Families of mutually isospectral Riemannian orbifolds

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Abstract

In this paper, we consider three arithmetic families of isospectral non-isometric Riemannian orbifolds and in each case derive an upper bound for the size of the family which is polynomial as a function of the volume of the orbifolds. The first family that we consider are those constructed by Vignéras' method. The second and third families are those whose covering groups are the minimal covolume arithmetic subgroups and maximal arithmetic subgroups of $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$.

1. *Introduction*

Let M be a complete, orientable Riemannian orbifold of finite volume. The spectrum of M is the set of eigenvalues of the Laplace–Beltrami operator acting on $L_2(M)$. It is known that this spectrum is discrete and that every eigenvalue occurs with a finite multiplicity. Two orbifolds are said to be *isospectral* if their spectra coincide. A natural problem is to determine the extent to which the spectrum of M determines its geometry and topology. It is well known for instance, that both volume and scalar curvature are spectral invariants. Isometry class, on the other hand, is not a spectral invariant. This was first shown by Milnor [17], who exhibited sixteen-dimensional flat tori which were isospectral and non-isometric. For an excellent survey of the history of constructing isospectral non-isometric Riemannian manifolds and orbifolds, we refer the reader to [11].

In this paper, we examine two systematic methods for constructing isospectral non-isometric manifolds. The first is due to Vignéras [23] and was used to exhibit isospectral non-isometric hyperbolic 2- and 3-manifolds. Vignéras' method is based upon the arithmetic of orders in quaternion algebras defined over algebraic number fields. The second method, due to Sunada [22], is extremely versatile and, along with its variants and generalizations, accounts for the majority of the known examples of isospectral non-isometric manifolds. Using Sunada's method, Brooks, Gornet and Gustafson [4] have constructed arbitrarily large families of isospectral non-isometric Riemann surfaces. In fact, they prove the existence of a constant $c > 0$ and an infinite sequence $g_i \to \infty$ such that for each i there exist at least $g_i^{c \log(g_i)}$ mutually isospectral non-isometric Riemann surfaces of genus g_i. We note that similar results have recently been obtained by McReynolds [16] in the context of complex hyperbolic 2-space and real hyperbolic n-space.

It is a result of McKean [15] that for a fixed Riemann surface S, there are at most finitely many Riemann surfaces which are isospectral to S and pairwise non-isometric. A similar result was proved by Kim [12] for hyperbolic 3-manifolds. In his book, Buser [6] provides an explicit bound for the number of surfaces which are isospectral to a fixed surface S of genus g and pairwise non-isometric; there are at most $\exp(720g^2)$ such surfaces.

In this paper, we consider various natural families of Riemannian orbifolds and provide upper bounds (in terms of volume) for the cardinality of a set of pairwise isospectral non-isometric

orbifolds whose elements lie in these families. All of the families we consider are arithmetic, and it is significant that in each case the upper bound we deduce is polynomial as a function of volume. We do not expect that the exponents which occur in these results are best possible. Rather, we have attempted to indicate a specific exponent for which each result holds and whose proof is not so technical as to obscure the underlying ideas.

Our first result considers families of isospectral non-isometric orbifolds constructed via Vignéras' method. Because Vignéras' method can be applied (and generalized) to a wide variety of settings, we emphasize that throughout this paper when we refer to Vignéras' method we refer to the setting which Vignéras treated in her paper [**23**]: groups arising from the elements of norm 1 in maximal orders of quaternion algebras defined over number fields.

THEOREM 1.1.   *The cardinality of a family of pairwise isospectral non-isometric Riemannian orbifolds of volume V constructed via Vignéras' method is at most $242e^9V^2$ for all $V > 0$.*

In Sections 4 and 5, we consider families of maximal arithmetic subgroups of $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$. These groups comprise an extremely important class of arithmetic subgroups. It is known, for instance, that the hyperbolic 3-orbifold and 3-manifold of smallest volume are both arithmetic [**7**, **9**]. In Section 4, we consider maximal arithmetic subgroups which have minimal covolume in their commensurability class and prove the following theorem.

