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Automorphisms of the compression body graph

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Abstract

Let S be a closed, orientable surface. We show that the automorphism group of the compression body graph of S is the (extended) mapping class group $\mathrm{Mod}(S)$. Here, vertices are compression bodies with exterior boundary S, and edges connect compression bodies when one can be realized as a submanifold of the other.

1. *Introduction*

A *compression body* is a compact, orientable, irreducible 3-manifold C with a distinguished 'exterior' boundary component $\partial_+ C$, such that the inclusion $\partial_+ C \longrightarrow C$ is π_1-surjective. Fixing a surface S, an *S-compression body* is a pair (C, f), where C is a compression body and $f : S \to \partial_+ C$ is a homeomorphism.

Any S-compression body can be constructed as follows (see Lemma 2.1). Starting with $S \times [0, 1]$, attach 2-handles along a collection of disjoint essential annuli in $S \times \{0\}$ and then glue a 3-ball onto every resulting spherical boundary component. Here, the exterior boundary is $S \times \{1\}$, which clearly π_1-surjects, and has a natural identification with S. Two extreme examples of this construction occur when the collection of annuli is empty, in which case we obtain the *trivial compression body* $S \times [0, 1]$, and when the collection is large enough so that after attaching the 2-handles, *every* interior boundary component is a sphere, in which case $\partial C = S \times \{1\}$ and C is a *handlebody*.

We say that (C, f) and (D, g) are *isomorphic* if there is a homeomorphism $H : C \longrightarrow D$ such that $H \circ f = g$. We also say that (C, f) *is contained in* (D, g) if there is an embedding $H : C \longrightarrow D$ such that $H \circ f = g$. Two S-compression bodies are isomorphic if and only if each is contained in the other (see Section 2).

The *compression body graph*, written $\mathcal{CB}(S)$, is the graph whose vertices are isomorphism classes of nontrivial S-compression bodies, and where $(C, f), (D, g)$ are adjacent if either

$$(C, f) \subset (D, g) \quad \text{or} \quad (D, g) \subset (C, f).$$

The *mapping class group* of S, written $\mathrm{Mod}(S)$, is the group of isotopy classes of self-homeomorphisms ϕ of S. (Sometimes this is called the extended mapping class group.) It acts on $\mathcal{CB}(S)$ by precomposing the markings:

$$(C, f) \xrightarrow{\phi} (C, f \circ \phi^{-1}).$$

Theorem 1.1. *When $g(S) \geqslant 2$, the natural map* $\mathrm{Mod}(S) \longrightarrow \mathrm{Aut}(\mathcal{CB}(S))$ *is a surjection.*

Here, $g(S)$ is the genus of the surface S. Note that when S is a torus the theorem is false, since then $\mathcal{CB}(S)$ is an infinite graph with no edges.

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The action of Mod(S) is faithful except when S has genus 2, in which case the kernel is generated by the hyperelliptic involution. This follows from the analogous statement about the action of the mapping class group on the complex of curves, since any simple closed curve α on S gives a *small compression body* $S[a]$ obtained by attaching a 2-handle along an annulus framing α on $S \times \{0\}$, and Mod(S) acts on these small compression bodies via the defining curves (see Section 2).

The compression body graph is an example of a *comparability graph*, where an edge joins vertices that are comparable in a partial order. As such, it is *perfect*, that is, the chromatic and clique numbers of all subgraphs agree. Such graph invariants come up briefly below; for instance, Lemma 3.6 implies that the chromatic and clique numbers of $\mathcal{CB}(S)$ are $2g-1$.

The inspiration for Theorem 1.1 is the theorem of Ivanov [**7**], see also Luo [**10**], that the automorphism group of the curve graph is Mod(S). Here, the *curve graph* is the graph $\mathcal{C}(S)$ whose vertices are isotopy classes of simple closed curves on S, and edges connect isotopy classes that admit disjoint representatives. Ivanov used his theorem to give a geometric proof that the isometry group of Teichmüller space, regarded with the Teichmüller metric, is also Mod(S) (originally proved by Royden [**13**] and Earle–Kra [**5**]), and the outer automorphism group of the mapping class group is trivial. Since then, there have been a number of papers proving similar rigidity results for complexes associated to a surface S, for example, the complex of nonseparating curves [**6**], and the pants complex [**11**].

The action of the mapping class group on $\mathcal{CB}(S)$ encodes a wealth of information about the interaction of mapping classes and 3-manifolds. For instance, an element $\phi \in \mathrm{Mod}(S)$ fixes an S-compression body (C, f) if and only if the homeomorphism $f \circ \phi \circ f^{-1}$ of $\partial_+ C$ extends to a homeomorphism of C. Extension into compression bodies has been previously studied by Casson–Long [**4**], Long [**8**, **9**], Biringer–Johnson–Minsky [**2**], and Ackermann [**1**], among others. In studying the cobordism group of surface automorphisms, Bonahon [**3**, Proposition 5.1] shows that when a homeomorphism of a surface S extends to a 3-manifold M with $\partial M = S$, it also extends to a 3-manifold in which all the nonperiodic action happens on the union of a compression body and an interval bundle.

There are a number of similar 'graphs of compression bodies' that are quasi-isometric to $\mathcal{CB}(S)$. In particular, Maher–Schleimer have shown that a related *handlebody graph* is δ-hyperbolic and has infinite diameter, which implies the same facts about $\mathcal{CB}(S)$. (Maher–Schleimer should be credited as the first to study the implied notion of distance between handlebodies or compression bodies.) One could also connect two compression bodies with an edge if they differ by a single compression, or alternatively by a single *minimal compression* (see Section 2.2), and one could also consider directed versions of all these graphs. For directed graphs of all compression bodies, our proof goes through verbatim, and some of the work in Section 3 can actually be avoided. We thought it interesting to characterize the automorphism group of the undirected graph in the case of $\mathcal{CB}(S)$, though, which requires a bit more work. In general, we find $\mathcal{CB}(S)$ (and its directed version) to be the most natural of these: it is a comparability graph, the edge relation can be seen transparently within the curve complex of S, and its structure is natural when considering the extension of homeomorphisms of S into compression bodies. For instance, [**2**] implies that if the attracting lamination of a pseudo-Anosov map $f : S \longrightarrow S$ is a limit of meridians in C, then there is a finite f-orbit in the link of $C \in \mathcal{CB}(S)$.

For the proof of Theorem 1.1, we introduce an auxiliary simplicial complex, which is of independent interest. The *torus complex*, denoted by $\mathcal{TC}(S)$, is the simplicial complex whose vertices are isotopy classes of nonseparating simple closed curves, and where a collection of vertices $\{a_0, \dots, a_k\}$ spans a k-simplex if there exists a punctured torus $T \subset S$ such that a_i can be isotoped to be contained in T for all $1 \leqslant i \leqslant k$.

THEOREM 1.2. *For* $g(S) \geqslant 2$, *the natural map* $\mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{TC}(S))$ *is a surjection.*

So, every bijection of the set of nonseparating simple closed curves on S that preserves when curves lie in a punctured torus is given by a mapping class.

As in Theorem 1.1, this map is an isomorphism except when S has genus 2, in which case the kernel is generated by the hyperelliptic involution. The relationship between $\mathcal{TC}(S)$ and $\mathcal{CB}(S)$ is described in the following proof sketch.

1.1. *Sketch of the proof of Theorem* 1.1

We will outline here the proof of the main theorem, modulo results to be proved later. A full proof will be given at the end of the paper in Section 5.

Suppose that $f : \mathcal{CB}(S) \longrightarrow \mathcal{CB}(S)$ is an automorphism. In Proposition 3.1, we show that f preserves the set of small compression bodies $S[a]$, those that are obtained from S by compressing a single curve a. Moreover, f preserves whether the compressing curve is nonseparating or separating. Briefly, the idea is that small compression bodies are (among) those with small *height*, a notion of complexity introduced in Section 2.2, and the height of a compression body C is encoded in the chromatic number of certain subsets of the link of $C \in \mathcal{CB}(S)$. This is the subject of Section 3.

In particular, f acts on the set of nonseparating simple closed curves on S. This action has the property that it preserves when a set of nonseparating curves comes from a single punctured torus $T \subset S$. When $g(S) \geqslant 3$, this is because two nonseparating curves a, b lie in a punctured torus if and only if the compression bodies $S[a]$ and $S[b]$ contain a common subcompression body, while $g(S) = 2$ requires an additional argument. This leads us to consider the *torus complex* $\mathcal{TC}(S)$.

Section 4 is dedicated to proving Theorem 1.2 and is entirely separate from the rest of the paper.

Consequently, the action of f on the set of nonseparating small compression bodies agrees with the action of mapping class $\phi \in \mathrm{Mod}(S)$. We then show that the actions of f and ϕ agree on all of $\mathcal{CB}(S)$, using that a compression body is determined by the small compression bodies it contains.

