Computations of Mather Minimal Log Discrepancies

by

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To my parents and my wife

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ABSTRACT

Shokurov proved that certain conjectures for Minimal Log Discrepancy imply the termination of Minimal Model Program. The effort towards proving those conjectures directly has not been very successful. In this context, the notion of Mather Minimal Log Discrepancy was introduced by Ishii in recent papers, in a way similar to the usual Minimal Log Discrepancy. It is not surprising that their properties are closely related, as shown in some recent papers. In this thesis, we compute this invariant via jet schemes and arc spaces for toric varieties and very general hypersurfaces.

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CHAPTER I

Introduction

1.1 History and background

One of the fundamental objects of research in algebraic geometry is the classification of algebraic varieties up to isomorphism classes. However, the difficulty of this objective has made people consider variations and simplifications of this problem. Classification up to birational classes is one that has caught much attention from mathematicians.

The so-called Minimal Model Program (MMP) is exactly aimed at taking each algebraic variety to the ”simplest” birationally equivalent model through a conjecturally finite sequence of ”decreasing” birationally equivalent models. The program originates in the study of the classification of surfaces due to geometers of Italian School around 1900. However, the major modern concepts were introduced in the 1980s.

It was noticed that even if one is only interested in classifying smooth varieties, one has to also include algebraic varieties with mild singularities. One way to measure singularities is the notion of *minimal log discrepancy*, which was introduced by Shokurov in [Sho88]. It is connected to solving the termination problem of flips in the Minimal Model Program.

Recently, the notion of *Mather minimal log discrepancy* was introduced by Ishii in [Ish13]. It is closely related to the minimal log discrepancy and they share many similar properties. But the Mather minimal log discrepancy has the advantage of being easier to compute than the minimal log discrepancy. This thesis is concerned with the computation of Mather minimal log discrepancy in the context of toric varieties and very general hypersurfaces.

Let us start by introducing the minimal log discrepancy. Let X be a normal $\mathbb{Q}$-Gorenstein variety over an algebraically closed field k of characteristic zero and K_X be the canonical divisor on X. The $\mathbb{Q}$-Gorenstein condition means that there is some positive integer r such that rK_X is a Cartier divisor. Given a birational morphism $f : Y \longrightarrow X$ with Y normal, the *relative canonical divisor*, denoted by $K_{Y/X}$, is the unique $\mathbb{Q}$-divisor on Y which is supported on the exceptional locus and linearly equivalent to $K_Y - f^*(K_X)$. Here $f^*(K_X)$ is defined as $\frac{1}{r}f^*(rK_X)$ where r is as above. If both X and Y are smooth varieties, $K_{Y/X}$ is an effective divisor locally defined by the Jacobian determinant of f.

We also need the following notion. A *divisor over* X is a prime divisor on a normal variety Y with a birational morphism $f : Y \longrightarrow X$ (Y is called a *birational model* over X). This divisor E defines a discrete valuation ord_E on $K(Y) = K(X)$. Two divisors over X are equivalent if they define the same valuation on $K(X)$. In particular if $g : Z \longrightarrow Y$ is a birational morphism with Z normal, then E is equivalent to its strict transform on Z. For a nonzero ideal $\mathfrak{a}$ on X, by pursuing higher birational models, we may assume that E is a divisor on a smooth variety Y over X such that the birational morphism $f : Y \longrightarrow X$ factors through the blow-up along $\mathfrak{a}$. Hence $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$ for an effective divisor D on Y, and we have $\mathrm{ord}_E(\mathfrak{a})$ equals the coefficient of E in D. The *center* of E is the closure of $f(E)$ in X and is denoted by $c_X(E)$. For each divisor E over X as above, we denote by k_E the coefficient of E in $K_{Y/X}$; it is known to be independent of choice of Y and $E \subset Y$.

For a divisor E over X and an ideal $\mathfrak{a}$ in $\mathcal{O}_X$, the *log discrepancy* of $(X, \mathfrak{a})$ with respect to E is defined as

$$a(E; X, \mathfrak{a}) := k_E - \mathrm{ord}_E(\mathfrak{a}) + 1.$$

For each closed subset W in X, the minimal log discrepancy of $(X, \mathfrak{a})$ with respect to W is

$$\mathrm{mld}(W; X, \mathfrak{a}) := \min\{a(E; X, \mathfrak{a}) | c_X(E) \subset W\}.$$

It is well known that certain conjectures on the minimal log discrepancy imply the termination of MMP (see [Sho04]). However, not much about minimal log discrepancy is known compared to other invariants defined in similar settings such as the log canonical threshold. An introduction to minimal log discrepancies can be found in [Amb06].

The main tools of this thesis are jet schemes and arc spaces. The concepts were introduced by Nash in [Nas64]. They were further developed and attracted a lot of attention due to the theory of motivic integration by Kontsevich, Denef and Loeser in [Kon95] and [DL99].

We now describe our basic setting. Let k be an algebraically closed field of characteristic zero (later assumed to be $\mathbb{C}$) and X be an algebraic variety over k. For each nonnegative integer m, the m^{th} *jet scheme* of X, denoted by X_m, parameterizes all m-jets on X, that is, morphisms Spec $k[t]/(t^{m+1}) \longrightarrow X$. In particular, X_0 is isomorphic to X and X_1 is the total tangent space of X. The higher jet schemes X_m are higher order generalizations of the total tangent space. There are naturally truncation maps $\pi_{m,n} : X_m \longrightarrow X_n$ for every pair m, n with $m \geq n$ induced by the canonical truncation maps

$$k[t]/(t^{m+1}) \longrightarrow k[t]/(t^{n+1}).$$

When $n = 0$ we simply write π_m instead of $\pi_{m,0}$. Vaguely speaking the variety X has "good" singularities if the dimension of X_m is small. In fact, the dimension of X_m is at least $(m+1)\dim(X)$, which is attained when X is smooth.

The *arc space* X_∞ is the projective limit of the projective system $\{X_m\}_{0\leq m<\infty}$. It parameterizes germs of formal arcs on X, that is, morphisms Spec $k[\![t]\!] \longrightarrow X$. Unlike jet schemes, the arc space X_∞ is not usually of finite type (infinite-dimensional). We denote by ψ_m the induced map from X_∞ to X_m. When $m = 0$, we simply denote the map by π. Contrary to the intuition about X_m above, X has "good" singularities if the dimension of $\psi_m(X_\infty)$ is large. In fact, the dimension of $\psi_m(X_\infty)$ is at most $(m+1)\dim(X)$. When X is smooth, $\psi_m(X_\infty)$ is equal to X_m.

One is usually interested in a special type of subsets of X_∞ called *contact loci*, consisting of arcs with specified order along an ideal sheaf. More precisely, let $\mathfrak{a}$ be an ideal sheaf on X and γ : Spec $k[\![t]\!] \longrightarrow X$ be an arc on X. The *order* $\mathrm{ord}_\gamma(\mathfrak{a})$ of γ along an ideal $\mathfrak{a}$ is the t-order of the inverse image $\mathfrak{a}\cdot k[\![t]\!]$ of $\mathfrak{a}$ via γ. A contact locus of the form $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ is the subset of X_∞ consisting of those arcs γ with $\mathrm{ord}_\gamma(\mathfrak{a}) \geq m$. In this dissertation, we will compute the dimension of the image of special subsets of X_∞ in X_m by decomposing the subsets into contact loci.

Mustaţă first explored the link between jet schemes and arc spaces with certain invariants of singularities (more specifically, the log canonical threshold) in [Mus01] and [Mus02]. This link has been studied further afterwards. In particular, Ein, Mustaţă and Yasuda described the minimal log discrepancy in terms of jet schemes and arc spaces in [EMY02]. While this is a general description, it is effective in proving some of the existing conjectures on the minimal log discrepancy (Inversion of Adjunction and the semicontinuity conjectures) only in the case of smooth varieties (see [EMY02, Theorem 0.1, Theorem 0.3]), and more generally, in the case of local complete intersection varieties (see [EM04, Theorem 1.1, Theorem 1.2]).

The notion of Mather minimal log discrepancy was introduced in this context. Let

$f : Y \longrightarrow X$ be a resolution of singularities so that Y is a sufficiently "high" birational model over X (will be made clear in Chapter II). It is defined in a similar way to the usual minimal log discrepancy (by simply replacing the relative canonical divisor with the *Mather discrepancy divisor*) but it is much easier to describe in terms of jet schemes and arc spaces. The Mather minimal log discrepancy for a closed point x of a variety X is denoted by $\widehat{\mathrm{mld}}(x; X)$. Of the many nice properties of the Mather minimal log discrepancy, one of the most important is Inversion of Adjunction ([dFD11, Theorem 4.10] and [Ish13, Proposition 3.10]).

Unlike the usual relative canonical divisor, the Mather discrepancy divisor is defined for arbitrary varieties rather than just normal $\mathbb{Q}$-Gorenstein varieties. When both notion are defined, the two differ by the pull back of a certain ideal sheaf ([Ish13, 2.2]). In particular, Mather minimal log discrepancy is always larger than or equal to the usual minimal log discrepancy. Their relation has been further studied in [IR13], [EI15] and [dFT16]. We note that contrary to usual minimal log discrepancies, the variety has "good" singularities when Mather minimal log discrepancies are small (see [Ish13, Theorem 4.7] for more precise description). We recall these results in Chapter III.

The Mather minimal log discrepancy is related to jet schemes and arc spaces by [Ish13, Lemma 4.2]. We review this relation carefully in Chapter III. This allows us to compute Mather minimal log discrepancies by computing the dimension of certain subsets of jet schemes instead.

In this dissertation, we compute the Mather minimal log discrepancy via jet schemes and arc spaces for a closed point in two different classes of varieties: toric varieties and hypersurfaces. Our result for hypersurfaces requires a generality condition on the coefficients of the defining equations. It only gives a lower bound for the Mather minimal log discrepancy, which gives a precise formula in many examples.

1.2 Outline and main results

Now let us give an outline of the thesis.

In Chapter II we give an overview of jet schemes and arc spaces. We start with the definition of jet schemes. A large portion is devoted to proving the existence of jet schemes. The proof shows that the jet schemes of an affine variety X are also affine and we get explicit defining equations for X_m. In fact, if $X = \mathbb{V}(f_1, \ldots, f_r)$ in $\mathbb{A}^n$, the jet scheme X_m is the set of m-jets γ that correspond to morphisms of k-algebras

$$\gamma^* : k[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \longrightarrow k[t]/(t^{m+1}).$$

So if we write

$$f_l(\sum_{j=0}^{m} x_1^{(j)} t^j, \ldots, \sum_{j=0}^{m} x_n^{(j)} t^j) = \sum_{j=0}^{m} G_l^j t^j \text{ in } k[t]/(t^{m+1}),$$

where each G_l^j is a polynomial in $(x_i^{(j)})_{1 \leq i \leq n, 0 \leq j \leq m}$, then we deduce

$$X_m = \text{Spec } k\left[x_i^{(j)} | 1 \leq i \leq n, 0 \leq j \leq m\right] \Big/ \Big(G_l^j | 1 \leq l \leq r, 0 \leq j \leq m\Big).$$

This will be important for our analysis in Chapter V.

Next we review arc spaces and cylinders (especially contact loci). The arc space of a variety X is the projective limit of the projective system $\{X_m\}_{0 \leq m < \infty}$ of jet schemes and cylinders are inverse images of constructible subsets of X_m in the arc space X_∞.

In Chapter III we review the basics about Mather minimal log discrepancy. We start by defining the notion of Mather discrepancy divisor through Nash blow-ups. Then we prove the following proposition.

Proposition I.1. *([Ish13, Lemma 4.2]) Let X be a variety over an algebraically closed field k of characteristic zero. If x is a closed point of X, then we have*

$$\widehat{\text{mld}}(x;X) = \lim_{m\to\infty}((m+1)\dim(X) - \dim(\psi_m(\pi^{-1}(x)))).$$

We show that the above limit exists and is a finite number by showing that the sequence $m\dim(X) - \dim(\psi_m(\pi^{-1}(x)))$ is stationary for $m \gg 0$. The key point for the examples considered in Chapter IV and Chapter V, is to compute/bound $\dim(\psi_m(\pi^{-1}(x)))$ for m large enough.

Chapter IV is devoted to the study of the Mather minimal log discrepancy for toric varieties. Recall that a toric variety is a normal algebraic variety containing a torus T as an open dense subset such that the group action of the torus on itself extends to an action on the entire variety. The geometry of a toric variety is completely determined by the fan associated to the variety. Thus, it is not surprising that our formula for the Mather minimal log discrepancy is given in terms of the combinatorial data of this fan.

We compute the Mather minimal log discrepancy of a toric variety at a closed point x. The question is local so we assume $X = X(\sigma)$ is the affine toric variety associated to the cone $\sigma \subset N_{\mathbb{R}} := N\otimes_{\mathbb{Z}}\mathbb{R}$, where $N \cong \mathbb{Z}^n$ is the lattice of σ. We further assume that σ spans $N_{\mathbb{R}}$. First, we consider the case when x is a torus-invariant point. By Proposition III.22, the key is to compute $\dim(\psi_m(\pi^{-1}(x)))$ for m large enough. This space is decomposed into T_m-orbits. Here T_m is the m^{th} jet scheme of the torus T in X and it naturally acts on X_m. We use the fact that those orbits correspond to lattice points in the interior of σ. The characterization of orbits follows from the work of Ishii ([Ish04]).

The problem thus comes down to finding the dimension of each T_m orbit, which is in turn done by computing the dimension of the stabilizer.

In order to state our result, we introduce some notation. Let n be the dimension of X

and $M = N^\vee$ be the dual lattice. We define the dual space $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ and the dual cone $\sigma^\vee := \{u \in M_\mathbb{R} | \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}$.

With the above notation, we show the dimension of the T_m-orbit associated to a lattice point a in the interior of σ is equal to

$$(m+1)n - \min\Big\{\sum_{i=1}^{n} \langle a, u_i\rangle | u_1, \ldots, u_n \text{ span } M_\mathbb{R}, \text{ with } u_i \in M \cap \sigma^\vee \text{ for each } i\Big\},$$

where the minimum is run over all linearly independent sets of vectors $\{u_1, \ldots, u_n\}$ in $M \cap \sigma^\vee$. Now we just need to let a vary and take the maximum. Hence we get the following theorem:

Theorem I.2. *Let X be an affine toric variety associated to a cone σ of dimension n over an algebraically closed field k of characteristic zero. Let N be the lattice of σ and M be the dual lattice. If σ spans $N_\mathbb{R}$ and x is the torus-invariant point, then we have*

$$\widehat{\text{mld}}(x; X) = \min_{a \in \text{Int}(\sigma) \cap N} \Big\{ \min\Big\{\sum_{i=1}^{n} \langle a, u_i\rangle | u_1, \ldots, u_n \text{ span } M_\mathbb{R},\ u_i \in M \cap \sigma^\vee \text{ for each } i\Big\}\Big\},$$

where the second minimum is taken over all linearly independent sets of vectors $\{u_1, \ldots, u_n\}$ in $M \cap \sigma^\vee$.

We use the theorem to compute $\widehat{\text{mld}}(x; X)$ in some examples. For example, we show that if X is a toric surface, then $\widehat{\text{mld}}(x; X) = \dim(X)$ (which is 2). In higher dimension, the same conclusion holds if the torus-fixed point x is an isolated singularity point and X is simplicial. We also give some examples where $\widehat{\text{mld}}(x; X) \neq \dim(X)$.

We conclude Chapter IV by considering arbitrary closed point on a toric variety X. Recall that the set of closed points of X is a disjoint union of T-orbits associated to faces of the cone σ. Each orbit is generated by a distinguished point associated to the corresponding face. Therefore, the problem reduces to computing the Mather minimal log discrepancy at these distinguished points, and it is further reduced to the case of a torus-invariant point in the following sense:

Theorem I.3. *Let $X = X(\sigma)$ be an affine toric variety of dimension n over an algebraically closed field k of characteristic zero. Let τ be a face of σ of dimension $k < n$ and x_τ be the distinguished point associated to τ. If Y is the k-dimensional affine toric variety associated to the cone τ and y is the torus-invariant point of Y, then we have*

$$\widehat{\mathrm{mld}}(x_\tau; X) - n = \widehat{\mathrm{mld}}(y; Y) - k.$$

We consider the case of very general hypersurfaces in Chapter V. A hypersurface is a closed subvariety of an affine space that is defined by a single polynomial equation. Let $f = \sum_{i=1}^{N} a_{I^i} x^{I^i}$ be the defining equation of a hypersurface $X \subset \mathbb{A}^{n+1}$, where I^i are multi-indices and x^{I^i} stands for $\Pi_{j=1}^{n+1} x_j^{I_j^i}$. The *support* of f is the set $A := \{I^1, \ldots, I^N\} \subset \mathbb{Z}^{n+1}$. When $A \neq \emptyset$, the *dimension* of A is the dimension of the linear span over $\mathbb{Q}$ of the convex hull of $A - a$, for any $a \in A$. Following from the result of Yu ([Yu16, Theorem 3]), we deduce that for a support A such that $\dim(A) \geq 2$ or $\dim(A) = 1$ and the convex hull of A contains exactly two integral points, and for general coefficients a_{I^i}, X is an integral hypersurface. Then, under a certain generality condition, we give a lower bound for the Mather minimal log discrepancy of X at the origin. As in the case of toric varieties, we write $\psi_m(\pi^{-1}(0))$ as a disjoint union, up to the image of a thin set, of subsets C_α^m, with $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ running over all $(n+1)$-tuples of positive integers. For simplicity, we define the *product* of an $(n+1)$-tuple α with a multi-index I as $\alpha \cdot I := \sum_{j=1}^{n+1} \alpha_j I_j$. An $(n+1)$-tuple α is called *feasible* if $\min_{1 \leq i \leq N}\{\alpha \cdot I^i\}$ is attained by at least two different i's. We show that $C_\alpha^m = \emptyset$ if α is not feasible; when f has a fixed support and very general coefficients, $\dim(C_\alpha^m)$ is bounded above by

$$mn - \sum_{j=1}^{n+1}(\alpha_j - 1) - 1 + \min_{1 \leq i \leq N}\{I^i \cdot \alpha\} - \min_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n+1 \text{ with } I_j^i > 0}} \{I^i \cdot \alpha - \alpha_j\}.$$

By taking the maximum over all feasible α's, we obtain the following theorem:

Theorem I.4. *Let $f = \sum_{i=1}^{N} a_{I^i} x^{I^i}$ be a polynomial with a fixed support A such that f has no constant term and that f is not divisible by any x_i, and let X be the hypersurface defined by f. If A is 1-dimensional and the convex hull contains only two integral points, or if A has dimension ≥ 2, then for very general coefficients $(a_{I^i})_{1 \leq i \leq N}$, the hypersurface X is integral and we have*

$$\widehat{\mathrm{mld}}(0; X) \geq \min_{\alpha} \{ \sum_{j=1}^{n+1} (\alpha_j - 1) + 1 - \min_{1 \leq i \leq N} \{ I^i \cdot \alpha \} + \min_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n+1 \text{ with } I^i_j > 0}} \{ I^i \cdot \alpha - \alpha_j \} \} + n,$$

where the first minimum is taken over all feasible $(n+1)$-tuples α.

In spite of the fact that the theorem only gives a lower bound of Mather minimal log discrepancy, we can use the proof of the above result to show that the inequality is actually an equality in many cases. We end the chapter with various examples. These examples show that the lower bound can be attained in many cases, but we also see that the inequality in the theorem can be strict.

CHAPTER II

Jet Schemes and Arcs Spaces

In this chapter we review basic properties of jet schemes and arc spaces that we need in the following chapters. We mostly follow [EM09]. For more details, see [Mus14], [DL99] and [dF16].

2.1 Jet schemes

A variety is an integral, separated scheme of finite type over a field. Let k be an algebraically closed field of arbitrary characteristic and X be a scheme of finite type over k. For each nonnegative integer m, we define the m^{th} *jet scheme* of X, denoted by X_m, to be a scheme over k such that for every k-algebra A we have a functorial bijection

$$\mathrm{Hom}_{\mathsf{Sch}/k}(\mathrm{Spec}(A), X_m) \cong \mathrm{Hom}_{\mathsf{Sch}/k}(\mathrm{Spec}\ A[t]/(t^{m+1}), X). \tag{2.1}$$

Note that if jet schemes exist, then they are unique up to a canonical isomorphism since the bijection (2.1) describes the functor of points of X_m. In particular, each element of the left-hand side of the bijection (2.1) is an A-valued point of X_m, which is also called an A-valued m-jet of X. A k-valued point of X_m is simply called an m-jet of X. Clearly when $m = 0$ we have $X_0 \cong X$. Assuming the existence of all X_m, the canonical truncation map $A[t]/(t^{m+1}) \to A[t]/(t^{p+1})$ for $m > p$ induces the map

$$\mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X) \longrightarrow \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{p+1}), X).$$

This induces via the bijection (2.1) a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$. We denote this map by π_m when $p = 0$. These canonical projections satisfy the obvious compatibilities $\pi_{p,q} \circ \pi_{m,p} = \pi_{m,q}$ for $m > p > q$.

