A note on some sufficient conditions for mixed monotone systems

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***Abstract*— Mixed monotone systems form an important class of nonlinear systems that have recently received attention in abstraction-based control design area. Slightly different definitions exist in the literature, and it remains a challenge to verify mixed monotonicity of a system in general. In this paper, we first clarify the relation between different existing definitions of mixed monotone system, and then provide a new and more general sufficient condition for mixed monotonicity. Discussions are provided regarding to these two main contributions.**

I. Introduction

Mixed monotonicity is a property of a function that generalizes monotonicity. The latter one captures the property that the images of a function preserve the order of their pre-images, while the former one refers to the fact that a function can be decomposed into a monotonically increasing part and a monotonically decreasing part. Apparently a monotone function is trivially a mixed monotone function with either the decreasing part or the increasing part being constant and zero.

In this paper, we study a special class of nonlinear dynamical systems called mixed monotone systems. A notable property of such systems is that their flow maps are mixed monotone functions. With this property, we can efficiently approximate the system's states at any time (and hence the trajectories) according to the system's initial states. Previously, mixed monotonicity of a dynamical system is used for qualitative system analysis, including analyzing global stability [8], [3], and studying convergence relation between the solutions to a parabolic system and its corresponding elliptic system [7]. Recently, mixed monotone systems has attracted some attention in the area of abstraction-based controller synthesis [4], [9]. In these works, the mixed monotonicity of a system is used for quantitative reachability analysis and abstraction computation. Moreover, unlike the earlier works focusing on qualitative analysis, these new works study mixed monotone systems defined on some compact region not necessarily invariant under the dynamics.

Despite the usefulness of mixed monotonicity in both qualitative and quantitative analysis, the definition of mixed monotone system is not completely consistent in the literature. The authors notice that mixed monotone systems have two slightly different (but highly related) definitions in the literature [5], [4]. Moreover, it remains unclear how to verify the mixed monotonicity of a function or a system in general. Instead, there are only some sufficient conditions [3], [5] available for checking mixed monotonicity. Aimed at solving these challenges, we present two main contributions of this paper. We first clarify two different definitions of mixed monotone systems existing in the literature, then we give a more general sufficient condition that can be used to verify mixed monotonicity. This novel sufficient condition is especially useful when the considered function for which mixed monotonicity is to be verified is defined on a compact region. Some discussions are provided along the way in section III.

II. Preliminaries

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. By convention, we use bold font lower-case letter, e.g. $\mathbf{x}$, to denote a vector from $\mathbb{R}^n$ (or any other general vector space). Subscript i in $\mathbf{x}_i$ will be used to distinguish different vectors, while normal font x_i is used to denote the i^{th} component of a vector $\mathbf{x}$.

Next, we give some definitions and preliminary results related to mixed monotone function/system.

Definition 1: (Proper Cone [2]) Let $\mathcal{X}$ be a real vector space, a set $K \subseteq \mathcal{X}$ is a *cone* if it is closed under non-negative scaling, i.e.,

$$\mathbf{x} \in K, a \geq 0 \Rightarrow a\mathbf{x} \in K. \tag{1}$$

Furthermore, a cone K is said to be *proper* if it is:

1. convex: $\mathbf{x}_1, \mathbf{x}_2 \in K, a_1, a_2 \geq 0 \Rightarrow a_1\mathbf{x}_1 + a_2\mathbf{x}_2 \in K$[1];
2. pointed: $\mathbf{x} \in K, a < 0 \Rightarrow a\mathbf{x} \notin K$.
3. closed: $\{\mathbf{x}_n\}_{n=1}^{\infty} \subseteq K$ and $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ implies $\mathbf{x} \in K$;
4. solid: K has nonempty interior.

Definition 2: (Generalized Inequality) A proper cone $K \subseteq \mathcal{X}$ defines a partial order on $\mathcal{X}$ in the following sense:

$$\mathbf{x}, \mathbf{y} \in \mathcal{X} : \mathbf{x} \succeq \mathbf{y} \text{ iff } \mathbf{x} - \mathbf{y} \in K. \tag{2}$$

Similarly one can define $\preceq$.