2. *Compression bodies*

A *compression body* is a compact, orientable, irreducible 3-manifold C with a π_1-surjective boundary component $\partial_+ C$, called the *exterior boundary* of C. The complement $\partial C \smallsetminus \partial_+ C$ is called the *interior boundary* and is written $\partial_- C$. Note that the interior boundary is incompressible. For if an essential simple closed curve on $\partial_- C$ bounds a disk $D \subset C$, then $C \smallsetminus D$ has either one or two components, and in both cases, Van Kampen's Theorem implies that $\partial_+ C$, which is disjoint from D, cannot π_1-surject.

Let S be a closed, orientable surface. In the introduction, we defined an *S-compression body* as a pair (C, f) where $f : S \longrightarrow \partial_+ C$ is a homeomorphism. Throughout the remainder of the paper, we will suppress the marking f, and consider compression bodies whose exterior boundaries are implicitly identified with S. With this new language, two S-compression bodies are *isomorphic* if they are homeomorphic via a map that is the identity on their exterior boundaries, and an S-compression body C is *contained in* D, written $C \subset D$, if there is an embedding of C into D that is the identity on the exterior boundary. Often, we will just view C as a submanifold of D that shares its exterior boundary.

If $\{a_1, \dots, a_k\}$ is a collection of disjoint simple closed curves on S, let $S[a_1, \dots, a_k]$ be the S-compression body obtained by *compressing* each of the curves a_i. This means that we attach 2-handles to $S \times [0, 1]$ along a collection of annuli on $S \times \{0\}$ whose core curves are the a_i, fill in S^2-boundary components with balls, and identify S with $S \times \{1\}$. We will call $\{a_1, \dots, a_k\}$ a *compressing system* for $S[a_1, \dots, a_k]$.

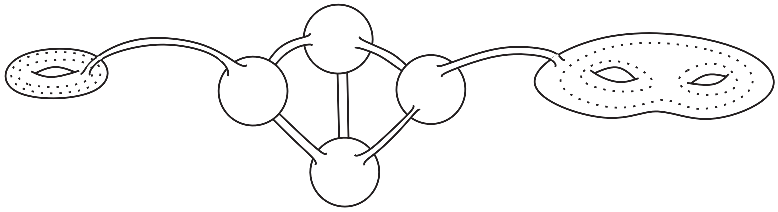

FIGURE 1. *A boundary-connected sum of four balls and interval bundles over a torus and a genus 2 surface. Here, the boundary-connected sum of the four balls is a genus 2 handlebody.*

A simple closed curve on S is called a *disk*, or *meridian*, of an S-compression body C if it bounds an embedded disk in C. The *disk set* of an S-compression body C, written $\mathcal{D}(C)$, is the set of (isotopy classes of) meridians of C.

LEMMA 2.1. *If $\{a_1, \ldots, a_k\} \subset \mathcal{D}(C)$ is a collection of disjoint meridians of C, then $S[a_1, \ldots, a_k] \subset C$. Moreover, if $\{a_1, \ldots, a_k\}$ is maximal, $S[a_1, \ldots, a_k] = C$.*

In particular, any compression body can be constructed from S as above, by compressing a collection of simple closed curves and filling in spheres with balls.

Proof. As the meridians $\{a_1, \ldots, a_k\}$ are disjoint, they bound a collection of disjoint disks D_i in C. By irreducibility, every 2-sphere boundary component of a neighborhood of the union $S \cup \bigcup_i D_i$ bounds a ball in C. So, filling in these boundary components gives a submanifold of C homeomorphic to $S[a_1, \ldots, a_k]$.

Now assume that the collection $\{a_1, \ldots, a_k\}$ is maximal. The interior boundary components of $S[a_1, \ldots, a_k]$ are then incompressible in C: if not, a simple closed curve on an interior boundary component that compresses in $C \setminus S[a_1, \ldots, a_k]$ can be homotoped to a new meridian on S that is disjoint from the collection $\{a_1, \ldots, a_k\}$. So, each component of $C \setminus S[a_1, \ldots, a_k]$ has a π_1-surjective, incompressible boundary component, so is an interval bundle by Waldhausen's cobordism theorem [**15**]. ☐

The following is an immediate consequence of Lemma 2.1.

COROLLARY 2.2. *Let C, D be S-compression bodies. Then C and D are isomorphic if and only if $\mathcal{D}(C) = \mathcal{D}(D)$, and $C \subseteq D$ if and only if $\mathcal{D}(C) \subseteq \mathcal{D}(D)$.*

In particular, this implies that C and D are isomorphic if and only if $C \subseteq D$ and $D \subseteq C$. Compression bodies can also be constructed as boundary-connected sum of closed balls and interval bundles $F_i \times [0,1]$, where the F_i are closed, orientable surfaces, and the boundary-connected sums are always performed along $F_i \times \{1\}$ (see Figure 1). Here, such sums of irreducible 3-manifolds are irreducible, and the union of the $F_i \times \{1\}$ with the boundaries of the balls and 1-handles is a π_1-surjective boundary component.

Lemma 2.1 shows that every compression body can be so constructed. If $C = S[a_1, \ldots, a_k]$, let F_i be the surfaces obtained by *surgering* S along disks $D_1, \ldots, D_k \subset C$ with boundary $a_1, \ldots, a_k$. (Here, if S_i is a component of $S \setminus (a_1 \cup \cdots \cup a_k)$, then F_i is obtained from S_i by attaching the adjacent disks to its boundary components.) Each $F_i \subset C$ bounds either a ball, if F_i is a sphere, or an interval bundle $F_i \times [0,1]$. These pieces are attached along the disks D_j, which expresses C as a boundary-connected sum.

COROLLARY 2.3. *A compression body is determined up to homeomorphism by the genera of its boundary components.*

Note that since the exterior boundary always has the largest genus, it is not necessary to label the genera as ‘exterior’ and ‘interior’ in the statement of the corollary.

Proof. We claim that any compression body C can be obtained by attaching interval bundles $F_i \times [0,1]$ with a *single* 1-handle to a handlebody H. To do this, think of a boundary-connected sum decomposition for C as a graph, where vertices are balls and interval bundles, and edges are 1-handles. The homeomorphism type of the compression body is unchanged if each interval bundle vertex in the graph is replaced by a ball vertex, and then that interval bundle is reattached to the new ball with an additional 1-handle. The result is a graph of balls and 1-handles, that is, a handlebody, attached to interval bundles.

The genera of the interior boundary components determine the interval bundles, while the difference between the genus of the exterior boundary and the total genus of the interior boundary is the genus of the handlebody. □

Finally, we end with a useful gluing construction.

COROLLARY 2.4 (Exterior-to-interior gluings). *Suppose that C is an S-compression body with an interior boundary component $F \subset \partial_- C$, and D is an F-compression body. Then the natural gluing $C \sqcup_F D$ is an S-compression body.*

Conversely, let $C \subset E$ be the S-compression bodies, and let $\partial_- C = F_1 \sqcup \ldots \sqcup F_n$. Then E is isomorphic to an S-compression body obtained by gluing to C a collection of (possibly trivial) F_i-compression bodies D_i, one for each i.

Proof. Represent C and D as the boundary-connected sums of balls and interval bundles. Gluing $F \times [0,1]$ to D does not change its homeomorphism type, so $C \sqcup_F D$ is a boundary-connected sum of the balls and interval bundles from D, together with all balls and interval bundles from C except $F \times [0,1]$.

For the second part, extend a compressing system $a_1, \ldots, a_k$ for C to a compressing system $a_1, \ldots, a_k, b_1, \ldots, b_l$ for $C \sqcup_F D$. The b_j are all disjoint from $a_1, \ldots, a_k$, so are homotopic to simple closed curves b'_j on the interior boundary of C. Then C_i is the compression body defined by the compressing system consisting of all b'_j that lie on F_i. □

2.1. *Small compression bodies*

Throughout this work, there will be a special class of compression bodies that we will consistently come back to, which we now define.

DEFINITION 1. A *small compression body* is a compression body C that can be written as $S[a]$ for some simple closed curve $a \subset S = \partial_+ C$.

A solid torus is an example of a small compression body — it has a unique meridian.

When S has genus at least 2, the disk set of a small compression body $S[a]$ has a unique meridian only when a is separating. We will prove this, but first we need some notation. Given $a, b \in \mathcal{C}(S)$, the *geometric intersection number*, denoted $i(a,b)$, is the minimal number of intersections between any two representatives of a and b. If $a, b \in \mathcal{C}(S)$ and $i(a,b) = 1$, the *band sum of a and b* is the separating curve

$$B(a,b) = \partial N(a \cup b),$$

where $N(a \cup b)$ is a regular neighborhood of $a \cup b$. Note that $B(a,b)$ is the boundary of a once-punctured torus, $N(a \cup b)$, that contains a. Conversely, any curve that bounds a once-punctured torus T containing a can be expressed as a band sum $B(a,b)$, by taking b to be any curve in T that intersects a once.