Remarks II.1. Assuming for now the existence of jet schemes, the following facts follow easily from the definition:

(i) If $f : X \to Y$ is a morphism of schemes of finite type over k, then there is an induced morphism of jet schemes $f_m : X_m \to Y_m$. Note that the induced maps f_m are compatible with the canonical projections $\pi_{p,q}$, i.e. $\pi_{m,p} \circ f_m = f_p \circ \pi_{m,p}$.

(ii) For schemes X and Y of finite type over k, there is a canonical isomorphism

$$(X \times Y)_m \cong X_m \times Y_m,$$

for every $m \geq 0$.

(iii) If G is a group scheme over k acting on a scheme X of finite type over k, then G_m is also a group scheme over k and it acts on X_m.

We now prove the existence of jet schemes. The ingredients of the proof will be important to our analysis in the following chapters. We begin with the following lemma:

Lemma II.2. *Suppose that X_m exists and let $\pi_m : X_m \longrightarrow X$ be the canonical projection. If U is an open subset of X, then U_m also exists and is isomorphic to $\pi_m^{-1}(U)$.*

Proof. Let A be a k-algebra and consider an A-valued m-jet $\gamma : \text{Spec }(A[t]/(t^{m+1})) \to X$. Then $\pi_m(\gamma)$ is an A-valued point of X obtained by composing with the map

$$\text{Spec }(A) \to \text{Spec }(A[t]/(t^{m+1})).$$

Clearly γ factors through U if and only if $\pi_m(\gamma)$ factors through U. This establishes the bijection (2.1) for $\pi_m^{-1}(U)$. Hence U_m exists and is isomorphic to $\pi_m^{-1}(U)$. □

Proposition II.3. *If X is a variety over k, then the jet scheme X_m exists for every $m \geq 0$. Moreover, X_m is of finite type over k.*

Proof. First we assume that X is affine. Choose an embedding $X \hookrightarrow \mathbb{A}^n$ and let $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$ be generators for the ideal defining X. For a k-algebra A, consider an A-valued m-jet γ of X represented by γ : Spec $A[t]/t^{m+1} \to X$. Giving γ is equivalent to giving a morphism of k-algebras

$$\gamma^* : k[x_1, \ldots, x_n]/[g_1, \ldots, g_r] \longrightarrow A[t]/(t^{m+1}).$$

Let us write

$$\gamma^*(x_i) = \sum_{j=0}^{m} x_i^{(j)} t^j, \text{ for } 1 \leq i \leq n.$$

They should satisfy $g_l(\gamma^*(x_1), \ldots, \gamma^*(x_n)) = 0$ in $k[t]/t^{m+1}$ for $1 \leq l \leq r$. If we write

$$g_l(\sum_{j=0}^{m} x_1^{(j)} t^j, \ldots, \sum_{j=0}^{m} x_n^{(j)} t^j) = \sum_{j=0}^{m} G_l^{(j)}(\underline{x}) t^j \pmod{t^{m+1}},$$

we see that

(2.2) $$X_m \cong \text{Spec } k[x_i^{(j)} | 1 \leq i \leq n, 1 \leq j \leq m]/(G_l^{(j)} | 1 \leq l \leq r, 0 \leq j \leq m).$$

In particular, we conclude the jet schemes of an affine scheme are also affine schemes, of finite type over k.

Consider now the general case, when X is an arbitrary scheme of finite type over k. Let $\{U_i\}_i$ be an affine open cover of X. According to what we just showed, for each U_i there is a jet scheme $(U_i)_m$. Moreover, Lemma II.2 shows that for each i and j, one has a canonical isomorphism $(\pi_m^{U_i})^{-1}(U_i \cap U_j) \to (\pi_m^{U_j})^{-1}(U_i \cap U_j)$, since both are isomorphic to $(U_i \cap U_j)_m$. The scheme X_m is constructed by gluing the schemes $(U_i)_m$ along these isomorphisms. In addition, the projections $\pi_m^{U_i}$ glue together to give $\pi_m^X : X_m \to X$. □

Remark II.4. It follows from the proof that the canonical projections $\pi_{m,p} : X_m \to X_p$ are affine morphisms.

Remark II.5. Another consequence of the above proof is that if $X \hookrightarrow \mathbb{A}^n$ is a closed immersion, then the induced morphism of jet schemes $X_m \hookrightarrow (\mathbb{A}^n)_m$ is also a closed immersion. Moreover, we deduce from the explicit description of the equations of X_m in $(\mathbb{A}^n)_m$ that more generally, if $X \hookrightarrow Y$ is a closed immersion then so is the induced map $X_m \to Y_m$.

Example II.6. The simplest (but important) example is $X = \mathbb{A}^n$. It follows immediately from equation (2.2) that $(\mathbb{A}^n)_m \cong \mathbb{A}^{(m+1)n}$. Furthermore, the canonical projections $\pi_{m,p}$ are just projections along certain coordinate planes.

Lemma II.7. *([EM09, Lemma 2.9]) If $f : X \to Y$ is an étale morphism, then for every $m \geq 0$ the following commutative diagram is Cartesian:*

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ {\scriptstyle \pi_m^X}\downarrow & & {\scriptstyle \pi_m^Y}\downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

Corollary II.8. *If X is a smooth variety of dimension n, then the canonical projections $\pi_{m,p}$ are locally trivial fibrations with fiber $\mathbb{A}^{(m-p)n}$. In particular, X_m is smooth of dimension $(m+1)n$.*

Proof. For every point $x \in X$, one can find an open subset $x \in U$ and an étale morphism $U \to \mathbb{A}^n$. Using Lemma II.7, the assertion is reduced to the case of an affine space, which follows from Example II.6. □

Suppose X is a closed subvariety of $\mathbb{A}^n$, whose ideal is generated by $g_1, \ldots, g_r$. For a k-algebra A, an A-valued m-jet is a morphism γ : Spec $A[t]/(t^{m+1}) \to X$. Giving this is equivalent to giving the corresponding morphism of k-algebras

$$\gamma^* : k[x_1, \ldots, x_n]/(g_1, \ldots, g_r) \longrightarrow A[t]/(t^{m+1}). \tag{2.3}$$

We often write the image of x_i under the above morphism as $\sum_{j=0}^{m} x_i^{(j)} t^j$ as in the proof of the existence of jet schemes.

2.2 Arc spaces and cylinders

We work in the same settings as in the previous section. Given a scheme X of finite type over k, we have a projective system

$$\ldots \longrightarrow X_m \longrightarrow X_{m-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 = X,$$

in which all morphisms are affine. Therefore, the projective limit exists in the category of k-schemes. The projective limit is denoted by X_∞ and it is called the *arc space* of X. Unlike the jet schemes, the arc space is typically not of finite type over k. We denote by ψ_m the canonical map $X_\infty \to X_m$. We also write $\pi := \psi_0 : X_\infty \to X_0 = X$ for the projection to the original scheme X.

It follows from the definition of jet schemes and projective limit that for every field extension K of k, we have functorial isomorphisms

$$\text{Hom}(\text{Spec}(K), X_\infty) \cong \varprojlim \text{Hom}(\text{Spec } K[t]/t^{m+1}, X) \cong \text{Hom}(\text{Spec } K[\![t]\!], X).$$

A k-valued point of X_∞ is called an arc on X and is represented by

$$\gamma : \text{Spec } k[\![t]\!] \longrightarrow X. \tag{2.4}$$

For every field extension K of k, a K-valued point of X_∞ is called an K-valued arc of X. From now on, whenever we deal with X_m and X_∞ we will restrict to their k-valued points. Since the jet schemes are of finite type over k this causes no ambiguity. Note that since we only consider the k-valued points, X_∞ is the set-theoretic projective limit of the X_m and the Zariski topology on X_∞ is the projective limit topology.

Remark II.9. As in the case of jet schemes, if $f : X \to Y$ is a morphism of schemes of finite type over k, then we have an induced map on the arc spaces $f_\infty : X_\infty \to Y_\infty$ that is compatible with canonical projections.

Remark II.10. For schemes X and Y of finite type over k, there is a canonical isomorphism $(X \times Y)_\infty \cong X_\infty \times Y_\infty$ and we have the following commutative diagram:

$$\begin{array}{ccc} (X \times Y)_\infty & \xrightarrow{\cong} & X_\infty \times Y_\infty \\ \downarrow{\scriptstyle \psi_m^{X\times Y}} & & \downarrow{\scriptstyle \psi_m^X \times \psi_m^Y} \\ (X \times Y)_m & \xrightarrow{\cong} & X_m \times Y_m. \end{array}$$

We now define the notion of cylinders. Recall that a *constructible set* in a scheme of finite type over k is a finite union of locally closed subsets. A *cylinder* in X_∞ is a subset of the form $C = \psi_m^{-1}(S)$, for some nonnegative integer m and some constructible subset S of X_m. The arc spaces are typically not of finite type over k. So far most study on arc spaces has been focusing on cylinders and their irreducible components.

There is a special type of cylinders, the *contact loci*, that will play an important role in what follows. To an ideal sheaf $\mathfrak{a}$, we associate subsets of arcs with prescribed vanishing order along $\mathfrak{a}$. More precisely, if $\gamma : \text{Spec } k[\![t]\!] \to X$ is an arc, the inverse image of $\mathfrak{a}$ is an ideal in $k[\![t]\!]$ generated by t^r, for some r (if the ideal is not zero). This r is *the order of* γ *along* $\mathfrak{a}$, denoted by $\text{ord}_\gamma(\mathfrak{a})$. When the inverse image is zero, we put $\text{ord}_\gamma(\mathfrak{a}) = \infty$. A contact locus is a subset of X_∞ of one of the following forms:

$$\text{Cont}^e(\mathfrak{a}) := \{\gamma \in X_\infty | \text{ord}_\gamma(\mathfrak{a}) = e\},$$

or

$$\text{Cont}^{\geq e}(\mathfrak{a}) := \{\gamma \in X_\infty | \text{ord}_\gamma(\mathfrak{a}) \geq e\}.$$

We can similarly define subsets of X_m with specified order along $\mathfrak{a}$, namely $\text{Cont}^e(\mathfrak{a})_m$ and $\text{Cont}^{\geq e}(\mathfrak{a})_m$, for $m \geq e$. It is clear that for every $m \geq e$, we have

$$\text{Cont}^e(\mathfrak{a}) = \psi_m^{-1}(\text{Cont}^e(\mathfrak{a})_m),\ \text{Cont}^{\geq e}(\mathfrak{a}) = \psi_m^{-1}(\text{Cont}^{\geq e}(\mathfrak{a})_m).$$

This implies that $\mathrm{Cont}^e(\mathfrak{a})$ is a locally closed set and $\mathrm{Cont}^{\geq e}(\mathfrak{a})$ is a closed set.

Definition II.11. Let X be a scheme of finite type over k of pure dimension d. A subset $A \subset X_\infty$ is *thin* if there is some closed subvariety S of X whose dimension is strictly less than d such that $A \subset S_\infty$. If a subset A is not thin, it is *fat*.

We need the following result for our discussion in the following chapters:

Lemma II.12. *([EM05, Proposition 5.10]) Let X be a variety over k of dimension d. Then*

(1) For every $m \geq 0$, we have

$$\dim(\psi_m(X_\infty)) \leq (m+1)d.$$

(2) For every $m,\ n \geq 0$ with $m \geq n$, the fibers of $\psi_m(X_\infty) \to \psi_n(X_\infty)$ are of dimension $\leq (m-n)d$.

Proof. Clearly assertion (1) follows from assertion (2) since $\psi_0(X_\infty) = X$. Moreover, it suffices to prove assertion (2) when $m = n+1$. The question is local, hence we may assume that X is affine.

Let X be a closed subscheme of $\mathbb{A}^N = \mathrm{Spec}\ k[x_1, \ldots, x_N]$ defined by an ideal generated by $g_1, \ldots, g_r$. Consider $\gamma \in X_n$ given by $u = (u_1, \ldots, u_N)$ where each $u_i \in k[t]$ has degree $\leq n$. Let $T = \mathrm{Spec}\ k[t]$ and Z be the subscheme of $T \times \mathbb{A}^N$ defined by the ideal

$$I_Z = (g_1(u + t^{n+1}x), \ldots, g_r(u + t^{n+1}x)).$$

Let Z' be a subscheme of $T \times \mathbb{A}^N$ defined by

$$I_{Z'} = (f | hf \in I_Z \text{ for some nonzero } h \in k[t]).$$

Then clearly $Z' \subset Z$, and by construction $Z = Z'$ over the generic point of T. Moreover, we claim that Z' is flat over T. It suffices to show that for every prime ideal $\mathfrak{p} = (P)$ of $k[t]$, $(\mathcal{O}_{T\times\mathbb{A}^N}/I_{Z'})_\mathfrak{p}$ is flat over $k[t]_\mathfrak{p}$. This comes down to computing

$$\mathrm{Tor}_1^{k[t]_\mathfrak{p}}((\mathcal{O}_{T\times\mathbb{A}^N}/I_{Z'})_\mathfrak{p}, k[t]_\mathfrak{p}/(f)k[t]_\mathfrak{p})$$

for every nonzero $f \in k[t]$ according to the local criterion for flatness ([Eis13, Theorem 6.8]) and the fact that $k[t]$ is a PID. Since

$$\begin{aligned} \operatorname{Tor}_1^{k[t]_{\mathfrak{p}}}((\mathcal{O}_{T\times\mathbb{A}^N}/I_{Z'})_{\mathfrak{p}}, k[t]_{\mathfrak{p}}/(f)k[t]_{\mathfrak{p}}) &\cong \operatorname{Tor}_1^{k[t]_{\mathfrak{p}}}(k[t]_{\mathfrak{p}}/(f)k[t]_{\mathfrak{p}}, (\mathcal{O}_{T\times\mathbb{A}^N}/I_{Z'})_{\mathfrak{p}}) \\ &= \{b \in (\mathcal{O}_{T\times\mathbb{A}^N}/I_{Z'})_{\mathfrak{p}} | fb = 0\} = 0, \end{aligned}$$

we conclude that Z' is flat over T.

Note that the generic fiber of Z over T is isomorphic to $X \times_k k(t)$. So the fiber of Z' over the origin is either empty or has dimension d. On the other hand, every element in the fiber of $\psi_{n+1}(X_\infty) \to \psi_n(X_\infty)$ over γ is the $(n+1)$-jet of an arc in X given by $u + t^{n+1}w$ for some $w \in (k[\![t]\!])^N$. Thus $g_i(u + t^{n+1}w) = 0$ for every i. By definition of $I_{Z'}$, we see that if $f \in I_{Z'}$, then $f(t, w) = 0$. This shows that the fiber over γ is embedded in the fiber of Z' over the origin of T. Hence its dimension is $\leq d$. $\square$

CHAPTER III

Mather Minimal Log Discrepancy

In this chapter we introduce the Mather minimal log discrepancy following [Ish13]. The definition is very similar to the usual minimal log discrepancy. Details on usual minimal log discrepancy and its relation to arc spaces can be found in [EMY02]. In Section 3.1 we define Mather minimal log discrepancy and in Section 3.2 we relate it to jet schemes and arc spaces.

3.1 Mather minimal log discrepancy

We start by introducing some notion.

Definition III.1. Let X be a variety over a field k and $f : Y \to X$ be a proper birational morphism of varieties, with Y normal. Each prime divisor E on Y gives a valuation ord_E on $K(Y) = K(X)$. Here E is called a *divisor over* X and we equate two divisors on two normal varieties over X if they give rise to the same valuation on X. The *center* of E is the closure of the image of E on X. A *divisorial valuation* on X is one of the form $v = q \cdot \text{ord}_E$ where q is a positive integer and E is a divisor over X.

Let X be a variety of dimension d over an algebraically closed field k of characteristic zero. For simplicity we write Ω_X for the sheaf of relative differentials $\Omega_{X/k}$. The projection

$$\pi : \mathbb{P}_X(\wedge^d \Omega_X) \longrightarrow X$$

is an isomorphism over the smooth locus $X_{\text{reg}} \subset X$. In particular, there is a section $\sigma : X_{\text{reg}} \to \mathbb{P}_X(\wedge^d \Omega_X)$.

Definition III.2. The *Nash blow-up* of X is the closure of the image of σ, and is denoted by $\hat{X}$. It is a variety over k with a projective morphism $\pi|_{\hat{X}} : \hat{X} \to X$ that is an isomorphism over the smooth locus of X. The line bundle

$$\widehat{K}_X := \mathcal{O}_{\mathbb{P}_X(\wedge^d \Omega_X)}(1)|_{\hat{X}}$$

is called the *Mather canonical line bundle* of X.

Remark III.3. If X is smooth, then clearly $\hat{X} = X$ and $\widehat{K}_X$ is just the canonical line bundle of X. More generally, the Nash blow-up can be thought of as the parameter space of limits of all tangent directions at smooth points of X.

One can always find a resolution of singularities $f : Y \to X$ that factors through the Nash blow-up. Then the image of the $f^*(\wedge^d \Omega_X)$ under the canonical homomorphism

$$\wedge^d df : f^*(\wedge^d \Omega_X) \to \wedge^d \Omega_Y$$

is of the form $J \wedge^d \Omega_Y$ where J is an invertible ideal sheaf on Y ([dFEI07, Proposition 1.7]). Let $\widehat{K}_{Y/X}$ be the effective divisor defined by J. This is supported on the exceptional locus of f and it is called the *Mather discrepancy divisor*. For each prime divisor E on Y, we define $\hat{k}_E := \text{ord}_E(\widehat{K}_{Y/X})$. If $v = q \cdot \text{ord}_E$ is a divisorial valuation, we write $\hat{k}_v := q \cdot \hat{k}_E$.

Definition III.4. Let $(X, \mathfrak{a})$ be a pair where X is a variety over k and $\mathfrak{a}$ is a nonzero ideal in $\mathcal{O}_X$. For a closed subset W of X, the Mather minimal log discrepancy of $(X, \mathfrak{a})$ along W is defined as

$$\widehat{\text{mld}}(W; X, \mathfrak{a}) := \inf\{\hat{k}_E - \text{ord}_E(\mathfrak{a}) + 1 | E \text{ is a divisor over } X \text{ with center in } W\}.$$

When $\dim(X) = 1$ and the infimum is negative, we make the convention that

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a}) = -\infty.$$

Remark III.5. If $\dim(X) \geq 2$ and $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}) < 0$, then $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}) = -\infty$ (see [Ish13, Remark 3.4]). This is why we make the convention for the case when $\dim(X) = 1$.

When $W = \{x\}$ for some closed point $x \in X$ and $\mathfrak{a} = \mathcal{O}_X$, we denote the Mather minimal log discrepancy by $\widehat{\mathrm{mld}}(x; X)$ for simplicity. The Mather minimal log discrepancy has many nice properties. We quote a few important ones here without proof.

Proposition III.6. *[Ish13, Corollary 4.3] For every closed point x of a variety X of dimension d, we have*

$$\widehat{\mathrm{mld}}(x; X) \geq d.$$

Proposition III.7. *[IR13, Proposition 1.2] Let X be a variety of dimension d. Then, for every closed point $x \in X$ and effective $\mathbb{R}$-Cartier divisor B on X, we have*

$$\widehat{\mathrm{mld}}(x; X, \mathcal{J}_X \cdot \mathcal{O}_X(B)) \leq d,$$

where $\mathcal{J}_X$ is the Jacobian ideal of X. Moreover, equality holds if and only if x is a nonsingular point and $B = 0$ around x.

Proposition III.8. *[Ish13, Corollary 3.14] Let X be a variety of dimension d and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$. Then the function $x \mapsto \widehat{\mathrm{mld}}(x; X, \mathfrak{a} \cdot \mathcal{J}_X)$, for $x \in X$ a closed point, is lower semicontinuous.*

The Mather minimal log discrepancy can be computed via jet schemes and arc spaces as follows:

Proposition III.9. *[Ish13, Proposition 3.7] Let X be a variety of dimension d and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_X$. If W is a proper closed subset of X and I_W is the (reduced) ideal of W, then we have the following:*

(1) $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}) = \inf_{m \in \mathbb{N}} \{\mathrm{codim}(\mathrm{Cont}^{m}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq 1}(I_W)) - m\}.$

(2) $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}) = \inf_{m \in \mathbb{N}} \{\mathrm{codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq 1}(I_W)) - m\}.$

Probably the most important property is Inversion of Adjunction:

Proposition III.10. *[Ish13, Proposition 3.10] Let X be a variety of dimension d and A be a non-singular variety containing X as a closed subvariety of codimension c, with ideal I_X. Let W be a proper closed subset of X and $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be an ideal such that $\mathfrak{a} := \tilde{\mathfrak{a}} \cdot \mathcal{O}_X \subset \mathcal{O}_X$ is a nonzero ideal. Then we have*

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a} \cdot \mathcal{J}_X) = \widehat{\mathrm{mld}}(W; A, \tilde{\mathfrak{a}} \cdot I_X^c).$$

As their definitions suggest, the Mather minimal log discrepancy and the usual minimal log discrepancy are closely related. The following result on minimal log discrepancy was proved using Mather minimal log discrepancy:

Proposition III.11. *[dFT16, Theorem 1.5] Let X be a normal $\mathbb{Q}$-Gorenstein variety of dimension d and $x \in X$ be a closed point such that the exceptional divisor of the normalized blow-up of X at x has a generically reduced irreducible component. Then*

$$\mathrm{mld}(x; X, \mathcal{O}_X) \leq d,$$

and equality holds if and only if X is smooth at x.

