Gromov-Witten invariants of symmetric products of projective space

by

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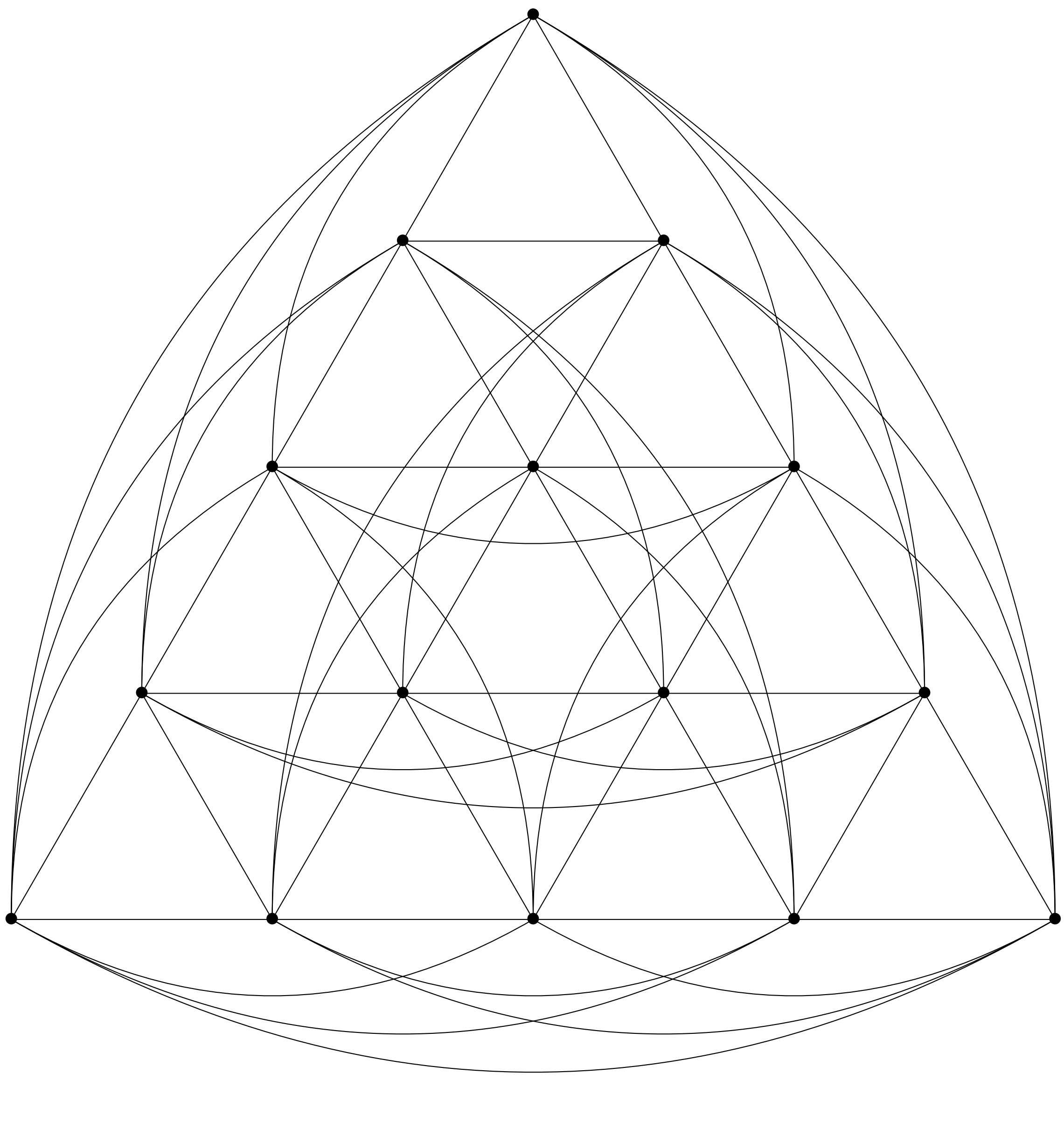

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My thesis is dedicated to the amazing Rohini Ramadas. It is impossible to describe the ways in which Rohini has contributed to this thesis and my life over the last six years. More than anyone else, Rohini has taught me how to learn math, explain math, enjoy math, do research, and navigate academia. Rohini is the center of my life and my perfect partner. I am lucky.

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ABSTRACT

Gromov-Witten invariants are numbers that roughly count curves of a fixed type on an algebraic variety X. For example, for 3 general points and 6 general lines in $X = \mathbb{P}^3$, there are exactly 190 twisted cubics intersecting all of them, so 190 is a Gromov-Witten invariant of $\mathbb{P}^3$. Gromov-Witten invariants appear in algebraic geometry and string theory. In the special case when X is a toric variety, Kontsevich found a method to compute any Gromov-Witten invariant of X. Givental and Lian-Liu-Yau used Kontsevichs algorithm to prove a mirror theorem, which states that Gromov-Witten invariants of X have an interesting rigid structure predicted by physicists. The main result of this thesis is a mirror theorem for the nontoric orbifold $X = \mathrm{Sym}^d(\mathbb{P}^r)$, the symmetric product of projective space, which parametrizes unordered d-tuples of points in $\mathbb{P}^r$.

CHAPTER 1

Introduction

Over the last 20 years, following predictions from string theory [10], mathematicians have proven a series of results known as *mirror theorems*. These theorems reveal elegant patterns and structures embedded in the collection of (usually genus-zero) Gromov-Witten invariants, or counts of curves satisfying geometric constraints, of a fixed target variety or orbifold X. They also allow for indirect computation of these invariants in many cases where direct computation is combinatorially difficult. However, the scope of these results is essentially limited to the world of toric geometry; in every case, X is either a complete intersection in a toric stack, or a stack admitting a toric degeneration.

The goal of this thesis is to develop tools for mirror symmetry outside of toric geometry. The main theorem (Theorem 6.1.2) is a genus-zero mirror theorem for the orbifold $\mathrm{Sym}^d \mathbb{P}^r$ parametrizing unordered d-tuples of points in projective r-space. Because $\mathrm{Sym}^d \mathbb{P}^r$ is nontoric, and further nonabelian, this result is not possible using existing techniques. Theorem 6.1.2 gives an explicit formula for a generating function of Gromov-Witten invariants of $\mathrm{Sym}^d \mathbb{P}^r$. This is the only known mirror theorem for a nonabelian orbifold, besides single-point orbifolds $[\bullet/G]$.

1.1 Enumerating curves

From an algebro-geometric perspective, Gromov-Witten stems from enumerative geometry of curves in varieties, motivated by questions like these[1]:

1. How many plane conic curves pass through five general specified points?

2. How many twisted cubic curves in 3-space pass through three general specified points and intersect six general specified lines?

3. How many lines are contained in a general cubic surface in 3-space?

[1] Here "plane" refers to $\mathbb{CP}^2$ and "3-space" refers to $\mathbb{CP}^3$. We work over $\mathbb{C}$ throughout.

Classically, the answers are 1, 190, and 27 respectively. A natural kind of question generalizing these is the following:

Question 1.1.1. Fix[2] a smooth complex projective variety X, a genus $g \in \mathbb{Z}_{\geq 0}$, a homology class $\beta \in H_2(X, \mathbb{Z})$, and a list of cohomology classes $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{Z})$. How many smooth genus g curves in X of homology class β intersect generic subvarieties $V_1, \ldots, V_n$ representing the classes $\gamma_1, \ldots, \gamma_n$?

Note that in the case $X = \mathbb{P}^2$, $g = 0$, $\beta = 2[L]$, $n = 5$, and $\gamma_1 = \cdots = \gamma_5 = [pt]$, Question 1.1.1 is precisely Question (1) above, and in the case $X = \mathbb{P}^3$, $g = 0$, $\beta = 3[L]$, $n = 9$, $\gamma_1 = \gamma_2 = \gamma_3 = [pt]$, and $\gamma_4 = \cdots = \gamma_9 = [L]$ it is precisely Question 2. For a general cubic surface $X = S$, with $g = 0$, $\beta = [L]$, and $n = 0$, Question 1.1.1 is the same as question 3.

In practice, these questions are usually very difficult to answer. In Gromov-Witten theory, the objects of study are not the correct answers, but certain natural incorrect answers called *Gromov-Witten invariants*.

In detail, the counting problem can be set up as an intersection theory problem on a moduli space $\mathcal{M}_g(X, \beta)$ parametrizing genus g curves of degree β in X, as follows. The condition that a curve intersects V_i defines a closed subvariety $\mathcal{M}_i \subseteq \mathcal{M}_g(X, \beta)$, where

$$\mathcal{M}_i = \{C \in \mathcal{M}_g(X, \beta) : C \cap V_i \neq \emptyset\}.$$

The intersection $\bigcap_{i=1}^n \mathcal{M}_i$ is the set of genus g, degree d curves in X that intersect all of the subvarieties $V_1, \ldots, V_n$. Therefore the number of points in this intersection is the correct answer to Question 1.1.1.

A strategy to compute this number is the following. $\mathcal{M}_g(X, \beta)$ is not compact. We compactify $\mathcal{M}_g(X, \beta)$ to get a space $\overline{\mathcal{M}}_g(X, \beta)$, and we obtain compactified subvarieties $\overline{\mathcal{M}}_i \subseteq \overline{\mathcal{M}}_g(X, \beta)$ such that $\overline{\mathcal{M}}_i \cap \mathcal{M}_g(X, \beta) = \mathcal{M}_i$. We calculate the class of the intersection $\bigcap_{i=1}^n \overline{\mathcal{M}}_i$ using the intersection product $\prod_{i=1}^n [\overline{\mathcal{M}}_i]$ in the cohomology ring of $\overline{\mathcal{M}}_g(X, \beta)$. The answer is some multiple of the class of a point, and this number is a Gromov-Witten invariant, denoted $\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,\beta}^X$. The three motivational questions correspond to the Gromov-Witten invariants:

$$\langle [pt], [pt], [pt], [pt], [pt] \rangle_{0,5,2[L]}^{\mathbb{P}^2}, \quad \langle [pt], [pt], [pt], [L], [L], [L], [L], [L], [L] \rangle_{0,9,3[L]}^{\mathbb{P}^3}, \text{ and } \quad \langle \rangle_{0,0,[L]}^S.$$

The reason that $\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,\beta}^X$ may be an incorrect[3] answer to Question 1.1.1 is that even if the subspaces $\overline{\mathcal{M}}_i$ intersect transversely for general choices of V_i, the intersection $\bigcap_{i=1}^n \overline{\mathcal{M}}_i$ may

[2] To be precise, this is not exactly a generalization of the question (3), since (3) works with a *generic* cubic surface, whereas here we have fixed a variety X. Making sense of this discrepancy (i.e. making the answer invariant under deformations of X) leads to the "virtual" intersection theory discussed in Section 2.4.

[3] Indeed, $\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,\beta}^X$ is a rational number, not even an integer. This is because $\overline{\mathcal{M}}_g(X, \beta)$ is an Deligne-Mumford stack/orbifold rather than a variety.

contain points of the boundary $\overline{\mathcal{M}}_g(X,\beta) \smallsetminus \mathcal{M}_g(X,\beta)$.

The compactification $\overline{\mathcal{M}}_g(X,\beta)$ and the marked versions $\overline{\mathcal{M}}_{g,n}(X,\beta)$ are introduced in Section 2.4; they are spaces of *maps* $(f : C \to X)$ from (possibly singular) curves to X, called *stable maps*. (The interior $\mathcal{M}_g(X,\beta) \subseteq \overline{\mathcal{M}}_g(X,\beta)$ consists of *embeddings* of *smooth* curves, i.e. actual smooth curves contained in X.) In fact, the spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ are not really compactifications, in that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ may contain irreducible components other than the Zariski closure of $\mathcal{M}_{g,n}(X,\beta)$. Similarly, the subvariety $\overline{\mathcal{M}}_i$ may be larger than the Zariski closure of $\mathcal{M}_i$.

In this thesis, the main case of interest is $X = \mathrm{Sym}^d \mathbb{P}^r$, which is not a smooth projective variety; it is a smooth complex orbifold. The entire discussion above works with little modification (see Section 2.4) for X an orbifold.

1.2 Mirror theorems

A mirror theorem for a smooth variety or orbifold X states that a generating function $\mathbf{f}$ of Gromov-Witten invariants of X is equal to an explicit analytic function I_X (an "I-function"). Such a statement asserts that there is a rigid structure to the various Gromov-Witten invariants of X. Because a Gromov-Witten invariant has many inputs (i.e. the genus g, the class β, and the list of cohomology classes $\gamma_1, \dots, \gamma_n$), the generating function $\mathbf{f}$ and the explicit function I will have many variables.

The statement of a mirror theorem usually involves generating functions of *descendant Gromov-Witten invariants*, a generalization of Gromov-Witten invariants. These generating functions are easier to compute with than generating functions of usual Gromov-Witten invariants. A descendant Gromov-Witten invariant has extra inputs $a_1, \dots, a_n$, and is denoted $\langle \psi^{a_1}\gamma_1, \dots, \psi^{a_n}\gamma_n\rangle_{g,n,\beta}$. The symbols $\psi^{a_1}, \dots, \psi^{a_n}$ represent certain cohomology classes $\psi_1^{a_1}, \dots, \psi_n^{a_n} \in H^*(\overline{\mathcal{M}}_{g,n}(X,\beta), \mathbb{Q})$ (see Definition 2.4.3), and the descendant Gromov-Witten invariant is defined as an intersection product with extra factors (see Section 2.5 for the precise definition):

$$\langle \psi^{a_1}\gamma_1, \dots, \psi^{a_n}\gamma_n\rangle_{g,n,\beta} := \prod_{i=1}^{n} [\overline{\mathcal{M}}_i] \cdot \psi_i^{a_i}.$$

Gromov-Witten invariants as discussed in the previous section are the case $a_1 = \cdots = a_n = 0$ of descendant Gromov-Witten invariants. From now on, we use the term "Gromov-Witten invariants" to include descendant Gromov-Witten invariants.

Example 1.2.1. Givental [27] (and Lian-Liu-Yau [45] in slightly different language) proved the following mirror theorem for projective space.

Theorem 1.2.2 (Givental, Lian-Liu-Yau)**.** *Let*

$$J(t,Q,z) := z + tH + \sum_{n,\beta\geq 0}\sum_{i=0}^{r}\frac{Q^\beta}{n!}\langle tH,\ldots,tH,\frac{H^i}{z-\psi}\rangle_{0,n,\beta}^{\mathbb{P}^r}H^{r-i}. \tag{1.1}$$

Here the symbol $\frac{1}{z-\psi}$ *means the series expansion* $\sum_{m\geq 0}\frac{\psi^m}{z^{m+1}}$. *The series* $J(t,Q,z)$ *is a cohomology-valued generating function; explicitly,* $J(t,Q,z)\in H^*(\mathbb{P}^r,\mathbb{Q})[[Q]]((z^{-1}))$. *Then*

$$J(t,Q,z) = I(t,Q,z) := z\sum_{\beta\geq 0}\frac{e^{t(H/z+\beta)}Q^\beta}{\prod_{\gamma=1}^{\beta}(H+\gamma z)^{r+1}}. \tag{1.2}$$

We should note that this example (unlike the next one, Example 1.2.3) is not useful from the point of view enumerating curves in $\mathbb{P}^r$, for two reasons. First, the cohomology classes tH appearing in the coefficients of $J(t,Q,z)$ mean that we are counting curves that intersect a fixed hyperplane — but in projective space, every curve meets every hyperplane, so this is no restriction at all. Second, almost all of the coefficients are descendant Gromov-Witten invariants, which do not have a clear enumerative significance.

In fact, this theorem can be significantly generalized; the equality (1.2) is obtained from the equality on the bottom of page 20 in [14] by setting many variables equal to zero. The generating function in that generalization *does* have enumerative significance, since every nondescendant Gromov-Witten invariant appears as a coefficient. (For example, the number 190 of twisted cubics intersecting 6 general lines and 3 general points in space is a coefficient, when $r=3$.) We did not state it here because the statement requires the formalism of the Givental cone, which we discuss next.

Usually, mirror theorems cannot be stated as simply as Theorem 1.2.2; instead, they are usually stated using a framework introduced by Givental to encode genus-zero Gromov-Witten invariants of X. Givental's insight was to focus on a special class of generating functions, those of the form:

$$\mathbf{f} := -z + \mathbf{t}(t,Q,z) + \sum_{\substack{n\geq 0,\beta\text{ effective}\\ \beta\neq 0\text{ or }n\geq 2}}\frac{Q^\beta}{n!}\langle\mathbf{t}(t,Q,\psi),\ldots,\mathbf{t}(t,Q,\psi),\frac{\gamma_\phi}{-z-\psi}\rangle_{0,n,\beta}^{X}\gamma^\phi. \tag{1.3}$$

Some explanation is required. The formal variable Q keeps track of the degree $\beta\in H_2(X,\mathbb{Z})$. Here $\mathbf{t}(t,Q,z)$ is a polynomial in z with coefficients in $H^*(X,\mathbb{Q})[[t,Q]]$. Note that the coefficients are well-defined (sums of) (descendant) Gromov-Witten invariants, where $\mathbf{t}(t,Q,\psi)$ means that ψ is formally substituted into $\mathbf{t}(t,Q,z)$. In Theorem 1.2.2, $J(t,Q,-z)$ is of the form (1.3), with $\mathbf{t}(t,Q,z)=tH$.

