HAUSDORFF DIMENSION AND QUASICONFORMAL MAPPINGS

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Dedicated to the memory of A. S. Besicovitch

1. Introduction. In this paper we study what happens to the Hausdorff dimension of a set \( A \), denoted by \( H\dim A \), under an \( n \)-dimensional quasiconformal mapping \( f : D \to D' \) with \( A \subset D \). It is clear that

\[ H\dim f[A] = H\dim A \quad (1) \]

if \( f \) is a diffeomorphism or, more generally, if \( f \) and \( f^{-1} \) are locally Lipschitzian. We show first, however, that (1) need not hold if \( f \) is a general quasiconformal mapping. Next we give bounds for \( H\dim f[A] \) in terms of \( H\dim A \), \( n \), and the maximal dilatation of \( f \). In particular, we prove that \( H\dim A = 0 \) implies \( H\dim f[A] = 0 \), and we conjecture that \( H\dim A = n \) implies \( H\dim f[A] = n \), or equivalently that \( H\dim A < n \) implies \( H\dim f[A] < n \). We establish this conjecture for the case where \( n = 2 \) and then prove that, for general \( n \), \( H\dim f[A] < n \) whenever \( A \) is contained in an \( m \)-dimensional hyperplane with \( m < n \). An example shows that \( H\dim f[A] \) can be arbitrarily close to \( n \), even when \( A \) is a 1-dimensional segment.

2. Notation. We shall use the terminology and notation for quasiconformal mappings given in [16]. Moreover, since we are concerned only with local properties which are invariant under Möbius transformations, we shall consider only quasiconformal mappings \( f : D \to D' \) where \( D \) and \( D' \) are domains in the non-compact \( n \)-dimensional Euclidean space \( \mathbb{R}^n \).

For \( a \in (0, \infty) \), the Hausdorff \( a \)-dimensional outer measure of a set \( A \subset \mathbb{R}^n \) is defined as

\[ H_a(A) = \lim_{d \to 0} \left( \inf \sum_{i} \text{dia} (A_i)^a \right), \quad (2) \]

where the infimum is taken over all countable coverings of \( A \) by sets \( A_i \) with \( \text{dia} (A_i) < d \). The Hausdorff dimension of \( A \) is then given by

\[ H\dim A = \inf \{ a : H_a(A) = 0 \}. \quad (3) \]

Clearly \( 0 \leq H\dim A \leq n \).

3. We shall need the following generalization of a result due to Mori [13; Lemma 4].

**Lemma.** Suppose that \( f : D \to D' \) is an \( n \)-dimensional \( K \)-quasiconformal mapping, that \( U \) is a bounded domain with \( U \subset D \), and that \( x \in U \). Let

\[ M = \max_{y \in \partial U} |y-x|, \quad m = \min_{y \in \partial U} |y-x|, \quad L = \max_{y \in \partial U} |f(y)-f(x)|, \quad l = \min_{y \in \partial U} |f(y)-f(x)|. \]

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If the ball $B^n(f(x), L)$ is contained in $D'$, then
\[ L \leq C l, \]  
where $C$ is a finite constant which depends only on $n$, $K$, and $M/m$.

Proof. Suppose that $l < L$ and let $R$ denote the image under $f^{-1}$ of the spherical ring
\[ R' = \{ y : l < |y-f(x)| < L \} \subset D'. \]
Then $R$ is a ring which separates $x$ and a point $y \in \partial U$ from $\infty$ and a point $z \in \partial U$. Hence if $\Gamma$ is the family of arcs joining the components of $C(R)$ in $R$, it follows from the extremal property of the Teichmüller ring in $R^n$[4 and 14], or from [16; 11.9] that
\[ M(\Gamma) \geq h_n \left( \frac{z-x}{|y-x|} \right) \geq h_n \left( \frac{M}{m} \right), \]
where $h_n : (0, \infty) \to (0, \infty)$ is positive and decreasing. Then since $f$ is $K$-quasiconformal
\[ M(\Gamma) \leq KM(f(\Gamma)) = K\omega_{n-1} \left( \log \frac{L}{l} \right)^{1-n}, \]
and (4) follows from (5) and (6).