Remark 1: The order $\succeq$ induced by a proper cone K is indeed a partial order. First note that $\mathbf{0} \in K$ by letting $a = 0$ in Eq. (1), hence $\mathbf{x} - \mathbf{x} = \mathbf{0} \in K$, which means $\mathbf{x} \succeq \mathbf{x}$ and the induced order is reflexive. By convexity of K, the induced order is transitive, i.e., $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$ implies that $\mathbf{x} \succeq \mathbf{z}$. By pointedness, the induced order is antisymmetric, i.e., $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{x}$ implies $\mathbf{x} = \mathbf{y}$. Moreover, if cone K is closed, the induced $\succeq$ is preserved under limitation; if K is solid, the it allow us to define *strict inequality* as $\mathbf{x} \succ\succ \mathbf{y}$ iff $\mathbf{x} - \mathbf{y} \in \text{int}(K)$.

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[1] Note that together with Eq. (1), this is the same as usual convexity of a set, where $a_1 \in [0, 1]$ and $a_2 = 1 - a_1$.

Definition 3: (Monotone Mapping) Let $f : \mathcal{X} \to \mathcal{T}$ be a mapping, and let $\succeq_\mathcal{X}$ and $\succeq_\mathcal{T}$ be the partial orders induced by some cones defined on $\mathcal{X}$ and $\mathcal{T}$. Mapping f is said to be *monotone* if it is order preserving, that is,

$$\mathbf{x}, \mathbf{y} \in \mathcal{X}, \mathbf{x} \succeq_\mathcal{X} \mathbf{y} \Rightarrow f(\mathbf{x}) \succeq_\mathcal{T} f(\mathbf{y}). \tag{3}$$

Definition 4: (Mixed Monotone Mapping) Mapping $f : \mathcal{X} \to \mathcal{T}$ is *mixed monotone* if there exists $g : \mathcal{X} \times \mathcal{X} \to \mathcal{T}$ satisfying the following:

1. f is "embedded" on the diagonal of g, i.e., $g(\mathbf{x}, \mathbf{x}) = f(\mathbf{x})$;
2. g is monotone increasing in terms of the first argument, i.e., $\mathbf{x}_1 \succeq_\mathcal{X} \mathbf{x}_2, \Rightarrow g(\mathbf{x}_1, \mathbf{y}) \succeq_\mathcal{T} g(\mathbf{x}_2, \mathbf{y})$;
3. g is monotone decreasing in terms of the second argument, i.e., $\mathbf{y}_1 \succeq_\mathcal{X} \mathbf{y}_2 \Rightarrow g(\mathbf{x}, \mathbf{y}_1) \preceq_\mathcal{T} g(\mathbf{x}, \mathbf{y}_2)$.

g is called a *decomposition function* of f.

Usually, monotonicity and mixed monotonicity are defined in terms of the so called *positive cone* [4], [1], definition of which is very similar to that of a proper cone, except that a positive cone is not required to be closed or solid. The results that are to be presented hold for mixed monotone systems defined by a positive cone. In many important applications, however, the cones used to define the orders also turn out to be proper.

The following theorem allows us to approximate the values of a mixed monotone function in some region, using its decomposition function.

Proposition 1: (Theorem 1 in [4]) Let $f : \mathcal{X} \to \mathcal{T}$ be a mapping, $\succeq_\mathcal{X}$ and $\succeq_\mathcal{T}$ be the partial orders induced by some cones defined on $\mathcal{X}$ and $\mathcal{T}$, and $X = \{\mathbf{x} \in \mathcal{X} \mid \underline{\mathbf{x}} \preceq_\mathcal{X} \mathbf{x} \preceq_\mathcal{X} \overline{\mathbf{x}}\}$. Assume f is mixed monotone with decomposition function $g : \mathcal{X} \times \mathcal{X} \to \mathcal{T}$, then

$$g(\underline{\mathbf{x}}, \overline{\mathbf{x}}) \preceq_\mathcal{T} f(\mathbf{x}) \preceq_\mathcal{T} g(\overline{\mathbf{x}}, \underline{\mathbf{x}}), \forall \mathbf{x} \in X. \tag{4}$$

Definition 5: (Mixed Monotone System) For simplicity, consider autonomous system governed by differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$, where $\mathbf{x} \in X \subseteq \mathbb{R}^n$ is the state. Let $\Phi_t : X \to X$ be the flow that maps the initial state at time instant 0 to final state at time instant t. The system is called mixed monotone if its flow map Φ_t is mixed monotone (i.e., satisfying Definition 4) for all t such that Φ_t is defined.

To this point, we have all the definitions needed in this paper regarding to mixed monotone functions and systems. Note that these concepts are defined in general ordered real vector spaces. In many cases, however, the space we consider is $\mathbb{R}^n$ and the partial order is induced by an orthant in $\mathbb{R}^n$. In particular, if the orthant is the positive quadrant, then the induced order is simply element-wise $\leq$ in $\mathbb{R}^n$. In what follows, we will only consider (mixed) monotone functions/systems in $\mathbb{R}^n$ w.r.t. orthant-induced orders.

