that just as for graded connected bialgebras, the existence of a unique coinverse is automatic. Thus Condition Afgl1 may be replaced by the requirement that R is a topological Hopf k-algebra, and Condition Afgl2 by the requirement that θ is a map of topological Hopf k-algebras.

(iv) As in the non-equivariant case, one may also give a coordinate-free definition (that is, a definition without specifying the orientation $y(\varepsilon)$): this gives the notion of an equivariant formal group. We defer further discussion to [**9**].

By contrast with the classical case, the underlying ring R is usually not a power series ring. We shall show in § 13 that a choice of a complete flag in our universe gives rise to an additive basis of R. Once we have such a basis, we may make calculations, and we develop the necessary machinery for this. The abstract fruit of this is that there is a universal ring for equivariant formal group laws.

EXAMPLES 11.3. (i) The motivation in Part I shows that if $E_A^*(\cdot)$ is a complex oriented theory, we may obtain an A-equivariant formal group law by taking $k = E_A^*$, $R = E_A^*(\mathbb{C}P^\infty)$ and $y(\varepsilon)$ to be the complex orientation. This class of examples is very general and we refer the reader to Part II for specific instances.

(ii) A 1-equivariant formal group law is a formal group law in the classical sense.

(iii) Any formal group law over k gives an A-equivariant formal group law over k with trivial Euler classes by taking the map θ to be the composite of the counit and the diagonal $k \to k^{A^*}$.

(iv) A formal group law over a complete local ring k defines a group structure on the maximal ideal $\mathfrak{m}$. If we also specify a group homomorphism e: $A^* \to \mathfrak{m}$ so that the image generates $\mathfrak{m}$ as an ideal, this gives an A-equivariant formal group law over k, with $R = k[[y]]$, $\theta(\alpha)(y) = e(\alpha)$ and orientation $y(\varepsilon) = y$. This is an algebraic version of the equivariant formal group law arising from Borel cohomology as in § 8. This type of algebraic structure is considered in [**12**] and [**11**].

(v) A formal group law over k, together with units $e(\alpha)$ in k for $\alpha \neq \varepsilon$, allows us to define an equivariant formal group law $R = k[[y]]^{A^*}$. The coproduct on R combines the coproduct on $k[[y]]$ with multiplication on A^*. The augmentation on $k[[y]]$ gives the augmentation on R. One possible orientation $y(\varepsilon)$ has εth factor y and αth factor $e(\alpha)$ for $\alpha \neq \varepsilon$. This is an algebraic version of the equivariant formal group law arising from cohomology of fixed points as in § 9.

12. *The definition relative to a flag*

In this section we give another definition motivated by the topological case. Here we use the fact that an orientation, together with a decomposition of $\mathbb{C}P^\infty$ into Schubert cells, defines an additive basis of $E_A^*(\mathbb{C}P^\infty)$ when $E_A^*(\cdot)$ is complex oriented. Thus, if we choose a particular flag F, we may express all available structures with respect to the resulting basis. This idea comes from the first author's thesis [**3**], where it is applied to the case of a cyclic group, and the obvious type of periodic complete flag. We find it essential to have the flexibility to discuss bases arising from different flags, and have used completely different notation from [**3**] to avoid confusion, since our indexing conventions are different. In particular, we view $y(V^i)$ as having a superscript, and use the summation convention to determine the position of decorations in other notation.

The definition is in terms of a topological module with additional structure, so the following notation is convenient.

NOTATION 12.1. Given a ring k and a complete flag

$$F = (0 = V^0 \subset V^1 \subset V^2 \subset \dots),$$

we write

$$k\{\{F\}\} = k\{\{1, y(V^1), y(V^2), \dots\}\}$$

for the topological k-module obtained as the inverse limit of the free k-modules with basis 1, $y(V^1), y(V^2), \dots, y(V^s)$.

DEFINITION 12.2. An (A, F)-*formal group law* over a commutative ring k is the topological k-module $k\{\{F\}\}$ with a continuous product, a continuous coproduct and a continuous action of A^* satisfying Conditions (R), (A), (T), (Flag), and (Ideal) below.

To describe the conditions, note that the product, action and coproduct are specified with respect to the topological basis by formulae

$$y(V^i)y(V^j) = \sum_{s \geqslant 0} b_s^{i,j} y(V^s),$$

$$l_\alpha y(V^i) = \sum_{s \geqslant 0} d(\alpha)_s^i y(V^s),$$

$$\Delta y(V^i) = \sum_{s,t \geqslant 0} f_{s,t}^i y(V^s) \otimes y(V^t),$$

for suitable structure constants $b_s^{i,j}$, $d(\alpha)_s^i$ and $f_{s,t}^i$ in k. The continuity of the structure maps may be made explicit in terms of the structure constants.

Continuity conditions:

(1) for fixed i, s the coefficients $b_s^{i,j}$ are zero for j sufficiently large, and similarly with i and j exchanged,

(2) for fixed α, s the coefficients $d(\alpha)_s^i$ are zero for i sufficiently large, and

(3) for fixed s, t the coefficients $f_{s,t}^i$ are zero for i sufficiently large.

The continuity conditions are necessary in some of the conditions to ensure the sums in the following statements are finite. We have resisted the temptation to write out the formulae explicitly in terms of structure constants, but the reader is encouraged to do this at least once; the properties are listed separately to assist with this. The main formal properties are as follows.

(R) The product is

(1) commutative,

(2) associative, and

(3) unital.

(A) The action is
  (1) through ring homomorphisms,
  (2) associative, and
  (3) unital.

(T) The coproduct is
  (1) through ring homomorphisms,
  (2) equivariant in the sense that $\Delta \circ l_{\alpha\beta} = (l_\alpha \widehat{\otimes} l_\beta) \circ \Delta$,
  (3) commutative,
  (4) associative and
  (5) unital.

There are also two normalization conditions.

(Flag) $y(\alpha_{j+1})y(V^j) = y(V^{j+1})$.

(Ideal) For each i, the ideal $(y(V^i))$ has additive topological basis $y(V^i), y(V^{i+1}), y(V^{i+2}), \ldots$.

EXAMPLE 12.3. (i) The motivation in Part I shows that if $E_A^*(\cdot)$ is a complex oriented theory, we may obtain an (A, F)-formal group law for any complete flag F by taking $k = E_A^*$ and $y(\varepsilon)$ to be the complex orientation. This class of examples is very general and we refer the reader to Part II for specific instances.

(ii) If A is the trivial group, an equivariant formal group law is simply a classical one-dimensional commutative formal group law, specified by a coproduct on the power series ring $k[[y]]$, orientation $y(\varepsilon) = y$ and $y(V^i) = y^i$.

It is not easy to construct examples of (A, F)-formal group laws directly, but we will see in § 13 that any A-equivariant formal group law gives an (A, F)-formal group law.

13. *Comparison*

We shall show that an A-equivariant formal group law (as defined in Definition 11.1), together with a flag F, is equivalent to an (A, F)-formal group law (as defined in Definition 12.2). This gives a means for calculation with A-equivariant formal group laws. It also proves that an (A, F)-formal group law is essentially independent of the flag F; later we give formulae showing how the structure constants for an (A, F)-formal group law are related to those of an (A, F')-formal group law.

LEMMA 13.1. *An (A, F)-formal group law $k\{\{F\}\}$ is an A-equivariant formal group law.*

Proof. For simplicity assume F begins with ε. We take $R = k\{\{F\}\}$, and use the elements $y(\alpha) = l_\alpha y(\varepsilon)$ as the notation suggests. The comultiplication is explicitly part of the structure of an (A, F)-formal group, and required to have the properties stated in Afgl1.