PROPOSITION 2.5 (Disk sets of small compression bodies). *Suppose that S is a closed, orientable surface and a is a simple closed curve on S. If S is a torus or a is separating,*

$$\mathcal{D}(S[a]) = \{a\},$$

while if the genus $g(S) \geqslant 2$ and a is nonseparating, then

$$\begin{aligned}\mathcal{D}(S[a]) &= \{a\} \cup \{B(a,b) : b \in \mathcal{C}(S),\ i(a,b) = 1\}\\ &= \{a\} \cup \{\partial T : T \subset S \text{ a punctured torus with } a \subset T\}.\end{aligned}$$

In the remainder of the paper, we will call a small compression body $C = S[a]$ *separating* or *nonseparating* depending on the type of the compressed curve $a \subset S$.

An S-compression body C is called *minimal* if it does not contain any nontrivial subcompression bodies. Any minimal compression body must be small, but if a is nonseparating and $g(S) \geqslant 2$ then $S[a]$ is not minimal, since compressing any separating meridian gives a nontrivial subcompression body. On the other hand, all other small compression bodies have a single meridian, so are certainly minimal. In summary:

COROLLARY 2.6. *An S-compression body is minimal if and only if it is a solid torus or a small compression body obtained by compressing a separating curve.*

Before proving Proposition 2.5, we need the following lemma. Although we are only concerned with compression bodies, we might as well state it more generally.

LEMMA 2.7. *Suppose that S is a boundary component of a compact 3-manifold C, and a, b are meridians on S with $i(a,b) > 0$. Then the intersections with a divide b into a collection of arcs, one of which, say b', has the following properties:*

(i) *both intersections of b' with a happen on the same side of a;*
(ii) *the union of b' with either of the two arcs of a with the same endpoints is a meridian, which is disjoint from (after isotopy) but not isotopic to a.*

Proof. Pick two transverse disks D_a and D_b with boundaries a and b, and assume that the number of components of the intersection $D_a \cap D_b$ is minimal.

Let γ be an arc of $D_a \cap D_b$ that is *innermost* in D_b, meaning that one of the two components, say X, of $D_b \smallsetminus \gamma$ has no intersections with D_a. The boundary arc $b' = X \cap \partial D_b$ is disjoint from a except at its endpoints. These two intersections happen on the same side of a, since the side of the disk D_a that X is on cannot flip while traversing γ. (Properly embedded disks in 3-manifolds are always two-sided, since they are simply connected.)

Gluing X to either of the two components of $D_a \smallsetminus \gamma$ gives a disk $D \subset C$ whose boundary is the union of b' with an arc a' of a, as desired. Note that ∂D is an essential simple closed curve in S, since if it were inessential b' and a' would be homotopic rel endpoints, and then a and b would not be in minimal position. Also, since the intersections of b' with a happen on the same side of a, ∂D can be isotoped to be disjoint from a.

Hoping for a contradiction, assume that ∂D is isotopic to a. By a small isotopy, ∂D can be made disjoint from a; more carefully, perform the isotopy by pushing a' slightly away from a while keeping its endpoints on b'. After the isotopy, ∂D and a cobound an annulus $A \subset S$. If A and b' approach a from the same side, then b' is homotopic to $a \smallsetminus a'$ rel endpoints (see Case 1, Figure 2), so as before a, b cannot be in minimal position.

If A and b' approach a from opposite sides, then a is nonseparating and up to the action of the mapping class group of S, exactly as shown in Case 2, Figure 2. However, it is impossible to extend the b' in this figure to a closed curve that is in minimal position with respect to a.

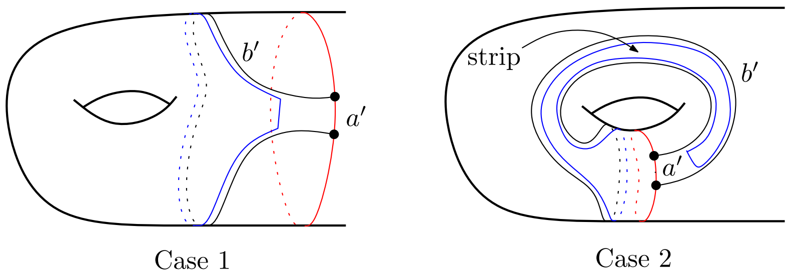

FIGURE 2 (colour online). *The curve a is the vertical circle containing the segment a'. The curve ∂D is obtained by isotoping $a' \cup b'$ to be disjoint from a. In both cases, the annulus A bounded by a and ∂D starts from the left of a.*

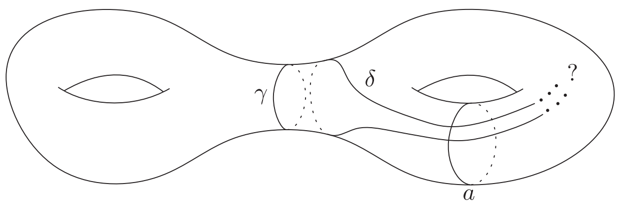

FIGURE 3. *A surgery of a curve δ along a cannot produce a curve that is a band sum of a.*

For continuing from the endpoints of b', since it cannot turn immediately back to intersect a again, the curve b would be forced to wind infinitely many times through the thin 'strip' indicated in the figure, and could never close up. □

We are now ready to prove Proposition 2.5, which characterizes the disk sets of small compression bodies.

Proof. Suppose first that $a \subset S$ is a separating curve. If D is disk in $S[a]$ with boundary a, and S_i is a component of $S \setminus a$, then the union $D \cup S_i$ is a closed surface isotopic in C to an interior boundary component of $S[a]$. As remarked at the beginning of Section 2, this means that $D \cup S_i$ is incompressible in $S[a]$. So, the only essential simple closed curves on S_i that are compressible in $S[a]$ are isotopic to the boundary, a. In other words, there are no other meridians of $S[a]$ that are disjoint from a. *A priori*, there could be meridians that intersect a, but Lemma 2.7 converts these to meridians disjoint from a, so in fact a is the only meridian.

Now, suppose that $a \subset S$ is nonseparating. Form a closed surface S' by attaching to $S \setminus a$ two copies of the disk D. As before, S' is incompressible in $S[a]$.

Assume that γ is a meridian in $S \setminus a$. Then, γ bounds a disk in S'. The intersection of this disk with $S \setminus a$ is a twice punctured disk with γ as a boundary component, and reidentifying the two copies of a gives a punctured torus $T \subset S$ bounded by γ that contains a.

If δ is a meridian of $S[a]$ that intersects a, then by Lemma 2.7, there is a surgery on δ that produces a meridian disjoint from a, which is then a band sum γ by the previous paragraph. However, Figure 3 shows that this is impossible. □

2.2. *Height of a compression body*

When C is an S-compression body, a *sequence of minimal compressions* for C is a chain

$$S \times [0,1] = C_0 \subset C_1 \subset \cdots \subset C_k = C \tag{2.1}$$

of compression bodies in which each C_i is created from C_{i-1} by gluing a minimal F_i-compression body to some component $F_i \subset \partial_- C_{i-1}$. Recall that a compression body is *minimal* if it does not contain any nontrivial subcompression bodies (in Corollary 2.6), we saw that these are exactly the solid tori and separating small compression bodies.

Sequences of minimal compressions are exactly chains (C_i) as in (2.1) that are maximal, in the sense that they are not properly contained in a larger chain.

As an example, let $C = S[a]$, where a is nonseparating and $g(S) \geqslant 2$. By Proposition 2.5, any separating meridian b for $S[a]$ bounds a punctured torus containing a, so a is isotopic to a curve a' on a torus $T \subset \partial_- S[b]$. The compression body $S[a]$ is obtained by attaching a solid torus to $S[b]$ along T so that the meridian is identified with a'. So here,

$$S \times [0,1] \subset S[b] \subset S[a]$$

is a sequence of minimal compressions for any separating meridian b in $S[a]$.

More generally, we have the following lemma.

LEMMA 2.8. *Suppose that $C = S[a_1, \ldots, a_k]$, and for each i let S_i be the component of $S \setminus a_1 \cup \cdots \cup a_{i-1}$ containing a_i. If*

$(*)$ for each i, either we have $g(S_i) = 1$, or we have $g(S_i) \geqslant 2$ and a_i separates S_i,

then $C_i = S[a_1, \ldots, a_i]$ defines a sequence of minimal compressions for C. Conversely, any sequence of minimal compressions for C can be written as $C_i = S[a_1, \ldots, a_i]$ for some collection $a_1, \ldots, a_k$ satisfying $(*)$.

Whenever $C = S[a_1, \ldots, a_k]$, the (a_i) can be altered to satisfy $(*)$. For if $g(S_i) = 0$, then a_i is already a meridian in $S[a_1, \ldots, a_{i-1}]$, so its inclusion is redundant and it can be removed. If $g(S_i) \geqslant 2$ and a_i is nonseparating in S_i, insert a new curve $b_i \subset S_i$ that bounds a punctured torus containing a_i between a_{i-1}, a_i in the sequence.