4. The Cantor sets $C_s^n$. For each integer $n \geq 1$ and each $s \in (0, \frac{1}{2})$ we define a family of Cantor sets $C_s^n$ as follows. Let $Q$ denote the closed unit cube
\[ Q = \{ x = (x_1, \ldots, x_n) : 0 \leq x_i \leq 1 \}, \]
choose a collection of $2^n$ disjoint closed cubes $Q_i$ of side $s$ in int $Q$, $1 \leq i \leq 2^n$, oriented so that for each $i$ there exists a similarity mapping
\[ g_i(x) = sx + a_i, \quad a_i \in Q, \]
which maps $Q$ onto $Q_i$. Such collections of cubes $Q_i$ obviously exist for each $s \in (0, \frac{1}{2})$. Next for each $j \geq 1$ let
\[ F_j = \bigcup_{i_1, \ldots, i_j = 1}^{2^n} g_{i_1} \circ \cdots \circ g_{i_j} \{ Q \}. \]
Then $\{F_j\}$ is a decreasing sequence of compact sets, and each set $F_j$ is the union of $2^{jn}$ disjoint closed cubes of side $s^j$. Hence
\[ C_s^n = \bigcap_{i=1}^{\infty} F_j \]
is a compact set, and
\[ H\text{-dim} C_s^n = n \frac{\log \frac{1}{s}}{\log s} \]
by, for example, [1; Theorem 3] or [12; Theorem III]. In particular,
\[ 0 < H\text{-dim} C_s^n < n \]
and 

$$\lim_{s \to 0} \text{H-dim } C_s^n = 0, \quad \lim_{s \to \frac{1}{2}} \text{H-dim } C_s^n = n. \quad (8)$$

5. **Theorem.** For each integer $n \geq 2$ and each pair of such Cantor sets $C_s^n$ and $C_t^n$ there exists a quasiconformal mapping $f: R^n \to R^n$ which maps $C_s^n$ onto $C_t^n$.

**Proof.** Let $g_i$ and $F_j$, $g_i'$ and $F_j'$ denote respectively the similarity mappings and sets corresponding to the constructions for $C_s^n$, $C_t^n$ given in §4. Then it is not difficult to see that there exists a piecewise linear homeomorphism $f_1: R^n \to R^n$ such that $f_1(x) = x$ if $x \in R^n \sim Q$ and such that for each $i$

$$f_1(x) = g_i' \circ g_i^{-1} (x)$$

if $x \in g_i[Q]$. Then $f_1$ is $K$-quasiconformal for some $K$ and $f_1[F_1] = F_1'$. Next define $f_2: R^n \to R^n$ by setting $f_2(x) = f_1(x)$ if $x \in R^n \sim F_1$ and

$$f_2(x) = g_i' \circ f_1 \circ g_i^{-1}(x)$$

if $x \in g_i[Q]$. Then $f_2$ is a piecewise linear $K$-quasiconformal mapping, $f_2[F_2] = F_2'$, and for each $i$ and $j$

$$f_2(x) = g_j' \circ g_j \circ g_j^{-1} \circ g_i^{-1}(x)$$

if $x \in g_i \circ g_j[Q]$. Continuing in this way, we obtain a sequence of piecewise linear $K$-quasiconformal mappings $f_j: R^n \to R^n$ such that $f_{j+1}(x) = f_j(x)$ in $R^n \sim F_j$ and $f_j[F_j] = F_j'$. This sequence converges to a $K$-quasiconformal mapping $f: R^n \to R^n$ which maps $F_j$ onto $F_j'$ for each $j$. Hence $f$ maps $C_s^n$ onto $C_t^n$.