III. Main Results

In this section, we present the main results in this paper. We first clarify the relation between two different definitions of mixed monotone systems in the literature, and then give a sufficient condition for mixed monotonicity of a function.

A. On the Relation Between Two Different Definitions of Mixed Monotone Systems

This section tries to clarify the relation between mixed monotone systems (as defined in Section II) and systems with mixed monotone vector fields. Note that the two types of systems are different conceptually: the former ones are defined to have mixed monotone flow map, while the latter ones have mixed monotone vector field. The authors notice that both type of systems are called mixed monotone by some works in literature. [5], [4], [3]. On the other hand, however, there is a nice result in [1] showing monotonicity of the vector field implies that of the flow map. Therefore, an analogue question to ask is: for a given system $\dot{\mathbf{x}} = f(\mathbf{x})$, does the fact that the vector field f is mixed monotone also implies the flow map Φ_t to be mixed monotone?

To answer this question, we have the following result.

Theorem 1: Given system $\dot{\mathbf{x}} = f(\mathbf{x})$, where state $\mathbf{x} \in X \subseteq \mathbb{R}^n$ and vector field f is defined on some open set $\widetilde{X}$ containing set X, assume that f is locally Lipschitz on $\widetilde{X}$ and is mixed monotone, the system is forward complete, and the domain X is positively invariant under the considered dynamics. Then, the flow map Φ_t is mixed monotone.

Proof: Denote the system by $\Sigma : \dot{\mathbf{x}} = f(\mathbf{x})$. Let f be a mixed monotone map and g be f's decomposition function. We prove Theorem 1 by constructing a decomposition function for Φ_t using g.

We start by the standard trick of constructing an "embedding system" [6], i.e., consider the following system

$$\Sigma^{\mathrm{E}} : \begin{cases} \dot{\mathbf{x}} = g(\mathbf{x}, \mathbf{y}) \\ \dot{\mathbf{y}} = g(\mathbf{y}, \mathbf{x}) \end{cases} \tag{5}$$

where g is a decomposition function of f. For system (5), one can make the following observations:

(i) the embedding system has monotone vector field, under the following order defined on $\mathcal{X} \times \mathcal{X}$:

$$(\mathbf{x}_1, \mathbf{y}_1) \succeq_{\mathcal{X}\times\mathcal{X}} (\mathbf{x}_2, \mathbf{y}_2) \text{ iff } \mathbf{x}_1 \succeq_\mathcal{X} \mathbf{x}_2 \text{ and } \mathbf{y}_1 \preceq_\mathcal{X} \mathbf{y}_2; \tag{6}$$

(ii) diagonal $D := \{(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{X} \mid \mathbf{x} = \mathbf{y}\}$ is invariant under Φ_t, the flow map of Σ;

(iii) when state $(\mathbf{x}, \mathbf{y})$ stays on the diagonal D, we have $\dot{\mathbf{x}} = f(\mathbf{x}) = \dot{\mathbf{y}} = f(\mathbf{y})$.

In other words, the dynamics of the system Σ is "embedded" on the diagonal of that of Σ^{E}.

Since Σ^{E} has monotone vector field (observation (i)), by the infinitesimal characterization of monotone systems given by [1], Σ^{E} has monotone flow Ψ_t under the same order in the state space $\mathcal{X} \times \mathcal{X}$, i.e.,

$$(\mathbf{x}_1, \mathbf{y}_1) \succeq_{\mathcal{X}\times\mathcal{X}} (\mathbf{x}_2, \mathbf{y}_2) \Rightarrow \Psi_t(\mathbf{x}_1, \mathbf{y}_1) \succeq_{\mathcal{X}\times\mathcal{X}} \Psi_t(\mathbf{x}_2, \mathbf{y}_2). \tag{7}$$

Moreover, by observations (ii) and (iii), we have

$$\Phi_t(\mathbf{x}) = \Psi_t(\mathbf{x}, \mathbf{x}) \tag{8}$$

Now combining (6), (7) and (8) leads to the fact that Ψ_t is a decomposition function of Φ_t. Hence Φ_t is a mixed monotone map and Σ is a mixed monotone system by definition. ■

A few remarks are in order. The usefulness of Theorem 1 lies in that one can obtain a decomposition function of the vector field for a time discretization of a system from that of the associated continuous-time system. To be specific, given a system $\dot{\mathbf{x}} = f(\mathbf{x})$ satisfying the hypotheses in Theorem 1, let g be the decomposition function of f. The discrete-time system with sampling time Δ is simply governed by difference equation $\mathbf{x}^+ = \Phi_t(\mathbf{x})$, where Φ_t is the flow map; and $t = n\Delta$ are the sampling time instants. If one can somehow find Ψ_t, the flow map of the embedding system, one automatically obtains a decomposition function for Φ_t, which is the right-hand-side of the time-discretized system equation.