Proof. Suppose $C = S[a_1, \ldots, a_k]$ and (a_i) satisfies $(*)$. Then for each i,

$$C_i = C_{i-1} \sqcup_{F_i} F_i[a_i'],$$

where F_i is the component of $\partial_- C_{i-1}$ homotopic to the surface obtained by attaching disks to S_i, and $a_i' \subset F_i$ is the unique curve homotopic to a_i. Note that $g(S_i) = g(F_i)$, and a_i separates S_i if and only if a_i' separates F_i. Then $(*)$ says that $F_i[a_i']$ is minimal.

Conversely, if (C_i) is a sequence of minimal compressions, we can use Lemma 2.1 to iteratively extend compressing systems from C_{i-1} to C_i. The result is a compressing system $a_1, \ldots, a_k$ for C that satisfies $(*)$. □

The following is the main result of the section.

PROPOSITION 2.9. *If C is a compression body with $\partial_- C = F_1 \sqcup \cdots \sqcup F_n$, the length k of any sequence of minimal compression $S \times [0,1] = C_0 \subset C_1 \subset \cdots \subset C_k = C$ is*

$$\mathfrak{h}(C) := (2 \cdot g(S) - 1) - \sum_{i=1}^{n} (2 \cdot g(F_i) - 1). \tag{2.2}$$

We call $\mathfrak{h}(C)$ the *height* of C. A genus g handlebody has height $2g - 1$, so a solid torus has height 1. A separating small compression body also has height 1, as the genera of the two interior boundary components sum to the genus of the exterior boundary.

Proof. If C is a compression body and F is a component of $\partial_- C$, then $\mathfrak{h}$ adds when an F-compression body D is glued to C:

$$\mathfrak{h}(C \sqcup_F D) = \mathfrak{h}(C) + \mathfrak{h}(D),$$

since the only boundary component from C or D that is not referenced in $\mathfrak{h}(C \sqcup_F D)$ is F, but $2 \cdot g(F) - 1$ appears with opposite signs in $\mathfrak{h}(C)$ and $\mathfrak{h}(D)$. So, gluing on a solid torus or a separating small compression body increments height. □

As a consequence of Proposition 2.9, height is positive and increases under the inclusion

$$C \subset D \Longrightarrow \mathfrak{h}(C) < \mathfrak{h}(D).$$

Hence, the length of any chain (C_i) of subcompression bodies of C is at most $\mathfrak{h}(C)$, and any chain is contained in a maximal chain, that is, a sequence of minimal compressions.

Corollary 2.10 ('Short' compression bodies).

(i) *Height* 1 *compression bodies are solid tori and small compression bodies* $S[a]$, *where* a *is separating.*
(ii) *Height* 2 *compression bodies are small compression bodies* $S[a]$, *where* a *is nonseparating and* $g(S) \geqslant 2$, *and compression bodies of the form* $S[a_1, a_2]$, *where* a_1, a_2 *are disjoint, separating curves on* S.

Proof. By Proposition 2.9, height 1 compression bodies are minimal compression bodies, so (1) follows from Corollary 2.6.

A nonseparating small compression body $S[a]$, where $g(S) \geqslant 2$, has height 2, since its interior boundary is connected with genus 1 less than that of the exterior boundary.

A pair of disjoint separating curves $a, b \subset S$ separates S into three subsurfaces S_1, S_2, S_3, where $\sum_i g(S_i) = g(S)$. All of these have positive genus, so $S[a, b]$ has three interior boundary components, with these same genera. So, $\mathfrak{h}(S[a, b]) = 2$.

Finally, if a compression body C has height 2, then by Corollary 2.8 there must be a pair of disjoint curves a_1, a_2 satisfying $(*)$ with $S[a_1, a_2] = C$. It follows from $(*)$ that $g(S) \geqslant 2$, and that a_1 is separating. If a_2 is nonseparating, then $(*)$ implies that the component of $S \setminus a_1$ containing a_2 is a punctured torus, in which case $S = S[a_2]$. □

2.3. *Separating and nonseparating compression bodies*

In this section, it is shown that for every compression body, either there exists a compressing system consisting of entirely nonseparating curves, or one entirely of separating curves, and that these options are mutually exclusive. Furthermore, this dichotomy is determined by the presence or lack, respectively, of a nonseparating meridian.

Proposition 2.11. *For an* S*-compression body* C, *the following seven conditions are equivalent:*

(i) $C = S[a_1, \ldots, a_k]$, *where each* a_i *is separating;*
(ii) $H_1(S) \longrightarrow H_1(C)$ *is injective (and hence an isomorphism);*
(iii) *every meridian of* C *is separating;*
(iv) *solid tori are never used in any sequence of minimal compressions for* C*;*
(v) C *has a sequence of minimal compressions in which solid tori are never used;*
(vi) *the number of interior boundary components of* C *is* $\mathfrak{h}(C) + 1$*;*
(vii) *the genera of the interior boundary components of* C *sum to* $g(S)$*;*
(viii) C *is contained in a compression body that has* $g(S)$ *interior boundary components, all of which are tori, as shown in Figure 4.*

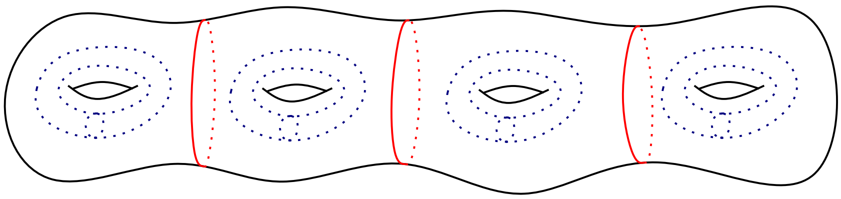

FIGURE 4 (colour online). *An S-compression body that has $g(S)$ interior boundary components, all of which are tori (outlined in dots). The vertical circles form a compressing system. All such compression bodies are isomorphic, although not all compressing systems look like the above.*

Proof. For (1) $\Longrightarrow$ (2) the kernel of $\pi_1 S \longrightarrow \pi_1 C$ is normally generated by the curves $a_1, \ldots, a_k$, which lie in the commutator subgroup $[\pi_1 S, \pi_1 S]$. So, the entire kernel lies in $[\pi_1 S, \pi_1 S]$, implying that the induced map $H_1(S) \longrightarrow H_1(C)$ is injective.

For (2) $\Longrightarrow$ (3), note that any nonseparating curve is nontrivial in $H_1(S)$, so by (2) cannot be trivial in $H_1(C)$.

(3) $\Longrightarrow$ (1) are trivial.

(3) $\Longrightarrow$ (4), since gluing a solid torus to an interior boundary component $F \subset \partial_- C$ compresses a nonseparating curve on F, which is then homotopic to a nonseparating meridian on S.

(4) $\Longrightarrow$ (5) is trivial.

(5) $\Longrightarrow$ (6), since gluing a separating small compression body onto an interior boundary component F removes F, but contributes two new interior boundary components.

(6) $\Longrightarrow$ (7), by the definition of height.

(7) $\Longrightarrow$ (8), since to each interior boundary component F of C, we can glue an F-compression body with $g(F)$ interior boundary components, all of which are tori.

For (8) $\Longrightarrow$ (2), note that the compression body in Figure 4 has a compressing system consisting of only separating curves, so all its meridians are separating by the fact that (1) $\Longrightarrow$ (2). The same is then true for any subcompression body. □

The next result is a corollary of Lemma 2.1 and Corollary 2.3.

COROLLARY 2.12. *An S-compression body C can be written as $S[a_1, \ldots, a_m]$ with each a_i nonseparating if and only if C contains a nonseparating meridian.*

Proof. The forward implication is obvious, so assume that C has a nonseparating meridian. As described in Corollary 2.3, we can construct C by attaching, for each component of $\partial_- C$, an interval bundle with a single 1-handle to a handlebody H. Since C has a nonseparating meridian, H must have positive genus, for otherwise H would admit a compressing system with only separating meridians, violating Proposition 2.11 (3).