Next, observe that $k\{\{F\}\}/(y(\varepsilon)) \cong k$: it is clear that $y(V^1), y(V^2), \dots$ lie in the ideal $(y(\varepsilon))$, and our Condition (Ideal) shows that they also span it. We may now show that $y(\varepsilon)$ is regular. Indeed, by definition of $k\{\{F\}\}$, the elements $1, y(V^2/\varepsilon), y(V^3/\varepsilon), \dots$ have independent images under the map $y(\varepsilon)\colon k\{\{F\}\} \to (y(\varepsilon))$, so it suffices to show that they span $k\{\{F\}\}$ as a topological k-module. This can be done by adapting the proof of Lemma 13.2 below.

We define $\theta\colon k\{\{F\}\} \to k^{A^*}$ by taking $\theta(r)(\alpha)$ to be the constant coefficient in $l_\alpha(r)$. This is a continuous map of rings since l_α is. Condition Afgl2 follows from the equivariance of the coproduct. It now follows that $y(\varepsilon)$ generates $\ker(\theta(\varepsilon))$ and hence that the topology on R is defined by the kernel of θ. This completes the proof.

LEMMA 13.2. *If R is an A-formal group law, we obtain an (A, F)-formal group law by defining $y(V) = y(\alpha_1)y(\alpha_2)\dots y(\alpha_n)$ where $V = \alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_n$.*

Proof. The main point is to show that if we choose a complete flag F, the elements $1, y(V^1), y(V^2), \dots$ form a topological basis. Indeed, the elements $y(V^i)$ define a continuous function $\lambda\colon k\{\{F\}\} \to R$ of k-modules, and we claim it is an isomorphism. This will immediately define the ring structure satisfying (R), and the coproduct structure. The action l_α is given by the composite

$$R \xrightarrow{\Delta} R \widehat{\otimes} R \xrightarrow{\theta} k^{A^*} \otimes R \xrightarrow{\pi_\alpha \otimes 1} \{\alpha\} \otimes R.$$

It is easy to verify that this is an action and that the coproduct is equivariant.

Surjectivity of λ follows by approximating elements of R by a convergent series in the image of λ. Indeed, by Condition Afgl3(ii), given $s \in R$ we may choose $r_0 \in k$ so that $s - r_0 \in (y(\alpha_1))$, say $s = r_0 + y(\alpha_1)s_1$. Similarly, $s_1 = r_1 + y(\alpha_2)s_2$ and so forth. To establish injectivity suppose $\lambda(\sum_i r_i y(V^i)) = 0$, and suppose there is a first non-zero coefficient r_{i_0}. Thus

$$\lambda\Bigg(\sum_i r_i y(V^i)\Bigg) = y(V^{i_0})\lambda\Bigg(\sum_{i \geqslant i_0} r_i y(V^i/V^{i_0})\Bigg).$$

Since $y(V^{i_0})$ is regular, this means that $\lambda(\sum_i r_i y(V^i/V^{i_0})) = 0$, but this is a contradiction since reducing modulo $y(\alpha_{i_0+1})$ recovers r_{i_0}.

Condition (Flag) is built into the definition, as are the continuity conditions. To prove Condition (Ideal) it suffices to show that $y(\alpha)y(V^i)$ has zero coefficients of $y(V^j)$ with $j < i$. If not,

$$y(\alpha)y(V^i) = y(\alpha)y(V^j)y(V^i/V^j) = y(V^j)z$$

with $V^i/V^j \neq 0$ and $z = z_0 + z_1 y(V^{j+1}/V^j) + \dots$ with $z_0 \neq 0$. Since $y(V^j)$ is regular by Condition Afgl3(i), $y(\alpha)y(V^i/V^j) = z$. Now reduce modulo $y(\alpha_{j+1})$, and by Condition Afgl3(ii) (or rather the consequence pointed out in Remark 11.2(i)), we contradict the fact $z_0 \neq 0$. This completes the proof.

This shows that $k\{\{F\}\}$ does not depend in an essential way on the flag F, so we allow ourselves to write $k\{\{F\}\} = k\{\{\mathscr{U}\}\}$. We extend this notation to finite-dimensional subspaces $V \subset \mathscr{U}$ by taking $k\{\{V\}\} = k\{\{\mathscr{U}\}\}/(y(V))$. Finally, we may extend it to infinite-dimensional subspaces $\mathscr{U}' \subseteq \mathscr{U}$ by taking $k\{\{\mathscr{U}'\}\}$ to be the inverse limit of the rings $k\{\{V'\}\}$ with the inverse limit topology. There is then an induced map $\mathrm{res}^{\mathcal{U}}_{\mathcal{U}'}\colon k\{\{\mathcal{U}\}\} \to k\{\{\mathcal{U}'\}\}$. We also have a commutative square

$$\begin{array}{ccc} k\{\{\mathcal{U}\}\} & \longrightarrow & k\{\{\mathcal{U}'\delta\}\} \\ {\scriptstyle l_\delta}\downarrow & & \downarrow{\scriptstyle l_\delta} \\ k\{\{\mathcal{U}\}\} & \longrightarrow & k\{\{\mathcal{U}'\}\} \end{array} \tag{13.3}$$

14. *The representing ring*

It is immediate that an (A, F)-formal group law is uniquely specified by structure constants $b^{i,j}_s$, $d(\alpha)^i_s$ and $f^i_{s,t}$ satisfying the continuity conditions, and so that $k\{\{F\}\}$ satisfies Conditions (R), (A), (T), (Flag), and (Ideal). We now want to form the representing ring $L_A(F)$ for (A, F)-formal group laws, as the $\mathbb{Z}$-algebra with generators $b^{i,j}_s$, $d(\alpha)^i_s$ and $f^i_{s,t}$ subject to the relations implied by these conditions. The remaining obstacle is that we must show the continuity conditions can be given by uniform formulae.

By way of motivation, consider the topological case. Various vanishing conditions result from the fact that $y(V)$ generates the ideal of elements restricting to zero on $\mathbb{C}P(V)$.

(VRs) The product $y(V')y(V'')$ is zero on restriction to $\mathbb{C}P(V')$ or $\mathbb{C}P(V'')$.

(VAs) The function $l_\alpha y(V) = y(V \otimes \alpha)$ vanishes on restriction to $\mathbb{C}P(V \otimes \alpha)$.

(VTs) The coproduct $\Delta y(V)$ is zero on restriction to $\mathbb{C}P(W_1) \times \mathbb{C}P(W_2)$ if $\mathbb{C}P(W_1) \times \mathbb{C}P(W_2)$ maps into $\mathbb{C}P(V)$ up to homotopy. A sufficient condition for this is given in terms of the dimensions of the fixed point sets by the CW-approximation theorem. We require that the inequality

$$\dim \mathbb{C}P(W_1)^B + \dim \mathbb{C}P(W_2)^B \leqslant \dim \mathbb{C}P(V)^B$$

holds for all subgroups $B \subseteq A$. It is easy but unilluminating to express this in terms of the representations V, W_1 and W_2.

The reader should now make explicit the vanishing of structure constants that these conditions imply in the topological case. For $b^{i,j}_s$ and $d(\alpha)^i_s$ the answer is precisely as in the following proposition. For the tensor product it states that the coefficient $f^i_{s,t}$ is zero if $\dim \mathbb{C}P(V^{s+1})^B + \dim \mathbb{C}P(V^{t+1})^B \leqslant \dim \mathbb{C}P(V^i)^B$ holds for all subgroups $B \subseteq A$. Since any fixed estimate suffices for our purpose, we shall be satisfied with a cruder one.

PROPOSITION 14.1. *For any (A, F)-formal group over k we have the following explicit vanishing conditions:*

(VR) $b^{i,j}_s = 0$ *if* $s < i$ *or* $s < j$;

(VA) $d(\alpha)^i_s = 0$ *if* $V^i \otimes \alpha \geqslant V^{s+1}$;

(VT) $f^i_{s,t} = 0$ *if* $V^i \geqslant \alpha_1^{-1} V^{s+1} \otimes V^{t+1}$.

REMARK 14.2. Note that because the flag F exhausts $\mathcal{U}$, the proposition gives explicit forms of the continuity conditions of § 12.