6. **Corollary.** For each integer $n \geq 2$ and each pair of numbers $\alpha, \beta \in (0, n)$, there exists a quasiconformal mapping $f: R^n \to R^n$ and a compact set $A \subset R^n$ such that

$$\text{H-dim } A = \alpha, \text{H-dim } f[A] = \beta. \quad (9)$$

**Proof.** By (7) and (8) we can choose $s, t \in (0, \frac{1}{2})$ so that for any of the corresponding Cantor sets $C_s^n$, $C_t^n$,

$$\text{H-dim } C_s^n = \alpha, \text{H-dim } C_t^n = \beta.$$ 

Theorem 5 then yields a quasiconformal mapping $f: R^n \to R^n$ which maps $C_s^n$ onto $C_t^n$, and (9) follows with $A = C_s^n$.

7. **Remark.** The above proof shows that for each $\alpha \in (0, n)$ there exists a set $A \subset R^n$ with $\text{H-dim } A = \alpha$ such that

$$\inf_f \text{H-dim } f[A] = 0, \sup_f \text{H-dim } f[A] = n, \quad (10)$$

where the infimum and supremum are taken over all quasiconformal mappings $f: D \to D'$ with $A \subset D$. We consider next what can be said if we take the infimum and supremum in (10) over the subclass of mappings $f: D \to D'$ which are $K$-quasiconformal for some fixed $K$.

8. **Theorem.** If $f: D \to D'$ is an $n$-dimensional $K$-quasiconformal mapping and if $A \subset D$ with $\text{H-dim } A \geq \alpha > 0$, then $\text{H-dim } f[A] \geq \beta > 0$, where

$$\beta = \alpha K^{1/(1-n)} \geq \alpha/K. \quad (11)$$

**Proof.** Since $A$ is the countable union of sets with compact closure in $D$, we may assume that $A$ is contained in a compact subset of $D$. Then since $f^{-1}$ is locally
Hölder continuous with exponent $K^{1/(1-n)}$ in $D'$ ([5; Corollary 6] or [10; 3.2]), there exists a positive constant $c$ such that
\[ |f(x) - f(y)| \geq c|x - y|^{K^{1/(n-1)}} \]  
(12)
for all $x, y \in A$. If $b > \text{H-dim } f[A]$, then (2), (3), and (12) imply that $H_a(A) = 0$, where $a = b K^{1/(n-1)}$. Hence $a \geq \alpha$ and (11) follows.

9. COROLLARY. If $f : D \to D'$ is an $n$-dimensional quasiconformal mapping and if $A \subset D$ with $\text{H-dim } A = 0$, then $\text{H-dim } f[A] = 0$.

10. CONJECTURE. If $f : D \to D'$ is an $n$-dimensional $K$-quasiconformal mapping and if $A \subset D$ with $\text{H-dim } A \leq \alpha < n$, then $\text{H-dim } f[A] \leq \beta < n$, where $\beta$ depends only on $\alpha, n,$ and $K$.

11. We shall establish this conjecture for the case where $n = 2$. The proof is based on the following important result due to Bojarski [7; p. 226].

**THEOREM.** If $f$ is a 2-dimensional $K$-quasiconformal mapping, then its Jacobian $J_f$ is locally $L^2$-integrable for $q \in [1, p(K)]$, where $p(K) > 1$ depends only on $K$.

It is easy to see that $p(K) \leq K/(K-1)$, and it has been conjectured that Theorem 11 holds with $p(K) = K/(K-1)$.

12. **THEOREM.** If $f : D \to D'$ is a 2-dimensional $K$-quasiconformal mapping and if $A \subset D$ with $\text{H-dim } A \leq \alpha < 2$, then $\text{H-dim } f[A] \leq \beta < 2$, where
\[ \beta = \frac{2p(K)\alpha}{2(p(K)-1)+\alpha} \]  
(13)
and $p(K)$ is the constant given in Theorem 11.