Each 1-handle intersects ∂H in a disk; let $\{D_1, \ldots, D_n\}$ be the collection of these disks. Choose any pants decomposition $\{a_1, \ldots, a_m\}$ for the punctured surface $\partial H \setminus (\bigcup_i D_i)$ in which all the a_i are nonseparating (as in Figure 5); we claim that $C = S[a_1, \ldots, a_k]$. To see this, observe that $\{a_1, \ldots, a_m, \partial D_1, \ldots, \partial D_n\}$ is a maximal set of disjoint meridians for C, and for each $1 \leqslant i \leqslant n$ there exists $1 \leqslant j \neq k \leqslant m$ such that ∂D_i bounds a pair of pants with some a_j and a_k. It follows that

$$C = S[\partial D_1, \ldots, \partial D_n, a_1, \ldots, a_k] = S[a_1, \ldots, a_k],$$

where the first equality is due to Lemma 2.1. □

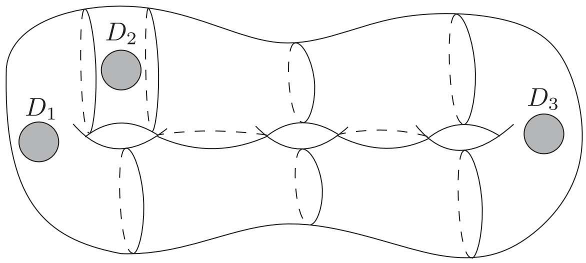

FIGURE 5. *A handlebody with spots D_1, D_2, D_3 corresponding to the attachment of* 1*-handles. The nonseparating meridians shown on the surface give a pants decomposition of the punctured surface obtained by deleting the spots from the boundary of the handlebody.*

3. *Small compression bodies are invariant*

The main goal of this section is to show that an automorphism of $\mathcal{CB}(S)$ sends small compression bodies to small compression bodies with the same type:

PROPOSITION 3.1. *Suppose that f is an automorphism of the compression body graph $\mathcal{CB}(S)$. If a is a simple closed curve on S, then $f(S[a]) = S[b]$ for some simple closed curve b. Moreover, a is separating if and only if b is separating.*

Recall from Section 2.2 that the *height*, written $\mathfrak{h}(C)$, of an S-compression body C is the length k of any sequence of minimal compressions

$$S \times [0,1] = C_0 \subset \cdots C_k = C.$$

Height 1 compression bodies are solid tori and small compression bodies $S[a]$, where a is separating. Height 2 compression bodies are of the form $S[a]$, where a is nonseparating, or of the form $S[a,b]$, where a, b are separating. See Corollary 2.10.

The bulk of the work in Proposition 3.1 is in the following result.

PROPOSITION 3.2. *Every automorphism of the compression body graph preserves height.*

The proof of Proposition 3.2 will occupy most of this section; the idea is that the height of C is encoded in the chromatic numbers of certain subsets of the link of C in $\mathcal{CB}(S)$. We describe the structure of links in Section 3.1, and finish the proof of height preservation in Section 3.2.

The invariance of small compression bodies almost follows from Proposition 3.2, but one must also show that nonseparating small compression bodies are not sent by f to compression bodies $S[a,b]$, where a, b are separating. For this, one can use the following lemma.

LEMMA 3.3. *Any height preserving automorphism $f : \mathcal{CB}(S) \longrightarrow \mathcal{CB}(S)$ preserves the set of compression bodies C with only separating meridians.*

Such C admit a number of different characterizations, see Proposition 2.11. In particular, by (1) $\Longrightarrow$ (3), a height 2 compression body $S[a,b]$, with a, b separating, has only separating meridians. A nonseparating small compression body clearly does not.

Proof. Note that f preserves containment, since one can distinguish between the edge relations $C \subset D$ and $D \subset C$ using height. It must then preserve the set of chains $C_1 \subset C_2 \subset \cdots$ of compression bodies, and therefore the set of maximal such chains, that is, sequences of minimal compressions. Also f sends handlebodies to handlebodies, since these are exactly the compression bodies with height $2g(S)-1$.

It follows that f preserves the set of compression bodies C that have only torus interior boundary components, as these C can be characterized by the fact that after choosing a handlebody $H \supset C$, there are only finitely many sequences of minimal compressions from C to H. Here, solid tori can be attached to the interior boundary components of C in any order, but the attachment maps are prescribed by H. If C has a higher genus interior boundary component F, then there are infinitely many intermediate compressions to choose from when attaching a handlebody to F.

In particular, f must preserve the height $g(S)-1$ compression bodies C that have only torus interior boundary components, as shown in Figure 4. Here, the number of boundary components is $g(S)$, since g additional compressions are required to reach a handlebody, which has height $2g(S)-1$. By Proposition 2.11, the compression bodies that are contained in such C are exactly those that have only separating meridians. □

The remainder of the section presents the proof of Proposition 3.2, which states that automorphisms of the compression body graph preserve height.

3.1. *Links in $\mathcal{CB}(S)$*

Given a graph Γ we denote the edge relation in Γ by $\sim_\Gamma$. We will simply use $\sim$ when the graph is clear from context. The *join* of two graphs Γ and Γ', written $\Gamma + \Gamma'$, is the graph with vertex set $\Gamma \sqcup \Gamma'$, and where vertices v, w are adjacent if either:

(i) $v, w \in \Gamma$ and $v \sim_\Gamma w$, or
(ii) $v, w \in \Gamma'$ and $v \sim_{\Gamma'} w$, or
(iii) $v \in \Gamma$ and $w \in \Gamma'$.

In particular, note that if two vertices in a graph join are not connected by an edge, they must lie in the same factor. Here is a useful consequence.

Fact 3.4 (Uniqueness of join). Suppose Γ, Γ' are anticonnected graphs, and Δ, Δ' are arbitrary. If $\Gamma + \Gamma' = \Delta + \Delta'$, then up to exchanging factors, $\Gamma = \Delta$ and $\Gamma' = \Delta'$.

Here, a graph is *anticonnected* if any two vertices can be connected by an antipath, that is, a sequence of vertices (v_i) where $v_i \not\sim v_{i+1}$ for all i. To prove the fact, just assume there is some $v \in \Delta \cap \Gamma$ and note that any antipath starting at v stays in Δ.

As mentioned above, the link of a compression body $C \in \mathcal{CB}(S)$ decomposes as a join

$$\mathrm{Link}(C) = \mathrm{Link}^-(C) + \mathrm{Link}^+(C),$$

where

$$\mathrm{Link}^+(C) = \{D \in \mathcal{CB}(S) : C \subset D\}$$

and

$$\mathrm{Link}^-(C) = \{D \in \mathcal{CB}(S) : D \subset C\}$$

are the *uplink* and *downlink* of C, respectively.

Lemma 3.5. *If $C \in \mathcal{CB}(S)$, the graphs $\mathrm{Link}^+(C)$ and $\mathrm{Link}^-(C)$ are anticonnected.*

So by Fact 3.4, the only way to write $\mathrm{Link}(C)$ as a join is using the uplink and downlink.

Proof. We will say two vertices v, w are *antiadjacent* if $v \not\sim w$ in $\mathcal{CB}(S)$.

We first deal with $\mathrm{Link}^+(C)$. Pick a handlebody $H \supset C$ that lies in Γ. By Corollary 2.4, H is obtained from C by attaching handlebodies (with smaller genus) to the components of $\partial_- C$. In particular, there is a nonseparating simple closed curve a on some component $F \subset \partial_- C$ that bounds a disk in H.

Choose a simple closed curve b on F with $i(a, b) = 1$, and let $C[b]$ be the compression body obtained from C by compressing b. Then $C[b]$ and H are antiadjacent, since two meridians in a handlebody cannot intersect once. Moreover, every compression body containing $C[b]$ is antiadjacent to H as well.

Any two compression bodies with the same height are antiadjacent. The compression body $C[b]$ and its uplink represent all heights in $\mathrm{Link}^+(C)$, except $\mathfrak{h}(C) + 1$ when $g(F) \geqslant 2$. In this last case, though, there is a separating simple closed curve c on F that is not a band sum with b, and then $C[c]$ is a height $\mathfrak{h}(C) + 1$ compression body that is antiadjacent to $C[b]$. Therefore, the uplink $\mathrm{Link}^+(C)$ is anticonnected.

The argument for $\mathrm{Link}^-(C)$ is similar. Start with a compression body $D \subset C$ with height $\mathfrak{h}(C) - 1$, and pick some meridian a of C that is not a meridian in D. Then every subcompression body of C that contains a is antiadjacent to D. These fill out all heights in $\mathrm{Link}^-(C)$, except height 1 if a is not separating. In this latter case, $\mathfrak{h}(C) - 1 \geqslant 2$, so C must have a separating meridian b that is not a band sum with a. Then $S[b]$ is a height 1 compression body that is antiadjacent to $S[a]$, and the downlink is anticonnected. □

A *clique* in a graph Γ is a complete subgraph and the *clique number*, written $\omega(\Gamma)$, is the number of vertices in a maximal clique. A *proper coloring* of a graph is a labeling of the vertices such that vertices connected by an edge are assigned different labels. The minimal number of colors required to give a proper coloring of Γ is the *chromatic number*, written $\chi(\Gamma)$. It is clear that

$$\omega(\Gamma) \leqslant \chi(\Gamma). \tag{3.1}$$

LEMMA 3.6. *If $C \in \mathcal{CB}(S)$, the clique and chromatic numbers satisfy*

$$\omega(\mathrm{Link}^-(C)) = \chi(\mathrm{Link}^-(C)) = \mathfrak{h}(C) - 1$$

and

$$\omega(\mathrm{Link}^+(C)) = \chi(\mathrm{Link}^+(C)) = 2g - 1 - \mathfrak{h}(C).$$