Proof. Note that (VR) is equivalent to (Ideal): indeed, it is clear by (Flag) that all elements $y(V^j)$ with $j \geqslant i$ lie in the ideal $(y(V^i))$. Both conditions are equivalent to requiring that for all j, $y(V^i)y(V^j)$ has zero coefficient of $y(V^k)$ for all $k < i$.

Next we note that (VA) follows. Indeed, if $\alpha V^i \geqslant V^{s+1}$ then $l_\alpha y(V^i) = y(\alpha V^i)$ is a multiple of $y(V^{s+1})$ by (Flag). We have just observed that (VR) shows that any such element has zero coefficient of $y(V^k)$ with $k < s+1$.

For (VT) we use the counit condition and the fact that Δ is a ring map, together with (Ideal). For an element $u = \sum_{p,q} a_{p,q} y(V^p) \otimes y(V^q)$ we say that u *vanishes up to* (s, t) if $a_{p,q} = 0$ if $p \leqslant s$ and $q \leqslant t$. We shall find an $i_0 = i_0(s, t)$ so that $\Delta y(V^i)$ vanishes up to (s, t) whenever $i \geqslant i_0$: we do not attempt to find the best possible i_0.

Note that by (Ideal), if u vanishes up to (s, t) then any product uv vanishes up to (s, t). Furthermore, we may ensure vanishing up to $(s+1, t)$ by using suitable elements v. Again by (Ideal), the only terms in u which can contribute to non-vanishing up to $(s+1, t)$ in uv are $a_{s+1,j} y(V^{s+1}) \otimes y(V^j)$ for $j \leqslant t$. Thus multiplying by $v_1 = y(\alpha_{s+2}) \otimes z$ or $v_2 = z \otimes y(V^{t+1})$ ensures vanishing up to $(s+1, t)$.

The counit condition states that

$$\Delta(y(V)) = 1 \otimes y(V) + y(V) \otimes 1 \mod (y(V^1) \otimes y(V^1)).$$

In particular, $\Delta(y(V^{t+1}))$ vanishes up to $(0, t)$, proving (VT) for $s = 0$; we prove the general case by induction on s.

For the inductive step we apply equivariance $(l_\alpha \otimes 1) \circ \Delta = \Delta \circ l_\alpha$ to the counit condition, and find that

$$\Delta(y(\alpha V)) = 1 \otimes y(V) + y(\alpha V) \otimes 1 \mod (y(\alpha V^1) \otimes y(V^1)).$$

Thus, taking $\alpha = \alpha_{s+2}\alpha_1^{-1}$ and $W = \alpha V$ we see that if u vanishes up to (s, t) then $\Delta(y(W))u$ vanishes up to $(s+1, t)$ provided W contains $\alpha_{s+2}\alpha_1^{-1}V^{t+1}$. This completes the proof.

We may now proceed to form the representing ring.

COROLLARY 14.3. *There is a representing ring $L_A(F)$ for (A, F)-formal group laws, constructed as the $\mathbb{Z}$-algebra with generators $b_s^{i,j}$, $d(\alpha)_s^i$ and $f_{s,t}^i$ subject to the relations implied by* (R), (A), (T), (Flag), (VR), (VA) *and* (VT).

REMARK 14.4. Since Lemma 13.1 gives a canonical way to view an (A, F)-formal group law as an A-formal group law and Lemma 13.2 gives a canonical way to view this as an (A, F')-formal group law, there is a canonical isomorphism $L_A(F') \xrightarrow{\cong} L_A(F)$.

Note that there is massive redundancy in the generating set: we shall see in the next few sections that the ring is generated by the elements $d(\alpha)_0^1$ and $f_{s,t}^1$.

Finally, we comment briefly on the representing ring for objects analogous to orientable complex stable theories: A-equivariant formal group laws over k with specified Euler classes. The Euler classes are specified by a function $\chi\colon A^* \to k$ and we let $L_A^{\text{strict}}(F, k, \chi)$ denote the representing ring when the strict Euler classes are required to agree with χ. Note that this will not necessarily contain k: for example, the condition $\chi(\varepsilon) = 0$ is imposed. Understanding A-equivariant formal group laws where the strict Euler classes are unit multiples of specified Euler classes (that is, orientability and orientations of complex stable theories) seems more subtle.

15. *Some relations in the representing ring: leading terms*

After § 13 we have enough structure to work in the algebraic setup just as if it arose from topology. The purpose of this section and the next is to establish relations amongst the structure constants for equivariant formal group laws. Because the leading terms are the most significant, and in view of the confusing forest of superscripts and subscripts, we have decided to present discussion of the first few terms separately, as a motivation for the general case. The reader may also find it helpful to refer to Appendix C where some calculations are done when A is of order 2. As a matter of logic the present section may be omitted.

We shall need to discuss various different flags, so when necessary we write $b(F)_s^{i,j}$ or $(b_F)_s^{i,j}$ to emphasize that we are working with the flag F, and similarly for the d and f functions. The essential difficulty is in the form of the product, so we comment on $y(V^i)y(V^j)$ a little further. We show that to express it in terms of the flag basis, it is sufficient to understand the action of A^* in that basis. The idea is that the path $\alpha_j, \alpha_{j+1}, \alpha_{j+2}, \ldots$ also defines a complete flag F/V^j, and the corresponding basis is $y(V^j/V^j) = 1, y(V^{j+1}/V^j), y(V^{j+2}/V^j), \ldots$. The point of this is that $y(V^j)y(V^{j+n}/V^j) = y(V^{j+n})$, which is an element of the original basis. Thus we simply express $y(V^i)$ in terms of the new basis as $y(V^i) = \sum_k \overline{b}_{j+k}^{i,j} y(V^{j+k}/V^j)$ and then $y(V^i)y(V^j) = \sum_k \overline{b}_{j+k}^{i,j} y(V^{j+k})$. Thus $\overline{b}_s^{i,j} = b_s^{i,j}$, explaining the notation. In practice we do this one step at a time, since $y(V^i) = y(\alpha_1)\ldots y(\alpha_i)$. It therefore suffices to identify $y(\alpha)y(V^j)$, for all α and j. For this we note that $y(\alpha) = l_{\alpha\alpha_{j+1}^{-1}} y(\alpha_{j+1})$, and work with respect to the F/V^j basis.

We also write $F\delta$ for the flag $(V^0\delta \subset V^1\delta \subset V^2\delta \subset \ldots)$ whose associated path is $\alpha_1\delta, \alpha_2\delta, \alpha_3\delta, \ldots$. We refer to $F\delta$ as the *rotated flag*. It is clear by applying l_δ to the formula defining d_F that

$$d_{F\delta}(\alpha\delta)_s^i = d_F(\alpha)_s^i.$$

It is thus no loss of generality to work with flags beginning with $V^1 = \varepsilon$, and in this section we assume the flag path begins $\varepsilon, \alpha, \beta, \gamma, \ldots$. We show that for any complete flag F' the action coefficients $d_{F'}(*)_*^1$ can be deduced from the coefficients $d_F(*)_*^1$. It then follows that all product coefficients $(b_F)_*^{*,*}$ can be deduced, and all the higher d_F. Since we are just dealing with the first few terms, we simplify notation; we already know that for $r \in A^*$ we have $e(r) = d(r)_0^1$, and we define $\lambda(r) = d(r)_1^1$, $\mu(r) = d(r)_2^1$, and $\nu(r) = d(r)_3^1, \ldots$. Thus by definition,

$$y(r) = e(r) + \lambda(r)y(\varepsilon) + \mu(r)y(\varepsilon)y(\alpha) + \nu(r)y(\varepsilon)y(\alpha)y(\beta) + \ldots .$$

The first observation is that the coefficient $d_F(r)_i^1$ only depends on the flag as far as V^{i+1}. This is because the coefficient can be recovered after reduction modulo $y(V^{i+1}) = y(\varepsilon)y(\alpha_2)\ldots y(\alpha_{i+1})$. Accordingly, we write $e(r), \lambda_\alpha(r), \mu_{\alpha,\beta}(r), \ldots$ to emphasize this.