Givental gave a geometric interpretation of generating functions of the form (1.3). Namely, there is a (germ of an) infinite-dimensional scheme $\mathcal{L}_X$, called the *Givental cone* of X (see Section 2.6), such that generating functions $\mathbf{f}$ as in (1.3) are in natural bijection with $\mathbb{C}[[t,Q]]$-valued points of $\mathcal{L}_X$. Using this setup, the typical statement of a mirror theorem is of the form, "the following explicit function $I_X(t,Q,-z) \in H^*(X,\mathbb{Q})[[t,Q]]((z^{-1}))$ is a $\mathbb{C}[[t,Q]]$-valued point of the Givental cone $\mathcal{L}_X$." One may then write $I_X(t,Q,-z)$ as a generating function of Gromov-Witten invariants of the form (1.3), where $-z+\mathbf{t}(t,Q,z)$ is the part of $I_X(t,Q,-z)$ with nonnegative powers of z in the Laurent expansion of $I_X(t,Q,-z)$ in z^{-1}.

Example 1.2.3. Givental [28] and Lian-Liu-Yau [44] also proved a mirror theorem for the general quintic threefold $\mathscr{Q} \subseteq \mathbb{P}^4$. Stating this theorem requires the language of the Givental cone.

Theorem 1.2.4 (Givental,Lian-Liu-Yau). *Let*

$$I(t,Q,z) := z \sum_{\beta \geq 0} e^{t(H/z+\beta)} Q^\beta \frac{\prod_{\gamma=1}^{5\beta}(5H+\gamma z)}{\prod_{\gamma=1}^{\beta}(H+\gamma z)^5}, \tag{1.4}$$

where H is the pullback of the hyperplane class on $\mathbb{P}^4$. Then $I(t,Q,-z)$ is a $\mathbb{C}[[t,Q]]$-valued point of the Givental cone $\mathcal{L}_{\mathscr{Q}}$.

By the definition of the $\mathcal{L}_{\mathscr{Q}}$, $I(t,Q,-z)$ is equal to a generating function of Gromov-Witten invariants, but in this example it is difficult to say which one. In order to find out, we need to identify the terms with nonnegative power of z, in the Laurent expansion of $I(t,Q,-z)$ in z^{-1}. Let $-z+\mathbf{t}(t,Q,z) \in H^*(\mathscr{Q},\mathbb{Q})[[t,Q,z]]$ be the sum of all such terms. We calculate the coefficient of Q in $\mathbf{t}(t,Q,z)$ as follows. The first terms of $I(t,Q,z)$ are (using $H^4=0$):

$$\begin{aligned} I(t,Q,z) &= ze^{tH/z} + zQe^{t(H/z+1)}\frac{(5H+z)(5H+2z)(5H+3z)(5H+4z)(5H+5z)}{(H+z)^5} + O(Q^2) \\ &= ze^{tH/z} + zQe^{t(H/z+1)}\frac{10625H^3z+5625H^2z^2+1370Hz^3+120z^4}{(H+z)^4} + O(Q^2) \end{aligned}$$

Writing $\frac{1}{(H+z)^5} = \frac{1}{z^5} - \frac{5H}{z^6} + 15\frac{H^2}{z^7} - 35\frac{H^3}{z^8}$, and $e^{t(H/z+1)} = e^t(1+tH/z+O(z^{-2}))$ we have

$$I(t,Q,-z) = -z + tH + Qe^t(-120z + (770+120t)H) + O(z^{-1}) + O(Q^2).$$

In other words, $\mathbf{t}(t,Q,z) = tH + Qe^t(-120z+(770+120t)H) + O(Q^2)$. Let $\mathbf{t}_1(t,Q,z) =$

$tH + Qe^t(-120z + (770+120t)H)$. Then by the definition of $\mathcal{L}_{\mathscr{Q}}$, we find

$$
\begin{aligned}
I(t,Q,-z) &= -z + \mathbf{t}_1(t,Q,z) \\
&\quad + \sum_{\substack{n,\beta\geq 0 \\ \beta>0 \text{ or } n\geq 2}} \sum_{i=0}^{3} \frac{Q^\beta}{n!} \langle \mathbf{t}_1(t,Q,\psi),\ldots,\mathbf{t}_1(t,Q,\psi), \frac{H^i}{-z-\psi}\rangle^{\mathscr{Q}}_{0,n+1,\beta} \frac{H^{3-i}}{5} \\
&\quad + O(Q^2).
\end{aligned} \tag{1.5}
$$

Repeating this process with the Q^2 term of $I(t,Q,-z)$ (and subsequently higher order terms), we can calculate $\mathbf{t}(t,Q,z)$ to arbitrarily high order in Q. That is, we can identify $I(t,Q,-z)$ as a generating function of Gromov-Witten invariants.

The invariants in this generating function do not immediately appear to have enumerative significance, similarly to the last example. However, one can use the fact that $\mathscr{Q}$ is a Calabi-Yau threefold to express any degree β Gromov-Witten invariant of $\mathscr{Q}$ explicitly in terms the invariant $\langle\rangle^{\mathscr{Q}}_{0,0,\beta}$. (This is the Gromov-Witten invariant corresponding to the enumerative question of how many degree β rational curves lie on $\mathscr{Q}$. This is essentially the only enumerative question one can ask about genus zero curves on $\mathscr{Q}$.) The process uses certain recursive identities called the *string equation*, *dilaton equation*, and *divisor equation*, see Section 26.3 of [32]. This allows one to recover all "enumerative" (non-descendant) Gromov-Witten invariants, in particular the invariants $\langle\rangle^{\mathscr{Q}}_{0,0,\beta}$, from Theorem 1.2.4. (This is how Givental and Lian-Liu-Yau verified the predictions of [10].) We illustrate this is Example 2.6.5, which computes $\langle\rangle^{\mathscr{Q}}_{0,0,1} = 2875$ and $\langle\rangle^{\mathscr{Q}}_{0,0,2} = 609250 + 2875/8$.

So far, mirror theorems have been proven in genus zero for:

- Quintic threefolds [28, 44],
- Complete intersections in toric varieties [27, 45],
- Complete intersections in homogeneous spaces [39],
- Toric bundles [8],
- Toric orbifolds/stacks and certain complete intersections, and [12, 18, 17]
- Toric stack bundles [35].

There are also some results in higher genus, see e.g. [57, 43, 25]. The proofs of all of these theorems rely on toric geometry, specifically the fact that toric varieties and orbifolds admit a torus action with isolated fixed points and 1-dimensional orbits. (See Section 1.5 and 4.2.)

1.3 Statement of main theorem

The main result is a mirror theorem for $X = \mathrm{Sym}^d \mathbb{P}^r$, the orbifold parametrizing unordered d-tuples of points of $\mathbb{P}^r$. Using the language of Section 1.2, the statement is as follows.

Theorem 6.1.2. Introduce formal variables Q, z, $\{t_i\}_{0\leq i\leq r}$, and $\{x_\varpi\}_{\varpi\in\mathrm{Part}(d)}$, where $\mathrm{Part}(d)$ is the set of partitions of d. Let

$$I_{\mathrm{Sym}^d\mathbb{P}^r}(t,\mathbf{x},Q,z) := z\sum_{\sigma\in\mathrm{Part}(d)} 1_\sigma \sum_{\beta\geq 0}\exp\left(\sum_{i=0}^r t_i([H_{\sigma,i}]/z+\beta)\right)Q^\beta \sum_{\substack{\mathbb{Z}_{>0}\text{-labels } L=(L_\eta) \text{ of}\\ \text{the parts of } \sigma \text{ with sum } \beta}} \left(\sum_{\substack{\mathbf{k}=(k_\varpi)_{\varpi\in\mathrm{Part}(d)}\\ k_\varpi\geq 0}} H(\sigma\prod_\varpi \varpi^{k_\varpi})\prod_{\varpi\in\mathrm{Part}(d)}\frac{x_\varpi^{k_\varpi}}{z^{k_\varpi}k_\varpi!}\right) \cdot\left(\frac{|S_\sigma|}{|S_{\sigma,L}|}\right)\left(\prod_{\eta\in\sigma}\frac{1}{\prod_{\gamma=1}^{L_\eta}\prod_{i=0}^r\left(H_{\sigma,\eta,i}+\frac{\gamma}{\eta}z\right)}\right). \tag{1.6}$$

Then $I_{\mathrm{Sym}^d\mathbb{P}^r}(t,\mathbf{x},Q,-z)$ is a $\mathbb{C}[[\{t_i\},Q,\{x_\varpi\}]]$-valued point of the Givental cone $\mathcal{L}_{\mathrm{Sym}^d\mathbb{P}^r}$. Here we use the notation:

- $1_\sigma \in H^*_{CR,T}(\mathrm{Sym}^d\mathbb{P}^r,\mathbb{Q})$ is the Chen-Ruan cohomology class of the twisted sector corresponding to the partition σ (see Section 3.2.2),
- $[H_{\sigma,i}]$ and $[H_{\sigma,\eta,i}]$ are hyperplane classes defined in Section 3.1.3,
- $H(\sigma\prod_\varpi \varpi^{k_\varpi})$ is the number of ways of factoring $1\in S_d$ as a product $a_1\cdots a_{1+\sum k_\varpi}$, where the conjugacy classes (i.e. partitions) of the permutations a_j are given by the list $(\sigma\prod_\varpi\varpi^{k_\varpi})$, and
- S_σ and $S_{\sigma,L}$ are automorphism groups, of the partition S_σ and the labeled partition $S_{\sigma,L}$, respectively (see Section 1.7).

As in Example 1.2.3, we may write $I_{\mathrm{Sym}^d\mathbb{P}^r}(t,\mathbf{x},Q,-z)$ as a generating function of Gromov-Witten invariants by calculating the terms $-z+\mathbf{t}(t,\mathbf{x},Q,z)$ of its Laurent expansion in z^{-1} with nonnegative powers of z. We get

$$\mathbf{t}(t,\mathbf{x},Q,z)=\theta := \sum_\sigma\sum_{i=0}^r t_i[H_{\sigma,i}] + \sum_{\varpi\in\mathrm{Part}(d)} x_\varpi 1_\varpi. \tag{1.7}$$

The following corollary is immediate:

Corollary 6.3.1. Let

$$J_{\mathrm{Sym}^d \mathbb{P}^r}(Q,\theta,z) = z + \theta + \sum_{\beta,n} \frac{Q^\beta}{n!} \left\langle \theta, \ldots, \theta, \frac{\gamma_\phi}{z-\overline{\psi}} \right\rangle_{0,n+1,\beta}^{\mathrm{Sym}^d \mathbb{P}^r, T} \gamma^\phi,$$

where θ is as in (1.7). Then $I_{\mathrm{Sym}^d \mathbb{P}^r}(t, \mathbf{x}, Q, z) = J_{\mathrm{Sym}^d \mathbb{P}^r}(t, \mathbf{x}, Q, z)$.

Corollary 6.3.1 allows us to recover some, but not all, non-descendant Gromov-Witten invariants. This is because, when we vary the coefficients t_i and x_ϖ, θ takes values in a proper subspace of $H^*_{CR,T}(\mathrm{Sym}^d \mathbb{P}^r, \mathbb{Q})$. In the future we hope to generalize Theorem 6.1.2 along the lines of Ciocan-Fontanine and Kim's "big" mirror theorem for toric varieties [14]. For such a theorem, there would be no analog of Corollary 6.3.1, since the I-function would contain arbitrarily higher powers of z.

1.4 Motivation: the crepant resolution conjecture

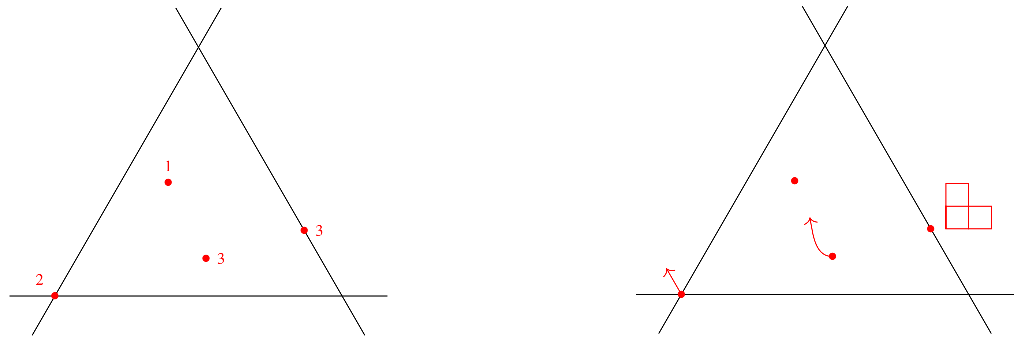
Mirror theorems show that the Gromov-Witten invariants of an orbifold X have *internal* structure, a natural question is how Gromov-Witten invariants of X are related to those of *another* orbifold Y, when there is a geometric relationship between X and Y. We focus on one example of such a relationship.

An orbifold X has an underlying *singular* variety (or algebraic space) $\underline{X}$, called the coarse moduli space of X, with a natural map $X \to \underline{X}$ (see Section 2.3). Since $\underline{X}$ is not smooth, it does not have Gromov-Witten invariants. However, suppose Y is a crepant resolution of singularities of $\underline{X}$, i.e. a resolution such that $K_{\underline{X}}$ pulls back to K_Y :

$$\begin{tikzcd} X \arrow[dr, "coarse\ moduli"'] & & Y \arrow[dl, "crepant"] \\ & \underline{X} & \end{tikzcd}$$

(Such resolutions have long been studied in algebraic geometry, see [52].) Physicists predicted that X and Y are equivalent from the point of view of mirror symmetry. This claim was formulated as a precise mathematical conjecture, the crepant resolution conjecture, by Ruan and others ([53, 9, 20]). The conjecture states that the the Gromov-Witten invariants of X are related in an explicit way to those of Y: there is a certain reversible transformation that turns a generating function of Gromov-Witten invariants of X into a generating function of Gromov-Witten invariants of Y.

The crepant resolution conjecture has been proved for toric orbifolds X, when Y is a toric crepant resolution [19]. The proof is via the mirror theorem for toric orbifolds [18]; the mirror theorem provides explicit analytic functions I_X and I_Y, which are directly compared. The conjecture has also been proven for the non-proper orbifolds $\mathrm{Sym}^d \mathbb{C}^2$ [9] (whose moduli spaces of stable maps $\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d \mathbb{C}^2, \beta)$ exist only for $\beta = 0$), and by explicit computations for $\mathrm{Sym}^2 \mathbb{P}^2$ [56].



Unordered 9-tuple of points in $\mathbb{P}^2$

Schematic picture of the ideal:
$(x+y, x^2) \cap (yz - (x-2z)^2 + z^2, x^3)$
$\cap (x-z, y-2z) \cap (z^2, z(x-2y), (x-2y)^2)$

Figure 1.1: A point of $\mathrm{Sym}^9 \mathbb{P}^2$ (or $\underline{\mathrm{Sym}^9 \mathbb{P}^2}$) and a point of $\mathrm{Hilb}^{(9)}(\mathbb{P}^2)$ mapping to it

We are interested in the following case of this conjecture. The Hilbert scheme $\mathrm{Hilb}^{(d)}(\mathbb{P}^2)$ of points in $\mathbb{P}^2$ is a smooth scheme that is a moduli space for length d subschemes of $\mathbb{P}^2$. Such a subscheme S is a disjoint union of (at most d) subschemes $S_1, \ldots, S_k$, of lengths $d_1 + \cdots + d_k = d$, concentrated at distinct single points $P_1, \ldots, P_k$. This defines an unordered d-tuple of points of $\mathbb{P}^2$, consisting of d_i copies of P_i for $i = 1, \ldots, k$. (See Figure 1.1.) This does not define a regular map $\mathrm{Hilb}^{(d)}(\mathbb{P}^2) \to \mathrm{Sym}^d \mathbb{P}^2$, but it does define a regular (birational) map $\mathrm{Hilb}^{(d)}(\mathbb{P}^2) \to \underline{\mathrm{Sym}^d \mathbb{P}^2}$, and one can show that this map is a crepant resolution. This gives a diagram

$$\begin{tikzcd}
\mathrm{Sym}^d \mathbb{P}^r \arrow[dr, "coarse\, moduli"'] & & \mathrm{Hilb}^{(d)}(\mathbb{P}^2) \arrow[dl, "crepant"] \\
& \underline{\mathrm{Sym}^d \mathbb{P}^r} &
\end{tikzcd}$$

(More generally, such a resolution exists for symmetric products of any smooth surface. For symmetric products of higher-dimensional varieties X, there is still a map $\mathrm{Hilb}^{(d)}(X) \to \underline{\mathrm{Sym}^d X}$, but is not generally birational, nor is $\mathrm{Hilb}^{(d)}(X)$ smooth.) The goal is to prove the crepant resolution conjecture in this case using the method of [19], but to do so one must prove two mirror theorems; one for $\mathrm{Sym}^d \mathbb{P}^2$ and one for $\mathrm{Hilb}^{(d)}(\mathbb{P}^2)$. The case $r = 2$ of Theorem 6.1.2 is the first of these.

1.5 Proof techniques

There is a tool called Atiyah-Bott torus localization, which can be used to simplify computations of integrals on an orbifold $\mathcal{M}$ when $\mathcal{M}$ admits an action of a torus T (see Chapter 4). Specifically, torus localization expresses any integral over $\mathcal{M}$ as an integral over the T-fixed locus $\mathcal{M}^T$ (See Section 4.1.

In Gromov-Witten theory, if an orbifold X admits a T-action, then so does the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Specifically, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a space of maps to X, and T acts by postcomposition. Since Gromov-Witten invariants are defined as intersection products (integrals) on $\overline{\mathcal{M}}_{g,n}(X,\beta)$, torus localization can be applied. Using the natural $T=(\mathbb{C}^*)^{r+1}$-action on $\mathbb{P}^r$, Kontsevich [41] developed an algorithm to compute any Gromov-Witten invariant of a complete intersection in $\mathbb{P}^r$. The algorithm involves a combinatorial description of the geometry of the T-fixed locus $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,\beta)^T$.