**Proof.** As in the proof of Theorem 8, we may assume that $A$ is contained in an open set with compact closure $F$ in $D$. Then for each $a \in (\alpha, 2)$ and each $q \in (1, p(K))$ we must show that $H_a(f[A]) = 0$, where
\[ b = \frac{2qa}{2(q-1)+a} . \]

Choose $\epsilon > 0$ and $d > 0$. Then $H_a(A) = 0$ and by [8; Lemma 1] we can choose a covering of $A$ by non-overlapping squares $Q_i$ of side $s_i$ such that $Q_i \subset F$, 
\[ \text{dia } (f[Q_i]) < d, \]
and 
\[ \sum_i s_i^a < \epsilon. \]  
(14)

Let $x_i$ denote the centre of $Q_i$ and set
\[ L_i = \max_{y \in \partial Q_i} |f(y) - f(x_i)|, \quad l_i = \min_{y \in \partial Q_i} |f(y) - f(x_i)| . \]

By choosing $d$ sufficiently small, we may assume that the disks $B^2(f(x_i), L_i)$ all lie in $D'$. Then Lemma 3 implies that $L_i \leq Cl_i$, where $C$ is a finite constant which depends only on $K$. Hence
\[ \text{dia } (f[Q_i]) \leq 2L_i \leq 2Cl_i \leq C_l m(f[Q_i])^{1/2}, \]
where $C_1 = 2C\pi^{-1/2}$. On the other hand,

$$m(f[Q_i]) = \int_{Q_i} J_f \, dm \leq s_i^{2(q^{-1})/q} \left( \int_{Q_i} J_f^q \, dm \right)^{1/q}$$

by Hölder's inequality. Thus

$$\sum \text{dia} (f[Q_i])^b \leq C_1^b \sum s_i^{b(q^{-1})/q} \left( \int_{Q_i} J_f^q \, dm \right)^{b/2q},$$

and a second application of Hölder's inequality yields

$$\sum \text{dia} (f[Q_i])^b \leq C_1^b \left( \sum s_i^q \right)^{b(q^{-1})/aq} \left( \int_{Q_i} J_f^q \, dm \right)^{b/2q}.$$  \hspace{1cm} (15)

Finally, since $d$ can be chosen arbitrarily small, (14) and (15) imply that

$$H_b(f[A]) \leq C_1^b \left( \int_{Q_i} J_f^q \, dm \right)^{b/2q} e^{b(q^{-1})/aq},$$

and letting $\varepsilon \to 0$ yields $H_b(f[A]) = 0$.

13. COROLLARY. If $f : D \to D'$ is a 2-dimensional quasiconformal mapping and if $A \subset D$ with $\text{H-dim} A = 2$, then $\text{H-dim} f[A] = 2$.

14. Remark. If the conjecture that Theorem 11 holds with $p(K) = K/(K-1)$ is correct, then Theorem 12 would imply that

$$\frac{2\alpha}{2K-(K-1)\alpha} \leq \text{H-dim} f[A] \leq \frac{2K\alpha}{2+(K-1)\alpha}$$

for each 2-dimensional $K$-quasiconformal mapping $f : D \to D'$ and each set $A \subset D$ with $\text{H-dim} A = \alpha$. These bounds are asymptotic to those implied by Theorem 8 as $\alpha \to 0$.

15. Suppose that $f : D \to D'$ is an $n$-dimensional quasiconformal mapping and that $J_f$ is locally $L^q$-integrable for $q \in (1, p)$ where $p > 1$. Then the proof for Theorem 12 shows that

$$\text{H-dim} f[A] \leq \beta = \frac{np\alpha}{n(p-1)+\alpha} < n$$

for each $A \subset D$ with $\text{H-dim} A \leq \alpha < n$. Unfortunately it is not known whether the analogue of Theorem 11 holds in higher dimensions, and hence we cannot use this argument to establish Conjecture 10 for general $n$.