We also need the coefficients $e_F^n(r), \lambda_F^n(r), \mu_F^n(r), \ldots$ in $l_r y(V^n)$, and we show by induction on n that these can be deduced from the coefficients $e, \lambda_\alpha, \mu_{\alpha,\beta}, \ldots$.

Indeed $y(V^{n+1}) = y(V^n)y(\alpha_{n+1})$, so we have

$$l_r y(V^{n+1}) = e^n_F(r)y(\alpha_{n+1}r) + \lambda^n_F(r)y(V^1)y(\alpha_{n+1}r) + \mu^n_F(r)y(V^2)y(\alpha_{n+1}r) + \dots .$$

The trick for calculating the ith term is to express $y(\alpha_{n+1}r)$ in the F/V^i basis. Of course this uses the coefficients $e_{F/V^i}, \lambda_{F/V^i}, \mu_{F/V^i}, \dots$. This looks dangerously close to being circular, so we show explicitly that it is not.

LEMMA 15.1. *The leading term is given by*

$$e^n_F(r) = e(V^n r).$$

Proof. The proof here and in the following two lemmas is to expand the right-hand side of the equation

$$l_r y(V^{n+1}) = l_r[y(V^n)]y(\alpha_{n+1}r).$$

To do this we first expand $l_r[y(V^n)]$ to obtain

$$e^n_F(r)y(\alpha_{n+1}r) + \lambda^n_F(r)y(\varepsilon)y(\alpha_{n+1}r) + \mu^n_F(r)y(\varepsilon)y(\alpha)y(\alpha_{n+1}r) + \dots .$$

Now expand $y(\alpha_{n+1}r)$ with respect to the $\varepsilon, \alpha, \beta, \dots$ basis in the first term, with respect to the $\alpha, \beta, \gamma, \dots$ basis in the second term, with respect to the $\beta, \gamma, \delta, \dots$ basis in the third term, and so forth. This gives the required formula.

One important consequence is that if r^{-1} occurs in V^n, this leading term vanishes. Since F is a complete flag, we thus see that $e^n_F(r) = 0$ if n is sufficiently large. Note also that for $n \geqslant 2$ this coefficient depends on more than the first subquotient of F.

For the higher terms we cannot expect a closed formula, but a recursive algorithm is quite sufficient. The proofs are precisely like those for Lemma 15.1 above, and the general case is presented in detail in the next section.

LEMMA 15.2. *The first term is given recursively in terms of Euler classes and* $\lambda_\alpha(r)$ *by the formula*

$$\lambda^{n+1}_F(r) = e(V^n r)\lambda_\alpha(\alpha_{n+1}r) + \lambda^n_F(r)e(\alpha^{-1}\alpha_{n+1}r).$$

We note here that, once n is large enough that $e(V^n r) = 0$, the recursion states that $\lambda^{n+1}_F(r) = \lambda^n_F(r)e(\alpha^{-1}\alpha_{n+1}r)$. Thus if $\alpha_{n+1} = \alpha r^{-1}$, we obtain zero. Once again we see that $\lambda^n_F(r) = 0$ if n is sufficiently large. Finally, we note that if $n \geqslant 2$ then $\lambda^n_F(r) = 0$ modulo Euler classes.

It is instructive to record one further instance explicitly.

LEMMA 15.3. *The second term is given recursively in terms of Euler classes and* $\lambda_\alpha(r)$ *by the formula*

$$\mu^{n+1}_F(r) = e(V^n r)\mu_{\alpha,\beta}(\alpha_{n+1}r) + \lambda^n_F(r)\lambda_{\alpha^{-1}\beta}(\alpha^{-1}\alpha_{n+1}r) + \mu^n_F(r)e(\beta^{-1}\alpha_{n+1}r).$$

The new feature here is that the coefficient λ is associated to a different flag. We had therefore better ensure that we can deduce it before we need to apply this recursive formula.

Suppose then that F' begins $\varepsilon, \zeta, \dots$. Thus $y(r) = e(r) + \lambda_\zeta(r)y(\varepsilon)$ modulo

$y(\varepsilon)y(\zeta)$. Now, find the first occurrence of ζ in F, and express $y(r)$ in terms of the F basis:

$$y(r) = e(r) + \lambda_\alpha(r)y(\varepsilon) + \mu_{\alpha,\beta}(r)y(\varepsilon)y(\alpha) + \dots;$$

it is sufficient to work modulo $y(\varepsilon)y(\alpha)\dots y(\zeta)$. Now we expand $y(\alpha), y(\alpha)y(\beta), y(\alpha)y(\beta)y(\gamma), \dots$ in terms of the F'/ε basis. In the present case we only seek the coefficient of $y(\varepsilon)$ so we only need the constant term in the expansions, and this is easily calculated. Thus we obtain

$$\lambda_\zeta(r) = \lambda_\alpha(r) + \mu_{\alpha,\beta}(r)e(\alpha\zeta^{-1}) + \nu_{\alpha,\beta,\gamma}(r)e(\alpha\zeta^{-1})e(\beta\zeta^{-1}) + \dots,$$

and the sum is finite, since $e(\zeta\zeta^{-1}) = 0$.

Note in particular that this shows that $\lambda_\alpha(r) = \lambda_\zeta(r)$ modulo Euler classes, so taking $\zeta = r$ we see that $\lambda_\alpha(r) = 1$ modulo Euler classes.

16. *Some relaitons in the representing ring: the general case*

We extend the calculations of the previous section to higher coefficients. The discussion is directed towards understanding the universal ring $L_F(A)$. Our first task is to give a reasonably efficient set of generators. The reader may also find it helpful to refer to Appendix C where some calculations are done when A is of order 2.

THEOREM 16.1. *The representing ring $L_F(A)$ is generated as an algebra by the Euler classes $e(\alpha)$ and the coefficients $f(F)^1_{j,k}$. The representing ring $L_A^{\text{strict}}(k, F, \chi)$ is generated as a k-algebra by the elements $f^1_{s,t}$.*

The strategy of proof is as follows, where we write $x \leftarrow (y, z)$ to mean that x can be expressed in terms of y and z, together with some self-explanatory abbreviations.

Process 0: $d_{F'}(*)^1_{n+1} \leftarrow (d_{F'}(*)^1_n, d^1_F)$ (see Lemma 16.4).

Process 1: $f^{n+1} \leftarrow (f^n, d^*)$
from the fact that $(\Delta(y(V^{n+1})) = \Delta(y(V^n))\Delta\ (y(\alpha_{n+1}))$.

Process 2: $d^{n+1} \leftarrow (d^{\leqslant n}, d^1)$ (see Lemma 16.3).

Process 3: $d^n \leftarrow (f^n, e)$ (see Lemma 16.7).

Here, Process 0 has been used to avoid specifying the flags used for the generators d^1 in Process 2. Using Process 3 we obtain the generators d^1 from the coefficients f^1 and e. Then, using Process 2 recursively, we obtain all the generators d^n. Finally, Process 1 can be used recursively to obtain all the coefficients f^n. We have already seen how the coefficients b can be obtained from the generators d. We now turn to the detailed implementation of the strategy.

First, let us write the definition of the coefficients d_F in longhand:

$$y(r) = d_F(r)^1_0 + d_F(r)^1_1 y(V^1) + d_F(r)^1_2 y(V^2) + d_F(r)^1_3 y(V^3) + \dots.$$

The first observation is that the coefficient $d_F(r)^1_i$ only depends on the flag as far as V^{i+1}. This is because the coefficient is determined by reducing modulo $y(V^{i+1}) = y(\varepsilon)y(\alpha_2)\dots y(\alpha_{i+1})$. Accordingly we write $d_{V^1}(r)_0, d_{V^2}(r)_1, d_{V^3}(r)_2, \dots$ to emphasize this; note that we have also omitted the superscript 1. By contrast with the previous section we have not normalized the flag to begin with ε, so the start of the flag is an essential piece of information.