THEOREM 1.2.   *Fix a commensurability class $\mathcal{C}(k, B)$ of cocompact arithmetic lattices in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$. The cardinality of a family of pairwise isospectral non-isometric arithmetic Riemannian orbifold quotients of $\mathcal{H} = \mathbb{H}_2^a \times \mathbb{H}_3^b$ which have minimal volume V in this commensurability class is at most $242V^{18}$ for all $V > 0$.*

In Section 5, we consider the family of all maximal arithmetic subgroups and prove a bound which is again polynomial in volume. In fact, our result is somewhat stronger in the sense that we prove that within a fixed commensurability class, the set of all maximal arithmetic subgroups which have covolume less than V is bounded by a function polynomial in V. Although the existence of such a bound is implicit in the proof of Theorem 1.6 of Belolipetsky, Gelander, Lubotzky and Shalev [**1**], no explicit bound is currently known. To that end, we prove the following theorem.

THEOREM 1.3.   *Fix a commensurability class of cocompact arithmetic lattices in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$. The number of conjugacy classes of maximal arithmetic lattices with covolume at most V in this commensurability class is less than $242V^{20}$ for all $V > 0$.*

The proofs of our theorems make extensive use of Borel's volume formula [**2**], the work of Chinburg and Friedman [**7**], a refinement of the Odlyzko discriminant bounds [**18**] due to Poitou [**19**] and the Brauer–Seigel theorem. Along the way, we prove a number theoretic result of independent interest which bounds the class number of an algebraic number field in terms of the absolute value of the discriminant of the field.

2. *Notation*

Throughout this paper, k will denote a number field of degree n with signature (r_1, r_2). That is, k has r_1 real places and r_2 complex places. As a consequence, $n = r_1 + 2r_2$. We will denote

by V_∞ the set of archimedean places of k. The class number of k will be denoted by h_k and the absolute value of the discriminant of k will be denoted by d_k. If $\mathfrak{p}$ is a prime of k, then we will denote by $N(\mathfrak{p})$ the norm of $\mathfrak{p}$.

For a quaternion algebra B over k, we define $\mathrm{Ram}(B)$ to be the set of primes of k (possibly infinite) which ramify in B, $\mathrm{Ram}_f(B)$ to be the set of finite primes of k lying in $\mathrm{Ram}(B)$ and $\mathrm{Ram}_\infty(B)$ to be the set of archimedean places of k lying in $\mathrm{Ram}(B)$. The discriminant of B, which we will denote by $\mathfrak{D}$, is defined to be the product of the primes lying in $\mathrm{Ram}_f(B)$.

3. *Isospectral orbifolds obtained via Vignéras' method*

We begin by describing Vignéras' construction [**23**] of isospectral non-isometric Riemannian orbifolds. For a more detailed exposition, see [**14**, Chapter 12].

Let k be a degree n number field of signature (r_1, r_2) and let B be a quaternion algebra over k which is not ramified at all archimedean places of k. There exists an isomorphism

$$B \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{H}^r \times \mathrm{M}_2(\mathbb{R})^s \times \mathrm{M}_2(\mathbb{C})^{r_2}, \quad r + s = r_1,$$

which induces an embedding

$$B^\times \hookrightarrow \prod_{\nu \in V_\infty \setminus \mathrm{Ram}_\infty(B)} B_\nu^\times.$$

This embedding in turn induces an embedding $\rho : B^\times/k^\times \hookrightarrow G = \mathrm{PGL}_2(\mathbb{R})^s \times \mathrm{PGL}_2(\mathbb{C})^{r_2}$. Let K be a maximal compact subgroup of G so that $G/K = \mathbb{H}_2^s \times \mathbb{H}_3^{r_2}$ is a product of two- and three-dimensional hyperbolic spaces. The Riemannian orbifolds constructed by Vignéras are of the form $\Gamma \backslash G/K$ for Γ a discrete subgroup of isometries of G/K.

Let $\mathcal{O}_k$ be the ring of integers of k and $\mathcal{O}$ be a maximal $\mathcal{O}_k$-order of B. Denote by $\mathcal{O}^1$ the multiplicative group of elements of $\mathcal{O}$ with reduced norm 1. Then $\Gamma_\mathcal{O}^1 = \rho(\mathcal{O}^1)$ is a discrete subgroup of isometries of G/K of finite covolume which is cocompact if B is a division algebra.