In many control applications, system modeling is done in continuous-time, with the system equation derived by some governing physical principles, while there are controller design techniques developed for discrete-time models. In such cases, Theorem 1 can be used to leverage mixed monotonicity in the design procedure.

B. A New Sufficient Condition for Mixed Monotonicity

In this part we give a sufficient condition for a function to be mixed monotone. Particularly, we prove its sufficiency by constructing a decomposition function.

Theorem 2: Assume $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, and

$$\frac{\partial f_i}{\partial x_j}(x) \in [a_{ij}, b_{ij}], \forall \mathbf{x} \in X \subseteq \mathbb{R}^n, \tag{9}$$

where a_{ij} and b_{ij} are finite real numbers, then f is mixed monotone on X, under element-wise order $\leq$ on $\mathbb{R}^n$ and $\mathbb{R}^m$.

Proof: We prove Theorem 2 by constructing a decomposition function for f, then f is mixed monotone by definition.

By assumption $\frac{\partial f_i}{\partial x_j}(\mathbf{x}) \in [a_{ij}, b_{ij}]$ for all $\mathbf{x} \in X$, the interval $[a_{ij}, b_{ij}]$ must satisfy at least one of the following four cases:

case 1: sign-stable positive $a_{ij} \geq 0$
case 2: sign-unstable "positive" $a_{ij} \leq 0$, $b_{ij} \geq 0$, $|a_{ij}| \leq |b_{ij}|$
case 3: sign-unstable "negative" $a_{ij} \leq 0$, $b_{ij} \geq 0$, $|a_{ij}| \geq |b_{ij}|$
case 4: sign-stable negative $b_{ij} \leq 0$.

According to the above cases, define $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ as

$$\forall i \in 1\ldots, m : \quad g_i(\mathbf{x}, \mathbf{y}) = f_i(\mathbf{z}) + (\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i)^T(\mathbf{x} - \mathbf{y}), \tag{10}$$

where $\mathbf{z} = [z_1, \ldots, z_n]^T$, $\boldsymbol{\alpha}_i = [\alpha_{i1}, \ldots, \alpha_{in}]^T$, $\boldsymbol{\beta}_i = [\beta_{i1}, \ldots, \beta_{in}]^T$ are n vectors defined as follows

$$z_j = \begin{cases} x_j & \text{case 1,2} \\ y_j & \text{case 3,4} \end{cases}, \tag{11}$$

$$\alpha_{ij} = \begin{cases} 0 & \text{case 1,3,4} \\ |a_{ij}| + \epsilon & \text{case 2} \end{cases}, \tag{12}$$

$$\beta_{ij} = \begin{cases} 0 & \text{case 1,2,4} \\ -|b_{ij}| - \epsilon & \text{case 3} \end{cases}, \tag{13}$$

where ϵ is a small positive number.

Next we show that g is a decomposition function of f.

1. Obviously $g(\mathbf{x}, \mathbf{x}) = f(\mathbf{x})$ by equations (10) and (11).
2. $\mathbf{x}_1 \geq \mathbf{x}_2 \Rightarrow g(\mathbf{x}_1, \mathbf{y}) \geq g(\mathbf{x}_2, \mathbf{y})$ because

$$\begin{aligned} \forall i : \frac{\partial g_i}{\partial x_j} &= \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k}\frac{\partial z_k}{\partial x_j} + (\alpha_{ij} - \beta_{ij}) \\ &= \frac{\partial f_i}{\partial z_j}\frac{\partial z_j}{\partial x_j} + (\alpha_{ij} - \beta_{ij}) \\ &= \begin{cases} \frac{\partial f_i}{\partial x_j} & \text{case 1} \\ \frac{\partial f_i}{\partial x_j} + |a_{ij}| + \epsilon & \text{case 2} \\ |b_{ij}| + \epsilon & \text{case 3} \\ 0 & \text{case 4} \end{cases} \\ &\geq 0. \end{aligned} \tag{14}$$

