Dualities Arising from Borcea-Voisin Threefolds

by

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ABSTRACT

In the early 1990s, Borcea-Voisin orbifolds were some of the earliest examples of Calabi-Yau threefolds shown to exhibit mirror symmetry, but at the quantum level this has been poorly understood. Here the enumerative geometry of this family is placed in the context of a gauged linear sigma model which encompasses the threefolds' Gromov-Witten theory and three companion theories (FJRW theory and two mixed theories). For certain Borcea-Voisin orbifolds of Fermat type, all four genus zero theories are calculated explicitly. Furthermore, the I-functions of these theories are related by analytic continuation and symplectic transformation. In particular, it is shown that the relation between the Gromov-Witten and FJRW theories can be viewed as an example of the Landau-Ginzburg/Calabi-Yau correspondence for complete intersections of toric varieties. For certain mirror families, the corresponding Picard-Fuchs systems are then derived and the I-functions are shown to solve them, thus demonstrating that the mirror symmetry of Borcea-Voisin orbifolds extends to the quantum level.

CHAPTER 1

Introduction

1.1 Background

Mirror symmetry has been a driving force in geometry and physics for more than twenty years. The first mirror symmetric phenomenon to be discovered mathematically was purely at the level of Hodge numbers. That is, it was noticed that many Calabi-Yau manifolds pair up in such a way that the Hodge diamonds of one is the Hodge diamond of the other rotated by a right angle. One of the earliest sets of examples of this broad phenomenon was discovered by Borcea [6] and Voisin [27], now known as Borcea-Voisin manifolds; distinguishing them by their K3 surfaces, these form a class of 92 members which provides many pairs exhibiting cohomological mirror symmetry. These are given as resolutions of certain quotients of products of elliptic curves and certain admissible K3 surfaces $(\widetilde{E \times K})/\mathbb{Z}_2$. The deeper interest in mirror symmetry on the quantum level originates in the fact that in many cases, the physical observables in string theories defined on mirror background Calabi-Yau manifolds are identical. The first example of this to be demonstrated mathematically was the use of mirror symmetry to predict the number of rational curves on certain Calabi-Yau threefolds. In particular, Givental [17] and Lian, Liu and Yau [22] related the Gromov-Witten invariants from enumerative geometry of one manifold (encapsulated in a 'J-function') to the periods of the Picard-Fuchs equation of its mirror (encapsulated in an 'I-function'). The methods used in Givental's formalism have been applied to establish this deeper form of mirror symmetry for several pairs of families of manifolds [17], but not yet the Borcea-Voisin manifolds whose example had in fact preceded it.

One method of attack is another physical duality altogether called the Landau-Ginzburg (LG)/Calabi-Yau (CY) correspondence. Different aspects of this physical correspondence have been formalised in a few ways, but the one of most importance here is that produced by [16] between Gromov-Witten theory on the Calabi-Yau side, and Fan-Jarvis-Ruan-Witten (FJRW) theory on the Landau-Ginzburg side, both defined for a hypersurface of weighted projective space. Such a relationship has been established already for a number of examples, including the quintic [9] mirror quintic [26], general Calabi-Yau hypersurfaces [8], and in a more general form for the classic Calabi-Yau three-fold complete intersections [11], and hypersurfaces of Fano and general type [1]. There is a corresponding mirror symmetry for FJRW theory [21], known as Bergland-Hübsch-Krawitz (BHK) mirror symmetry, forming a square of dualities. It appears that FJRW theory is often somewhat easier to compute than the Calabi-Yau theory, and that a promising method of attack to prove Calabi-Yau mirror symmetry may be via BHK mirror symmetry and the Landau-Ginzburg duality.

In the case of hypersurfaces of weighted projective spaces, there is a direct duality between the Calabi-Yau 'phase' and the Landau-Ginzburg phase. Borcea-Voisin orbifolds are not hypersurfaces of weighted projective spaces, but can be given as complete intersections in quotients of products of weighted projective spaces. For more general complete intersections of toric varieties, Witten proposed an important set of physical models called *Gauged Linear Sigma Models* (GLSM). A general GLSM has a far more complex and interesting phase structure, divided into several chambers, where wall-crossing can be viewed as a generalisation of the LG/CY correspondence. This has been put on a mathematical footing by Fan, Jarvis and Ruan [15]. In the case of Borcea-Voisin orbifolds, this produces four different *curve-counting theories*, the original Gromov-Witten and FJRW theories being two of them. This should give an alternative approach to finding the quantum mirror structure of Borcea-Voisin orbifolds, via BHK mirror symmetry.

Gromov-Witten theory and its companion theories all come with a state space $\mathcal{H}^\circ = \bigoplus_h \phi_h \mathbb{C}$, where $\circ$ stands for the GW, FJRW or mixed theories. It is endowed with an inner product analogous to the Poincaré pairing, and a multiplication with identity ϕ_0. Furthermore they are each assigned a moduli space $\mathcal{M}^\circ$ of marked curves endowed with extra structure satisfying certain stability conditions. In Gromov-Witten theory, the curves are endowed with stable maps to $[\mathcal{X}/G]$, where $\mathcal{X} = \{W = 0\}$. In FJRW theory, they are endowed with line bundles satisfying conditions depending on the polynomial W and the group G. In the case of interest in this paper, $\mathcal{X}$ is the complete intersection defined by polynomials W_1, W_2. The mixed theories come from considering the Gromov-Witten theory of one of the W_i and the FJRW theory of the other, subject to compatibility conditions; the moduli spaces classify marked curves endowed with a stable map to $[\{W_i = 0\}/G]$ and line bundles subject to conditions depending on W_j, $j \neq i$, and that stable map. All of these moduli space come equipped with virtual classes.

In all these theories we integrate over the virtual class of the moduli space to define certain intersection numbers

$$\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle^{\circ}_{0,n[,\beta]},$$

for $\tau_{a_i}(\phi_{h_i}) = \psi_i^{a_i}\phi_{h_i}$, where the ψ-classes are defined in the usual way as the first Chern classes of the Hodge bundle over the moduli space, itself defined fibre-wise as the space of holomorphic differentials over the base curve. The data for Gromov-Witten theory and the mixed theories include stable maps from the source curve, and it is helpful for our purposes to specify the homology class β of the image of this stable map; this is not included for the purely FJRW invariants.

We may encapsulate the enumerative information of each theory by defining corresponding *J-functions* over $\mathcal{H}^{\circ}((z^{-1}))$ by setting $J^{\circ}(\sum_h t_0^h \phi_h, z)$ to be

$$z\phi_0 + \sum_h t_0^h \phi_h + \sum_{\substack{n\geq 0 \\ (h_1,\ldots,h_n) \\ [\beta \geq 0]}} \sum_{\varepsilon, k} \frac{t_0^{h_1} \ldots t_0^{h_n}}{n! z^{k+1}} \langle \tau_0(\phi_{h_1}), \ldots \tau_0(\phi_{h_n}), \tau_k(\phi_\varepsilon) \rangle^{\circ}_{0,n+1[,\beta]}.$$

More precisely, we shall consider the *ambient* or *narrow* J-functions, which restrict to the ambient or narrow classes, classes induced from the ambient product of weighted projective spaces, and which have far more manageable enumerative geometry.

The original quantum mirror theorems relate the J-function of an orbifold, from the curve-counting A-side, to its *I-function*, a fundamental solution of the orbifold's corresponding Picard-Fuchs equations, by means of a *mirror map* $\tau(\mathbf{t})$. For $I(\mathbf{t}, z) = f(\mathbf{t})z + \mathbf{g}(\mathbf{t}) + \mathcal{O}(z^{-1})$, the relation is given by

$$J(\tau(\mathbf{t}, z)) = \frac{I(\mathbf{t}, z)}{f(\mathbf{t})},$$

where $\tau(\mathbf{t}) = \frac{\mathbf{g}(\mathbf{t})}{f(\mathbf{t})}$. The mirror orbifold then swaps the roles of the I- and J-functions.

In this paper, we will find the J-functions of each theory, but it will be simpler to demonstrate the correspondence in terms of I-functions. We later justify our nomenclature by finding the Picard-Fuchs equations of certain mirror partners and showing that they are solved by the GW I-function.

For certain coefficients $K_{\mathbf{b}}, L_{\mathbf{b}}$ and functions $F_{\mathbf{b}}, G_{\mathbf{b}}$, we find the (narrow, genus zero)

Gromov-Witten I-function to be

$$I_{\mathrm{GW}}(\mathcal{Y}) = ze^{(2D_E t_1 + D_E(t_2+t_3)+\sum_{i=1}^4 w_i D_K t_{4+i})/z} \times$$
$$\sum_{\mathbf{b}\in\mathrm{Box}(\mathcal{Y})} K_{\mathbf{b}} L_{\mathbf{b}} \sum_{c\in\frac{1}{2}\mathbb{N}_0} \sum_{\substack{(a,b,k_1,\dots,k_m)\in\mathbb{N}_0^m \\ a\geq -c/2, b\geq -c/w_0 \\ v^S(a,b,c,k_1,\dots,k_m)=\mathbf{b}}} (q_1^a q_2^b q_3^c \prod_{j=1}^m x_j^{k_j})$$
$$e^{a(2t_1+t_2+t_3)+b(\sum_{i=0}^3 w_i t_i)+c(t_1+t_4+2t_8)} F_{\mathbf{b}}(a,c,\mathbf{k}) G_{\mathbf{b}}(b,c,\mathbf{k}) \mathbf{1}_{\mathbf{b}}.$$

For example, for $E = \{X^2+Y^4+Z^4=0\}$ in $\mathbb{P}(2,1,1)$, $K=\{x^2+y^6+z^6+w^6=0\}\subseteq\mathbb{P}(3,1,1,1)$, and $\sigma : (X,x)\mapsto(-X,-x)$, we find:

$$I_{\mathrm{GW}}^{[E\times K/\langle\sigma\rangle]}(\mathbf{t},z) = ze^{(2D_E t_1 + D_E(t_2+t_3)+3D_K t_4 + D_K(t_5+t_6+t_7))/z} \times$$
$$\Bigg(\sum_{c\in\mathbb{N}_0} \sum_{\substack{a,b\in\mathbb{Z} \\ a\geq -c/2 \\ b\geq -c/3}} q_1^a q_2^b q_3^c e^{(2a+c)t_1 + a(t_2+t_3)+(3b+c)t_4 + b(t_5+t_6+t_7)+2ct_8} \times$$
$$\frac{\Gamma(2D_E/z+1)\Gamma(D_E/z+1)^2\Gamma(3D_K/z+1)\Gamma(D_K/z+1)^3}{\Gamma(2D_E/z+2a+c+1)\Gamma(D_E/z+a+1)^2\Gamma(3D_K/z+3b+c+1)\Gamma(D_K/z+b+1)^3} \times$$
$$\frac{\Gamma(4D_E/z+4a+2c+1)\Gamma(6D_K/z+6b+2c+1)}{\Gamma(2c+1)\Gamma(4D_E/z+1)\Gamma(6D_K/z+1)}$$
$$+z^{-1} \sum_{c\in\frac{1}{2}\mathbb{N}_0\setminus\mathbb{N}_0} \sum_{\substack{a,b\in\mathbb{Z} \\ a\geq -c/2 \\ b\geq -c/3}} q_1^a q_2^b q_3^c e^{(2a+c)t_1 + a(t_2+t_3)+(3b+c)t_4 + b(t_5+t_6+t_7)+2ct_8} \times$$
$$\frac{\Gamma(2D_E/z+\frac{1}{2})\Gamma(D_E/z+1)^2\Gamma(3D_K/z+\frac{1}{2})\Gamma(D_K/z+1)^3}{\Gamma(2D_E/z+2a+c+1)\Gamma(D_E/z+a+1)^2\Gamma(3D_K/z+3b+c+1)\Gamma(D_K/z+b+1)^3}) \times$$
$$\frac{\Gamma(4D_E/z+4a+2c+1)\Gamma(6D_K/z+6b+2c+1)}{\Gamma(2c+1)\Gamma(4D_E/z+1)\Gamma(6D_K/z+1)} \mathbf{1}_\sigma \Bigg).$$

For our cases of interest we find the I-function for the first mixed theory to be

$$I_{\mathrm{FJRW,\,GW}}(\mathbf{t},z) = ze^{(w_0 t_4 + w_1 t_5 + w_2 t_6 + w_3 t_7)D_K/z} e^{-z}$$
$$\times \sum_{n_3,n_\sigma\in\mathbb{N}_0^3} 2\frac{\Gamma(\frac{1}{4}+\frac{n_3}{2}+\frac{n_\sigma}{4})^2 t_3^{n_3} t_\sigma^{n_\sigma}}{\Gamma(\frac{1}{4}+\langle\frac{n_3}{2}+\frac{n_\sigma}{4}\rangle+1)^2 n_3! n_\sigma!} z^{\lfloor\frac{n_3}{2}+\frac{n_\sigma}{4}\rfloor-(n_3+n_\sigma)} \sum_{\mathbf{b}\in\mathrm{Box}([K/\langle\sigma_K\rangle])} L_{\mathbf{b}}$$
$$\times \sum_{\substack{(b,c,\mathbf{k})\in\Lambda E^S_{\mathbf{b}}([K/\langle\sigma_K\rangle]) \\ 2c+\sum_{j=l+1}^m k_j = n_\sigma}} (q_2^b q_3^c \prod_{j=1}^m x_j^{k_j}) e^{b(\sum_{i=1}^n w_i t_i)+c(t_4+2t_8)} G_{\mathbf{b}}(b,c,\mathbf{k}) \phi_{h_{n_3,n_\sigma}} \mathbf{1}_{\mathbf{b}}$$
$$=: \sum_{n_1,n_3,n_\sigma} \sum_{\mathbf{b}\in\mathrm{Box}([K/\langle\sigma_K\rangle])} \omega^{\mathrm{FJRW,\,GW}}_{n_3,n_\sigma,\mathbf{b}} z^{\lfloor\frac{n_3}{2}+\frac{n_\sigma}{4}\rfloor-(n_1+n_3+n_\sigma)} \phi_{n_1,n_3,n_\sigma} \mathbf{1}_{\mathbf{b}}.$$

For our cases of interest we find the I-function for the second mixed theory to be

$$\begin{aligned}
&I_{\mathrm{GW,\,FJRW}}(\mathbf{t},z) = ze^{(2t_1+t_2+t_3)D_E/z}e^{-z} \\
&\times \sum_{n_3,n_\sigma\in\mathbb{N}_0^3} 2\frac{\Gamma(\frac{1}{6}+\frac{n_3}{3}+\frac{n_\sigma}{6})^3 t_3^{n_3} t_\sigma^{n_\sigma}}{\Gamma(\frac{1}{6}+\langle\frac{n_3}{3}+\frac{n_\sigma}{6}\rangle+1)^3 n_3! n_\sigma!} z^{\lfloor\frac{n_3}{3}+\frac{n_\sigma}{6}\rfloor-(n_1+n_3+n_\sigma)} \\
&\times \sum_{\mathbf{b}\in\mathrm{Box}([E/\langle\sigma_E\rangle])} K_{\mathbf{b}} \sum_{\substack{(a,c)\in\Lambda E^S_{\mathbf{b}}([E/\langle\sigma_E\rangle]) \\ 2c=n_\sigma}} q_2^a q_3^c e^{a(2t_1+t_2+t_3)+c(t_1+2t_8)} F_{\mathbf{b}}(a,c)\mathbf{1}_{\mathbf{b}}\phi_{h_{n_3,n_\sigma}} \\
=:& \sum_{n_1,n_3,n_\sigma} \sum_{\mathbf{b}\in\mathrm{Box}([E/\langle\sigma_E\rangle])} \omega^{\mathrm{GW,\,FJRW}}_{\mathbf{b},n_3,n_\sigma} z^{\lfloor\frac{n_3}{3}+\frac{n_\sigma}{6}\rfloor-(n_3+n_\sigma)}\mathbf{1}_{\mathbf{b}}\phi_{n_1,n_3,n_\sigma}.
\end{aligned}$$

The FJRW I-function for $W = X^2+Y^4+Z^4+x^2+y^6+z^6+w^6 = 0$ we find to be

$$\begin{aligned}
I^{\mathrm{nar}}_{FJRW}(\mathbf{t},z) =& \sum_{M,N,C\geq 0} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}+\frac{M}{2}+\frac{C}{4})^2\Gamma(\frac{1}{2})\Gamma(\frac{1}{6}+\frac{N}{3}+\frac{C}{6})^3}{\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{4}+\langle\frac{M}{2}+\frac{C}{4}\rangle+1)^2\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{6}+\langle\frac{N}{3}+\frac{C}{6}\rangle+1)^3} \\
&\times \frac{T_1^M T_2^N T_3^C}{M!N!C!} z^{1-M-N-C+2\lfloor\frac{M}{2}+\frac{C}{4}\rfloor+3\lfloor\frac{N}{3}+\frac{C}{6}\rfloor}\phi_{h(M,N,C)} \\
=:& \sum_h \sum_{(M,N,C):h(M,N,C)=h} \omega^{\mathrm{FJRW}}_h z^{1-M-N-C+2\lfloor\frac{M}{2}+\frac{C}{4}\rfloor+3\lfloor\frac{N}{3}+\frac{C}{6}\rfloor}\phi_h.
\end{aligned}$$

where $h(M,N,C)$ is given in $(\mathbf{C}^*)^7$ by

$$(-1, e^{2\pi i(\frac{M}{2}+\frac{C}{4}+\frac{1}{4})}, e^{2\pi i(\frac{M}{2}+\frac{C}{4}+\frac{1}{4})}, -1, e^{2\pi i(\frac{N}{3}+\frac{C}{6}+\frac{1}{4})}, e^{2\pi i(\frac{N}{3}+\frac{C}{6}+\frac{1}{4})}, e^{2\pi i(\frac{N}{3}+\frac{C}{6}+\frac{1}{4})}).$$

1.2 Structure of this Paper and Statement of Main Results

Borcea-Voisin orbifolds are some of the first examples of non-trivial Calabi-Yau threefolds, each given as $E\times K/\mathbb{Z}_2$, for E an elliptic curve, K a K3 surface, and $\mathbb{Z}_2$ generated by the product of anti-symplectic involutions on both factors. We consider special cases where both factors can be given as algebraic hypersurfaces in weighted projective space, and the involution acts by negating the first coordinate of each.

In this paper we first give an overview of Gromov-Witten theory, FJRW theory and the hybrid theories as defined in [15]. Each of these is encapsulated in a state space (which provides cohomological information) and an I-function (which provides enumerative information) defined from a certain moduli space. On the quotient of a variety cut out by polynomials $W_1,\ldots,W_n$ by a group G, Gromov-Witten theory gives invariants which count curves going through subvarieties in given homology classes, with certain corrections to allow integration over the moduli space to be well-defined. FJRW theory gives invariants counting curves endowed with line bundles of given multiplicities at certain marked points, subject to conditions depending on the W_i and G. In our case, we have two polynomials W_1, W_2, and this allows us to define two intermediate theories, similar to the Gromov-Witten theory for one and the FJRW theory for the other. In this paper, we find the state spaces and I-functions for all of these theories in certain cases. We restrict to genus-zero curves, and *ambient* and *narrow* theories, considering only those classes induced from the cohomology of the ambient space.

We then prove the following:

Theorem 1.2.1. **(1)** *For Borcea-Voisin orbifolds of Fermat type with quartic elliptic curve, the state spaces of all four theories are isomorphic as graded inner product spaces.*

(2) *For Borcea-Voisin orbifolds of Fermat type, the narrow/ambient state spaces of all four theories are isomorphic as graded inner product spaces.*

(3) *For Borcea-Voisin orbifolds of Fermat type, a mirror theorem holds in the sense that the narrow I-functions found in this paper, if written in the form $I^\circ(\mathbf{t}, z) = f(\mathbf{t})z + \mathbf{g}(\mathbf{t}) + \mathcal{O}(\mathbf{t}^2)$, are related to the narrow J-functions of their respective theories by*

$$J^\circ(\tau(\mathbf{t}), z) = \frac{I^\circ(\mathbf{t}, z)}{f(\mathbf{t})}, \text{ where } \tau(\mathbf{t}) = \frac{\mathbf{g}(\mathbf{t})}{f(\mathbf{t})}.$$

(4) *For Borcea-Voisin orbifolds of Fermat type with quartic elliptic curve, the mixed theory I-functions $I_{\mathrm{FJRW,GW}}(\mathbf{t}, z)$ are related to $\mathrm{I}_{\mathrm{GW}}(\mathbf{t}, z)$ by analytic continuation and symplectic transformation.*

(5) *For Borcea-Voisin orbifolds of Fermat type with K3 surface $\{x^2 + y^6 + z^6 + w^6 = 0\}$, the mixed theory I-functions $I_{\mathrm{GW,FJRW}}(\mathbf{t}, z)$ are related to $\mathrm{I}_{\mathrm{GW}}(\mathbf{t}, z)$ by analytic continuation and symplectic transformation.*

(6) *For $E = \{X^2 + Y^4 + Z^4 = 0\}$ in $\mathbb{P}(2, 1, 1)$, $K = \{x^2 + y^6 + z^6 + w^6 = 0\} \subseteq \mathbb{P}(3, 1, 1, 1)$, and $\sigma : (X, x) \mapsto (-X, -x)$, $I_{\mathrm{FJRW}}(\mathbf{t}, z)$ is related to $\mathrm{I}_{\mathrm{GW}}(\mathbf{t}, z)$ by analytic continuation and symplectic transformation.*

After this, we prove certain cases of the mirror theorem:

Theorem 1.2.2. ***(A quantum mirror theorem for Borcea-Voisin threefolds).*** *Let $\mathcal{Y}$ be a Borcea-Voisin threefold $(E \times K)/\langle\sigma\rangle$ with a well-defined twisted hypersurface birational model (as defined below) and a topological mirror Borcea-Voisin threefold $\check{\mathcal{Y}}$. Then $\check{\mathcal{Y}}$ may be given as the fibre over 0 of a 3-parameter family $\check{\mathcal{Y}}_{\psi,\varphi,\chi}$.*

Let $J_{\mathcal{Y}}(z,t)$ be the ambient, genus-0 Gromov-Witten J-functions for the sectors generated by E, K and $\mathbf{1}_\sigma$. Then there is a corresponding I-function $I_{\check{\mathcal{Y}}}(t,z)$ that satisfies certain equations system of Picard-Fuchs equations generated by the 3-parameter family. In particular:

1. *For $\mathcal{Y} = X(19,1,1), \check{\mathcal{Y}} = X(1,1,1)$ in Borcea's list, $I_{\check{\mathcal{Y}}}(\psi,\varphi,\chi)$ compiles the solutions for the full Picard-Fuchs system.*

2. *In all other cases, $I_{\check{\mathcal{Y}}}$ solves at least one Picard-Fuchs equation, as well as 2 other equations generalised from equations from $X(19,1,1)$.*

3. *For $\mathcal{Y} = X(6,4,0), \check{\mathcal{Y}} = X(14,4,1)$, the I-function 'slices'*

$$I_{X(6,4,0)}(\psi,0,0), I_{X(6,4,0)}(0,\varphi,0), I_{X(6,4,0)}(0,0,\chi)$$

   satisfy the one-parameter Picard Fuchs equations for the three 1-parameter families containing the twisted birational model of $X(14,4,0)$ parametrised by ψ,φ,χ respectively.

Furthermore, $I_{\check{\mathcal{Y}}}$ and $J_{\mathcal{Y}}$ are related by

$$J_{\mathcal{Y}}(\tau(\mathbf{t}),z) = \frac{I_{\check{\mathcal{Y}}(t,z)}(\mathbf{t},z)}{F(\mathbf{t})},$$

where $I_{\check{\mathcal{Y}}} = F(t)z + G(t) + O(z^{-1})$ and $\tau(t) = G(t)/F(t)$ is the classical topological mirror map.

Finally, without restricting to the above three sectors, in most cases $I_{\check{\mathcal{Y}}}$ satisfies a certain convenient class of further Picard-Fuchs equations coming from a family of dimension > 3.

CHAPTER 2

Preliminaries

2.1 Borcea-Voisin Orbifolds

Any elliptic curve E is endowed with an involution σ_E whose induced map on $H^2(E)$ is $-\mathrm{id}$, most simply given as that induced by the map $z \mapsto -z$ in $\mathbb{C}$, if E is considered as the quotient of $\mathbb{C}$ by a lattice.

Similarly, several K3 surfaces K are also endowed with involutions σ_K such that the induced map on $H^2(K)$ is also $-\mathrm{id}$. These 'anti-symplectic involutions' were explored and mostly classified by Nikulin in [25]. The fixpoint sets of such involutions are unions of curves, which are either empty, have at least one of genus more than one, or are the union of exactly two curves of genus one. [27]

$E \times K$ is Calabi-Yau, as the product of two Calabi-Yau manifolds. It has an involution $\sigma := \sigma_E \times \sigma_K$, whose induced map on cohomology is now the identity. The quotient $[E \times K/\langle\sigma\rangle]$, in general, has singularities (unless the fixpoint set of σ_K was empty, in which case we have the Enriques surface). We may resolve these canonically, and the corresponding manifold $\widetilde{E \times K}/\mathbb{Z}_2$, known as a *Borcea-Voisin manifold*, is also Calabi-Yau. To avoid considering the resolution of singularities separately, we treat the quotient itself as a *Borcea-Voisin orbifold* $\mathcal{Y} = [E \times K/\mathbb{Z}_2]$, the main objects of study of this paper.

Suppose $E = E_f := \{X^2 + f(Y, Z) = 0\} \subset \mathbb{P}(v_0, v_1, v_2)$ and $K = K_g := \{x^2 + g(y, z, w)\} \subset \mathbb{P}(w_0, w_1, w_2, w_3)$, and $\gcd(v_0, w_0) = 1$. As in [2] define the *twist map*

$$T : (X, Y, Z, x, y, z, w) \mapsto ((\frac{x}{X})^{\frac{v_1}{v_0}} Y, (\frac{x}{X})^{\frac{v_2}{v_0}} Z, y, z, w).$$

The image of $E \times K$ under this map is the hypersurface

$$\overline{\mathcal{Y}} = \{f(Y, Z) - f(y, z, w) = 0\} \subset \mathbb{P}(w_0 v_1, w_0 v_2, v_0 w_1, v_0 w_2, v_0 w_3).$$

$T|_{E \times K}$ is generically a double cover; the quotient map induces a birational equivalence

$\mathcal{Y} \dashrightarrow \overline{\mathcal{Y}}$ fitting into the commutative diagram

$$\begin{tikzcd} E \times K \arrow[d] \arrow[dr, dashed, "T"] & \\ \mathcal{Y} = (E \times K)/\mathbb{Z}_2 \arrow[r, dashed, "\overline{T}"] & \overline{\mathcal{Y}} \end{tikzcd}$$

2.2 The Chen-Ruan cohomology of Borcea-Voisin Orbifolds

$H^*_{CR}(\mathcal{Y})$ decomposes into two parts: first, there is a part coming from σ-invariant classes in $H^*(E \times K)$, which in turn decomposes into $(H^+(E) \otimes H^+(K)) \oplus (H^-(E) \otimes H^-(K))$, where $H^\pm$ denotes the eigenspace of σ_E or σ_K respectively with eigenvalue ± 1. Let the fixed point set of σ_K be $\Sigma = \coprod_{i=1}^N C_i$, where C_i is connected and has genus g_i. Let $N' = \sum g_i$. We compute $\chi(K/\sigma_K) = 12 + N - N'$. From this, and degree considerations, we have

$$H^+(E) = \begin{matrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{matrix}\ , \quad H^-(E) = \begin{matrix} & 0 & \\ 1 & & 1 \\ & 0 & \end{matrix}$$

$$H^+(K) = \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & a & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}\ , \quad H^-(K) = \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & b & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{matrix}$$

where $a = 10 + N - N', b = 10 - N + N'$. The total invariant part of the cohomology is then

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a+1 & & 0 & \\ 1 & & b+1 & & b+1 & & 1 \\ & 0 & & a+1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

Second, there is a part coming from the twisted orbifold sectors, or classically from the fixed point locus of the involution. In the Chen-Ruan formalism, this is given by the cohomology of the fixed point sets of the conjugacy classes of the group with the index 'twisted' by a number called the age. In our case, we have only one non-trivial conjugacy class $\{\sigma\}$, and $\mathrm{Fix}(\sigma) = 4\coprod_{i=1}^{N} C_i$. The normal bundle of Σ has rank 2, on which the involution acts with eigenvalue $e^{\frac{1}{2}(2\pi i)}$. We therefore include the cohomology of Σ:

$$\begin{array}{ccc} & 4N & \\ 4N' & & 4N' \\ & 4N & \end{array}$$

but shift the degree by 1, ceding

$$H^*_{CR}(\mathcal{Y}) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h_{1,1} & & 0 & \\ 1 & & h_{2,1} & & h_{2,1} & & 0 \\ & 0 & & h_{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}, \tag{2.1}$$

where $h_{1,1} = 11 + 5N - N'$, $h_{2,1} = 11 + 5N' - N$. It turns out that for every Nikulin involution of a K3 surface whose fixpoint set has N components whose genera sum to N', there is another with N' components whose genera sum to N, [27] These therefore correspond to mirror Borcea-Voisin orbifolds in the Hodge diamond sense, with $N = 0, N' = 0$ and $N = 2, N' = 2$ corresponding to self-mirror orbifolds.