If X is a normal $\mathbb{Q}$-Gorenstein variety, one can show that $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}) \geq \mathrm{mld}(W; X, \mathfrak{a})$ for every closed subset W and nonzero ideal $\mathfrak{a}$. To see this, let r be a positive integer such that rk_X is a Cartier divisor on X and $f : Y \to X$ be a resolution of singularities that factors through the Nash blow-up. If we write the image of the homomorphism

$$(\wedge^d \Omega_X)^{\otimes r} \longrightarrow \mathcal{O}_X(rK_X)$$

as $I_r \cdot \mathcal{O}_X(rK_X)$, where I_r is an ideal of $\mathcal{O}_X$, then clearly we have

$$f^*(I_r) \cdot \mathcal{O}_Y(r\widehat{K}_{Y/X}) = \mathcal{O}_Y(rK_{Y/X}).$$

In particular, we have $\widehat{K}_{Y/X} \geq K_{Y/X}$. Then $\widehat{\mathrm{mld}}(W; X, \mathfrak{a}) \geq \mathrm{mld}(W; X, \mathfrak{a})$ follows by definition.

More results on the relation between Mather minimal log discrepancy and the usual minimal log discrepancy can be found at [EI15] and [IR13]. For more details on Mather minimal log discrepancy, we refer to [Ish13], [EI15] and [IR15].

3.2 Relation to jet schemes and arc spaces

Let k be an algebraically closed field of characteristic zero and X be a variety over k of dimension d. We specialize to the case when $W \subset X$ is a closed point and $\mathfrak{a} = \mathcal{O}_X$, and establish the relation between Mather minimal log discrepancy and jet schemes and arc spaces. For that purpose, we need to introduce the notion of maximal divisorial sets.

Definition III.12. If v is a divisorial valuation on a variety X, the maximal divisorial set corresponding to v, denoted $C_X(v)$, is defined as the Zariski closure of

$$\{\gamma \in X_\infty | \mathrm{ord}_\gamma = v\}.$$

Definition III.13. If X and Y are varieties over k, and $A \subset X$ and $B \subset Y$ are constructible subsets. Then a map $f : A \to B$ is a *piecewise trivial fibration with fiber* F, if there exists a finite partition of B into locally closed subsets S of Y such that $f^{-1}(S)$ is isomorphic to $S \times F$ and $f|_{f^{-1}(S)}$ is the projection $S \times F \to S$ under the isomorphism.

Recall that for a scheme X of finite type over k, there are canonical morphisms $\pi : X_\infty \to X$ and $\psi_m : X_\infty \to X_m$ for every $m \geq 0$. A cylinder $A \subset X_\infty$ is *stable* at level $n \in \mathbb{N}$ if it is a union of fibers of ψ_n and if $\psi_{m+1}(X_\infty) \to \psi_m(X_\infty)$ is a piecewise trivial fibration over $\psi_m(A)$ with fiber $\mathbb{A}^{\dim(X)}$ for each $m \geq n$. A is called stable if it is stable at some level. If A is stable at level n, then the *codimension* of A is $(m+1)\dim(X) - \dim(\psi_m(A))$ for any $m \geq n$. Clearly this is independent of choice of m.

We quote the following results without proof:

Proposition III.14. *([Ish13, Proposition 2.9]) Let X be an affine variety and $\alpha_i \subset \mathcal{O}_X$ for $1 \leq i \leq r$ be non-zero ideals. Then for any $e_1, \ldots, e_r \in \mathbb{N}$, every fat irreducible component of* $\mathrm{Cont}^{\geq e_1}(\alpha_1) \cap \ldots \cap \mathrm{Cont}^{\geq e_r}(\alpha_r)$ *is a maximal divisorial set.*

Proposition III.15. *([Ish13, Proposition 2.13]) For a divisorial valuation $q \cdot \mathrm{val}_E$ on a variety X, the corresponding maximal divisorial set is stable and its codimension is given by*

$$\mathrm{codim}(C_X(q \cdot \mathrm{val}_E)) = q(\hat{k}_E + 1).$$

More details on divisorial valuations and maximal divisorial sets can be found in [dFEI07].

Definition III.16. Fix a closed point x of X. Recall that from Section 2.2 we have canonical morphisms $\psi_m : X_\infty \to X_m$, and in particular, $\pi = \psi_0 : X_\infty \to X$. For every $m \geq 0$ we define

$$\lambda_m(x) := md - \dim \psi_m(\pi^{-1}(x)).$$

When there is no confusion we simply write λ_m instead of $\lambda_m(x)$.

Remark III.17. Corollary II.8 shows that when X is a smooth variety and x is a closed point of X, we have for each $m \geq 0$,

$$\lambda_m(x) = m\dim(X) - \dim(\psi_m(\pi^{-1}(x))) = 0.$$

Proposition III.18. *For every $m \geq 0$, we have $\lambda_m \geq 0$ and $\lambda_{m+1} \geq \lambda_m$.*

Proof. The first assertion is the consequence of the second one and the fact that $\lambda_0 = 0$. So we only need to prove the second assertion. According to Lemma II.12, the map

$$\psi_{m+1}(\pi^{-1}(x)) \longrightarrow \psi_m(\pi^{-1}(x))$$

has fibers of dimension at most d. Thus, we have

$$\dim(\psi_{m+1}(\pi^{-1}(x))) \leq \dim(\psi_m(\pi^{-1}(x))) + d.$$

This shows that $\lambda_{m+1} \geq \lambda_m$ for $m \geq 0$. □

The sequence λ_m is nondecreasing, hence it has a limit (finite or infinite). The limit is an integer if the sequence is stationary when m is large enough. We first quote the following result:

Proposition III.19. *([dFEI07, Proposition 3.5]) If X is a variety over k, then the number of irreducible components of a cylinder on X_∞ is finite.*

Lemma III.20. *Let X be a variety over k of dimension d and x be a closed point of X. Then for $m \gg 0$, λ_m is constant.*

Proof. The question is local so we assume that X is affine. Since $\pi^{-1}(x)$ is a contact locus, by Proposition III.19 it has a finite number of irreducible components $C_1, \ldots, C_r,\ Z_1, \ldots, Z_s$, where C_i's are fat and Z_j's are thin. Each thin component Z_j is contained in the arc space of some subvariety of X of dimension $\leq d-1$. By Lemma II.12, we have

$$\dim(\psi_m(Z_j)) \leq (m+1)(d-1),$$

for every $m \geq 0$ and $1 \leq j \leq s$. On the other hand, by Proposition III.14, each C_i is a maximal divisorial set $C_X(q_i \cdot \mathrm{val}_{E_i})$ for some positive integer q_i and divisor E_i over X. Hence, according to Proposition III.15, we have

$$\dim(\psi_m(C_i)) = (m+1)d - q_i(\hat{k}_{E_i} + 1),$$

for every $1 \leq i \leq r$ and $m \gg 0$. We conclude that for $m \gg 0$, we have

$$\dim(\psi_m(\pi^{-1}(x))) = \max_{1\leq i\leq r}\{\dim(\psi_m(C_i))\} = md - \lambda,$$

where $\lambda = \min_{1\leq i\leq r}\{q_i(\hat{k}_{E_i} + 1)\} - d$. □

Definition III.21. Let X be a variety of dimension d and x be a closed point of X. The sequence $(\lambda_m(x))_{m\geq 0}$ introduced above is stationary for $m \gg 0$. So $\lim_{m\to\infty} \lambda_m(x)$ exists and it is equal to $\lambda_m(x)$ for all m large enough. We denote this limit by $\lambda(x)$. When there is no confusion, we also write this limit as λ.

The following result from [Ish13] describes the Mather minimal log discrepancy in terms of jet schemes and arc spaces. It is a special case of Proposition III.9.

Proposition III.22. *If X is a variety over k of dimension d and x is a closed point of X, then*

$$\lambda(x) = \widehat{\mathrm{mld}}(x; X) - d.$$

Proof. By the proof of Lemma III.20, we have

$$\pi^{-1}(x) = \bigcup_{1\leq i\leq r} C_i$$

up to a thin set in X_∞. We see that $C_i = C_X(q_i \cdot \mathrm{val}_{E_i})$, where each $q_i \cdot \mathrm{val}_{E_i}$ is a divisorial valuation centered at $\{x\}$. For each i, since C_i is an irreducible component of $\pi^{-1}(x)$, we have $q_i = 1$. On the other hand, every divisor E over X with center $\{x\}$ is equivalent to one of the E_i's since an arc γ with $\mathrm{ord}_\gamma = \mathrm{ord}_E$ must be contained in one of C_i's.

By definition $\lambda(x) = \mathrm{codim}(\pi^{-1}(x)) - d = \min_{1\leq i\leq r}\{\mathrm{codim}(C_X(q_i \cdot \mathrm{ord}_{E_i})\} - d$. According to Proposition III.15, $\min_{1\leq i\leq r}\{\mathrm{codim}(C_X(q_i \cdot \mathrm{val}_{E_i})\} = \min_{1\leq i\leq r}\{\hat{k}_{E_i} + 1\}$. It follows from the above analysis and the definition of Mather minimal log discrepancy that this minimum is equal to $\widehat{\mathrm{mld}}(x; X)$. □

Remark III.23. If X is smooth and $x \in X$ is a closed variety, by Remark III.17 we have $\lambda_m(x) = 0$ for every $m \geq 0$. Hence, by Proposition III.22 we have $\widehat{\mathrm{mld}}(x; X) = \dim(X)$. The same conclusion holds if we only assume x is a smooth point of X because in this case we may replace X by a smooth open neighborhood of x. Therefore, we only consider singular points in the following chapters.

CHAPTER IV

Mather Minimal Log Discrepancy of Toric Varieties

This chapter is devoted to the computation of the Mather minimal log discrepancy associated to a closed point on a toric variety. We will first do the computation for a torus-invariant point, and then show the computation generalizes to an arbitrary closed point.

The computation depends only on local properties of the toric variety so we assume throughout the chapter that $X = X(\sigma)$ is an affine toric variety associated to a cone σ over an algebraically closed field of characteristic zero.

We write x_σ for the torus-invariant point in X (when it exists), and therefore according to Proposition III.22, computing the Mather minimal log discrepancy associated to x_σ is equivalent to computing the dimension of $C^m := \psi_m(\pi^{-1}(x_\sigma))$ for m large enough. More precisely, $\widehat{\mathrm{mld}}(x_\sigma; X) = mn - \dim(C^m)$ when $m \gg 0$. We will decompose C^m into orbits under the T_m-action, where T is the torus in X, and compute the dimension of C^m by computing the dimension of these orbits instead.

4.1 Quick review

In this section we provide a quick review of basic facts on toric varieties. For details we refer the reader to [Ful93].

Let k be an algebraically closed field of characteristic zero. An affine *toric variety* of dimension n is defined using a *lattice* $N \cong \mathbb{Z}^n$ and a cone σ in $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. A *cone*

σ is a rational convex cone in $N_{\mathbb{R}}$ containing no nonzero linear subspace and which is generated by finitely many lattice vectors.

Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the *dual lattice* and we put $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $\langle \cdot, \cdot \rangle$ the canonical pairing $M \times N \to \mathbb{Z}$. The affine toric variety associated to the cone σ is defined as

$$X(\sigma) := \text{Spec } k[M \cap \sigma^{\vee}],$$

where $\sigma^{\vee}$ is the *dual cone* contained in $M_{\mathbb{R}}$, i.e. $\sigma^{\vee} = \{u \in M_{\mathbb{R}} | \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$. The semigroup algebra $k[M \cap \sigma^{\vee}]$ is defined as $\bigoplus_{u \in M \cap \sigma^{\vee}} k \cdot \chi^u$, with $\chi^u \cdot \chi^v = \chi^{u+v}$. Then clearly for some elements $u_1, \ldots, u_s \in M \cap \sigma^{\vee}$, we have $\chi^{u_1}, \ldots, \chi^{u_s}$ generate $k[M \cap \sigma^{\vee}]$ if and only if $u_1, \ldots, u_s$ generate $M \cap \sigma^{\vee}$ as a semigroup.

A k-valued point x of $X(\sigma)$ corresponds to a homomorphism of k-algebras

$$x^* : k[\sigma^{\vee} \cap M] \longrightarrow k.$$

We put $\sigma^{\perp} := \{u \in M_{\mathbb{R}} | \langle u, v \rangle = 0 \text{ for all } v \in \sigma\}$. A *face* of σ is a subset of σ of the form $\{v \in \sigma | \langle u, v \rangle = 0\}$, for some $u \in \sigma^{\vee}$. The *distinguished point* x_{τ} corresponding to a face τ of σ is defined by $x_{\tau}^*(\chi^u) = 1$ if $u \in \tau^{\perp}$, and $x_{\tau}^*(\chi^u) = 0$ otherwise.

Remark IV.1. The point x_{σ} exists if $N_{\mathbb{R}}$ is the linear span of σ. This point will play a special role in what follows. The computation when $N_{\mathbb{R}}$ is not spanned by σ can be easily reduced to this case, since $X(\sigma)$ will be a product of a torus with a lower-dimensional toric variety that contains a torus-invariant point. So from now on, we assume that σ spans $N_{\mathbb{R}}$.

The toric variety $X = X(\sigma)$ contains the torus $T = \text{Spec } k[M] \cong (k^*)^n$ and the group action of T on itself extends to an action on X. More precisely, the T-action on X is given by $G : T \times X \to X$, which is equivalent to the following morphism of k-algebras:

$$k[M \cap \sigma^{\vee}] \longrightarrow k[M] \otimes k[M \cap \sigma^{\vee}], \chi^u \longmapsto \chi^u \otimes \chi^u. \tag{4.1}$$

It is a general fact that the T-orbit $O(\tau)$ that contains x_τ is of dimension equal to the codimension of τ in σ. In particular, the point x_σ is the unique torus-invariant point. The toric variety X is the disjoint union of the orbits $O(\tau)$ with τ varying over all faces of σ. Therefore, any point of X lies in the same orbit with one of the x_τ's.

A torus-invariant prime divisor D is the closure of the orbit associated to a one-dimensional face. Let us call this one-dimensional face τ. Then we have

$$D = V(\tau) := \text{Spec } k[M \cap \sigma^\vee \cap \tau^\perp].$$

4.2 Characterization of orbits in C^m

Definition IV.2. Recall that for each variety X over k there are canonical morphisms $\psi_m : X_\infty \to X_m$ and $\pi : X_\infty \to X$. For every $m \geq 1$, we define C^m a subset of X_m as

$$C^m := \psi_m(\pi^{-1}(x_\sigma)).$$

Let T be the torus in $X = X(\sigma)$. It follows from Remark II.1 (iii) that there is a natural group action of T_m on X_m. In this section, we approximate C^m by a union of T_m-orbits and show that these orbits can be represented by lattice points in the interior of σ. This section builds on the work of Ishii [Ish04] who gave a similar description for the T_∞-orbits in X_∞. We denote by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers.

Let γ : Spec $k[t]/(t^{m+1}) \longrightarrow X$ be an m-jet inside C^m and let δ : Spec $k[\![t]\!] \longrightarrow X$ be an arc on X which lifts γ. Then we have the following commutative diagram:

$$\begin{array}{ccc} k[M \cap \sigma^\vee] & \xrightarrow{\delta^*} & k[\![t]\!] \\ & \overset{\gamma^*}{\searrow} & \downarrow \\ & & k[t]/(t^{m+1}), \end{array}$$

where the vertical map is the canonical truncation.

Let $\tau_1, \ldots, \tau_d$ be the one-dimensional faces of σ and $D_i := V(\tau_i)$ be the corresponding torus-invariant prime divisors of X. We assume that δ is not in the arc space of any D_i.

Equivalently, $\delta^*(\chi^u) \neq 0$ for every $u \in M \cap \sigma^\vee$. Thus the order in t of $\delta^*(\chi^u) \in k[\![t]\!]$ is well-defined. We put $\text{ord}_\delta(u) := \text{ord}_t(\delta^*(\chi^u))$ for each $u \in M \cap \sigma^\vee$. Let S_m be the set $\{0, 1, \ldots, m, \infty\}$ and $Tr_m : \mathbb{Z}_{\geq 0} \to S_m$ be the obvious truncation map that takes any number larger than m to ∞. We define ord_γ to be the composition of ord_δ with Tr_m. Then we get the following commutative diagram:

$$\begin{array}{ccc} M \cap \sigma^\vee & \xrightarrow{\text{ord}_\delta} & \mathbb{Z}_{\geq 0} \\ & \searrow^{\text{ord}_\gamma} & \downarrow{\scriptstyle Tr_m} \\ & & S_m. \end{array}$$

Note that for each $u \in M \cap \sigma^\vee$, the value of $\text{ord}_\gamma(u)$ only depends on $\gamma^*(\chi^u)$. Thus ord_γ is independent of choice of δ and we call it the order map of γ.

Since ord_δ is an additive map that takes lattice points in the cone $\sigma^\vee$ to nonnegative integers, it corresponds uniquely to a lattice point in σ. Now we give a first description of some of the T_m-orbits of C^m.

Lemma IV.3. *With the above notation, $\psi_m(X_\infty \setminus \bigcup_i (D_i)_\infty)$ is preserved by the T_m-action and its orbits are in one-to-one correspondence with the maps $M \cap \sigma^\vee \to S_m$ that can be lifted to additive maps $M \cap \sigma^\vee \to \mathbb{Z}_{\geq 0}$. The corresponding map is exactly the order map of any element in the orbit.*

Moreover, $\psi_m(\pi^{-1}(x_\sigma) \setminus \bigcup_i (D_i)_\infty)$ is also preserved by the T_m-action and its orbits are in one-to-one correspondence with the maps $M \cap \sigma^\vee \to S_m$ that can be lifted to additive maps $M \cap \sigma^\vee \to \mathbb{Z}_{\geq 0}$, such that the inverse image of $\{0\}$ is $\{0\}$.

Proof. First let us describe the T_m-action on X_m and the T_∞-action on X_∞. Let

$$g : \text{Spec } k[t]/(t^{m+1}) \to \text{Spec } k[M]$$

be a point of T_m and

$$\gamma : \text{Spec } k[t]/(t^{m+1}) \to \text{Spec } k[M \cap \sigma^\vee]$$

be a point of X_m. Then $g \cdot \gamma$ is a morphism Spec $k[t]/(t^{m+1}) \to$ Spec $k[M \cap \sigma^\vee]$ that is equal to $G \circ (g, \gamma)$. By equation (4.1), for each $u \in M \cap \sigma$ we have

$$(g \cdot \gamma)^*(\chi^u) = g^*(\chi^u) \cdot \gamma^*(\chi^u).$$

Similarly, if δ : Spec $k[\![t]\!] \to k[M]$ is a point in T_∞ and α : Spec $k[\![t]\!] \to k[M \cap \sigma^\vee]$ is a point of X_∞, then for each $u \in M \cap \sigma^\vee$ we have

$$(\delta \cdot \alpha)^*(\chi^u) = \delta^*(\chi^u) \cdot \alpha^*(\chi^u).$$

Note that both $g^*(\chi^u)$ and $\delta^*(\chi^u)$ above are units since χ^u has an inverse χ^{-u} in $k[M]$.

Now let γ be an m-jet in $\psi_m(X_\infty \backslash \bigcup_i (D_i)_\infty)$ with a lifting δ in $X_\infty \backslash \bigcup_i (D_i)_\infty$. For each $\alpha \in T_m$, there is a lifting $\xi \in T_\infty$ of α by smoothness of T. Since $\xi^*(\chi^u)$ is a unit for each $u \in M$, $(\xi \cdot \delta)^*(\chi^u) \neq 0$ for each $u \in M \cap \sigma^\vee$. Hence $\xi \cdot \delta$ is also in $X_\infty \backslash \bigcup_i (D_i)_\infty$. Therefore, $\alpha \cdot \gamma = \psi_m(\xi \cdot \delta)$ is in $\psi_m(X_\infty \backslash \bigcup_i (D_i)_\infty)$. This shows that $\psi_m(X_\infty \backslash \bigcup_i (D_i)_\infty)$ is preserved by the T_m-action.