Suppose now that B is as above and that $\mathcal{O}$ and $\mathcal{O}'$ are maximal orders of B. It is clear that if there exists $\alpha \in B^\times$ such that $\mathcal{O}' = \alpha \mathcal{O} \alpha^{-1}$ then $\Gamma_\mathcal{O}^1$ and $\Gamma_{\mathcal{O}'}^1$ will be conjugate and consequently that $\Gamma_\mathcal{O}^1 \backslash G/K$ and $\Gamma_{\mathcal{O}'}^1 \backslash G/K$ will be isometric. What is not obvious and is the content of [**23**, Théorème 3] is that if $\Gamma_\mathcal{O}^1 \backslash G/K$ and $\Gamma_{\mathcal{O}'}^1 \backslash G/K$ are isometric, then $\mathcal{O}$ and $\mathcal{O}'$ are conjugate by an element of $B^\times$. Furthermore, Vignéras [**23**, Théorème 7] showed that when a certain technical number theoretic condition is satisfied, the spectrum of $\Gamma_\mathcal{O}^1 \backslash G/K$ is independent of the choice of maximal order $\mathcal{O}$. Although we will not state this number theoretic condition explicitly because of the burdensome notation that it would require, we do remark that if $\mathrm{Ram}_f(B)$ is non-empty, then this condition will be satisfied.

Before stating and proving this section's main theorem, we prove a lemma which will be used throughout this paper and which is of independent interest.

LEMMA 3.1. *Let k be a number field of signature (r_1, r_2) with class number h_k and absolute value of discriminant d_k. Then $h_k \leqslant 242 d_k^{3/4}/(1.64)^{r_1}$.*

Proof. The analytic class number formula and the proof of the Brauer–Siegel theorem [**13**, pp. 300, 322] imply that for any real $s > 0$ we have

$$h_k \leqslant \frac{\omega_k s(s-1) \Gamma(s)^{r_2} \Gamma(s/2)^{r_1} \zeta_k(s) d_k^{s/2}}{2^{r_1} \mathrm{Reg}_k\, 2^{r_2 s} \pi^{ns/2}}, \tag{3.2}$$

where ω_k denotes the number of roots of unity contained in k, $\zeta_k(s)$ is the Dedekind zeta function of k and Reg_k is the regulator of k.

Using the well-known estimate $\zeta_k(s) < \zeta(s)^n$, we obtain, for $s = 1.5$, the inequality

$$h_k < \frac{3\omega_k \, (2.62)^n \, d_k^{3/4}}{4 \operatorname{Reg}_k 2^{3r_2/2} \pi^{3n/4}}.$$

Friedman [**10**, p. 620] has shown that $\operatorname{Reg}_k \geqslant 0.0031\omega_k \exp(0.241n + 0.497r_1)$. Combining this and the fact that $\exp(0.497) > 1.64$ with the estimate above and simplifying gives

$$h_k < \frac{242 d_k^{3/4}}{(1.64)^{r_1} 2^{3r_2/2}},$$

from which the proposition easily follows. $\square$

REMARK 3.3.   We note that the exponent $t = \frac{3}{4}$ occurring in Lemma 3.1 is a consequence of evaluating (3.2) at $s = \frac{3}{2}$. Evaluating (3.2) at a different value of $s > 0$ would lead to a different exponent t in the lemma's statement. It would be interesting to find the infimum of all t for which there is an upper bound of the form $h_k < c_0(d_k^t/c_1^{r_1} c_2^{r_2})$ where c_0, c_1, c_2 are constants satisfying $c_0 > 0$ and $c_1, c_2 \geqslant 1$.

REMARK 3.4.   Lemma 3.1 generalizes a result of Borel and Prasad [**3**, p. 143] which shows that $h_k \leqslant 10^2(\pi/12)^n d_k$.

We now prove Theorem 1.1.

Proof.   All of the orbifolds obtained via Vignéras' method have covering groups which are of the form $\Gamma_{\mathcal{O}}^1$, for $\mathcal{O}$ a maximal order in a quaternion algebra B defined over a number field k. If $\mathcal{O}'$ is another maximal order of B, then $\Gamma_{\mathcal{O}'}^1$ will be conjugate to $\Gamma_{\mathcal{O}}^1$ precisely when $\mathcal{O}$ and $\mathcal{O}'$ are conjugate in B. It therefore follows that the number of elements in a family of pairwise isospectral non-isometric orbifolds constructed via Vignéras' method is bounded above by the number of conjugacy classes of maximal orders in B. It is well known that this latter number (often referred to as the *type number* of B) coincides with the degree of the maximal abelian extension of k which has 2-elementary Galois group, is unramified outside of the real places in $\operatorname{Ram}(B)$ and in which all finite primes of $\operatorname{Ram}(B)$ split completely [**8**, pp. 37, 39]. This latter field extension is contained in the strict class field of k, which has degree at most $2^{r_1} h_k$.