Proof. Labeling a vertex of $\mathrm{Link}^-(C)$ by its height is a proper coloring, so $\chi(\mathrm{Link}^-(C)) \leqslant \mathfrak{h}(C) - 1$. The induced subgraph on any maximal chain of subcompression bodies $C_1 \subset \cdots \subset C_{\mathfrak{h}(C)-1} = C$ is a complete graph, so $\omega(\mathrm{Link}^-(C)) \geqslant \mathfrak{h}(C) - 1$. Therefore,

$$\omega(\mathrm{Link}^-(C)) = \chi(\mathrm{Link}^-(C)) = \mathfrak{h}(C) - 1$$

by (3.1). The case for uplinks is similar. □

3.2. *Automorphisms preserve height: the proof of Proposition 3.2*

Fix an automorphism $f \in \mathrm{Aut}(\mathcal{CB}(S))$. If $C \in \mathcal{CB}(S)$, we can write

$$\mathrm{Link}(f(C)) = \mathrm{Link}^+(f(C)) + \mathrm{Link}^-(f(C)) = f(\mathrm{Link}^+(C)) + f(\mathrm{Link}^-(C)).$$

The uniqueness of the join (Lemma 3.5) that implies that either:

(i) $f(\mathrm{Link}^+(C)) = \mathrm{Link}^+(f(C))$ and $f(\mathrm{Link}^-(C)) = \mathrm{Link}^-(f(C))$; or
(ii) $f(\mathrm{Link}^+(C)) = \mathrm{Link}^-(f(C))$ and $f(\mathrm{Link}^-(C)) = \mathrm{Link}^+(f(C))$.

Lemma 3.7. *Either* (i) *holds for all* $C \in \mathcal{CB}(S)$, *or* (ii) *holds for all* $C \in \mathcal{CB}(S)$.

Proof. Since the graph $\mathcal{CB}(S)$ is connected, it suffices to show that when two compression bodies are adjacent, either (i) holds for both or (ii) holds for both. For convenience, let

$$\mathrm{Link}^{\geqslant}(C) = \{C\} \cup \mathrm{Link}^{+}(C) \quad \text{and} \quad \mathrm{Link}^{\leqslant}(C) = \{C\} \cup \mathrm{Link}^{-}(C).$$

Then if $C \subset D$, the only inclusions that are present between the four sets

$$\mathrm{Link}^{\geqslant}(C),\ \mathrm{Link}^{\leqslant}(C),\ \mathrm{Link}^{\geqslant}(D),\ \mathrm{Link}^{\leqslant}(D)$$

are that $\mathrm{Link}^{\geqslant}(C) \supset \mathrm{Link}^{\geqslant}(D)$ and $\mathrm{Link}^{\leqslant}(C) \subset \mathrm{Link}^{\leqslant}(D)$. Here, it is necessary to add in C to the up- and downlinks, for if H is a handlebody, $\mathrm{Link}^{+}(H)$ is empty, and is included in all four sets above. Similarly, $\mathrm{Link}^{-}(H)$ is empty when C is a separating small compression body.

In conclusion, it cannot be the case that up- and downlinks are switched for C, but preserved for D (or *vice versa*), since then when considering $f(C)$ and $f(D)$, one would see some $\mathrm{Link}^{\geqslant}$ included in a $\mathrm{Link}^{\leqslant}$ (or *vice versa*). □

By Lemma 3.6, the height of a compression body can be calculated from the chromatic number of its uplink, or of its downlink. So, in light of Lemma 3.7, f is either height preserving (in which case we are done) or 'height reversing', that is,

$$\mathfrak{h}(f(C)) = 2g - \mathfrak{h}(C), \quad \text{for all } C \in \mathcal{CB}(S). \tag{3.2}$$

Assuming (3.2), we break into cases. When $g(S) \geqslant 3$, every compression body of height $2g-2$ has a single torus interior boundary component, so by Corollary 2.3, they are all homeomorphic. Therefore, $\mathrm{Mod}(S)$ acts transitively on height $2g-2$ compression bodies. This action is conjugated by f to a transitive, height preserving action on height 2 compression bodies. We claim that this is a contradiction. Indeed, suppose the automorphisms act transitively on compression bodies of height 2. Then, Corollary 2.10 implies that there exists an automorphism f sending a compression body of the form $S[a,b]$, where a,b are separating and disjoint, to a nonseparating small compression body. By Lemma 3.7, f is height-preserving implying $f(S[a,b])$ cannot contain a nonseparating meridian by Proposition 2.11 and Lemma 3.3, a contradiction.

When S has genus 2, the argument above fails since a genus 2 surface does not admit a pair of disjoint separating curves. Here, there are three possible heights:

($\mathfrak{h} = 1$) separating small compression bodies;
($\mathfrak{h} = 2$) nonseparating small compression bodies;
($\mathfrak{h} = 3$) handlebodies.

As f is height reversing, the set of nonseparating small compression bodies is left invariant, while separating small compression bodies and handlebodies are exchanged. Note that since f reverses height, it reverses the order of containment: $C \subset D \Longrightarrow f(C) \supset f(D)$.

Let a, b be the curves indicated in Figure 6, and let

$$f(S[a]) = S[a'], \quad f(S[b]) = S[b'].$$

As $S[a]$ and $S[b]$ are contained in a common handlebody, both $S[a']$ and $S[b']$ contain the same separating small compression body, that is, there are punctured tori T_a and T_b containing a', b' with $\partial T_a = \partial T_b$. If $T_a = T_b$, then a', b' have nonzero algebraic intersection number, so $S[a']$ and $S[b']$ cannot be contained in the same handlebody, contradicting that both a, b contain a common separating small compression body. Therefore, we must have $T_a \neq T_b$, in which case the configuration of a', b' is the same, up to homeomorphism, as that of a, b. But this is a contradiction, since then $S[a'], S[b']$ contain infinitely many common separating small compression bodies, while $S[a], S[b]$ are contained in only a single handlebody.

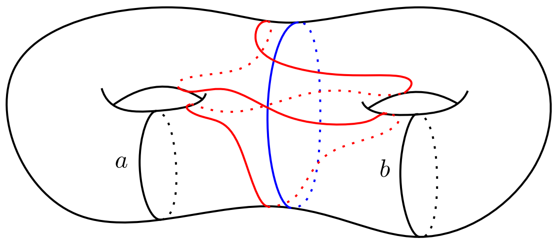

FIGURE 6 (colour online). *The two curves above are both meridians in only a single handlebody, while there are infinitely many separating curves that are disjoint from both.*

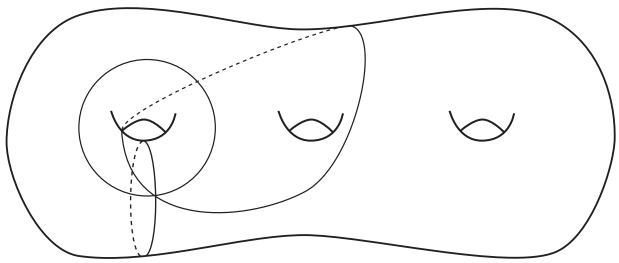

FIGURE 7. *An example of three curves forming an empty triangle in $\mathcal{TC}(S)$.*

4. *The torus complex*

The *torus complex* associated to a surface S, written $\mathcal{TC}(S)$, is the complex whose vertices are isotopy classes of simple nonseparating curves and where the collection of vertices $\{a_0, \ldots, a_k\}$ spans a k-dimensional simplex if there exists a punctured torus $T \subset S$ with $a_i \subset T$ for every $1 \leqslant i \leqslant k$ (see Figure 7). Note that this is an infinite dimensional complex.

As described in the introduction, the automorphism group of the compression body graph can be determined using the following theorem.

THEOREM 1.2. *The natural map* $\mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{TC}(S))$ *is a surjection.*

As with automorphisms of the curve complex and $\mathcal{CB}(S)$, the map is an isomorphism unless S has genus 2, in which case the kernel is generated by the hyperelliptic involution. We will prove surjectivity by showing that an automorphism of $\mathcal{TC}(S)$ induces an automorphism of the *Schmutz graph*. Here, the Schmutz graph $\mathcal{N}(S)$ has the same vertex set as $\mathcal{TC}(S)$, but edges connect pairs of vertices that intersect once. Schaller [14] proved that every automorphism of $\mathcal{N}(S)$ is induced by a mapping class.

We will need the following definition.

DEFINITION 2. A *triangle* in $\mathcal{TC}(S)$ is a triple of vertices a, b, c, where each pair of vertices is connected by an edge. A triangle is *empty* if it does not bound a 2-simplex.