For $\psi_m(\pi^{-1}(x_\sigma) \backslash \bigcup_i (D_i)_\infty)$, one applies the same argument and observes that an arc δ lies above the torus-invariant point x_σ if and only if $\delta^*(\chi^u)$ has positive order whenever $u \neq 0$, which is equivalent to $\mathrm{ord}_\delta^{-1}(0) = \{0\}$. Since $\xi^*(\chi^u)$ is a unit for each $u \in M$ and $\xi \in T_\infty$, $\xi \cdot \delta$ also lies above 0. This shows that $\psi_m(\pi^{-1}(x_\sigma) \backslash \bigcup_i (D_i)_\infty)$ is also preserved by the T_m-action.

Pick two m-jets $\alpha \in T_m$ and $\gamma \in \psi_m(X_\infty \backslash \bigcup_i (D_i)_\infty)$. The morphism α^* takes any χ^u, with $u \in M$, to a unit. Therefore, multiplying γ by α does not change the order map ord_γ. In other words, $\mathrm{ord}_\gamma = \mathrm{ord}_{\alpha \cdot \gamma}$. This shows that the order map is the same for all points in a T_m-orbit.

Now we show that two m-jets in $\psi_m(X_\infty \backslash \bigcup_i (D_i)_\infty)$ with the same order map are in the same T_m-orbit. Let γ be in $\psi_m(X_\infty \backslash \bigcup_i (D_i)_\infty)$ and ϕ be its order map. We define the special m-jet γ_ϕ whose associated morphism is

$$\gamma_\phi^* : k[M \cap \sigma^\vee] \longrightarrow k[t]/(t^{m+1}),\ \gamma_\phi^*(\chi^u) = t^{\phi(u)},$$

with the convention that $t^\infty = 0$. If we write $\phi(a) + \phi(b) = \infty$ whenever the sum is $\geq m+1$, then we have $\phi(a) + \phi(b) = \phi(a+b)$ for any $a, b \in M \cap \sigma^\vee$. Therefore, γ_ϕ^* is a homomorphism of k-algebras.

Let $\delta \in X_\infty \backslash \underset{i}{\cup} (D_i)_\infty$ be a lifting of γ and ψ be the order map of δ. Then we may define an arc δ_ψ such that

$$\delta_\psi^* : k[M \cap \sigma^\vee] \longrightarrow k[\![t]\!],\ \delta_\psi^*(\chi^u) = t^{\psi(u)}.$$

Obviously, δ_ψ lifts γ_ϕ, and it has the same order map as δ. Hence we have a morphism of k-algebras as follows:

$$\alpha^* : k[M \cap \sigma^\vee] \longrightarrow k[\![t]\!],\ \alpha^*(\chi^u) = \delta^*(\chi^u)/\delta_\psi^*(\chi^u).$$

α^* extends to the entire $k[M]$ since $M \cap \sigma^\vee$ spans M and since $\alpha^*(\chi^u)$ is a unit for each $u \in M \cap \sigma^\vee$. Hence $\alpha \in T_\infty$ and clearly we have $\alpha \cdot \delta_\psi = \delta$, and therefore, $\psi_m(\alpha) \cdot \gamma_\phi = \gamma$. This shows that γ is in the same T_m-orbit as the special m-jet γ_ϕ, and so is any other m-jet with the same order map.

Finally, we show that each map $\phi : M \cap \sigma^\vee \to S_m$ that can be lifted to an additive map $\psi : M \cap \sigma^\vee \to \mathbb{Z}_{\geq 0}$ is the order map of some m-jet in $\psi_m(X_\infty \backslash \underset{i}{\cup} (D_i)_\infty)$. Define the special m-jet γ_ϕ and the arc δ_ψ that lifts γ_ϕ in the same way as above. Then we have $\delta_\psi \in X_\infty \backslash \underset{i}{\cup} (D_i)_\infty$ because $\delta_\psi^*(\chi^u)$ has finite order for each $u \in M \cap \sigma^\vee$. Therefore, γ_ϕ is an m-jet in $\psi_m(X_\infty \backslash \underset{i}{\cup} (D_i)_\infty)$ and clearly $\mathrm{ord}_{\gamma_\phi} = \phi$. Hence we have produced a T_m-orbit whose corresponding order map is equal to the map ϕ that we started with. □

As mentioned above, an additive map $M \cap \sigma^\vee \to \mathbb{Z}_{\geq 0}$ corresponds uniquely to a lattice point in σ. Denote by φ_a the additive map corresponding to the lattice point a and $\bar{\varphi}_a$ the composition of φ_a with the truncation map Tr_m. Then clearly every order map in Lemma IV.3 is equal to $\bar{\varphi}_a$ for some $a \in \sigma \cap N$. In particular, the order map takes only 0 to 0 if the lattice point a is contained in Int(σ), the interior of σ. However, there could be more than one such a. To understand the additive maps $M \cap \sigma^\vee \to \mathbb{Z}_{\geq 0}$ better, we first study the semigroup $M \cap \sigma^\vee$ and show that there is a unique minimal set of generators.

Definition IV.4. An element $u \in M \cap \sigma^\vee$ is called *irreducible* if it cannot be written as the sum of two nonzero elements of $M \cap \sigma^\vee$.

Lemma IV.5. *The semigroup $M \cap \sigma^\vee$ has a unique minimal set of generators consisting of all the irreducible elements.*

Proof. First, since $\sigma^\vee$ is a convex polyhedral cone, $M \cap \sigma^\vee$ is finitely generated. Therefore, there exists a minimal set of generators.

Second, we show that any element of $M \cap \sigma^\vee$ can be generated by irreducible elements. Pick an element $v \in$ Int$(\sigma) \cap N$. Then $\langle u, v\rangle$ is a positive integer for any $u \in M \cap \sigma^\vee$. We claim that for each $u \in M \cap \sigma^\vee$, u can be written as the sum of at most $\langle u, v\rangle$ irreducible elements. If $\langle u, v\rangle = 1$, then u must irreducible. Otherwise, there are nonzero elements $u_1, u_2 \in M \cap \sigma^\vee$ such that $u = u_1 + u_2$. But $\langle u_1, v\rangle$ and $\langle u_2, v\rangle$ are both positive integers since $v \in$ Int$(\sigma) \cap N$. This is not possible as they add up to $\langle u, v\rangle = 1$. Inductively, suppose our claim holds for all u such that $\langle u, v\rangle \leq p$. Pick $u \in M \cap \sigma^\vee$ such that $\langle u, v\rangle = p+1$. If u is irreducible, then we are done. Otherwise, there are nonzero elements $u_1, u_2 \in M \cap \sigma^\vee$ such that $u = u_1 + u_2$. Both $\langle u_1, v\rangle$ and $\langle u_2, v\rangle$ are $\leq p$. By assumption, u_1 and u_2 can be written as the sum of at most $\langle u_1, v\rangle$ and $\langle u_2, v\rangle$ irreducible elements respectively. Therefore, u can be written as the sum of at most $\langle u_1, v\rangle + \langle u_2, v\rangle = p + 1$ irreducible elements.

Finally, note that any set of generators must contain all irreducible elements by definition. We conclude that the set of irreducible elements form the unique minimal set of generators for $M \cap \sigma^\vee$. □

Remark IV.6. If $u_1, \dots, u_s$ form the unique minimal set of generators of $M \cap \sigma^\vee$, then $\chi^{u_1}, \dots, \chi^{u_s}$ also form the unique minimal set of monomial generators of $k[M \cap \sigma^\vee]$.

The following lemma makes a connection between the set of order maps and the set of lattice points.

Lemma IV.7. *Fix an integer $m \geq 1$ and let $\chi^{u_1}, \chi^{u_2}, \dots, \chi^{u_s}$ be the minimal set of monomial generators of $k[M \cap \sigma^\vee]$. For each integer $c \geq 0$ we define*

$$P_c := \left\{ a \in \sigma \cap N \middle| \text{the set } \{u_i | \varphi_a(u_i) \leq m + c\} \text{ spans } M_{\mathbb{R}} \right\}.$$

Then the following hold:

(1) For any two different $a, b \in P_0$, $\bar{\varphi}_a \neq \bar{\varphi}_b$.

(2) There exists some $c_0 \in \mathbb{Z}^+$ such that for any $a \in \sigma \cap N$ one can find $b \in P_{c_0}$ with $\bar{\varphi}_a = \bar{\varphi}_b$.

Proof. First let's assume we have $a, b \in P_0$ and $\bar{\varphi}_a = \bar{\varphi}_b$. Define

$$\Gamma_0 := \{u_i | \varphi_a(u_i) \leq m\}.$$

Since $\bar{\varphi}_a = \bar{\varphi}_b$, we deduce that φ_a and φ_b take the same values on Γ_0. By definition of P_0, Γ_0 spans $M_{\mathbb{R}}$. Thus we conclude that $\varphi_a = \varphi_b$, which implies that $a = b$.

For (2), we choose a positive integer c_0 large enough such that for any subset $S \subset \{u_1, u_2, \dots, u_s\}$ that does not span $M_{\mathbb{R}}$, there is some $v \in N$ satisfying

$$\varphi_v(u_i) = 0, \text{ for all } u_i \in S, \text{ and } 1 \leq \max_{u_i \notin S} \{\varphi_v(u_i)\} \leq c_0. \tag{4.2}$$

Such a number c_0 exists because there are only finitely many subsets of $\{1, 2, \ldots, s\}$. For each point $b \in \sigma \cap N$ we put $S_b := \big\{u \in \{u_1, \ldots, u_s\} | \varphi_b(u) \leq m + c_0\big\}$. If there is some b such that $\bar{\varphi}_a = \bar{\varphi}_b$ and such that S_b spans $M_{\mathbb{R}}$, then we are done.

Now suppose there is no such b. We pick a point b such that $\bar{\varphi}_a = \bar{\varphi}_b$ and such that S_b is maximal. By relabeling we may write $S_b = \{u_1, \ldots, u_l\}$ for some integer $l < s$. By assumption S_b does not span $M_{\mathbb{R}}$, so we can fine $v \in N$ that satisfies (4.2) with S replaced by S_b. Clearly there is some positive integer k such that

$$\varphi_{b-kv}(u_i) > m, \text{ for all } i > l,$$

$$\varphi_{b-kv}(u_{i_0}) \leq m + c_0, \text{ for some } i_0 > l.$$

Notice that $\bar{\varphi}_{b-kv} = \bar{\varphi}_b = \bar{\varphi}_a$, and hence $b - kv \in \sigma \cap N$. But clearly we have

$$S_b \subsetneq \{u_1, \ldots, u_l, u_{i_0}\} \subset S_{b-kv}.$$

This contradicts the maximality of S_b. So we conclude that there must be some $b \in P_{c_0}$ such that $\bar{\varphi}_a = \bar{\varphi}_b$. □

Remark IV.8. We have proved that for each $a \in \sigma \cap N$, the map $\bar{\varphi}_a$ corresponds to a T_m-orbit in $\psi_m(X_\infty \backslash \cup_i (D_i)_\infty)$. We denote this orbit by $T_{m,a}$.

Remark IV.9. For each $a \in \text{Int}(\sigma) \cap N$, φ_a is an additive map $M \cap \sigma^\vee \to \mathbb{Z}_{\geq 0}$ such that $\varphi_a^{-1}(0) = \{0\}$. According to Lemma IV.3 the corresponding orbits $T_{m,a}$ are all the T_m-orbits contained in $\psi_m(\pi^{-1}(x_\sigma) \backslash \cup_i (D_i)_\infty)$.

Corollary IV.10. *The sets $\psi_m(X_\infty \backslash \bigcup_i (D_i)_\infty)$ and $\psi_m(\pi^{-1}(x_\sigma) \backslash \bigcup_i (D_i)_\infty)$ contain only finitely many T_m-orbits.*

Proof. According to Lemma IV.3, we just need to show there are finitely many order maps $\bar{\varphi}_a$ for $a \in \sigma \cap N$. By Lemma IV.7, there is a positive integer c_0 such that every order map is equal to $\bar{\varphi}_a$ for some $a \in P_{c_0}$. Therefore, it suffices to show that P_{c_0} is compact.

For any $u_{j_1}, \dots, u_{j_n} \subset \{u_1, \dots, u_s\}$ that span $M_{\mathbb{R}}$, we define

$$K_{j_1, j_2, \dots, j_n} := \{a \in \sigma \cap N | \varphi_a(u_{j_i}) \leq m + c_0 \text{ for } 1 \leq i \leq n\}.$$

Then P_{c_0} is the union of all $K_{j_1, \dots, j_n}$ as $(j_i)_{1 \leq i \leq n}$ varies such that $u_{j_1}, \dots, u_{j_n}$ span $M_{\mathbb{R}}$. Since this is a finite union, it suffices to show that each $K_{j_1, \dots, j_n}$ is compact.

By relabeling let us assume that $j_i = i$ for $1 \leq i \leq n$. Let $v_1, \dots, v_l$ be a minimal set of generators of $\sigma \cap N$. Since $u_1, \dots, u_n$ span $M_{\mathbb{R}}$, for each v_i there exists some u_j with $1 \leq j \leq n$ such that $\langle v_i, u_j \rangle$ is a positive integer. Therefore,

$$K_{1,2,\dots,n} \subset \{a \in \sigma \cap N | a = \sum_{i=1}^{l} c_i v_i, \text{ with } 0 \leq c_i \leq m + c_0 \text{ for each } i\}.$$

This shows that $K_{1,2,..,n}$ is compact. □

Remark IV.11. The structure of the jet schemes of toric varieties is in general very hard to describe unlike the case of arc spaces. One can find a description of jet schemes of toric surfaces in [Mou11]. Instead of the entire jet schemes, we only describe the structure of images of the arc space in the m^{th} jet scheme.

4.3 Main results

In this section we compute the dimension of the orbit $T_{m,a}$ by computing the dimension of the corresponding stabilizer. Denote by $H_{m,a}$ the stabilizer of any element of $T_{m,a}$ under the T_m-action. We start with the following lemma.

Lemma IV.12. *Let $u_1, \dots, u_n \in M$ be elements that generate $M_{\mathbb{R}}$ over $\mathbb{R}$. For every $a_{i,j} \in k$ with $1 \leq i \leq n$ and $0 \leq j \leq m$ such that $a_{i,0} \neq 0$ for all i, the set of elements $\alpha \in T_m$ such that*

$$\alpha^*(\chi^{u_i}) = \sum_{j=0}^{m} a_{i,j} t^j \text{ for } 1 \leq i \leq n \tag{4.3}$$

is nonempty and finite.

Proof. Consider the subgroup M' of M generated by $u_1, \ldots, u_n$ and the corresponding torus $T' = \text{Spec } k[M']$. Note that we have an induced morphism $f\colon T \to T'$. It is well-known that in characteristic 0, this map is finite and étale. This follows, for example, by choosing a basis $w_1, \ldots, w_n$ of M such that $d_1 w_1, \ldots, d_n w_n$ is a basis of M', for some positive integers $d_1, \ldots, d_n$. In this case, it follows from Lemma II.7 that

$$T_m \simeq T'_m \times_{T'} T.$$

In particular, the induced morphism $T'_m \to T_m$ is finite and étale and its fibers are non-empty and finite. Since it is clear that there is a unique $\beta \in T'_m$ such that $\beta^*(\chi^{u_i}) = \sum_{j=0}^m a_{i,,j} t^j$ for all i, we deduce the assertion in the lemma. □

Definition IV.13. For each $a \in \text{Int}(\sigma) \cap N$, we define

$$\Phi(a) := \min \Big\{ \sum_{i=1}^{n} \langle a, u_i \rangle | u_1, \ldots, u_n \text{ span } M_{\mathbb{R}}, \text{ with } u_i \in M \cap \sigma^\vee \text{ for each } i \Big\}, \tag{4.4}$$

where the minimum is run over all linearly independent sets of vectors $\{u_1, \ldots, u_n\}$ in $M \cap \sigma^\vee$.

Remark IV.14. Clearly if the minimum in (4.4) is attained at some elements $u_1, \ldots, u_n$, each u_i must be irreducible. We show in the following one way to find elements $u_1, \ldots, u_n$ at which the above minimum is achieved.

Fix $a \in \text{Int}(\sigma) \cap N$. Let $u_1, \ldots, u_s$ be the minimal set of generators of the semigroup $M \cap \sigma^\vee$ and let $S_0 := \{u_1, \ldots, u_s\}$. We first choose $u_{j_1} \in S_0$ such that $\varphi_a(u_{j_1}) = \langle a, u_{j_1} \rangle$ is minimal and define $S_1 := S_0 \backslash \text{Span}(u_{j_1})$. Recursively, for each $1 \leq i \leq n-1$, assuming $u_{j_1}, \ldots, u_{j_i}$ are chosen and $S_i = S_0 \backslash \text{Span}(u_{j_1}, \ldots, u_{j_i})$, we choose $u_{j_{i+1}} \in S_i$ such that $\varphi_a(u_{j_{i+1}}) = \langle a, u_{j_{i+1}} \rangle$ is minimal and define $S_{i+1} := S_0 \backslash \text{Span}(u_{j_1}, \ldots, u_{j_{i+1}})$. Once $u_{j_1}, \ldots, u_{j_n}$ are all chosen, it is clear that they span $M_{\mathbb{R}}$.

Lemma IV.15. *For each* $a \in \mathrm{Int}(\sigma) \cap N$ *and* $u_{j_1}, \ldots, u_{j_n}$ *chosen as above, we have*

$$\sum_{k=1}^{n} \langle a, u_{j_k} \rangle = \Phi(a).$$

Proof. By Remark IV.14 we can find $i_1, \ldots, i_n$ such that $u_{i_1}, \ldots, u_{i_n}$ span $M_{\mathbb{R}}$ and they compute $\Phi(a)$. If the set $\{u_{i_1}, \ldots, u_{i_n}\}$ is equal to $\{u_{j_1}, \ldots, u_{j_n}\}$, the claim in the lemma follows immediately. Hence we assume that by relabeling, there exists some k, with $1 \leq k \leq n$, such that $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}$ and $i_k \neq j_k$. If $k = n$, we have $\langle a, u_{j_n} \rangle \leq \langle a, u_{i_n} \rangle$ by the choice of u_{j_n}. Hence

$$\sum_{k=1}^{n} \langle a, u_{j_k} \rangle \leq \sum_{k=1}^{n} \langle a, u_{i_k} \rangle.$$

This proves the claim in the lemma.

Now suppose the conclusion holds when $k > k_0$ for some $k_0 < n$, and we consider the case when $k = k_0$. We claim there exists some l, with $k_0 \leq l \leq n$, such that

$$u_{j_{k_0}} \notin \mathrm{Span}(u_{i_1}, \ldots, \hat{u}_{i_l}, \ldots, u_{i_n}).$$

Otherwise, we have

$$\begin{aligned} u_{j_{k_0}} &\in \bigcap_{l=k_0}^{n} \mathrm{Span}(u_{i_1}, \ldots, \hat{u}_{i_l}, \ldots, u_{i_n}) \\ &= \mathrm{Span}(u_{i_1}, \ldots, u_{i_{k_0-1}}) \\ &= \mathrm{Span}(u_{j_1}, \ldots, u_{j_{k_0-1}}). \end{aligned}$$

But this contradicts the fact that $u_{j_1}, \ldots, u_{j_n}$ span $M_{\mathbb{R}}$.

The above claim implies that if we replace u_{i_l} by $u_{j_{k_0}}$, $u_{i_1}, \ldots, u_{i_n}$ still span $M_{\mathbb{R}}$. It also shows that

$$u_{i_l} \notin \mathrm{Span}(u_{j_1}, \ldots, u_{j_{k_0-1}}),$$

and hence by the choice of $u_{j_{k_0}}$, we have $\langle a, u_{j_{k_0}} \rangle \leq \langle a, u_{i_l} \rangle$. We conclude that if we replace u_{i_l} by $u_{j_{k_0}}$, the question is reduced to the case when $k \geq k_0 + 1$, and we are done by induction. □

Theorem IV.16. *Fix a lattice point $a \in \mathrm{Int}(\sigma)$. Let $\chi^{u_1}, \chi^{u_2}, \ldots, \chi^{u_s}$ be the minimal set of monomial generators of $k[M \cap \sigma^\vee]$. If $H_{m,a}$ is the stabilizer of any element of $T_{m,a}$ under the T_m-action, then the following hold:*

(1) We have $\dim(H_{m,a}) = \Phi(a)$ for

$$m \geq \max\{\max_{1 \leq i \leq n} \langle a, u_{j_i} \rangle\}, \tag{4.5}$$

where the maximum is taken over all possible choices of n vectors $u_{j_1}, ..., u_{j_n}$ among $u_1, u_2, ..., u_s$ that span $M_{\mathbb{R}}$, and such that the minimum in (4.4) is attained.

(2) If m does not satisfy the inequality (4.5), then we have either $\dim(H_{m,a}) = \Phi(a)$ or $m \leq \dim(H_{m,a}) \leq \Phi(a)$.