If E and K can be given by equations inside weighted projective spaces $\mathbb{P}(\mathbf{w}_E), \mathbb{P}(\mathbf{w}_K)$, we define the *ambient space* to be $\mathcal{X} = [(\mathbf{P}(\mathbf{w}_E) \times \mathbf{P}(\mathbf{w}_K)/\tilde{\sigma}]$, where $\tilde{\sigma}$ lifts σ. We define the *ambient cohomology* $H^{\mathrm{amb}}(\mathcal{Y}) = i^*(H^*_{\mathrm{CR}}(\mathcal{X})) \subseteq H^*_{\mathrm{CR}}(\mathcal{Y})$, induced by the inclusion $i : \mathcal{Y} \hookrightarrow \mathcal{X}$. From basic weighted projective geometry this is always of the form

$$H^*_{CR}(\mathcal{X}) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{\mathrm{amb}}_{1,1} & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & h^{\mathrm{amb}}_{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array},$$

for $h^{\mathrm{amb}}_{1,1} \leq h_{1,1}$.

If a Borcea-Voisin threefold $(E_f \times K_g)/\mathbb{Z}_2$ has a Borcea-Voisin mirror, then that mirror

Table 2.1: Polynomials for example mirror pairs

$g(y,z,w)$	K_g	$\check{g}(y,z,w)$	$K_{\check{g}}$
$y^6+z^6+w^6$	$\mathbb{P}(3,1,1,1)[6]$	$y^5+yz^5+zw^6$	$\mathbb{P}(25,10,8,7)[50]$
$y^5+z^5+w^{10}$	$\mathbb{P}(5,2,2,1)[10]$	$w^8+wz^4+zy^5$	$\mathbb{P}(16,5,7,4)[32]$

Table 2.2: Example mirror pairs

X(1, 1, 1)	X(19, 1, 1)
X(6, 4, 0)	X(14, 4, 0)

may be given by $[(E_f \times K_{\check{g}})/\mathbb{Z}_2]$ in the same way. E_f is self-mirror, and $K_{\check{g}}$ is the mirror K3 surface of K_g. A list of Borcea-Voisin mirror pairs may be found expressed concisely in [3]. We list certain examples in Table 2.1 that will be important later.

Borcea [6] classified the Borcea-Voisin threefolds by invariants (r, a, δ), where r is the rank of the Picard lattice of the fixpoint set, a is its dual 2-torsion, and δ is either 0 or 1 depending on an extra condition. For our purposes $r = 10 - N' + N$, $a = 12 - N' - N$. For easy reference, in Borcea's notation, the examples correspond respectively to the Borcea-Voisin orbifolds in Table 2.2. For the first mirror pair, a twist map is defined when E is modelled by $\{X^2+Y^4+Z^4=0\}$, and for the second when E is modelled by $\{X^2+Y^3+Z^6=0\}$.

2.3 Gromov-Witten theory

A marked (possibly orbifold) curve $(C, p_1, p_2, \ldots, p_n)$ is *stable* if C is connected, compact, at worst nodal, has no marked point as a node, and has finitely many automorphisms which fix the special points (marked points and nodes). This last condition is equivalent to every genus 0 irreducible component containing at least three special points and every genus 1 component having at least one special point.

Given an orbifold $\mathcal{Y}$, we call a map $f : (C, p_1, p_2, \ldots, p_n) \to \mathcal{Y}$ *stable* if and only if every component of every fibre is stable; thus, the map can only be constant on an irreducible component of C if that component is stable.

There is a well-defined projective moduli stack $\overline{\mathfrak{M}}_{g,n}(\mathcal{Y}, \beta)$ of stable maps to $\mathcal{Y}$, where the source curves are of genus g with n marked points, and the image of the maps lie in the class $\beta \in H_2(\mathcal{Y})$. [4] There are subtleties about integrating over this moduli space. Instead of the fundamental class we generally integrate over a specified *virtual fundamental class*

$[\overline{\mathfrak{M}}_{g,n}(\mathcal{Y},\beta)]^{\mathrm{vir}}$ whose definition may be found in [5].

An orbifold comes with an *inertial manifold* $I\mathcal{Y} = \coprod_{(g)} \mathrm{Fix}(g)$ which serves to decompose the orbifold into parts corresponding to different group conjugacy classes at the orbifold points. This moduli space is endowed with evaluation maps $\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,n}(\mathcal{Y},\beta) \to I\mathcal{Y}$, given in the manifold case by $\mathrm{ev}_i : f \mapsto f(p_i)$. In the orbifold case, the inertia orbifold allows us to keep track of which twisted sector the evaluation map sends a marked point to. There is a natural line bundle $\mathbb{L}_i \to \overline{\mathfrak{M}}_{g,n}(\mathcal{Y},\beta)$ whose fibre at each map f in the moduli space is the cotangent line at $f(p_i)$. (At orbifold points this differs from the corresponding tangent space for the underlying space by a factor of the multiplicity of the point.) We define $\psi_i = c_1(\mathbb{L}_i)$. Then the *descendant Gromov-Witten invariants* are given by

$$\langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n) \rangle_{g,n}^{\beta} = \int_{[\overline{\mathfrak{M}}_{g,n}(\mathcal{Y},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\psi^{a_i}.$$

All these invariants may be packaged into

$$\langle \mathbf{t}, \ldots, \mathbf{t} \rangle_{g,n}^{\beta} = \sum_{k_1,\ldots,k_n \geq 0} \langle \tau_{k_1}(t_{k_1}), \ldots, \tau_{k_n}(t_{k_n}) \rangle,$$

for $\sum_{i\geq 0} t_i z^i \in H^*_{\mathrm{CR}}(\mathcal{Y})[[z]]$. This is the generating function for the genus-g *descendant potential*

$$\mathcal{F}^g_{\mathcal{Y}}(\mathbf{t}) = \sum_{n\geq 0} \sum_{\beta \in H^2_{\mathrm{CR}}(\mathcal{Y})} \frac{q^d}{n!} \langle \mathbf{t}, \mathbf{t}, \ldots \mathbf{t} \rangle_{g,n}^{\beta},$$

which takes values over the Novikov ring $\mathbb{C}[[H_2(\mathcal{Y})]]$ (or, in our case, the subring involving only effective classes $\mathbb{C}[[H_2(\mathcal{Y}) \cap NE(\mathcal{Y})]]$).

We generally specify a degree-two generating set $\{\phi_\alpha\}$ of $H^*_{\mathrm{CR}}(\mathcal{Y})$, and write $\mathbf{t} = \sum_{k\geq 0} \sum_\alpha t_k^\alpha \phi_\alpha z^k$.

2.4 FJRW Theory

This subsection closely follows [16]. A Landau-Ginzburg model is given by quasi-homogeneous polynomial function $W : \mathbb{C}[x_1, x_2, \ldots x_N] \to \mathbb{C}$ with weights $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N$ and degree d, and a group G which leaves W invariant. We denote by q_i, $1 \leq i \leq n$ the *charges* $\frac{\bar{w}_i}{d}$. The hypersurface X_W is Calabi-Yau if and only if $\sum_i q_i = 1$. We shall work exclusively in the *Fermat* case, in which $W = \sum_{i=1}^{N} x^{a_i}$, where $a_i = \frac{d}{\bar{w}_i} = \frac{1}{q_i}$.

The *maximal group* $G_{\max}$ of all diagonal symmetries leaving W invariant is given by

$$\{(e^{\frac{2\pi i r_1}{a_1}}, \dots, e^{\frac{2\pi i r_N}{a_N}}) | 1 \leq r_i a_i\}.$$

There is a special group element $J := (e^{\frac{2\pi i}{a_1}}, \dots, e^{\frac{2\pi i}{a_N}})$.

Then $(W, \langle J \rangle)$ corresponds to the hypersurface $X_W = \{W = 0\} \subseteq (\mathbb{C}^N \backslash \{0\})/\langle J \rangle = \mathbb{P}(w_1, \dots, w_N)$.

More generally, a well-defined FJRW theory may be given by any group such G such that $\langle J \rangle \subseteq G \subseteq G_{\max}$ (by a result of Krawitz, [21], these correspond to the *admissible* groups of [16]), and this morally corresponds to the orbifold $[X_W/(G/\langle J \rangle)]$).

It is easy to check that if W_1, W_2 share no variables in common, then $X_{W_1} \times X_{W_2}$ corresponds to $(W_1 + W_2, \langle J_1, J_2 \rangle)$. In our case, we will consider a Landau-Ginzburg theory of the form $(W_1 + W_2, \langle J_1, J_2, \sigma \rangle)$.

For $h \in G$, let N_h be the dimension of the fixpoint subspace $\mathrm{Fix}(h) \subseteq \mathbb{C}^N$, let $W_h = W|_{\mathrm{Fix}(h)}$, and let $W_h^{+\infty} = (\mathrm{Re} W_h)^{-1}(]\rho, +\infty[)$ for $\rho >> 0$.

The state space is given by

$$\mathcal{H}_{\mathrm{FJRW}}(W, G) = \bigoplus_{h \in G} \mathcal{H}_h,$$

where

$$\mathcal{H}_h = H^{N_h}(\mathrm{Fix}(h), W_h^{\infty}; \mathbb{C})^G.$$

The sectors corresponding to h for which $\mathrm{Fix}(h) = \{0\}$ are termed *narrow* sectors, and we denote their union $\mathcal{H}_{\mathrm{nar}}$. These will be seen to correspond to the ambient classes in Gromov-Witten theory. All other sectors are termed *broad*.

Analogously to Chen-Ruan cohomology, we define the *age* of the action of g with eigenvalues $e^{2\pi i \Theta_k(h)}$ to be $\sum_k \Theta_k(h)$, and the Hodge bidegree of H_g is then shifted by $(\mathrm{age}(g) - 1, \mathrm{age}(g) - 1)$, giving total degree $\deg_W(\alpha) = N_h + 2(\mathrm{age}(g) - 1)$.

An alternative construction can be given by the theory of the Milnor ring

$$\mathcal{Q}_W := \mathbb{C}[x_1, x_2, \dots, x_N]/\mathcal{J}_W$$

where $\mathcal{J}_W$ is the Jacobian ideal

$$(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots \frac{\partial W}{\partial x_N}).$$

This is a complex vector space of dimension

$$\mu = \prod_{i=1}^{N} (\frac{1}{q_i} - 1),$$

where it is a straightforward exercise to show that the highest degree attained is that of $\mathrm{Hess}(W)$, equal to $\sum_{i=1}^{N} (1 - 2q_j)$, which we label the *central charge*, denoted by $\hat{c}$.

Then we can alternatively define the FJRW state space sector-wise by

$$\mathcal{H}_g = \Omega^{N_g}(\mathrm{Fix}(g))/(\mathrm{d}, W|_{\mathrm{Fix}(g)} \wedge \Omega^{N_g - 1}) \cong \mathcal{Q}_{W|_{\mathrm{Fix(g)}}} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_N.$$

Let ϕ_g be the fundamental class in $\mathcal{H}_g$, and write $\phi^g = \phi_{g^{-1}}$.

There is a natural pairing $\eta(\cdot, \cdot)$ given in the context of the Milnor ring, given by

$$fg = \frac{\langle f, g \rangle}{\mu} \mathrm{Hess}(W) + \text{lower terms}.$$

FJRW theory associates a moduli stack $\mathcal{W}_{g,n}^{(W,G)}$ of curves endowed with line bundles and some further structure to each Landau-Ginzburg model (W, G). It was originally defined to give an appropriate context for the solution of the *Witten equation* defined in [28]

$$\overline{\partial} u_i + \overline{\frac{\partial W}{\partial u_i}} = 0,$$

where W is a quasi-homogeneous polynomial and and u_i is a section of a line bundle over some complex curve C. A full treatment can be found in [16].

Instead of considering maps from the curves to an ambient space, we consider specified sets of line bundles which each loosely correspond to coordinates.

A *d-stable W-spin orbicurve* is a marked orbicurve $(C, p_1, \ldots, p_n)$ with at worst nodal singularities endowed with line bundles $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_N$ such that for each monomial term $W_i = \prod_{j=1}^{N} x_j^{a_{ij}}$ of W we have

$$\bigotimes_{j=1}^{N} \mathcal{L}_j^{\otimes a_{ij}} \cong \omega_{\log} := \omega_C \otimes (\sum_{i=1}^{n} p_i),$$

and isomorphisms

$$\phi_i : \mathcal{L}_i^{\otimes d} \to \omega_{\log}.$$

There is a proper Deligne-Mumford stack $\mathcal{W}_{g,n}$ of n-marked W-spin curves of genus g. [16]

We represent $h \in G_{\max}$ by $(e^{2\pi i\Theta_1(h)}, \ldots, e^{2\pi i\Theta_N(h)})$. Then the moduli space decomposes into $\coprod_{\mathbf{h}\in G^n} W_{g,n}(\mathbf{h})$, where $\Theta_k(h_i) = \mathrm{mult}_{p_i}(\mathcal{L})/d$. For a curve in $\mathcal{W}_{g,n}(\mathbf{h})$, the corresponding line bundle over the coarse space $|\mathcal{L}_k|$ has degree

$$q_k(2g-2+n) - \sum_{i=1}^{n} \Theta_k(h_i),$$

which must be an integer - this must be true for all k for $\mathcal{W}_{g,n}(\mathbf{h})$ to be non-empty.

For admissible [16, 21] $G \subseteq G_{\max}$, there is some quasi-homogeneous polynomial Z such that $G_{\max}(W+Z) = G$. Then $\mathcal{W}_{g,n,G} \subseteq \mathcal{W}_{g,n}$ is the (proper [16]) substack of $(W+Z)$-orbicurves.

There is a virtual cycle $[W_{g,n,G}]^{\mathrm{vir}}$ of degree $2(\hat{c}-3)(1-g)+n-\sum_{i=1}^{n} n\iota(h_i)$, and there are ψ-classes defined similarly to the Gromov-Witten case. [16]

The FJRW invariants are given by

$$\langle \psi^{a_1}\phi_{h_1}, \psi^{a_2}\phi_{h_2}, \ldots, \psi^{a_n}\phi_{h_2}\rangle_{g,n}^{(W,G)} = (\prod_k q_k)^{g-1} \int_{[\mathcal{W}_{g,n,G}(h_1,\ldots,h_n)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{a_i}.$$

This is defined via the virtual class of the moduli space, which is given by $-R\pi_*(\bigoplus_{k=1}^{N} \mathcal{L}_k)^{\vee}$.

We also have a product structure $\cup$ given by

$$\eta_(\alpha \cup \beta, \gamma) = \langle \alpha, \beta, \gamma \rangle.$$

For several reasons it is less complicated to consider only the FJRW theory of the narrow sectors, and this paper will only compute the FJRW invariants involving $\phi_h \in \mathcal{H}_{\mathrm{nar}}$. However, it will also help to be able to use the whole group G. We define the *extended narrow state space*

$$\mathcal{H}^{\mathrm{ext}} := \mathcal{H}^{\mathrm{nar}} \oplus \bigoplus_{h\in G\setminus\mathcal{N}} \mathbb{C}\phi_h,$$

where $\mathcal{N}$ is the set of $h \in G$ giving narrow sectors.

Finally, we define our twisted invariants, corresponding to an integral not over the whole ambient space, but rather the sub-variety of interest. This is also given in [16], based on the formalism given for Gromov-Witten invariants in [17].

2.5 The Fan-Jarvis-Ruan GLSM

This subsection is a very quick overview of the required vocabulary of the theory of the Gauged Linear Sigma Model defined in [15]. Let G be a reductive group in $\mathrm{GL}(V)$, for V some vector space. Let $\theta \in \hat{G}$ be some character of G, and let L_θ be the induced line bundle over V, given by $V \times \mathbb{C}$ with G acting by $g : (\mathbf{v}, \tilde{z}) \mapsto (g \cdot \mathbf{v}, \theta(g)\tilde{z})$. A point $\mathbf{v}$ of V is *semistable* under the action of G and a choice of θ if there is some positive integer m and some G-invariant section $f \in H^0(V, L^{\otimes m}; \mathbb{C})^G$ for which $f(\mathbf{v}) \neq 0$. We let $V^{ss}(\theta)$ be the set of (G, θ)-semistable points. A semistable point is *stable* if it has finite stabiliser and closed G-orbit, and we label the set of stable points $V^s(\theta)$. In general, this removes a zero-standard-measure set of 'bad points' which cause the quotient $[V/G]$ to be non-separated. In our cases of interest, $V^s(\theta) = V^{ss}(\theta)$. We define the GIT quotient $[V//_\theta G]$ to be $[V^{ss}(\theta)/G]$. This plays the role of the 'ambient space' of the theory.

This theory does not solely consider the action of G, but rather the action of an extension of G which generalises the weights. Let $\mathbb{C}^*_R$ be $\mathbb{C}^*$ acting on V by

$$\lambda : (v_1, \ldots, v_n) \mapsto (\lambda^{c_1} v_1, \ldots, \lambda^{c_n} v_n).$$

The c_i are not necessarily positive. If the action of $\mathbb{C}^*_R$ commutes with the action of G and $G \cap \mathbb{C}^*_R = \langle J \rangle$, where $J = (e^{2\pi i c_1/d}, \ldots, e^{2\pi i c_n/d})$. Let $\Gamma = \langle G, \mathbb{C}^*_R \rangle \leq \mathrm{GL}(V)$. Let ϑ be a lift of θ from G to Γ. It is clear that $V^{ss}(\vartheta) \subset V^{ss}(\theta)$. If equality holds, we call such a lift ϑ *good*. In our case, we shall only be considering the trivial lift.

For the last piece of input data, fix a non-degenerate G-invariant polynomial W defined on V of degree d and charges $q_i = c_i/d$. This shall be the *superpotential* of the theory. The image of the critical set $\mathcal{CR}$ of W in the GIT-quotient, $[\mathcal{CR}//_\theta G]$, is of special interest. If W is non-degenerate, then $\mathcal{CR}$ is compact.

The theory is endowed with a state space directly generalising that of FJRW theory. When G is abelian, we define $\mathcal{H}_{\mathrm{GLSM}} = \bigoplus_{\alpha \in \mathbb{Q}} H^{\alpha+2q}_{\mathrm{CR}}([V//_\theta G], W^\infty; \mathbb{C})$, where $W^\infty = (\mathrm{Re}W)^{-1}(]0, \rho[)$ for $\rho >> 0$, and $q = \sum_i q_i$. This plays the role of the ambient state space.

If an element $g \in G$ has compact inertia stack component (that is, fixpoint set) then it is *narrow*. $\mathcal{H}_{\mathrm{nar}} = \bigoplus_{g,\,\mathrm{narrow}} \mathcal{H}^{(g)}_{\mathrm{CR}}$. If an element of $\mathcal{H}_{\mathrm{GLSM}}$ is Poincaré-dual to a substack of $\mathcal{CR}$, then it is *critical*. Let $\mathcal{H}_{\mathrm{GLSM,\,comp}}$ be the span of the narrow and critical elements. This will encompass the narrow and ambient sectors in FJRW and GW theory, respectively.

The moduli space of the theory classifies *stable Landau-Ginzburg quasimaps*. These are given by tuples

$$(\mathcal{C}, y_1, \ldots, y_n, \mathcal{P}, \sigma, \kappa),$$

such that $(\mathcal{C}, y_1, y_2, \ldots, y_n)$ is an n-pointed orbicurve, $\mathcal{P} : \mathcal{C} \to B\Gamma$ is a representable principal Γ-bundle, $\sigma : \mathcal{C} \to \mathcal{E} = \mathcal{P} \times_\Gamma V$ is a global section, and $\kappa : \zeta^*\mathcal{P} \to \omega^\circ_{\log,\mathcal{C}}$ is an isomorphism of principal $\mathbb{C}^*$-bundles, where in turn $\zeta : \Gamma \to \mathbb{C}^*_{\mathrm{R}}$ is the group homomorphism sending G to 1 and $(\lambda^{c_1}, \ldots, \lambda^{c_n})$ to λ^d, and $\omega^\circ_{\log,\mathcal{C}}$ is the induced principal $\mathbb{C}^*$-bundle associated to $\omega_{\log,\mathcal{C}}$. They moreover satisfy certain technical stability conditions detailed in [15] which depend on the lift ϑ and a rational number ϵ, which restricts the behaviour of σ on certain points of $\mathcal{C}$. In our case, we shall also require that σ induces a map $[\sigma] : \mathcal{P} \to V$ with image in $\mathcal{CR}(W)$. They define the notion of the *degree* of a quasimap, which coincides with the degree of the image in the Gromov-Witten case. We let $\mathrm{LGQ}^{\epsilon,\vartheta}_{g,n}([\mathcal{CR}^{\mathrm{ss}}/\mathcal{G}], \beta)$ be the moduli space of such stable n-pointed genus-g LG quasimaps for given ϑ, ϵ and degree β.

By the main theorems of [15], this moduli space is a Deligne-Mumford stack.

We define the corresponding inertia stack to be

$$\mathbb{I}[\mathcal{CR}//_\theta G] = [\{(v, g) \in V^{\mathrm{ss}}(\theta) \times G \mid gv = v\}/G]$$

where G acts on the second factor. This moduli space has a well-defined virtual class, ψ-classes, and evaluation maps $ev_i : \mathrm{LGQ}^{\epsilon,\vartheta}_{g,n}([\mathcal{CR}^{\mathrm{ss}}/\mathcal{G}]) \to \mathbb{I}[\mathcal{CR}//_\theta G]$ which allow us to define invariants

$$\langle \tau_{l_1}(\alpha_1), \ldots, \tau_{l_n}(\alpha_n)\rangle = \int_{[\mathrm{LGQ}^{\epsilon,\vartheta}_{g,n}([\mathcal{CR}^{\mathrm{ss}}/\mathcal{G}],\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} ev_i^*(\alpha_i)\psi_i^{l_i}.$$

In our case of interest, the possible weights for the character θ will divide the GLSM into four chambers, one of which will correspond to Gromov-Witten theory, one to FJRW theory, and the other two to certain 'mixed' theories.

2.6 Givental's Formalism for Mirror Symmetry

The definitions of Gromov-Witten theory, FJRW theory and the mixed theories have already been analogous in several ways. For this section, the treatment is identical and follows for all *generalised* Gromov-Witten theories (see [12]). Let $\circ$ stand for any GW, FJRW or mixed theory under discussion. So far, each have a state space endowed with an inner product and invariants which can be compiled into genus-g generating functions $\mathcal{F}^\circ_g = \sum_{n\geq 0} \frac{1}{n!}\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\rangle$. Let the *total genus descendant potential* be

$$\mathcal{D}^\circ := e^{\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}^\circ_g}.$$

Define the symplectic vector space $\mathcal{V}^\circ$ to be $\mathcal{H}^\circ((z^{-1}))$ endowed with the symplectic form

$$\Omega^\circ(f,g) = \mathrm{Res}_{z=0}\langle f(-z), g(z)\rangle^\circ,$$

defined via the induced inner product. We choose a polarisation $\mathcal{V}^\circ_+ = H^\circ[z], \mathcal{V}^\circ_- = z^{-1}H^\circ[[z^{-1}]]$. This gives us Darboux coordinates $(\mathbf{p},\mathbf{q})$ corresponding to

$$\sum_{k\geq 0}\sum_{i\in I} q^i_k \phi_i z^k + \sum_{k\geq 0}\sum_{i\in I} p_{k,i}\phi^i(-z)^{-k-1}.$$

The dilaton shift is a slight adjustment fitting the $\mathbf{t}$-coordinates to our framework:

$$q^0_1 = t^0_1 - 1, q^i_k = t^i_k.$$

We define the *Givental Lagrangian cone*

$$\mathcal{L}_\circ = \{\mathbf{p} = \mathrm{d}_\mathbf{q}\mathcal{F}^\circ_0\},$$

which is Lagrangian with respect to Ω° and by some basic generalised Gromov-Witten theory [18] can be shown to be a cone. Its elements may be written

$$-\phi_0 z + \sum_{\substack{k\geq 0\\ i\in I}} t^i_k\phi_i z^k + \sum_{\substack{a_1,\ldots,a_n,a\geq 0\\ i_1,\ldots,i_n,i\in I}} \frac{t^{i_1}_{a_1}\ldots t^{i_n}_{a_n}}{n!(-z)^{a+1}}\langle\psi^{a_1}\phi_{i_1},\ldots,\psi^{1_n}\phi_{i_n},\psi^a\phi_i\rangle^\circ_{0,n+1}\phi^i.$$

By some further Gromov-Witten theory [12] it can be shown that

$$\mathcal{L}\cap T_f\mathcal{L} = zT_f\mathcal{L},$$

where $\mathcal{L}$ is ruled by the $zT_f\mathcal{L}$ over all f, and there is a filtration

$$T_f\mathcal{L}\supset zT_f\mathcal{L}\supset z^2T_f\mathcal{L}\supset\ldots$$

The image of a function $f(\mathbf{t})$ (with $\mathbf{t}\in\mathcal{H}^\circ$) corresponds to a 'slice' of $\mathcal{L}$. In this way $\mathcal{L}^\circ\cap(-\phi_0 z\oplus\mathcal{H}^\circ\oplus\mathcal{V}^\circ_-)$ corresponds to the J-function:

$$J^\circ(\mathbf{t},z) = \phi_0 z + \mathbf{t} + \sum_{n\geq 0}\sum_{a\geq 0, i\in I}\frac{1}{n!z^{a+1}}\langle\mathbf{t},\ldots,\mathbf{t},\phi_i\psi^a\rangle^\circ_{0,n+1}\phi^i.$$

Givental's version of mirror symmetry relates the J-function to the *I-function* $I(\mathbf{t},z)$, which in the Calabi-Yau sense provides solutions to the Picard-Fuchs equations of the

Calabi-Yau family. 'Quantum' mirror symmetry *in the sense of Givental* conjectures that there is a *mirror map* $\tau : \mathcal{H} \to \mathcal{H}^\vee$ so that

$$J^\circ(\tau(\mathbf{t}), -z) = I^\circ(\mathbf{t}, -z),$$

where on the Calabi-Yau side, $\mathcal{H}^\vee$ is the state space of the mirror orbifold. That is, there is an invertible mirror map between the cohomologies of two mirror manifolds that not only rotates the Hodge diamond by 90° but also swaps their I- and J-functions. Based on the types of string theory to which they are associated, it is common to refer to the Kähler geometry relating to the J-function and the (1, 1)-sector of the Hodge diamond relating to it as the 'A-side', and the complex geometry relating to the I-function, Picard-Fuchs equations and the (2, 1)-sector as the 'B side'. Thus a mirror pair have their A- and B-sides swapped.