Suppose a, b are adjacent vertices in $\mathcal{TC}(S)$. We claim that if $i(a, b) = 1$, there exists a punctured torus $T \subset S$ such that there are infinitely many empty triangles a, b, c with a, c contained in T, while if $i(a, b) \neq 1$, there are at most three empty triangles for every T. As this characterization concerns just the simplicial structure of $\mathcal{TC}(S)$, it proves that intersection number one is preserved by simplicial automorphisms. So by Schaller [14], any automorphism of $\mathcal{TC}(S)$ is induced by a mapping class.

Assuming $i(a,b)=1$, the empty triangles are easy to construct. Choose any simple closed curve d in S that intersects a once, is disjoint from b, but is not homotopic into a regular neighborhood N of $a \cup b$. If T is a regular neighborhood of $a \cup d$, then twisting a around d gives infinitely many simple closed curves c in T that lie in punctured tori with both a and b. Any such c determines an empty triangle with vertices a, b, c.

For the other direction, we will need the following result.

PROPOSITION 4.1. *At most one edge of an empty triangle in $\mathcal{TC}(S)$ can connect simple closed curves on S that intersect more than once.*

Deferring the proof for a moment, let us finish the characterization of pairs of vertices that intersect once. Suppose $i(a,b) > 1$. If a, b, c is an empty triangle, then by Proposition 4.1, we must have $i(a,c) = i(b,c) = 1$. But if T is a punctured torus containing a, there are at most three simple closed curves $c \subset T$ that intersect both a, b once.

This is easiest to see in coordinates. After picking a basis for the homology of T, proper arcs and curves in T can be labeled by extended rational numbers $\frac{p}{q} \in \mathbb{Q} \cup \infty$. Here, the $\frac{p}{q}$-arc is the unique arc that is disjoint from the $\frac{p}{q}$-curve (see Figure 9(a)). The intersection number of the $\frac{p}{q}$-curve and $\frac{m}{n}$-curve is $|pn - mn|$, and the same formula holds for intersections of arcs and curves (although for intersections of two arcs, it is off by one). So, assume a is the $\frac{1}{0}$-curve, and some component of $b \cap T_3$ is the $\frac{p}{q}$-arc. As $i(a,b) > 0$, we can assume that this arc intersects a, that is, $q \neq 0$. Then if the $\frac{m}{n}$-curve in T intersects a and b once, it intersects the $\frac{p}{q}$-arc at most once, so

$$|1 \cdot n - 0 \cdot m| = 1, \quad |p \cdot n - q \cdot m| \leqslant 1.$$

These conditions are only satisfied when $q = \pm 1$, in which case $\frac{m}{n}$ must be $\frac{p}{q}$ or $(p \pm 1)/q$.

4.1. *The proof of Proposition* 4.1

We require the following lemma.

LEMMA 4.2. *If $a_1, a_2 \in \mathcal{TC}(S)$ are in minimal position and contained in a punctured torus $T \subset S$, then for any punctured torus $T' \neq T$ in S containing a_1,*

$$a_2 \cap T' = \alpha_1 \sqcup \cdots \sqcup \alpha_n,$$

where α_j is a simple proper arc of T' satisfying $i(a_1, \alpha_j) \leqslant 1$ for each $1 \leqslant j \leqslant n$.

Proof. Suppose $i(a_1, \alpha_j) \geqslant 2$ for some $1 \leqslant j \leqslant n$, then we may perform the surgery shown and described in Figure 8. The resulting curve a satisfies $i(a_1, a) = 1$ implying it is in minimal position with respect to a_1 allowing us to conclude that $a \neq a_1$. As this surgery occurred in T', it is clear that $a \subset T'$. Furthermore, as a is obtained from a surgery on a_1 and a_2, $a \subset T$. Now T and T' share two distinct simple closed curves implying $T = T'$, which is a contradiction. □

Proof of Proposition 4.1. Label the vertices of the empty triangle as $a_1, a_2, a_3 \in \mathcal{TC}(S)$. We will show that if $i(a_1, a_2) > 1$, then $i(a_1, a_3) = 1$.

Let $T_i \subset S$ be the punctured torus containing a_j and a_k for $i \neq j \neq k \in \{1, 2, 3\}$. Then in light of Lemma 4.2, it is enough to show that $a_3 \cap T_3$ has a single component.

Note that we can guarantee that the collection of curves $\{a_i\}$ are in pairwise minimal position in S as well as in all the T_i by taking geodesic representatives of each a_i and ∂T_i in a complete hyperbolic metric on S.

As above, we will work in coordinates, labeling arcs and curves in T_3 by extended rational numbers $\frac{p}{q} \in \mathbb{Q} \cup \infty$. We will assume a_1 is the $\frac{1}{0}$-curve in T_3. Since $i(a_1, a_3) > 0$, Lemma 4.2

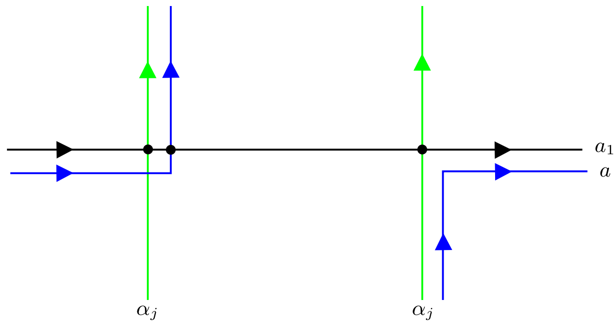

FIGURE 8 (colour online). *Shown here are two consecutive intersections between a_1 and α_j and the curve a resulting from the following surgery: Begin at the intersection on the left, follow α_j to the second intersection, and then following a_1 back to the first intersection. Note that $i(a_1, a) = 1$.*

guarantees that there exists a component α of $a_3 \cap T_3$ that intersects a_1 exactly once; without loss of generality, we may assume α is the $\frac{0}{1}$-arc. As $i(a_1, a_2) \neq 1$, we know a_2 is not the $\frac{0}{1}$-curve; in particular, $i(a_2, \alpha) = 1$ by Lemma 4.2 as $a_2, a_3 \subset T_1$. This forces a_2 to be the $\frac{1}{m}$-curve for some $m \in \mathbb{Z}$ with $|m| \geqslant 2$.

We now want to rule out the existence of other components β of $a_3 \cap T_3$. Any such β intersects both a_1 and a_2 once. For if $i(a_1, \beta) = 0$, then β is the $\frac{1}{0}$-arc, implying $i(a_2, \beta) > 1$, contradicting Lemma 4.2. So, we can conclude $i(a_1, \beta) = 1$, again by Lemma 4.2. Similarly, we have $i(a_2, \beta) = 1$.

Suppose first that β is not isotopic to α. Then as $i(a_1, \beta) = 1$, we know β is the $\frac{n}{1}$-arc for some $n \neq 0 \in \mathbb{Z}$. We now have two integers m, n satisfying

$$|m \cdot n - 1| = 1,$$

as $i(a_2, \alpha) = 1$. Since $n \neq 0$,

$$(m, n) \in \{(2, 1), (-2, -1)\}$$

so that $|m| \geqslant 2$.

Consider the case $(m, n) = (2, 1)$, where a_2 is the $\frac{1}{2}$-curve and β is the $\frac{1}{1}$-arc (see Figure 9(b)). Choose orientations for a_1, a_2, and a_3. As a_1 and a_3 live in a punctured torus, we know that the orientations of the intersections of α and β with a_1 must agree (as in Figure 9(b)). However, this forces the orientations of the intersections of α and β with a_2 to disagree, which contradicts a_2 and a_3 being contained in the punctured torus T_1. A similar argument implies that $(m, n) \neq (-2, -1)$.

We have now shown that all the components of $a_3 \cap T_3$ are isotopic to the $\frac{0}{1}$-arc in T_3. When ∂T_2 and ∂T_3 are put in minimal position, the intersection $T_2 \cap T_3$ must then be exactly as shown in Figure 9(c), since any component of $T_2 \cap T_3$ must contains some component of $a_3 \cap T_3$. Then $R = T_2 \cap (S \smallsetminus T_3)$ is a rectangle, and $a_3 \cap R$ is a collection of parallel arcs. Since S is orientable, the parallel arcs in $a_3 \cap T_3$ and $a_3 \cap R$ glue to a collection of parallel loops. But a_3 is supposed to be a simple closed curve, so $a_3 \cap T_3$ must have a single component intersecting a_1 once. □

5. *Automorphisms as mapping classes*

In this section, we complete the proof of Theorem 1.1, that is to say that the natural homomorphism $\mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{CB}(S))$ is a surjection.