We begin by summarizing the results of the previous section in the general notation: note that we have rotated the flags to the natural position.

LEMMA 16.2. (i) *The leading term is given by*

$$d_F(r)_0^n = e(\alpha_1^{-1}V^n r).$$

This is zero if n is sufficiently large.

(ii) *The first term is given recursively in terms of Euler classes and $d_{V^2}(*)_*^1$ by the formula*

$$d_F(r)_1^{n+1} = d_F(r)_0^n d_{V^2}(\alpha_1^{-1}\alpha_{n+1}r)_1 + d_F(r)_1^n d_{V^2/V^1}(\alpha_2^{-1}\alpha_{n+1}r)_0.$$

This is zero if n is sufficiently large, and if $n \geqslant 2$ then $d(r)_1^n = 0$ modulo Euler classes.

(iii) *The second term is given recursively in terms of the coefficients $d_F(*)_*^1$ by the formula*

$$\begin{aligned} d_F(r)_2^{n+1} &= d_F(r)_0^n d_{V^3}(\alpha_1^{-1}\alpha_{n+1}r)_2 + d_F(r)_1^n d_{V^3/V^1}(\alpha_2^{-1}\alpha_{n+1}r)_1 \\ &\quad + d_F(r)_2^n d_{V^3/V^2}(\alpha_3^{-1}\alpha_{n+1}r)_0. \end{aligned}$$

It is not hard to write down the general recursion, and it should now be possible to understand what it means.

LEMMA 16.3.

$$\begin{aligned} d_F(r)_k^{n+1} &= d_F(r)_0^n d_{V^{k+1}}(\alpha_1^{-1}\alpha_{n+1}r)_k + d_F(r)_1^n d_{V^{k+1}/V^1}(\alpha_2^{-1}\alpha_{n+1}r)_{k-1} \\ &\quad + d_F(r)_2^n d_{V^{k+1}/V^2}(\alpha_3^{-1}\alpha_{n+1}r)_{k-2} \\ &\quad + \ldots + d_F(r)_{k-1}^n d_{V^{k+1}/V^{k-1}}(\alpha_k^{-1}\alpha_{n+1}r)_1 \\ &\quad + d_F(r)_k^n d_{V^{k+1}/V^k}(\alpha_{k+1}^{-1}\alpha_{n+1}r)_0. \end{aligned}$$

Proof. First note that $l_r y(V^{n+1}) = [l_r y(V^n)]y(\alpha_{n+1}r)$. When calculating the term $d_F(r)_i^n y(V^i)y(\alpha_{n+1}r)$, we need to express $y(\alpha_{n+1}r)$ in terms of the F/V^i flag, so the jth coefficient is $d_{F/V^i}(\alpha_{i+1}^{-1}\alpha_{n+1}r)_j^1 = d_{V^{i+j+1}/V^i}(\alpha_{i+1}^{-1}\alpha_{n+1}r)_j$. This gives the stated formulae.

Now, suppose by induction that all coefficients $d_{F'}(*)_l^1$ with $l < k$ and all coefficients $d_F(*)_l^*$ with $l \leqslant k$ can be expressed in terms of the coefficients $d_F(*)_*^1$. Noting that the only occurrence of $d(*)_k^1$ on the left-hand side uses the F basis, we see that the lemma shows that the coefficients $d_F(*)_k^{n+1}$ can also be so expressed.

It thus follows by induction that if all coefficients $d_{F'}(*)_l^1$ with $l < k$ can be expressed in terms of the coefficients $d_F(*)_l^1$ with $l \leqslant k$ then all coefficients $d_F(*)_l^*$ with $l \leqslant k$ can be so expressed.

The following lemma completes the justification of Process 0.

LEMMA 16.4. *The coefficient $d_{F'}(r)_k^1$ can be expressed in terms of the coefficients $d_{F'}(*)_l^1$ with $l < k$ and the coefficients $d_F(*)_*^1$.*

Proof. By rotation we may suppose that F' also begins with α_1. Now $d_{F'}(r)^1_k = d_{(V')^{k+1}}(r)_k$ is the coefficient of $y((V')^k)$ in the F' expansion of $y(r)$. We will give a way of calculating this coefficient in terms of the $d_F(*)^1_*$. Indeed, we also have $y(r) = \sum_i d_F(r)^1_i y(V^i)$. Furthermore, $y(V^i) = y(\alpha_1)y(V^i/V^1)$, and we may express $y(V^i/V^1)$ in terms of the F'/V^1 basis, and the coefficient of $y((V')^k/V^1)$ will give us the contribution to the coefficient in question. Furthermore, the contribution from $y(V^i)$ is zero once V^i contains $(V')^{k+1}$, since $y(V^i)$ is then already zero modulo $y((V')^{k+1})$; thus the number of terms is finite. It thus suffices to apply the following lemma with $W = V^i/V^1$ and $F'' = F'/V^1$.

LEMMA 16.5. *If we write $y(W)$ in the F'' basis as $y(W) = \sum_l [F'']^W_i y((V'')^i)$ then the coefficients $[F'']^W_l$ may be expressed in terms of coefficients $d_V(*)_m$ with $m \leqslant l$.*

Proof. The proof is by induction on the dimension of W. If $W = 0$, the result is trivial. The inductive step is to write $W = W' \oplus \beta$. Then we have $y(W) = y(W')y(\beta) = \sum_i [F'']^{W'}_i y((V'')^i)y(\beta)$, and in the ith term we write $y(\beta)$ in terms of the $F''/(V'')^i$ basis, using the coefficients $d_{F''/(V'')^i}(\beta(\alpha''_{i+1})^{-1})^1_j$. The contributing coefficients have $i + j = l$, so both $i, j \leqslant l$. This completes the proof of Lemma 16.5 and hence also the proof of Lemma 16.4.

COROLLARY 16.6. *Modulo Euler classes we have $d_F(r)^1_1 = 1$, and $d_F(r)^1_k = 0$ if $k \neq 1$.*

Proof. We have given explicit formulae when $k = 0$ or 1 in the previous section. In principle the previous two lemmas also give an explicit formula in terms of Euler classes, but it is perhaps worth giving a less cluttered proof.

Suppose then that $k \geqslant 2$, and that the result has been proved for $l < k$ for all flags. For the rest of the proof we work modulo Euler classes without comment. We prove that $d_F(r)^1_k = d_{F'}(r)^1_k$ for any flag F', and hence by taking F' to begin with r, that both vanish.

First note, by induction, that in any basis F'',

$$y(r) = y(s) + d_{F''}(sr^{-1})^1_k y((V'')^k) + \text{higher terms},$$

so that $y(V) = y(W)$ if $\dim(V) = \dim(W) < k$. Now take any two complete flags F, and F'. By rotation we assume that both F and F' begin with ε. Expanding $y(r)$ in the F' basis we have

$$y(r) = y(\varepsilon) + d_{F'}(r)^1_k y((V')^k) + \text{higher terms}.$$

Similarly, in the F basis we have

$$y(r) = y(\varepsilon) + d_F(r)^1_k y(V^k) + \text{higher terms}.$$

However $y(V^k) = y(\varepsilon)y(V^k/\varepsilon)$, and by the observation, $y(V^k/\varepsilon) = y((V')^k/\varepsilon)$ modulo $y((V')^{k+1}/\varepsilon)$. Similarly $y(V^{k+i}) = y(\varepsilon)y(V^k/\varepsilon)y(V^{k+i}/V^k) = 0$, modulo $y((V')^{k+1}/\varepsilon)$. Thus $d_F(r)^1_k = d_{F'}(r)^1_k$ as required.

Finally, we see that the equivariance of Δ allows us to deduce the coefficients $d_F(\alpha)^1_*$ from the coproduct and the Euler classes.