In order to calculate these J-functions (on either side), however, we define an equivariant theory first and take the non-equivariant limit. $\mathbb{C}^*$ acts on each $\mathcal{L}_k$ by multiplication on each fibre, and on each moduli space by performing this action pointwise. The equivariant theories are defined over the ground ring $R = H^*_{\mathbb{C}^*}(pt, \mathbb{C})[[s_0, s_1, \ldots]]$, and the invariants are given by cupping the integrand with (*twisting by*) the multiplicative characteristic class

$$(E) := \exp(\sum_k s_k \mathrm{ch}_k(E)).$$

If $s_d = 0$ for all d, this class is zero and we still have the *untwisted theory*. For

$$s_d = \begin{cases} -\ln\lambda, & d = 0 \\ \frac{(d-1)!}{\lambda^d}, & d > 0 \end{cases}$$

we will recover the dual of the equivariant Euler class of the virtual bundle in the non-equivariant limit.

Proof. For a line bundle $\mathcal{L}$, we have

$$\begin{aligned}
\exp(\sum_{d\geq 0} s_d \mathrm{ch}_d(-\mathcal{L})) &= \exp(\ln(\lambda)\,\mathrm{ch}_0(\mathcal{L}) - \sum_{d>0} \frac{(d-1)!}{\lambda^d}\mathrm{ch}_d(\mathcal{L})) \\
&= \exp(\mathrm{ch}_0(\check{\mathcal{L}}) - \sum_{d>0}(-1)^{d-1}\frac{(d-1)!}{\lambda^d}\mathrm{ch}_d(\check{\mathcal{L}})) \\
&= \lambda\exp(\ln(1 + \frac{c_1(\check{\mathcal{L}})}{\lambda})) \\
&= \lambda + c_1(\check{\mathcal{L}}).
\end{aligned}$$

Taking $\lambda \to 0$ gives the desired result for $\mathcal{L}$; as we are working over $\mathbb{C}$, extending by the splitting principle will give us the same result for all vector bundles. $\square$

We use the twisted invariants to define twisted generating functions $\mathcal{F}^{\mathrm{tw}}$, twisted potential $\mathcal{D}^{\mathrm{tw}}$, a twisted Lagrangian cone $\mathcal{L}^{\mathrm{tw}}$ and a twisted J-function J^{tw}. When appropriate we will denote their untwisted analogues $\mathcal{F}^{\mathrm{un}}, \mathcal{D}^{\mathrm{un}}, \mathcal{L}^{\mathrm{un}}, J^{\mathrm{tw}}$.

2.7 The Landau-Ginzburg/Calabi-Yau Correspondence and Mirror Symmetry

Beyond analogous definitions, mirror symmetry is a non-trivial duality relating the I-function and J-function. The LG/CY correspondence furthermore relates the state spaces and I-functions and J-functions of GW theory and FJRW theory:

Calabi-Yau side

$$\begin{tikzcd} \mathcal{H}^{GW}(\mathcal{Y}), J_{\mathrm{GW}}(\mathcal{Y}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathcal{H}^{GW}(\check{\mathcal{Y}}), I_{\mathrm{GW}}(\mathcal{Y}) \cong J_{\mathrm{GW}}(\check{\mathcal{Y}}) \arrow[d, leftrightarrow] \\ \mathcal{H}^{FJRW}(W,G), J_{\mathrm{GW}}(\mathcal{X}) \arrow[r, leftrightarrow] & \mathcal{H}^{FJRW}((\check{W,G})), I_{\mathrm{FJRW}}(W,G) \cong J_{\mathrm{FJRW}}(\check{W},\check{G}) \end{tikzcd}$$

Landau-Ginzburg side

We shall restrict to the narrow FJRW sectors of (W, G), which correspond to the cohomology sectors induced by the ambient space of $\mathcal{Y}$ on the GW side, and then relate $I^{\mathrm{amb}}_{\mathrm{GW}}(\mathbf{t}, -z)$ and $I^{\mathrm{nar}}_{\mathrm{FJRW}}(\mathbf{t}, -z)$, which can both be found by *twisting* the actual I-functions of the ambient space or its LG dual. On the Gromov-Witten B-side we take the I-function to provide solutions to the Picard-Fuchs equations of a family parametrised by ψ around the point $\psi = 0$. Then the Landau-Ginzburg I-function is taken at $\psi = \infty$. The actual LG/CY correspondence is thus given as the composition of an analytic continuation from $\psi = 0$ to ∞ and a symplectic transformation between $\mathcal{V}^{\mathrm{GW}}$ and $\mathcal{V}^{\mathrm{FJRW}}$.

As a final note, the bottom arrow represents a neat formulation of mirror symmetry on the LG side due to Krawitz [21] known as Bergland-Huebsch-Krawitz mirror symmetry. Where aspects of mirror symmetry are themselves difficult to prove on the Calabi-Yau side, it may be easier to rephrase them in terms of FJRW theory.

2.8 Picard Fuchs Equations and Gauss-Manin Connections

We follow the introduction given by Morrison [24]. Given a family of Calabi-Yau n-folds $\mathcal{M}_\psi$, a smoothly varying family of holomorphic n-forms ω_ψ, and holomorphically varying classes $\gamma_i(\psi)$ which generate the middle homology of M_ψ, the *period integrals* of the family are defined to be $\int_{\gamma_i(\psi)} \omega_\psi$. Where $\gamma_i(\psi)$ varies along a non-contractible path in the base of the family, monodromy considerations must be taken into account, so that the period integrals demonstrate non-trivial behaviour.

We list all the period integrals for each γ_i into a vector

$$v(z) = (\int_{\gamma_0(\psi)} \omega_\psi, \ldots, \int_{\gamma_r(\psi)} \omega_\psi).$$

Consider the span of the first k ψ-derivatives of $v(z)$. For generic ψ, the dimension is constant, and bounded by $r+1$. Therefore, for some k, the first k ψ-derivatives are linearly dependent, so that we have $a_j(\psi)$ such that

$$\frac{\mathrm{d}^k}{\mathrm{d}\,\psi^k} v(\psi) + \sum_{j=0}^{k-1} a_j(\psi) \frac{\mathrm{d}^j}{\mathrm{d}\,\psi^j} v(\psi) = 0.$$

The Picard-Fuchs equation characterises the period integrals, so it is given by

$$\frac{\mathrm{d}^k}{\mathrm{d}\,\psi^k} f(\psi) + \sum_{j=0}^{k-1} a_j(\psi) \frac{\mathrm{d}^j}{\mathrm{d}\,\psi^j} f(\psi) = 0.$$

In order to avoid complications from singularities, it is common to rewrite this as

$$(\psi \frac{\mathrm{d}}{\mathrm{d}\,\psi^k})^k f(\psi) + \sum_{j=0}^{k-1} b_j(\psi) (\psi \frac{\mathrm{d}^j}{\mathrm{d}\,\psi})^j f(\psi) = 0.$$

The full space of complex structures of a given Calabi-Yau threefold $\mathcal{Y}$ is given by $H^{2,1}(\mathcal{Y})$. Thus the full Picard Fuchs equations are derived from varying Hodge structures within a family of dimension at most $h^{2,1}$. In the case of the quintic threefold and elliptic curves, $h^{2,1} = 1$, so we derive one equation in one parameter ψ.

In the more general case, including complete intersections and all Borcea-Voisin manifolds, the full story requires a multi-parameter model: in fact the simplest Borcea-Voisin case is a three-parameter model. Consider a Calabi-Yau threefold M given as the quotient of some hypersurface $\{\mathcal{Q}=0\}/G$ in some weighted projective space $\mathbb{P}^4(\mathbf{q})$.

The local ring $\mathcal{R}_{\text{loc}}$ of such Calabi-Yau quotients may be generated by the degree-d G-invariant monomials of the weighted projective space, identified with a subspace of the cohomology ring (in fact, that part induced from the ambient weighted projective space). That is, it has a degree-symmetric graded basis

$$\{m_0 = 1; m_1, m_2, \dots, m_{h^{2,1}}; m_{h^{2,1}+1}, \dots, m_{2h_{2,1}}; m_{2h^{2,1}+1}\}.$$

It is therefore given as a quotient $\mathbb{C}[x,y,z,w,v]^G/\mathcal{I}$, where $\mathcal{I}$ is generated as an ideal by relations among the m_i.

Varying the complex structure of M along these monomials produces a family

$$M_{\psi_1,\dots,\psi_{h^{2,1}}} = \{\mathcal{Q} + \sum_{i=1}^{h^{2,1}} \psi_i m_i = 0\}/G.$$

The Picard-Fuchs equations are then given by the generating relations of $\mathcal{I}$, replacing m_i by $\frac{\partial}{\partial\psi_i}$, and depending on convention changing variables to some appropriate power of ψ. Furthermore, often only a cursory examination of the structure of $\mathcal{R}_{\text{loc}}$ is required to determine the number and order of the Picard-Fuchs equations. For example, for a Calabi-Yau threefold in a one-parameter family, its local ring has a basis $\{1, m, m^2, m^3\}$, so that there is one relation expressing m^4 as a linear combination of the rest, given a fourth-order equation. For a three-parameter model, there are six possible degree $2d$ products of m_1, m_2, m_3, but the degree-2 part of the local ring must have degree 3; therefore there must be three relations of degree 3, giving 3 Picard-Fuchs equations of order 2, as well as other more complicated equations of higher order.

Another understanding was originally provided by Manin [23]. Given a one-parameter family $\pi : V \to B$, the vector bundle $R^n\pi_*\mathbb{C} \otimes \mathcal{O}_B$ over B comes with a natural *Gauss-Manin* connection, $\nabla_{GM}(\psi)$, for $\psi \in B$. Let ω_ψ be a holomorphic form of appropriate degree to define the period integrals. Then relations

$$\sum_{k=0}^{n} f_k(b) \nabla_{GM}(\psi)^k \omega_\psi = 0$$

correspond to the Picard-Fuchs equations $\sum_{k=0}^{n} f_k(\psi)(\frac{\mathrm{d}^k}{\mathrm{d}\,\psi^k} f) = 0$, satisfied by the period integrals. Then the Picard-Fuchs equations can be seen in fact to be equivalent to the Gauss-Manin connection, in that they define the flat sections. An analogous situation holds for multi-parameter families.

In general, the solutions to the Picard-Fuchs equations (that is, the periods) may be compiled into an optimal solution, a multivariate function of all parameters, known as the $I - function$ of that family. This is the object of interest in quantum mirror symmetry.

For one-parameter families, the Dwork-Griffiths method provides a method to compute the Picard-Fuchs equation. Suppose we have a family of polynomials $\mathcal{Q}_\psi(x_0, x_1, \ldots, x_n)$ of degree d and weights $w_0, \ldots, w_n$. Maximal-degree differentials on $\mathbb{P}(w_0, \ldots, w_n)$ are given by $P\Omega/Q$ with matching degrees for the numerator and denominator, where

$$\Omega = \sum_{i=0}^{n} (-1)^i w_j x_j \bigwedge_{j \neq i} \mathrm{d}x_j.$$

Such forms may be associated to linear forms on $H_n(\mathbb{P}(w_0, \ldots, w_n))$ by the map $\gamma \mapsto \int_\gamma \frac{P\Omega}{Q^r}$. This sets up a correspondence between $H^n(\mathbb{P}(w_0, \ldots, w_n))$ and the forms $\frac{P\Omega}{Q^r}$ modulo $J_{\mathcal{Q}_\psi}$.

The key to the algorithm is Griffiths' reduction of pole order formula modulo $J_{\mathcal{Q}_\psi}$. Choose arbitrary polynomials P_i of degree $w_i + rd - \sum_{i=1}^n w_i$. Then the exterior derivative of

$$\sum_{i<j} \frac{w_i x_i P_j - w_j x_j P_i}{Q^r} \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}\hat{x}_i \wedge \ldots, \wedge \mathrm{d}\hat{x}_j \wedge \ldots \wedge \mathrm{d}x_n$$

is

$$\frac{r \sum_{i=0}^n P_i \frac{\partial Q}{\partial x_i} \Omega}{Q^{r+1}} - \frac{\sum_{i=0}^n \frac{\partial P_i}{\partial x_i} \Omega}{Q^l}.$$

It follows that

$$\frac{r \sum_{i=0}^n P_i \frac{\partial Q}{\partial x_i} \Omega}{Q^{r+1}} \cong_{J_{\mathcal{Q}_\psi}} \frac{\sum_{i=0}^n \frac{\partial P_i}{\partial x_i} \Omega}{Q^l}.$$

That is, whenever the numerator of a rational n-form is in $J_{\mathcal{Q}_\psi}$, it is possible to reduce the pole order explicitly. Starting with such an n-form and taking its derivatives, we may perform this reduction as many times as needed to determine linear relation among them. This will give the Picard-Fuchs equation.

CHAPTER 3

The Gromov-Witten side

3.1 The State Space

There are two elliptic curves of Fermat type $X^2 + P(Y, Z)$ with anti-symplectic involution given by $X \mapsto -X$, tabulated below.

We will assume that $E = \{X^2 + Y^4 + Z^4 = 0\}$ throughout. The second curve produces complications, to be discussed later.

For the sake of simplifaction, from here on E will be the quartic curve $\{X^2 + Y^4 + Z^4 = 0\}$ in $\mathbb{P}(2, 1, 1)$,with the corresponding involution $\sigma_E : X \mapsto -X$ unless otherwise specified. The choice of elliptic curve does not change the full state space, and we shall see that the changes to the narrow part of the state space are minor. Explicit equations and Nikulin involutions for K3 surfaces in weighted projective space given by polynomials of the form $x^2 + P(y, z, w)$ are tabulated in [19] by invariants (r, a), where $(N, N') = (1 + \frac{r-a}{2}, 11 - \frac{r+a}{2})$. Those of Fermat type are listed in Table 3.2.

Note that in all of the above cases, only the first has all w_i pairwise relatively prime, and in all other cases any common factor for any two w_i, w_j divides only those two, and all such common factors are prime. Define $d = \mathrm{lcm}(w_0, w_1, w_2, w_3)$.

We consider the K3 surface $\{x^2 + y^6 + z^2 + w^2 = 0\} \subseteq \mathbb{P}(3, 1, 1, 1)$, with involution $\sigma_K : x \mapsto -x$. (We use upper case for the coordinates corresponding to the elliptic curve, and lower case for those corresponding to the K3 surface).

Table 3.1: Elliptic curves with anti-symplectic involution $X \mapsto -X$

$P(X, Y, Z)$	Ambient space $\mathbb{P}(v_0, v_1, v_2)$
$Y^4 + Z^4$	$\mathbb{P}(2, 1, 1)$
$Y^3 + Z^6$	$\mathbb{P}(3, 2, 1)$

Table 3.2: Fermat polynomials for K3 surfaces

$P(x,y,z)$	Ambient space $\mathbb{P}(w_0,w_1,w_2,w_3)$	N	N'
$y^6+z^6+w^6$	$\mathbb{P}(3,1,1,1)$	1	10
$y^5+z^5+w^{10}$	$\mathbb{P}(5,2,2,1)$	2	6
$y^3+z^{10}+w^{15}$	$\mathbb{P}(15,10,3,2)$	4	4
$y^3+z^7+w^{42}$	$\mathbb{P}(21,14,6,1)$	6	6
$y^3+z^9+w^{18}$	$\mathbb{P}(9,6,2,1)$	3	7
$y^4+z^8+w^8$	$\mathbb{P}(4,2,1,1)$	1	9
$y^4+z^5+w^{20}$	$\mathbb{P}(10,5,4,1)$	2	6
$y^4+z^6+w^{12}$	$\mathbb{P}(6,3,2,1)$	1	7
$y^3+z^{12}+w^{12}$	$\mathbb{P}(6,4,1,1)$	2	10
$y^3+z^8+w^{24}$	$\mathbb{P}(12,8,3,1)$	3	7

We shall also denote the ambient space $[(\mathbb{P}_{2,1,1}\times\mathbb{P}_{w_0,w_1,w_2,w_3})/\mathbb{Z}_2]$ by $\mathcal{X}$.

In this case, from (2.1), we have

$$H^*_{\mathrm{CR}}(\mathcal{Y}) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 6 & & 0 & \\ 1 & & 60 & & 60 & & 0 \\ & 0 & & 6 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We will be specifically considering the *ambient classes*, that is those induced from classes from Y.

We write the ambient space $\mathcal{X} = (\mathbb{P}_{2,1,1}\times\mathbb{P}_{3,1,1,1})/\mathbb{Z}_2$ as $((\mathbb{C}^3\backslash\{0\})\times(\mathbb{C}^4\backslash\{0\})\times\mathbb{C}^*)/(\mathbb{C}^*)^3$ where $(\mathbb{C}^*)^3$ acts by

$$(\lambda_1,\lambda_2,\lambda_3):(X,Y,Z,x,y,z,w,\alpha)\mapsto(\lambda_1^2\lambda_3X,\lambda_1Y,\lambda_1Z,\lambda_2^3w_0\lambda_3x,\lambda_2^{w_1}y,\lambda_2^{w_2}z,\lambda_2^{w_3}w,\lambda_1^2\alpha).$$

Thus if α is chosen to be 1, then $\lambda_3=\pm1$, and the action is that of σ.

We consider which $(\lambda_1,\lambda_2,\lambda_3)$ have fixpoints - each of these group elements corresponds to a non-empty component of the inertia stack. For example, for $\mathbb{P}(3,1,1,1)$, we consider whether or not $Y,Z=0$, or $y=z=w=0$, and find 18 elements:

$$(\pm1,\sqrt[3]{1},1),(\pm i,\sqrt[3]{-1},-1),(\pm i,1,-1),(1,\sqrt[3]{-1},-1),(1,1,-1).$$

However, those which require $Y = Z = 0$ or $y = z = w = 0$ do not intersect $E \subseteq \mathbb{P}_{2,1,1}$ or $K \subseteq \mathbb{P}_{w_0,w_1,w_2,w_3}$, as this would require $X = 0$ or $x = 0$. Only two group elements remain: $(1, 1, 1)$, corresponding to the identity, and $(1, 1, -1)$, corresponding to the involution itself.

Thus the untwisted and σ-sectors in $H^{\text{amb}}_{\mathbf{CR}}(\mathcal{Y})$ are generated by the Poincaré-dual classes to

$$\begin{gathered} \{pt\} \\ E,\, H_K,\, pt_\sigma \\ E \times H_K,\, K,\, \Sigma \\ \mathcal{Y} \end{gathered}$$

where H_K is the intersection of a hyperplane in $\mathbb{P}_{w_0,w_1,w_2,w_3}$ with K, and Σ is the fixpoint set of σ, isomorphic to four copies of $\Sigma_{(2w_0-1)(w_0-1)}$, (by the degree-genus formula and the fact that the Calabi-Yau condition and involution imply that $d = 2w_0$). We will find it useful to write the classes in a slightly different way. Let $D_i = \{X_i = 0\}$ be the i-th coordinate X_i, $i = 1, \ldots, 8$. We will express these divisors in terms of the toric divisors $D_E = H_E \times K$ and $D_K = E \times H_K$. We have $D_E = K, D_K = E \times H_K, D_K \cup D_K = E, D_E \cup D_K = H_K, D_E \cup D_K \cup D_K = \{pt\}, D_E^2 = D_K^2 = 0$. We write $\mathbf{1}_\sigma$ for the identity class on the σ-sector, and the point class on that sector is given by $D_K \cup \mathbf{1}_\sigma$ after scaling (which can be seen from the intersection form on a resolution of the orbifold, and Bézout's theorem). However, $D_E \cup \mathbf{1}_\sigma$ can be shown in both of these ways to be zero, so that $D_E, D_K, \mathbf{1}_\sigma$ are multiplicative generators of a basis of the cohomology. For the untwisted sector it is the same as the usual intersection product, and for the twisted sector we have $\mathbf{1}_\sigma D_E = 0$, so that we may take the (one-dimensional) twisted part of degree 4 to be generated by $\mathbf{1}_\sigma D_K$. For $\mathbb{P}(3, 1, 1, 1)$, the story ends here.

For the other cases, when w_i, w_j share a common prime factor p, there is also a fixed subspace $\mathbb{P}(\frac{w_i}{p}, \frac{w_j}{p})$, one copy each corresponding to elements of a subgroup $\mathbb{Z}_p$. There are two cases: first, one of i, j, WLOG $i = 0$, in which case the intersection with K is $\frac{d}{w_j}\, \mathcal{B}\mathbb{Z}_p$, but does not intersect $\mathrm{Fix}(\sigma)$; we thus have $(p-1)$ dimensions of ambient classes of degree 2 and, multiplying by $H_E \times K$, another $(p-1)$ dimensions of ambient classes of degree 4. Otherwise, neither of $i, j = 0$, in which case we have that $\mathbb{P}(\frac{w_i}{p}, \frac{w_j}{p}) \cap K \subseteq \mathrm{Fix}(\sigma)$, contributing the non-trivial degree-2 sectors from $\mathcal{B}\mathbb{Z}_p$, which, cupping with $D_E = H_E \times K$, contributes just as many of degree 4. Furthermore, multiples of $H_E \times K$ are the only classes that cup with these classes non-trivially.

For $p_{i,j} > 1$ $(i \neq j)$, we must include all $(a, b, c) \in \mathbb{Z} \times \frac{1}{p_{i,j}}\mathbb{Z} \times \mathbb{Z}$. Under the valuation map such (a, b, c) correspond to the sectors generated by $\mathbf{1}_{g^r_{i,j}}$ where $b \equiv_1 \frac{r}{p_{i,j}}$. These all have degree 2, and we have as many classes again given by $D_E \mathbf{1}_{g^r_{i,j}}$.

For $p_{0,j} > 1$ and $j > 0$, $\mathrm{Fix}((\lambda_1, \lambda_2, -1))$ for non-trivial λ_2 may also be non-trivial: if $w_i = 2p$ does not divide w_0, we have $(\zeta_{2p}^k)^{2p} = 1$ for all k and $(-1)(\zeta_{2p}^k)^{p\frac{w_0}{p}} = 1$ for *odd* k. This means that both x and the coordinate corresponding to w_i are both fixed, and so by Bézout's theorem we have a non-empty zero-dimensional intersection of this one-dimensional weighted projective subspace with K, adding a new point twisted sector for each odd k from $1, \ldots, 2p$: there are p of these, denoted $\mathbf{1}_{\sigma g'_{\frac{r}{2p_{0,i}}}}$.

The final possible situation giving non-empty twisted sectors arises when w_i, w_j do not divide w_0 but share a common factor for $i \neq 0$ (which must be 2): this only occurs for $(5,2,2,1)$, $(21,14,6,1)$ and $(15,10,3,2)$. We have $\mathrm{Fix}(1,-1,-1) = \{w = 0\}$, which also has non-trivial intersection with K and gives $(2-1) = 1$ twisted copy of $\mathbb{P}(\frac{w_i}{2}, \frac{w_j}{2})$. This contributes just as many new sectors: $(p-1)+(q-1)$ point sectors of the form $\mathbf{1}_{\sigma g'_{\frac{r}{2p_{0,i}}}}, \mathbf{1}_{\sigma g'_{\frac{r}{2p_{0,j}}}}$, and one 1-dimensional sector arising when $\mathbf{1}_{\sigma g'_{\frac{r}{2p_{0,i}}}} = \mathbf{1}_{\sigma g'_{\frac{r}{2p_{0,j}}}}$, which we shall denote $\mathbf{1}_{\tilde{g}}$, and the generic point in the same sector equal to $D_K \mathbf{1}_{\tilde{g}}$. Half of these classes, including $\mathbf{1}_{\tilde{g}}$, have degree 2, and the rest have degree 4.

Let

$$S_j^\alpha = \{\frac{r}{2p_{0,j}} : 2 \not| r \; r \leq 2p_{0,j} - 1, \, \deg_{\mathrm{CR}}(-1, \zeta_{2p}^r, -1) = \alpha\},$$

and let

$$S(\alpha) = \bigcup_{j : \, w_j \not| w_0} S_j^\alpha.$$

Under the CR-pairing, $(\mathbf{1}_g)^{-1} = D_E \mathbf{1}_{g^{-1}}, \; (\mathbf{1}_{\sigma g})^{-1} = \mathbf{1}_{\sigma g^{-1}}$ for $g \neq \tilde{g}$, and $(\mathbf{1}_{\sigma\tilde{g}})^{-1} = D_K \mathbf{1}_{\sigma\tilde{g}}$.

Then the ambient part of $\mathcal{H}^{\mathrm{amb}}(\mathcal{Y})$ is in general given by the following sets of generators, ordered by degree:

$$\{\mathbf{1}\}$$
$$\{D_E, D_K, \mathbf{1}_\sigma\} \cup \bigcup_{\substack{p_{i,j}>1 \\ 1 \leq r \leq p_{i,j}-1}} \{\mathbf{1}_{g_{i,j}^r}\} \cup \bigcup_{\frac{r}{p} \in S(2)} \{\mathbf{1}_{g_{\frac{r}{2p}}}\}$$
$$\{D_E D_K, D_K^2, D_K \mathbf{1}_\sigma\} \cup \bigcup_{\substack{p_{i,j}>1 \\ 1 \leq r \leq p_{i,j}-1}} \{D_E \mathbf{1}_{g_{i,j}^r}\} \cup \bigcup_{\frac{r}{p} \in S(4)} \mathbf{1}_{\sigma g'_{\frac{r}{p}}} [\cup \{D_K \mathbf{1}_{\sigma\tilde{g}}\}]$$
$$\{u\}$$

where the sector $D_K \mathbf{1}_{\sigma\tilde{g}}$ is included when $w_1 = \frac{d}{3}$.

For $E = \{X^2 + Y^3 + Z^6 = 0\}$, similar arguments hold as for K, and there is an extra σ_E-twisted sector, doubling the number of σ-twisted sectors.

3.2 Enumerative Geometry

We treat $\mathcal{X}$ as a Deligne-Mumford toric stack in the sense of [7].

$\mathcal{X}$ corresponds to the stacky fan $(N, \mathbf{\Sigma}, \rho)$ where

$$N = \mathbb{Z}^5 + \langle(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)\rangle + \langle(0, 0, \frac{w_1}{w_0}, \frac{w_2}{w_0}, \frac{w_3}{w_0})\rangle + \langle(\frac{1}{4}, \frac{1}{4}, \frac{w_1}{2w_0}, \frac{w_2}{2w_0}, \frac{w_3}{2w_0})\rangle \subset \mathbb{Z}^5 \otimes \mathbb{Q},$$

with $\rho : \mathbb{Z}^8 \to N$ given by

$$\begin{pmatrix} -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & -\frac{w_1}{w_0} & 1 & 0 & 0 & \frac{w_1}{2w_0} \\ 0 & 0 & 0 & -\frac{w_2}{w_0} & 0 & 1 & 0 & \frac{w_2}{2w_0} \\ 0 & 0 & 0 & -\frac{w_3}{w_0} & 0 & 0 & 1 & \frac{w_3}{2w_0} \end{pmatrix}$$

where the columns $\rho_1, \rho_2, \ldots, \rho_8$ and the maximal cones of Σ are those generated by all but one of $\{\rho_1, \rho_2, \rho_3\}$, and all but one of $\{\rho_4, \rho_5, \rho_6, \rho_7\}$ (and excludes ρ_8, which appears only due to the factor $\mathbb{C}^*$ which is solely included to represent the $\mathbb{Z}_2$-action as toric).