Givental found a recursive structure in Kontsevich's algorithm, which implies that when X is a complete intersection in $\mathbb{P}^r$ (or more generally a smooth projective toric variety), any generating function of the form (1.3) satisfies a certain recursion relation. Furthermore, Givental showed that an element of $H^*(X,\mathbb{Q})[[t,Q]]((z^{-1}))$ is of the form (1.3) *if any only if* it satisfies this recursion. In other words, the recursion relation exactly characterizes $\mathbb{C}[[t,Q]]$-valued points of $\mathcal{L}_X$.

Using predictions from string theory, Givental produced I-functions $I_X(t,Q,z)$, and showed via a direct check that $I_X(t,Q,-z)$ satisfies the recursion relation. This proved Theorem 1.2.4, and after finding the initial terms $\mathtt{t}(t,Q,z)$, proved Theorem 1.2.2.

This method has been streamlined since Givental's proof, by Kim [39], Brown [8] (while proving the mirror theorem for toric bundles), and Coates-Corti-Iritani-Tseng [18] (while proving the mirror theorem for toric orbifolds). The argument in [18] relies on calculations of Johnson [36] and Liu ([46], Section 9) involving toric orbifolds, which are analogous to (but much more subtle than) Kontsevich's calculations in projective space.

We use torus localization to prove Theorem 6.1.2, and our proof has the same overall structure as that of the mirror theorem for toric stacks [18]. We use the natural diagonal action of $T=(\mathbb{C}^*)^{r+1}$ on $\mathrm{Sym}^d\,\mathbb{P}^r$, which as above induces a T-action on the moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta)$ for all g, n, and β (though we need only $g=0$).

We immediately run into one of the main obstacles in the proof of Theorem 6.1.2. Applying Kontsevich's algorithm or Givental's recursive argument requires a description of the fixed locus $\overline{\mathcal{M}}_{0,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta)^T$. (In particular, one must be able to effectively compute integrals over the fixed locus, and be able to compute the normal bundle.) For X a toric orbifold, Kontsevich's description of $\overline{\mathcal{M}}_{0,n}(X,\beta)^T$ was completely combinatorial. This is essentially due to the fact that T-fixed stable maps to X are closely related to 1-dimensional T-orbits in X, and in a toric orbifold (with its dense torus action) these orbits are isolated. However, in $\mathrm{Sym}^d\,\mathbb{P}^r$ with its diagonal $T=(\mathbb{C}^*)^{r+1}$-action, the 1-dimensional T-orbits are not isolated; they come in positive dimensional families. (In other words, one can find points that are very close together, whose T-orbits are 1-dimensional and *distinct*.) In order to carry out the argument, we must first describe each family explicitly (see Theorem 4.5.23). Our description applies in more generality, for example to symmetric products of toric varieties.

In Chapters 4 and 5, we carry out the analog of the Johnson and Liu's toric computations, which are needed to compute the relevant integrals over the fixed loci $\overline{\mathcal{M}}_{0,n}(\operatorname{Sym}^d \mathbb{P}^r, \beta)^T$. Altogether, the description of the T-fixed locus and these calculations allow us to prove the analog of Givental's recursion, which says that an element of $H^*_{CR,T}(\operatorname{Sym}^d \mathbb{P}^r, \mathbb{Q})[[t, Q]]((z^{-1}))$ is a point of $\mathcal{L}_{\operatorname{Sym}^d \mathbb{P}^r}$ if and only if it satisfies a certain recursion (Theorem 5.1.4).

Finally, we show directly, via Tseng's orbifold quantum Riemann-Roch theorem [54], that the series $I_{\operatorname{Sym}^d \mathbb{P}^r}(t, \mathbf{x}, Q, -z)$ also satisfies these recursions.

It is natural to ask where these complicated functions (1.2), (1.4), and (1.6) come from. Besides preditions by physicists, there is recent work of Ciocan-Fontanine, Kim, and others [15, 13, 14, 12] (with origins in Bertram's [7] for $X = \mathbb{P}^r$) showing that in many cases, the I-function can be written in terms of *quasimap invariants* [13]. In particular, the authors produce I-functions for a large class of targets X, including toric orbifolds and certain complete intersections in toric orbifolds. However, we could not make rigorous sense of the method for $X = \operatorname{Sym}^d \mathbb{P}^r$. In Section 6.4, we informally discuss how we found $I_{\operatorname{Sym}^d \mathbb{P}^r}$, and conjecture the existence of a particular moduli space that would allow one to produce a candidate function I_X for quite general orbifolds X.

Finally, we would like to draw the reader's attention to a technical aspect of the recursion in Theorem 5.1.4, condition (**II**). The recursion expresses Laurent coefficients in $\mathbf{f}$ with negative exponents in terms of those with positive exponents. We have not seen this type of recursive structure before — it is a certain generalization of the one in Theorem 41 of [18] — but suspect it will be useful in more generality. We give a more detailed discussion of this point in Section 5.2.

1.6 Organization of the thesis

Section 1.7 sets up notational and combinatorial conventions. Chapter 2 gives background on orbifolds and orbifold Gromov-Witten theory. Chapter 3 defines and gives important properties of symmetric products, especially of projective space. In Chapter 4, we introduce Atiyah-Bott torus localization and virtual localization for moduli spaces of stable maps, and carry out the various calculations required to apply localization to compute Gromov-Witten invariants. Chapter 5 proves the recursive characterization (Theorem 5.1.4) of the Givental cone $\mathcal{L}_{\operatorname{Sym}^d \mathbb{P}^r}$. Finally, Chapter 6 defines the explicit series $I_{\operatorname{Sym}^d \mathbb{P}^r}$, and proves that it lies on $\mathcal{L}_{\operatorname{Sym}^d \mathbb{P}^r}$.

1.7 Notation and conventions

1.7.1 Multisets, partitions, and multipartitions

A *finite multiset* ϖ is an unordered finite collection of elements a (we write $a \in \varpi$), possibly appearing more than once. Multisets are denoted with braces, e.g. $\{a, a, b\}$, and we will clarify if an object is a set or a multiset when necessary. We write $\mathrm{Mult}(\varpi, a)$ for the number of times that a appears in ϖ. If a multiset is indexed by another multiset, we write e.g. $\{a_i\}_{i \in I}$. A *submultiset* $\varpi' \subseteq \varpi$ is a multiset such that for $\mathrm{Mult}(\varpi', a) \leq \mathrm{Mult}(\varpi, a)$ for $a \in \varpi'$. Unions are always taken to be disjoint, e.g. $\{a, a, b\} \cup \{a, c\} = \{a, a, a, b, c\}$. The *cardinality* or *length* of a multiset is the number of elements, including multiplicities. We take sums and products over multisets with multiplicity, e.g. $\sum_{\eta \in \{1,2,2\}} \eta = 5$.

For $d \in \mathbb{Z}_{>0}$, a *partition of* d is a multiset of positive integers whose sum (with multiplicities) is d. The (finite) set of partitions of d is denoted $\mathrm{Part}(d)$. The *ones partition of* d is the multiset $\{1, \ldots, 1\}$ of size d. A *nonnegative ordered partition of* d is an ordered tuple of nonnegative integers whose sum is d. The (finite) set of nonnegative ordered partitions of d of length r is denoted $\mathrm{ZPart}(d, r)$.

If D is a tuple of positive integers, a *multipartition of* D is a tuple $(\varpi_d)_{d \in D}$, with ϖ_d a partition of d. The (finite) set of multipartitions of D is denoted $\mathrm{MultiPart}(D)$. The *ones multipartition of* D is the multipartition of D each of whose elements ϖ_d is the ones partition of d. We think of a multipartition as a multiset, via the forgetful map $\mathrm{MultiPart}(D) \to \mathrm{Part}(\sum_{d \in D} d)$ sending $(\varpi_d)_{d \in D}$ to $\bigcup_{d \in D} \varpi_d$. For example, we write $\eta \in (\varpi_d)_{d \in D}$ to mean that η is a part of some ϖ_d.

If ϖ is a partition, we write S_ϖ for the group of automorphisms of the partition; e.g. for the partition $\varpi = \{1, 1, 1, 2, 2\}$ of 7, we have $S_\varpi \cong S_3 \times S_2$. For $\sigma = (\varpi_d)_{d \in D}$ a multipartition of D, we define $S_\sigma := \prod_{d \in D} S_\varpi$.

Let A be a set, and let $\sigma = (\varpi_d)_{d \in D}$ be a multipartition of D. An *A-labeling L of σ* is an assignment $\{L_\eta\}_{\eta \in \sigma}$ of an element of A to each part η of each ϖ_d. Precisely, it is the data of a tuple $\tilde{\sigma} = (\tilde{\varpi}_d)_{d \in D}$, where $\tilde{\varpi}_d$ is a multiset of pairs (η, a) with $\eta \in \mathbb{Z}_{>0}$ and $a \in A$, such that the tuple $\sigma = (\varpi_d)_{d \in D}$ obtained by forgetting the second entry of each pair in each $\tilde{\varpi}_d$ is equal to σ. We define $S_{\sigma, L}$ to be the subgroup of S_σ of permutations that preserve labels.

1.7.2 Graphs

Let Γ be a graph. We write $V(\Gamma)$ for the set of vertices and $E(\Gamma)$ for the set of edges. A *flag* of Γ is a pair (v, e) with e incident to v; we write $F(\Gamma)$ for the set of flags of Γ. We write $E(\Gamma)_v$ for the set of edges incident to v, and $E(\Gamma)_{v_1, v_2}$ for the set of edges connecting v_1 to v_2.

A map of graphs $\phi : \Gamma \to \Gamma'$ is a pair of maps $\phi_V : V(\Gamma) \to V(\Gamma')$ and $\phi_E : E(\Gamma) \to E(\Gamma')$, such that $\phi_E(E(\Gamma)_{v_1,v_2}) \subseteq E(\Gamma')_{\phi_V(v_1),\phi_V(v_2)}$.

CHAPTER 2

Orbifold Gromov-Witten theory

This chapter is intended as an introduction to the parts of orbifold Gromov-Witten theory that we require. Since this material can be found elsewhere [54, 46, 36, 2], we concentrate on an intuitive elementary discussion of the behaviors of orbifolds/stacks, as compared to varieties. Of course, this material can also be found elsewhere [4].

2.1 Orbifolds and orbifold maps

Question 2.1.1. What does the moduli space $\mathcal{M}$ of unordered pairs of real numbers look like?

By calling $\mathcal{M}$ a moduli space, one thing we mean is that any point of $\mathcal{M}$ corresponds to an unordered pair of real numbers, and that for any unordered pair of real numbers, there is a corresponding point of $\mathcal{M}$. We may visualize $\mathcal{M}$ as follows.

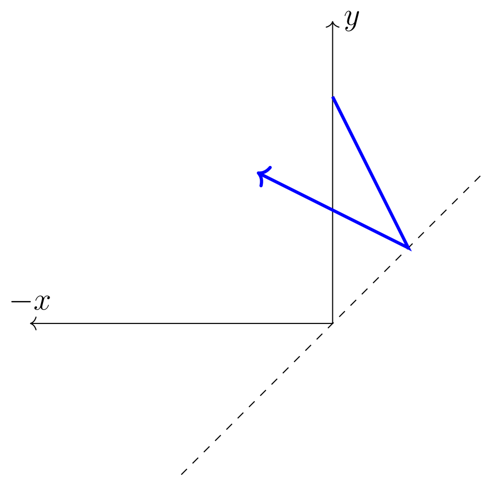
For any unordered pair $\{a, b\}$ of real numbers, we may uniquely put the pair in (weakly) ascending order. In other words, there is a bijective map P from $\mathcal{M}$ to the half-plane $\{(x, y) \in \mathbb{R}^2 : y \geq x\}$.

Is "manifold with boundary" the correct geometric structure on $\mathcal{M}$? We claim that $\mathcal{M}$ comes with extra structure along the line $y = x$. Intuitively, this line behaves as a mirror. To see this, we must consider not only single unordered pairs (points of $\mathcal{M}$), but also 1-dimensional *families* of unordered pairs (paths in $\mathcal{M}$).

Consider the family $\{t, 3 - 2t\}$ of unordered pairs. Via the map P, this defines a path in the half-plane $\{y \geq x\}$: here are some values.

$$\begin{array}{ll} t = 0 & (0, 3) \\ t = 1 & (1, 1) \\ t = 2 & (-1, 2) \end{array}$$

Between $t = 0$ and $t = 2$, here is the path:



This is what we meant by mirror behavior. An equivalent way of saying this is the following. Rather than choosing an ordering, we could simply keep track of both orderings. For a pair $\{a, b\}$, we obtain two (unordered) points of $\mathbb{R}^2$, namely (a, b) and (b, a). These points are permuted by an action of S_2 on $\mathbb{R}^2$, where the nontrivial element σ of S_2 acts by reflecting over the line $x = y$. In other words, $\mathcal{M}$ is in bijection with the *space of orbits* $\mathbb{R}^2/S_2$.

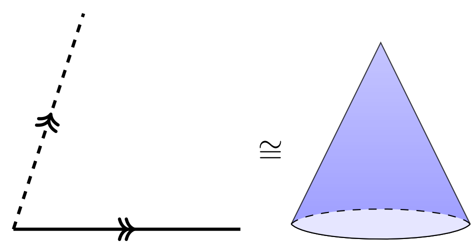
As a topological space, $\mathbb{R}^2/S_2$ is isomorphic to the half-plane $y \geq x$. However, $\mathbb{R}^2/S_2$ has extra structure, giving way to the "mirror" behavior above. In particular, there is a notion of a smooth curve on $\mathbb{R}^2/S_2$; given a smooth curve in $\mathbb{R}^2$, we just take its image under the quotient map. The path above is smooth, since it is the image of a line in $\mathbb{R}^2$.

In general, "well-behaved" moduli spaces behave like the quotient space of a manifold V by a group G acting by automorphisms — at least locally. An *orbifold* is the correct geometric object to capture this behavior. An orbifold has a cover by charts U whose data is a manifold together with a group action, and the transition functions must respect the group actions. (See [4], Section 1.1 for the precise definition.) As an orbifold, $\mathcal{M}$ "remembers" the group S_2, in that any orbifold chart containing a point on the line $y = x$ can be identified with a small neighborhood of that point in $\mathbb{R}^2$, together with the action of S_2 fixing that point. We say that S_2 is the *isotropy group* at that point. In general, for a point x of an orbifold X, we write G_x for the isotropy group at x. If G_x is nontrivial we call x an *orbifold point*.[1]

The simplest orbifolds are quotients of a manifold by a finite group; these are called *global quotients*. We write $X = [V/G]$ for such an orbifold, where V is the manifold and G the group. The main topic of this thesis, the example $\mathrm{Sym}^d \mathbb{P}^r := [(\mathbb{CP}^r)^d/S_d]$, is one of these. Also, any manifold is a global quotient orbifold, taking G to be trivial. In Section 2.2 we will see orbifolds that are not global quotients.

[1] Since G_x depends upon a choice of orbifold chart, it is only defined up to isomorphism.

Example 2.1.2. Consider $[\mathbb{R}^2/(\mathbb{Z}/n/Z)]$, where $a \in \mathbb{Z}/n\mathbb{Z}$ acts by rotation on $\mathbb{R}^2$ by $2\pi a/n$. The result is in bijection with a half-open sector in $\mathbb{R}^2$, with the edges identified. (As a topological space, this isomorphic to $\mathbb{R}^2$, but not as an orbifold.) We may picture $[\mathbb{R}^2/(\mathbb{Z}/n\mathbb{Z})]$ as a cone, with cone angle $2\pi/n$, and isotropy group $\mathbb{Z}/n\mathbb{Z}$ at the cone point:



Note that (unlike in some literature) the group G need not act faithfully. (For a general orbifold, this translates to G_x being nontrivial for every point x.[2]) As an extreme example, one of the orbifolds we will often need to consider is BG, the quotient of a point by the trivial action of the group G. Another example is $X \times BG$ for any orbifold X and finite group G; this is the same as the quotient $[X/G]$ by the trivial action of G.[3] These orbifolds are sometimes called *ineffective orbifolds*, or *G-gerbes,* where G is the isomorphism class of the generic point of X.

Definition 2.1.3. A structure that can be defined on a manifold can usually be defined on an orbifold. We can make sense of the notion of complex (or Riemannian or symplectic) structures, vector bundles/sheaves, tangent spaces, maps of orbifolds, covering maps, fundamental groups, homology and cohomology of orbifolds, etc. For global quotient orbifolds, these all simplify:

- A complex structure on $[V/G]$ is a complex structure on V such that G acts by biholomorphisms.
- A vector bundle E on $[V/G]$ is a G-equivariant vector bundle $\tilde{E}$ on V. (Note: For $v \in V$, G_v acts on the fiber $\tilde{E}_v$. This is called the *monodromy action*.)
- The tangent space at $x \in [V/G]$ is the disjoint union of the tangent spaces at all points of the corresponding G-orbit in V, modulo the derivative of the action of G. This has the structure of a vector space modulo a group acting by linear automorphisms, or equivalently a representation (of G_x).
- A map of orbifolds $f : [V/G] \to [V'/G']$ is a map[4] $\tilde{f} : V \to V'$, together with a map $\phi : G \to G'$, such that $\tilde{f}(g \cdot v) = \phi(g) \cdot \tilde{f}(v)$ for all $v \in V$ and $g \in G$. (Two maps $(\tilde{f}_1, \phi_1)$

[2]Here and elsewhere, we assume X is connected.

[3]It is straightforward to define the quotient of an *orbifold* by a finite group; the result is another orbifold.