We now relate $2^{r_1} h_k$ to the discriminant d_k of k, which we will in turn relate to the volume V. Lemma 3.1 shows that $h_k \leqslant 242(1.64)^{-r_1} d_k^{3/4}$, hence $2^{r_1} h_k \leqslant 242(1.22)^{r_1} d_k^{3/4}$.

Recall the formula for the covolume of $\Gamma_{\mathcal{O}}^1$ (see Borel [**2**, 7.3], Maclachlan and Reid [**14**, Section 11.1], Vignéras [**24**, Corollaire IV.1.8] and, for the totally real case, Shimizu [**21**, Appendix]):

$$\operatorname{vol}(\mathcal{H}/\Gamma_{\mathcal{O}}^1) = \frac{2(4\pi)^s d_k^{3/2} \zeta_k(2)\Phi(\mathfrak{D})}{(4\pi^2)^{r_1}(8\pi^2)^{r_2}}, \tag{3.5}$$

where $\Phi(\mathfrak{D}) = N(\mathfrak{D}) \prod_{\mathfrak{p} | \mathfrak{D}}(1 - 1/N(\mathfrak{p}))$.

Applying the trivial estimates $\zeta_k(2), \Phi(\mathfrak{D}), (4\pi)^s \geqslant 1$, we see that

$$V \geqslant \frac{d_k^{3/2}}{(4\pi^2)^{r_1}(8\pi^2)^{r_2}}. \tag{3.6}$$

In order to get rid of the dependence on r_1 and r_2 in (3.6), we apply the discriminant bounds of Odlyzko [**18**] and Poitou [**19**] (see also [**5**, Section 2]). In particular, there exists an absolute constant C such that $\log(d_k) \geqslant r_1 + n(\gamma + \log(4\pi)) - C$, where $\gamma = 0.57721 \ldots$ is the Euler–Mascheroni constant. As $n = r_1 + 2r_2$, we deduce that $\log(d_k) \geqslant 4r_1 + 6r_2 - C$, and because

$4 > \log(4\pi^2)$ and $6 > \log(8\pi^2)$, we conclude from (3.6) that $Ve^C \geqslant d_k^{1/2}$. Easy computations based on the formulas in [**5**, Section 2] show that one may take $C = 4.5$, hence $Ve^{4.5} \geqslant d_k^{1/2}$.

We have already shown that the cardinality of a family of pairwise isospectral non-isometric Riemannian orbifolds of volume V constructed via Vignéras' method is at most $242(1.22)^{r_1} d_k^{3/4}$. We have also seen that $\log(d_k) \geqslant 4r_1 - 4.5$. It follows that $d_k \geqslant \exp(4r_1)/\exp(4.5)$. To ease notation slightly, set $f(r_1) = \exp(4r_1)/\exp(4.5)$. Then it is easy to see that $(1.22)^{r_1} \leqslant f(r_1)^{1/4}$ for all $r_1 > 1$. It follows that $242(1.22)^{r_1} \leqslant 242f(r_1)^{1/4} \leqslant 242d_k^{1/4}$, hence the cardinality of a family of pairwise isospectral non-isometric Riemannian orbifolds of volume V constructed via Vignéras' method is at most $242d_k \leqslant 242e^9 V^2$. The theorem follows. $\qquad\square$

REMARK 3.7. The exponent of 2 which appears in Theorem 1.1 is likely not best possible and can be improved. It would be interesting to find the infimum of all exponents for which the statement of Theorem 1.1 holds.

4. *Isospectral minimal volume orbifolds*

We begin this section by setting up the notation needed to define the maximal arithmetic lattices in the commensurability class $\mathcal{C}(k, B)$ defined by a quaternion algebra B over k. Our description will necessarily be brief. For more details, we refer the reader to Borel [**2**], Chinburg and Friedman [**8**, p. 41] and Maclachlan and Reid [**14**, Section 11].