LEMMA 16.7. *The following formula gives the action coefficients in terms of the cogroup coefficients and the Euler classes*

$$d_F(\beta)^i_k = \sum_j f^i_{j,k} e(\beta V^1 \otimes V^j).$$

Note that the $j = 0$ *term is* $f^i_{0,k} = \delta^i_k$.

Proof. Firstly, we see from the equivariance of Δ that

$$l_\beta = (\mathrm{res}^{\mathcal{U}}_{\beta^{-1}} \widehat{\otimes} 1)\Delta.$$

Applying this to $y(V^i)$, we see that the coefficient of $y(V^k)$ on the left is $d_F(\beta)^i_k$. To identify the coefficient of $y(V^k)$ on the right we recall that, by definition, $\mathrm{res}^{\mathcal{U}}_{\beta^{-1}} = \mathrm{res}^{\mathcal{U}}_{V^1} l_{\beta V^1}$, and calculate

$$\begin{aligned}
(\mathrm{res}^{\mathcal{U}}_{\beta^{-1}} \widehat{\otimes} 1)\Delta y(V^i) &= (\mathrm{res}^{\mathcal{U}}_{V^1} l_{\beta V^1} \widehat{\otimes} 1) \sum_{j,k} f^i_{j,k} y(V^j) \otimes y(V^k) \\
&= (\mathrm{res}^{\mathcal{U}}_{V^1} \widehat{\otimes} 1) \sum_{j,k} f^i_{j,k} \sum_l d(\beta V^1)^j_l y(V^l) \otimes y(V^k) \\
&= \sum_{j,k} f^i_{j,k} d(\beta V^1)^j_0 y(V^k).
\end{aligned}$$

This completes the proof of Lemma 16.7 and hence also the proof of Theorem 16.1.

Appendix A. The additive and multiplicative group laws

Let us consider two special cases: the additive and multiplicative laws. It is easy to pick these out since we do not need to say much about which flag we are considering. For simplicity, we work in the ungraded setting. For further details and extensions of these results see [**8**].

The additive law is given by

$$\Delta^*_a(y(\varepsilon)) = y'(\varepsilon) + y''(\varepsilon)$$

and the multiplicative law by

$$\Delta^*_m(y(\varepsilon)) = y'(\varepsilon) + y''(\varepsilon) - y'(\varepsilon)y''(\varepsilon).$$

The instructive thing here is how this imposes restrictions on the Euler classes. In particular, the additive law implies that the Euler classes are all $\mathbb{Z}$-torsion when A is finite; this shows that we cannot expect to use logarithms in the same way as the non-equivariant case, since Euler classes are not generally of this form. However, results of tom Dieck [**7**] suggest that (at least for cyclic groups) we may hope to use the multiplicative logarithm as in [**1**, I.6.7].

For convenience of calculation we shall assume the flag begins with ε. Thus the additive case has $f^1_{1,0} = f^1_{0,1} = 1$ and $f^1_{j,k} = 0$ otherwise. Thus, from Lemma 16.7 we see that

$$y(\beta) = e(\beta) + y(\varepsilon).$$

Rotating by α we find that $e(\alpha\beta) = e(\alpha) + e(\beta)$, that is, $e\colon A^* \to k$ is a group homomorphism, and in particular if $\beta^n = \varepsilon$, we see that $ne(\beta) = 0$. This identifies the universal ring for additive equivariant formal group laws.

PROPOSITION A.1. *The free commutative ring* $\mathrm{Symm}(A^*)$ *on the abelian group* A^* *is universal for additive formal group laws. If A is a torus, this is the coefficient ring of ordinary Borel cohomology.*

The multiplicative case has $f^1_{1,0} = f^1_{0,1} = 1$, $f^1_{1,1} = -1$ and $f^1_{j,k} = 0$ otherwise. Thus, from Lemma 16.7 we see that

$$y(\beta) = e(\beta) + [1 - e(\beta)]y(\varepsilon).$$

Rotating by α shows that $(1 - e(\alpha\beta)) = (1 - e(\alpha))(1 - e(\beta))$, that is, $1 - e\colon A^* \to k^\times$ is a group homomorphism. This identifies the universal ring for multiplicative group laws.

PROPOSITION A.2. *The group ring* $\mathbb{Z}[A^*]$ *is the universal ring for multiplicative group laws. Since A is abelian,* $\mathbb{Z}[A^*] \cong R(A)$*, and the coefficient ring of K-theory is universal for multiplicative formal group laws.*

Appendix B. The coinverse

We show here that any A-equivariant formal group law has a unique coinverse, and is thus a Hopf algebra. This is exactly analogous to the fact that a connected bialgebra has a coinverse, and the proof is analogous to that of [**15**, 8.2], although substantially more complicated. The idea is that the formal group law is equipped with a filtration: the subquotients are controlled by the well-understood Hopf algebra k^{A^*} and the filtration is complete.

PROPOSITION B.1. *Given an A-formal group law* $(R, \Delta, \theta, \{y(\alpha)\}_{\alpha \in A^*})$*, there exists a unique algebra homomorphism* $\gamma\colon R \to R$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} R & \xrightarrow{\Delta} & R \widehat{\otimes}_k R & \xrightarrow{1 \otimes \gamma} & R \widehat{\otimes}_k R \\ {\scriptstyle \theta(\varepsilon)}\downarrow & & & & \downarrow{\scriptstyle m} \\ k & & \xrightarrow[j]{\hspace{6em}} & & R \end{array}$$

Moreover,

(1) $\gamma \circ \gamma = 1_R$,
(2) $(\gamma \otimes \gamma) \circ \Delta = \Delta \circ \gamma$,
(3) $l_\alpha \circ \gamma = \gamma \circ l_{\alpha^{-1}}$,
(4) $\theta(\alpha) \circ \gamma = \theta(\alpha^{-1})$.

Proof. Let $0 \subset V^1 \subset V^2 \subset \ldots$ be a complete flag in $\mathcal{U}$ with $V^{i+1} = V^i \oplus \alpha_{i+1}$ as usual. We assume for convenience that $\alpha_1 = V^1 = \varepsilon$. If γ exists, we may express it in terms of the flag basis

$$\gamma(y(V^i)) = \sum_{j=0}^{\infty} c^i_j y(V^j),$$

and if it is to be continuous, we require that for any fixed j we have $c^i_j = 0$ for large i.

We shall construct γ by finding the coefficients $c_0^i, c_1^i, c_2^i, \ldots$ in turn. We therefore take

$$\gamma_n(y(V^i)) = \sum_{j=0}^{n} c_j^i y(V^j),$$

and suppose inductively that γ_n: $R \to R$ is a k-module homomorphism with the property that

$$m \circ (1 \otimes \gamma_n) \circ \Delta(x) \equiv j \circ \theta(\varepsilon)(x) \mod (y(V^{n+1}))$$

for any $x \in R$. To start the induction, set $\gamma_0 = \theta(\varepsilon)$.