[4]There is a subtle technical detail here. It may be possible to write $[V_1/G_1] \cong [V_2/G_2]$, but the set of maps $[V_1/G_1] \to [V'/G']$ is not in bijection with the set of maps $[V_2/G_2] \to [V'/G']$ (in our definition). This is fixed by taking a "common refinement", i.e. a presentation $[V_0/G_0]$ where G_0 contains G_1 and G_2. In general, one defines orbifold maps using *Morita equivalences*, see [4].

and $(\tilde{f}_2, \phi_2)$ are identified if there exists $h \in G'$ such that $\tilde{f}_1(v) = h \cdot \tilde{f}_2(v)$ for all $v \in V$; in this case we have

$$\tilde{f}_2(g \cdot v) = h^{-1}\tilde{f}_1(g \cdot v) = h^{-1}\phi_1(g)\tilde{f}_1(v) = h^{-1}\phi_1(g)h \cdot \tilde{f}_2(v),$$

i.e. ϕ_2 is the postcomposition of ϕ_1 with conjugation by h.) The map $\phi : G \to G'$, which is well-defined up to postcomposition by conjugation, is called the *monodromy map*.

- A covering map $f : [V/G] \to [V'/G']$ of orbifolds is a map such that $\tilde{f} : V \to V'$ is a covering map. In particular, the natural quotient map $V \to [V/G]$ (of orbifolds) given by $\tilde{f} = \mathrm{id}$ is a covering map for any V and G.

- The universal cover $\tilde{V}$ of V is also the universal cover of $[V/G]$, and the (orbifold) fundamental group of $[V/G]$ is the group of deck transformations of $\tilde{V}$ over $[V/G]$.

- The homology and cohomology of $[V/G]$ are canonically isomorphic to the G-equivariant homology and G-equivariant cohomology of V.

The following fact is helpful for computing intersection products: if x, x' are points of a (connected) orbifold X, then $|G_x|\,[x] = |G_{x'}|\,[x'] \in H_0(X, \mathbb{Q})$. In particular, $[x]$ is $\frac{1}{|G_x|}$ times the class of a nonorbifold point of X.

There is an important construction associated to any orbifold X, called its *inertia orbifold* IX. (See Example 2.5 in [4].)

Definition 2.1.4. Let $X = [V/G]$ be a global quotient orbifold. The inertia orbifold $I[V/G]$ is the quotient of $\{(v, g) \in V \times G | g \cdot v = v\}$ by G, where $h \in G$ acts by $h \cdot (v, g) = (h \cdot v, hgh^{-1})$.

For a general orbifold X, this construction glues to form an orbifold IX. As a set, IX consists of pairs $(x, (g))$ with $x \in X$ and (g) a conjugacy class in G_x.

Example 2.1.5. Let $X = BG$, for G a finite group. Then IX is the quotient of G (as a set of points) by itself, acting by conjugation. Points of IX are in bijection with conjugacy classes (g) of G, and the isotropy group of a point (g) is $C_G(g)$. There is a distinguished conjugacy class (e), which corresponds to a point of IX with isotropy group BG (since $C_G(e) = G$). For example, if $G = S_3$, then IX is the disjoint union of three orbifold points $(e) \cong BS_3$, $(\mathrm{tr}) \cong B(\mathbb{Z}/2\mathbb{Z})$, and $(\mathrm{cy}) \cong B(\mathbb{Z}/3\mathbb{Z})$.

Example 2.1.6. Let $X = [\mathbb{R}^2/S_2]$ as above. Then IX is the quotient of $\mathbb{R}^2 \times \{e\} \cup \Delta \times \{\sigma\} \subseteq \mathbb{R}^2 \times S_2$ by S_2, where Δ is the diagonal $\{x = y\}$, and $\sigma \in S_2$ is the transposition. Thus, IX is the union of a copy of X with the quotient $[\Delta/S_2]$, where S_2 acts trivially.

Definition 2.1.7. In general, IX has a copy of X as a connected component, corresponding to pairs $(v, e) \in V \times G$. (These automatically glue in the case where X is not a global quotient.) This connected component is called the *untwisted sector*, and other connected components are called *twisted sectors*.

2.2 Orbifold curves

Besides $\mathrm{Sym}^d \mathbb{P}^r$, the other orbifolds we will use most often are (complex) *orbifold curves*, or *orbifold Riemann surfaces*. These are similar to Riemann surfaces, except that instead of being locally isomorphic to $\mathbb{C}$ with biholomorphic transition functions, they are locally isomorphic to $[\mathbb{C}/\mu_n]$ (with its complex structure) for some n, where μ_n is the group of nth roots of unity acting by multiplication. (Again, the biholomorphic transition functions have to be compatible with the group actions on charts.) The point $0 \in [\mathbb{C}/\mu_n]$ has isotropy group μ_n, and all other points in the chart have trivial isotropy group; this local behavior also appeared in Example 2.1.2.

There is a natural map $[\mathbb{C}/\mu_n] \to \mathbb{C}$ that sends the orbit $\{a, \zeta_n a, \zeta_n^2 a, \ldots, \zeta_n^{n-1} a\}$ to a^n, which implies that there is a natural holomorphic map from an orbifold Riemann surface X to a Riemann surface $\underline{X}$. (This is a special case of the "coarse moduli space" map mentioned in the introduction; see Section 2.3.) We may think of X as being the data of $\underline{X}$ endowed with "extra structure" at finitely many points P, namely the data of the size r_P of the isotropy group at P.

Remark 2.2.1. There is a close relationship between covering maps of orbifold curves and ramified covers of (nonorbifold) curves. Let $f : C \to D$ be a degree d ramified cover of curves, and let $y \in D$. The ramification profile of f at y is a partition $\varpi_f \in \mathrm{Part}(d)$, and for each part η of ϖ_f, there is a point x_η of C mapping to y with local degree η.

We modify C and D. We replace a small neighborhood U of y with a copy of $[\mathbb{C}/\mu_N]$, where $N = \mathrm{lcm}(\varpi_f)$ is the least common multiple $\mathrm{lcm}(\varpi_f)$ of all parts $\eta \in \varpi_f$. For each $x_\eta \in C$, we replace the neighborhood $f^{-1}(U)$ with a copy of $[\mathbb{C}/\mu_{N/\eta}]$, mapping down to $[\mathbb{C}/\mu_N] \subseteq D$ by the pair $(\tilde{f}, \phi)$, with $\tilde{f}(z) = z$ and $\phi(\zeta) = \zeta^{-1}$.

After doing this operation for all ramification points of D, we are left with a canonical covering map of orbifold curves (in the sense of Definition 2.1.3, at least locally). The original map is recovered as the induced map on coarse moduli spaces.

Some orbifold curves are global quotient orbifolds. For example, there is a curve, constructed below, whose coarse moduli space is isomorphic to $\mathbb{P}^1$, with whose only orbifold points are over 0 and ∞, both with isotropy group isomorphic to μ_n. This orbifold curve is the quotient of $\mathbb{P}^1$ by μ_n, where μ_n acts by scaling on $\mathbb{P}^1$. However, if we had introduced only one orbifold point, the result would not be a global quotient. (If it were, then Remark 2.2.1 would assign to the nontrivial covering map $V \to [V/G]$ a ramified cover of $\mathbb{P}^1$, ramified at a single point. No such ramified cover exists.) Similarly, if the isotropy groups at the two orbifold points had different sizes, then the curve would not be a global quotient.

These curves are constructed as follows. The complement $[\mathbb{C}^*/\mu_n]$ of the cone point in $[\mathbb{C}/\mu_n]$ is isomorphic to $\mathbb{C}^*$, via the diagram

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "\tilde{f}"] \arrow[d, "quotient"'] & \mathbb{C}^* \arrow[d, "\mathrm{id}"] \\ {[\mathbb{C}^*/\mu_n]} & \mathbb{C}^* \end{tikzcd}$$

with $\tilde{f} = z \mapsto z^n$ (and $\phi : \mu_n \to 1$ trivial). Therefore we may glue $[\mathbb{C}^*/\mu_n]$ to $[\mathbb{C}^*/\mu_m]$ by identifying the respective copies of $\mathbb{C}^*$ via the isomorphism $z \mapsto z^{-1}$.

2.3 Orbifolds and stacks

We will need to consider singular objects, so we use the algebraic analog of orbifolds: Deligne-Mumford stacks. For the remainder of the thesis, we use the terms "orbifold" and "smooth Deligne-Mumford stack" interchangeably. Working with stacks requires dealing with unfortunate technical difficulties. For example, morphisms of stacks do not behave like arrows in a category, but rather like 1-arrows in a 2-category.[5] Luckily one can usually avoid these issues. For more details, see e.g. [21].

One way to define a (separated) Deligne-Mumford stack (over $\mathbb{C}$) is by mimicking the last section, using (étale) charts and transition functions, where each chart is the quotient of an affine scheme by a finite group (Lemma 2.2.3 of [3]). This definition is difficult to state precisely, and it is also more natural to define Deligne-Mumford stacks via maps from schemes, along the lines of the "functor of points" perspective. That is, a stack X is usually defined as a category $\mathfrak{C}_X$, with a functor $F_X : \mathfrak{C}_X \to \mathrm{Sch}$ to the category of schemes, that satisfies certain properties. The objects η of $\mathfrak{C}_X$ with $F_X(\eta) = S$ are thought of as "maps from S to X." Morphisms of stacks are functors that commute with F_X, and 2-morphisms are natural transformations. Any scheme is naturally a stack, and indeed, considering a scheme S as a stack, the category of 1-morphisms $f : S \to X$ is equivalent to the subcategory $F_X^{-1}(S)$ of $\mathfrak{C}_X$.

Example 2.3.1. For V a scheme and G a group acting on V by automorphisms, $[V/G]$ is a Deligne-Mumford stack. (This is obvious using the first definition of a Deligne-Mumford stack. To use the

[5]This means e.g. that morphisms may themselves have automorphisms. In fact this already came up in Definition 2.1.3, when we identified certain morphisms via an action of G. One has to treat these identifications not as an equivalence relation, but as a collection of 2-arrows.

second definition, we need to define a map $S \to [V/G]$. Such a map is defined to be a principal G-bundle $\tilde{S} \to S$, together with a G-equivariant map $\tilde{S} \to V$.) Similarly any orbifold curve has the natural structure of a Deligne-Mumford stack.

As with orbifolds, properties of a scheme can usually be extended to Deligne-Mumford stacks [21]:

- A Deligne-Mumford stack is reduced/smooth if it has a covering of charts $[U_i/G_i]$, where U_i is a reduced/smooth scheme.
- A Deligne-Mumford stack is separated if its diagonal morphism is proper after base change to any scheme. (Part of the defition of a Deligne-Mumford stack is that this base change is a map of schemes.)
- A separated Deligne-Mumford stack is proper if it admits a surjective map from a proper scheme.

As mentioned in the introduction, an orbifold has an underlying variety. More precisely and generally, any separated Deligne-Mumford stack X admits a *coarse moduli space map* $X \to \underline{X}$, where $\underline{X}$ is an algebraic space, and the map is bijective on $\mathbb{C}$-points and universal with respect to maps to algebraic spaces (see [38].) If $\{[U_i/G_i]\}_i$ is an atlas for X with $U_i = \operatorname{Spec}(A_i)$, then $\underline{X}$ has an étale atlas consisting of the charts $\{U_i/G_i = \operatorname{Spec}(A_i^G)\}$. This allows us to think of a separated Deligne-Mumford as an algebraic space, endowed with some extra structure, in the spirit of the first example (Question 2.1.1).

Example 2.3.2. It will be important later that even if X is smooth, $\underline{X}$ is usually singular. (Exceptions include orbifold curves, or more generally Deligne-Mumford stacks whose only isotropy is cyclic and along a divisor.) Let $X = [\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})]$, where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $1 \cdot (x, y) = (-x, -y)$. Then X is a smooth Deligne-Mumford stack by definition. By the description of coarse moduli spaces on charts, we have

$$\underline{X} \cong \operatorname{Spec}(\mathbb{C}[x, y]^{\mathbb{Z}/2\mathbb{Z}}) \cong \operatorname{Spec}(\mathbb{C}[x^2, xy, y^2]) \cong \operatorname{Spec}(\mathbb{C}[a, b, c]/(b^2 - ac)).$$

The latter is a quadric cone in $\mathbb{C}^3$, a singular variety.

A map $X \to Y$ of Deligne-Mumford stacks is *representable* if for every map $S \to Y$ from a scheme S, the base change $X \times_Y S$ is a scheme. Equivalently, the map of isotropy groups at every point is injective.

Finally, we see a few more examples.

Example 2.3.3. Orbifold curves are exactly the smooth, proper, connected, 1-dimensional Deligne-Mumford stacks.

Example 2.3.4. A *balanced twisted curve* or *prestable orbifold curve* C is a connected, proper, 1-dimensional Deligne-Mumford stack that is (étale) locally isomorphic to:

1. isomorphic to $[\mathbb{C}/\mu_n]$ for some n, where μ_n acts by multiplication, or
2. isomorphic to $[\mathbb{V}(xy)/\mu_n]$, where $\mathbb{V}(xy) \subseteq \mathbb{C}^2$ is the union of the coordinate axes, and μ_n acts by multiplication by opposite roots of unity on x and y.

We often consider these curves with labeled marked points $b_1, \ldots, b_n$, where every smooth orbifold point is marked, and no nodes are marked. If every rational component has at least three nodes or markings (collectively known as special points), then we say C is *stable*. Prestable orbifold curves are the building blocks of orbifold stable maps.

Remark 2.3.5. One can generalize Remark 2.2.1 to covers of prestable orbifold curves, as follows. One may check that any connected étale cover of a node $[\mathbb{V}(xy)/\mu_n]$ from Example 2.3.4 is of the form $[\mathbb{V}(xy)/\mu_{n/a}] \to [\mathbb{V}(xy)/\mu_n]$ for some a, where the map is the quotient by the action of μ_a by multiplication by opposite roots of unity. The induced map on coarse moduli spaces is the map $\mathbb{V}(xy) \to \mathbb{V}(xy)$ given by $(x, y) \mapsto (x^a, y^a)$. This is the local (*balanced*) structure of an *admissible cover* in the sense of [30]. (An admissible cover is a map of nodal curves whose restriction to the smooth locus of the source is a ramified cover, and whose restriction to an étale neighborhood of a node is that just described.)

Indeed, given an étale cover of a prestable orbifold curve, the coarse moduli map is an admissible cover, by Lemma 4.2.1 of [1]. However, [1] shows that unlike in Remark 2.2.1, the process of taking the coarse moduli spaces is not quite reversible. In particular, the induced map from the moduli stack of étale covers of prestable orbifold curves to the moduli stack of admissible covers is the normalization map. (The moduli stack of admissible covers is in general singular, whereas the moduli stack of étale covers of prestable orbifold curves is smooth.)

Example 2.3.6. Consider the quotient $X = [V(xy, yz, xz)/(\mathbb{Z}/3\mathbb{Z})]$, where $V(xy, yz, xz)$ is the union of the coordinate axes in $\mathbb{CP}^3$, and μ_3 acts by cycling the coordinates x, y and z. Then X is birational to $\mathbb{P}^1$, and has a single singular point whose tangent space has dimension 3. One can check that the coarse moduli space of X is isomorphic to $\mathbb{P}^1$.

Similarly, we could replace μ_3 by S_3, acting on the coordinates by permutations. The result X' is an S_2-gerbe over X. (Incidentally, the normalization of X' is the "nonbanded" gerbe $[\mathbb{P}^1/S_3]$ in Example II.9 of [36].)

Example 2.3.7. Consider the quotient of the scheme $V = \operatorname{Spec}(\mathbb{C}[\epsilon]/(\epsilon^n))$ by the group $G = \mu_2$, acting by multiplication on ϵ. This is a nasty nonreduced Deligne-Mumford stack, that might arise as a substack of a smooth stack, based at a point with isotropy group μ_2. We can calculate its coarse moduli space:

$$\underline{[V/G]} \cong \operatorname{Spec}((\mathbb{C}[\epsilon^2]/(\epsilon^n))^{\mu_2}) \cong \operatorname{Spec} \mathbb{C}[\delta]/(\delta^{\lceil n/2 \rceil}).$$

2.4 Moduli spaces of stable maps

The fundamental objects of Gromov-Witten theory are the moduli spaces of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$, developed by Kontsevich and Manin. Fix X a smooth projective variety, nonnegative integers g and n, and a class $\beta \in H_2(X, \mathbb{Z})$. The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parametrizes the data of:

- A genus g connected curve C with at worst nodal singularities,
- A collection of n distinct labeled marked smooth points $b_1, \ldots, b_n$ on C, and
- A map $f : C \to X$ with $f_*[C] = \beta$,

subject to the condition that if $C' \subseteq C$ is a rational irreducible component, and $f|_{C'}$ is a constant map, then C' is *stable*, i.e

$$(\text{\# nodes on } C') + (\text{\# marked points on } C') \geq 3.$$

By work of Kontsevich [41], and Fulton-Pandharipande [26], $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a proper and separated Deligne-Mumford stack. (It may be viewed as a "compactification" of the space $\mathcal{M}_{g,n}(X, \beta)$ of smooth genus g curves in β, but in reality may have extra irreducible or connected components, possibly of large dimensions; see Example 2.4.1.)

There is a natural universal curve C over $\overline{\mathcal{M}}_{g,n}(X, \beta)$, with sections $s_1, \ldots, s_n$, and a universal map f :

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

There are "evaluation maps" $\mathrm{ev}_i = f \circ \sigma_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$.