Let S be a finite set of primes of k which is disjoint from $\mathrm{Ram}_f(B)$. For each prime $\mathfrak{p} \in S$, let $\{M_{\mathfrak{p}}^1, M_{\mathfrak{p}}^2\}$ represent an edge in the tree of maximal orders of $\mathrm{M}_2(k_{\mathfrak{p}})$. Given this notation, we define $\Gamma_{S,\mathcal{O}}$ to be the image in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$ of

$$\{\overline{x} \in B^\times/k^\times : x\mathcal{O}_{\mathfrak{p}}x^{-1} = \mathcal{O}_{\mathfrak{p}} \text{ for } \mathfrak{p} \notin S \text{ and } x \text{ fixes } \{M_{\mathfrak{p}}^1, M_{\mathfrak{p}}^2\} \text{ for } \mathfrak{p} \in S\}.$$

When $S = \emptyset$, we define $\Gamma_{\mathcal{O}} = \Gamma_{\emptyset,\mathcal{O}}$. These groups were first defined by Borel [**2**], who showed that while all of these groups are not maximal arithmetic subgroups of $\mathcal{C}(k, B)$, every maximal arithmetic subgroup of $\mathcal{C}(k, B)$ is conjugate to some $\Gamma_{S,\mathcal{O}}$. Borel also derived formulas for the covolumes of these groups, showing in particular that the group $\Gamma_{\mathcal{O}}$ has minimal covolume in its commensurability class.

We now prove Theorem 1.2.

Proof. If M is an arithmetic orbifold quotient of $\mathcal{H}$ which has minimal volume in its commensurability class, then there exists a number field k, quaternion algebra B over k and maximal order $\mathcal{O}$ of B such that the covering group of M is conjugate to $\Gamma_{\mathcal{O}}$. If $\mathcal{O}'$ is another maximal order of B, then $\Gamma_{\mathcal{O}'}$ is conjugate to $\Gamma_{\mathcal{O}}$ if and only if $\mathcal{O}$ and $\mathcal{O}'$ are conjugate in B. As in the proof of Theorem 1.1, it follows that the maximum number of orbifolds which are all isospectral to M and are mutually non-isometric is bounded above by $242(1.22)^{r_1} d_k^{3/4}$.

We now derive a lower bound for the volume $\mathrm{vol}(\mathcal{H}/\Gamma_{\mathcal{O}})$ of M. In order to do so, we note that by the results of Borel [**2**, Sections 8.4 and 8.5] and Chinburg and Friedman [**7**, Lemma 2.1] we have that $[\Gamma_{\mathcal{O}} : \Gamma_{\mathcal{O}}^1] \leqslant 2^{n+|\mathrm{Ram}_f(B)|} h_k$. From this, we deduce that

$$V = \mathrm{vol}(\mathcal{H}/\Gamma_{\mathcal{O}}) \geqslant \frac{2(4\pi)^s d_k^{3/2} \zeta_k(2) \Phi(\mathfrak{D})}{(4\pi^2)^{r_1}(8\pi^2)^{r_2} 2^{n+|\mathrm{Ram}_f(B)|} h_k}. \tag{4.1}$$

Let $\omega_2(B)$ denote the number of primes of k which have norm 2 and ramify in B. Then $\Phi(\mathfrak{D})/2^{|\mathrm{Ram}_f(B)|} \geqslant (\frac{1}{2})^{\omega_2(B)}$ and $\zeta_k(2) \geqslant (4/3)^{\omega_2(B)}$ (the latter is an immediate consequence of the Euler product expansion of $\zeta_k(s)$). Using this, Lemma 3.1, the trivial bound $\omega_2(B) \leqslant n$

and simplifying, we see that

$$V \geqslant \frac{d_k^{3/4}}{25^{r_1}(8\pi^2)^{r_2}3^n} \tag{4.2}$$

$$\geqslant \frac{d_k^{3/4}}{(75)^n}. \tag{4.3}$$

An easy consequence of [**7**, Lemma 4.3] is that $3\log(V) \geqslant n$. Equation (4.3) now implies that $d_k \leqslant V^{22}$. As $1.22 < \exp(\frac{1}{4})$, we see that $242(1.22)^{r_1} \leqslant 242\exp(n/4) \leqslant 242\exp(\log(V^3)/4) = 242V^{3/4}$. Putting all of this together, we see that $242(1.22)^{r_1}d_k^{3/4} \leqslant 242V^{3/4}V^{33/2} \leqslant 242\,V^{18}$, finishing the proof. $\square$

5. Isospectral maximal arithmetic lattices

In Section 4, we derived an upper bound for the cardinality of a family of pairwise isospectral non-isometric arithmetic Riemannian orbifolds whose members all have covering groups of the form $\Gamma_\mathcal{O}$. In this section, we consider Riemannian orbifolds whose covering groups belong to the broader class of maximal arithmetic lattices in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$.