We can, however, establish a weaker form of Conjecture 10 for general $n$ by a different method. We require some additional notation. Suppose that $f : D \to D'$ is an $n$-dimensional homeomorphism. If $\overline{B}(x, r) \subset D$, we set

$$L(x, f, r) = \max_{|y-x|=r} |f(y) - f(x)|, I(x, f, r) = \min_{|y-x|=r} |f(y) - f(x)|.$$

Next we say that a closed cube $Q \subset D'$ is $f$-admissible if for each $x \in f^{-1}[Q],

$$\overline{B}(x, d) \subset D, \overline{B}(f(x), l(x, f, d)) \subset D'.$$
where $d = \text{dia}(f^{-1}[Q])$. Since $f$ is a homeomorphism, each point of $D'$ is contained in the interior of some $f$-admissible cube $Q$.

16. Lemma. Suppose that $f: D \to D'$ is an $n$-dimensional $K$-quasiconformal mapping, that $T$ is an $(n-1)$-plane in $\mathbb{R}^n$, and that $Q$ is an $f$-admissible closed cube of side $s$ in $D'$. Then there exists an integer $p \geq 2$, which depends only on $n$ and $K$, such that the subdivision of $Q$ into $p^n$ congruent closed cubes of side $s/p$ contains a cube which does not meet $f[D \cap T]$.

Proof. Let $C = C(n, K)$ denote the number given by Lemma 3 when $M/m = 1$. We shall show that the assertion is true for $p > \max(6, 3Cn^{1/2})$.

Fix such an integer $p$, let $Q_0$ denote a cube of the corresponding subdivision which contains the centre of $Q$, and let $S = f[D \cap T]$. If $S \cap Q_0 = \emptyset$, we are finished. Otherwise choose a point $z \in S \cap Q_0$, let $y = f^{-1}(z)$, and let $e$ denote a unit normal to $T$. Then $B = B^n(z, s/3) \subset Q$ and we can choose $r > 0$ so that

$$x = y + re \in f^{-1}[\partial B] \subset f^{-1}[Q].$$

Since $Q$ is $f$-admissible, Lemma 3 implies that

$$l(x, f, r) \geq (1/C) L(x, f, r) \geq (1/C) |f(y) - f(x)| = s/(3C).$$

Next since $T$ is an $(n-1)$-plane, $B^n(x, r) \cap T = \emptyset$ while

$$B^n(f(x), s/(3C)) \subset f[B^n(x, r)].$$

Hence the ball $B^n(f(x), s/(3C))$ does not meet $S$, and since this ball contains a cube of the subdivision, the proof is complete.

17. Definition. A set $S \subset \mathbb{R}^n$ is said to be a $K$-quasiconformal $m$-ball if there is a neighbourhood $D$ of $S$ and an $n$-dimensional $K$-quasiconformal mapping $f: D \to D'$ such that $f[S]$ is an ordinary $m$-dimensional (open or closed) ball. When $m = 1$, $S$ is also said to be a $K$-quasiconformal arc. Finally $S$ is said to be a quasiconformal $m$-ball if it is a $K$-quasiconformal $m$-ball for some $K$.

18. Theorem. If $S$ is a $K$-quasiconformal $m$-ball in $\mathbb{R}^n$ and if $m < n$, then

$$m \leq H\text{-dim } S \leq \beta < n,$$

where $\beta$ depends only on $n$ and $K$.

Proof. Since $S$ is homeomorphic to an ordinary $m$-ball, $S$ has topological dimension $m$, and the lower bound in (16) follows from [6; p. 107].