While $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is singular, it has a structure called a *perfect obstruction theory* (see [6] and [42]), which gives rise to a *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_{\mathrm{vd}}(\overline{\mathcal{M}}_{g,n}(X, \beta)$, where

$$\mathrm{vd} = (1 - g)(\dim X - 3) + n + \int_\beta c_1(TX).$$

There is a useful long exact sequence of sheaves on $\overline{\mathcal{M}}_{g,n}(X, \beta)$:

$$\begin{aligned} 0 \to \mathrm{Aut}(C) \to R^0\pi_* f^*TX \to \mathcal{T}^1 \to \\ \to \mathrm{Def}(C) \to R^1\pi_* f^*TX \to \mathcal{T}^2 \to 0, \end{aligned}$$

where $\mathrm{Aut}(C)$ and $\mathrm{Def}(C)$ parametrize infinitesimal automorphisms and deformations of the marked curve $(C, b_1, \ldots, b_n)$, $R^0\pi_* f^*TX$ and $R^1\pi_* f^*TX$ parametrize infinitesimal deformations and obstructions of the map $f : C \to X$, and $\mathcal{T}^1$ and $\mathcal{T}^2$ are the "tangent sheaf" and "obstruction sheaf" of $\overline{\mathcal{M}}_{g,n}(X, \beta)$. One reason this is useful is that in the special case when $\mathcal{T}^1$ and $\mathcal{T}^2$ are vector bundles, we have $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} = e(\mathcal{T}^2)$ (using Poincaré duality).

Example 2.4.1. We describe $\overline{\mathcal{M}}_{0,0}(\mathcal{Q}, 2)$, where $\mathcal{Q} \subseteq \mathbb{P}^4$ is a general quintic threefold. It is a classical fact that $\mathcal{Q}$ contains exactly 2875 nonintersecting lines (see also Example 2.6.5 below), and it was shown in [37] that $\mathcal{Q}$ contains exactly 609250 smooth conic curves. This implies that there are three types of genus zero, degree 2, unmarked stable maps to $\mathcal{Q}$:

1. Maps $f : \mathbb{P}^1 \to \mathcal{Q}$ that map isomorphically to a conic in $\mathcal{Q}$ (there are 609250 of these),
2. Maps $f : \mathbb{P}^1 \to \mathcal{Q}$ that map to a line in $\mathcal{Q}$ as a degree 2 cover, ramified over two distinct points, and
3. Maps $f : C \to \mathcal{Q}$, where C consists of two copies of $\mathbb{P}^1$ joined at a node, and f maps each $\mathbb{P}^1$ with degree 1 to a line in $\mathcal{Q}$.

One can check by explicit construction that stable maps of type 3 are precisely the limits of those of type 2 in families when the two ramification points collide. Thus $\overline{\mathcal{M}}_{0,0}(\mathcal{Q}, 2)$ consists of 609250 points (from maps of type 1), together with 2875 copies of $[\mathbb{P}^2/S_2]$, where S_2 acts trivially on $\mathbb{P}^2$. Here an open subset (the complement of the discriminant conic) of each $\mathbb{P}^2$ parametrizes maps of type 2 up to automorphism, and the discriminant conic parametrizes maps of type 3 (which are uniquely defined by the image point of the node). The reason for the S_2 action is that each of these maps has a symmetry, namely switching the two branches of the source curve.

One can also compute the virtual fundamental class $[\overline{\mathcal{M}}_{0,0}(\mathcal{Q}, 2)]^{\mathrm{vir}}$ (see [32], Chapter 27): it is the sum of the fundamental classes of the 609250 points, plus the sum of $[pt]/8$ over each copy of $[\mathbb{P}^2/S_2]$. (Here $[pt]$ is the class of a nonorbifold point, i.e. the pushforward of the fundamental class along any map $\operatorname{Spec} \mathbb{C} \to [\mathbb{P}^2/S_2]$.) The total degree of the class is $609250 + 2875/8$; this number is the Gromov-Witten invariant $\langle\rangle_{0,0,2}^{\mathcal{Q}}$ defined in Section 2.5. (This invariant is calculated via the Theorem 1.2.4 in Example 2.6.5.)

Example 2.4.2. Another important case is $X = \mathbb{P}^r$. If $g = 0$, then the spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, \beta)$ are smooth irreducible Deligne-Mumford stacks, whose coarse moduli spaces are projective varieties.

There is a natural inclusion $j : \overline{\mathcal{M}}_{0,n}(\mathscr{Q}, \beta) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, \beta)$, and one can use j to compute intersection numbers on $\overline{\mathcal{M}}_{0,n}(\mathscr{Q}, \beta)$. In particular, the cohomology of the ambient space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, \beta)$ is relatively well-understood, so after calculating $j_*[\overline{\mathcal{M}}_{0,n}(\mathscr{Q}, \beta)]^{\mathrm{vir}}$, one can compute any intersection product on $\overline{\mathcal{M}}_{0,n}(\mathscr{Q}, \beta)$ of classes pulled back from $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, \beta)$.

In general (especially in high genus), one should not expect $\overline{\mathcal{M}}_{g,n}(X, \beta)$ to be smooth, though it happened to be smooth in the previous two examples. (In fact, there is a precise sense in which it can be "arbitrarily singular," even for $X = \mathbb{P}^r$ [55].)

Definition 2.4.3. We define natural classes in $H^2(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q})$:

$$\psi_i := c_1(s_i^*(T^*_{C|\overline{\mathcal{M}}_{g,n}(X,\beta)})),$$

where $T^*_{C|\overline{\mathcal{M}}_{g,n}(X,\beta)}$ is the relative cotangent bundle of π.

2.4.1 Orbifold stable maps

All of the above can be extended to the case where X is a smooth proper Deligne-Mumford stack, due to Abramovich-Graber-Vistoli [3, 2]. There are a few modifications:

(i) The curve C is allowed to have orbifold structure at marked points and nodes: it must be a prestable orbifold curve as in Example 2.3.4.

(ii) The map $f : C \to X$ is required to be representable.

(iii) The maps ev_i take values in the inertia stack IX rather than in X. Intuitively, at a marked point $P \in C$, there is a canonical generator $g \in G_P$, and the monodromy map takes g to a well-defined conjugacy class in $G_{f(P)}$, namely a point of IX.

(iv) Depending on the definition one uses, the maps s_i may not exist, due to the possibility that the marked points form a *nontrivial gerbe* over the moduli space. We adopt the convention as in Section 7.1 of [46], that marked gerbes come with the data of a trivialization.[6] Because of this convention, the classes ψ_i may be defined as above.

Example 2.4.4. The case $X = BS_d$ (and more generally $X = BG$) is studied in detail [1]. Spaces of orbifold stable maps to BS_d are closely related to Hurwitz spaces and Harris-Mumford spaces of admissible covers [30]. An orbifold stable map $f : C \to BS_d$ is (by definition) a principal

[6] If we did not take this convention, then the maps ev_i would land in a *rigidification* of IX, i.e. a stack that looks the same as IX geometrically, but has smaller isotropy groups.

S_d-bundle $\mathcal{P} \to C$. There is a correspondence between principal S_d-bundles and degree d étale covers, that sends

$$(\mathcal{P} \to C) \mapsto (\mathcal{P} \times_{S_d} \{1, \ldots, d\} \to C).$$

Example 2.2.1 says that there is, in turn, a correspondence between these degree d *étale* maps $D \to C$ of *orbifold* covers, and degree d *ramified* covers $\underline{D} \to \underline{C}$ of *nonorbifold* curves. These two correspondences identify an open subset of $\overline{\mathcal{M}}_{g,n}(BS_d, 0)$ with a Hurwitz space parametrizing (possibly disconnected) degree d ramified covers of genus g nonorbifold curves, with arbitrary ramification profiles at marked points.

The monodromy map in Definition 2.1.3 at a marked point of C with isotropy group μ_n is a map $\mu_n \to S_d$, up to conjugacy. These maps are in bijection with conjugacy classes of S_d, which in turn are in bijection with partitions of d. Indeed, the partition of d associated to an orbifold marked point of C by the monodromy map is precisely the ramification profile at the corresponding nonorbifold marked point of $\underline{C}$.

2.5 Gromov-Witten invariants and twisted Gromov-Witten invariants

A *Gromov-Witten invariant* is an integral of the form

$$\langle \overline{\psi}_1^{a_1} \gamma_1, \ldots, \overline{\psi}_n^{a_n} \gamma_n \rangle_{g,n,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{j=1}^{n} \overline{\psi}_j^{a_j} \operatorname{ev}_j^* \gamma_j \in \mathbb{Q}, \tag{2.1}$$

where

- $\overline{\psi}_j$ is the jth cotangent class on $\overline{\mathcal{M}}_{g,n}(X, \beta)$, coming from the cotangent space to the *coarse moduli space of* C,[7] and
- the "insertions" γ_j are in the Chen-Ruan cohomology $H^*_{CR}(X, \mathbb{Q}) := H^*(IX, \mathbb{Q})$ (see item (**iii**) of Section 2.4.1).[8]

If X has an action of a torus T, it induces a natural T-action on IX and $\overline{\mathcal{M}}_{g,n}(X, \beta)$, and $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}}$, $\overline{\psi}_j$, and $\operatorname{ev}_j^* \gamma_j$ are naturally equivariant classes (where $\gamma_j \in H^*_{CR,T}(X, \mathbb{Q})$). In this case (2.1) defines an *equivariant Gromov-Witten invariant* (an element of $H^*_T(\operatorname{Spec} \mathbb{C}, \mathbb{Q})$, denoted by $\langle \cdots \rangle_{g,n,\beta}^{X,T}$) via T-equivariant integration.

[7] Note that $\overline{\psi}_j = r_j \psi_j$, where r_j is the size of the isotropy group at the mark b_j, and ψ_j is the usual ψ class.

[8] In enumerative terms, we may think of these insertions as not only specifying incidence conditions (as in Section 1.1), but also specifying the failure of a curve, locally in a chart $[V/G] \subseteq X$, to lift to V.

There is also a notion of a *twisted Gromov-Witten invariant*. Let E be a vector bundle on X, equipped with the linear action of a torus T (that acts trivially on X). Consider the class $R\pi_* f^* E \in K_T^0(\overline{\mathcal{M}}_{g,n}(X,\beta))$. This has a well-defined invertible Euler class $e_T(R\pi_* f^* E)$, see [33]. We define twisted Gromov-Witten invariants:

$$\langle \overline{\psi}_1^{a_1}\gamma_1, \dots, \overline{\psi}_n^{a_n}\gamma_n\rangle_{g,n,\beta}^{X,T,E} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{j=1}^{n} \overline{\psi}_j^{a_j} \operatorname{ev}_j^* \gamma_j \cup e_T^{-1}(R\pi_* f^* E).$$

2.6 The Givental cone

For the rest of the thesis, we restrict to the case $g = 0$.

Notation 2.6.1. We write $H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C},\mathbb{Q})$ for the fraction field of $H^*_T(\operatorname{Spec}\mathbb{C},\mathbb{Q})$, and

$$H^*_{T,\mathrm{loc}}(X,\mathbb{Q}) := H^*_T(X,\mathbb{Q}) \otimes_{H^*_T(\operatorname{Spec}\mathbb{C},\mathbb{Q})} H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C},\mathbb{Q}).$$

Definition 2.6.2. Following [18], the *T-equivariant Novikov ring of X* is

$$\Lambda_T^{\mathrm{nov}} := H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C},\mathbb{Q})[[\mathrm{NE}(X) \cap H_2(X,\mathbb{Z})]],$$

We write $\mathbf{Q} := \{Q_i\}_i$ for a set of generators of $\mathrm{NE}(X) \cap H_2(X,\mathbb{Z})$, and use the shorthand

$$H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C},\mathbb{Q})[[\mathbf{Q}]] := H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C},\mathbb{Q})[[\mathrm{NE}(X) \cap H_2(X,\mathbb{Z})]].$$

When $X = \operatorname{Sym}^d \mathbb{P}^r$, there is a single generator, denoted Q. (See Section 3.2.2.)

Definition 2.6.3. *Givental's symplectic*[9] *vector space* is

$$\mathcal{H} := H^*_{CR,T,\mathrm{loc}}(X,\mathbb{Q})[[\mathbf{Q}]]((z^{-1})) = \mathcal{H}^+ \oplus \mathcal{H}_-,$$

where $\mathcal{H}^+ = H^*_{CR,T,\mathrm{loc}}(X,\mathbb{Q})[[\mathbf{Q}]][z]$ and $\mathcal{H}^- = z^{-1}H^*_{CR,T,\mathrm{loc}}(X,\mathbb{Q})[[\mathbf{Q}]][[z^{-1}]]$.

Inside $\mathcal{H}$, there is a special subscheme $\mathcal{L}_{\operatorname{Sym}^d \mathbb{P}^r}$ called the *Givental cone* of $\operatorname{Sym}^d \mathbb{P}^r$, which encodes the genus-zero Gromov-Witten invariants of $\operatorname{Sym}^d \mathbb{P}^r$. (Precisely, $\mathcal{L}_{\operatorname{Sym}^d \mathbb{P}^r}$ is a formal germ of a subscheme, defined at $-1 \cdot z$, where $1 \in H^*_{CR,T,\mathrm{loc}}(\operatorname{Sym}^d \mathbb{P}^r,\mathbb{Q})$ is the fundamental class of the untwisted sector.) Fix a basis γ_ϕ of $H^*_{CR,T,\mathrm{loc}}(\operatorname{Sym}^d \mathbb{P}^r,\mathbb{Q})$, with Poincaré dual basis γ^ϕ.

[9] The symplectic structure is not relevant to us, so we do not define it.

Definition 2.6.4. A $\Lambda_{\mathrm{nov}}^T[[x]]$-valued point of $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ is defined to be a formal Laurent series

$$-1z + \mathbf{t}(z) + \sum_{n=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\phi} \frac{Q^\beta}{n!} \left\langle \mathbf{t}(\overline{\psi}), \dots, \mathbf{t}(\overline{\psi}), \frac{\gamma_\phi}{-z-\overline{\psi}} \right\rangle_{0,n+1,\beta}^{\mathrm{Sym}^d \mathbb{P}^r, T} \gamma^\phi \in \mathcal{H}[[x]],$$

where $\mathbf{t}(z) \in (x, Q) \subseteq \mathcal{H}_+[[x]]$. (For brevity, we suppress the variables x and Q in $\mathbf{t}$.)

$\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ has several important geometric properties that follow from relations between Gromov-Witten invariants: see Appendix B of [16], which also defines $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ rigorously as a non-Noetherian formal scheme. For example, it is a cone in a certain sense, hence the name (Proposition B.2 of [16]). It is also (formal) Lagrangian with respect to the symplectic structure on $\mathcal{H}$, so is often called the *Lagrangian cone* of X.

Example 2.6.5. We illustrate how the Theorem 1.2.4, the mirror theorem for quintic threefolds, can be used to recover Gromov-Witten invariants. In the discussion after Theorem 1.2.4, we saw that

$$\begin{aligned} I(t,Q,-z) &= -z + \mathbf{t}_1(t,Q,z) \\ &\quad + \sum_{\substack{n,\beta\geq 0 \\ \beta>0 \text{ or } n\geq 2}} \sum_{i=0}^{3} \frac{Q^\beta}{n!} \langle \mathbf{t}_1(t,Q,\psi), \dots, \mathbf{t}_1(t,Q,\psi), \frac{H^i}{-z-\psi} \rangle_{0,n+1,\beta}^{\mathcal{Q}} \frac{H^{3-i}}{5} \\ &\quad + O(Q^2), \end{aligned} \tag{2.2}$$

where $\mathbf{t}_1(t,Q,z) = tH + Qe^t(-120z + (770+120t)H)$ and $I(t,Q,z)$ is defined in (1.4). For simplicity, we consider (2.2) with t set to 0. Equating the coefficients of Q in (2.2) gives

$$\begin{aligned} &-z\frac{(5H-z)(5H-2z)(5H-3z)(5H-4z)(5H-5z)}{(H-z)^5} \\ &\qquad = -120z + 770H + \sum_{i=0}^{3} \langle \frac{H^i}{-z-\psi} \rangle_{0,1,1}^{\mathcal{Q}} \frac{H^{3-i}}{5} \end{aligned} \tag{2.3}$$

By construction, the coefficients of 1 and H agree on both sides. The remaining terms on the left side of (2.3) are

$$-\frac{575H^2}{z} - \frac{1150H^3}{z^2}. \tag{2.4}$$

The moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{Q},\beta)$ is n-dimensional. A dimension count implies that the coefficients of H^2 and H^3 on the right side of (2.3) are equal to:

$$\langle \frac{\psi}{z^2} \rangle_{0,1,1}^{\mathcal{Q}} \frac{H^3}{5} + \langle \frac{H}{-z} \rangle_{0,1,1}^{\mathcal{Q}} \frac{H^2}{5}. \tag{2.5}$$

Using the dilaton and divisor equations, (2.5) simplifies to

$$\frac{-2}{z^2}\langle\rangle^{\mathscr{Q}}_{0,0,1}\frac{H^3}{5}+\frac{1}{-z}\langle\rangle^{\mathscr{Q}}_{0,0,1}\frac{H^2}{5}.$$

Setting this equal to (2.4) gives $\langle\rangle^{\mathscr{Q}}_{0,0,1}=2875$.

We also calculate $\langle\rangle^{\mathscr{Q}}_{0,0,2}$, since the computation will show that there is some recursive structure in the Gromov-Witten invariants of $\mathscr{Q}$. Again setting $t=0$, we compute that the Q^2-coefficient of $\mathbf{t}(0,Q,z)$ is $\mathbf{t}_2(0,Q,z):=-113400z+810225H$. Equating the coefficients of $\frac{Q^2H^2}{-5z}$ in (2.2) gives

$$-z\frac{\prod_{\gamma=1}^{10}(5H-\gamma z)}{\prod_{\gamma=1}^{2}(H-\gamma z)^5}=\frac{1}{2}\langle-120\psi+770H,-120\psi+770H,H\rangle^{\mathscr{Q}}_{0,3,0}+\langle-120\psi+770H,H\rangle^{\mathscr{Q}}_{0,2,1} \\ +\langle H\rangle^{\mathscr{Q}}_{0,1,2}$$

Applying the dilaton and divisor equations yields:

$$\frac{21040875}{4}=296450\langle H,H,H\rangle^{\mathscr{Q}}_{0,3,0}+120\langle\rangle^{\mathscr{Q}}_{0,0,1}+770\langle\rangle^{\mathscr{Q}}_{0,0,1}+2\langle\rangle^{\mathscr{Q}}_{0,0,2}$$

Plugging in the known values $\langle H,H,H\rangle^{\mathscr{Q}}_{0,3,0}=5$ (a translation of that fact that $H^3=5[pt]\in H^6(\mathscr{Q},\mathbb{Q})$) and $\langle\rangle^{\mathscr{Q}}_{0,0,1}=2875$ then gives:

$$\langle\rangle^{\mathscr{Q}}_{0,0,2}=\frac{4876875}{8}=609250+\frac{2875}{8},$$

as claimed in Example 2.4.1.