Now suppose γ_n has been constructed: we wish to find coefficients $c_{n+1}^i \in k$ so as to define a map γ_{n+1} that has the inverse property modulo $(y(V^{n+2}))$. For any $x \in R$ we define $r_x \in k$ by

$$m \circ (1 \otimes \gamma_n) \circ \Delta(x) \equiv j \circ \theta(\varepsilon)(x) + r_x y(V^{n+1}) \mod (y(V^{n+2})).$$

We need to choose the coefficients c_{n+1}^i in such a way that

$$m \circ (1 \otimes (\gamma_{n+1} - \gamma_n)) \circ \Delta(x) \equiv -r_x y(V^{n+1}) \mod (y(V^{n+2}))$$

for any $x \in R$. It suffices to establish this as x runs through the topological basis $y(\alpha_{n+2}) = y(V^1\alpha_{n+2}), y(V^2\alpha_{n+2}), y(V^3\alpha_{n+2}), \ldots$. We write the coproduct Δ: $R \to R \widehat{\otimes}_k R$ as

$$\Delta(y(V^i)) = \sum f_{s,t}^i y(V^s) \otimes y(V^t).$$

Recall that for fixed s, t, $f_{s,t}^i = 0$ for large i and that the unital condition of the A-equivariant formal group law implies that

$$f_{s,0}^i = f_{0,s}^i = \begin{cases} 1 & \text{if } s = i, \\ 0 & \text{if } s \neq i. \end{cases}$$

Using the A^*-equivariant property of Δ, we see that

$$\Delta(y(V^i\alpha_{n+2})) = \sum f_{s,t}^i y(V^s\alpha_{n+2}) \otimes y(V^t).$$

Thus

$$m \circ (1 \otimes (\gamma_{n+1} - \gamma_n)) \circ \Delta(y(V^i\alpha_{n+2})) = \sum f_{s,t}^i c_{n+1}^t y(V^s\alpha_{n+2}) y(V^{n+1}).$$

Since $\alpha_1 = \varepsilon$, we see that $y(V^{n+2})$ divides $y(V^s\alpha_{n+2})y(V^{n+1})$ if $s \geqslant 1$. Since $f_{0,t}^i$ is 1 when $t = i$ and is 0 otherwise we find that

$$m \circ (1 \otimes (\gamma_{n+1} - \gamma_n)) \circ \Delta(y(V^i\alpha_{n+2})) \equiv c_{n+1}^i y(V^{n+1}) \mod (y(V^{n+2})).$$

Thus we define $c_{n+1}^i = -r_{y(V^i\alpha_{n+2})}$. It remains only to check the continuity condition that these coefficients must vanish for large i. This follows since $m \circ (1 \otimes \gamma_{n+1}) \circ \Delta$ gives the continuous homomorphism $j \circ \theta(\varepsilon)$: $R \to R/(y(V^{n+2}))$.

So far we have proved the existence of a k-module homomorphism γ: $R \to R$ that makes the diagram commute. It is clear from the way we constructed it that γ is unique with this property, but in any case this follows from an easy formal argument: if γ and γ' are both inverses then the composite

$$R \xrightarrow{\Delta} R \widehat{\otimes}_k R \widehat{\otimes}_k R \xrightarrow{\gamma \otimes 1 \otimes \gamma'} R \widehat{\otimes}_k R \widehat{\otimes}_k R \xrightarrow{m} R$$

must agree simultaneously with γ and γ'.

To show that γ is a k-algebra homomorphism and to establish the other claims we argue similarly by uniqueness of inverse. Let S denote $R \widehat{\otimes}_k R$. Note that S has a coproduct Δ_S: $S \to S \widehat{\otimes}_k S$ defined by the composite

$$R \widehat{\otimes}_k R \xrightarrow{\Delta \otimes \Delta} R \widehat{\otimes}_k R \widehat{\otimes}_k R \widehat{\otimes}_k R \xrightarrow{1 \otimes \tau \otimes 1} R \widehat{\otimes}_k R \widehat{\otimes}_k R \widehat{\otimes}_k R,$$

where τ denotes the twist map. The ring S has a unit map j_S: $k \to S$ defined by $j_S = j \otimes j$ and an augmentation θ_S: $S \to k^{A^*}$ defined as the composite

$$R \widehat{\otimes}_k R \xrightarrow{\theta \otimes \theta} k^{A^*} \widehat{\otimes}_k k^{A^*} \xrightarrow{m} k^{A^*}.$$

In fact, we may regard S as a two-dimensional A-formal group over k. Now by diagram chasing it can be shown that $m \circ (\gamma \otimes \gamma)$ and $\gamma \circ m$ are both inverse to the multiplication m: $S \to R$ in the sense that if ϕ denotes either of $m \circ (\gamma \otimes \gamma)$ or $\gamma \circ m$, then the diagram

$$\begin{array}{ccccc} S & \xrightarrow{\Delta_S} & S \widehat{\otimes}_k S & \xrightarrow{m \otimes \phi} & R \widehat{\otimes}_k R \\ \theta_S(\varepsilon) \downarrow & & & & \downarrow m \\ k & & \xrightarrow[j]{\hspace{6em}} & & R \end{array}$$

commutes. It follows formally that $m \circ (\gamma \otimes \gamma) = \gamma \circ m$ and hence that γ is an algebra homomorphism. Dually, $(\gamma \otimes \gamma) \circ \Delta$ and $\Delta \circ \gamma$ must agree since a diagram chase shows that they are both inverse to Δ. Similarly, $\gamma \circ \gamma$ and 1 agree since they are both inverse to γ. For claim (3) we use the equivariant property of Δ and observe that

$$\begin{aligned} m \circ (1 \otimes (l_\alpha \circ \gamma \circ l_\alpha)) \circ \Delta &= m \circ ((l_\alpha \circ l_{\alpha^{-1}}) \otimes (l_\alpha \circ \gamma \circ l_\alpha)) \circ \Delta \\ &= m \circ (l_\alpha \otimes l_\alpha) \circ (1 \otimes \gamma) \circ (l_{\alpha^{-1}} \otimes l_\alpha) \circ \Delta \\ &= l_\alpha \circ m \circ (1 \otimes \gamma) \circ \Delta \\ &= l_\alpha \circ j \circ \theta(\varepsilon) \\ &= j \circ \theta(\varepsilon). \end{aligned}$$

Thus $l_\alpha \circ \gamma \circ l_\alpha = \gamma$ since they are both inverse to 1. One can check directly from our construction of γ that $\gamma(y(\varepsilon)) \in (y(\varepsilon))$. Hence $\theta(\varepsilon) \circ \gamma = \theta(\varepsilon)$. It follows that

$$\theta(\alpha) \circ \gamma = \theta(\varepsilon) \circ l_\alpha \circ \gamma = \theta(\varepsilon) \circ \gamma \circ l_{\alpha^{-1}} = \theta(\varepsilon) \circ l_{\alpha^{-1}} = \theta(\alpha^{-1}).$$

Appendix C. The group of order 2

In this appendix we make some of the formulae explicit for the group $A = C_2$ of order 2. We use the flag with alternating subquotients $\varepsilon, \eta, \varepsilon, \eta, \varepsilon, \eta, \varepsilon, \eta, \ldots$. Since there is only one non-trivial element of A^*, we write simply $d_j^i = d(\eta)_j^i$, and continue with the convention that we drop i if $i = 1$. It is convenient to declare $d_j^i = 0$ if $j < 0$. As we give formulae it will become ever more apparent that there is a sense in which an equivariant formal group is a deformation of a non-equivariant formal group with the Euler class $e = d_0$ as deformation parameter [**9**].

First note that since

$$y(\eta) = d_0 + d_1 y(\varepsilon) + d_2 y(\varepsilon) y(\eta) + \ldots,$$

by rotation we have

$$y(\varepsilon) = d_0 + d_1 y(\eta) + d_2 y(\eta) y(\varepsilon) + \ldots,$$

and thus, premultiplying both sides by $y(\varepsilon)$ we have an expression for $y(\varepsilon)^2$ in the standard basis:

$$b_k^{1,1} = d_{k-1}.$$

Thus

$$k\{\{\mathscr{U}\}\} = k[[x]][y]/(y^2 = xp(x) + yq(x))$$

where $y = y(\varepsilon)$, $x = y(\varepsilon)y(\eta)$, and where $p(x) = d_1 + d_3 x + d_5 x^2 + \ldots$ and $q(x) = e + d_2 x + d_4 x^2 + \ldots$. Now consider $y(V^i)y(V^j)$; if i or j is even, it is clearly $y(V^{i+j})$, and if both are odd it is $y(V^{i-1})y(V^{j-1})y(\varepsilon)^2 = y(V^{i+j-2})y(\varepsilon)^2$. Combining these into a single formula we have

$$b_k^{i,j} = d(\eta^{ij})_{k-i-j+1}.$$

For d_i^n note that if n is even, $l_\eta y(V^n) = y(V^n)$, so that $d_i^n = \delta_i^n$. The same observation when n is odd gives $d_k^n = d_{k-n+1}$. Also, the fact that $l_\eta = (\mathrm{res}_\eta \otimes 1) \circ \Delta$ gives

$$d_k = \delta_k^1 + ef_{1,k}^1.$$

Altogether we have

$$d_k^n = \begin{cases} \delta_k^n & \text{if } n \text{ is even,} \\ \delta_k^n + ef_{1,k-n+1}^1 & \text{if } n \text{ is odd.} \end{cases}$$