The *box* of $\mathbf{\Sigma}$ is

$$\mathrm{Box}(\mathcal{X}) := \{\sum_{i:\rho_i \in \kappa} a_i \rho_i \text{ for some } \kappa \in \mathbf{\Sigma},\ 0 \leq a_i < 1\},$$

and is in one-to-one correspondence with the set of sectors of the inertia orbifold of $\mathcal{X}$. We are interested in the subset of those which induce non-empty sectors in $\mathcal{Y}$, which we shall denote $\mathrm{Box}(\mathcal{Y})$. We will also denote the elements of the box corresponding to $\mathbf{1}_{g^r_{i,j}}$ and $\mathbf{1}_{g_{\sigma\frac{r}{2p}}}$ by $\mathbf{b}_{g^r_{i,j}}$ and $\mathbf{b}_{\sigma g'_{\frac{r}{2p}}}$ respectively.

The twisted sectors coming from $\mathbb{P}(w_0, w_1, w_2, w_3)$ are given by the following. We note that from the construction of the toric fan of weighted projective space, $w_0\rho_4 + w_1\rho_5 + w_2\rho_6 + w_3\rho_7 = 0$, so that for each $p_{i,j} := \gcd(w_i, w_j) > 1$,

$$\frac{rw_i}{p_{i,j}}\rho_{4+i} + \frac{rw_j}{p_{i,j}}\rho_{4+j} + \sum_{k \neq i,j} \lceil \frac{rw_k}{p_{i,j}} \rceil \rho_{4+k} = \sum_{k \neq i,j} \langle -\frac{rw_k}{p_{i,j}} \rangle \rho_{4+k}$$

is an element of the box, since it is given as a member of N on the left and its ρ_i-coordinates are strictly bounded by 1 in the expression on the right.

For $p_{0,i} > 1$, we refer to [7] to note that when we have $c_1, c_2 \in \kappa \in \mathbf{\Sigma}$, then $\mathbf{1}_{c_1}\mathbf{1}_{c_2} = \mathbf{1}_{c_1+c_2}$. $\mathbf{1}_\sigma$ is represented by $\rho_8 = \frac{1}{2}(\rho_1 + \rho_4)$, and so the sectors generated by elements of the form $\mathbf{1}_{\sigma g'_{\frac{r}{2p}}}$ are given by $\frac{1}{2}(\rho_1+\rho_4)+[g'_{\frac{r}{2p}}]$, the latter being the fan representation given above; we consider those inducing non-zero sectors in $\mathrm{Box}(\mathcal{Y})$ as before.

We extend $\mathbf{\Sigma}$ by the sectors represented by $s_1,\dots,s_l,s_{l+1},\dots,s_m$, where the sectors $s_1,\dots,s_l$ represent the $\mathbf{1}_{g^r_{i,j}}$, and $s_{l+1},\dots,s_m$ correspond to the sectors given by the elements of $S(2)$. The representations found above give coefficients $s_{i,j}$, $0\le s_{i,j}<1$ such that $\sum_{i:\rho_i\in\sigma(j)} s_{i,j}\rho_i = s_j$, where $\sigma(j)$ is the cone containing s_j (we set all other coefficients for each j to be zero). Furthermore, such $s_{i,j}$ are unique. Note that $s_{1,j}=s_{4,j}=0$ for $1\le j\le l$ and $s_{1,j}=s_{4,j}=-\frac{1}{2}$ for $l+1\le j\le m$.

Let $1\le m\le M:=|\mathrm{Box}(\mathcal{Y})|$ and $S=\{1,2,\dots,m\}$. Then choose an injective function $S\mapsto \mathrm{Box}(\mathcal{Y})$, given by $i\mapsto s_i$, $1\le i\le m$. We define the *S-extended stacky fan* $\mathbf{\Sigma}^S$ by extending $\rho:\mathbb{Z}^8\to N$ to $\rho^S:\mathbf{Z}^{8+m}\to N$ by setting $\rho_{8+i}=S(i)$. It represents the same orbifold as Σ [14]. Following [7], we tautologically lift ρ to $R=I$ with projection ρ:

$$\begin{tikzcd} & \mathbb{Z}^8 \arrow[d, "\pi=\rho"] \\ \mathbb{Z}^8 \arrow[ur, "R=I"] \arrow[r, "\rho"] & N \end{tikzcd}$$

with kernel $\mathbb{L}\xhookrightarrow{Q}\mathbb{Z}^8$, where

$$Q=\begin{pmatrix} 2 & 0 & 1\\ 1 & 0 & 0\\ 1 & 0 & 0\\ 0 & w_0 & 1\\ 0 & w_1 & 0\\ 0 & w_2 & 0\\ 0 & w_3 & 0\\ 0 & 0 & 2 \end{pmatrix}.$$

Taking the Gale dual we find $\check{\mathbb{L}}\cong\mathbb{Z}^3$ and

$$\check{\rho}:\mathbb{Z}^{3+8}/\ker\mathrm{Im}\left([RQ]^*\right)\to\ker\mathrm{Im}\left([RQ]^*\right)$$

is given by the same matrix, so that the group

$$G=\mathrm{Hom}\left(\check{\mathbb{L}},\mathbb{C}^*\right)=(\mathbb{C}^*)^3$$

acts by

$$\alpha = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_0 & w_1 & w_2 & w_3 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix},$$

the desired weights. The columns here correspond to D_i in $\mathbb{L}$ [14]. From this the Chen-Ruan cohomology may also be computed from this construction via [7]: this agrees with our outline in the previous subsection.

From [7] we can relate the Chen-Ruan cohomology to the stacky fan construction. The twisted sector generated by $\mathbf{1}_\sigma$ is given by $\rho_8 = \frac{w_1}{2w_0}\rho_1 + \frac{w_2}{2w_0}\rho_2 + \frac{w_3}{2w_0}\rho_3$.

The corresponding kernel $\mathbb{L}^S$ under S-extension fits into the exact sequence

$$0 \to \mathbb{L} \to \mathbb{L}^S \to \mathbb{Z}^m.$$

Considering these inside their tensor products with $\mathbb{Q}$, this sequence splits via the map $e_j \mapsto e_{8+j} - \sum_i s_{j,i}e_i$. Therefore $\mathbb{L}^S \otimes \mathbb{Q}$ is the image of the map sending $(a, b, c, k_1, \ldots, k_m)$ to

$$a(2\rho_1 + \rho_2 + \rho_3) + b(\sum_{i=1}^{4} w_i\rho_{4+i}) + c(\rho_1 + \rho_4 + 2\rho_8) + \sum_{j=1}^{m} k_j(s_j - \sum_{i=0}^{8} s_{j,i}\rho_i).$$

The Mori cone $\mathrm{NE}(\mathcal{X}) \subset \mathbb{L}$ is $\sum_{\kappa\in\mathbf{\Sigma}} \mathbb{R}_{\geq 0}\check{C}_\tau$, where $\check{C}_\kappa$ is the dual cone of $C_\kappa = \sum_{i\notin\kappa} \mathbb{R}_{\geq 0}\rho_i$. Sorting through the cases, we get

$$\mathrm{NE}(\mathcal{X}) = \{(a, b, c) \in \mathbb{L} \,|\, c \geq 0, 2a + c \geq 0, w_0 b + c \geq 0\}.$$

This is generated by rays

$$\begin{aligned} \{a + \frac{c}{2} \geq 0, b + \frac{c}{w_0} = 0, c = 0\} &= \langle (1, 0, 0) \rangle, \\ \{a + \frac{c}{2} = 0, b + \frac{c}{w_0} \geq 0, c = 0\} &= \langle (0, 1, 0) \rangle, \\ \{a + \frac{c}{2} = 0, b + \frac{c}{w_0} = 0, c \geq 0\} &= \langle (-\frac{1}{2}, -\frac{1}{w_0}, 1) \rangle. \end{aligned}$$

The S-extended Mori cone is given by $NE^S(\mathcal{X}) := NE(\mathcal{X}) \times \mathbb{R}^m_{\geq 0}$.

Using the notation of [14], we define $\Lambda^S_\kappa = \{\sum_i^{8+m} \lambda_i e_i \in \mathbb{L} \otimes \mathbb{Q} \,|\, i \notin \kappa \implies \lambda_i \in \mathbb{Z}; \lambda_i \in \mathbb{Z}, i > 8\}$, and $\Lambda^S = \bigcup_{\kappa\in\Sigma} \Lambda^S_\kappa$. That is, those elements of $\mathbf{L}^S$ for which the following must hold: $2c \in \mathbb{Z}$, one of $2a + c - \sum_j k_j s_{j,1} \in \mathbb{Z}$ or $a \in \mathbb{Z}$, and one of

$w_0b + c - \sum_j k_j s_{j,4} \in \mathbb{Z}$ or $w_i b - \sum_j k_j s_{j,i} \in \mathbb{Z}$, for some $i = 5, 6, 7$. The *valuation map* is then defined to be

$$v^S : \Lambda \to \mathrm{Box}(\mathbf{\Sigma})$$

by

$$\lambda \mapsto \sum_{i=1}^{8} \lceil \lambda_i \rceil \rho_i = \sum_{i=1}^{8} \langle -\lambda_i \rangle \rho_i,$$

the latter inequality holding by equations defining $\mathbb{L}$. We set $\Lambda_{\mathbf{b}} = (v^S)^{-1}(\mathbf{b})$, $\Lambda E(\mathcal{X}) = \Lambda \cap \mathrm{NE}(\mathcal{X})$ and $\Lambda E_{\mathbf{b}}(\mathcal{X}) = \Lambda_{\mathbf{b}} \cap \mathrm{NE}(\mathcal{X})$.

The untwisted I-function is given in [14] by

$$I_{\mathrm{GW}}(\mathbf{t}, z) = ze^{\sum_i D_i t_i/z} \sum_{\mathbf{b} \in \mathrm{Box}(\mathbf{\Sigma})} \sum_{\lambda \in \Lambda E_{\mathbf{b}}} q^{\lambda} \prod_{i}^{8+m} e^{(D_i \cdot \lambda) t_i} \frac{\prod_{\langle d \rangle = \langle \lambda_i \rangle,\, d \leq 0} (D_i + dz)}{\prod_{\langle d \rangle = \langle \lambda_i \rangle, d \leq \lambda_i} (D_i + dz)} \mathbf{1}_b,$$

where q^{λ} are the Novikov variables recording the class λ, and the D_i are the divisor classes given above for $i \leq 8$, and zero for $i > 8$.

Denote each term corresponding to $(\mathbf{b}, \lambda)$ by $I_{\mathbf{b},\lambda}(t, z)$. $\mathcal{Y}$ is the generic zero section of the bundle $\mathcal{E} = \mathcal{O}(4, 0) \oplus \mathcal{O}(0, 2w_0)$. The *twisted* I-function of this bundle is then given in [13] by

$$\sum_{\mathbf{b},\lambda} I_{\mathbf{b},\lambda}(t, z) M_{\mathbf{b},\lambda}(t, z),$$

via the modification factors

$$M_{\mathbf{b},\lambda}(t, z) = \prod_{k=1}^{2\cdot(2a+c+\frac{1}{2}\sum_{j=l+1}^{m} k_j)} (4D_E + kz) \prod_{l=1}^{2\cdot(w_0b+c+\frac{1}{2}\sum_{j=l+1}^{m} k_j)} (2w_0 D_K + lz).$$

Note that considering the cases when $\lambda_i \in \mathbb{Z}$ and $\lambda_i \notin \mathbb{Z}$ separately, we may write

$$\frac{\prod_{\langle d \rangle = \langle \lambda_i \rangle,\, d \leq 0} (D_i + dz)}{\prod_{\langle d \rangle = \langle \lambda_i \rangle, d \leq \lambda_i} (D_i + dz)}$$

as

$$z^{-\lceil \lambda_i \rceil} \frac{\Gamma(D_i/z + \langle\langle \lambda_i \rangle\rangle)}{\Gamma(D_i/z + \lambda_i + 1)},$$

where $\langle\langle \lambda \rangle\rangle := 1 - \langle 1 - \lambda \rangle$. We rewrite the modification factor similarly. For $i = 8$, this gives a factor of $\frac{1}{(2c)!}$, and for $i = 8 + j$, this gives a factor of $\frac{1}{k_j!}$. This re-expression will allow us to extend the function analytically.

We now have all the ingredients to write the I-function in our case explicitly. Collect all the factors constant in $m, n, c, \mathbf{k}$ as $K_{\mathbf{b}}, L_{\mathbf{b}}$ for each $\mathbf{b} \in \mathrm{Box}(\mathcal{Y})$:

$$
\begin{aligned}
K_0 = K_{g_{i,j}^r} &= \frac{\Gamma(2D_E/z+1)\Gamma(D_E/z+1)^2}{\Gamma(4D_E/z+1)} \\
K_\sigma = K_{\sigma g'_{\frac{r}{2p}}} &= z^{-1/2}\frac{\Gamma(2D_E/z+\frac{1}{2})\Gamma(D_E/z+1)^2}{\Gamma(4D_E/z+1)} \\
L_0 &= \frac{\Gamma(w_0D_K/z+1)\prod_{\nu=1}^3\Gamma(w_\nu D_K/z+1)}{\Gamma(2w_0D_K/z+1)}, \\
L_\sigma &= z^{-1/2}\frac{\Gamma(w_0D_K/z+\frac{1}{2})\prod_{\nu=1}^3\Gamma(w_\nu D_K/z+1)}{\Gamma(2w_0D_K/z+1)}, \\
L_{g_{i,j}^r} &= z^{\sum_\nu((\mathbf{b}_{g_{i,j}^r})_{4+\nu}-\lceil(\mathbf{b}_{g_{i,j}^r})_{4+\nu}\rceil)}\frac{\prod_{i=0}^3\Gamma(w_iD_K/z+(\mathbf{b}_{g_{p_{i,j}^r}})_{4+\nu}+1)}{\Gamma(2w_0D_K/z+1)}, \\
L_{\sigma g'_{\frac{r}{2p}}} &= z^{\sum_\nu((\mathbf{b}_{g'_{\frac{p}{2r}}})_{4+\nu}-\lceil((\mathbf{b}_{g'_{\frac{p}{2r}}})_{4+\nu}\rceil)}\frac{\prod_{\nu=0}^3\Gamma(w_\nu D_K/z+(\mathbf{b}_{\sigma g'_{\frac{r}{2p}}})_{4+\nu})}{\Gamma(2w_0D_K/z+1)}.
\end{aligned}
$$

Then collect the factors depending on $m, n, c, \mathbf{k}$ as $F(a, c, \mathbf{k}), G(b, c, \mathbf{k})$:

$$
\begin{aligned}
F(a,c,\mathbf{k}) &= \frac{\Gamma(4D_E/z+4a+2c-2\sum_{\mu=1}^m k_\mu s_{\mu,1}+1)}{\Gamma(2D_E/z+2a+c-\sum_{\mu=1}^m k_\mu s_{\mu,1}+1)\Gamma(D_E/z+a+1)^2(2c+1)!}, \\
G(b,c,\mathbf{k}) &= \frac{\Gamma(2w_0D_K/z+2w_0b+2c-2\sum_{\mu=1}^m k_\mu s_{\mu,4}+1)}{\Gamma(w_0D_K/z+w_0b+c-\sum_{\mu=1}^m k_\mu s_{\mu,4}+1)} \\
&\quad\times\frac{1}{\prod_{\nu=1}^3\Gamma(w_\nu D_K/z+w_\nu b-\sum_{\mu=1}^m k_\mu s_{\mu,4}+1)\prod_{\mu=1}^m k_\mu!}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&I_{\mathrm{GW}}(\mathcal{Y}) = ze^{(2D_Et_1+D_E(t_2+t_3)+\sum_{i=1}^4 w_iD_Kt_{4+i})/z}\times \\
&\sum_{\mathbf{b}\in\mathrm{Box}(\mathcal{Y})}K_{\mathbf{b}}L_{\mathbf{b}}\sum_{c\in\frac{1}{2}\mathbb{N}_0}\sum_{\substack{(a,b,k_1,\dots,k_m)\in\mathbb{N}_0^m\\ a\geq -c/2,\, b\geq -c/w_0\\ v^S(a,b,c,k_1,\dots,k_m)=\mathbf{b}}}(q_1^aq_2^bq_3^c\prod_{j=1}^m x_j^{k_j}) \\
&e^{a(2t_1+t_2+t_3)+b(\sum_{i=0}^3 w_it_i)+c(t_1+t_4+2t_8)}F(a,c,\mathbf{k})G(b,c,\mathbf{k})\mathbf{1}_{\mathbf{b}}.
\end{aligned}
$$

Note that all other powers of z cancel, and that this is a direct consequence of the Calabi-Yau condition that the charges sum to 1 in each factor space (the $(4a+2c)+(6b+2c)$ coming from the vector bundle precisely matches the $a+a+2a+c+b+b+b+3b+c+2c$ coming from the ambient space). Note also that setting all the divisors to degree 1 and all variables other than z to degree 0, our function is homogeneous, and only the first multiple sum contributes the lowest powers of z. Upon expanding in z we find our I-function to be of the form

$$z + (2t_1 + t_2 + t_3)D_E + (3t_4 + t_5 + t_6 + t_7)D_K + \sum_{\mu=1}^{m} L_\mu x_\mu \mathbf{1}_\mu + O(z^{-1});$$

thus the condition $S\sharp$ as defined in [14] holds, so that the mirror theorem found there applies, as follows.

Proposition 3.2.1.

$$J^{\mathcal{Y}}_{\mathrm{GW}}((2t_1+t_2+t_3)D_E+(3t_4+t_5+t_6+t_7)D_K+\sum_{\mu=1}^{m} L_\mu x_\mu \mathbf{1}_\mu, z) = I^{\mathcal{Y}}_{\mathrm{GW}}(\mathbf{t}, x_1, \ldots, x_m, z).$$

It will be simpler to demonstrate the correspondence in terms of these I-functions defined above, but in this thesis we do not address the Picard-Fuchs equations, so our nomenclature for the I-functions originates solely by analogy from its role in the above mirror theorem; our computations are all in terms of the A side.

For clarity, in our first example $K = \{x^2 + y^6 + z^6 + w^6 = 0\} \subseteq \mathbb{P}(3,1,1,1)$, the I-function reduces to:

$$\begin{aligned}
&I^{\mathcal{Y}}_{\mathrm{GW}}(\mathbf{t}, z) = z e^{(2D_E t_1 + D_E(t_2+t_3) + 3D_K t_4 + D_K(t_5+t_6+t_7))/z} \\
&\times\Bigg(\sum_{c\in\mathbb{N}_0} \sum_{\substack{a,b\in\mathbb{Z} \\ a\geq -c/2 \\ b\geq -c/3}} q_1^a q_2^b q_3^c e^{(2a+c)t_1 + a(t_2+t_3) + (3b+c)t_4 + b(t_5+t_6+t_7) + 2ct_8} \\
&\times \frac{\Gamma(2D_E/z+1)\Gamma(D_E/z+1)^2\Gamma(3D_K/z+1)\Gamma(D_K/z+1)^3}{\Gamma(2D_E/z+2a+c+1)\Gamma(D_E/z+a+1)^2\Gamma(3D_K/z+3b+c+1)\Gamma(D_K/z+b+1)^3} \\
&\times \frac{\Gamma(4D_E/z+4a+2c+1)\Gamma(6D_K/z+6b+2c+1)}{\Gamma(2c+1)\Gamma(4D_E/z+1)\Gamma(6D_K/z+1)} \\
&+ z^{-1} \sum_{c\in\frac{1}{2}\mathbb{N}_0\setminus\mathbb{N}_0} \sum_{\substack{a,b\in\mathbb{Z} \\ a\geq -c/2 \\ b\geq -c/3}} q_1^a q_2^b q_3^c e^{(2a+c)t_1 + a(t_2+t_3) + (3b+c)t_4 + b(t_5+t_6+t_7) + 2ct_8} \\
&\times \frac{\Gamma(2D_E/z+\frac{1}{2})\Gamma(D_E/z+1)^2\Gamma(3D_K/z+\frac{1}{2})\Gamma(D_K/z+1)^3}{\Gamma(2D_E/z+2a+c+1)\Gamma(D_E/z+a+1)^2\Gamma(3D_K/z+3b+c+1)\Gamma(D_K/z+b+1)^3}) \\
&\times \frac{\Gamma(4D_E/z+4a+2c+1)\Gamma(6D_K/z+6b+2c+1)}{\Gamma(2c+1)\Gamma(4D_E/z+1)\Gamma(6D_K/z+1)} \mathbf{1}_\sigma\Bigg).
\end{aligned}$$

Finally, for $E = \{X^2+Y^3+Z^6 = 0\}$, we have double the number of σ-twisted sectors, and the I-function may be found in precisely the same way.

CHAPTER 4

The Pure Landau-Ginzburg Side

4.1 The FJRW State Space

In this subsection we compute the narrow FJRW state spaces for $W = X^2+Y^4+Z^4+x^2+P(y,z,w)$, where $P(y,z,w)$ has degree d and weights w_0, w_1, w_2, w_3 and $G = \langle J_1, J_2, \sigma\rangle$. We let q_k be the associated charges of W. (Note that $w_i = \bar{w}_{4+i}$ in our notation from §1). First, it is instructive to compute the full state space for an example. Below we find the full state space for $P(y,z,w) = y^6 + z^6 + w^6$.

- **Case I.** $W|_g = X^2 + Y^4 + Z^4 + x^2 + y^6 + z^6 + w^6$. For e, the fixpoint set is the whole space, and we have

  $$\begin{aligned}\mathcal{H}^{\mathrm{I}}_{\mathrm{FJRW}} &= \mathcal{Q}_{W_{\mathrm{I}}} \cdot \mathrm{d}X \wedge \mathrm{d}Y \wedge \mathrm{d}Z \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w \\ &= \mathbb{C}[X,Y,Z,x,y,z,w]/\langle X, Y^3, Z^3, x, y^5, z^5, w^5\rangle.\end{aligned}$$

  This has generators

  $$Y^b Z^c y^e z^f w^g \mathrm{d}Y \wedge \mathrm{d}Z \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w,$$

  for $0 \le b, c \le 2$ and $0 \le e, f, g \le 4$. These are all invariant under σ. We find that such a form is invariant under J_1 iff $4 \mid b + c$ and under J_2 iff $6 \mid e + f + g$. We find that $\mathcal{H}^{\mathrm{I}}_{\mathrm{FJRW}} \cong \mathbb{C}^{42}$.

- **Case I*.** $W|_g = Y^4 + Z^4 + y^6 + z^6 + w^6$. The standard generators of the Milnor ring are given by

  $$Y^b Z^c y^e z^f w^g \mathrm{d}Y \wedge \mathrm{d}Z \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w,$$

  such that $4 \mid b + c + 2$ and $6 \mid e + f + g + 3$. This gives 3×20 possibilities, so that $\mathcal{H}^{\mathrm{I*}}_{\mathrm{FJRW}} \cong \mathbb{C}^{60}$.

Table 4.1: Breakdown of polynomial terms fixed by group elements for K3 weights (6, 1, 1, 1)

	g			σg	
g	N_g	$\deg_W(g)$	Case	$N_{\sigma g}$	Case
e	7	3	I	5	I*
J_1	4	2	II	4	II*
J_1^2	5	3	III	3	III*
J_1^3	4	4	II	4	II*
J_2	3	1	IV	3	IV*
J_1J_2	0	0	V	2	V*
$J_1^2J_2$	1	1	IV	1	VI*
$J_1^3J_2$	0	2	V	2	V*
J_2^2	4	2	VII	2	VII*
$J_1J_2^2$	1	1	VI*	1	VI
$J_1^2J_2^2$	2	2	V*	0	V
$J_1^3J_2^2$	1	3	VI*	1	VI
J_2^3	3	3	IV	3	IV*
$J_1J_2^3$	0	2	V	2	V*
$J_1^2J_2^3$	1	3	VI	1	VI*
$J_1^3J_2^3$	0	4	V	2	V*
J_2^4	4	4	VII	2	VII*
$J_1J_2^4$	1	3	VI*	1	VI
$J_1^2J_2^4$	2	4	V*	0	V
$J_1^3J_2^4$	1	5	VI*	1	VI
J_2^5	3	5	IV	3	IV*
$J_1J_2^5$	0	4	V	2	V*
$J_1^2J_2^5$	1	5	VI	1	VI*
$J_1^3J_2^5$	0	6	V	2	V*

- **Cases II, II*, IV, IV*, VI, VI*.** Here W_g has only one X^2 or x^2 appearing, and so to be invariant under σ we would require the forms over $\mathbb{C}$ to be 0. Thus these cases do not contribute to $\mathcal{H}_{\mathrm{FJRW}}$.
- **Cases III, V*, VII.** These W_g have only one of $Y^4 + Z^4$ or $y^6 + z^6 + w^6$ appearing (WLOG $Y^4 + Z^4$), but also have $X^2 + x^2$ appearing. Thus, invariance under both J_2 (otherwise J_1) will require the forms to be zero. Again, there is no contribution.
- **Case III*.** Here $W|_g = y^6 + z^6 + w^6$, with Milnor generators $y^e z^f w^g \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w$. We require $6|e + f + g + 3$, giving $\mathcal{H}_{\mathrm{FJRW}}^{\mathrm{III}*} \cong \mathbb{C}^{20}$.
- **Case V.** These are the narrow sectors, each isomorphic to $\mathbb{C}$.
- **Case VII*.** $W_g = Y^4 + Z^4$ Similarly to case III*, we find that $\mathcal{H}_{\mathrm{FJRW}}^{\mathrm{VII}*} \cong \mathbb{C}^3$.

Putting this together and sorting by bi-degree, we find that

$$\bigoplus_{g \in G} \mathcal{H}_{\mathrm{FJRW}}^{g} \cong \mathcal{H}_{\mathrm{GW}}(\mathcal{Y})$$

as desired.

We similarly find that $\mathcal{H}_{\mathrm{FJRW}}^{\mathrm{nar}} \cong \mathcal{H}_{\mathrm{GW}}^{\mathrm{amb}}$. The narrow sectors have generators corresponding to $g \in G$ for which $\mathrm{Fix}(g) = \{0\}$ (those of type V above). Ordered by degree, these are:

$$\begin{gathered} J_1 J_2 \\ J_1^3 J_2,\ J_1 J_2^3,\ \sigma J_1^2 J_2^2 \\ J_1^3 J_2^3,\ J_1 J_2^5,\ \sigma J_1^2 J_2^4 \\ J_1^3 J_2^5 \end{gathered}$$

4.2 The Narrow Sectors

In general, if w_0, w_1, w_2, w_3 are all relatively prime, our group has elements $\sigma^t J_1^r J_2^s$, for $t = 0, 1,\ 0 \le r \le 3,\ 0 \le s \le d - 1$. We wish to find the narrow sectors, which correspond to those group elements h for which $\Theta_k(h) \neq 0$ for all k. First we consider the case where $t = 0$. Then we must have $r = 1, 3$, and s must not be divisible by any $\frac{d}{w_i}$. For each i, there are $w_i - 1$ values of s which are divisible by d/w_i not counting 0 (giving broad sectors), and these are the ones we exclude. The Gorenstein condition for the K3 surface tells us that $d := \mathrm{lcm}(w_0, w_1, w_2, w_3) = \sum_i w_i$, so it follows that there are in total $3 = d - \sum_{i=0}^{3} (\frac{d}{d/w_i} - 1) - 1$ possible s. (Two of these will always be precisely $s = 1, d - 1$; from the bounds given by the degree formula, these are the only ones which can ever have degree 0 or 6 respectively.) This gives at least 6 sectors, as above. If $t = 1$, then $r = 2$, and we require s to be even, and we argue similarly, but just for w_1, w_2, w_3: this gives two possible s giving narrow sectors. The sectors $\sigma J_1^2 J_2^2$ and $\sigma J_1^2 J_2^{d-2}$ are included among these, so this completes the set.