There is also a notion of a *twisted Givental cone* $\mathcal{L}^E_X$. As above, let E be a T-equivariant vector bundle over X, where T acts trivially on X. A $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point of $\mathcal{L}^E_X$ is defined to be

$$-1z+\mathbf{t}(z)+\sum_{n=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\phi}\frac{Q^\beta}{n!}\left\langle\mathbf{t}(\overline{\psi}),\ldots,\mathbf{t}(\overline{\psi}),\frac{\gamma_\phi}{-z-\overline{\psi}}\right\rangle^{X,T,E}_{0,n+1,\beta}\gamma^\phi, \tag{2.6}$$

for some $\mathbf{t}(z)\in\langle x,Q\rangle\subseteq\mathcal{H}_+[[x]]$.

Remark 2.6.6. We may allow $\mathbf{t}(z)$ to be a power series in z, because $\overline{\psi}$ is nilpotent (as the T-action on X is trivial). We will need this to make sense of Condition (**III**) of Theorem 5.1.4. Also, here γ_ϕ and γ^ϕ are dual bases of $H^*_T(X,\mathbb{Q})$ under the *twisted* Poincaré pairing, see [18].

CHAPTER 3

Properties of $\operatorname{Sym}^d \mathbb{P}^r$

This chapter introduces symmetric product stacks/orbifolds $\operatorname{Sym}^d X$ and discusses geometric properties. Specifically, we describe the tangent bundle of $\operatorname{Sym}^d X$ (Lemma 3.1.1), and compute the Chen-Ruan cohomology of $\operatorname{Sym}^d X$ with rational coefficients (Section 3.1.3). We make this computation more explicit in the case $X = \mathbb{P}^r$ (Section 3.2.2).

3.1 Symmetric product stacks

3.1.1 Two definitions of $\operatorname{Sym}^d X$

Let X be a scheme. The dth power X^d has a natural action of S_d. The dth symmetric product $\operatorname{Sym}^d X$ is the orbifold $[X^d/S_d]$. That is, for S a scheme, an object $f : S \to \operatorname{Sym}^d X$ of $\operatorname{Sym}^d X$ over S is a principal S_d-bundle $\tilde{S}$ over S, together with an S_d-equivariant map $\tilde{f} : \tilde{S} \to X^d$. A morphism $(f : S \to \operatorname{Sym}^d X) \to (g : T \to \operatorname{Sym}^d X)$ over $S \to T$ is a diagram

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] \arrow[rr, bend left, "\tilde{f}"] & \tilde{T} \arrow[r, "\tilde{g}"'] \arrow[d] & X^d \\ S \arrow[r] & T & \end{tikzcd}$$

such that the square is cartesian and the triangle commutes. If X is smooth, then so is $\operatorname{Sym}^d X$, since it has an étale cover by the smooth scheme X^d.

There is an equivalent characterization of $\operatorname{Sym}^d X$, which will allow us to sidestep some of the complications of working with stacks. We define a stack $\widetilde{\operatorname{Sym}}^d X$ that is naturally isomorphic to $\operatorname{Sym}^d X$. Roughly, rather than parametrizing "d ordered points of X up to reordering", $\widetilde{\operatorname{Sym}}^d X$ will parametrize "Maps $P : d(\bullet) \to X$," where $d(\bullet) = \bigcup_{j=1}^d \operatorname{Spec} \mathbb{C}$. Precisely, an object $f : S \to \widetilde{\operatorname{Sym}}^d X$ over S is an étale map $\rho : S' \to S$ of degree d (i.e. a bundle with fiber $d(\bullet)$), together with a map $f' : S' \to X$. A morphism $(f : S \to \widetilde{\operatorname{Sym}}^d X) \to (g : T \to \widetilde{\operatorname{Sym}}^d X)$ over $S \to T$ is a diagram

$$
\begin{tikzcd}
S' \arrow[r] \arrow[d] \arrow[rr, bend left, "f'"] & T' \arrow[r, "g'"'] \arrow[d] & X \\
S \arrow[r] & T &
\end{tikzcd}
$$

There is a map $\operatorname{Sym}^d X \to \widetilde{\operatorname{Sym}}^d X$ that sends:

$$
\left(\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{f}"] \arrow[d] & X^d \\
S &
\end{tikzcd}\right) \mapsto \left(\begin{tikzcd}
\tilde{S} \times_{S_d} \{1, \dots, d\} \arrow[r] \arrow[d] & X \\
S &
\end{tikzcd}\right),
$$

where the S_d-action on $\{1, \dots, d\}$ is the obvious one, and the map $\tilde{S} \times_{S_d} \{1, \dots, d\} \to X$ sends $(\tilde{s}, i) \mapsto \mathrm{pr}_i \circ \tilde{f}(s)$, where pr_i denotes the ith projection $X^d \to X$. It is easy to check that this is S_d-equivariant, and defines a morphism of stacks.

In the other direction, we may send:

$$
\left(\begin{tikzcd}
S' \arrow[r, "f'"] \arrow[d] & X \\
S &
\end{tikzcd}\right) \mapsto \left(\begin{tikzcd}
\mathcal{I}som_S(S', \{1, \dots, d\}) \arrow[r] \arrow[d] & X^d \\
S &
\end{tikzcd}\right),
$$

where $\mathcal{I}som_S(S', \{1, \dots, d\})$ is the principal S_d-bundle given on small (étale) open sets $U \to S$ by the set of isomorphisms $S' \times_S U \to \{1, \dots, d\} \times U$. Given such an isomorphism, f' determines a U-valued point of X^d. It is again straightforward to check that this defines a map of stacks $\widetilde{\operatorname{Sym}}^d X \to \operatorname{Sym}^d X$, and that it is an inverse to the previous map. For the rest of the thesis we will use the descriptions interchangeably and denote them both by $\operatorname{Sym}^d X$. It is useful to keep in mind the following diagram, where the cube is Cartesian and the left and right faces consist of étale maps:

$$
\begin{tikzcd}
\tilde{S} \times \{1, \dots, d\} \arrow[rr] \arrow[dr] \arrow[dd] & & X^d \times \{1, \dots, d\} \arrow[dr] \arrow[dd] \arrow[ddrr, bend left] & & \\
& \tilde{S} \arrow[rr, "\tilde{f}"] \arrow[dd, "\mathrm{pr}"] & & X^d \arrow[dd, "\mathrm{pr}"] & \\
S' = \tilde{S} \times_{S_d} \{1, \dots, d\} \arrow[rr] \arrow[dr, "\rho"] & & X^d \times_{S_d} \{1, \dots, d\} \arrow[rr, "P"] \arrow[dr, "\rho"] & & X \\
& S \arrow[rr, "f"] & & \operatorname{Sym}^d X &
\end{tikzcd} \tag{3.1}
$$

(The composition $S' \to X^d \times_{S_d} \{1, \dots, d\} \xrightarrow{P} X$ is f'.)

3.1.2 The tangent bundle to $\operatorname{Sym}^d X$

Now we assume X is a smooth scheme. The two definitions of $\operatorname{Sym}^d X$ in Section 3.1.1 give two descriptions of the tangent bundle $T \operatorname{Sym}^d X$. First, since $\operatorname{Sym}^d X$ is isomorphic to $[X^d/S_d]$, $T \operatorname{Sym}^d X$ is the vector bundle on $\operatorname{Sym}^d X$ corresponding to the S_d-equivariant vector bundle $T(X^d)$ on X^d, where S_d acts by the derivative. Consider the portion of Figure (3.1), with more maps named:

$$\begin{tikzcd}
X^d \times \{1,\dots,d\} \arrow[dd, "\mathrm{pr}'"] \arrow[dr, "\tilde{\rho}"] \arrow[ddrr, bend left] & & \\
& X^d \arrow[dd, "\mathrm{pr}"] & \\
X^d \times_{S_d} \{1,\dots,d\} \arrow[rr, "P"] \arrow[dr, "\rho"] & & X \\
& \operatorname{Sym}^d X &
\end{tikzcd}$$

Lemma 3.1.1. *There is a natural isomorphism* $T \operatorname{Sym}^d X \cong \rho_*(P^*TX)$.

Proof. Since the square is cartesian and the maps are étale, we have

$$\mathrm{pr}^*(\rho_*(P^*TX)) \cong \tilde{\rho}_*((\mathrm{pr}')^*(P^*TX)) = \tilde{\rho}_*((\mathrm{pr}' \circ P^*TX)).$$

Recall that $\mathrm{pr}' \circ P$ is simply the "universal coordinate map," so since $\tilde{\rho}$ is a trivial étale cover, there is a canonical isomorphism

$$\tilde{\rho}_*((\mathrm{pr}' \circ P^*TX)) \cong \bigoplus_{\ell=1}^{d} P_\ell^* TX \cong T(X^d).$$

Since $\tilde{\rho}$ is S_d-equivariant, there is an induced S_d-action on $T(X^d)$ which agrees with the usual one. Thus the isomorphism descends to give $\rho_*(P^*TX) \cong T \operatorname{Sym}^d X$. $\square$

3.1.3 The inertia stack and Chen-Ruan cohomology of $\operatorname{Sym}^d X$

Again, let X be a smooth variety. We describe the inertia stack of $\operatorname{Sym}^d X$, see Definition 2.1.4. (See also Section 3 of [2]).

To a map $d(\bullet) \xrightarrow{P} X$, we may assign a partition $\sigma \in \mathrm{Part}(d)$, where parts correspond to points of $\mathrm{Im}(P)$. This gives a stratification of $\operatorname{Sym}^d X$ into locally closed strata $(\operatorname{Sym}^d X)_\sigma$ indexed by $\sigma \in \mathrm{Part}(d)$.

Proposition 3.1.2. *For each $\sigma \in \mathrm{Part}(d)$,*

$$(\mathrm{Sym}^d X)_\sigma \cong \left(\prod_{i \geq 1} \mathrm{Sym}^{\mathrm{Mult}(\sigma,i)} X \right) \smallsetminus \Delta$$

up to generic stabilizer, where Δ is the union of all diagonals. Each point of $(\mathrm{Sym}^d X)_\sigma$ has isotropy group isomorphic to $\prod_{\eta \in \sigma} S_\sigma$.

Proof. On $(\mathrm{Sym}^d X)_\sigma$, there is a natural operation of contracting fibers of each map $S' \xrightarrow{P} X$, and remembering the number of points in each contracted fiber. This induces a well-defined map

$$(\mathrm{Sym}^d X)_\sigma \to \left(\prod_{i \geq 1} \mathrm{Sym}^{\mathrm{Mult}(\sigma,i)} X \right) \smallsetminus \Delta.$$

The "obvious" scheme-theoretic inverse fails to be a functor, but it is easy to check that the map above is a rigidification in the sense of [1]. The isotropy group at a point $d(\bullet) \xrightarrow{P} X$ is simply the group of automorphisms of $d(\bullet)$ commuting with P, which by definition of $(\mathrm{Sym}^d X)_\sigma$ is $\prod_{\eta \in \sigma} S_\sigma$. $\square$

It follows from Proposition 3.1.2 that components of $I \,\mathrm{Sym}^d X$ are indexed by $\mathrm{Part}(d)$. We denote the component associated to $\sigma \in \mathrm{Part}(d)$ by $(I \,\mathrm{Sym}^d X)_\sigma$. We have, up to generic stabilizer,

$$I(\mathrm{Sym}^d X)_\sigma \cong \overline{(\mathrm{Sym}^d X)_\sigma} \cong \prod_{i \geq 1} \mathrm{Sym}^{\mathrm{Mult}(\sigma,i)} X.$$

From Example 2.5 of [4], the generic stabilizer is isomorphic to the centralizer of (any representative of) σ in S_d. Explicitly, this is the group (see Section 1.7):

$$C(\sigma) = S_\sigma \ltimes \prod_{\eta \in \sigma} \mu_\eta. \tag{3.2}$$

The (nonorbifold) cohomology with rational coefficients of each component $I(\mathrm{Sym}^d X)_\sigma$ may be computed explicitly from that of X, using the fact that

$$H^*(\mathrm{Sym}^d X, \mathbb{Q}) \cong H^*(X^d, \mathbb{Q})^{S_d} \cong (H^*(X, \mathbb{Q})^{\otimes d})^{S_d}.$$

In particular, there are isomorphisms:

$$H^2(\mathrm{Sym}^d X, \mathbb{Q}) \cong (H^2(X, \mathbb{Q})^{\oplus d})^{S_d} \cong H^2(X, \mathbb{Q}).$$

3.2 Symmetric products of projective space

In this section we expand on some of the above in the case $X = \mathbb{P}^r$, and discuss the natural torus action on $\mathrm{Sym}^d \mathbb{P}^r$. First, we introduce some notation to be used throughout.

3.2.1 Notation for projective space

We denote a point of $\mathbb{P}^r = \mathbb{P}(\mathbb{C}^{r+1})$ by $[x_0 : x_1 : \cdots : x_r]$ with $x_i \in \mathbb{C}$. We denote the coordinate points of $\mathbb{P}^r$ by $P_0, P_1, \ldots, P_r$, where P_i is the point where all coordinates except x_i vanish. We denote by $L_{\{i_1,i_2\}}$ (where $0 \leq i_1, i_2 \leq r$ and $i_1 \neq i_2$) the coordinate line passing through P_{i_1} and P_{i_2}. We write $P_{\{i_1,i_2\}}$ for the midpoint of this line, i.e. the point where $x_{i_1} = x_{i_2}$ and $x_i = 0$ for $i \neq i_1, i_2$. We also write H_i for the ith coordinate hyperplane $\{x_i = 0\} \subseteq \mathbb{P}^r$.

Throughout, we use the action of $T := (\mathbb{C}^*)^{r+1}$ on $\mathbb{P}^r$ by scaling the coordinates. We have $H_T^*(\mathrm{Spec}\,\mathbb{C}, \mathbb{Q}) = \mathbb{Q}[\alpha_0, \ldots, \alpha_r]$, and $H_{T,\mathrm{loc}}^*(\mathrm{Spec}\,\mathbb{C}, \mathbb{Q}) = \mathbb{Q}(\alpha_0, \ldots, \alpha_r)$, where $-\alpha_i$ is the weight of the character $T \to \mathbb{C}^*$ defined by $(\lambda_0, \ldots, \lambda_r) \mapsto \lambda_i$.

We denote by $[P_i]$, (resp. $[L_{\{i_1,i_2\}}]$, $[H_i]$) the class in $H_T^*(\mathbb{P}^r, \mathbb{Q})$ identified with P_i (resp. $L_{\{i_1,i_2\}}$, H_i) under the equivariant Poincaré duality isomorphism.

3.2.2 The Chen-Ruan cohomology of $\mathrm{Sym}^d \mathbb{P}^r$

In Section 3.1.3, we saw that there exists an isomorphism $H_T^2(\mathrm{Sym}^d X, \mathbb{Q}) \cong H_T^2(X, \mathbb{Q})$. For convenience, we now fix such an isomorphism. If $\alpha \in H_T^2(X, \mathbb{Q})$, then we write $\alpha \in H_T^2(\mathrm{Sym}^d X, \mathbb{Q})$ for the class that pulls back to

$$\sum_{j=1}^{d} \mathrm{pr}_j^* \alpha \in H_T^2(X^d, \mathbb{Q}).$$

Using this isomorphism, we write $[H_i] \in H_T^2(\mathrm{Sym}^d \mathbb{P}^r, \mathbb{Q})$, which gives a distinguished set of generators for $H_T^2(\mathrm{Sym}^d \mathbb{P}^r, \mathbb{Q})$. (Via the Poincaré pairing, this defines a natural notion of the degree of a curve on $\mathrm{Sym}^d \mathbb{P}^r$.)

From Section 3.1.3, the components of $I\,\mathrm{Sym}^d \mathbb{P}^r$ are indexed by $\mathrm{Part}(d)$, and are given by

$$(I\,\mathrm{Sym}^d \mathbb{P}^r)_\sigma \cong \prod_{i \geq 1} \mathrm{Sym}^{\mathrm{Mult}(\sigma,i)} \mathbb{P}^r.$$

We then have

$$H_T^*((I\,\mathrm{Sym}^d \mathbb{P}^r)_\sigma, \mathbb{Q}) \cong \bigoplus_{i \geq 1} H_T^*(\mathrm{Sym}^{\mathrm{Mult}(\sigma,i)} \mathbb{P}^r, \mathbb{Q}).$$

For $\eta \in \sigma$, there is a corresponding factor $H_T^*(\operatorname{Sym}^{\operatorname{Mult}(\sigma,i)} \mathbb{P}^r, \mathbb{Q})$, and we denote by $[H_{\sigma,\eta,i}]$ the class $[H_i]$ in this factor. We write $[H_{\sigma,i}]$ for $\sum_\eta [H_{\sigma,\eta,i}]$.

The Chen-Ruan cohomology is

$$H^*_{CR,T}(\operatorname{Sym}^d \mathbb{P}^r, \mathbb{Q}) := \bigoplus_{\sigma \in \operatorname{Part}(d)} H_T^*((I \operatorname{Sym}^d \mathbb{P}^r)_\sigma, \mathbb{Q}),$$

with an appropriate shift in grading that we will not need.