We now prove Theorem 1.3.

Proof. Let S be a finite set of primes of k which are disjoint from $\mathrm{Ram}_f(B)$ and let $\mathcal{O}$ be a maximal order of B. Then $\Gamma_{S,\mathcal{O}}$ is an arithmetic lattice in $\mathcal{C}(k,B)$ and, while $\Gamma_{S,\mathcal{O}}$ may not be maximal, every maximal arithmetic lattices in $\mathcal{C}(k,B)$ is of this form. If $\mathcal{O}'$ is another maximal order of B, then $\Gamma_{S,\mathcal{O}}$ is conjugate to $\Gamma_{S,\mathcal{O}'}$ if and only if $\mathcal{O}$ and $\mathcal{O}'$ are conjugate in B. It follows that the number of conjugacy classes of maximal arithmetic lattices of volume at most V in $\mathcal{C}(k,B)$ is less than the type number of B times the number of choices for the set S. We have already seen that the type number of B is less than $242(1.22)^{r_1}d_k^{3/4}$.

Before proceeding, we require a definition. Let $\Gamma_1, \Gamma_2 \in \mathcal{C}(k,B)$. We define the *generalized index* $[\Gamma_1 : \Gamma_2] \in \mathbb{Q}$ to be the quotient $[\Gamma_1 : \Gamma_1 \cap \Gamma_2]/[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$.

Maclachlan and Reid [**14**, Theorem 11.5.1] have shown that there exists an integer $0 \leqslant m \leqslant |S|$ such that $[\Gamma_\mathcal{O} : \Gamma_{S,\mathcal{O}}] = 2^{-m}\prod_{\mathfrak{p}\in S}(N(\mathfrak{p})+1)$. Set $\mathcal{H} = \mathbb{H}_2^a \times \mathbb{H}_3^b$ so that

$$\frac{V}{\mathrm{vol}(\mathcal{H}/\Gamma_\mathcal{O})} \geqslant \frac{\mathrm{vol}(\mathcal{H}/\Gamma_{S,\mathcal{O}})}{\mathrm{vol}(\mathcal{H}/\Gamma_\mathcal{O})}$$

$$\geqslant \prod_{\mathfrak{p}\in S}\frac{(N(\mathfrak{p})+1)}{2}$$

$$\geqslant \prod_{\substack{\mathfrak{p}\in S \\ N(\mathfrak{p})\neq 2}}N(\mathfrak{p})^{1/3}.$$

In the proof of Theorem 1.2, we proved that $\mathrm{vol}(\mathcal{H}/\Gamma_\mathcal{O}) \geqslant d_k^{1/22}$, hence the above inequalities show that

$$\frac{V^3}{d_k^{3/22}} \geqslant \prod_{\substack{\mathfrak{p}\in S \\ N(\mathfrak{p})\neq 2}}N(\mathfrak{p}). \tag{5.1}$$

Equation (5.1) shows that the number of choices for the set S is less than the number of integral ideals of norm less than $V^3/d_k^{3/22}$, which by [**1**, Lemma 3.4] is less than $(\pi^2/6)^n(V^6/d_k^{3/11})$. Therefore, the number of conjugacy classes of maximal arithmetic lattices of volume at most V in $\mathcal{C}(k,B)$ is less than $242(1.22)^{r_1}(\pi^2/6)^n d_k^{21/44}V^6$.

To ease notation, let $V' = \text{vol}(\mathcal{H}/\Gamma_{\mathcal{O}})$. Using the fact that $3\log(V') \geqslant n$ (see [**7**, Lemma 4.3]), we deduce that the number of conjugacy classes of maximal arithmetic lattices of volume at most V in $\mathcal{C}(k, B)$ is less than $242V'^3 V^6 d_k^{21/44}$. As we have seen that $d_k^{1/22} \leqslant V'$ and trivially have $V' \leqslant V$, we deduce that $d_k^{21/44} \leqslant V'^{21/2}$, hence $242V'^3 V^6 d_k^{21/44} \leqslant 242V'^{27/2}V^6 \leqslant 242\,V^{20}$. $\qquad\square$

Because isospectral Riemannian orbifolds must have the same volume, the following is an immediate consequence of Theorem 1.3.