However, when w_i, w_j have a common factor $p_{i,j}$, we note that the broad sectors corresponding to $k\frac{d}{p_{i,j}}$ for $k = 1, \ldots, p_{i,j} - 1$ have been excluded twice in the above method, since they are divisible by both $\frac{d}{w_i}$ and $\frac{d}{w_j}$. So to avoid double-counting, we must exclude $p_{i,j} - 1$ fewer sectors above when $t = 0$; that is, we add another $p_{i,j} - 1$ possible s. All of the corresponding $J_1 J_2^s$ have degree 2 (that is, excluding $s = 1, d-1$), and all the corresponding $J_1^3 J_2^s$ have degree 4. Furthermore, $J_1 J_2^s$ is dual to $J_1 J_2^{d-r}$ with respect to the FJRW-pairing.

When $t = 1$, for the sectors to be narrow we require s to be even. Thus for each i, we must find k such that $2k(\frac{w_i}{d}) = k\frac{w_i}{w_0}$ is not an integer for any i. We count first those which give integers for some i. Since $w_1 + w_2 + w_3 = w_0 = \frac{d}{2}$, it follows that each $w_i = p_{0,i}$ (either 1 or a prime) or $2p_{0,i}$. If $w_j = p_{0,j}$ (in which case $w_j | w_0$), or $w_j = 2p_{0,j}$ and $w_i | w_0$, then $k\frac{w_j}{w_0}$ is an integer for any k: there are then $\frac{w_0}{w_j}$ possible k for each w_i, $i = 1, 2, 3$, and if they are all relatively prime, then this gives a total of $w_0 - \sum_{i=1}^{3}(w_i - 1) - 1 = 2$ narrow sectors accounting for $k = 0$. If $w_j = 2p_{0,j}$ does not divide w_0, then $k\frac{w_j}{w_0}$ is only an integer for *even* k: thus we have $p_{0,j}$ fewer broad sectors, so for each such case we must add a further $p_{0,j}$ broad sectors. If w_i, w_j share a common factor for $i = 1, 2, 3$, then that common factor is 2, and it follows that this situation adds no further narrow sectors. Alternatively one may just check the Θ values of $\sigma J_1^2 J_2^s$ for the two cases $(5, 2, 2, 1)$ and $(21, 14, 6, 1)$ to see that these do contribute $2 + 1 + 1$ and $2 + 7 + 3$ narrow sectors with $t = 1$, respectively.

From the FJRW degree formula for the narrow sectors, with $N = 6$, we see that the degree is clearly even, can only be 0 for $J_1 J_2$, and therefore 6 only for $(J_1 J_2)^{-1}$. Otherwise since $\deg_W(h) = 6 - \deg_W(h^{-1})$, for the rest the degree is equi-distributed between 2 and 4.

Ordering by degree, the narrow FJRW state space is then generated by sectors coming from

- $J_1 J_2$
- $J_1^3 J_2, \sigma J_1^2 J_2^2, 1 + \sum_{i,j:\gcd(i,j)>1}(p_{i,j} - 1)$ sectors of the form $J_1 J_2^s$,
  $\frac{1}{2}\sum_{w_j \nmid w_0} p_{0,j}$ sectors of the form $\sigma J_1^2 J_2^s$
- $J_1 J_2^{d-1}, 1 + \sum_{i,j:\gcd(i,j)>1}(p_{i,j} - 1)$ sectors of the form $J_1^3 J_2^s, \sigma J_1^2 J_2^{d-2}$,
  $\frac{1}{2}\sum_{w_j \nmid w_0} p_{0,j}$ sectors of the form $\sigma J_1^2 J_2^s$

- $J_1^3 J_2^{d-1}$

For example, for $(w_0, w_1, w_2, w_3) = (5, 2, 2, 1)$ we have narrow sectors coming from

$$\begin{gathered} J_1 J_2 \\ J_1^3 J_2, , J_1 J_2^3, J_1 J_2^7, \sigma J_1^2 J_2^2, \sigma J_1^2 J_2^6, \\ J_1 J_2^9, J_1^3 J_2^3, J_1^3 J_2^7, \sigma J_1^2 J_2^8, \sigma J_1 J_2^4 \\ J_1^3 J_2^9 \end{gathered}$$

Here, $J_1 J_2$ is the unit of our product structure $\mathbf{1}$ (and, in fact, carries all the *unstable* enumerative data).

For $(3, 1, 1, 1)$, things simplify yet further. The cup product can be found entirely by three facts, all found in [16]:

1. $\langle \alpha \cup \beta, \mathbf{1} \rangle = \langle \alpha \cdot \beta \cdot \mathbf{1} \rangle = \eta(\alpha, \beta)$;
2. For $\langle \alpha \cdot \beta \cdot \gamma \rangle = \eta(\alpha \cup \beta, \gamma)$ to be non-zero, we need their degrees to sum to 6;
3. For the previous condition to hold we need all the line bundles to be integral. For three-point FJRW-invariants this implies $q_j - \sum_{i=1}^3 \Theta_i(h_j) \in \mathbb{Z}$.

All non-zero 3-point invariants (and hence the cup product) are determined from the pairing except for two: $\langle \phi_{\sigma J_1^2 J_2^2}, \phi_{\sigma J_1^2 J_2^2}, \phi_{J_1 J_2^3} \rangle$ and $\langle \phi_{J_1^3 J_2}, \phi_{J_1 J_2^3}, \phi_{J_1 J_2^3} \rangle$, which we have freedom to set to 1. In terms of the cup product, this is exactly the ring structure of Chen-Ruan cohomology under the identification

$$\begin{aligned} D_E &\mapsto \phi_{J_1^3 J_2}, \\ D_K &\mapsto \phi_{J_1 J_2^3}, \\ \Sigma \mathbf{1}_\sigma &\mapsto \phi_{\sigma J_1^2 J_2^2}. \end{aligned}$$

4.3 The FJRW I-function

In genus zero, the virtual class takes a simple form.

Lemma 4.3.1. *In genus zero, we have*

$$[\mathcal{W}_{0,n}^{(W,G)}]^{\mathrm{vir}} = R^1 \pi_* (\bigoplus_{k=1}^{7} \mathcal{L}_k).$$

Proof. By definition $[\mathcal{W}_{0,n}^{(W,G)}]^{\mathrm{vir}} = (-R^0 + R^1)\pi_*(\bigoplus_{k=1}^{8} \mathcal{L}_k)$, and over each point

$$(\mathcal{C}, p_1, \ldots, p_n, \mathcal{L}_1, \ldots, \mathcal{L}_N, \varphi_1, \varphi_2, \ldots, \varphi_N),$$

the fibre of $R^0\pi_*(\bigoplus_{k=1}^{N} \mathcal{L}_k)$ is given by $\bigoplus_{k=1}^{N} H^0(\mathcal{C}, \mathbb{L}_k)$. It is thus sufficient to show that, for all k, $\mathbb{L}_k$ has no non-trivial global sections.

We proceed by induction on the dual graph Γ of $\mathcal{C}$. Vertices of Γ correspond to irreducible components of $\mathcal{C}$, and edges correspond to nodes. Each vertex v is marked by g_v, the genus of the component, and S_v, the set of marked points or nodes, which has cardinality k_v.

From [16], the degree of the pushforward of $\mathcal{L}_i$ to the coarse curve is

$$\deg(|\mathcal{L}_k|) = q_k(k_v - 2) - \sum_{i=1}^{n} \Theta_k(h_i),$$

where h_i records the multiplicity of $\mathcal{L}_i$ at the marked points. For the narrow sectors, $\Theta_k(h_i) > 0$ for all i, k. If $\mathcal{C}$ is irreducible (so there are no nodes) we have $\Theta_k(h_i) = m_{i,k}q_k$ for $m_{i,k} \in \mathbb{Z}_{>0}$, and so

$$\deg(|\mathcal{L}_k|) = -2q_k + q_k \sum_{i=1}^{n}(1 - m_{i,k}) < 0,$$

so that $H^0(\mathcal{C}, \mathcal{L}_k) = 0$.

Otherwise, since $\mathcal{C}$ is a compact connected curve of genus zero, Γ is a finite tree where every irreducible component has at least one node. Let σ_k be any global section of $\mathcal{L}_k$ on $\mathcal{C}$. Let n_v be the number of edges (nodes) attached to v. Then we have

$$\deg(\mathcal{L}_k|_v) \leq q_k(n_v - 2) < n_v - 1.$$

If v is a leaf of Γ corresponding to irreducible component $\mathcal{C}_v$, then $n_v = 1$ and $\deg(\mathcal{L}_k|_v) < 0$ and so $\sigma_k|_v = 0$. We proceed by induction: if at each stage we remove all the leaves and consider the leaves of the new graph we obtain, these correspond to vertices of the original graph all of whose adjacent vertices but one have been removed earlier; that is, to components all of whose nodes but one are connected to components on which σ is known to be zero. We must show that σ is zero on all the leaves. This follows from the the above inequality, as σ has more zeroes on the given component than its degree there, so must be constantly zero. The result follows. $\square$

Definition 4.3.1.1. For $h \in G$, let $i_k(h) = \langle \Theta_k(h) - q_k \rangle$.

Note that each k such that $\Theta_k(h) = 0$ increases N by 1, so we have $\deg_W(h) = 2\sum_k i_k(h)$.

We have already discussed the twisted variants in §2. These correspond to a full twisted FJRW theory, which corresponds to considering a slightly different set of line bundles. Intuitively, we may separate out the line multiplicities by the greatest common divisors of their multiplicities at the points; accounting for the fact that d-th roots may be lower degree roots too. First, note that over marked curves, to give the n-th root $\mathcal{L}$ of $\omega_{\log}$ is equivalent to giving an n-th root $\tilde{\mathcal{L}}$ of $\omega_{\log}(-D)$ for some divisor $0 \leq D < \sum_i nD_i$, with multiplicity 0 at p_i. This correspondence can be given by $\mathcal{L} \mapsto p^* p_* \mathcal{L}$, where p is the map which forgets the monodromy at the marked points. Then from this perspective we can separate our narrow sectors from the rest in a slightly modified moduli space $\tilde{\mathcal{W}}_{g,n}$. This can be constructed for one line bundle at time, for each component corresponding to a series of multiplicities $\mathbf{h}$, and then given in full as a union of all possible fibre products of such moduli spaces across all 7 line bundles, satisfying conditions specified by the group action. Thus the moduli space decomposes into a disjoint union of moduli spaces $\tilde{\mathcal{W}}(\mathbf{h})$. We will not dwell on the full machinery here, but see [16] for details.

We wish to show that considering these modified line bundles (and thus defining our invariants over this twisted moduli space instead) removes the issue of broad sectors in a clean way for the twisted invariants in genus zero.

Lemma 4.3.2. ***(Ramond Vanishing)*** *Over $\tilde{\mathcal{W}}_{0,n}(\mathbf{h})$, we have that $\pi_*(\sum_{k=1}^7 \mathcal{L}_k) = 0$ and $R^1\pi_*(\bigoplus_{k=1}^7 \mathcal{L}_k)$ is locally free.*

Proof. This is clear if all sectors in $\mathbf{h}$ are narrow. Assume there is a broad sector h_i. Then $\Theta_k(h_i) = 0$ for some k. The short exact sequence

$$0 \to \tilde{\mathcal{L}}_k \to \tilde{\mathcal{L}}_k(D_i) \to \tilde{\mathcal{L}}_k(D_i)|_{D_i} \to 0$$

induces a long exact sequence

$$\begin{aligned} 0 &\to \pi_*\tilde{\mathcal{L}}_k \to \pi_*\tilde{\mathcal{L}}_k(D_i) \to \pi_*\tilde{\mathcal{L}}_k(D_i)|_{D_i} \\ &\to R^1\pi_*\tilde{\mathcal{L}}_k \to R^1\pi_*\tilde{\mathcal{L}}_k(D_i) \to R^1\pi_*\tilde{\mathcal{L}}_k(D_i)|_{D_i}. \end{aligned}$$

The last term is zero by dimension considerations, and the first two terms are zero by a similar argument to lemma 4.3.1, but with one extra subtlety: if $\mathcal{C}$ is reducible and we follow through the same argument for the irreducible component $\mathcal{C}_v$ containing p_i, we may conclude that $\deg\tilde{\mathcal{L}}_k(D_i)|_{\mathcal{C}_{p_i}} < n_v$ (the number of nodes on $\mathcal{C}_v$), not $< n_v - 1$. But since in our inductive step we are considering *non*-tails it must be connected to at least two other components, and since we can prove this step for all other components it follows for C_v too. Thus our long exact sequence is reduced to

$$0 \to \pi_* \tilde{\mathcal{L}}_k(D_i)|_{D_i} \to R^1\pi_*\tilde{\mathcal{L}}_k \to R^1\pi_*\tilde{\mathcal{L}}_k(D_i) \to 0,$$

and so

$$c_{\text{top}}(R^1\pi_*\tilde{\mathcal{L}}_k) = c_{\text{top}}(\pi_*\tilde{\mathcal{L}}_k(D_1)|_{D_1}) \cdot c_{\text{top}}(R^1\pi_*\tilde{\mathcal{L}}_k(D_1)).$$

Note that $\tilde{\mathcal{L}}_k(D_i)|_{D_i} \cong \mathcal{L}_k|_{D_i}$, and since by assumption our multiplicity is zero here, this is a root of $\omega_{\log}|_{D_1}$, and this is trivial, we have that $c_{\text{top}}(\pi_*(\tilde{\mathcal{L}}_k(D_1)|_{D_1}) = 0$, and so $c_{\text{top}}(R^1\pi_*\tilde{\mathcal{L}}_k) = 0$. □

The derivation of the I-function will follow similar reasoning to that of the analogous theorem in [9]. First, we compute the untwisted J-function. For

$$\langle \phi_{h_1}, \ldots, \psi^a\phi_h\rangle^{\text{untwisted}}_{0,n+1} = \int_{[\mathcal{W}_{0,n,+1,G}(h_1,\ldots,h_n,h)]^{\text{vir}}} \psi^a,$$

to be non-zero we require the moduli space to be non-empty, which when all sectors are narrow requires that

$$-2q_k + \sum_i i_k(h_i) = i_k(h)$$

be an integer, so that $i_k(h) = \langle 2q_k - \sum_{i=1}^n i_k(h_i)\rangle$. Accordingly, for $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ we define $h_{\mathbf{n}}$ by $i_k(h_{\mathbf{n}}) = \langle -2q_k + \sum_{i=1}^n i_k(h_{n_i})\rangle$. We further require that the degree of ψ^a matches the dimension of the moduli space; that is, $a = n - 2$. In this case, by the string equation proved in [16], we find that (4.3) is 1.

Therefore

$$J^{\text{un}}(\mathbf{t}, z) = \sum_{\mathbf{n}:G\to\mathbb{N}_0} \frac{1}{z^{|\mathbf{n}|-1}} \prod_{h\in G} \frac{(t^h)^{\mathbf{n}(h)}}{\mathbf{n}(h)!}\phi_{h_{\mathbf{n}}},$$

where we label the term in the sum corresponding to $\mathbf{n}$ by $J^{\text{un}}_{\mathbf{n}}(\mathbf{t}, z)$.

We define a *twisting map* $\Delta : \mathcal{V}^{\text{un}} \to \mathcal{V}^{\text{tw}}$ which takes the *untwisted* Lagrangian cone $\mathcal{L}^{\text{un}}$ defined via the untwisted invariants to the *twisted* Lagrangian cone $\mathcal{L}^{\text{tw}}$ defined similarly via the twisted invariants.

Proposition 4.3.3. *If we set*

$$\Delta = \bigoplus_h \prod_{k=1}^7 \exp\Big(\sum_{d\geq 0} s_d \frac{B_{d+1}(i_k(h)+q_k)}{(d+1)!} z^d\Big),$$

where $B_d(x)$ is the d-th Bernoulli polynomial, then Δ is a symplectic transformation and

$\Delta(\mathcal{L}^{\mathrm{un}}) = \mathcal{L}^{\mathrm{tw}}$.

Proof. First, we show that Δ is symplectic. Consider $\Omega(\Delta f, \Delta g)$ for $f, g \in \mathcal{V}^{\mathrm{un}}$. Expanding f, g we note that the only non-zero terms in the product are given by pairing terms in ϕ_h in f with terms in $\phi_{h^{-1}}$ in g. We write $f = \sum_h f_h \phi_h$ and $g = \sum_h g_h \phi_h$. Note that in this case $i_k(h^{-1}) + q_k = 1 - (i_k(h) + q_k)$ and since the Bernoulli polynomials satisfy $B_{d+1}(1-x) = (-1)^{d+1} B_d(x)$, it follows that $\Omega(\Delta f, \Delta g)$ is the residue at $z = 0$ of the sum given for each h by

$$\begin{aligned}
&\langle \prod_k e^{\sum_{d\geq 0} s_d \frac{B_{d+1}(i_k(h)+q_k)}{(d+1)!} z^d} f_h(z)\phi_h, \prod_k e^{\sum_{d\geq 0} s_d \frac{B_{d+1}(i_k(h^{-1})+q_k)}{(d+1)!} (-z)^d} g_{h^{-1}}(-z)\phi_{h^{-1}} \rangle \\
&= e^{\sum_{k,d} (s_d \frac{B_{d+1}(i_k(h)+q_k)}{(d+1)!} z^d + s_d \frac{(-1)^{d+1} B_{d+1}(i_k(h)+q_k)}{(d+1)!} (-1)^d z^d)} \langle f_h(z)\phi_h, g_{h^{-1}}(-z)\phi_{h^{-1}} \rangle \\
&= \langle f_h(z)\phi_h, g_{h^{-1}}(-z)\phi_{h^{-1}} \rangle,
\end{aligned}$$

which gives $\mathrm{Res}_{z=0} \bigoplus_h \langle f_h(z)\phi_h, g_{h^{-1}}(-z)\phi_{h^{-1}} \rangle = \Omega(f, g)$, as required.

The potential is defined in terms of a quantum expansion; we show that $\hat{\Delta}\mathcal{D}^{\mathrm{un}} = \mathcal{D}^{\mathrm{tw}}$, where $\hat{\Delta}$ is the quantisation of Δ (see the detailed exposition in [12] for details). This is given by first forming the *Hamiltonian* corresponding to $\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}$ by $h_F(v) = \frac{1}{2}\langle \sum_{g\geq 0} \hbar^{g-1}\mathcal{F}v, v\rangle$, and then quantising the coordinates of $\mathcal{V}^{\mathrm{un}}, \mathcal{V}^{\mathrm{tw}}$ by replacing them with the operators

$$\begin{aligned}
\hat{q}^h &= q^h, \\
\hat{p}^h &= \hbar \frac{\partial}{\partial q^h},
\end{aligned}$$

we obtain $\hat{h}_F$, and define $\hat{\mathcal{D}} = e^{\hat{h}_F/\hbar}$.

We show that $\hat{\Delta}\mathcal{D}^{\mathrm{un}} = \mathcal{D}^{\mathrm{tw}}$.

Remembering the dilaton shift (2.6), it is straightforward to check term by term that $\hat{\Delta}\mathcal{D}^{\mathrm{un}}$ is a solution to

$$\begin{aligned}
\frac{\partial \Phi}{\partial s_d} &= \sum_{k=0}^{7} P_d^{(k)} \Phi, \\
P_d^{(k)} &= \frac{B_{d+1}(q_k)}{(d+1)!} \frac{\partial}{\partial t_{d+1}^{J_1 J_2}} - \sum_{\substack{a\geq 0\\ h\in G}} \frac{B_{d+1}(i_k(h)+q_k)}{(d+1)!} t_a^h \frac{\partial}{\partial t_{d+a}^h} \\
&\quad + \frac{\hbar}{2} \sum_{\substack{a+a'=d-1\\ h,h'\in G}} (-1)^{a'} \eta^{h,h'} \frac{B_{d+1}(i_k(h)+q_k)}{(d+1)!} \frac{\partial^2}{\partial t_a^h \partial t_{a'}^{h'}}.
\end{aligned}$$

Here $\eta^{h,h'} = e^{s_0}\delta_{h',h^{-1}}$, the inverse of the pairing.

We wish to show that and $\mathcal{D}^{\text{tw}}$ is also a solution to the above equation; the equality will then follow. Expanding $\mathcal{D}^{\text{tw}}$ and remembering the integral definition of the invariants and $\mathbf{c}(E)$, after some calculation we find that $\mathcal{D}^{\text{tw}}$ is a solution if and only if

$$\sum_{n=0}^{\infty}\frac{1}{n!}\langle\mathbf{t}(\psi),\mathbf{t}(\psi),\ldots,\mathbf{t}(\psi),\mathrm{ch}_d(R\pi_*(\mathcal{L}_k))\cdot\mathbf{c}(R\pi_*(\bigoplus_l\mathcal{L}_l))\rangle_{0,n}$$
$$=P_d^{(k)}\mathcal{F}_g^{\text{tw}}+\frac{\hbar}{2}\sum_{\substack{a+a'=d-1\\h,h'\in G}}(-1)^{a'}\eta^{h,h'}\frac{B_{d+1}(i_k(h)+q_k)}{(d+1)!}\frac{\partial\mathcal{F}_g^{\text{tw}}}{\partial t_a^h}\frac{\partial\mathcal{F}_g^{\text{tw}}}{\partial t_{a'}^{h'}}$$

Here we appeal to [10], where this equation was proved in some generality.

Therefore we have $\hat{\Delta}(\mathcal{D}^{\text{un}}) = \mathcal{D}^{\text{tw}}$. Taking the semi-classical limit $\hbar\to 0$, the conclusion follows. □

We set

$$D_h = t^h\frac{\partial}{\partial t_0^h},$$
$$D^k = \sum_{h\in G} i_k(h)D_h,$$
$$\mathbf{s}(x) = \sum_{d\geq 0} s_d\frac{x^d}{d!},$$
$$G_y(x,z) = \sum_{l,m\geq 0} s_{l+m-1}\frac{B_m^*(y)}{m!}\frac{x^l}{l!}z^{m-1}.$$

Note that

$$D_h J_{\mathbf{n}}^{\text{un}}(\mathbf{t},z) = \mathbf{n}(h)J_{\mathbf{n}}^{\text{un}}(\mathbf{t},z),$$
$$G_y(x,z) = G_0(x+yz,z),$$
$$G_0(x+z,z) = g_0(x,z)+\mathbf{s}(x).$$

We will build our twisted I-function from these functions, and prove that it lies on $\mathcal{L}^{\text{tw}}$ as given above. We set

$$J^{\mathbf{s}}(\mathbf{t},z) = \exp\left(\sum_{k=1}^{7}(-G_{q_k}(zD^k,z))\right)J^{\mathbf{un}}(\mathbf{t},z).$$

Note that we may write

$$J^{\mathrm{s}}(\mathbf{t}, z) = \sum_{\mathbf{n}} \prod_{k=1}^{7} \exp(-G_{q_k}((\sum_{h\in G} \mathbf{n}(h) i_k(h))z, z) J^{\mathrm{un}}_{\mathbf{n}}(\mathbf{t}, z),$$
$$\Delta = \prod_{k=1}^{7} \bigoplus_{h\in G} \exp(G_{q_k}(i_k(h)z, z)).$$

Lemma 4.3.4. *$J^{\mathrm{s}}(\mathbf{t}, z)$ lies on $\mathcal{L}^{\mathrm{un}}$.*

Proof. We follow the proof found in [9]. We write elements of $\mathcal{V}^{\mathrm{tw}}$ in standard form:

$$f = -z\phi_{J_1 J_2} + \sum_{l\geq 0} \mathbf{t}_l z^l + \sum_{l\geq 0} \frac{\mathbf{p}_l(f)}{(-z)^{l+1}}$$

for some $\mathbf{p}_l(f) = \sum_{h\in G} p_{l,j}(f)\phi^h$. Define

$$E_l(f) = \mathbf{p}_l(f) - \sum_{\substack{n\geq 0\\ h\in G}} \frac{1}{n!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), \psi^l \phi_h \rangle J^{\mathrm{un}}_{0,n+1} \phi^h.$$

Then

$$\mathcal{L}^{\mathrm{un}} = \{f \in \mathcal{V}^{\mathrm{un}} \mid E_l(f) = 0\}.$$

For the purposes of an enumeration for induction we set $\deg(s_d) = d + 1$ and perform induction on the terms of $J^{\mathrm{s}}(\mathbf{t}, z)$, ordered by that degree, treating it as a power series in $s_1, s_2, \ldots$. (The s_d are mixed since they appear in the exponential factor, so we do not simply perform induction on d.) The degree zero part has no equivariant terms and reduces to $J^{\mathrm{un}}(\mathbf{t}, z)$, which is of course in $\mathcal{L}^{\mathrm{un}}$. This is the base case. Now assume that the degree n part vanishes. Then we can find some 'corrected' $\tilde{J}^{\mathrm{s}}(\mathbf{t}, z)$ which *is* in $\mathcal{L}^{\mathrm{un}}$ which has the same degree-n part as J^{s}.