For the upper bound, there exists, by hypothesis, an $n$-dimensional $K$-quasi-conformal mapping $f: D \to D'$ such that $f[S]$ is an ordinary $(n-1)$-plane $T \subset \mathbb{R}^n$. Choose $p = p(n, K)$ as in Lemma 16 and set

$$a = (1-p^{-n})^{1/2} < 1.$$

Then

$$ap^n > a^2 p^n = p^n - 1,$$

and we may choose $\beta \in (0, n)$ so that $ap^\beta = p^n - 1$. Then $\beta$ depends only on $n$ and $K$, and it suffices to show that $H_\beta(S) = 0$. Moreover since $S$ can be covered by a countable collection of $f$-admissible cubes, it suffices to prove that $H_\beta(S \cap Q) = 0$ for each $f$-admissible closed cube $Q \subset D'$.

Let $Q$ denote such a cube with side $s$, subdivide $Q$ into $p^n$ congruent closed cubes
of side $s/p$, and let $Q_1, \ldots, Q_q$ denote the cubes of this subdivision which meet $S$.
Since $Q$ is $f$-admissible, Lemma 16 implies that $q \leq p^n - 1$ and hence that

$$\sum_{i=1}^{q} \text{dia} \ (Q_i)^{\theta} = q((s/p) n^{1/2})^{\theta} \leq a(sn^{1/2})^{\theta}.$$  

Next subdivide each cube $Q_i$ into $p^n$ congruent closed cubes of side $s/p^2$, and let $Q_{i1}, \ldots, Q_{i_{q_i}}$ denote the cubes of this subdivision which meet $S$. Then since each $Q_i$ is $f$-admissible, Lemma 16 implies that $q_i \leq p^n - 1$ for each $i$ and hence that

$$\sum_{i=1}^{q_i} \left( \sum_{j=1}^{q_i} \text{dia} \ (Q_{ij})^{\theta} \right) = \sum_{i=1}^{q_i} q_i((s/p^2) n^{1/2})^{\theta} \leq a^2 (sn^{1/2})^{\theta}.$$  

Continuing in this way, we see that for each integer $j \geq 1$, $S \cap Q$ can be covered by a finite number of closed cubes $Q_{i_k}$ of side $s/p^j$ such that

$$\sum_i \text{dia} \ (Q_{i_k})^{\theta} \leq a^j(sn^{1/2})^{\theta}.$$  

Then letting $j \to \infty$ we conclude that $H_\beta (S \cap Q) = 0$.

19. COROLLARY. If $f : D \to D'$ is an $n$-dimensional $K$-quasiconformal mapping and if $A \subset D$ is contained in a countable union of $K$-quasiconformal $(n-1)$-balls, then $H\text{-dim} f [A] \leq \beta < n$, where $\beta$ depends only on $n$ and $K$.

Proof. Since $f$ is $K$-quasiconformal, $f [A]$ is contained in a countable union of $K^2$-quasiconformal $(n-1)$-balls, and the conclusion follows from Theorem 18.

20. THEOREM. For each pair of integers $n \geq 2$ and $m \in [1, n-1]$ and each number $\beta \in [m, n)$, there exists a quasiconformal $m$-ball $S \subset R^n$ with $H\text{-dim} S = \beta$.

Proof. Let $A = Q \cap T$, where $Q$ is the closed unit cube and $T$ is the $m$-plane

$$T = \{x = (x_1, \ldots, x_n) : x_{m+1} = \ldots = x_n = \frac{1}{2}\},$$  

and set $s = (4^n+1)^{-1}$. Then we can find $2^n$ disjoint oriented closed cubes $Q_i$ of side $s$ in $\text{int} Q$ with centres in $A$. Following the construction in §4 for the corresponding Cantor set $C_s^n$, we see that for each $j$, the $2^{jn}$ disjoint closed cubes of side $s^j$ in $F_j$ all have their centres in $A$. Hence

$$C_s^n = \bigcap_{j=1}^{\infty} F_j \subset A. \quad (18)$$

Now choose $\epsilon \in (0, \frac{1}{2})$ and a Cantor set $C_\epsilon^n$ so that $H\text{-dim} C_\epsilon^n = \beta$, let $f : R^n \to R^n$ be the quasiconformal mapping given in the proof of Theorem 5 which maps $C_s^n$ onto $C_\epsilon^n$, and set $S = f [A]$. Since $A$ is obviously a quasiconformal $m$-ball, so is $S$. Then (18) implies that