COROLLARY 5.2. *The cardinality of a family of commensurable, pairwise isospectral non-isometric arithmetic Riemannian orbifold quotients of $\mathcal{H} = \mathbb{H}_2^a \times \mathbb{H}_3^b$ with covering groups which are maximal in their commensurability class and have volume V is at most $242\,V^{20}$ for all $V > 0$.*

REMARK 5.3. Reid [**20**] has shown that isospectral arithmetic hyperbolic 2- and 3-orbifolds are always commensurable, hence when $a + b = 1$ the commensurability assumption in Theorem 1.2 and Corollary 5.2 can be omitted.

References

1. M. BELOLIPETSKY, T. GELANDER, A. LUBOTZKY and A. SHALEV, 'Counting arithmetic lattices and surfaces', *Ann. of Math.* (2) 172 (2010) 2197–2221.

2. A. BOREL, 'Commensurability classes and volumes of hyperbolic 3-manifolds', *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) 8 (1981) 1–33.

3. A. BOREL and G. PRASAD, 'Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups', *Inst. Hautes Études Sci. Publ. Math.* 69 (1989) 119–171.

4. R. BROOKS, R. GORNET and W. H. GUSTAFSON, 'Mutually isospectral Riemann surfaces', *Adv. Math.* 138 (1998) 306–322.

5. S. BRUEGGEMAN and D. DOUD, 'Local corrections of discriminant bounds and small degree extensions of quadratic base fields', *Int. J. Number Theory* 4 (2008) 349–361.

6. P. BUSER, *Geometry and spectra of compact Riemann surfaces*, Modern Birkhäuser Classics (Birkhäuser Boston Inc., Boston, MA, 2010). Reprint of the 1992 edition.

7. T. CHINBURG and E. FRIEDMAN, 'The smallest arithmetic hyperbolic three-orbifold', *Invent. Math.* 86 (1986) 507–527.

8. T. CHINBURG and E. FRIEDMAN, 'An embedding theorem for quaternion algebras', *J. London Math. Soc.* (2) 60 (1999) 33–44.

9. T. CHINBURG, E. FRIEDMAN, K. N. JONES and A. W. REID, 'The arithmetic hyperbolic 3-manifold of smallest volume', *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* (4) 30 (2001) 1–40.

10. E. FRIEDMAN, 'Analytic formulas for the regulator of a number field', *Invent. Math.* 98 (1989) 599–622.

11. C. GORDON, 'Survey of isospectral manifolds', *Handbook of differential geometry*, vol. I (North-Holland, Amsterdam, 2000) 747–778.

12. I. KIM, 'Isospectral finiteness on hyperbolic 3-manifolds', *Comm. Pure Appl. Math.* 59 (2006) 617–625.

13. S. LANG, *Algebraic number theory*, 2nd edn, Graduate Texts in Mathematics 110 (Springer, New York, 1994).

14. C. MACLACHLAN and A. W. REID, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics 219 (Springer, New York, 2003).

15. H. P. MCKEAN, 'Selberg's trace formula as applied to a compact Riemann surface', *Comm. Pure Appl. Math.* 25 (1972) 225–246.

16. D. B. MCREYNOLDS, 'Isospectral locally symmetric manifolds', *Indiana Univ. Math. J.* (2) 63 (2014) 533–549.

17. J. MILNOR, 'Eigenvalues of the Laplace operator on certain manifolds', *Proc. Natl Acad. Sci. USA* 51 (1964) 542.

18. A. M. ODLYZKO, 'Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results', *Sém. Théor. Nombres Bordeaux* (2) 2 (1990) 119–141.

19. G. Poitou, 'Sur les petits discriminants', *Séminaire Delange-Pisot-Poitou, 18e année: (1976/77), Théorie des nombres, Fasc. 1 (French)*, pages Exp. No. 6, 18 (Secrétariat Math., Paris, 1977).
20. A. W. Reid, 'Isospectrality and commensurability of arithmetic hyperbolic 2- and 3-manifolds', *Duke Math. J.* 65 (1992) 215–228.
21. H. Shimizu, 'On zeta functions of quaternion algebras', *Ann. of Math.* (2) 81 (1965) 166–193.
22. T. Sunada, 'Riemannian coverings and isospectral manifolds', *Ann. of Math.* (2) 121 (1985) 169–186.
23. M.-F. Vignéras, 'Variétés riemanniennes isospectrales et non isométriques', *Ann. of Math.* (2) 112 (1980) 21–32.
24. M.-F. Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics 800 (Springer, Berlin, 1980).

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