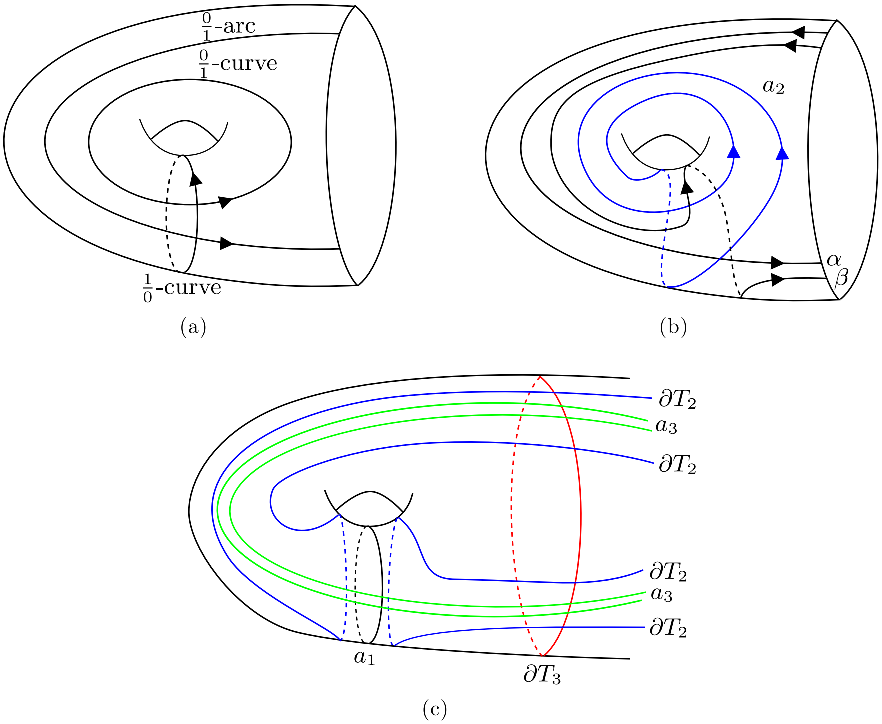

FIGURE 9 (colour online). (a) *A reference torus.* (b) *Drawn here is a_2, α, and β with a_2 given an arbitrary orientation. As a_1 and a_3 live in a torus, the orientations of the intersections of α and β with the $\frac{1}{0}$-curve must agree.* (c) *The region containing a_1 that is bounded by the ∂T_2 and ∂T_3 is the intersection $T_2 \cap T_3$.*

Let f be an automorphism of $\mathcal{CB}(S)$. By Proposition 3.1, f permutes the small compression bodies $S[a] \in \mathcal{CB}(S)$, so the formula $f(S[a]) = S[f_*(a)]$ defines a map

$$f_* : \{\text{simple closed curves on } S\} \longrightarrow \{\text{simple closed curves on } S\}.$$

Moreover, Proposition 3.1 says that a is nonseparating if and only if $f_*(a)$ is. Since a nonseparating curve $a \subset S$ is contained in a punctured torus $T \subset S$, if and only if $S[\partial T] \subset S[a]$, f_* preserves when a collection of nonseparating curves is contained in a punctured torus. So, f_* extends to an automorphism of the torus complex $\mathcal{TC}(S)$. As every automorphisms of $\mathcal{TC}(S)$ agrees with a mapping class (Theorem 1.2), the action of f on the set of nonseparating small compression bodies agrees with a mapping class.

By postcomposing f with a mapping class, we obtain an automorphism of $\mathcal{CB}(S)$ fixing all nonseparating small compression bodies. We claim the following.

PROPOSITION 5.1. *The only automorphism of $\mathcal{CB}(S)$ that fixes all nonseparating small compression bodies is the identity.*

This will imply that our f above agrees with a mapping class, and will finish the proof of Theorem 1.1. To prove Proposition 5.1, we must set up some terminology.

Let Σ be an orientable finite-type surface, possibly with boundary. The *curve graph* $\mathcal{C}(\Sigma)$ is the graph whose vertices are isotopy classes of essential nonperipheral simple closed curves in Σ and where edges connect pairs of simple closed curves that intersect minimally. (Note that unless S is a torus, a punctured torus or a 4-holed sphere, there are pairs of disjoint curves on

S, so then 'intersect minimally' means disjoint.) The curve graph $\mathcal{C}(\Sigma)$ has a natural metric $d_{\mathcal{C}}$, determined by setting each edge to have length one. Masur–Minksy [**12**, Proposition 4.6] have shown that $\mathcal{C}(\Sigma)$ has infinite diameter. Let F be an interior boundary component of a separating small compression body $S[a]$ and let D_a be a properly embedded disk bounded by a. Let $\Sigma_F \subset S$ be the component of $S \smallsetminus a$ such that $\Sigma_F \cup D_a$ is isotopic to F within $S[a]$. Note that as these surfaces are incompressible in $S[a]$, the isotopy gives a canonical identification

$$\mathcal{C}(\Sigma_F \cup D_a) \xrightarrow{\cong} \mathcal{C}(F).$$

The *interior boundary projection* from S to F is the multivalued function

$$\pi_F : \mathcal{C}(S) \longrightarrow \mathcal{C}(\Sigma_F \cup D_a) \cong \mathcal{C}(F)$$

defined as follows: let $b \in \mathcal{C}(S)$ and assume b is in minimal position with a. Now,

(i) if $b \in \mathcal{C}(\Sigma_F) \subset \mathcal{C}(S)$, let $\pi_F(b) = b$;
(ii) if $b \cap \partial\Sigma_F$ is nonempty, then for each arc β of $b \cap \Sigma_F$, let $\beta_1, \beta_2 \in \mathcal{C}(\Sigma_F)$ be the two components of the boundary of a regular neighborhood of $\beta \cup \partial\Sigma_F$; then,

$$\pi_F(b) = \bigcup_{\beta \subset b \cap \Sigma_F} \{\beta_1, \beta_2\};$$

(iii) otherwise $\pi_F(b) = \emptyset$.

For the familiar, this is the same as the *subsurface projection* from S to Σ_F, except that at the end we cap off the boundary of Σ_F with a disk. The only fact we will need is the following.

LEMMA 5.2. *If $S[a], S[b]$ are both contained in an S-compression body C, there is component $F \subset \partial_- S[a]$ and an element $m \in \pi_F(b)$ that bounds a disk in $C \smallsetminus S[a]$.*

Proof. The surgery of b and $\partial\Sigma_F$ presented in Lemma 2.7(2) gives a meridian for C that is contained in Σ_F, and isotoping this to F gives an element of $\pi_F(b)$. □

We can now prove Proposition 5.1.

Proof. The proof will proceed in three stages: first, we show that our automorphism f fixes all compression bodies that contain a nonseparating meridian, then we show that this implies that f fixes all small compression bodies (including the separating ones), and then we show that f is the identity.

LEMMA 5.3. *An automorphism of $\mathcal{CB}(S)$ fixing every nonseparating small compression body fixes every compression body containing a nonseparating meridian.*

Proof. If C is an S-compression body containing a nonseparating meridian, then $C = S[a_1, \ldots, a_m]$, where each a_i is nonseparating (Corollary 2.12). If $f \in \mathrm{Aut}(\mathcal{CB}(S))$ fixes every nonseparating small compression body, then $S[a_i] \subset f(C)$ for $1 \leqslant i \leqslant m$. In particular, $C \subset f(C)$, but $\mathfrak{h}(C) = \mathfrak{h}(f(C))$ (Proposition 3.2) forcing $C = f(C)$. □

LEMMA 5.4. *An automorphism of $\mathcal{CB}(S)$ that fixes every compression body that contains a nonseparating meridian fixes every small compression body.*

Proof. By Proposition 3.1, the set of small compression bodies is invariant, so we must only show that separating small compression bodies are not nontrivially permuted. So, we claim that if $S[a]$ and $S[b]$ are distinct separating small compression bodies, there is a compression body C that has a nonseparating meridian such that

$$S[a] \subset C, \quad \text{but} \quad S[b] \not\subset C.$$

This will prove the lemma, since if $f(C) = C$, then we cannot have $f(S[a]) = S[b]$.

From the definition, it is easy to see that $\pi_F(b)$ has finite diameter in $\mathcal{C}(F)$, so that we can choose a nonseparating curve $c \in \mathcal{C}(F)$ satisfying

$$d_{\mathcal{C}}(c, \pi_F(b)) \geqslant 2. \tag{5.1}$$

Let C be obtained by gluing $F[c]$ to $S[a]$ (see Corollary 2.4). If $S[b] \subset C$, Lemma 5.2 gives some $m \in \pi_F(b)$ that bounds a disc in $C \setminus S[a] = F[c]$. (Note that the m given must lie on F, since the other component of $\partial_- S[a]$ is incompressible in C.) But by Lemma 2.5, every meridian in $F[c]$ is disjoint from c, so this would violate (5.1). □

Finally, recall that the *disk set* of a compression body C, denoted $\mathcal{D}(C)$, is the collection of all (isotopy classes of) meridians in C. Then,

(a) $S[a] \subset C$ if and only if $a \in \mathcal{D}(C)$; and
(b) $\mathcal{D}(C) = \mathcal{D}(D)$ if and only if C and D are isomorphic (Corollary 2.2).

Therefore, an automorphism fixing every small compression body is the identity. □

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