Proof. For simplicity we write φ_m for $\min\{m, \bar{\varphi}_a\}$, and φ^m for $\min\{m+1, \bar{\varphi}_a\}$. Let $H_{m,a}$ be the stabilizer of the special jet $\gamma_{\bar{\varphi}_a}$ defined in Lemma IV.3. Then an m-jet $\alpha \in T_m$ is contained in $H_{m,a}$ if and only if

$$\alpha^*(\chi^{u_i}) \cdot t^{\varphi^m(u_i)} = t^{\varphi^m(u_i)} \text{ in } k[t]/(t^{m+1}) \text{ for } 1 \leq i \leq s. \tag{4.6}$$

This is clearly equivalent to

$$\begin{aligned} \alpha^*(\chi^{u_i}) &= 1 + \sum_{j=m+1-\varphi^m(u_i)}^{m} a_{i,j} t^j, \text{ if } \varphi^m(u_i) \leq m, \\ \alpha^*(\chi^{u_i}) &= \sum_{j=0}^{m} a_{i,j} t^j, \text{ if } \varphi^m(u_i) = m+1, \end{aligned} \tag{4.7}$$

for each i and for some $a_{i,j} \in k$, with the condition that $a_{i,0} \neq 0$ when $\varphi^m(u_i) = m+1$.

Choose any n vectors from $\{u_1, \ldots, u_s\}$ that span $M_{\mathbb{R}}$. By relabeling, let us assume they are $u_1, \ldots, u_n$. We define $A := \mathbb{A}^{\sum_{i=1}^n \varphi^m(u_i)}$ and the map

$$\pi : H_{m,a} \longrightarrow A, \ \pi(\alpha) = (a_{i,j})_{1 \leq i \leq n, m+1-\varphi^m(u_i) \leq j \leq m}.$$

Then Lemma IV.12 implies that π has finite fibers. Therefore, we have

$$\dim(H_{m,a}) \leq \dim(A) = \sum_{i=1}^{n} \varphi^m(u_i).$$

By letting $u_1, \dots, u_n$ vary so that they span $M_{\mathbb{R}}$, we conclude that

$$\dim(H_{m,a}) \leq \min\{\sum_{i=1}^{n} \varphi^m(u_{j_i}) | u_{j_1}, \dots, u_{j_n} \text{ span } M_{\mathbb{R}}\} \leq \Phi(a).$$

In what follows, we assume that after relabeling, $u_1, \dots, u_n$ are chosen as in Lemma IV.15. We claim that

$$\dim(H_{m,a}) \geq \sum_{i=1}^{n} \varphi_m(u_i). \tag{4.8}$$

Consider the subgroup M' of M generated by $u_1, \dots, u_n$ and the corresponding torus $T' = \text{Spec } k[M']$. By the proof of Lemma IV.12, the commutative diagram

$$\begin{array}{ccc} T_m & \longrightarrow & T'_m \\ {\scriptstyle \pi^T_m}\downarrow & & {\scriptstyle \pi^{T'}_m}\downarrow \\ T & \longrightarrow & T'. \end{array}$$

is Cartesian. Hence, for each $\alpha' \in T'_m$ that lies over $(1, \dots, 1) \in T'$, there is a unique $\alpha \in T_m$ lying over $(1, \dots, 1) \in T$ such that α is mapped to α'. We claim that for each $\alpha' \in T'_m$ lying over $(1, \dots, 1)$ that satisfies (4.7) for $1 \leq i \leq n$, the corresponding $\alpha \in T_m$ is an element in $H_{m,a}$.

To prove this, we just need to show that α satisfies conditions (4.7) for $1 \leq i \leq s$. Since α maps to α', it automatically satisfies (4.7) for $1 \leq i \leq n$. Now pick an integer z such that $n+1 \leq z \leq s$. Then there exist integers $l > 0$, d_i and $q \leq n$ such that

$$lu_z = \sum_{i=1}^{q} d_i u_i, \tag{4.9}$$

where $d_q \neq 0$. By applying α^* on both sides, we get

$$\alpha^*(\chi^{u_z})^l = \prod_{i=1}^{q} \alpha^*(\chi^{u_i})^{d_i}.$$

By using (4.7) for $1 \leq i \leq n$, we see that the t-order of $\prod_{i=1}^{q} \alpha^*(\chi^{u_i})^{d_i} - 1$, hence also that of $\alpha^*(\chi^{u_z})^l - 1$, is at least $\min_{1\leq i\leq q}\{m+1-\varphi_m(u_i)\}$. Since α lies over $(1,\ldots,1)$, this implies that the t-order of $\alpha^*(\chi^{u_z}) - 1$ is at least $\min_{1\leq i\leq q}\{m+1-\varphi_m(u_i)\}$. On the other hand, equation (4.9) implies that

$$u_z \in \{u_1,\ldots,u_s\}\backslash \text{Span}(u_1,\ldots,u_k),$$

for each k with $1 \leq k \leq q-1$. By the construction of $u_1,\ldots,u_n$, we have $\varphi_a(u_z) \geq \varphi_a(u_k)$ for each $1 \leq k \leq q$. Hence, we have

$$m+1-\varphi^m(u_z) \leq \min_{1\leq i\leq q}\{m+1-\varphi^m(u_i)\}.$$

So the t-order of $\alpha^*(\chi^{u_z}) - 1$ is $\geq m+1-\varphi^m(u_z)$. This, however, implies condition (4.7) for $i = z$. Since z is arbitrary, α satisfies conditions (4.7) for $1 \leq i \leq s$, and hence $\alpha \in H_{m,a}$.

Define the affine space A and the map $\pi : H_{m,a} \to A$ as above with respect to $u_1,\ldots,u_n$. Let $Y \subset A$ be the subspace defined by $a_{i,0} = 1$ for $1 \leq i \leq n$ such that $\varphi^m(u_i) = m+1$. Then the above discussion shows that Y is contained in the image of π. We conclude that

$$\dim(H_{m,a}) \geq \dim(Y) = \sum_{i=1}^{n} \varphi_m(u_i).$$

According to Lemma IV.15, the minimum in (4.4) is achieved by $u_1,\ldots,u_n$. Hence, the condition (4.5) guarantees that $m \geq \varphi_a(u_i)$ for each $1 \leq i \leq n$. Under this condition, we have

$$\dim(H_{m,a}) \geq \sum_{i=1}^{n} \varphi_m(u_i) = \sum_{i=1}^{n} \varphi_a(u_i) = \Phi(a).$$

This completes the proof of (1).

For (2), we consider two cases. If $m \geq \max_{1\leq i\leq n} \varphi_a(u_i)$, then (4.8) implies that $\dim(H_{m,a}) \geq \Phi(a)$ as in (1). If there is some i, with $1 \leq i \leq n$, such that $m < \varphi_a(u_i)$, then $\varphi_m(u_i) = m$. So (4.8) implies that $\dim(H_{m,a}) \geq \varphi_m(u_i) = m$. Since we have proved that $\dim(H_{m,a})$ is always $\leq \Phi(a)$, the conclusions in (2) follow. □

Corollary IV.17. *With the same assumptions as in Theorem IV.16 and for all $m \geq 0$, the dimension of the orbit $T_{m,a}$ satisfies one of the following:*

$$\dim(T_{m,a}) = (m+1)n - \Phi(a), \text{ or} \tag{4.10}$$

$$(m+1)n - \Phi(a) \leq \dim(T_{m,a}) \leq (m+1)n - m. \tag{4.11}$$

Proof. Observe that T is smooth of dimension n. Hence by Corollary II.8, $\dim(T_m) = n(m+1)$. The conclusions follow immediately from Theorem IV.16. □

Now we can prove our main result. Recall that the Mather minimal log discrepancy can be computed in terms of the invariant λ defined in Definition III.21, via Property III.22. According to Lemma III.20, this in turn can be computed from the dimension of C^m (defined in Definition IV.2), when m is large enough. We have seen that C^m can be approximated by a union of explicit T_m-orbits. Thus computing the dimension of C^m boils down to computing the dimension of these T_m-orbits.

Theorem IV.18. *For m large enough we have*

$$\dim(C^m) = n(m+1) - \min_{a \in \mathrm{Int}(\sigma) \cap N} \Phi(a), \tag{4.12}$$

where Φ is defined in Definition IV.13.

Proof. First of all, note that $T_{m,a}$ lies over the torus-fixed point x_σ if and only if a is in the interior of the cone σ (see Lemma IV.3). Therefore C^m is the union of finitely many T_m-orbits $T_{m,a}$ (by Lemma IV.3 and Corollary IV.10), for a in the interior of σ, and of the orbits contained in the image of the $(D_i)_\infty$. But $\dim(\psi_m((D_i)_\infty) \leq (n-1)(m+1)$ by Lemma II.12. When m is large enough, the dimension of these orbits contained in the image of the $(D_i)_\infty$ is smaller than $mn - \lambda(x_\sigma)$. Thus, we only need to compute $\max_{a \in \text{Int}(\sigma) \cap N} \dim(T_{m,a})$ when m is large enough. Note that even though Int$(\sigma) \cap N$ is an infinite set, we are actually taking maximum over the finite set of T_m-orbits.

By Lemma III.20 we thus see if m is large enough, then

$$mn - \lambda(x_\sigma) = \dim(C^m) = \max_{a \in \text{Int}(\sigma) \cap N} \dim(T_{m,a}).$$

Let us fix such m such that, in addition, $m > n + \lambda(x_\sigma)$. From Corollary IV.17, we see two cases (4.10) and (4.11). If $\dim(T_{m,a}) \leq (m+1)n - m$, then we have

$$n(m+1) - \Phi(a) \leq \dim(T_{m,a}) \leq (m+1)n - c \cdot m < mn - \lambda(x_\sigma).$$

Therefore, replacing these $\dim(T_{m,a})$ by $n(m+1) - \Phi(a)$ does not change the maximum of $\dim(T_{m,a})$. So we get

$$\begin{aligned} & \max_{a \in \text{Int}(\sigma) \cap N} \dim(T_{m,a}) \\ = & \max_{a \in \text{Int}(\sigma) \cap N} \Big\{ (m+1)n - \Phi(a) \Big\} \\ = & \ n(m+1) - \min_{a \in \text{Int}(\sigma) \cap N} \Phi(a). \end{aligned}$$

The last formula gives the assertion in the theorem. □

Corollary IV.19. *Let X be an affine toric variety over k of dimension n associated to a cone σ. Let N be the lattice and M be the dual lattice. If σ spans $N_\mathbb{R}$ and $x_\sigma \in X$ is the torus-invariant point, the invariant $\lambda(x_\sigma)$ defined in Definition III.21 is computed by the following formula*

$$\lambda(x_\sigma) = \min_{a \in \text{Int}(\sigma) \cap N} \Phi(a) - n,$$

where the function Φ is defined in Definition IV.13.

The following is a direct corollary of Corollary IV.19 and Proposition III.22.

Corollary IV.20. *With the same assumptions as in Corollary IV.19, we have*

$$\widehat{\mathrm{mld}}(x_\sigma; X) = \min_{a \in \mathrm{Int}(\sigma)\cap N} \Big\{ \min \big\{ \sum_{i=1}^{n} \langle a, u_i\rangle | u_1, \ldots, u_n \text{ span } M_{\mathbb{R}},\, u_i \in M \cap \sigma^\vee \text{ for each } i \big\} \Big\},$$

where the second minimum is run over all linearly independent sets of vectors $\{u_1, \ldots, u_n\}$ *in* $M \cap \sigma^\vee$.

4.4 Examples

The conclusions of Theorem IV.18 and Corollary IV.19 involve two minima. It is not clear whether the formula can be simplified in the case of an arbitrary toric variety. However, we can simplify this formula in some special cases. In this section, we provide some examples of computations of the invariant λ.

Example IV.21. Suppose $\sigma \subset \mathbb{R}^2$ is the two dimensional cone generated by $2e_1 - e_2$ and e_2, where e_1 and e_2 form the standard basis of N. Then $\sigma^\vee$ is a cone in $M_{\mathbb{R}}$ generated by e_1^* and $e_1^* + 2e_2^*$, where e_1^* and e_2^* form the dual basis. It's easy to see that $u_1 = e_1^*$, $u_2 = e_1^* + e_2^*$ and $u_3 = e_1^* + 2e_2^*$ form the minimal set of generators of $M \cap \sigma^\vee$.

For each $a \in N$ we write $a = (x, y)$, where x, y are coordinates with respect to the standard basis. In order that $a \in \mathrm{Int}(\sigma) \cap N$, we need to have $x > 0$ and $x + 2y > 0$. Therefore, according to Corollary IV.19 we have

$$\begin{aligned} \lambda(x_\sigma) &= \min_{x>0, x+2y>0} \min\{x + (x + y), x + (x + 2y), (x + y) + (x + 2y)\} - 2 \\ &= \min_{x>0, x+2y>0} \min\{2x + y, 2x + 3y\} - 2. \end{aligned}$$

It's easy to see that the minimum is equal to 0, which is attained when $x = 1$ and $y = 0$, and hence $\widehat{\mathrm{mld}}(x_\sigma; X) = \dim(X) = 2$.

In fact, we have the following general result:

Proposition IV.22. *If the torus-invariant point* x_σ *is an isolated singularity of a simplicial toric variety* X*, then* $\lambda(x_\sigma) = 0$*, and hence* $\widehat{\mathrm{mld}}(x_\sigma; X) = \dim(X)$.

Proof. First we claim that if x_σ is an isolated singularity, then all facets (faces of codimension 1) of σ are nonsingular. Suppose that there is a proper face τ of σ that is singular. Recall that

$$O(\tau) = \text{Spec } k[M \cap \tau^\perp] \cong (k^*)^{n-\dim(\tau)}$$

is the T-orbit that contains the distinguished point x_τ. Denote by N_τ the subgroup of N generated by $N \cap \tau$. Then we may choose a splitting of N and write

$$N = N_\tau \oplus N', \ \tau = \tau' \oplus \{0\},$$

where τ' is a cone in $(N_\tau)_\mathbb{R}$. Dually, we can decompose $M = M_\tau \oplus M'$. Let

$$U_\tau = \text{Spec } k[M \cap \tau^\vee],$$

and let $U_{\tau'}$ be the affine toric variety corresponding to the cone τ' and lattice N_τ. With this notation, we have

$$U_\tau \cong \text{Spec } k[M_\tau \cap \tau'^\vee] \times \text{Spec } k[M'] \cong U_{\tau'} \times (k^*)^{n-\dim(\tau)}. \tag{4.13}$$

Note that U_τ is an open subset of X that contains $O(\tau)$. Since τ' is a singular cone, the torus-fixed point $x_{\tau'} \in U_{\tau'}$ is a singular point. In this case, the orbit $O(\tau)$, which corresponds via the above isomorphism to $\{x_{\tau'}\} \times \text{Spec } k[M']$ is a subset of dimension $n - \dim(\tau)$ contained in the singular locus of X and that contains 0 in its closure. This contradicts the fact that 0 is an isolated singular point of X. So we conclude that all facets are nonsingular.

Since X is simplicial, the cone σ has only n one-dimensional faces. Assume that $v_1, \ldots, v_n$ are the primitive lattice vectors on these one-dimensional faces. Then $v_1, \ldots, v_{n-1}$ span a facet of σ, and is therefore nonsingular. By applying an automorphism on N one may assume that $v_1 = e_1, \ldots,\ v_{n-1} = e_{n-1}$ and $v_n = a_1e_1 + \cdots + a_{n-1}e_{n-1} + te_n$ with $0 \le a_i < t$. Define $a = e_1 + \cdots + e_n$.

Note that $o_i := te_i^* - a_i e_n^*$ is orthogonal to the facet spanned by $v_1, \ldots, \hat{v}_i, \ldots, v_n$ for every i, with $1 \le i \le n-1$. In fact, the dual cone $\sigma^\vee$ is spanned by $o_1, \ldots, o_{n-1}, e_n^*$. Since $\langle o_i, a \rangle = t - a_i > 0$ and $\langle e_n^*, a \rangle = 1$, a is in the interior of σ.

Clearly $e_1^*, \ldots, e_n^*$ are all in the dual cone $\sigma^\vee$. In fact, each e_i^* is on the face spanned by o_i and e_n^*. Since $\varphi_a(e_i^*) = 1$, we have $\Phi(a) \le n$ and hence $\lambda(x_\sigma) = 0$. By Proposition III.22 we get $\widehat{\mathrm{mld}}(x_\sigma; X) = \dim(X)$. □

Corollary IV.23. *If X is a two-dimensional affine toric variety, then $\lambda(x_\sigma) = 0$.*

Proof. Observe that every two-dimensional affine toric variety is simplicial, and that every facet is one-dimensional, hence nonsingular. Thus x_σ is an isolated singularity of a simplicial toric variety. The conclusion follows immediately from Proposition IV.22. □

The above examples might suggest that λ is always 0, or $\widehat{\mathrm{mld}}$ is always equal to $\dim(X)$, for any toric variety. But this is not true in general, as we will see shortly. Now let us look at an example of a different type. We discuss this class of examples in detail in the next chapter (see Example V.23 for details); we refer to this section for the proof of the formula that we use.

Example IV.24. Let $X \subset \mathbb{A}^{n+1}$ be the hypersurface defined by the binomial function

$$f = x_1 x_2 \cdots x_n - y^{n-1}$$

for some $n \ge 3$. The dimension of X is n while the dimension of X_{sing} is $n-2$. Since X is Cohen-Macaulay, being a hypersurface, it follows from Serre's criterion that X is normal. It is a general fact that X is a toric variety if it is normal and defined by binomials ([Stu95, Lemma 1.1]). By applying formula (5.17) below from Chapter V, we immediately get $\lambda = 1$.

4.5 Extension to arbitrary closed points

Corollary IV.19 gives the formula that computes the invariant λ associated to the torus-invariant point for an affine toric variety X over k that corresponds to a cone that spans $N_{\mathbb{R}}$. In this section, we show how this computation is generalized to an arbitrary closed point of a toric variety X. We start by proving the following proposition:

Proposition IV.25. *Let X and Y be two varieties over k such that Y is smooth. For any closed points $x \in X$ and $y \in Y$, the invariant λ for the closed point $(x, y) \in X \times Y$ is equal to $\lambda(x)$.*

Proof. By definition, we have

$$\lambda((x, y)) = (\dim(X) + \dim(Y))m - \dim(\psi_m^{X\times Y}((X \times Y)_\infty),$$

for m large enough. According to Remark II.10, this is equal to

$$(\dim(X) + \dim(Y))m - \dim(\psi_m^X(X_\infty)) - \dim(\psi_m^Y(Y_\infty)) = \lambda(x) + \lambda(y).$$

According to Remark III.23, $\lambda(y) = 0$. Therefore, $\lambda((x, y)) = \lambda(x)$. □

The key fact is that any closed point is in the orbit of the distinguished point of a face of σ. More precisely, let X be the affine toric variety associated to a cone σ and $O(\tau)$ be the T-orbit that contains the distinguished point x_τ for some face τ of σ. Then $X = \cup_\tau O(\tau)$ with τ varying over all faces of σ. We refer the reader to [Ful93, Chapter 3] for details. Then $\lambda(p) = \lambda(x_\tau)$ for every p in $O(\tau)$, because there is an element $t \in T$ that maps p to x_τ, and such that multiplication by t gives an automorphism of X. Therefore, it is enough to compute $\lambda(x_\tau)$, where τ is a face of σ.

Following the notation in the proof of Proposition IV.22, we have an open subset $U_\tau \cong U_{\tau'} \times (k^*)^{n-\dim(\tau)}$ of X that contains $O(\tau)$. The point $x_\tau \in O(\tau)$ is mapped to $(x_{\tau'}, \underline{1})$ by the isomorphism, where $x_{\tau'}$ is the torus-invariant point in $U_{\tau'}$. Therefore, we are reduced to computing $\lambda((x_{\tau'}, \underline{1}))$ and we obtain the following corollary by using Proposition IV.25:

Corollary IV.26. *With the above notation, if $X = X(\sigma)$ is an affine toric variety over k of dimension n and τ is a face of σ, then we have $\lambda(x_\tau) = \lambda(x_{\tau'})$. Equivalently, we have*

$$\widehat{\mathrm{mld}}(x_\tau; X) - n = \widehat{\mathrm{mld}}(x_{\tau'}; U_{\tau'}) - \dim(\tau).$$

CHAPTER V

Mather Minimal Log Discrepancy of Hypersurfaces

This chapter is devoted to the computation of the Mather minimal log discrepancy of the origin on a hypersurface whose defining equation has very general coefficients.

5.1 Basic setup

Throughout this chapter, we assume that X is an integral hypersurface in

$$\mathbb{A}^{n+1} = \text{Spec } \mathbb{C}[x_1, \ldots, x_{n+1}],$$

for some positive integer n, over the field of complex numbers. We have $\dim(X) = n$. After a change of coordinates, we may and will assume that the origin of $\mathbb{A}^{n+1}$ is contained in X.