The idea is that taking the derivative with respect to each s_d knocks down the degree of each term by at least 1, and we can use our inductive hypothesis and properties of the Lagrangian cone to prove the hypothesis for the degree $n + 1$ part. By the chain rule, viewing E_l as a function of J^{s}:

$$\frac{\partial}{\partial s_d}(E_j(J^{\mathrm{s}}(\mathbf{t}, -z)) = (d_{J_{\mathrm{s}}} E_l) \circ (z^{-1} P_d J^{\mathrm{s}}(\mathbf{t}, z)),$$

where from the definition of J^{s} we have

$$P_d = \sum_{m=0}^{d+1} \frac{1}{m!(i+1-m)!} z^m B_m(q_k)(zD^k)^{d+1-m}.$$

But then up to degree n,

$$\frac{\partial}{\partial s_d}(E_l J^{\mathrm{s}}) = \frac{\partial}{\partial s_d}(E_l \tilde{J}^{\mathrm{s}}).$$

Since $zT_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}} = \mathcal{L}^{\mathrm{un}} \cap T_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}}$, and $\mathcal{L}^{\mathrm{un}}$ is a cone, we have that operating on any point of $\mathcal{L}^{\mathrm{un}}$ by zD gives us a point of $zT_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}} \subset T_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}}$. By repeated application a polynomial in such operators still keeps the point in this space, since $z^k T_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}} \subset zT_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}}$, so $P_d\tilde{J}^{\mathrm{s}}$ is still in $zT_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}}$; and thus $z^{-1}P_d\tilde{J}^{\mathrm{s}}$ lies in $T_{J^{\mathrm{s}}}\mathcal{L}^{\mathrm{un}}$. Therefore the whole derivative vanishes, so that the hypothesis holds up to degree $n+1$, and our conclusion follows. $\square$

It follows from the two previous lemmas that $\Delta(J^{\mathrm{s}})$ lies in the twisted Lagrangian cone; this will be our equivariant I-function. We have

$$I^{\mathrm{tw}}(\mathbf{t}, z) := \Delta(J^{\mathrm{s}}(\mathbf{t}, -z)) = \sum_{\mathbf{n}} \mathbf{M}_{\mathbf{n}}(z) J^{\mathrm{un}}_{\mathbf{n}}(\mathbf{t}, z),$$

where

$$\begin{aligned}
\mathbf{M}_{\mathbf{n}}(z) &= \prod_{k=1}^{7} \exp(G_{q_k}(\langle \sum_{h\in G} \mathbf{n}(h) i_k(h)\rangle z, z) - G_{q_k}(\sum_{h\in G} \mathbf{n}(h) i_k(h) z, z)) \\
&= \prod_{\substack{1\le k\le 7 \\ 0\le b<\lfloor \sum_{h\in G}\mathbf{n}(h)i_k(h)\rfloor}} \exp(-\sum_{d\ge 0} s_d \frac{(-q_k z - \langle \sum_{h\in G} \mathrm{n}(h) i_k(h)\rangle z - bz)^d}{d!}) \\
&= \prod_{\substack{1\le k\le 7 \\ 0\le b<\lfloor \sum_{h\in G}\mathbf{n}(h)i_k(h)\rfloor}} \exp(\ln(\lambda) + \sum_{d>0} (-1)^{d+1} \frac{(q_k z + \langle \sum_{h\in G} \mathrm{n}(h) i_k(h)\rangle z + bz)^d}{d!}) \\
&= \prod_{\substack{1\le k\le 7 \\ 0\le b<\lfloor \sum_{h\in G}\mathbf{n}(h)i_k(h)\rfloor}} (\lambda + q_k z + \langle \sum_{h\in G} \mathrm{n}(h) i_k(h)\rangle z + bz)
\end{aligned}$$

similarly to the argument in [9], and likewise as $\lambda \to 0$ we obtain the non-equivariant

I-function, which in our case we restrict to the narrow sectors of degree at most 2, so that

$$
\begin{aligned}
I_{FJRW}^{\mathrm{nar}}(\mathbf{t},z) &= \sum_{\mathbf{n}:\mathcal{N}_{\deg\leq 2}\to\mathbb{N}_0} \Bigg(\prod_{\substack{1\leq k\leq 7\\ 0\leq b<\lfloor\sum_{h\in G}\mathbf{n}(h)i_k(h)\rfloor}} \Big(q_k + \langle\sum_h \mathrm{n}(h)i_k(h)\rangle + b\Big)\Bigg)\\
&\times\ z^{\sum_k\lfloor\sum_h \mathrm{n}(h)i_k(h)\rfloor-|\mathbf{n}|+1}\prod_h \frac{\mathbf{t}^{\mathbf{n}(h)}}{\mathbf{n}(h)!}\phi_{h_{\mathbf{n}}}\\
&= \sum_{\mathbf{n}:\mathcal{N}_{\deg\leq 2}\to\mathbb{N}_0}\Bigg(\prod_{1\leq k\leq 7}\frac{\Gamma\Big(q_k+\sum_h \mathrm{n}(h)i_k(h)\Big)}{\Gamma\Big(q_k+\langle\sum_h \mathrm{n}(h)i_k(h)\rangle+1\Big)}\Bigg) z^{\sum_k\lfloor\sum_h \mathrm{n}(h)i_k(h)\rfloor-|\mathbf{n}|+1}\prod_h\frac{\mathbf{t}^{\mathbf{n}(h)}}{\mathbf{n}(h)!}\phi_{h_{\mathbf{n}}}.\\
&=: \sum_{h\in\mathcal{N}_{\deg\leq 2}} \omega_h^{\mathrm{FJRW}} z^{\sum_k\lfloor i_k(h)\rfloor-|\mathbf{n}|+1}\phi_h.
\end{aligned}
$$

Separating out J_1J_2 contributes a constant factor of $\sum_{n_{J_1J_2}=0}^{\infty}\frac{z^{-n_{J_1J_2}}}{n_{J_1J_2}!}=e^{-z}$.

This expression may be most easily expanded for $W=X^2+Y^4+Z^4+x^2+y^6+z^6+w^6=0$:

$$
\begin{aligned}
(i_k(J_1^3J_2))_k &= (0,\frac{1}{2},\frac{1}{2},0,0,0,0),\\
(i_k(J_1J_2^3))_k &= (0,0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3}),\\
(i_k(\sigma J_1^2J_2^2))_k &= (0,\frac{1}{4},\frac{1}{4},0,\frac{1}{6},\frac{1}{6},\frac{1}{6}).
\end{aligned}
$$

We re-express the sum over $\mathbf{n}$ as a sum over the triple $(\mathbf{n}(J_1^3J_2),\mathbf{n}(J_1J_2^3),\mathbf{n}(\sigma J_1^2J_2^2))=(M,N,C)$, as follows:

$$
\begin{aligned}
I_{FJRW}^{\mathrm{nar}}(\mathbf{t},z) &= \sum_{M,N,C\geq 0}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}+\frac{M}{2}+\frac{C}{4})^2\Gamma(\frac{1}{2})\Gamma(\frac{1}{6}+\frac{N}{3}+\frac{C}{6})^3}{\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{4}+\langle\frac{M}{2}+\frac{C}{4}\rangle+1)^2\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{6}+\langle\frac{N}{3}+\frac{C}{6}\rangle+1)^3}\\
&\times\ \frac{T_1^MT_2^NT_3^C}{M!N!C!}z^{1-M-N-C+2\lfloor\frac{M}{2}+\frac{C}{4}\rfloor+3\lfloor\frac{N}{3}+\frac{C}{6}\rfloor}\phi_{h(M,N,C)}\\
&=: \sum_h\sum_{(M,N,C):h(M,N,C)=h}\omega_h^{\mathrm{FJRW}}z^{1-M-N-C+2\lfloor\frac{M}{2}+\frac{C}{4}\rfloor+3\lfloor\frac{N}{3}+\frac{C}{6}\rfloor}\phi_h.
\end{aligned}
$$

It is easily seen that the exponent of z depends only on $(M \bmod 2, N \bmod 3, C \bmod 12)$, and that for each h there are only 6 cases. For other orbifolds of our type, there are similarly few cases, which depend similarly on the lowest common multiples of w_i.

In the general case, consider for which terms the exponent of z can be 1. This would require

$$\sum_k \mathbf{n}(h) = \sum_k \lfloor \sum_h \mathbf{n}(h) \rfloor,$$

but since it is always true that $i_k(h) < 1$, we must have $i_k(h_\mathbf{n}) = \langle \sum_h \mathbf{n}(h) i_k(h) \rangle = 0$ for all k, whence $h_\mathbf{n} = J_1 J_2$. Since we require that $\langle \sum_h \mathbf{n}(h) i_k(h) \rangle = 0$, the linear coefficient of $\mathbf{t}$ must be zero, so it follows that the term in z^1 must be $f(\mathbf{t}) z \phi_{J_1 J_2}$, where $f(\mathbf{t}) = f_0(t^{J_1 J_2}) + \mathcal{O}(\mathbf{t}^2)$.

Then we may write $I_{\mathrm{FJRW}}(z, \mathbf{t}) = f(\mathbf{t}) z \phi_{J_1 J_2} + \mathbf{g}(\mathbf{t}) + \mathcal{O}(z^{-1})$. Then if we set $\tau(\mathbf{t}) = \mathbf{g}(t)/f(t)$, we find that

$$\frac{I_{\mathrm{FJRW}}^{(W,G)}(\mathbf{t}, z)}{f(\mathbf{t})}$$

lies on $\mathcal{L}^{(W,G)}$ and is of the form $z\phi_{J_1 J_2} + \mathbf{t} + \mathcal{O}(z^{-1})$. Since the J-function is unique with respect to this property, this gives us the following FJRW 'mirror theorem'.

Proposition 4.3.5.

$$J_{\mathrm{FJRW}}^{(W,G)}(\tau(\mathbf{t}), z) = \frac{I_{\mathrm{FJRW}}^{(W,G)}(\mathbf{t}, z)}{f(\mathbf{t})}.$$

CHAPTER 5

The Intermediate Mixed Theories

We consider the Borcea-Voisin orbifolds as arising as complete intersections $\mathcal{Y}$ in GIT quotients

$$\mathcal{X}_\theta := [V//_\theta(\mathbb{C}^*)^3], V = (\mathbb{C}^3 \times \mathbb{C}^4 \times \mathbb{C} \times \mathbb{C}^2)$$

for some character θ of $(\mathbb{C}^*)^3$, where $(\mathbb{C}^*)^3$ acts by

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & w_0 & w_1 & w_2 & w_3 & 0 & 0 & -2w_0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

The potential for this theory is given by $p_1W_1 + p_2W_2$ where here the variables p_1, p_2 give coordinates of the last factor $\mathbb{C}^2$ to fit in with the notation of [15]. The critical locus of this potential is therefore

$$\{0 = W_1 = W_2 = p_1\frac{\partial W_1}{\partial X} = \ldots = p_1\frac{\partial W_1}{\partial Z} = p_2\frac{\partial W_2}{\partial x} = \ldots = p_2\frac{\partial W_2}{\partial w}\}.$$

Since W_1, W_2 are non-degenerate, either $p_1 = W_1 = 0$ or $p_1 \neq 0$, $(X, Y, Z) = 0$, and either $p_2 = W_2 = 0$ or $p_2 \neq 0$, $(x, y, z, w) = 0$.

We split into cases according to the characters $\theta : G \to \mathbb{C}^*$, which acts on the total space of L_θ by

$$g = (\lambda_1, \lambda_2, \lambda_3) : (\mathbf{v}, \tilde{z}) \mapsto (g \cdot \mathbf{v}, \lambda_1^{e_1}\lambda_2^{e_2}\lambda_3^{e_3}\tilde{z}).$$

In each case we shall consider $e_3 > 0$.

1. If $e_1 < 0$ and $e_2 < 0$, then the semi-stable points require some section f of $L_\theta^{\otimes k}$ to be G-invariant and non-zero there. Since the weights are negative for p_1, p_2, p_3 but positive for the other coordinates, to ensure invariance we require each monomial in f to have at least some non-zero X, Y, Z (for $\lambda_1 \neq 1$) and some non-zero x, y, z, w (for $\lambda_2 \neq 1$) to cancel the negative weights from θ. Otherwise, we have complete

freedom to choose k and f. Therefore,

$$V^{ss}(\theta) = \{(X, Y, Z), (x, y, z, w) \neq 0\}.$$

We choose ϑ to be the trivial lift, which is clearly good. The intersection with the critical locus is then

$$[\frac{(\mathbb{C}^3\backslash 0) \times (\mathbb{C}^4\backslash 0) \times (\mathbb{C}\backslash 0) \times \{0\} \times \{0\}}{(\mathbb{C}^*)^3}].$$

This is the geometric phase, which cedes Gromov-Witten theory.

2. If $e_1 > 0$ and $e_2 < 0$, then the semistable locus is

$$[V^{ss}//_\theta G] = [(\mathbb{C}^3 \times (\mathbb{C}^4\backslash 0) \times \mathbb{C}^* \times \mathbb{C})/(\mathbb{C}^*)^3].$$

   The ambient space can then be viewed as a non-trivial $B\mathbb{Z}_2$-gerbe over $[K/\langle\sigma_K\rangle]$. This is the first mixed theory.

3. If $e_1 < 0$ and $e_2 > 0$, then the semistable locus is

$$[V^{ss}//_\theta G] = [((\mathbb{C}^3\backslash 0) \times \mathbb{C}^4 \times \mathbb{C} \times \mathbb{C}^*)/(\mathbb{C}^*)^3].$$

   The ambient space can then be viewed as a non-trivial $B\mathbb{Z}_2$-gerbe over $[E/\langle\sigma_E\rangle]$. This is the second mixed theory.

4. If $e_1 > 0$ and $e_2 > 0$ then the semistable locus is

$$[V^{ss}//_\theta G] = [(\mathbb{C}^3 \times \mathbb{C}^4 \times \mathbb{C}^* \times \mathbb{C}^*)/(\mathbb{C}^*)^3].$$

   This is the FJRW theory. In our case, we have chosen $p_1 = p_2 = 1$ and the group action is the same as that of the group $\langle J_1, J_2, \sigma\rangle$ considered in the FJRW section.

We shall calculate the I-functions for the two mixed theories; it will be convenient to express the state space in a way that relates to the GW and FJRW theories. We first compute the first mixed theory, which we label by $(FJRW, GW)$, in that it is related to the FJRW theory for the elliptic curve part, and the Gromov-Witten theory for the K3 part. The situation for the other mixed theory is entirely similar.

Consider the induced actions of $\sigma \in G$ on $H_{\mathrm{CR}}(\mathbb{C}^3, W_1^{+\infty})$ and $H_{\mathrm{CR}}(\{x^2+P(x,y,z) = 0\})$. These split into positive and negative eigenspaces, $H^{\pm}_{\mathrm{FJRW},E}$ and $H^{\pm}_{\mathrm{GW},K}$ respectively.

If $\mathrm{Re}\, W$ takes on value greater than 2ρ, then either W_1 (the polynomial defining the elliptic curve) or W_2, the polynomial defining the K3 surface) must have real part greater than ρ. Taking ρ arbitrarily large, this allows us to decompose $W^{+\infty}$ and apply the Künneth theorem for relative cohomology to the GLSM state space $H_{\mathrm{CR}}(\mathcal{X}_\theta, W^{+\infty}; \mathbb{C})$. Then the part of the total Chen-Ruan cohomology untwisted by σ is given by the singular Künneth theorem as

$$(\mathcal{H}^{+}_{\mathrm{FJRW},E} \otimes \mathcal{H}^{+}_{\mathrm{GW},K}) \oplus (\mathcal{H}^{-}_{\mathrm{FJRW},E} \otimes \mathcal{H}^{-}_{\mathrm{GW},K}).$$

To this we must add the part twisted by σ, which is given as the G-invariant relative Chen-Ruan cohomology of

$$\mathrm{Fix}(\sigma) = \mathrm{Fix}(\langle J_1, \sigma_E \rangle) \times \mathrm{Fix}(\sigma_K) \subseteq \mathbb{C}^3 \times ([\{x^2 + P(x,y,z) = 0\}/\langle J_2 \rangle]).$$

By the Künneth theorem this is the tensor product of the σ-twisted parts $H^{\sigma_E}_{\mathrm{FJRW},E} \otimes H^{\sigma_K}_{\mathrm{GW},K}$.

We are interest in the sectors of compact type. These are by definition the spans of the narrow sectors (which are induced by the narrow sectors of $\mathcal{H}_{\mathrm{FJRW},E}$), and the critical sectors (which are induced by the ambient sectors of $\mathcal{H}_{\mathrm{GW},K}$). Thus we have

$$\mathcal{H}^{\mathrm{comp}}_{\mathrm{FJRW,\,GW}} = (H^{\mathrm{nar},+}_{\mathrm{FJRW},E} \otimes H^{\mathrm{amb},+}_{\mathrm{GW},K}) \oplus (H^{\mathrm{nar},-}_{\mathrm{FJRW},E} \otimes H^{\mathrm{amb},-}_{\mathrm{GW},K}) \oplus (H^{\sigma_E,\mathrm{nar}}_{\mathrm{FJRW},E} \otimes H^{\sigma_K,\mathrm{amb}}_{\mathrm{GW},K}).$$

We have a basis of $\mathcal{H}^{\mathrm{comp}}_{\mathrm{FJRW,\,GW}}$ given by elements we may write of the form:

- $\phi_{J_1} \mathbf{1}_0$
- $\phi_{J_1^3} \mathbf{1}_0, \phi_{\sigma_E J_1^2} \mathbf{1}_{\sigma_K}, 1 + \sum_{i,j:\gcd(i,j)>1} (p_{i,j} - 1)$ sectors of the form $\phi_{J_1} \mathbf{1}_{g^r_{i,j}}$, $\frac{1}{2} \sum_{w_j \nmid w_0} p_{0,j}$ sectors of the form $\sigma J_1^2 \mathbf{1}_{g_{\frac{r}{2p}}}$
- $\phi_{J_1} D_K^2, 1 + \sum_{i,j:\gcd(i,j)>1} (p_{i,j} - 1)$ sectors of the form $\phi^3_{J_1} \mathbf{1}_{g^r_{i,j}}, \phi_{\sigma_E J_1^2} D_K \mathbf{1}_{\sigma_K}$, $\frac{1}{2} \sum_{w_j \nmid w_0} p_{0,j}$ sectors of the form $\phi_{\sigma_E J_1^2} \mathbf{1}_{g_{\frac{r}{2p}}}$, and a sector $\phi_{J_1^3} D_K \mathbf{1}_{\tilde{g}}$ when $w_1 = \frac{d}{3}$
- $\phi_{J_1^3} D_K^2$

The new GLSM theory unifies the notion of degree, as well as the pairing. We have $\deg(\phi_h \beta) = \deg(\phi_h) + \deg(\beta)$. This gives an isomorphism between all narrow state spaces.

In our case, we re-express the Landau-Ginzburg quasimaps for the case of toric stacks as given in [15], decomposing the line bundle $\mathcal{P}$ and the sections σ, κ into summands. Then, for $E \subseteq \mathbb{P}(2,1,1)$, the moduli space $LGQ^{\vartheta,\epsilon}_{g,n}(\mathcal{X}_\theta, \beta)$ is given by

$$\begin{aligned}
&\{(\mathcal{C}, z_1, \ldots, z_n, \mathcal{L}_1, \mathcal{L}_2, \mathcal{T}, s_X, s_Y, s_Z, s_x, s_y, s_z, s_w, s_{p_1}, s_{p_2}) : \\
&s_X \in H^0(\mathcal{L}_1^{\otimes 3} \otimes \mathcal{T}), s_Y, s_Z \in H^0(\mathcal{L}_1), s_x \in H^0(\mathcal{L}_2^{\otimes w_0} \otimes \mathcal{T}), \\
&s_y \in H^0(\mathcal{L}_2^{\otimes w_1}), s_y \in H^0(\mathcal{L}_2^{\otimes w_2}), s_z \in H^0(\mathcal{L}_2^{\otimes w_3}), \\
&s_{p_1} \in H^0(\mathcal{L}_1^{-4} \otimes \omega_{\log}), s_{p_2} \in H^0(\mathcal{L}_2^{-2w_0} \otimes \omega_{\log}), \mathcal{T}^{\otimes 2} \cong \mathcal{O}\},
\end{aligned}$$

where the stability conditions are satisfied.

For Gromov-Witten theory, the sections s_i induce a map $f : \mathcal{C} \to \mathcal{X}_\vartheta$, so this is the moduli space of stable maps. For FJRW theory, this is the moduli space of spin curves subject to the conditions provided by these sections. For the mixed theories, we exploit the fact that the moduli space decomposes. Note that there are two possible 2-torsion bundles $\mathcal{T}$: one trivial and one not. Suppressing the markings, each LG quasimap $(\mathcal{C}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{T}, \sigma_X, \ldots, \sigma_x \ldots, \sigma_{p_1}, \sigma_{p_2})$ gives a pair of LG quasimaps

$$(\mathcal{C}, \mathcal{L}_1, \mathcal{T}, \sigma_X, \sigma_Y, \sigma_Z, \sigma_{p_1}), (\mathcal{C}, \mathcal{L}_2, \mathcal{T}, \sigma_x, \sigma_y, \sigma_z, \sigma_w, \sigma_{p_2}).$$

$\mathcal{T}$ depends on β, which corresponds to a member of the state space. Thus, provided this condition is satisfied, that moduli space presents as a fibre product over $\mathcal{M}_{g,n}$.

In the Gromov-Witten case, this is to say that giving a stable map $f : \mathcal{C} \to [E \times K/\mathbb{Z}_2]$ is equivalent to giving one stable map $f_1 : \mathcal{C} \to [E/\langle \sigma_E \rangle]$ and another $f_2 : \mathcal{C} \to [K/\langle \sigma_K \rangle]$, provided that the images of f_1, f_2 either both lie in the untwisted component of their inertia stacks, or both in the twisted components (giving a well-defined class β). In the FJRW case, we see that the we get separate line bundles subject to conditions equivalent to the FJRW stability conditions, i.e. the structure of W-spin curves.

For the first mixed theory, for $\mathbf{h} \in \mathcal{H}^{\mathrm{FJRW}}(W_1, \langle J_1, \sigma_E \rangle), \beta \in \mathcal{H}^{\mathrm{GW}}([K/\langle \sigma_K \rangle])$ such that $\mathbf{h}\beta \in \mathcal{H}_{\mathrm{FJRW,GW}}$, the genus-0 n-pointed moduli space for $\mathbf{h}\beta$ may be written

$$\overline{\mathcal{M}}^{\mathrm{FJRW,GW}}_{0,n}(\mathbf{W}, G, \beta\mathbf{h}) = \mathcal{W}^{\mathrm{FJRW}}_{0,n,\mathbf{h}}(W_1, \langle J_1, \sigma_E \rangle) \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{M}^{\mathrm{GW}}}_{0,n}([K/\langle \sigma_K \rangle], \beta).$$

This justifies the subscript notation for the mixed theories.

The Hodge bundle of this theory decomposes similarly, and we have

$$\psi_{\mathrm{FJRW,\,GW},i} = \psi_{\mathrm{FJRW},i} \psi_{\mathrm{GW},i},$$

so we see that the invariants above in fact decompose as

$$\langle \tau_{a_1}(h_1), \ldots, \tau_{a_n}(h_n)\rangle_{0,n}^{(W_1,\langle J_1,\sigma_E\rangle)} \langle \tau_{a_1}(\alpha_n), \ldots, \tau_{a_n}(\alpha_n)\rangle_{0,n}^{([K/\langle\sigma_K\rangle],\beta)}.$$

Composing the evaluation maps with the two projections gives a similar composition there. The only difficulty in determining the invariants lies in the fact that the virtual class does not necessarily decompose naturally.

We compute the untwisted invariants as follows. Since we are working with toric stacks, we may let $\rho : LGQ_{0,n}^{\vartheta,\epsilon}(\mathcal{X}_\theta, \beta) \to \overline{M_{0,n}}(\mathcal{X}_\theta, \beta')$ be the natural map sending a stable LG-quasimap to the induced stable map to $\mathcal{X}_\theta$, where β' is the induced homology class. For genus 0, the virtual class is given by the cosection construction in both cases, and is preserved. The untwisted invariants are therefore given by

$$\langle \tau_{a_1}(\alpha_1), \ldots, \tau_{a_n}(\alpha_n)\rangle_{0,n}^{\vartheta} = \int_{\rho^*([\overline{M_{0,n}}(\mathcal{X}_\theta,\beta')]^{\mathrm{vir}})} \prod_{i=1}^{n} \mathrm{ev}_i(\alpha_i)\psi_i^{a_i}$$

which by substitution are the genus-0 Gromov-Witten invariants of the 'ambient' $\mathcal{X}_\theta$. This is a toric stack, and as such can be computed from the main theorem of [14], as before, after naturally identifying the Chen-Ruan basis with the mixed basis.

For example, we can see that the untwisted FJRW theory is equivalent to the Gromov-Witten theory of $\mathcal{B}G$, and indeed that the untwisted FJRW J-function from the previous section may be given by $ze^{\sum_{h\in G} t_h\phi_h/z}$, identifying the FJRW basis elements ϕ_h with the fundamental classes of the h-sectors.

The string equation allows us to write this as

$$\begin{aligned} J^{\mathrm{un}}_{\mathrm{FJRW,GW}}(\mathbf{t}, z) =& e^{(\phi_{J_1}\mathbf{1}_0 t_{J_1} + \sum_{i=0}^3 w_i\phi_0 D_K t_{4+i} + \phi_{\sigma_E}\mathbf{1}_{\sigma_K} + \sum_{g:\deg(\phi_{\sigma_E}\mathbf{1}_{\sigma_K g})=2} \phi_{\sigma_E}\mathbf{1}_{\sigma_K g})/z)} \\ & \times \frac{\prod_{i=1}^4 \Gamma(w_i D_K/z+1)}{\Gamma(w_0 D_K + w_0 b + c + 1)\prod_{i=1}^3 \Gamma(w_i D_K/z + w_i b + 1)}. \end{aligned}$$

The twisted I-function may be found by the same methods of quantisation Givental given in the previous section and the orbifold Grothendieck-Riemann-Roch theorem, and from the work of Tseng, as detailed in [13]. We express it slightly differently, exploiting the decomposition of the moduli space and the additivity of the functor $c_{\mathrm{top}}(R^1\pi_*\bullet)$. As in [14], let $\beta \in \mathrm{Box}([K/\sigma_K])$ correspond to the fundamental class of a component of the inertia stack of $[K/\langle\sigma_K\rangle]$, let $\mathcal{K}$ be the line bundle whose first Chern class corresponds to K, and for $(p, g) \in I[K/\langle\sigma\rangle]$, let g act on $\mathcal{K}|_p$ by $e^{2\pi i f(b)}$ for given b. Then we have the following decomposition.

$$\Delta = \prod_{k=4}^{7} \bigoplus_{h\in\langle J_1,\sigma_E\rangle} \exp(G_{q_k}(i_k(h)z,z)) \bigoplus_{\substack{\beta\in\mathrm{Box}([K/\sigma_K])\\ \mathbf{h}\beta\in\mathcal{H}_{\mathrm{FJRW,GW}}}} \exp(G_{f(b)}(E,z)),$$

where $\mathbf{h}\beta$ may be represented as (b,c) analogously to the Gromov-Witten case.

Then we have

$$\begin{aligned}
&I_{\mathrm{FJRW,\,GW}}(\mathbf{t},z) = ze^{(w_0t_4+w_1t_5+w_2t_6+w_3t_7)D_K/z}e^{-z}\times\\
&\sum_{n_3,n_\sigma\in\mathbb{N}_0^3} 2\frac{\Gamma(\frac14+\frac{n_3}{2}+\frac{n_\sigma}{4})^2 t_3^{n_3}t_\sigma^{n_\sigma}}{\Gamma(\frac14+\langle\frac{n_3}{2}+\frac{n_\sigma}{4}\rangle+1)^2 n_3!n_\sigma!} z^{\lfloor\frac{n_3}{2}+\frac{n_\sigma}{4}\rfloor-(n_3+n_\sigma)} \sum_{\mathbf{b}\in\mathrm{Box}([K/\langle\sigma_K\rangle])} L_{\mathbf{b}}\times\\
&\sum_{\substack{(b,c,\mathbf{k})\in\Lambda E^S_{\mathbf{b}}([K/\langle\sigma_K\rangle])\\ 2c+\sum_{j=l+1}^m k_j=n_\sigma}} (q_2^bq_3^c\prod_{j=1}^m x_j^{k_j})e^{b(\sum_{i=1}^n w_it_i)+c(t_4+2t_8)}G_{\mathbf{b}}(b,c,\mathbf{k})\phi_{h_{n_3,n_\sigma}}\mathbf{1_b}\\
=:&\sum_{n_1,n_3,n_\sigma}\sum_{\mathbf{b}\in\mathrm{Box}([K/\langle\sigma_K\rangle])} \omega^{\mathrm{FJRW,\,GW}}_{n_3,n_\sigma,\mathbf{b}} z^{\lfloor\frac{n_3}{2}+\frac{n_\sigma}{4}\rfloor-(n_1+n_3+n_\sigma)}\phi_{n_1,n_3,n_\sigma}\mathbf{1_b}
\end{aligned}$$

where a factor involving c appears only once. Here we have extracted a factor of $\sum_{n_1=0}^{\infty}\frac{z^{-n_1}}{n_1!} = e^{-z}$ to suppress the term-wise dependence on n_1, and of course $2=\frac{\Gamma(\frac12)}{\Gamma(1+\frac12)}$.