3.2.3 The action of $(\mathbb{C}^*)^{r+1}$ on $\operatorname{Sym}^d \mathbb{P}^r$

The natural action of $T := (\mathbb{C}^*)^{r+1}$ on $\mathbb{P}^r$ induces a diagonal T-action on $(\mathbb{P}^r)^d$. This commutes with the action of S_d, hence acts on $\operatorname{Sym}^d \mathbb{P}^r$. It is easy to check that this action agrees with that on $\widetilde{\operatorname{Sym}}^d \mathbb{P}^r$ defined by postcomposition of $f' : S \to \mathbb{P}^r$ with the action on $\mathbb{P}^r$. The T-action on $\operatorname{Sym}^d \mathbb{P}^r$ induces an action on $\overline{\mathcal{M}}_{0,n}(\operatorname{Sym}^d \mathbb{P}^r, \beta)$ for all n, and β.

CHAPTER 4

Localization and virtual localization

In this chapter, we give a detailed description of $T := (\mathbb{C}^*)^{r+1}$-equivariant geometry of $\operatorname{Sym}^d \mathbb{P}^r$ and $\overline{\mathcal{M}}_{0,n}(\operatorname{Sym}^d \mathbb{P}^r, \beta)$. We focus on the aspects of this geometry required to compute Gromov-Witten invariants. In particular, we describe the T-fixed points and 1-dimensional T-orbits in $\operatorname{Sym}^d \mathbb{P}^r$. We use this to give a complete description (Theorem 4.5.23) of the T-fixed locus in $\overline{\mathcal{M}}_{0,n}(\operatorname{Sym}^d \mathbb{P}^r, \beta)$, and we compute the T-equivariant Euler class of the (virtual) normal bundle to the T-fixed locus (Proposition 4.6.2). We also give an example-heavy discussion (Section 4.2) of what can be said for general targets (smooth projective varieties).

4.1 The localization and virtual localization theorems

If $T = (\mathbb{C}^*)^r$ acts on a Deligne-Mumford stack X, the equivariant cohomology $H_T^*(X, \mathbb{Q})$ is a module over $H_T^*(\operatorname{Spec} \mathbb{C}, \mathbb{Q}) \cong \mathbb{Q}[\alpha_0, \ldots, \alpha_r]$, where $-\alpha_i$ is the weight of the character $T \to \mathbb{C}^*$ defined by $(\lambda_0, \ldots, \lambda_r) \mapsto \lambda_i$. We write $H_{T,\text{loc}}^*(\operatorname{Spec} \mathbb{C}, \mathbb{Q})$ for the localization $\mathbb{Q}(\alpha_0, \ldots, \alpha_r)$, and more generally $H_{T,\text{loc}}^*(X, \mathbb{Q}) := H_T^*(X, \mathbb{Q}) \otimes_{H_T^*(\operatorname{Spec} \mathbb{C}, \mathbb{Q})} H_{T,\text{loc}}^*(\operatorname{Spec} \mathbb{C}, \mathbb{Q})$. We will often use the Atiyah-Bott *localization theorem*, as well as Graber-Pandharipande's generalization, the *virtual localization theorem*.

Theorem 4.1.1 (Atiyah-Bott [5], see Edidin-Graham [22] for statement in the Chow ring)**.** *Let T be a torus acting on a smooth compact manifold X, with fixed point set F. Then the map $(\iota_F)_* : H_{T,\text{loc}}^*(F, \mathbb{Q}) \to H_{T,\text{loc}}^*(X, \mathbb{Q})$ is an isomorphism, where $(\iota_F)_*$ is the Gysin map associated to the inclusion $F \hookrightarrow X$. The inverse map is $\iota_F^*/e_T(N_{F|X})$, where $e_T(N_F)$ is the equivariant Euler class of the normal bundle to F. In particular, for $\alpha \in H_{T,\text{loc}}^*(X, \mathbb{Q})$, we have*

$$\int_X \alpha = \int_X (\iota_F)_* \left(\frac{\iota_F^* \alpha}{e_T(N_F)} \right) = \int_F \frac{\iota_F^* \alpha}{e_T(N_F)}.$$

Theorem 4.1.2 (Graber-Pandharipande [29])**.** *Let X be a Deligne-Mumford stack with a T-action and a T-equivariant perfect obstruction theory $E^\bullet$. Again, let $\iota_F : F \hookrightarrow X$ denote the inclusion of* *the fixed locus. Let* $[X]^{\mathrm{vir}}$ *denote the virtual fundamental class associated to* $E^\bullet$*. The* T*-fixed part of* $E^\bullet$ *defines a perfect obstruction theory on* F*, with virtual fundamental class* $[F]^{\mathrm{vir}}$*. The* virtual normal bundle N_F^{vir} *to* F *is the* T*-moving part of* $E^\bullet$*. Then*

$$\int_{[X]^{\mathrm{vir}}} \alpha = \int_{[F]^{\mathrm{vir}}} \frac{\iota_F^* \alpha}{e_T(N_F^{\mathrm{vir}})}. \tag{4.1}$$

Remark 4.1.3. The proof in [29] requires that X have an equivariant embedding into a smooth Deligne-Mumford stack, but this condition was removed in [11]. Also, it is usually convenient to write F as a union of connected components (or a union of open and closed substacks) F_j, in which case (4.1) becomes

$$\int_{[X]^{\mathrm{vir}}} \alpha = \sum_j \int_{[F_j]^{\mathrm{vir}}} \frac{\iota_{F_j}^* \alpha}{e_T(N_{F_j}^{\mathrm{vir}})}.$$

4.2 The T-fixed locus in $\overline{\mathcal{M}}_{0,n}(X, \beta)$, nonorbifold case

If X has a T-action, then postcomposition induces a T-action on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ for all n and β. In this section, we describe the behavior of torus-fixed stable maps to X when X is a smooth projective variety. The main purpose of the section is to emphasize, via examples, that this behavior is often very complicated, and one needs a lot of information about X and its torus action in order to write down $\overline{\mathcal{M}}_{0,n}(X, \beta)$ explicitly. This section contains no original results.

Definition 4.2.1. Let X be a smooth projective variety with an action of a torus T. The T*-graph* $\Gamma_{X,T}$ of X is the graph with a vertex v for every connected component X_v^T of X^T, and an edge $e = \{v_1, v_2\}$ for every connected component of the union of all 1-dimensional T-orbits joining v_1 and v_2.

As an example, the T-graphs for $\mathbb{P}^2 \times \mathbb{P}^2$ with two different torus actions are shown in Figure 4.1.

Remark 4.2.2. By Lemma 5 in [50], $\Gamma_{X,T}$ contains no self-edges. We do not know whether or not $\Gamma_{X,T}$ may contain multiple edges between two fixed vertices.

Example 4.2.3. In the case where T acts on X with isolated fixed points and 1-dimensional orbits, $\Gamma_{X,T}$ is identified (as a topological space) with the *1-skeleton* of X, i.e. the union of 0-dimensional and 1-dimensional T-orbits. This condition holds, for example, in any toric variety, e.g. Figure

When the T-graph of X is known, $\overline{\mathcal{M}}_{0,n}(X, \beta)^T$ may be written as a union of combinatorially indexed pieces. If one is lucky, it may be clear how these pieces fit together into connected

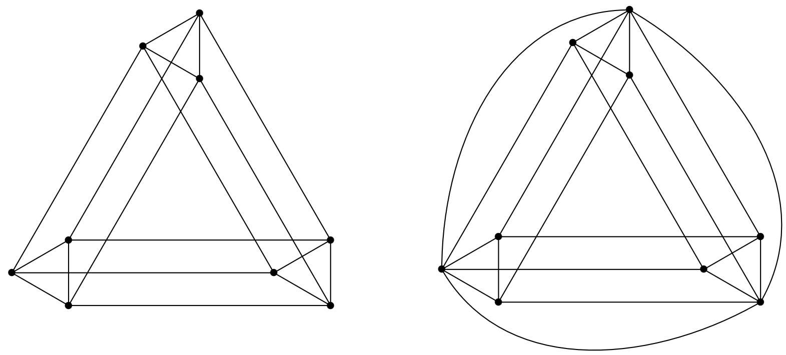


Figure 4.1: The T-graph of $\mathbb{P}^2 \times \mathbb{P}^2$ with the action of its dense torus, and the diagonal action of $(\mathbb{C}^*)^3$, respectively.

components, or unions of connected components, of $\overline{\mathcal{M}}_{0,n}(X, \beta)$. (Computing these may still be difficult.) For $e \in E(\Gamma_{X,T})$ a union of 1-dimensional T-orbits, we denote by $H_2(X, \mathbb{Z})_e$ the subgroup of $H_2(X, \mathbb{Z})$ generated by the classes of these 1-dimensional orbits. (They need not all be linearly equivalent, as we see in Example 4.2.14.)

Definition 4.2.4. An *n-marked (X, T)-decorated tree* $\tilde{\Gamma} = (\Gamma, \mathrm{Mark}, \mathbb{f}, d, \mathrm{vdeg})$ is

- A tree Γ,
- A marking map $\mathrm{Mark} : \{1, \ldots, n\} \to V(\Gamma)$,
- A map $\mathbb{f}$ of graphs $\Gamma \to \Gamma_{X,T}$,
- An edge degree map $\beta_{edge} : E(\Gamma) \to H_2(X, \mathbb{Z})_e$, and
- A vertex degree map $\mathrm{vdeg} : V(\Gamma) \to H_2(\mathbb{f}(v), \mathbb{Z})$. (Recall that $\mathbb{f}(v)$ is a connected component of X^T.)

We will often refer to an n-marked (X, T)-decorated tree as simply a "decorated tree" when the the meaning is clear.

Definition 4.2.5. The *degree* $\beta(\tilde{\Gamma})$ of an n-marked (X, T)-decorated tree is

$$\beta(\tilde{\Gamma}) := \sum_{e \in E(\Gamma)} \beta_{edge}(e) + \sum_{v \in V(\Gamma)} \mathrm{vdeg}(v) \in H_2(X, \mathbb{Z}).$$

Notation 4.2.6. We denote the set of n-marked (X, T)-decorated trees of degree β by $\mathrm{Trees}_{0,n}(X, \beta)$.

Proposition 4.2.7. *There is a natural map*

$$\Psi : (\overline{\mathcal{M}}_{0,n}(X, \beta))^T \to \mathrm{Trees}_{0,n}(X, \beta).$$

Proof. If $(f : C \to X) \in \overline{\mathcal{M}}_{0,n}(X, \beta)^T$, then on each irreducible component C, the restriction of f either

1. factors through X^T or
2. is a finite map to the closure of a 1-dimensional T-orbit.

We define a graph Γ_f whose vertex set $V(\Gamma_f)$ is the set of connected components of $f^{-1}(X^T)$ (which are unions of components of type 1), and whose edge set $E(\Gamma_f)$ is the set of irreducible components of type 2. Any irreducible component of type 2 contains exactly two distinct points that map to X^T, so $E(\Gamma_f)$ makes sense as a set of edges. The fact that Γ_f is a tree follows from the fact that C is genus zero.

There is a natural map $\mathbb{f}_f$ of graphs from Γ_f to $\Gamma_{X,T}$, which sends a connected component C_0 of $f^{-1}(X^T)$ to the component of X^T containing $f(C)$, and sends an irreducible component $C_1 \in E(\Gamma_f)$ to the component of the union of 1-dimensional T-orbits that contains $f(C_1)$.

Since $(f : C \to X)$ is T-fixed, each marked point is mapped to X^T. We define a marking map $\mathrm{Mark}_f : \{1, \dots, n\} \to V(\Gamma_f)$ that sends i to the connected component of $f^{-1}(X^T)$ containing b_i. We define a map $\beta_{edge,f} : E(\Gamma) \to \mathbb{N}$ that takes an edge e to $(f|_{C_e})_*[C_e]$, and a map $\mathrm{vdeg}_f : V(\Gamma) \to H_2(\mathbb{f}(v), \mathbb{Z})$ that sends a vertex v to $(f|_{C_v})_*[C_v]$. We then define $\Psi(f : C \to X) = (\Gamma_f, \mathbb{f}_f, \mathrm{Mark}_f, \beta_{edge,f}, \mathrm{vdeg}_f)$. $\square$

Notation 4.2.8. Let $\tilde{\Gamma}$ be a decorated tree, and let $(f : C \to X) \in \Psi^{-1}(\tilde{\Gamma})$. If $v \in V(\Gamma)$, then from Proposition 4.2.7, v corresponds to a union of irreducible components of C, or possibly a single point. We denote this by C_v. Similarly, an edge $e \in E(\Gamma)$ corresponds to an irreducible component of C, and we denote it by C_e. For $(v, e) \in F(\Gamma)$, we write $\xi(v, e)$ for the point $C_v \cap C_e \in C$. We say (v, e) is a *special flag* if $\xi(v, e)$ is a special point (mark or node), equivalently if $\mathrm{val}(v) > 1$, $\mathrm{vdeg}(v) > 0$, or $\mathrm{Mark}^{-1}(v) \neq \emptyset$.

For any $(f : C \to X) \in \Psi^{-1}(\tilde{\Gamma})$, the tangent space to C_e at $\xi(v, e)$ is naturally a (fractional) character of T. We denote the weight of this character by $w(v, e) \in H^2_T(\mathrm{Spec}(\mathbb{C}), \mathbb{Q})$. It is straightforward to check that this does not depend on the choice of $(f : C \to X)$.

We adapt the following notation from [46], Definition 53:

$$\begin{aligned}
V^1(\tilde{\Gamma}) &= \{v \in V(\Gamma) |\, \mathrm{val}(v) = 1, \mathrm{vdeg}(v) = 0, \left|\mathrm{Mark}^{-1}(v)\right| = 0\} \\
V^{1,1}(\tilde{\Gamma}) &= \{v \in V(\Gamma) |\, \mathrm{val}(v) = 1, \mathrm{vdeg}(v) = 0, \left|\mathrm{Mark}^{-1}(v)\right| = 1\} \\
V^2(\tilde{\Gamma}) &= \{v \in V(\Gamma) |\, \mathrm{val}(v) = 2, \mathrm{vdeg}(v) = 0, \left|\mathrm{Mark}^{-1}(v)\right| = 0\} \\
V^S(\tilde{\Gamma}) &= V(\Gamma) \smallsetminus (V^1(\tilde{\Gamma}) \cup V^{1,1}(\tilde{\Gamma}) \cup V^2(\tilde{\Gamma})).
\end{aligned}$$

We call vertices in $V^S(\tilde{\Gamma})$ *stable*. A vertex v is stable if and only if C_v is 1-dimensional (a prestable curve, rather than a single point).

For $v \in V^1(\tilde{\Gamma}) \cup V^{1,1}(\tilde{\Gamma})$, we will usually write $E(\Gamma)_v = \{e_v = \{v, v'\}\}$. For $v \in V^2(\tilde{\Gamma})$, we will usually write $E(\Gamma)_v = \{e_v^1 = \{v, v_1\}, e_v^2 = \{v, v_2\}\}$.

Example 4.2.9. If T acts on X with isolated fixed points and 1-dimensional orbits, then the fibers $\mathcal{M}_{\tilde{\Gamma}} := \Psi^{-1}(\tilde{\Gamma})$ are easy to describe. In this case a decorated tree $\tilde{\Gamma}$ determines irreducible components C_e and maps $f|_{C_e}$. The only things that are not determined are the curves C_v, which are stable genus zero curves with marking sets $\mathrm{Mark}^{-1}(v) \cup E(\Gamma)_v$. (The map $f|_{C_v}$ is the constant map from C_v to the single point $\mathbb{f}(v)$.) Thus we have[1]:

$$\mathcal{M}_{\tilde{\Gamma}} \cong \left(\prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{0,\mathrm{Mark}^{-1}(v) \cup E(\Gamma)_v} \right) \Big/ \mathrm{Aut}(\tilde{\Gamma}) \tag{4.2}$$

(The automorphism group $\mathrm{Aut}(\tilde{\Gamma})$ consists of automorphisms of Γ that commute with the maps Mark, $\mathbb{f}$, and β_{edge}.)

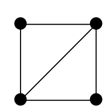
Since these fibers are proper, and there are finitely many of them for fixed n and β, they are in fact open and closed substacks of $\overline{\mathcal{M}}_{0,n}(X, \beta)^T$. (Since they are connected, they are exactly the connected components of $\overline{\mathcal{M}}_{0,n}(X, \beta)^T$.) In other words, if T acts on X with isolated fixed points and 1-dimensional orbits, then we have completely described the structure of $\overline{\mathcal{M}}_{0,n}(X, \beta)^T$: it is a disjoint union, over index set $\mathrm{Trees}_{0,n}(X, \beta)$, of spaces $\mathcal{M}_{\tilde{\Gamma}}$ as in (4.2).

Remark 4.2.10. It is important to note that this example is very special: in general, the fibers of Ψ^{-1} are only locally closed substacks of $\overline{\mathcal{M}}_{0,n}(X, \beta)^T$, as we see in the following example. Also, in general one should not expect $\mathcal{M}_{\tilde{\Gamma}}$ to be connected.

Example 4.2.11. Let $\mathbb{C}^*$ act on $(\mathbb{P}^1)^2$ diagonally. The fixed points of the action are:

$$(0, 0), (0, \infty), (\infty, 0), (\infty, \infty).$$

(We identify $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$ for now.) The complement of these points is a union of (1-dimensional) orbits. The T-graph is:

$\Gamma_{(\mathbb{P}^1)^2, \mathbb{C}^*} =$ 

The outer edges of the square correspond to unique orbits, whereas the diagonal edge corresponds

[1] To be precise, one must further take the quotient by the trivial action of a group of automorphisms of the ramified cover $C_e \to f(C_e)$ for each edge e.