Let f be the defining equation of X in $\mathbb{A}^{n+1}$. We can write $f = \sum_{i=1}^{N} a_{I^i} x^{I^i}$, where $I^i = (I_1^i, I_2^i, \ldots, I_{n+1}^i)$ are multi-indices and x^{I^i} stands for $\prod_{j=1}^{n+1} x_j^{I_j^i}$. Suppose that all the coefficients a_{I^i} are nonzero. Then N, the number of monomials in the polynomial f, is at least 2 by the integrality assumption, unless X is a coordinate plane. We denote by $\mathbb{Z}_+$ the set of positive integers and by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers.

Definition V.1. The *support* of a multi-index I^i is $|I^i| := \{j | I_j^i > 0\}$. Given an $(n+1)$-tuple $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{Z}^{n+1}$ and a multi-index I, we define the *product* as $\alpha \cdot I :=$

$\sum_{j=1}^{n+1} \alpha_j I_j$. The *support* of a polynomial $f = \sum_{i=1}^{N} a_{I^i} x^{I^i}$, with all $a_{I^i} \neq 0$, is the set

$$A = \{I^1, \dots, I^N\} \subset (\mathbb{Z}_{\geq 0})^{n+1}.$$

The *dimension* of A is defined by $\dim(A) = \dim_{\mathbb{Q}}(\text{Span}_{\mathbb{Q}}\{A - a\})$, for any $a \in A$.

Remark V.2. Clearly the dimension of a support A is independent of the choice of a. By our assumption on f, the support A of f has at least two points, hence $\dim(A) \geq 1$.

Remark V.3. We may assume without loss of generality that

$$\cup_{1 \leq i \leq N} |I^i| = \{1, 2, \dots, n+1\}.$$

Otherwise, X is the product of an affine space with a hypersurface of lower dimension. Then by Proposition IV.25, computing λ for the origin in X is reduced to computing the corresponding $\lambda(0)$ on the lower dimensional hypersurface.

Remark V.4. If 0 is a smooth point, then the invariant $\lambda(0)$ is trivially zero by Remark III.23. So we focus on the case where 0 is a singular point of X. In particular, we assume that X is not a hyperplane.

In order that the hypersurface X contains the origin, we require that f is a polynomial in $(x_1, x_2, \dots, x_n) \cdot \mathbb{C}[x_1, x_2, \dots, x_{n+1}]$, or equivalently, the point $(0, 0, \dots, 0)$ is not in the support of f. By requiring that X is irreducible and is not a hyperplane, we see that f is not divisible by x_i for each i. This means that the support of f contains at least one point in each coordinate plane $x_i = 0$. We first characterize those A, such that a general polynomial with support A defines an integral hypersurface. We denote by conv(A) the convex hull of A. The following result is a simplified version of [Yu16, Theorem 3]:

Theorem V.5. *Let $R = \mathbb{C}[x_1^{\pm 1}, \dots, x_{n+1}^{\pm 1}]$ be the Laurent polynomial ring in $n+1$ variables. Then a general polynomial f with support A generates a proper prime ideal in R if and only if one of the following holds:*

(1) $\dim(A) \geq 2$*, or*

(2) $\dim(A) = 1$ *and* conv(A) *contains only two integral points.*

Lemma V.6. *Let* $R = \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$ *be the Laurent polynomial ring in* $n+1$ *variables and* f *be a polynomial in* $\mathbb{C}[x_1, \ldots, x_{n+1}]$ *that is not divisible by any* x_i*. If* f *generates a prime ideal in* R*, then* f *also generates a prime ideal in* $\mathbb{C}[x_1, \ldots, x_{n+1}]$.

Proof. The assertion follows from the fact that

$$f \cdot \mathbb{C}[x_1, \ldots, x_{n+1}] = f \cdot \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}] \cap \mathbb{C}[x_1, \ldots, x_{n+1}].$$

This follows easily, using the fact that $\mathbb{C}[x_1, \ldots, x_{n+1}]$ is a UFD, from the fact that f is not divisible by any x_i. □

Definition V.7. A finite subset A of $(\mathbb{Z}_{\geq 0})^{n+1}$ is called *integral* if the following conditions hold:

(1) A contains at least one point in each coordinate plane $x_i = 0$,

(2) A does not contain the origin $(0, \ldots, 0)$, and

(3) $\dim(A) \geq 2$, or $\dim(A) = 1$ and conv(A) contains only two integral points.

Let $|A|$ be the cardinality of A. We denote by $F(A) \subset (\mathbb{C}^*)^{|A|}$ the set of coefficients, such that a polynomial f with support A and these coefficients generates a prime ideal in the Laurent polynomial ring R.

By Theorem V.5, the set $F(A)$, for each integral subset $A \subset (\mathbb{Z}_{\geq 0})^{n+1}$, contains a nonempty open subset of $(\mathbb{C}^*)^{|A|}$. The following is a direct corollary of Lemma V.6:

Corollary V.8. *If* f *is a polynomial with an integral support* A *and coefficients in* $F(A)$*, then* f *defines an integral hypersurface in* $\mathbb{A}^{n+1}$ *containing the origin.*

In what follows, we fix an integral subset $A \subset (\mathbb{Z}_{\geq 0})^{n+1}$ with cardinality $N \geq 2$, and assume that the defining equation f has support A and coefficients in $F(A)$. Recall that we write $f = \sum_{i=1}^{N} a_{I^i} x^{I^i}$ with all $a_{I^i} \neq 0$. In this case we have $A = \{I^1, \ldots, I^N\}$.

Definition V.9. For each positive integer m, we define as in the previous chapter the sets $C^m := \psi_m(\pi^{-1}(0))$ in the m^{th} jet scheme of X, where 0 is the origin of $\mathbb{A}^{n+1}$. For each $(n+1)$-tuple $\alpha \in \mathbb{Z}^{n+1}$ such that $1 \leq \alpha_j \leq m$ for each j, we define

$$C_{\alpha}^{m} := C^m \cap (\cap_{1 \leq j \leq n+1} \mathrm{Cont}^{\alpha_j}(x_j)_m).$$

In other words, C_{α}^{m} is a subset of C^m with prescribed order along each x_j.

Now we fix α with $\alpha_j \geq 1$ for each j. Let $n_0(\alpha) = \min_{1 \leq i \leq N}\{\alpha \cdot I^i\}$. After relabeling we may assume that the minimum is attained by precisely those i with $1 \leq i \leq k$ for some $k \geq 1$. Then the image of f under the map

$$\text{(5.1)} \quad \mathbb{C}[x_1, x_2, \ldots, x_{n+1}] \longrightarrow (\mathbb{C}[x_j^{(s)} | 1 \leq j \leq n+1, s \geq \alpha_j])[\![t]\!],\ x_j \longmapsto \sum_{s=\alpha_j}^{\infty} x_j^{(s)} t^s,$$

has t-order $\geq n_0(\alpha)$ and the coefficient of $t^{n_0(\alpha)}$ is

$$\text{(5.2)} \qquad P_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) := \sum_{i=1}^{k} a_{I^i} \Pi_{j=1}^{n+1} (x_j^{(\alpha_j)})^{I_j^i}.$$

P_0 is an element in $\mathbb{C}[x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}]$. We define the condition Δ^{α}, which will be used in the statements of the main result in this chapter, as follows:

Condition V.10. We say that the condition Δ^{α} holds for f if

$$\bigcap_{j=1}^{n+1} \mathbb{V}\Big(\frac{\partial P_0}{\partial x_j^{(\alpha_j)}}\Big) \cap \mathbb{V}(P_0) = \emptyset$$

in the torus

$$(\mathbb{C}^*)^{n+1} = \mathrm{Spec}\ \mathbb{C}[x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}]_{(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})}.$$

Definition V.11. The *weight* of a monomial $\prod_{j=1}^{n+1}\prod_{i=1}^{b_j} x_j^{(\beta_i^j)}$ is the sum of the superscripts $\sum_{i,j}\beta_i^j$; the *weight* of a polynomial in $(x_j^{(u)})_{1\leq j\leq n+1;\ u>0}$ is the smallest weight among its monomials.

Remark V.12. It is easy to see that for every s, each monomial in the coefficient of t^s in the image of a polynomial under the map (5.1) has weight s.

5.2 Main results

We begin with a few lemmas.

Lemma V.13. *For a fixed* $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{Z}_+)^{n+1}$*, the set*

$$F_\alpha := \{(a_{I^i})_{1\leq i\leq N} \in (\mathbb{C}^*)^N | \text{condition } \Delta^\alpha \text{ is satisfied}\}$$

contains a nonempty open subset of $(\mathbb{C}^*)^N$.

Proof. We use the Kleiman-Bertini Theorem in characteristic zero, which states that the general element of a linear system of divisors on a variety Y is nonsingular away from the base locus of the linear system and the singular locus of Y.

Let Y be the affine space $\mathbb{C}^{n+1} = \text{Spec } \mathbb{C}[x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}]$ and Z be the hypersurface in Y defined by the polynomial $P_0 = \sum_{i=1}^k a_{I^i}\Pi_{j=1}^{n+1}(x_j^{(\alpha_j)})^{I_j^i}$. Note that the left-hand side of the equation in condition Δ^α is the singular locus of Z. Therefore, it suffices to show that for a general choice of coefficients, Z will be nonsingular away from the coordinate planes.

The linear system of divisors $\mathbb{H}$ on Y consisting of hypersurfaces defined by polynomials of the form $p = \sum_{i=1}^k a_{I^i}\Pi_{j=1}^{n+1}(x_j^{(\alpha_j)})^{I_j^i}$, for $a_{I^i} \in \mathbb{C}$, is clearly base point free away from the coordinate planes because each monomial x^{I^i} is already so. By the Kleiman-Bertini Theorem, the hypersurface Z is nonsingular away from the coordinate planes for a general choice of $(a_{I^i})_{1\leq i\leq N} \in (\mathbb{C}^*)^N$. □

We now study the image of f under the map (5.1). It suffices to study the image of each monomial of f. The image of f is just the sum of the images of all the monomials of f.

Lemma V.14. *Fix $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{Z}_+)^{n+1}$. For each $s \geq 1$, the coefficient of t^s in the image of $x_1^{b_1} x_2^{b_2} \ldots x_{n+1}^{b_{n+1}}$ under the map (5.1) is equal to*

$$\sum_{c_{i,j}} \Big(\prod_{i=1}^{n+1} b_i! \cdot \prod_{j \geq \alpha_i} \frac{(x_i^{(j)})^{c_{i,j}}}{c_{i,j}!} \Big), \tag{5.3}$$

where the sum is over all $c_{i,j}$ with $1 \leq i \leq n+1$ and $j \geq \alpha_i$ such that

$$\sum_{i,j} j \cdot c_{i,j} = s \text{ and } \sum_{j \geq \alpha_i} c_{i,j} = b_i \text{ for all } i.$$

Proof. By considering the weight of a monomial $\prod_{i=1}^{n+1} \prod_{j \geq \alpha_i} (x_i^{(j)})^{c_{i,j}}$, it is clear that if this monomial appears with nonzero coefficient in the coefficient of t^s, then we have $\sum_{i,j} j \cdot c_{i,j} = s$. If such a monomial shows up in the image of $x_1^{b_1} x_2^{b_2} \ldots x_{n+1}^{b_{n+1}}$, then it clearly satisfies $\sum_{j \geq \alpha_i} c_{i,j} = b_i$ for all i. Moreover, it follows from the multinomial formula that if these conditions are satisfied, then the coefficient of the above monomial is

$$\prod_{i=1}^{n+1} \frac{b_i!}{\prod_{j \geq \alpha_i} c_{i,j}!}$$

□

This lemma shows that the images of two different monomials under the map (5.1) do not mix together. Thus, the number of monomials in the coefficient of t^s of the image of f is the sum of the numbers of monomials in the coefficient of t^s for the image of each of the monomials of f. Similarly, the highest superscript in the coefficient of t^s of the image of f is equal to the maximum of the highest superscript in the coefficient of t^s that appears in the images of all the monomials of f.

Lemma V.15. *Fix $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{Z}_+)^{n+1}$ and a monomial $x_1^{b_1} x_2^{b_2} \ldots x_{n+1}^{b_{n+1}}$. Then for each $s \geq \sum_{i=1}^{n+1} b_i \alpha_i$, the largest superscript appearing in the coefficient of t^s in the* *image of* $x_1^{b_1}x_2^{b_2}\dots x_{n+1}^{b_{n+1}}$ *under the map (5.1) is equal to* $s-\mu$ *for some fixed number* μ. *Moreover, this largest superscript only appears in the monomials of the form*

$$(x_1^{(\alpha_1)})^{b_1}(x_2^{(\alpha_2)})^{b_2}\dots(x_j^{(\alpha_j)})^{b_j-1}\dots(x_{n+1}^{(\alpha_{n+1})})^{b_{n+1}}\cdot x_j^{(s-\mu)},$$

for some j *such that* $\alpha_j=\max_{1\le i\le n+1}\alpha_i$, *and we have* $\mu=\sum_{i=1}^{n+1}b_i\alpha_i-\alpha_j$.

Proof. According to Lemma V.14, the coefficient of t^s in the image of the monomial $x_1^{b_1}x_2^{b_2}\dots x_{n+1}^{b_{n+1}}$ consists of monomials of the form $\prod_{i=1}^{n+1}\prod_{j=1}^{b_i}x_i^{(\beta_j^i)}$, with $\beta_j^i\ge\alpha_i$ for every i and j and such that $\sum_{i,j}\beta_j^i=s$. When $s<\sum_{i=1}^{n+1}b_i\alpha_i$, there is no such monomial. Hence we require that $s\ge\sum_{i=1}^{n+1}b_i\alpha_i$. When $s=\sum_{i=1}^{n+1}b_i\alpha_i$, there is only one monomial

$$(x_1^{(\alpha_1)})^{b_1}(x_2^{(\alpha_2)})^{b_2}\dots(x_j^{(\alpha_j)})^{b_j}\dots(x_{n+1}^{(\alpha_{n+1})})^{b_{n+1}}$$

in the coefficient of t^s. Hence the largest superscript that shows up in this case is equal to $\max_i\alpha_i$. In what follows, we assume that $s>\sum_{i=1}^{n+1}b_i\alpha_i$. Then the largest superscript that appears in the coefficient of t^s is given by the following optimization problem:

$$\begin{aligned}\max\quad&\max_{i,j}\{\beta_j^i\}\\ \text{s.t.}\quad&\beta_j^i\ge\alpha_i\text{ for each }i\\ \text{and}\quad&\sum_{i,j}\beta_j^i=s.\end{aligned}$$

Let $(\bar\beta_j^i)_{i,j}$ give the optimal solution to this optimization problem. We claim that there exist i_0 and j_0, with $1\le i_0\le n+1$ and $1\le j_0\le b_{i_0}$, such that $\bar\beta_j^i=\alpha_i$ if and only if $(i,j)\ne(i_0,j_0)$. First we show that if the maximum is attained by $\bar\beta_{j_0}^{i_0}$, then we have $\bar\beta_{j_0}^{i_0}>\alpha_{i_0}$. By relabeling, we assume that $\alpha_1=\max_i\alpha_i$. Consider another feasible solution $(\tilde\beta_j^i)$ to the optimization problem, with $\tilde\beta_1^1=\alpha_1+s-\sum_{i=1}^{n+1}b_i\alpha_i$ and $\tilde\beta_j^i=\alpha_i$ for $(i,j)\ne(1,1)$. Then clearly $\max_{i,j}\{\tilde\beta_j^i\}=\tilde\beta_1^1>\max_i\alpha_i$. Hence

$$\bar\beta_{j_0}^{i_0}=\max_{i,j}\{\bar\beta_j^i\}\ge\max_{i,j}\{\tilde\beta_j^i\}>\alpha_{i_0}.$$

Now suppose contrary to our claim, that we have another pair $(i_1, j_1) \neq (i_0, j_0)$, such that $\bar{\beta}^{i_1}_{j_1} > \alpha_{i_1}$. We define $\beta^i_j = \bar{\beta}^i_j$ if $(i, j) \neq (i_0, j_0), (i_1, j_1)$, $\beta^{i_0}_{j_0} = \bar{\beta}^{i_0}_{j_0} + 1$ and $\beta^{i_1}_{j_1} = \bar{\beta}^{i_1}_{j_1} - 1$. Then clearly (β^i_j) is also feasible, while $\max_{i,j}\{\beta^i_j\} > \max_{i,j}\{\bar{\beta}^i_j\}$. This contradicts our choice of $(\bar{\beta}^i_j)$.

The above discussion shows that the largest superscript that appears in a monomial in the coefficient of t^s shows up only in monomials of the form

$$(x_1^{(\alpha_1)})^{b_1}(x_2^{(\alpha_2)})^{b_2}\ldots(x_j^{(\alpha_j)})^{b_j-1}\ldots(x_{n+1}^{(\alpha_{n+1})})^{b_{n+1}}\cdot x_j^{(\beta_j(s))}.$$

Moreover, such monomials appear in the coefficient of t^s for all j.

By considering the weight of such a monomial, we get $s = \beta_j(s) - \alpha_j + \sum_{i=1}^{n+1} b_i\alpha_i$. This implies that when $\beta_j(s)$ is the largest superscript, $\alpha_j = \max_i \alpha_i$, and

$$\beta_j(s) = s - \sum_{i=1}^{n+1} b_i\alpha_i + \alpha_j.$$

This proves the lemma with $\mu = \sum_{i=1}^{n+1} b_i\alpha_i - \max_i \alpha_i$. □

Remark V.16. With the same proof as above, one can show that for each fixed index j, with $1 \leq j \leq n+1$, the largest superscript for x_j appearing in the coefficient of t^s in the image of $x_1^{b_1}x_2^{b_2}\ldots x_{n+1}^{b_{n+1}}$ under the map (5.1) appears in the monomials of the form

$$(x_1^{(\alpha_1)})^{b_1}(x_2^{(\alpha_2)})^{b_2}\ldots(x_j^{(\alpha_j)})^{b_j-1}\ldots(x_{n+1}^{(\alpha_{n+1})})^{b_{n+1}}\cdot x_j^{(s-\mu)},$$

where $\mu = \sum_{i=1}^{n+1} b_i\alpha_i - \alpha_j$.

Combining Lemma V.14 and Lemma V.15, we see that if $P = x_1^{b_1}\ldots x_{n+1}^{b_{n+1}}$ and if $\alpha_{j_0} = \max_i \alpha_i$, then for $s > \sum_{i=1}^{n+1} b_i\alpha_i$, the term with the highest superscript in the coefficient of t^s for the image of P under the map (5.1) is equal to

$$\frac{\partial P}{\partial x_{j_0}}(x_1^{(\alpha_1)},\ldots,x_{n+1}^{(\alpha_{n+1})})\cdot x_{j_0}^{(s-\mu)},$$

with $\mu = \sum_{i=1}^{n+1} b_i\alpha_i - \alpha_{j_0}$. When $s = \sum_{i=1}^{n+1} b_i\alpha_i$, the coefficient of t^s is equal to $P(x_1^{(\alpha_1)}, \ldots, x_{n+1}^{(\alpha_{n+1})})$. This is the smallest s such that the coefficient of t^s is nonzero. Similarly, for each fixed index j, the term with the highest superscript of x_j is equal to

$$\frac{\partial P}{\partial x_j}(x_1^{(\alpha_1)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \cdot x_j^{(s-\mu')},$$

with $\mu' = \sum_{i=1}^{n+1} b_i\alpha_i - \alpha_j$.

Since the image of different monomials of f do not mix, by Lemma V.15 the coefficient of t^s in the image of f under the map (5.1), for $s > \max_{1\le i\le N}\{\alpha \cdot I^i\}$, is of the form

$$\begin{aligned} &T_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})x_{j_0}^{(s-\mu_{j_0})} \\ &+Q_s(x_j^{(\beta_j)} | \alpha_j \le \beta_j \le s-\mu_{j_0}; \alpha_{j_0} \le \beta_{j_0} < s - \mu_{j_0}), \end{aligned} \tag{5.4}$$

for some index j_0, some $\mu_{j_0} > 0$, and polynomials T_0 and Q_s. In other words, the highest superscript is $s - \mu_{j_0}$ and it is attained at the index j_0. Recall that $f = \sum_{i=1}^{N} a_{I^i}x^{I^i}$ and $n_0(\alpha) = \min_i\{\alpha \cdot I^i\}$ and this minimum is attained by all $1 \le i \le k$. For each i with $I^i_{j_0} > 0$, we compute the product $\alpha \cdot I^i$ and define

$$n_0(\alpha)' := \min_{I^i_{j_0}>0}\{\alpha \cdot I^i\} \text{ and } \sigma := \{1 \le i \le N | I^i_{j_0} > 0,\ \alpha \cdot I^i = n_0(\alpha)'\}.$$

Clearly we have $n_0(\alpha)' \ge n_0(\alpha)$. According to the discussion for a monomial above, $n_0(\alpha)'$ is the smallest integer such that the coefficient of $t^{n_0(\alpha)'}$ contains a monomial divisible by $x_{j_0}^{(q)}$ for some q (in fact it is divisible by $x_{j_0}^{(\alpha_{j_0})}$). Moreover, the coefficient of $t^{n_0(\alpha)'}$ is equal to

$$P_1(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) + \text{ other terms without } x_{j_0}, \tag{5.5}$$

where $P_1(x_1, x_2, \ldots, x_{n+1}) = \sum_{i\in\sigma} a_{I^i}x^{I^i}$. Clearly if $n_0(\alpha)' = n_0(\alpha)$, then $\sigma \subset \{1, 2, \ldots, k\}$. Otherwise, if $n_0(\alpha)' > n_0(\alpha)$, then $\sigma \subset \{k+1, \ldots, N\}$. By the discussion for the case of a monomial, we have

$$T_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) = \frac{\partial P_1(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})}{\partial x_{j_0}^{(\alpha_{j_0})}}. \tag{5.6}$$

For each fixed index j, similar arguments for the highest superscript of x_j also hold.