In a single formula

$$d_k^n = \delta_k^n + e(\eta^n) f_{1,k-n+1}^1,$$

and

$$b_k^{i,j} = \delta_k^{i+j} + e(\eta^{ij}) f_{1,k-i-j+1}^1.$$

We can obtain useful relations from the statement $l_\eta l_\eta = 1$ (or alternatively by using the recursive formula for d^{n+1}). Thus

$$y(\eta) = (d_0 + d_1 y(\varepsilon)) + y(V^2)(d_2 + d_3 y(\varepsilon)) + y(V^4)(d_4 + d_5 y(\varepsilon)) + \ldots,$$

and applying l_η again we have

$$\begin{aligned} y(\varepsilon) = \quad & (d_0 + d_1 d_0) y(V^0) + d_1 d_1 y(V^1) + d_1 d_2 y(V^2) + d_1 d_3 y(V^3) + \ldots \\ & + (d_2 + d_3 d_0) y(V^2) + d_3 d_1 y(V^3) + d_3 d_2 y(V^4) + d_3 d_3 y(V^5) + \ldots \\ & + (d_4 + d_5 d_0) y(V^4) + d_5 d_1 y(V^5) + d_5 d_2 y(V^6) + d_5 d_3 y(V^7) + \ldots \\ & + (d_6 + d_7 d_0) y(V^6) + d_7 d_1 y(V^7) + d_7 d_2 y(V^8) + d_7 d_3 y(V^9) + \ldots . \end{aligned}$$

Comparing coefficients of $y(V^0)$ and $y(V^1)$ we find that

$$d_0 + d_1 d_0 = 0 \quad \text{and} \quad d_1 d_1 = 1.$$

Similarly, comparing coefficients of $y(V^{2n})$ and $y(V^{2n+1})$ for $n \geqslant 1$ we find that

$$d_1 d_{2n} + d_3 d_{2n-2} + \ldots + d_{2n+1} d_0 + d_{2n} = 0$$

and

$$d_1 d_{2n+1} + d_3 d_{2n-1} + \ldots + d_{2n+1} d_1 = 0.$$

Writing the first two in terms of $f^1_{1,*}$ we obtain

$$e(2+ef^1_{1,1})=0 \quad \text{and} \quad (1+ef^1_{1,1})^2=1.$$

The first of these allows us to omit $e(2+ef^1_{1,1})f^1_{1,2n}$ from the subsequent even terms to obtain

$$e^2[f^1_{1,3}f^1_{1,2n-2}+f^1_{1,5}f^1_{1,2n-4}+\ldots+f^1_{1,2n+1}f^1_{1,0}]=0$$

and

$$2e(1+ef^1_{1,1})f^1_{1,2n+1}+e^2[f^1_{1,3}f^1_{1,2n-1}+f^1_{1,5}f^1_{1,2n-3}+\ldots+f^1_{1,2n-1}f^1_{1,3}]=0.$$

We really want to examine the consequences of these relations e-adically and with e inverted. As a first step, note that modulo e^2 the first states that $2e=0$, and the rest are all consequences. Modulo 2, the first reads $e^2f^1_{1,1}=0$, and the second follows from it. The remaining even terms read

$$e^2[f^1_{1,3}f^1_{1,2n-2}+f^1_{1,5}f^1_{1,2n-4}+\ldots+f^1_{1,2n+1}f^1_{1,0}]=0,$$

and the remaining odd terms are trivial if $2n+1=3$ mod 4 and give

$$e^2(f^1_{1,2m+1})^2=0$$

in the remaining case (where $2n+1=4m+1$). If we instead invert e, the first relation states that $ef^1_{1,1}=-2$, the second is a consequence. The relation from $y(V^2)$ gives $f^1_{1,3}=0$, and the successive relations from $y(V^4), y(V^6),\ldots$ allow us to deduce that $f^1_{1,2n+1}=0$ for $n\geqslant 1$. The relation from $y(V^{2n+1})$ is automatically satisfied.

To obtain the recursive formula for f^{n+1} we use $\Delta y(V^{n+1})=\Delta y(V^n)\Delta y(\alpha_{n+1})$, and there are then two cases, since α_{n+1} is ε or η depending on whether $n+1$ is odd or even. The first case is straightforward, but the second uses $\Delta(y(\eta))=(1\otimes l_\eta)(\Delta(y(\varepsilon)))$; together these give the formulae

$$f^{n+1}_{s,t}=\begin{cases}\displaystyle\sum_{i,j,k,l} f^n_{i,j}f^1_{k,l}d(\eta^{ik})_{s-i-k+1}d(\eta^{jl})_{t-j-l+1} & \text{if } n+1 \text{ is odd,}\\ \displaystyle\sum_{i,j,k,l,m} f^n_{i,j}f^1_{k,l}d(\eta^{ik})_{s-i-k+1}d^l_m d(\eta^{jm})_{t-j-m+1} & \text{if } n+1 \text{ is even.}\end{cases}$$

Combining this with our expressions for the d coefficients we obtain

$$f^{n+1}_{s,t}=\sum_{i,j,k,l,m} f^n_{i,j}f^1_{k,l}[\delta^{i+k}_s+e(\eta^{ik})f^1_{1,s-i-k+1}][\delta^l_m+e(\eta^n)f^1_{1,m-l+1}]$$
$$\times[\delta^{j+m}_t+e(\eta^{jm})f^1_{1,t-j-m+1}].$$

It is perhaps worth making explicit the structure of the additive and multiplicative laws in this case. For an additive law, the action is given by

$$l_\eta y(V^n)=y(V^n)+e(\eta^n)y(V^{n-1})$$

and the product is given by

$$y(V^i)y(V^j)=y(V^{i+j})+e(\eta^{ij})y(V^{i+j-1}).$$

Let $x=y(V^2)=y(\varepsilon)y(\eta)$, and $y=y(\varepsilon)$. The ring $k\{\{F\}\}$ is the x-adic completion of the ring $k[x,y]/(y^2=x+ey)=k[y]$. The action by A^* is given by $y\mapsto e+y$, which is of order 2 since e is. Of course $x=y(e+y)$, so the completion is the one we are used to.

For a multiplicative law, the action is given by

$$l_\eta y(V^n) = [1 - e(\eta^n)]y(V^n) + e(\eta^n)y(V^{n-1})$$

and the product is given by

$$y(V^i)y(V^j) = [1 - e(\eta^{ij})]y(V^{i+j}) + e(\eta^{ij})y(V^{i+j-1}).$$

Let $x = y(V^2) = y(\varepsilon)y(\eta)$, and $y = y(\varepsilon)$. The ring $k\{\{F\}\}$ is the x-adic completion of $k[x, y]/(y^2 = (1-e)x + ey)$. The action by A^* is given by $y \mapsto e + (1-e)y$, which is of order 2 since $(1-e)^2 = 1$. Since $(1-e)$ is a unit, the ring is $k[y]$ again. The completion is with respect to $x = y(e + (1-e)y)$. We may let $z = 1 - y$, and it is reasonable to view $1 - e$ as the image of η in k, under the classifying map $R(A) \to k$ so that we obtain the completed ring $k[z]^\wedge_{(1-z)(1-\eta z)}$ in the form familiar from K-theory.

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