$$A = \left( \bigcup_{j=1}^{\infty} (A \sim F_j) \right) \cup C_s^n, \quad (19)$$

and hence that

$$S = \left( \bigcup_{j=1}^{\infty} f [A \sim F_j] \right) \cup C_\epsilon^n.$$
From the construction in the proof of Theorem 5, we see for each \( j \) that \( f(x) = f_j(x) \) in \( R^n \sim F_j \) and hence that \( f'(R^n \sim F_j) \) is piecewise linear. Thus

\[
\text{H-dim } f[A \sim F_j] = m \leq \beta
\]

for each \( j \) and \( \text{H-dim } S = \text{H-dim } C_r^n = \beta \).

21. **Remarks.** Theorem 20 shows that the upper bound \( \beta \) in Theorem 18 and Corollary 19 cannot be chosen so that it depends only on \( n \).

We can also apply Theorem 20 to the theory of quasiregular mappings. Suppose that \( f: D \to R^n \) is a quasiregular mapping, and let \( B_f \) denote the branch set of \( f \). Then \( B_f \) and \( f[B_f] \) are of \( n \)-dimensional measure zero \([9; 2.27 \text{ and } 8.3]\). Suppose next that \( B_f \neq \emptyset \). Then from \([11; 3.4]\) it follows that \( H_{n-2}(f[B_f]) > 0 \). Hence in this case

\[
n - 2 \leq \text{H-dim } [B_f] \leq n.
\]

On the other hand, by a result of Černavskii \([2]\) (see also \([15]\)), the topological dimension of \( B_f \) is at most \( n - 2 \), and the same is true for \( f[B_f] \) by \([3; 2.2]\).

22. **COROLLARY.** For each integer \( n \geq 3 \) and each pair of numbers \( \alpha, \beta \in [n - 2, n) \), there exists a quasiregular mapping \( f: R^n \to R^n \) such that

\[
\text{H-dim } B_f = \alpha, \text{H-dim } f[B_f] = \beta.
\]

**Proof.** Set \( m = n - 2 \) and let \( T \) be the \((n-2)\)-plane in \((17)\). Since \((19)\) holds with \( A \) replaced by \( T \), the proof of Theorem 20 shows that we can construct quasiconformal mappings \( g_1: R^n \to R^n \) and \( g_2: R^n \to R^n \) such that

\[
\text{H-dim } g_1[T] = \alpha, \text{H-dim } g_2[T] = \beta.
\]

Next define as in \([11; 3.19]\) a quasiregular winding mapping \( h: R^n \to R^n \) with \( B_h = h[B_h] = T \), and set \( f = g_2 \circ h \circ g_1^{-1} \). Then \( f: R^n \to R^n \) is quasiregular and \( B_f = g_1[T], f[B_f] = g_2[T] \).

23. **Final remarks.** The argument in the proof of Theorem 5 can be used to show that the lower bound in Theorem 8 is asymptotically sharp for sets of small Hausdorff dimension. More precisely, given \( K \in (1, \infty) \), one can construct for each \( \alpha \in (0, n) \) a \( K \)-quasiconformal mapping \( f_\alpha: R^n \to R^n \) and a compact set \( A_\alpha \subset R^n \) with

\[
\text{H-dim } A_\alpha = \alpha
\]

such that

\[
\lim_{\alpha \to 0} \frac{\text{H-dim } f_\alpha[A_\alpha]}{\alpha} = K^{1/(1-n)}.
\]

This argument also can be used to show that for each \( K \in (1, \infty) \) there exists a \( K \)-quasiconformal mapping \( f: R^n \to R^n \) and a compact set \( A \subset R^n \) with \( \text{H-dim } A = n \) such that \( f \) is differentiable with a vanishing Jacobian at each point of \( A \).

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