Remark V.17. Consider the weight of the first term in equation (5.5), we get $I^i \cdot \alpha = n_0(\alpha)'$ for each $i \in \sigma$. Hence each monomial in $T_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})$ has weight $n_0(\alpha)' - \alpha_{j_0} = I^i \cdot \alpha - \alpha_{j_0}$ for every $i \in \sigma$. Consider the weight of the first term in equation (5.4) we get $s = I^i \cdot \alpha - \alpha_{j_0} + s - \mu_{j_0}$ for each $i \in \sigma$, or equivalently, $\mu_{j_0} = I^i \cdot \alpha - \alpha_{j_0}$. But $s - \mu_{j_0}$ is the highest superscript appearing in the coefficient of t^s. Therefore, we have

$$\mu_{j_0} = \min_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n+1 \text{ with } I_j^i > 0}} \{I^i \cdot \alpha - \alpha_j\}, \tag{5.7}$$

and the minimum is attained when $i \in \sigma$ and $j = j_0$. The condition $I_j^i > 0$ is equivalent to $\frac{\partial P_1}{\partial x_j^{(\alpha_j)}} \neq 0$.

Recall that $C^m = \psi_m(\pi^{-1}(0))$ is a contact locus in X_m and

$$C_\alpha^m := C^m \cap (\cap_{1 \leq j \leq n+1} \mathrm{Cont}^{\alpha_j}(x_j)_m).$$

is a contact locus in C^m for each $(n+1)$-tuple α. The following lemma gives an upper bound to the dimension of C_α^m.

Lemma V.18. *Fix $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbb{Z}_+)^{n+1}$. Let m be an integer such that $\alpha_j \leq m$ for each j. If $\min_{1 \leq i \leq N}\{\alpha \cdot I^i\}$ is attained by a unique i, then $C_\alpha^m = \emptyset$. If $\min_{1 \leq i \leq N}\{\alpha \cdot I^i\}$ is attained by at least two different i's and if $(a_{I^i})_{1 \leq i \leq N} \in F_\alpha \cap F(A)$, where F_α is as defined in Lemma V.13 and $F(A)$ is as defined in Definition V.7, then we have*

$$\dim C_\alpha^m \leq mn - \sum_{j=1}^{n+1}(\alpha_j - 1) - 1 + \min_{1 \leq i \leq N}\{I^i \cdot \alpha\} - \min_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n+1 \text{ with } I_j^i > 0}} \{I^i \cdot \alpha - \alpha_j\}, \tag{5.8}$$

for all m large enough.

Remark V.19. An $(n+1)$-tuple $\alpha \in (\mathbb{Z}_+)^{n+1}$ is called *feasible* if $\min_{1 \leq i \leq N}\{\alpha \cdot I^i\}$ is attained by at least two different i's. Otherwise, it is called *non-feasible*. According to Lemma V.18, $C_\alpha^m = \emptyset$ if α is non-feasible.

Proof. Consider the affine space

$$\operatorname{Spec}\mathbb{C}[x_1^{(\alpha_1)}, x_1^{(\alpha_1+1)}, \ldots, x_1^{(m)}, \ldots, x_{n+1}^{(\alpha_{n+1})}, x_{n+1}^{(\alpha_{n+1}+1)}, \ldots, x_{n+1}^{(m)}] = \mathbb{A}^{m(n+1)-\sum_{i=1}^{n+1}(\alpha_i-1)}.$$

Clearly $x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}$ are regular functions on this affine space. We denote by U_m the open subset of $\mathbb{A}^{m(n+1)-\sum_{i=1}^{n+1}(\alpha_i-1)}$ where $x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}$ do not vanish. Let $m_0 = \max_{1\le j\le n+1}\{\alpha_j\}$. When $m \ge m_0$, C_α^m is naturally embedded in U_m.

Pick an arc γ in $X_\infty \cap (\cap_{1\le i\le n+1}\mathrm{Cont}^{\alpha_i}(x_i))$. Then γ is represented by a homomorphism of $\mathbb{C}$-algebras

$$\gamma^* : \mathbb{C}[x_1, \ldots, x_{n+1}] \longrightarrow \mathbb{C}[\![t]\!],\ \gamma^*(x_i) = \sum_{j=\alpha_i}^{\infty} x_i^{(j)} t^j, \tag{5.9}$$

such that $\gamma^*(f) = 0$ and $x_i^{(\alpha_i)} \neq 0$ for each i. Let us write G_s for the coefficient of t^s in $\gamma^*(f)$. Then by definition (equation (5.2)), we have $G_{n_0(\alpha)} = P_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})$. Hence, as long as $m > n_0(\alpha)$, we have $C_\alpha^m \subset \mathbb{V}(P_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}))$. But when $\min_{1\le i\le N}\{\alpha \cdot I^i\}$ is attained by a unique i, P_0 is a monomial. Thus, we have $\mathbb{V}(P_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})) = \emptyset$ in U_m, which implies that $C_\alpha^m = \emptyset$. This shows that $C_\alpha^m = \emptyset$ if α is non-feasible. In what follows, we assume that α is feasible and $(a_{I_i})_i \in F_\alpha \cap F(A)$, and show that in such a case, C_α^m is a finite union of locally closed subsets of U_m, all of them having dimension less than or equal to the right-hand side of (5.8).

For each $m \ge m_0$, we define $A_0 := C_\alpha^m \backslash \mathbb{V}(T_0)$. Since $(a_{I_i})_i \in F_\alpha$, we can find j such that

$$\frac{\partial P_0}{\partial x_j^{(\alpha_j)}}(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \neq 0.$$

Then for each $s \ge 1$, consider the highest superscript of x_j in the coefficient of $t^{n_0(\alpha)+s}$ and we get

$$G_{n_0(\alpha)+s} = \frac{\partial P_0}{\partial x_j^{(\alpha_j)}}(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \cdot x_j^{(\alpha_j+s)} + R_s, \tag{5.10}$$

where R_s is a polynomial in $\{x_i^{(t_i)}|\alpha_i \leq t_i \leq \alpha_i + s$ for all $i \neq j$, $\alpha_j \leq t_j < \alpha_j + s\}$. For each $s \geq 1$, if we consider the highest superscript among all x_i, for $1 \leq i \leq n+1$, in the coefficient of $t^{n_0(\alpha)'+s}$, we get

$$G_{n_0(\alpha)'+s} = T_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \cdot x_{j_0}^{(\alpha_{j_0}+s)} + \tilde{R}_s, \tag{5.11}$$

where $\tilde{R}_s$ is a polynomial in $\{x_i^{(t_i)}|\alpha_i \leq t_i \leq \alpha_{j_0} + s$ for all $i \neq j_0$, $\alpha_{j_0} \leq t_{j_0} < \alpha_{j_0} + s\}$.

Note that

$$G_{n_0(\alpha)'+m-\alpha_{j_0}} = T_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \cdot x_{j_0}^{(m)} + \tilde{R}_{m-n_0(\alpha)'}.$$

Every variable in the above expression of $G_{n_0(\alpha)'+m-\alpha_{j_0}}$ has a superscript $\leq m$. The same holds for G_k, with $n_0(\alpha) \leq k < n_0(\alpha)' + m - \alpha_{j_0}$. Hence each $\mathbb{V}(G_k)$, with $n_0(\alpha) \leq k \leq n_0(\alpha)' + m - \alpha_{j_0}$, can be considered as a closed subset of U_m.

We claim that if $m > n_0(\alpha)'$, then

$$A_0 = \mathbb{V}(G_{n_0(\alpha)}, G_{n_0(\alpha)+1}, \ldots, G_{n_0(\alpha)'+m-\alpha_{j_0}}) \backslash \mathbb{V}(T_0) \subset U_m.$$

In fact, if we embed A_0 naturally in $A_\infty :=$ Spec $\mathbb{C}[x_i^{(s_i)}|1 \leq i \leq n+1, s_i \geq \alpha_i]$, and consider each $\mathbb{V}(G_k)$ as a subset of A_∞, then we have

$$A_0 = U_m \cap (\cap_{s \geq 0} \mathbb{V}(G_{n_0(\alpha)+s})) \backslash \mathbb{V}(T_0).$$

Hence, A_0 is contained in $\mathbb{V}(P_0, G_{n_0(\alpha)+1}, \ldots, G_{n_0(\alpha)'+m-\alpha_{j_0}}) \backslash \mathbb{V}(T_0) \subset U_m$.

On the other hand, for each $s \geq 1$, if we fix

$$\{x_i^{(t_i)}|\alpha_i \leq t_i \leq \alpha_i + s \text{ for all } i \neq j,\ \alpha_j \leq t_j < \alpha_j + s\}$$

such that $\frac{\partial P_0}{\partial x_j^{(\alpha_j)}}(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \neq 0$, then the equation $G_{n_0(\alpha)+s} = 0$ has a unique solution for $x_j^{(\alpha_j+s)}$. Similarly, if we fix

$$\{x_i^{(t_i)}|\alpha_i \leq t_i \leq \alpha_{j_0} + s \text{ for all } i \neq j_0,\ \alpha_{j_0} \leq t_{j_0} < \alpha_{j_0} + s\}$$

such that $T_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \neq 0$, then the equation $G_{n_0(\alpha)'+s} = 0$ has a unique solution for $x_{j_0}^{(\alpha_{j_0}+s)}$. The existence of solutions for $G_{n_0(\alpha)+s} = 0$ and $G_{n_0(\alpha)'+s} = 0$, for each $s \geq 1$, shows that every element in $U_m \cap \mathbb{V}(P_0, G_{n_0(\alpha)+1}, \ldots, G_{n_0(\alpha)'+m-\alpha_{j_0}}) \backslash \mathbb{V}(T_0)$ can be lifted to an element in X_∞, and hence contained in A_0. Moreover, we see that each equation $G_{n_0(\alpha)+s} = 0$ or $G_{n_0(\alpha)'+s} = 0$ cuts down the dimension exactly by 1. This shows that the codimension of A_0 in U_m is exactly the number of equations unless $A_0 = \emptyset$. We conclude that if $A_0 \neq \emptyset$, then

$$\begin{aligned}\dim(A_0) &= m(n+1) - \sum_{i=1}^{n+1}(\alpha_i - 1) + n_0(\alpha) - (n_0(\alpha)' - \alpha_{j_0}) - 1 - m \\ &= mn - \sum_{i=1}^{n+1}(\alpha_i - 1) - 1 + n_0(\alpha) - \mu_{j_0}.\end{aligned}$$

Now suppose that $\psi_m(\gamma) \in C_\alpha^m \cap \mathbb{V}(T_0)$. Then the first term on the right-hand side of equation (5.11) vanishes. If we delete the first term and rearrange the equation to get a new highest-superscript term, we claim that for s sufficiently large (independent of m), the equation (5.11) becomes

$$G_{n_0(\alpha)'+s} = T_1 \cdot x_{j_1}^{(s-\mu_1)} + \text{Remaining Terms without } x_{j_1}^{(s-\mu_1)}$$

for some polynomial T_1 and some number μ_1, with $s - \mu_1$ being the highest superscript.

To prove this, let us consider the monomials in the expression of $G_{n(\alpha)'+s}$ after deleting $T_0 \cdot x_{j_0}^{(\alpha_{j_0}+s)}$. According to the proof of Lemma V.15, they are of the form $\prod_{j=1}^{n+1} \prod_{k=1}^{I_j^i} x_i^{(\beta_k^j)}$ for some i, with $\beta_k^j \geq \alpha_j$ and $\sum_{j,k} \beta_k^j = n_0(\alpha)' + s$. Hence, the number

$$\max_{s \geq 1}\{\max_{j,k}\{\beta_k^j\} - s\}$$

is bounded above. In fact, it is bounded above by α_{j_0}. Suppose that the maximum is attained by some $s_0 \geq 1$ and $(\bar{\beta}_k^j)_{j,k}$, with $\bar{\beta}_1^{j_1} = \max_{j,k}\{\bar{\beta}_k^j\}$. In other words, the highest superscript appears in the monomial

$$x_1^{(\bar{\beta}_1^1)} \ldots x_1^{\left(\bar{\beta}_{I_1^i}^1\right)} \ldots \widehat{x_{j_1}^{(\bar{\beta}_1^{j_1})}} \ldots x_{n+1}^{\left(\bar{\beta}_{I_{n+1}^i}^{n+1}\right)} \cdot x_{j_1}^{(\bar{\beta}_1^{j_1})}.$$

On the other hand, for each $s \geq s_0$, $G_{n_0(\alpha)'+s}$ contains the monomial

$$x_1^{(\bar\beta_1^1)} \dots x_1^{\left(\bar\beta^1_{I_1^i}\right)} \dots \widehat{x_{j_1}^{(\bar\beta_1^{j_1})}} \dots x_{n+1}^{\left(\bar\beta^{n+1}_{I^i_{n+1}}\right)} \cdot x_{j_1}^{(\bar\beta_1^{j_1}+s-s_0)}.$$

This shows that $\max_{s\geq 1}\{\max_{j,k}\{\beta_k^j\} - s\}$ is attained by all $s \geq s_0$ and the same j_1. Hence the highest superscript in $G_{n_0(\alpha)'+s}$ is equal to $s-\mu_1$ for some fixed μ_1 and for s sufficiently large. We also see that if $M \cdot x_{j_1}^{(s-\mu_1)}$ is a monomial in $G_{n_0(\alpha)'+s}$ that contains the highest superscript, then $M \cdot x_{j_1}^{(s'-\mu_1)}$ is a monomial in $G_{n_0(\alpha)'+s'}$ that contains the highest superscript for each $s' > s$. Moreover, for every such monomial, the weight of M is equal to $n_0(\alpha)' + \mu_1$. There are only finitely many monomials with a fixed weight. Hence, we get

$$G_{n_0(\alpha)'+s} = T_1 \cdot x_{j_1}^{(s-\mu_1)} + \text{Remaining Terms without } x_{j_1}^{(s-\mu_1)}$$

for s sufficiently large and for a fixed polynomial T_1. Clearly, we have $s - \mu_1 \leq \alpha_{j_0} + s$, or equivalently, $\mu_1 \geq -\alpha_{j_0}$.

Let m_1 be $\geq$ the largest superscript appearing in T_1 and $m_1 \geq m_0$. Then for each $m \geq m_1$, we define $A_1 := C_\alpha^m \cap \mathbb{V}(T_0)\backslash\mathbb{V}(T_1) \subset U_m$. With the same analysis as above we can show that $A_1 = \mathbb{V}(T_0, G_{n_0(\alpha)}, G_{n_0(\alpha)+1}, \dots, G_{n_0(\alpha)'+\mu_1+m})\backslash\mathbb{V}(T_1)$ and that each G_i cuts down dimension exactly by 1. Hence either $A_1 = \emptyset$ or

$$\begin{aligned}
\dim(A_1) &\leq m(n+1) - \sum_{i=1}^{n+1}(\alpha_i - 1) + n_0(\alpha) - 1 - n_0(\alpha)' - \mu_1 - m \\
&\leq mn - \sum_{i=1}^{n+1}(\alpha_i - 1) - 1 + n_0(\alpha) - (n_0(\alpha)' - \alpha_{j_0}) \\
&= mn - \sum_{i=1}^{n+1}(\alpha_i - 1) - 1 + n_0(\alpha) - \mu_{j_0}.
\end{aligned}$$

Inductively, suppose we have $A_k = C_\alpha^m \cap (\cap_{l\leq k-1}\mathbb{V}(T_l))\backslash\mathbb{V}(T_k)$ for each $m \geq m_k$, for some number $m_k \geq \max_{0\leq i\leq k-1}\{m_i\}$, and when $\psi_m(\gamma) \in A_k$ we have

$$G_{n_0(\alpha)'+s} = T_k \cdot x_{j_k}^{(s-\mu_k)} + \text{Remaining Terms}, \tag{5.12}$$

for s sufficiently large and some number $\mu_k \geq -\alpha_{j_0}$, where $s - \mu_k$ is the highest superscript.

Now suppose that $\psi_m(\gamma) \in C_\alpha^m \cap (\cap_{l \leq k} \mathbb{V}(T_l))$, then the first term of the right-hand side of equation (5.12) vanishes. If we delete the first term and rearrange the equation to get a new highest-superscript term, with the same proof as above we can show that

$$G_{n_0(\alpha)'+s} = T_{k+1} \cdot x_{j_{k+1}}^{(s-\mu_{k+1})} + \text{Remaining Terms},$$

for s sufficiently large (independent of m). Clearly, we have $\mu_{k+1} \geq \mu_k \geq -\alpha_{j_0}$.

Let m_{k+1} be $\geq$ the highest superscript in T_{k+1} and $m_{k+1} \geq m_k$. For each $m \geq m_{k+1}$, we define $A_{k+1} := C_\alpha^m \cap (\cap_{l \leq k} \mathbb{V}(T_l)) \backslash \mathbb{V}(T_{k+1})$. With the same analysis as in the case when $k = 0$ we can show that

$$A_{k+1} = \mathbb{V}(T_0, \ldots, T_k, G_{n_0(\alpha)}, G_{n_0(\alpha)+1}, \ldots, G_{n'_{k+1}+m}) \backslash \mathbb{V}(T_{k+1})$$

and that each G_i cuts down dimension exactly by 1. Hence either $A_{k+1} = \emptyset$ or

$$\begin{aligned} \dim(A_{k+1}) &\leq m(n+1) - \sum_{i=1}^{n+1} (\alpha_i - 1) + n_0(\alpha) - 1 - n_0(\alpha)' - \mu_{k+1} - m \\ &\leq mn - \sum_{i=1}^{n+1} (\alpha_i - 1) - 1 + n_0(\alpha) - \mu_{j_0}. \end{aligned}$$

We claim that there can be only finitely many such steps. First note that the highest superscript decreases, or equivalently, the number μ_k increases, by at least 1 as k increases by $n+1$ because we must have used the same subscript j_k during $n+2$ steps. Second, the decrease in highest superscript must eventually stop because $G_{n_0(\alpha)+s}$ contains the term $\frac{\partial P_0}{\partial x_j^{(\alpha_j)}}(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \cdot x_j^{(\alpha_j+s)}$ for some j, with $\frac{\partial P_0}{\partial x_j^{(\alpha_j)}}(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) \neq 0$, and for every $s \geq 1$. Hence, for each m large enough, we can decompose C_α^m into a finite union $\cup_{i \geq 0} A_i$, where each A_i either is empty or has dimension less than or equal to the number in the lemma. This completes the proof. $\square$

Remark V.20. From the proof of Lemma V.18 we see that for a fixed feasible α, coefficients $(a_{I_i})_i \in F_\alpha \cap F(A)$ and m large enough, if $A_0 \neq \emptyset$, then we have

$$\dim(C_\alpha^m) = mn - \sum_{j=1}^{n+1}(\alpha_j - 1) - 1 + \min_{1\le i\le N}\{I^i \cdot \alpha\} - \min_{\substack{1\le i\le N \\ 1\le j\le n+1 \text{ with } I^i_j>0}}\{I^i\cdot\alpha - \alpha_j\}. \tag{5.13}$$

We also see that $G_{n_0(\alpha)+s} = 0$ for each $s \geq 1$ has a solution as long as $T_0 \neq 0$. Therefore, $A_0 \neq \emptyset$ for m large enough if and only if $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ in the torus $(\mathbb{C}^*)^{n+1} \subset$ Spec $\mathbb{C}[x_1^{(\alpha_1)}, \ldots, x_{n+1}^{(\alpha_{n+1})}]$. Note that the condition $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ is independent of m.