For the second mixed theory, we compute the I-function for either of the two considered elliptic curves E and $W_2 = x^2+y^6+z^6+w^6$. Again, we may write a basis of $\mathcal{H}^{\mathrm{comp}}_{\mathrm{GW,\,FJRW}}$ as

- $\mathbf{1}_0\phi_{J_2}$
- $\mathbf{1}_0\phi_{J_2^3}, D_E\phi_{J_2}, \mathbf{1}_{\sigma_E}\phi_{\sigma_K J_2^2}[, \mathbf{1}_{\sigma_E g}\phi_{\sigma_K J_2^2}]$
- $\mathbf{1}_0\phi_{J_2^5}, D_E\phi_{J_2^3}, \mathbf{1}_{\sigma_E}\phi_{\sigma_K J_2^4}[, \mathbf{1}_{\sigma_E g}\phi_{\sigma_K J_2^4}]$
- $D_E\phi_{J_1^5}$

where the terms in square brackets come from the extra sector $\mathbf{1}_{\sigma g}$ that appears for $E = \{X^2+Y^3+Z^6=0\}$.

As we had before for W_1, let $h_{(n_1,n_3,n_\sigma)}$ be the unique element of $\langle J_2,\sigma_K\rangle$ such that

$$i_k(h_{(n_1,n_3,n_\sigma)}) = n_1 i_k(J_2) + n_3 i_k(J_2^3) + n_\sigma i_k(\sigma_K J_2^2).$$

Similarly to the first mixed theory, we find, for $E = \{X^2 + Y^4 + Z^4 = 0\}$,

$$
\begin{aligned}
&I_{\mathrm{GW,\,FJRW}}(\mathbf{t}, z) = z e^{(2t_1+t_2+t_3)D_E/z} e^{-z} \times \\
&\sum_{n_3, n_\sigma \in \mathbb{N}_0^3} 2 \frac{\Gamma(\frac{1}{6} + \frac{n_3}{3} + \frac{n_\sigma}{6})^3 t_3^{n_3} t_\sigma^{n_\sigma}}{\Gamma(\frac{1}{6} + \langle \frac{n_3}{3} + \frac{n_\sigma}{6} \rangle + 1)^3 n_3! n_\sigma!} z^{\lfloor \frac{n_3}{3} + \frac{n_\sigma}{6} \rfloor - (n_1+n_3+n_\sigma)} \times \\
&\sum_{\mathbf{b} \in \mathrm{Box}([E/\langle \sigma_E \rangle])} K_{\mathbf{b}} \sum_{\substack{(a,c) \in \Lambda E^S_{\mathbf{b}}([E/\langle \sigma_E \rangle]) \\ 2c = n_\sigma}} q_2^a q_3^c e^{a(2t_1+t_2+t_3)+c(t_1+2t_8)} F_{\mathbf{b}}(a,c) \mathbf{1}_{\mathbf{b}} \phi_{h_{n_3,n_\sigma}} \\
=: &\sum_{n_1,n_3,n_\sigma} \sum_{\mathbf{b} \in \mathrm{Box}([E/\langle \sigma_E \rangle])} \omega^{\mathrm{GW,\,FJRW}}_{\mathbf{b},n_3,n_\sigma} z^{\lfloor \frac{n_3}{3} + \frac{n_\sigma}{6} \rfloor - (n_3+n_\sigma)} \mathbf{1}_{\mathbf{b}} \phi_{n_1,n_3,n_\sigma}.
\end{aligned}
$$

For both of these mixed theories, the exponent of z is then only 1 for the term corresponding to the identity ($\phi_{J_1}\mathbf{1}_0$ or $\mathbf{1}\phi_{J_2}$, respectively), and the coefficient of z^0 is clearly linear in $\mathbf{t}$. If write

$$
\begin{aligned}
I_{\mathrm{FJRW,GW}}(z, \mathbf{t}) &= f_1(\mathbf{t}) z \phi_{J_1 \mathbf{1}_0} + \mathbf{g}_1(\mathbf{t}) + \mathcal{O}(z^{-1}), \\
I_{\mathrm{GW,FJRW}}(z, \mathbf{t}) &= f_2(\mathbf{t}) z \phi_{\mathbf{1}_0 J_2} + \mathbf{g}_2(\mathbf{t}) + \mathcal{O}(z^{-1}).
\end{aligned}
$$

Then for $i = 1, 2$, if we set $\tau_i(\mathbf{t}) = \mathbf{g}_i(t)/f_i(t)$, we find that

$$
\frac{I^{(W,G)}_{\mathrm{FJRW}}(\mathbf{t}, z)}{f(\mathbf{t})}
$$

lies on $\mathcal{L}^{(W,G)}$ and is of the form $z\phi_{J_1 J_2} + \mathbf{t} + \mathcal{O}(z^{-1})$. Since the J-function is unique with respect to this property, this gives us the following FJRW 'mirror theorems'.

Proposition 5.0.1.

$$
\begin{aligned}
J^{(W,G)}_{\mathrm{FJRW,GW}}(\tau_1(\mathbf{t}), z) &= \frac{I^{(W_1, \langle J_1, \sigma_E \rangle), [K/\langle \sigma_K \rangle]}_{\mathrm{FJRW,GW}}(\mathbf{t}, z)}{f_1(\mathbf{t})}, \\
J^{(W,G)}_{\mathrm{GW,FJRW}}(\tau_2(\mathbf{t}), z) &= \frac{I^{[E/\langle \sigma_E \rangle], (W_2, \langle J_2, \sigma_K \rangle)}_{\mathrm{GW,FJRW}}(\mathbf{t}, z)}{f_2(\mathbf{t})}.
\end{aligned}
$$

CHAPTER 6

The LG/CY Correspondence

6.1 The LG/CY State Space Correspondence

Artebani, Boissière and Sarti have proved in [2] that $\mathcal{H}_{\mathrm{FJRW}} \cong \mathcal{H}_{\mathrm{GW}}$ for all Borcea-Voisin orbifolds except those for which $6|w_0$, by constructing birational models for them. There are three such Fermat cases, where $(w_0, w_1, w_2, w_3) = (6, 3, 2, 1), (6, 4, 1, 1), (12, 8, 3, 1)$. We compute these too, along similar lines to how we found the state space for $(3, 1, 1, 1)$. The argument for $\mathcal{H}_{\mathrm{FJRW,\,GW}}$ and $\mathcal{H}_{\mathrm{GW,FJRW}}$ follow identically.

Consider the possible restrictions of W to fixpoint sets of elements of G which have non-zero-dimensional G-invariant Milnor ring. They must satisfy the following properties: that X^2 appears if and only if x^2 appears (since σ must be preserved), and that if X^2 (resp. x^2) appears, then other terms in Y, Z (resp. y, z, w) must appear (from the invariance under J_1, resp. J_2). Splitting the coordinates X, Y, Z from x, y, z, w and finding the corresponding fixpoint spaces, we list the possibilities these conditions leave and tabulate their contributions below. This gives a total dimension of 112. Checking the group elements of degree 2 with non-trivial restricted Milnor rings, we find they contribute dimension 9, so from the symmetries of the FJRW bi-degree with respect to inverses and swapping indices, we find the following diamond

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 9 & & 0 & \\
1 & & 45 & & 45 & & 1 \\
 & 0 & & 9 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

which is exactly the Hodge diamond on the Gromov-Witten side.

Table 6.1: Group elements fixing polynomial parts for K3 weights (6, 3, 2, 1)

W'	$\#\{g \in G \vert W\vert_{\mathrm{Fix}(g)} = W'\}$	$\dim(\mathcal{H}^g_{\mathrm{FJRW}})$
$X^2 + Y^4 + Z^4 + x^2 + y^4 + z^6 + w^{12}$	1	30
$X^2 + Y^4 + Z^4 + x^2 + y^4$	2	2
$X^2 + Y^4 + Z^4 + x^2 + z^6$	1	2
$Y^4 + Z^4 + y^4 + z^6 + w^{12}$	1	42
$Y^4 + Z^4 + y^4$	2	0
$Y^4 + Z^4 + z^6$	1	0
$Y^4 + Z^4$	2	3
$y^4 + z^6 + w^{12}$	1	14
$-$ (narrow)	14	1

Table 6.2: Group elements fixing polynomial parts for K3 weights (6, 4, 1, 1)

W'	$\#\{g \in G \vert W\vert_{\mathrm{Fix}(g)} = W'\}$	$\dim(\mathcal{H}^g_{\mathrm{FJRW}})$
$X^2 + Y^4 + Z^4 + x^2 + y^3 + z^{12} + w^{12}$	1	40
$X^2 + Y^4 + Z^4 + x^2 + y^3$	1	0
y^3	4	0
$Y^4 + Z^4 + y^3 + z^{12} + w^{12}$	1	60
$Y^4 + Z^4$	4	3
$y^3 + z^{12} + w^{12}$	1	20
- (narrow)	12	1

For $X^2+Y^4+Z^4+x^2+y^3+z^{12}+w^{12}$, we have the following. There are 11 sectors of degree 2, and we have the same symmetries from the degree formula. The FJRW diamond and the Hodge diamond on the Gromov-Witten side are both found to be as follows.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 11 & & 0 & \\
1 & & 59 & & 59 & & 1 \\
 & 0 & & 11 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Finally, for $X^2+Y^4+Z^4+x^2+y^3+z^8+w^{24}$, we have the following. There are 19 sectors of degree 2, and we have the same symmetries from the degree formula. Again, the FJRW

Table 6.3: Group elements fixing polynomial parts for K3 weights (12, 8, 3, 1)

W'	$\#\{g \in G \mid W\vert_{\mathrm{Fix}(g)} = W'\}$	$\dim(\mathcal{H}^g_{\mathrm{FJRW}})$
$X^2 + Y^4 + Z^4 + x^2 + y^3 + z^8 + w^{24}$	1	28
$X^2 + Y^4 + Z^4 + x^2 + y^3$	3	0
$X^2 + Y^4 + Z^4 + x^2 + z^8$	2	2
$Y^4 + Z^4 + y^3 + z^8 + w^{24}$	1	42
$Y^4 + Z^4 + y^3$	3	0
$Y^4 + Z^4 + z^8$	2	0
$Y^4 + Z^4$	6	3
$y^3 + z^8 + w^{24}$	1	14
y^3	6	0
z^8	2	0
- (narrow)	22	1

diamond and Hodge diamond on the Gromov-Witten side are both found to be as follows.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 19 & & 0 & \\
1 & & 43 & & 43 & & 1 \\
 & 0 & & 19 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The procedure and results are identical for $X^2+Y^3+Z^6$ in place of $X^2+Y^4+Z^4$ (though the narrow subspaces are larger, as computed previously).

The narrow, ambient and narrow mixed state spaces all certainly isomorphic as graded vector spaces with pairing, as has been made clear by computing explicit bases in the previous three sections.

6.2 The Quantum LG/CY Correspondence

Here we relate the I-functions of the Gromov-Witten, mixed, and FJRW theories. All I-functions in this section are taken to be the genus zero, narrow/ambient I-functions.

Proposition 6.2.1. ***(A partial Landau-Ginzburg/Calabi-Yau correspondence for Fermat-type Borcea-Voisin orbifolds, with respect to the elliptic curve)*** *Let $\mathcal{Y}$ be a Borcea-Voisin* *orbifold* $[E \times K/\mathbb{Z}_2]$ *given by* $E = \{X^2 + Y^4 + Z^4 = 0\}$, *and* K *a Fermat-type K3 surface with Nikulin involution, with* $1 \in \mathbb{Z}_2$ *acting by sending* $(X, x) \mapsto (-X, -x)$. *Then there exists an analytic continuation* ${I'}^{\mathcal{Y}}_{\mathrm{GW}}$ *of* $I^{\mathcal{Y}}_{\mathrm{GW}}$ *and a symplectic transformation* $\mathbb{U}_E : H^{(W_1,[K/\mathbb{Z}_2],G)}_{\mathrm{FJRW},\,\mathrm{GW}} \to H_{\mathrm{GW}}$ *sending* $I^{(W,[K/\mathbb{Z}_2],G)}_{\mathrm{FJRW},\,\mathrm{GW}}$ *to* ${I'}^{\mathcal{Y}}_{\mathrm{GW}}$.

Proof. It is clear that the GW and FJRW state spaces are isomorphic as graded inner product spaces, as are $\mathcal{H}^{\mathrm{amb}}_{\mathrm{GW}}$ and $\mathcal{H}^{\mathrm{nar}}_{\mathrm{FJRW}}$.

By convention, we set the Novikov variables $q_1, q_2, q_3 = 1$ and vary $\tilde{q}_1 = e^{2t_1+t_2+t_3}$, $\tilde{q}_2 = e^{3t_4+t_5+t_6+t_7}$, $\tilde{q}_3 = e^{t_1+t_4+2t_8}t_\sigma$. There are two fundamental issues here that must be addressed before performing the calculation itself.

Firstly there is the issue of convergence in these variables. Applying a ratio test to $I_{\mathrm{GW}}(\mathcal{Y})$ for a, holding b, c fixed, we get a radius of convergence $\tilde{q}_1 < \frac{1}{4^3}$. Similarly for b we find convergence for $\tilde{q}_2 < \frac{1}{6^4}$, and for c, k_j we have convergence for $|\tilde{q}_3|, |x_i| < 1$, $i = 1, \ldots, m$. Each term separates its dependence on a and b into separate factors, so we have convergence when all of these conditions hold.

We shall analytically continue in the variables $\tilde{q}_1$ (corresponding to a and the hyperplane divisor class in the elliptic curve factor), via the Mellin-Barnes method.

The function $\frac{1}{e^{2\pi i w}-1}$ has only simple poles at the integers, at each of which it has residue 1. Varying a then, and suppressing the dependence on $c, \mathbf{k}$ and z, we may write our I-function as a $\mathcal{H}_{\mathrm{GW}}[[z^{-1}]]$-linear combination of contour integrals of the form

$$\int_{C_E} \frac{1}{e^{2\pi i s_E} - 1} F(s_E)\, \mathrm{d}s_E.$$

where the cohomology classes are taken to be complex variables, and the contours are taken to be any curve in the s_E-plane, stretching from $i\infty$ to $-i\infty$ with a detour with all singularities of the Gamma functions appearing in F to the left, and another detour so that all non-negative integers are to the left, and all positive integers are to the right. The picture below illustrates this.

Then closing the curve on the left gives gives the corresponding term of I_{GW}, and we consider the integral upon closing the curve to the right. Here, we need to sum over all the negative integers; but from the functional equation of the Gamma function these are all multiples of $D_E^2 = 0$ and so these residues are zero. We are left with the residues at the s_E for which

$$4D_E/z + 4s_E + 2c + \sum_{j=l+1}^{m} k_j = -m,\ m \in \mathbb{Z}_{>0},$$

that is, $s_E = -\frac{m}{4} - \frac{c}{2} - \frac{1}{4}\sum_{\mu=1}^{m} k_\mu s_{\mu,1} - D_E/z$. Note that $\sum_{\mu=1}^{m} k_\mu s_{\mu,1} = -\frac{1}{2}\sum_{j=l+1}^{m} k_j$.

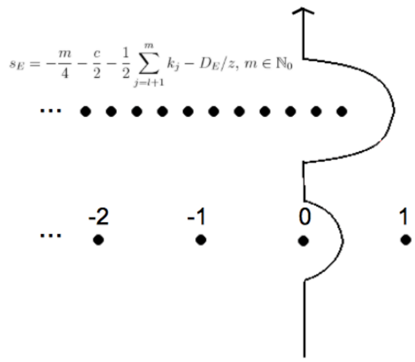

Figure 6.1: Contour of integration for Mellin-Barnes analytic continuation

The residues of the Gamma function are well known, and we have

$$\mathrm{Res}_{s_E=-\frac{m}{4}-\frac{c}{2}+\frac{1}{4}\sum_{j=l+1}^{m}k_j-D_E/z}\Gamma(2D_E/z+4a+2c+\sum_{j=l+1}^{m}k_j+1)=-\frac{1}{4}\frac{(-1)^m}{\Gamma(m)}.$$

The factor $1/(e^{2\pi i s_E}-1)$ is analytic at these points, but must be multiplied by a factor of $2\pi i$ from the residue theorem.

We may then rewrite the sum over a as a sum over m, with

$$-\frac{1}{4}\frac{(2\pi i)e^{2\pi i(\frac{m}{4}+\frac{c}{2}+\frac{1}{4}\sum_{j=l+1}^{m}k_j)}}{e^{-2\pi i D_E/z}-e^{2\pi i(\frac{m}{4}+\frac{c}{2}-\frac{1}{4}\sum_{j=l+1}^{m}k_j)}}\frac{(-1)^m}{\Gamma(m)\Gamma(1-\frac{m}{2})\Gamma(1-\frac{m}{4}-\frac{c}{2}-\frac{1}{4}\sum_{j=l+1}^{m}k_j)^2}$$

replacing all factors in the sum involving a, and the exponentials involving t_1, t_2, t_3 replaced by $e^{-\frac{m}{4}(2t_1+t_2+t_3)}e^{-\frac{c}{2}(t_2+t_3)}$.

Using the identity $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}$, we may put this all together as

$$-\frac{1}{4}(-1)^m\frac{\Gamma(\frac{m}{2})\sin(\frac{\pi m}{2})\Gamma(\frac{m}{4}+\frac{c}{2}+\frac{1}{4}\sum_{j=l+1}^{m}k_j)^2\sin(\pi(\frac{m}{4}+\frac{c}{2}+\frac{1}{4}\sum_{j=l+1}^{m}k_j))^2}{\pi^3}.$$

Clearly this is zero when m is even. Let $\mathbf{k}'=(k_{l+1},\ldots,k_m)$. Define

$$\mu := m \bmod 4, m = 4l_m + \mu$$
$$\tilde{\sigma} = (m + 2c + \sum_{j=l+1}^{m} k_j) \bmod 4.$$

We split the factors of the form $\sin(\pi x)/\pi$ in terms of Gamma functions again, and collect those other factors that depend only on $\mu, \tilde{\sigma}$ into $E_1(\mu, \tilde{\sigma})$:

$$-\frac{1}{4}\sum_{\substack{\mu=1,3\\ \tilde{\sigma}=1,2,3}} E_1(\mu,\tilde{\sigma})\frac{(-1)^{\mu}}{\Gamma(1-\frac{\mu}{2})\Gamma(1-\frac{\tilde{\sigma}}{4})^2}$$

$$\sum_{\substack{l_m,\in\mathbb{N}_0\\ m:=4l_m+\mu,\, c\in\frac{1}{2}\mathbb{N}_0,\,\mathbf{k}'\in\mathbb{N}_0^{m-l}\\ (m+2c+\sum_{j=l+1}^{m}k_j \bmod 4)=\tilde{\sigma}}} \frac{(-1)^{l_m}}{\Gamma(\frac{\mu}{2})\Gamma(\frac{\tilde{\sigma}}{4})^2}\frac{\Gamma(\frac{m}{2})\Gamma(\frac{m}{4}+\frac{c}{2}\sum_{j=l+1}^{m}k_j)^2}{\Gamma(m)}$$

Note that the terms where μ is even and $\tilde{\sigma}=0$ are zero.

Now applying the identity $\Gamma(\frac{m}{2})/\Gamma(m) = 2\sqrt{\pi}/(2^m\Gamma(\frac{m}{2}+\frac{1}{2}))$ we may rewrite this as

$$\begin{aligned}
&-\frac{\sqrt{\pi}}{2}\sum_{\substack{\mu=1,3\\ \tilde{\sigma}=1,2,3}} K_{\mathbf{b}_{\mu,\tilde{\sigma}}}E_1(\mu,\tilde{\sigma})\frac{(-1)^{\mu}}{\Gamma(1-\frac{\mu}{2})\Gamma(1-\frac{\tilde{\sigma}}{4})^2}e^{-\frac{m}{4}D_E(2t_1+t_2+t_3)}\\
&\times \sum_{\substack{l_m,\in\mathbb{N}_0\\ m:=4l_m+\mu,\\ c\in\frac{1}{2}\mathbb{N}_0,b\geq -c/3,\mathbf{k}\in\mathbb{N}_0^{m-l}\\ (m+2c+\sum_{j=l+1}^{m}k_j)\bmod 4=\tilde{\sigma}\\ v_K^S(b,c,\mathbf{k})=\mathbf{b}}} \frac{(-1)^{l_m}}{\Gamma(\frac{\mu}{2})\Gamma(\frac{\tilde{\sigma}}{4})^2}\frac{\Gamma(\frac{m}{2})\Gamma(\frac{m}{4}+\frac{c}{2}\sum_{j=l+1}^{m}k_j)^2}{2^m\Gamma(\frac{m}{2}+\frac{1}{2})\Gamma(2c+1)}\\
&\times L_{\mathbf{b}}(q_2^b q_3^c\prod_{j=1}^{m}x_j^{k_j})e^{b(\sum_{i=1}^{n}w_it_i)+c(t_4+2t_8)}G(b,c,\mathbf{k}')\mathbf{1}_{\mathbf{b}}.
\end{aligned}$$

where $\mathbf{b}_{\mu,\tilde{\sigma}}$ is $\mathbf{1}_0$ for $\tilde{\sigma}=\frac{1}{4},\frac{3}{4}$ and $\mathbf{1}_\sigma$ for $\tilde{\sigma}=\frac{1}{2}$.

For any $g\in\langle J_2\rangle$, factorise $\mathbf{1}_{\sigma g}$ as $\mathbf{1}_{\sigma_E}\mathbf{1}_{\sigma_K g}$. Identify the generators

$$\begin{aligned}
\phi_{J_1} &\mapsto \mathbf{1}_0,\\
\phi_{J_1^3} &\mapsto D_E,\\
\phi_{\sigma_E J_1^2} &\mapsto \mathbf{1}_{\sigma_E},
\end{aligned}$$

and

$$\begin{aligned}
2n_1+1 &\mapsto m,\\
n_\sigma &\mapsto 2c+\sum_{j=l+1}^{m}k_j.
\end{aligned}$$

Then if we set $\tilde{t}_1=\frac{1}{2}\tilde{q_1}^{-\frac{1}{4}}$, the interior sum above is exactly $\frac{1}{2}e^{-z}\omega_{n_3,n_\sigma,\mathbf{b}}^{\mathrm{FJRW,GW}}$. This gives a linear map

$$\mathbb{U}_E : \mathcal{V}_{\mathrm{FJRW,\,GW}} \to \mathcal{V}_{\mathrm{GW}}$$

sending $I_{\mathrm{FJRW,\,GW}}$ to I'_{GW}.

To represent this as a matrix, note that for each $\mathbf{b} \in \langle J_2 \rangle$, the map restricts to a 2×2 matrix between bases $\{\phi_{J_1}\mathbf{1}_b, \phi_{J_1^3}\mathbf{b}\}$ and $\{\mathbf{1}_b, D_E\mathbf{1}_b\}$. Each term in ϕ_h is given by $\tilde{\sigma} = \Theta_2(h)$, and both possibilities for μ appear. We must expand $K_{\mathbf{b}_{\mu,\tilde{\sigma}}}$ and $E(\mu, \tilde{\sigma})$ in terms of D_E/z. We do this with the following Taylor series, which we only need up to linear order for now, since $D_E^2 = 0$:

- $\Gamma(1 + x) = 1 - \gamma x$.
- $1/\Gamma(1 + x) = 1 + \gamma x$.
- $\Gamma(\frac{1}{2} + x) = \sqrt{\pi} - \sqrt{\pi}(2 \ln 2 + \gamma)x$.
- $\frac{1}{e^{-x} - k} = \frac{1}{1-k} + \frac{1}{(1-k)^2}x$,

where γ is the Euler-Mascheroni constant.

Let $\xi = e^{2\pi i\tilde{\sigma}}$. For given h, the corresponding $\tilde{\sigma}$ may be given by $\Theta_2(h)$. Then we have for $k = 1, 3$

$$\begin{aligned}\phi_{J_1^{\tilde{\sigma}}}\mathbf{1}_b \mapsto + \frac{2\sqrt{\pi}}{4}\frac{1}{2}(2\pi i)e^{-z}z^{\frac{\tilde{\sigma}-1}{2}}\Bigg(&\frac{\xi}{1-\xi}\sum_{\mu=1,3}\frac{1}{\Gamma(1-\frac{\mu}{2})\Gamma(1-\frac{\tilde{\sigma}}{4})^2} \\ &+\Big(\frac{2\pi i\xi}{(1-\xi)^2} + \frac{1}{(1-\xi)}(-2\gamma - \gamma - \gamma + 4\gamma)\Big)\sum_{\mu=1,3}\frac{1}{\Gamma(1-\frac{\mu}{2})\Gamma(1-\frac{\tilde{\sigma}}{4})^2}D_E/z\Bigg).\end{aligned}$$

The constant in front reduces to $\frac{\pi}{4}$ since

$$\frac{1}{\Gamma(\frac{1}{2})} + \frac{1}{\Gamma(-\frac{1}{2})} = \frac{1}{2\sqrt{\pi}}.$$

For the part corresponding to $\phi_{J_1}\mathbf{1}_\mathbf{b}, \phi_{J_1^3}\mathbf{1}_\mathbf{b}$, we get a (2×2)-matrix. After scaling, and using the identity $\Gamma(\frac{3}{4}) = \frac{\sqrt{2}\pi}{\Gamma(\frac{1}{4})}$, direct computation shows this to be symplectic and degree-preserving for $\deg z = 2$.

For $\phi_{\sigma_E}\mathbf{1}_{\sigma g}$, g possibly the identity, we get a (1×1)-matrix, since $D_E\mathbf{1}_\sigma = 0$, with entry simplifying drastically to $f(z) := -ie^{-z}$. $f(z)f^*(-z) = 1$, so this is also symplectic, and is clearly degree-preserving. Since it is the direct sum of symplectic matrices, the whole matrix $\mathbb{U}_E$ is also symplectic. $\square$

Remark 6.2.1.1. The situation is more complex for the other elliptic curve, $E = \{X^2 + Y^3 + Z^6 = 0\}$, since we may not so directly express this in the basis $\omega_{h,\mathbf{b}}^{\mathrm{FJRW,\,GW}}$ by the procedure above. This is because we cannot simply identify the appearances of $n_{J_1^3} i_k(J_1^3)$ with those of $2n_{J_1^3} q_k$ in $I_{\mathrm{FJRW,\,GW}}(\mathbf{t}, z)$ as before, since if $\Theta_k(h) = 2q_k + q_k$ then h cannot be narrow, since 3 appears as an exponent of Y.