3. $\mathbf{y}_1 \geq \mathbf{y}_2 \Rightarrow g(\mathbf{x}, \mathbf{y}_1) \leq g(\mathbf{x}, \mathbf{y}_2)$ because

$$\begin{aligned} \forall i : \frac{\partial g_i}{\partial y_j} &= \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k}\frac{\partial z_k}{\partial y_j} - (\alpha_{ij} - \beta_{ij}) \\ &= \frac{\partial f_i}{\partial z_j}\frac{\partial z_j}{\partial y_j} - (\alpha_{ij} - \beta_{ij}) \\ &= \begin{cases} 0 & \text{case 1} \\ -|a_{ij}| - \epsilon & \text{case 2} \\ \frac{\partial f_i}{\partial y_j} - |b_{ij}| - \epsilon & \text{case 3} \\ \frac{\partial f_i}{\partial y_j} & \text{case 4} \end{cases} \\ &\leq 0. \end{aligned} \tag{15}$$

It follows from definition 4 that g is a decomposition function of f and hence Theorem 2 is proved. ■

We now discuss some implications of this result. By Theorem 2, all differentiable functions with continuous partial derivatives are mixed monotone on a compact set, because the partial derivatives are bounded on the compact set, and hence satisfy the hypothesis of Theorem 2.

Theorem 2 is a natural extension of the result in [4], which only handles the case with sign-stable partial derivatives. The idea here is to use linear terms to create additional offset to overcome the sign-unstable partial derivatives, which leads to a decomposition. In the case where all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ are sign-stable, the decomposition function constructed by Theorem 2 gives a tight approximation in Proposition 1, that is, the inequality in equation (4) reduces to equality at some $x \in X$ [4]. However this is not true when there are sign-unstable partial derivatives. Thus in general the approximation given by Proposition 1 might be conservative when using the decomposition function constructed in Theorem 2. However, one can reduce such conservatism by dividing region X into smaller subregions and applying the same approximation on each subregion. Then the extremum function value over region X can be obtained by combing the extremum function values on those subregions. This divide-and-conquer approach, of course, requires more computational effort because one need to approximate the ranges of sign-unstable partial derivatives on each subregion.

Note that the construction of the decomposition function requires to approximate the ranges of the sign-unstable partial derivatives. Therefore, Theorem 2 together with Proposition 1 "shift" the difficulty of approximating the function value of f into approximating its partial derivatives $\frac{\partial f_i}{\partial x_j}$. By doing such, the difficulty may not be reduced in general. However, in many control applications, the considered systems including thermal system [9] and traffic network system [5], are naturally (mixed) monotone. If one can approximate the partial derivatives of system flow once and for all and prove its (mixed) monotonicity, such properties can be used to simplify the system analysis and design techniques.

IV. Conclusion

In this paper we studied mixed monotone functions and systems. The relation between different definitions of mixed monotone system in the literature were clarified, and a new sufficient condition for mixed monotonicity was proposed. By our new condition, all continuously differentiable functions defined on compact domains in $\mathbb{R}^n$ are mixed monotone with respect to orthants. However, the approximation technique by decomposition function is conservative when applied to the systems satisfying the new condition. Hence, finding better cones and better decompositions that would lead to tighter approximations for a mixed monotone function is still of interest.

References

[1] D. Angeli and E. D. Sontag. Monotone control systems. *IEEE Transactions on automatic control*, 48(10):1684–1698, 2003.

[2] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[3] T. Chu and L. Huang. Mixed monotone decomposition of dynamical systems with application. *Chinese science bulletin*, 43(14):1171–1175, 1998.

[4] S. Coogan and M. Arcak. Efficient finite abstraction of mixed monotone systems. In *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, pages 58–67. ACM, 2015.

[5] S. Coogan, M. Arcak, and A. A. Kurzhanskiy. Mixed monotonicity of partial first-in-first-out traffic flow models. In *Decision and Control (CDC), 2016 IEEE 55th Conference on*, pages 7611–7616. IEEE, 2016.

[6] J.-L. Gouzé and K. P. Hadeler. Monotone flows and order intervals. *Nonlinear World*, 1:23–34, 1994.

[7] V. Lakshmikantham and A. Vatsala. Monotone flows and mixed monotone reaction diffusion systems. In *Matematiča*, 1998.

[8] H. Smith. Global stability for mixed monotone systems. *Journal of Difference Equations and Applications*, 14(10-11):1159–1164, 2008.

[9] L. Yang, A. Karnik, B. Pence, M. T. B. Waez, and N. Ozay. Fuel cell thermal management: Modeling, specications and correct-by-construction control synthesis. In *Proceedings of American Control Conference*, 2017.