Recall that if we fix an integral support $A = \{I^1, \ldots, I^N\}$, then $F(A)$ and F_α are subsets of $(\mathbb{C}^*)^N$ defined in Definition V.7 and Lemma V.13 respectively. Each of them contains an open dense subset of $(\mathbb{C}^*)^N$. An $(n+1)$-tuple $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$ is called feasible if $\min_{1\le i\le N}\{\alpha \cdot I^i\}$ is attained by at least two different i's. For each feasible α, we define polynomials P_0 by equation (5.2) and T_0 by equation (5.4). Using the above lemmas we obtain the following theorem:

Theorem V.21. *Let $A = \{I^1, \ldots, I^N\}$ be an integral support. If*

$$(a_{I^i})_{1\le i\le N} \in \bigcap_{\text{feasible } \alpha} F_\alpha \cap F(A)$$

and X is the hypersurface in $\mathbb{A}^{n+1}$ defined by $f = \sum_{i=1}^N a_{I^i}x^{I^i}$, then X is an integral hypersurface containing the origin 0 and the invariant λ (defined in Definition III.21) for the origin satisfies

$$\lambda(0) \geq \min\{\sum_{j=1}^{n+1}(\alpha_j - 1) + 1 - \min_{1\le i\le N}\{I^i \cdot \alpha\} + \min_{\substack{1\le i\le N \\ 1\le j\le n+1 \text{ with } I^i_j>0}}\{I^i\cdot\alpha - \alpha_j\}\}, \tag{5.14}$$

where the first minimum is taken over all feasible $(n+1)$-tuples α.

Moreover, assume the first minimum is attained at some feasible α. If for this α, we have $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ in the torus $(\mathbb{C}^*)^{n+1} \subset$ Spec $\mathbb{C}[x_1^{(\alpha_1)}, \ldots, x_{n+1}^{(\alpha_{n+1})}]$, then the inequality (5.14) is in fact an equality.

Proof. Let f be fixed with $(a_{I^i})_{1\le i\le N} \in \cap_\alpha F_\alpha \cap F(A)$. By Corollary V.8, X is an integral hypersurface containing the origin 0. According to Proposition III.19, $\pi^{-1}(0)$ contains only finitely many irreducible components $C_1, \dots, C_p,\ Z_1, \dots, Z_q$, where each C_j is thin and each Z_i is fat. We have $\dim(C^m) = mn - \lambda(0)$ when m is large enough. For each thin irreducible component C_j of $\pi^{-1}(0)$, however, by Lemma II.12 we see that $\dim(\psi_m(C_j)) \le (m+1)(n-1)$. Thus, for m large enough, we have

$$\dim(C^m) = \max_{1\le i\le q} \dim(\psi_m(Z_i)).$$

By Lemma II.12, the fibers of $\psi_{m+1}(\pi^{-1}(x)) \to \psi_m(\pi^{-1}(x))$ have dimension $\le n$. Since $\dim(C^m) = mn - \lambda(0)$ for every m large enough, we have

$$\dim(C^{m+1}) = \dim(C^m) + n \text{ for } m \gg 0.$$

This also implies that there is some i such that

$$\dim(C^m) = \dim(\psi_m(Z_i)) \text{ for all } m \gg 0.$$

In fact, pick a positive integer M such that $\dim(C^{m+1}) = \dim(C^m) + n$ for all $m \ge M$. If $\dim(\psi_m(Z_i)) < \dim(C^m)$ for some $m \ge M$, then

$$\begin{aligned} \dim(\psi_{m+k}(Z_i)) &\le \dim(\psi_m(Z_i)) + nk \\ &< \dim(C^m) + nk \\ &= \dim(C^{m+k}), \end{aligned}$$

for every $k \ge 0$. It follows that if there exists $m_i \ge M$ for each i such that $\dim(\psi_{m_i}(Z_i)) < \dim(C^{m_i})$, we have $\dim(C^m) > \max_{1\le i\le q} \dim(\psi_m(Z_i))$ when $m > \max_{1\le i\le q}\{m_i\}$, a contradiction. Therefore, by relabeling we may assume that

$$\dim(C^m) = \dim(\psi_m(Z_1)) \text{ for all } m \ge M.$$

Since $X = \mathbb{V}(f)$ is irreducible and is not a hyperplane, we have $\mathbb{V}(x_i, f) \subsetneq \mathbb{V}(f)$ for each i. This implies that the fat component Z_1 is not contained in $\mathrm{Cont}^{\infty}(x_i)$ for each i, or equivalently, Z_1 does not have infinite order along any x_i. Choose an $(n+1)$-tuple $\alpha' = (\alpha'_1, \dots, \alpha'_{n+1}) \in (\mathbb{Z}_+)^{n+1}$ with

$$\alpha'_i = \min\{\mathrm{ord}_\gamma(x_i) | \gamma \in Z_1\}, 1 \leq i \leq n+1.$$

Then if $m \geq \max\{M, \alpha'_1, \dots, \alpha'_{n+1}\}$, $C^m_{\alpha'}$ contains a dense open subset of $\psi_m(Z_1)$, hence $\dim(C^m) = \dim(C^m_{\alpha'})$. By applying Lemma V.18, we get for $m \gg 0$,

$$\dim(C^m) \leq mn - \sum_{j=1}^{n+1}(\alpha'_j - 1) - 1 + \min_{1\leq i\leq N}\{I^i \cdot \alpha'\} - \min_{\substack{1\leq i\leq N \\ 1\leq j\leq n+1 \text{ with } I^i_j>0}}\{I^i \cdot \alpha' - \alpha'_j\}.$$

Therefore, for $m \gg 0$ we have

$$\begin{aligned}
\lambda(0) &= mn - \dim(C^m) \\
&\geq \sum_{j=1}^{n+1}(\alpha'_j - 1) + 1 - \min_{1\leq i\leq N}\{I^i \cdot \alpha'\} + \min_{\substack{1\leq i\leq N \\ 1\leq j\leq n+1 \text{ with } I^i_j>0}}\{I^i \cdot \alpha' - \alpha'_j\} \\
&\geq \min_{\text{feasible } \alpha}\Big\{\sum_{j=1}^{n+1}(\alpha_j - 1) + 1 - \min_{1\leq i\leq N}\{I^i \cdot \alpha\} + \min_{\substack{1\leq i\leq N \\ 1\leq j\leq n+1 \text{ with } I^i_j>0}}\{I^i \cdot \alpha - \alpha_j\}\Big\}.
\end{aligned}$$

Now suppose the first minimum in (5.14) is attained at some feasible α and that for this α we have

$$\dim(C^m_\alpha) = mn - \sum_{j=1}^{n+1}(\alpha_j - 1) - 1 + \min_{1\leq i\leq N}\{I^i \cdot \alpha\} - \min_{\substack{1\leq i\leq N \\ 1\leq j\leq n+1 \text{ with } I^i_j>0}}\{I^i \cdot \alpha - \alpha_j\}.$$

Since $C^m_\alpha \subset C^m$, we have $\dim(C^m_\alpha) \leq \dim(C^m)$. On the other hand, we have $\dim(C^m_\alpha) \geq \dim(C^m_{\alpha'}) = \dim(C^m)$ by the choice of α. This shows that

$$\begin{aligned}
\lambda(0) &= mn - \dim(C^m_\alpha) \\
&= \sum_{j=1}^{n+1}(\alpha_j - 1) + 1 - \min_{1\leq i\leq N}\{I^i \cdot \alpha\} + \min_{\substack{1\leq i\leq N \\ 1\leq j\leq n+1 \text{ with } I^i_j>0}}\{I^i \cdot \alpha - \alpha_j\}.
\end{aligned}$$

In other words, we obtain an equality if $\dim(C_\alpha^m)$ attains the upper bound in the statement of Lemma V.18 for some feasible α where the first minimum in (5.14) is attained. According to Remark V.20, this happens when $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ for such an α. $\square$

Combining Lemma V.13, Theorem V.21 and Proposition III.22, we get the following corollary:

Corollary V.22. *Let $A = \{I^1, \ldots, I^N\} \subset (\mathbb{Z}_{\geq 0})^{n+1}$ be a fixed integral subset (see Definition V.7). If X is a hypersurface in $\mathbb{A}^{n+1}$ defined by a very general polynomial with support A, then X is an integral hypersurface containing the origin 0 and we have*

(5.15)

$$\widehat{\mathrm{mld}}(0; X) \geq \min\{\sum_{j=1}^{n+1}(\alpha_j - 1) + 1 - \min_{1\leq i\leq N}\{I^i \cdot \alpha\} + \min_{\substack{1\leq i\leq N \\ 1\leq j\leq n+1 \text{ with } I^i_j>0}}\{I^i \cdot \alpha - \alpha_j\}\} + n,$$

where the first minimum is taken over all $(n+1)$-tuples α such that $\min\limits_{1\leq i\leq N}\{I^i \cdot \alpha\}$ is attained by at least two different i's.

Moreover, assume the first minimum is attained at some feasible α. If for this α, the polynomials P_0 (defined in equation (5.2)) and T_0 (defined in equation (5.4)) satisfy $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ in the torus $(\mathbb{C}^*)^{n+1} \subset \mathrm{Spec}\, \mathbb{C}[x_1^{(\alpha_1)}, \ldots, x_{n+1}^{(\alpha_{n+1})}]$, then the inequality (5.15) is in fact an equality.

5.3 Examples

There are many interesting examples of hypersurfaces where the inequality in Theorem V.21 turns out to be an equality. According to Theorem V.21, we just need to show that the coefficients are in $\bigcap_\alpha F_\alpha \cap F(A)$, and that $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ for certain feasible α. In these cases, the invariants λ and Mather mld are independent of the coefficients in the defining equations.

Example V.23. Let $X = \mathbb{V}(f) \subset \mathbb{A}^{n+1}$ be an integral variety of dimension n where f is a binomial. Note that X is not necessarily normal. So it might not be a toric variety. The irreducibility of X implies that we can write f in the form

$$f = ax_1^{\beta_1}x_2^{\beta_2}\cdots x_p^{\beta_p} - bx_{p+1}^{\beta_{p+1}}x_{p+2}^{\beta_{p+2}}\cdots x_{p+q}^{\beta_{p+q}},$$

where $p+q \leq n+1$. If $p+q < n+1$, X is the product of a lower dimensional binomial hypersurface with an affine space. The question is hence reduced to the case when $p+q = n+1$. By assuming $0 \in X$, we also require that $p \geq q \geq 1$. The support A contains $N = 2$ elements $(\beta_1, \beta_2, \ldots, \beta_p, 0, \ldots, 0)$ and $(0, \ldots, 0, \beta_{p+1}, \ldots, \beta_{p+q})$. By requiring that A is integral (see Definition V.7), we further assume that the line segment connecting these two points does not contain any other integral point. Hence X is integral if the coefficients $(a, b) \in F(A)$ according to Corollary V.8. On the other hand, by applying a coordinate change that takes x_1 to $c \cdot x_1$ and preserves all $x_2, \ldots, x_{p+q}$, we see that any two such hypersurfaces X and X', with different coefficients (a, b) and (a', b'), are isomorphic. Hence, we conclude that $F(A) = (\mathbb{C}^*)^2$. Clearly, an $(n+1)$-tuple $\alpha \in (\mathbb{Z}_+)^{n+1}$ is feasible (see Remark V.19) if and only if

(5.16) $$\sum_{i=1}^{p} \alpha_i\beta_i = \sum_{i=p+1}^{p+q} \alpha_i\beta_i.$$

For any feasible α, following the notation in the previous section, we have $n_0(\alpha) = \sum_{i=1}^{p} \alpha_i\beta_i$ and

$$P_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) = f(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})$$

is of weight $n_0(\alpha)$. Since $\frac{\partial P_0}{\partial x_j^{(\alpha_j)}}$ is a monomial for each j, $F_\alpha = (\mathbb{C}^*)^2$ for each feasible α.

Now fix a feasible α. Clearly if $\alpha_{j_0} = \max_{1\leq j\leq n+1}\{\alpha_j\}$, then we get

$$T_0(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})}) = \frac{\partial f(x_1^{(\alpha_1)}, x_2^{(\alpha_2)}, \ldots, x_{n+1}^{(\alpha_{n+1})})}{\partial x_{j_0}^{(\alpha_{j_0})}}.$$

In particular, P_0 is binomial while T_0 is a monomial. Hence $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ in the torus $(\mathbb{C}^*)^{n+1}$. According to Remark V.20, for each feasible α, we have

$$\begin{aligned}
\dim(C^m_\alpha) &= mn - \sum_{j=1}^{n+1}(\alpha_j - 1) - 1 + \min_{1\leq i\leq N}\{I^i\cdot\alpha\} - \min_{\substack{1\leq i\leq N \\ 1\leq j\leq n+1 \text{ with } I^i_j>0}}\{I^i\cdot\alpha - \alpha_j\} \\
&= mn - \sum_{j=1}^{n+1}(\alpha_j - 1) + \max_{1\leq i\leq n+1}\{\alpha_i - 1\}.
\end{aligned}$$

Therefore, Theorem V.21 implies that

$$\lambda = \min\{\sum_{i=1}^{n+1}\alpha_i - \max_{1\leq i\leq n+1}\alpha_i - n\}, \tag{5.17}$$

or equivalently,

$$\widehat{\text{mld}}(0;X) = \min\{\sum_{i=1}^{n+1}\alpha_i - \max_{1\leq i\leq n+1}\alpha_i\}, \tag{5.18}$$

where the first minimum is taken over all $\alpha \in (\mathbb{Z}_+)^{n+1}$ that satisfy equation (5.16).

Example V.24. Consider the Whitney Umbrella $X = \mathbb{V}(x^2 - y^2z)$. The nonsingular locus of X has codimension 1. Therefore it does not follow in the framework discussed in Chapter IV, since it is not normal. Nevertheless, we can use the formula (5.17) and conclude that $\lambda = 1$ and $\widehat{\text{mld}}(0;X) = 3$.

Remark V.25. The binomial hypersurfaces are nice examples where λ and Mather mld can be computed directly in a simple form. Note that the result is independent of coefficients a and b. This makes sense because we have seen that any two binomial polynomials with the same support define isomorphic hypersurfaces. However, this is not the case if f is more complicated, and then λ indeed depends on the coefficients.

Example V.26. Let X be a curve in $\mathbb{A}^2$ defined by $f = a_1x^2 + a_2y^2 + a_3xy + a_4y^3$. For a very general choice of coefficients a_i, λ has a lower bound given by equation (5.14). The lower bound is 0, which is achieved when $\alpha_1 = \alpha_2 = 1$.

First, assume all a_i are equal to 1. Then X is integral. For any choice of feasible α (see Remark V.19), it's clear that $P_0(x, y)$ can only be $x^2 + y^2 + xy$. Thus the condition Δ^α (see Condition V.10) is satisfied, or equivalently, we have $(1, 1, 1, 1) \in \cap_\alpha F_\alpha$. Since $T_0(x, y) = 2x$ or $2y$, we get $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$ in the torus $(\mathbb{C}^*)^4$. By Theorem V.21, we conclude that $\lambda = 0$, and the minimum in equation (5.14) is attained at the tuple α with $\alpha_1 = \alpha_2 = 1$.

Now instead we assume that $a_1 = a_2 = a_4 = 1$ and $a_3 = 2$. X is still integral. But condition Δ^α is no longer satisfied. By computing $\dim(\psi_m(\pi^{-1}(0)))$ directly from definition, it can be shown that $\lambda = 1$.

Example V.27. Let $X \subset \mathbb{A}^{n+1}$ be a hypersurface defined by $f = \sum_{i=1}^{n+1} x_i^{b_i}$, with $n \geq 2$. As Lemma V.18 suggests, we consider only feasible $(n + 1)$-tuples α (see Remark V.19). Clearly, α is feasible if and only if $\min\limits_{1\leq i\leq n+1} \{b_i\alpha_i\}$ is attained by at least two different i's.

Note that for any feasible α, $\frac{\partial P_0}{\partial x_j^{(\alpha_j)}}$ is always a monomial for each j. Thus, we have $\cap_{\text{feasible }\alpha} F_\alpha = (\mathbb{C}^*)^{n+1}$. Clearly, when $n \geq 2$, X is an integral hypersurface.

Similarly, T_0 is always a monomial for any feasible α. So we conclude $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$. According to Theorem V.21, we get

$$(5.19)\quad \lambda = \min\{\sum_{i=1}^{n+1}(\alpha_i - 1) + 1 - \min_{1\leq i\leq n+1}\{b_i\alpha_i\} + \min_{1\leq i\leq n+1}\{(b_i - 1)\alpha_i\}\}, \text{ or}$$

$$\widehat{\mathrm{mld}}(0; X) = \min\{\sum_{i=1}^{n+1}(\alpha_i - 1) + 1 - \min_{1\leq i\leq n+1}\{b_i\alpha_i\} + \min_{1\leq i\leq n+1}\{(b_i - 1)\alpha_i\}\} + n,$$

where the first minimum is taken over all feasible α.

There are many classical examples that fall into the category of Example V.27. A large portion of the following class of examples are of this type.

Example V.28. Consider here the ADE singularities. All the varieties here are integral.

(1) Singularities of type A_k: X is defined by $f = x_1^{k+1} + x_2^2 + \cdots x_n^2$ for $n \geq 3$.

Choose multi-index α with $\alpha_i = 1$ for $1 \leq i \leq n$. The minimum weight $n_0(\alpha)$ is attained by $n-1$ monomials if $k > 1$, or n monomials if $k = 1$. In both cases, α is feasible. Let $b_1 = k+1$ and $b_i = 2$ for $2 \leq i \leq n$. Then we have

$$\sum_{i=1}^{n+1}(\alpha_i - 1) + 1 - \min_{1\leq i\leq n+1}\{b_i\alpha_i\} + \min_{1\leq i\leq n+1}\{(b_i-1)\alpha_i\} = 0.$$

Hence according to equation (5.19), we get $\lambda = 0$ or $\widehat{\mathrm{mld}}(0;X) = n-1$.

(2) Singularities of type D_k: X is defined by $f = x_1^{k-1} + x_1x_2^2 + x_3^2 + \cdots + x_n^2$ with $k \geq 4$. One checks easily that the coefficients are in $\cap_\alpha F_\alpha$.

If $n \geq 4$, then there are at least two quadratic terms. Hence $\alpha = (1,\ldots,1)$ is feasible, which achieves the minimum 0 in equation (5.14). Note that we have

$$P_0 = (x_3^{(1)})^2 + \ldots + (x_n^{(1)})^2$$

and $T_0 = \frac{\partial P_0}{\partial x_3^{(1)}} = 2x_3^{(1)}$. Therefore, $\mathbb{V}(P_0)\backslash\mathbb{V}(T_0) \neq \emptyset$. By Theorem V.21, we get $\lambda = 0$ or $\widehat{\mathrm{mld}}(0;X) = n-1$.

When $n = 3$, the minimum 1 of equation (5.14) is achieved when $\alpha = (2,1,2)$. With similar analysis, we obtain $\lambda = 1$ or $\widehat{\mathrm{mld}}(0;X) = n = 3$.

(3) Singularities of type E_6: X is defined by $f = x_1^4 + x_2^3 + x_3^2 + \ldots + x_n^2$. This belongs to Example V.27. So we use equation (5.19).

If $n \geq 4$, with $\alpha = (1,\ldots,1)$, we get $\lambda = 0$ or $\widehat{\mathrm{mld}}(0;X) = n-1$. When $n = 3$, the minimum is achieved when $\alpha = (1,2,2)$, and we get $\lambda = 1$ or $\widehat{\mathrm{mld}}(0;X) = n = 3$.

(4) Singularities of type E_7: X is defined by $f = x_1^3x_2 + x_2^3 + x_3^2 + \ldots + x_n^2$.

This is very similar to case (2). Again one checks easily that the coefficients satisfy Condition Δ^α for all feasible multi-indices α.

If $n \geq 4$, $\alpha = (1,\ldots,1)$ is feasible. Similar to (2) we get $\lambda = 0$ or $\widehat{\mathrm{mld}}(0;X) = n-1$.

When $n = 3$, $\alpha = (2, 2, 3)$ is feasible and it gives minimum in equation (5.14). Simple analysis similar to the ones above shows that we have an equality and hence $\lambda = 2$ or $\widehat{\mathrm{mld}}(0; X) = n + 1 = 4$.

(5) Singularities of type E_8: X is defined by $f = x_1^5 + x_2^3 + x_3^2 + \ldots + x_n^2$. This belongs to Example V.27 so we can apply formula (5.19).

When $n \geq 4$, we get $\lambda = 0$ or $\widehat{\mathrm{mld}}(0; X) = n - 1$. The minimum is attained when $\alpha = (1, \ldots, 1)$. When $n = 3$, we have $\lambda = 2$ or $\widehat{\mathrm{mld}}(0; X) = n + 1 = 4$, and it is attained when $\alpha = (2, 2, 3)$.

5.4 Possible generalizations

We only treat the case when the hypersurface is defined by a very general polynomial with a fixed support. An obvious question is: what can we say if the hypersurface is defined by a general polynomial (so that it is integral) with a fixed support? Unfortunately, the polynomials P_0 defined in equation (5.2) and T_0 defined in equation (5.4) don't behave well and our method fails.

An obvious generalization of the results in this chapter is to treat the class of complete intersection varieties. However, our method doesn't work well when there are multiple defining equations.

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