Proposition 6.2.2. ***(A partial Landau-Ginzburg/Calabi-Yau correspondence for a Fermat-type Borcea-Voisin orbifolds, with respect to the K3 surface)*** *Let $\mathcal{Y}$ be the Borcea-Voisin orbifold $[E \times K/\mathbb{Z}_2]$ given by $E = \{X^2 + Y^4 + Z^4 = 0\}$, and $K = \{x^2 + y^6 + z^6 + w^6 = 0\}$, with $1 \in \mathbb{Z}_2$ acting by sending $(X, x) \mapsto (-X, -x)$. Then there exists an analytic continuation $I'^{\mathcal{Y}}_{\mathrm{GW}}$ of $I^{\mathcal{Y}}_{\mathrm{GW}}$ and a symplectic transformation $\mathbb{U}_K : H_{\mathrm{GW,\,FJRW}} \to H_{\mathrm{GW}}$ sending $I_{\mathrm{GW\,FJRW}}$ to $I'^{\mathcal{Y}}_{\mathrm{GW}}$.*

Proof. We shall analytically continue $I'_{\mathrm{GW}}(\mathbf{t}, z)$ from $\tilde{q}_2 = 0$ to $\tilde{q}_2 = 1$ as above, varying b to s_K and making the substitution

$$s_K = -\frac{n}{2w_0} - \frac{c}{w_0} - \frac{\sum_{j=1}^{m} k_j s_{j,4}}{w_0}$$

and obtain factors of the form

$$-\frac{1}{6} \sum_{\substack{\nu=1,3,5\\ \tilde{\sigma}=1,2,3}} E_2(\nu, \tilde{\sigma}) \frac{(-1)^{\nu}}{\Gamma(1-\frac{\nu}{2})\Gamma(1-\frac{\tilde{\sigma}}{4})^2}$$
$$\sum_{\substack{l_n, \in \mathbb{N}_0\\ n:=6l_n+\nu,\, c\in\frac{1}{2}\mathbb{N}_0\\ (n+2c) \bmod 4 = \tilde{\sigma}}} \frac{(-1)^{l_n}}{\Gamma(\frac{\nu}{2})\Gamma(\frac{\tilde{\sigma}}{4})^3} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{6}+\frac{c}{2})^3}{\Gamma(n)}$$

Let $\nu = n \bmod 6, \tilde{\sigma} = c \bmod 4$, and define $E_2(\nu, \tilde{\sigma})$ similarly to $E_1(\mu, \tilde{\sigma})$ above. After entirely similar manipulations, we obtain

$$-\frac{\sqrt{\pi}}{3} \sum_{\substack{\nu=1,3\\ \tilde{\sigma}=1,2,3}} L_{\mathbf{b}_{\nu,\tilde{\sigma}}} E_2(\nu, \tilde{\sigma}) \frac{(-1)^{\nu}}{\Gamma(1-\frac{\nu}{2})\Gamma(1-\frac{\tilde{\sigma}}{4})^3} e^{-\frac{n}{6} D_K(3t_4+t_5+t_6+t_7)}$$
$$\times \sum_{\substack{l_n, \in \mathbb{N}_0\\ n:=6l_n+\nu,\\ c\in\frac{1}{2}\mathbb{N}_0, a\geq -c/2\\ (n+2c) \bmod 6 = \tilde{\sigma}\\ v_E^S(a,c)=\mathbf{b}}} \frac{(-1)^{l_n}}{\Gamma(\frac{\nu}{2})\Gamma(\frac{\tilde{\sigma}}{4})^3} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{6}+\frac{c}{2})^2}{2^n \Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(2c+1)}$$
$$\times K_{\mathbf{b}} q_2^a q_3^c e^{a(2t_1+t_2+t_3)+c(t_4+2t_8)} F(b, c, \mathbf{k}') \mathbf{1}_{\mathbf{b}}.$$

Again, there is a factor of $\sin(\frac{n}{2}\pi)$ so only terms with odd ν appear.

If we identify the generators

$$\begin{aligned}
\phi_{J_2} &\mapsto \mathbf{1}_0, \\
\phi_{J_2^3} &\mapsto D_K, \\
\phi_{\sigma_K J_2^2} &\mapsto \mathbf{1}_{\sigma_K},
\end{aligned}$$

for the classes of degree 2, and the numbers

$$\begin{aligned}
2n_2 + 1 &\mapsto n, \\
n_\sigma &\mapsto 2c,
\end{aligned}$$

and set $\tilde{t}_2 = \frac{1}{2}\tilde{q}_2^{-\frac{1}{6}}$, the interior sum above is exactly $\frac{1}{2}e^{-z}\omega_{n_3,n_\sigma,\mathbf{b}}^{\mathrm{GW,\,FJRW}}$. This gives a linear map

$$\mathbb{U}_K : \mathcal{V}_{\mathrm{GW,\,FJRW}} \to \mathcal{V}_{\mathrm{GW}}$$

sending $I_{\mathrm{GW,FJRW}}$ to I'_{GW}. This time, we expand the analogous factors in D_K/z up to *quadratic* order:

- $\Gamma(1+x) = 1 - \gamma x + \frac{1}{2}(\gamma^2 + \frac{\pi^2}{6})$.
- $1/\Gamma(1+x) = 1 + \gamma x + \frac{1}{2}(\gamma^2 - \frac{\pi^2}{6})$.
- $\Gamma(\frac{1}{2}+x) = \sqrt{\pi} - \sqrt{\pi}(2\ln 2 + \gamma)x(\frac{1}{2}\pi^{\frac{5}{2}} + \sqrt{\pi}(2\ln 2+\gamma)^2)x^2$.
- $\frac{k}{e^{-x}-k} = \frac{k}{1-k} + \frac{k}{(1-k)^2}x + \frac{k(1+k)}{(1-k)^3}x^2$.

This allows us to write $\mathbb{U}_K$ as a direct sum of (3×3) blocks and (1×1) blocks, which may be verified to be symplectic just as before. Therefore, $\mathbb{U}_K$ is symplectic. $\square$

Remark 6.2.2.1. The case for $E = \{X^2 + Y^3 + Z^6 = 0\}$ proceeds entirely similarly, with extra terms coming from the sectors $\mathbf{1}_{\sigma g}$.

Theorem 6.2.3. ***(Two-parameter Landau-Ginzburg/Calabi-Yau correspondence for a Fermat-type Borcea-Voisin orbifold)*** *Let $\mathcal{Y}$ be the Borcea-Voisin orbifold $[E \times K/\mathbb{Z}_2]$ given by $E = \{X^2+Y^4+Z^4 = 0\}$, and $K = \{x^2+y^6+z^6+w^6 = 0\}$, with $1 \in \mathbb{Z}_2$ acting by sending $(X,x) \mapsto (-X,-x)$. Then there exists a two-parameter analytic continuation $I''^{\mathcal{Y}}_{\mathrm{GW}}$ of $I^{\mathcal{Y}}_{\mathrm{GW}}$ and a symplectic transformation $\mathbb{U} : H^{(W,G)}_{\mathrm{FJRW}} \to H_{\mathrm{GW}}$ sending $I^{(W,G)}_{\mathrm{FJRW}}$ to $I''^{\mathcal{Y}}_{\mathrm{GW}}$.*

Proof. We analytically continue $I_{\mathrm{GW}}(\mathbf{t}, z)$ with respect to $\tilde{q}_1$ and $\tilde{q}_2$ as in the previous two theorems. After precisely the same termwise manipulations, we find that $I''_{\mathrm{GW}}(\mathbf{t}, z)$ is given by

$$\frac{\pi}{6}\Big(K_0 L_0 \sum_{\substack{\mu=1,3\\ \nu=1,3,5\\ \tilde{\sigma}\in\mathbb{Z}_6}} + K_\sigma L_\sigma \sum_{\substack{\mu=1,3\\ \nu=1,3,5\\ \tilde{\sigma}\in\frac{1}{2}\mathbb{Z}_6\setminus\mathbb{Z}_6}}\Big)\frac{(-1)^{\mu+\nu}E_1(\mu,\tilde{\sigma})E_2(\nu,\tilde{\sigma})}{\Gamma(1-\frac{\mu}{2})\Gamma(1-(\frac{\mu}{4}+\frac{\tilde{\sigma}}{2}))^2\Gamma(1-\frac{\nu}{2})\Gamma(1-(\frac{\nu}{6}+\frac{\tilde{\sigma}}{2}))^3}$$

$$\times \sum_{\substack{l_m,l_n,l_c\geq 0\\ m:=4l_m+\mu\\ n:=6l_n+\nu\\ c:=4l_c+\tilde{\sigma}}} \frac{1}{2^m 2^n}\,\frac{(-1)^{l_m+l_n+l_c}\tilde{q}_1^{-m/4}\tilde{q}_2^{-n/6}\tilde{q}_3^{c}\Gamma(\frac{m}{4}+\frac{c}{2})^2\Gamma(\frac{n}{6}+\frac{c}{3})^3}{\Gamma(\frac{\mu}{2})\Gamma(\frac{\mu}{4}+\frac{\tilde{\sigma}}{2})^2\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu}{6}+\frac{\tilde{c}}{2})^3\Gamma(\frac{m}{2}+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(1+2c)}.$$

If we identify

$$\begin{aligned}
&t_{J_1^3 J_2} \mapsto \tilde{q}_1^{-1/4},\\
&t_{J_1 J_2^3} \mapsto \tilde{q}_2^{-1/6},\\
&t_{\sigma J_1^2 J_2^2} \mapsto \tilde{q}_3,\\
&2\mathbf{n}(J_1^3 J_2)+1 \mapsto m,\\
&2\mathbf{n}(J_1 J_2^3)+1 \mapsto n,\\
&n_{\sigma_K J_1^2} \mapsto 2c,
\end{aligned}$$

for the classes of degree 2, then the interior sum is exactly $\frac{1}{4}\omega^h_{\mathrm{FJRW}}$. This allows us to set up a linear map $\mathbb{U}$ as before, put together from $\mathbb{U}_E$ and $\mathbb{U}_K$. That is, splitting the FJRW basis vectors into their E- and K-parts as we have for the other two theories, this gives us $\mathbb{U}$ as a direct sum of (3×3) blocks (for $\phi_{J_2}, \phi_{J_2^3}, \phi_{J_2^5}$) and (2×2) blocks (for $\phi_{\sigma_K J_2^2}$ and $\phi_{\sigma_K J_2^4}$), which can all be checked to be symplectic. Therefore, $\mathbb{U}$ is also symplectic, and this verifies the Landau-Ginzburg/Calabi-Yau correspondence in our case. $\square$

CHAPTER 7

The Picard-Fuchs Systems

So far the I-functions have been treated solely as functions related to the J-functions by a homological map. It remains to justify this nomenclature by showing that the I-functions of $\mathcal{Y}$ compiles solutions of the Picard-Fuchs equations of $\hat{\mathcal{Y}}$. In this section we find the Picard-Fuchs systems.

7.1 The Variation of Hodge Structures

All computations in this section requiring Groebner bases were performed with the aid of Macaulay2. We work with the hypersurface birational model $\overline{\mathcal{Y}}$. For all cases, there are σ-invariant quasi-homogeneous monomials of the same degree as the defining polynomials of degree d given by

$$m_E = Y^2Z^2, \quad m_K = y^2z^2w^2, \quad m_\sigma = YZyzw.$$

More generally, there may be p extra independent monomials m_i', depending on $f(Y,Z)$ and $g(y,z,w)$. A suitable variation of Hodge structures is given by the polynomial family

$$\mathcal{Q}_{\psi,\varphi,\chi,\varphi_1,\dots,\varphi_k} = f(Y,Z) + g(y,z,w) + \psi Y^2Z^2 + \varphi y^2z^2w^2 + \chi YZyzw + \sum_{i=1}^{p} \varphi_i m_i'.$$

We seek homogeneous relations between these monomials modulo $J_{\mathcal{Q}}$, in degrees $2d,\ 3d$ and $4d$.

7.2 The Involution Equation

For all Borcea-Voisin manifolds for which the twist map is well defined, we have the 'involution relation' $m_E m_K = m_\sigma^2$, which may be thought of as characterising the involution.

This cedes one Picard-Fuchs equation, which we shall call the *involution equation*:

$$\partial_\psi \partial_\varphi f = \partial_\chi^2 f.$$

This equation holds for the period integrals of every Borcea-Voisin threefold for which a twist map exists.

7.3 3-parameter Picard-Fuchs equations for X(1, 1, 1)

We find the Picard-Fuchs equations for $\overline{\mathcal{Y}}$ when $E = \{X^2 + Y^4 + Z^4 = 0\}$ and $K = \{x^2 + y^6 + z^6 + w^6 = 0\}$, so that under the twist map

$$\mathcal{Q}_{\psi,\varphi,\chi} = Y^4 + Z^4 + y^6 + z^6 + w^6 + \psi Y^2 Z^2 + \varphi y^2 z^2 w^2 + \chi YZyzw.$$

We consider the monomials m_E, m_K, m_σ. We must find a complete generating set of independent homogeneous relations between them, up to degree $4d$. First, all calculations must be done modulo the Jacobian ideal $J_{\mathcal{Q}_\psi}$. A Groebner basis for the Jacobian is found, and then all calculations are performed via the normal form. (This Groebner basis calculation is the chief inhibitor in terms of computation time, especially when the defining equations have degrees higher than those examples dealt with in this paper.)

Degree 2d. There are 6 products of the m_E, m_K, m_σ of degree $2d$. By Poincaré duality they span a 3-dimensional subspace of the local ring. There are therefore 3 independent relations. One is of course provided by the involution relation $R_1 := -m_\sigma^2 - m_E m_K$. To find the other two, we first express the monomials in normal form with respect to the Gröbner basis of $J_{\mathcal{Q}}$, and reduce the resulting coefficient matrix. After a convenient choice of minors, we find the relations

$$\begin{aligned} R_2 &:= 4(4-\psi^2)m_E^2 + 4\psi\chi m_E m_\sigma + \chi^2 m_\sigma^2 = 0, \\ R_3 &:= 4(4-\psi^2)m_E m_\sigma + 4\psi\chi m_\sigma^2 + \chi^2 m_K m_\sigma = 0. \end{aligned}$$

The relation for m_E derived from the Dwork-Griffiths method may also be derived from these.

Degree 3d. There are 10 distinct products of m_E, m_K, m_σ of degree $3d$. By Poincaré duality, we require 9 independent relations to cut this down to rank 1; by multiplying the 3 degree-2 relations by m_E, m_K, m_σ we get 9 relations. Unfortunately, they are not all independent. This can be seen fairly quickly by noting that $m_3 R_2 - m_1 R_3 = 4(4 - \psi^2) m_3 R_1$. By some manipulation and factorisation, we can express all other monomials in terms of $m_E m_K^2, m_K^3, m_E m_K m_\sigma$ which are themselves subject to the further relations

$$\begin{aligned}
R_4 := {} & \chi(128\psi^3\varphi^3 - 48\psi^2\varphi^2\chi^2 + \chi^6 - 512\psi\varphi^3 + 192\varphi^2\chi^2 + 3456\psi^3 - 13824\psi)m_E m_K^2 \\
& + 4(32\psi^4\varphi^3 - 6\psi^2\varphi\chi^4 + \psi\chi^6 - 256\psi^2\varphi^3 + 24\varphi\chi^4 + 864\psi^4 + 512\varphi^3 \\
& - 6912\psi^2 + 13824)m_E m_K m_\sigma = 0, \\
R_5 := {} & 32(\varphi+3)(\varphi^2 - 3\varphi + 9)(32\psi^4\varphi^3 - 6\psi^2\varphi\chi^4 + \psi\chi^6 - 256\psi^2\varphi^3 + 24\varphi\chi^4 + 864\psi^4 \\
& + 512\varphi^3 - 6912\psi^2 + 13824)m_K^3 - \chi^2(1536\psi^4\varphi^4 - 1024\psi^3\varphi^3\chi^2 + 240\psi^2\varphi^2\chi^4 \\
& - 24\psi\varphi\chi^6 + \chi^8 - 12288\psi^2\varphi^4 + 4096\psi\varphi^3\chi^2 - 192\varphi^2\chi^4 - 20736\psi^4\varphi + 3456\psi^3\chi^2 \\
& + 24576\varphi^4 + 165888\psi^2\varphi - 13824\psi\chi^2 - 331776\varphi)m_E m_K^2 = 0.
\end{aligned}$$

Similarly to before, the relation for m_K derived from the Griffiths-Dwork method can be derived from these. All of these relations together may be checked to have rank 9, as required.

Degree 4d. We multiply all of the degree-$3d$ relations by m_E, m_K, m_σ to find 27 relations; the coefficient matrix of these relations in the product monomials of m_E, m_K, m_σ is of rank 15, which is the number of possible product monomials of degree $4d$. Therefore this is a complete set of monomials and the local ring is indeed 0 in all degrees higher than $3d$, and we have no further relations.

Therefore, the full set of Picard-Fuchs operators for the 3-parameter family is given by the following:

(i) $\partial_\chi^2 - \partial_\psi\partial_\varphi$,

(ii) $4(4-\psi^2)\partial_\psi^2 + 4\psi\chi\partial_\psi\partial_\chi + \chi^2\partial_\chi^2$,

(iii) $4(4-\psi^2)\partial_\psi\partial_\chi + 4\psi\chi\partial_\chi^2 + \chi^2\partial_\varphi\partial_\chi$,

(iv) $\chi(128\psi^3\varphi^3 - 48\psi^2\varphi^2\chi^2 + \chi^6 - 512\psi\varphi^3 + 192\varphi^2\chi^2 + 3456\psi^3 - 13824\psi)\partial_\psi\partial_\varphi^2$
$+ 4(32\psi^4\varphi^3 - 6\psi^2\varphi\chi^4 + \psi\chi^6 - 256\psi^2\varphi^3 + 24\varphi\chi^4 + 864\psi^4 + 512\varphi^3$
$- 6912\psi^2 + 13824)\partial_\psi\partial_\varphi\partial_\chi$,

(v) $32(\varphi+3)(\varphi^2 - 3\varphi + 9)(32\psi^4\varphi^3 - 6\psi^2\varphi\chi^4 + \psi\chi^6 - 256\psi^2\varphi^3 + 24\varphi\chi^4 + 864\psi^4$
$+ 512\varphi^3 - 6912\psi^2 + 13824)\partial_\varphi^3 - \chi^2(1536\psi^4\varphi^4 - 1024\psi^3\varphi^3\chi^2 + 240\psi^2\varphi^2\chi^4$
$- 24\psi\varphi\chi^6 + \chi^8 - 12288\psi^2\varphi^4 + 4096\psi\varphi^3\chi^2 - 192\varphi^2\chi^4 - 20736\psi^4\varphi$
$+ 3456\psi^3\chi^2 + 24576\varphi^4 + 165888\psi^2\varphi - 13824\psi\chi^2 - 331776\varphi)\partial_\psi\partial_\varphi^2$.

7.4 1-parameter Picard-Fuchs equations for X(14, 4, 0)

For the one-parameter families given by a given invariant monomial m_i, the Dwork-Griffiths method may be used to find linear relations between

$$\omega_i^{(r)} = \frac{(-1)^r (r-1)! \psi_i^r m_i^{r-1} \Omega}{Q^r}.$$

Varying m_E to give the family $Y^3 + Z^6 + y^5 + z^5 + w^{10} + \psi Y^2 Z^2$, we find the 2nd-order Picard-Fuchs operator

$$\text{(vi)} \quad (4\psi^3 + 27)\partial_\psi^2 + (-16\psi^3 + 54)\partial_\psi - 2\psi^3.$$

Varying m_K to give the family $Y^3 + Z^6 + y^5 + z^5 + w^{10} + \varphi y^2 z^2 w^2$, we find the 3th-order Picard-Fuchs operator

$$\text{(vii)} \quad (16\varphi^5 + 3125)\partial_\varphi^3 + (-216\varphi^5 + 28125)\partial_\varphi^2 - 312\varphi^5\partial_\varphi - 12\varphi^5.$$

Varying m_σ to give the family $Y^3 + Z^6 + y^5 + z^5 + w^{10} + \chi Y Z y z w$, we find the 4th-order Picard-Fuchs operator

$$\begin{aligned}\text{(viii)} \quad & (\chi^{30} - 11210083593750000000)\partial_\chi^4 - 40(\chi^{30} + 5605041796875000000)\partial_\chi^3 \\ & - 300(\chi^{10} - 6075000)(\chi^{20} + 6075000\chi^{10} + 36905625000000)\partial_\chi^2 - 360\chi^{30}\partial_\chi - 24\chi^{30}.\end{aligned}$$

7.5 Picard-Fuchs equations derived from other multiplication relations

There is one extra class of Picard-Fuchs equations which will not be difficult to check. Varying by other σ-invariant monomials corresponding to different Chen-Ruan classes, there are several other simple relations coming from multiplicative identities. For example, in degree 6, if we set $m_1' = y^6, m_2' = z^6, m_3' = y^3z^3$, we have the relation $m_1'm_2' = {m_3'}^2$, giving the Picard-Fuchs equation $\partial_{\varphi_1}\partial_{\varphi_2} = \partial_{\varphi_3}^2$.

7.6 Quantum Mirror Symmetry

We show that the I-functions already shown to transform to the Gromov-Witten J-functions under a mirror map compile solutions to the Picard-Fuchs equations, that is, the period integrals of their respective families. The equations are given in terms of $\psi, \phi, \chi, \varphi_i$. It will be easier to change the variables in the I-function than the Picard-Fuchs equations, by performing the same analytic continuation, as follows.

After analytic continuation in a we set

$$m = -4D_E/z - 4a - 2c,$$

so that we can rewrite the factors involving a in a sum over m as

$$\frac{-\frac{\pi i}{2} e^{2\pi i(\frac{m}{4}+\frac{c}{2})}}{e^{-2\pi i D_E/z} - e^{2\pi i(\frac{m}{4}+\frac{c}{2})}} \frac{(-1)^m}{\Gamma(m)\Gamma(1-\frac{m}{2})\Gamma(1-\frac{m}{4}-\frac{c}{2})^2}.$$

The identity $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$ allows us to write the second fraction as

$$(-1)^m \frac{\Gamma(\frac{m}{2})\sin(\frac{\pi m}{2})\Gamma(\frac{m}{4}+\frac{c}{2})^2 \sin(\pi(\frac{m}{4}+\frac{c}{2}))^2}{\pi^3}.$$

Similarly for the factors involving b, we set $n := -2w_0 D_K/z - 2w_0 b - 2c$, and we similarly rewrite these as

$$\frac{-\frac{\pi i}{2} e^{2\pi i(\frac{n}{2w_0}+\frac{c}{2})}}{e^{-2\pi i D_K/z} - e^{2\pi i(\frac{n}{2w_0}+\frac{c}{2})}} \frac{(-1)^n}{\Gamma(n)\Gamma(1-\frac{n}{2})\prod_{i=1}^{3}\Gamma(1-\frac{w_i n}{2w_0}-\frac{c}{2})}.$$

Finally, we make the substitution

$$\psi = \tilde{q}_E^2, \quad \varphi = \tilde{q}_K^{w_0}, \quad \chi = \tilde{q}_\sigma^{1/2}.$$

Proposition 7.6.1. *$I_{\mathcal{Y}}(\psi, \varphi, \chi)$ satisfies the involution equation, for all $\mathcal{Y}$ for which a twist map exists.*

Proof. Consider the coefficient of $\tilde{q}_E^{\frac{m}{2}} \tilde{q}_K^{\frac{n}{w_0}} \tilde{q}_\sigma^{2c}$ in $\partial_\chi^2 I_{\mathcal{Y}}(\psi, \varphi, \chi)$. To retrieve the corresponding coefficient in $\partial_\psi \partial_\varphi I_{\mathcal{Y}}(\psi, \varphi, \chi)$ we multiply by $\frac{m}{2}\frac{n}{w_0}$, divide by $(2c+1)(2c+2)$, and perform the shift

$$2c \mapsto 2(c+1), \quad \frac{m}{2} \mapsto \frac{m}{2} - 1, \quad \frac{n}{w_0} \mapsto \frac{n}{w_0} - 1.$$

Note that $\frac{m}{4} + \frac{c}{2}$ and $\frac{n}{2w_0} + \frac{c}{2}$ are preserved under this transformation. Absorbing the extra factors via the identity $x\Gamma(x) = \Gamma(x+1)$, we see that these coefficients are equal. □

Theorem 7.6.2 (A quantum mirror theorem for X(1,1,1))**.** *$I_{X(19,11)}(\psi, \varphi, \chi)$ solves the Picard-Fuchs equations for the three parameter family containing the twisted birational model of $X(1,1,1)$ parametrised by ψ, φ, χ.*

Proof. For $I_{X(19,1,1)}(\psi, \varphi, \chi)$, we have $(w_0, w_1, w_2, w_3) = (25, 10, 8, 7)$. We must check the equations for operators (i)-(v), term by term. Equation (i) has already been shown. Via the identity $\xi^2\partial_\xi^2 = (\xi\partial_\xi)^2 - \xi\partial_\xi$, operator (ii) may be rewritten

$$16\partial_\psi^2 - 4(\psi\partial_\psi)^2 + 4\psi\partial_\psi + 4(\psi\partial_\psi)(\chi\partial_\chi) + (\chi\partial_\chi)^2 - \chi\partial_\chi.$$

Noting that $\psi\partial_\psi$ and $\chi\partial_\chi$ have the effect of term-wise multiplication by $\frac{m}{2}$ and $2c$ respectively, we may compare the coefficients of $16\partial_\psi^2 I_{\mathcal{Y}}(\psi, \varphi, \chi)$ and

$$(4(\psi\partial_\psi)^2 - 4\psi\partial_\psi - 4(\psi\partial_\psi)(\chi\partial_\chi) - (\chi\partial_\chi)^2 + \chi\partial_\chi)I_{\mathcal{Y}}(\psi, \varphi, \chi).$$

The equation corresponding to operator (iii) holds similarly. (Note that entirely similar arguments for the above equations hold for any Borcea-Voisin threefold with a twist map, where E has weights $(2, 1, 1)$.)

After some labour, it is straightforward to check equations (iv) and (v) in the same way. With dimension considerations, $I_{X(19,1,1)}(\psi, \varphi, \chi)$ compiles these solutions. $\square$

Remark 7.6.2.1. Earlier we split the I-function into terms corresponding to $m \bmod 4$ and $n \bmod 6$. These separate terms are in fact themselves solutions compiling separate period integrals.

Theorem 7.6.3 (A quantum mirror theorem for 1-parameter families containing X(6, 4, 0))**.** *The I-functions $I_{X(6,4,0)}(\psi, 0, 0)$, $I_{X(6,4,0)}(0, \varphi, 0)$ and $I_{X(6,4,0)}(0, 0, \chi)$ satisfy the Picard Fuchs equations for the three 1-parameter families containing the twisted birational model of $X(14, 4, 0)$ parametrised by ψ, φ, χ respectively.*

Proof. In this case $(w_0, w_1, w_2, w_3) = (5, 2, 2, 1)$. Similarly to the proofs of the previous two propositions, it can be checked that $I_{X(6,4,0)}(\psi, 0, 0), I_{X(6,4,0)}(0, \varphi, 0)$ and $I_{X(6,4,0)}(0, 0, \chi)$ lie in the kernels of operators (vi), (vii), (viii) respectively. $\square$

Remark 7.6.3.1. In general, the I-function found earlier for each new extra ambient sector included corresponding to the monomial m'_j, the I-function acquires a factor of the form $x_j^{k_j}/k!$, and all occurrences of $\frac{c}{2}$ in (1) are replaced by $\frac{c}{2} + \frac{1}{2}k_j s_{j,4}$, where $s_{j,4}$ comes from the representation of that sector in the toric stacky structure. For any Picard-Fuchs equation coming from a product relation as in §5.4, the definition of $s_{j,4}$ implies that cancellation in fact occurs in exactly the same way as for the involution equation, and that this further equation holds.

Remark 7.6.3.2. With more processing power than that at our disposal, it should be possible to use the same procedure to find full sets of Picard-Fuchs equations for *all* Borcea-Voisin threefolds with twisted birational models and verify the quantum mirror theorem in genus zero for *all* ambient sectors. Another method of attack lies in the GKZ systems of [20].

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