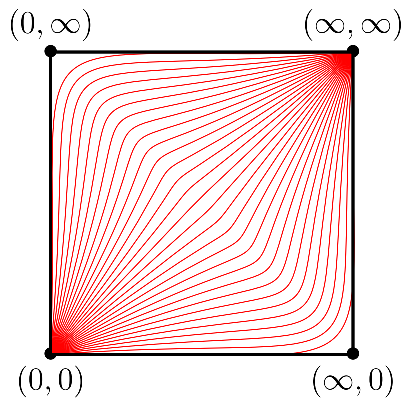


Figure 4.2: The torus orbits in $(\mathbb{P}^1)^2$

to the family of orbits of the form

$$\{(t, at) : t \in \mathbb{C}^*\}. \tag{4.3}$$

The base of this family, given by the a-coordinate, is isomorphic to $\mathbb{C}^*$. (See Figure 4.2; this family is shown in red.)

If $(f : C \to (\mathbb{P}^1)^2)$ is $\mathbb{C}^*$-fixed, then an irreducible component of C is either contracted to one of the four fixed points, or maps as a ramified cover of a 1-dimensional orbit, ramified over two of the fixed points.

Consider a degree d cover by $\mathbb{P}^1$ of an orbit of the form (4.3), ramified over $(0, 0)$ and (∞, ∞). As a stable map, this lies in $\Psi^{-1}(\tilde{\Gamma}) \subseteq \overline{\mathcal{M}}_{0,0}((\mathbb{P}^1)^2, (d, d))$, where Γ is the tree $v \bullet\!\xrightarrow{\;e\;}\!\bullet v'$, with

- $\mathbb{f}(v) = (0, 0)$,
- $\mathbb{f}(v') = (\infty, \infty)$,
- $\mathbb{f}(e)$ equal to the diagonal edge in $\Gamma_{(\mathbb{P}^1)^2, \mathbb{C}^*}$,
- $\beta_{edge}(e) = (d, d)$, and
- $\mathrm{vdeg}(v) = \mathrm{vdeg}(v') = 0$.

The family (4.3) of orbits induces a family $(f_a : \mathbb{P}^1 \to (\mathbb{P}^1)^2)$ of stable maps, corresponding to a map $\mathbb{C}^* \hookrightarrow \overline{\mathcal{M}}_{0,0}((\mathbb{P}^1)^2, (d, d))$. It is easy to compute that the limit as $a \to 0$ is the stable map $f_0 : C \to (\mathbb{P}^1)^2$, where C is isomorphic to two copies of $\mathbb{P}^1$ glued at a node, and f_0 sends the two components to the lines $\mathbb{P}^1 \times 0$ and $\infty \times \mathbb{P}^1$, each with degree d, each fully ramified over the two $\mathbb{C}^*$-fixed points.

The limit $(f_0 : C \to (\mathbb{P}^1)^2)$ lies in the fiber $\Psi^{-1}(\tilde{\Gamma}_0 = (\Gamma_0, \mathbb{f}_0, \beta_{edge,0}, \mathrm{vdeg}_0))$, where Γ_0 is the tree $\begin{tikzcd} v\,\bullet \arrow[r, dash, "e_1"] & \underset{v_1}{\bullet} \arrow[r, dash, "e_2"] & \bullet\, v' \end{tikzcd}$, with

- $\mathbb{f}_0(v) = (0, 0)$,
- $\mathbb{f}_0(v_1) = (\infty, 0)$,
- $\mathbb{f}_0(v') = (\infty, \infty)$,
- $\mathbb{f}_0(e_1)$ equal to the bottom edge of the square in $\Gamma_{(\mathbb{P}^1)^2,\mathbb{C}^*}$,
- $\mathbb{f}_0(e_2)$ equal to the rightmost edge of the square,
- $\beta_{edge,0}(e_1) = (d, 0)$,
- $\beta_{edge,0}(e_2) = (0, d)$, and
- $\mathrm{vdeg}_0(v) = \mathrm{vdeg}_0(v_1) = \mathrm{vdeg}_0(v') = 0$.

This shows that $\Psi^{-1}(\tilde{\Gamma}_0) \subseteq \overline{\Psi^{-1}(\tilde{\Gamma})} \subseteq \overline{\mathcal{M}}_{0,0}((\mathbb{P}^1)^2, (d, d))$. (The former is a single point, with some automorphism group.)

Note that if we had taken the limit $a \to \infty$ instead, we would have gotten a similar limit f_∞, but the two components would map to the top and leftmost edges of the square.

Remark 4.2.12. In Example 4.2.11, $\tilde{\Gamma}$ is obtained from $\tilde{\Gamma}_0$ by contracting a chain of edges $e_1 \cup e_2$, with intermediate vertex v_1, satisfying:

- $v_1 \in V^2(\tilde{\Gamma})$,
- $w(v_1, e_1) + w(v_1, e_2) = 0$.

This operation of "chain contraction" defines a partial ordering $\leq$ on $\mathrm{Trees}_{0,n}((\mathbb{P}^1)^2, \beta)$, where $\tilde{\Gamma} \leq \tilde{\Gamma}'$ if $\tilde{\Gamma}$ may be reached from $\tilde{\Gamma}'$ via a sequence of chain contractions. Generalizing the example, one may check that $\tilde{\Gamma} \leq \tilde{\Gamma}'$ if and only if $\Psi^{-1}(\tilde{\Gamma}') \subseteq \overline{\Psi^{-1}(\tilde{\Gamma})}$. Furthermore, $\leq$ has the property that if $\tilde{\Gamma}_1 \leq \tilde{\Gamma}'$ and $\tilde{\Gamma}_2 \leq \tilde{\Gamma}'$, then there exists $\tilde{\Gamma}$ satisfying $\tilde{\Gamma} \leq \tilde{\Gamma}_1$ and $\tilde{\Gamma} \leq \tilde{\Gamma}_2$.

It follows from these observations that the connected components of $\overline{\mathcal{M}}_{0,n}((\mathbb{P}^1)^2, \beta)$ are in bijection with decorated trees that are *minimal with respect to* $\leq$, namely those that admit no chain contractions, where the bijection identifies $\tilde{\Gamma}$ with $\overline{\mathcal{M}}_{\tilde{\Gamma}} := \overline{\Psi^{-1}(\tilde{\Gamma})}$.

In fact, using Example 4.2.11, one may explicitly write down $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ as a stack, in terms of certain moduli spaces of marked curves. This is closely related to the computation in Section 4.5 for $X = \mathrm{Sym}^d \mathbb{P}^r$.

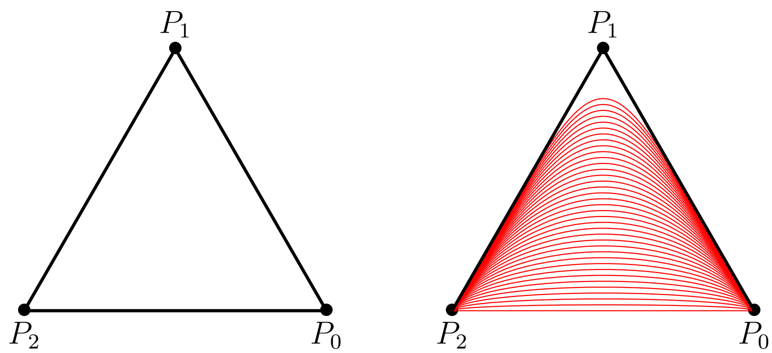


Figure 4.3: The graph $\Gamma_{\mathbb{P}^2,\mathbb{C}^*}$ and a picture of the 1-dimensional $\mathbb{C}^*$-orbits

Caution 4.2.13. Working from Example 4.2.11, one may be led to make all sorts of incorrect conjectures. The next few examples illustrate some of these.

Example 4.2.14. In Example 4.2.11, the ramification data of a T-fixed stable map was essentially constant in families. More specifically, when taking limits in $\overline{\mathcal{M}}_{0,n}((\mathbb{P}^1)^2, \beta)$, the only combinatorial difference between the special and general fibers is that irreducible components mapping as degree d fully ramified covers could be replaced with chains of rational curves, also mapping as degree d fully ramified covers. We see now that ramification data may change in a degeneration of T-fixed stable maps. Further, this change is not detected by our naive notion of decorated trees.

Consider $X = \mathbb{P}^2$, with $\mathbb{C}^*$ acting by $t \cdot [x_0 : x_1 : x_2] = [x_0 : tx_1 : t^2x_2]$. The curves $x_1^2 = ax_0x_2$ are 1-dimensional orbit closures for $a \in \mathbb{C}^*$. The graph $\Gamma_{\mathbb{P}^2,\mathbb{C}^*}$ is shown in Figure 4.3, along with a pictorial representation of the orbits. We may identify this family of curves with a family of stable maps, namely the inclusions of the 1-dimensional orbit closures $C_a \hookrightarrow \mathbb{P}^2$. Taking the limit as $a \to \infty$ gives a similar answer to that in Example 4.2.11: a nodal source curve C_∞, embedding in $\mathbb{P}^2$ as the union of the lines $L_{\{1,2\}}$ and $L_{\{0,1\}}$. (Recall from Section 3.2.1 that $L_{\{1,2\}} := \{x_0 = 0\}$ and $L_{\{0,1\}} = \{x_2 = 0\}$.) However, taking the limit as $a \to 0$ gives a degree 2 map f_0 from $C_0 \cong \mathbb{P}^1$ to the line $L_{\{0,2\}}$, ramified over the points P_2 and P_0.

In this example, the decorated tree is the same when $a \neq 0$ or $a = 0$, namely $\tilde{\Gamma} = \; v \bullet\!\overset{e}{\rule[0.5ex]{2em}{0.4pt}}\!\bullet v'$, with

- $\mathbb{f}(v) = P_2$,
- $\mathbb{f}(v) = P_0$,
- $\mathbb{f}(e)$ is the bottom edge of the triangle,
- $\beta_{edge}(e) = 2[L]$, and
- $\mathrm{vdeg}(v) = \mathrm{vdeg}(v') = 0$.

We note that the tangent weights at the points C_v and $C_{v'}$ are constant. To see this, note that the action on the tangent space at P_2 is equivariantly isomorphic to $t \cdot (x_0, x_1) = (t^{-2}x_0, t^{-1}x_1)$ on $\mathbb{C}^2$. For every $a \neq 0$, $f(C_e)$ is tangent to the line $L_{\{1,2\}}$, so the tangent weight $w(v, e)$ is $-\lambda$, where $\lambda \in H^2_{\mathbb{C}^*}(\mathrm{Spec}(\mathbb{C}), \mathbb{Q})$ is the canonical generator. When $a \to 0$, $f(C_e)$ is tangent to $L_{\{0,2\}}$, and is f is ramified at C_v with degree 2. Thus we have $w(v, e) = \frac{-2\lambda}{2} = -\lambda$.

The following example is modified from one pointed out to us by D. Speyer.

Example 4.2.15. In Example 4.2.11, we saw that if a graph $\tilde{\Gamma}'$ contained $v\bullet\!\overset{e_1}{\rule{2em}{0.4pt}}\!\underset{v_1}{\bullet}\!\overset{e_2}{\rule{2em}{0.4pt}}\!\bullet v'$ as a subgraph, where

- $v_1 \in V^2(\tilde{\Gamma}')$ and
- $w(v_1, e_1) + w(v_1, e_2) = 0$,

then any $(f' : C' \to (\mathbb{P}^1)^2) \in \Psi^{-1}(\tilde{\Gamma}')$ could be deformed to some $(f : C \to (\mathbb{P}^1)^2) \in \Psi^{-1}(\tilde{\Gamma})$, where $\tilde{\Gamma}$ is obtained from $\tilde{\Gamma}'$ by contracting the chain $e_1 \cup e_2$.

It is easy to see that the conditions $v_1 \in V^2(\tilde{\Gamma}')$ and $w(v_1, e_1) + w(v_1, e_2) = 0$ are necessary for such a deformation to exist. In general, a node of a curve may only be equivariantly smoothed if the tangent weights on the two branches add to zero. If $v_1 \notin V^2(\tilde{\Gamma}')$, then $v_1 \in V^S(\tilde{\Gamma}')$. Thus at the nodes $\xi(v_1, e_1)$ and $\xi(v_1, e_2)$, the tangent weight on one branch is zero, while the other is nonzero. Such nodes cannot deform.

However, we show that the two conditions are not sufficient. Consider $\mathbb{P}^2$, with the $\mathbb{C}^*$-action $t \cdot [x_0 : x_1 : x_2] = [x_0 : tx_1 : t^2x_2]$. We blow up $\mathbb{P}^2$ at P_1. Note that the $\mathbb{C}^*$-action lifts to this blowup, as pictured in Figure 4.4. It is easy to check that if we interpret the red curves as a family of stable maps and take the limit "upwards", the limit curve is a chain of three rational components, mapping to the three black lines. The middle component maps as a degree 2 ramified cover.

(One way to see this is by degree-counting. The generic map is degree $2H$, where H is the pullback of the hyperplane class. In the limit, the degree is $2(H - E) + kE$, where E is the exceptional divisor, $H - E$ is the class of the proper transforms of the lines $L_{\{1,2\}}$ and $L_{\{0,1\}}$, and k is the degree of the map on the middle component. For the degree to be constant, we must have $k = 2$. We can also check that the tangent weights on the middle component are double those on the axes.)

Now consider the following stable map. Let C be isomorphic to two copies of $\mathbb{P}^1$ glued at a node, and let f map one copy to (the proper transform of) the line $L_{\{1,2\}}$ isomorphically, and the second copy to the exceptional divisor as a degree 2 ramified cover. It is easy to check (and follows from the previous paragraph) that the two tangent weights at the node are opposites. On the other hand, f may not deform. Indeed, it would need to deform to a cover of an irreducible 1-dimensional

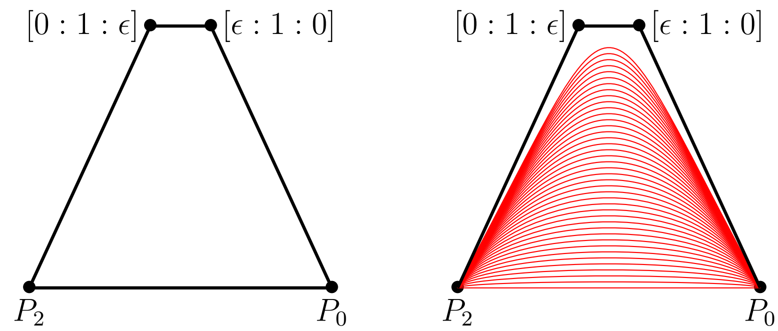


Figure 4.4: The graph $\Gamma_{\mathcal{B}\ell_{P_1}\mathbb{P}^2,\mathbb{C}^*}$ and a picture of the 1-dimensional $\mathbb{C}^*$-orbits

orbit closure, but we have already classified all such orbits. We say a chain contraction is *realizable* if it corresponds to a smoothing of a node on the source curve of a stable map. This example then shows that not every chain contraction is realizable.

Note that the characterization of the limit of generic 1-dimensional orbit closures also shows that there are realizable chain contractions where the two branches of the node being smoothed have different ramification degrees.

Definition 4.2.16. In [50], and later in [13], there are introduced (slightly incompatible) notions of *breaking nodes* and *breaking subcurves*. We use the definition from [13]; namely, a *breaking node* of a stable map $(f : C \to X)$ is one where the tangent weights of the two branches do not add up to zero.

By definition, $\Psi(f)$ contains a flag (or perhaps two) for every node, except those nodes on the interior of a subcurve that maps to X^T. These nodes are not breaking, since both tangent weights are zero. Therefore the set of breaking nodes includes into $F(\Gamma)$; we call a flag a *breaking flag* if it is in the image of this map.

Remark 4.2.17. Any flag (v, e) with $v \in V^S(\tilde{\Gamma})$ is a breaking flag.

Example 4.2.18. As we saw above, breaking nodes may not be equivariantly deformed. We saw in Example 4.2.15 that it may also be impossible to deform a nonbreaking node. We illustrate one more case of this, where the node is on the interior of a subcurve that maps to X^T.

First, consider the blowup Y of $\mathbb{P}^2$ at the points $[1:0:1]$, $[2:0:1]$, and $[3:0:1]$. The inclusion of the proper transforms of the x_0 and x_1 axes is a stable map to X. If it were to deform to a smooth curve, if would need to deform to the proper transform of a smooth plane conic passing through the three given points. But a smooth plane conic cannot pass through three colinear points. Therefore, the node cannot be smoothed.

To relate this to nonbreaking nodes, we need to realize Y as a component of the fixed locus of some variety X. We may simply take $X = Y \times \mathbb{P}^1$, with $\mathbb{C}^*$ acting on $\mathbb{P}^1$.

Having seeing these pathologies, we must settle for a very weak proposition, with no hope for any kind of converse.

Proposition 4.2.19. *Let $(f_0 : C_0 \to X)$ be in the closure of $\mathcal{M}_{\tilde{\Gamma}}$. Then $\tilde{\Gamma}$ can be obtained from $\Psi(f)$ by a sequence of chain contractions.*

Proof. Let $(f_t : C_t \to X) \in \mathcal{M}_{\tilde{\Gamma}}$ be a family of stable maps whose limit as $t \to 0$ is $(f_0 : C_0 \to X)$. Locally at $t = 0$, C_t is a smoothing of some collection of nodes of C_0. By equivariance, these nodes are nonbreaking, hence none of them connects an edge curve C_e to a vertex curve C_v. Thus smoothing these nodes corresponds to deleting vertices in $V^2(\tilde{\Gamma}_0)$, i.e. a sequence of chain contractions. □

4.3 The T-fixed locus in $\overline{\mathcal{M}}_{0,n}(X, \beta)$, orbifold case

If X is an orbifold, the situation is much more subtle than that above, and we know very little about the general case. As above, we may use decorated trees to isolate certain open and closed substacks of $\overline{\mathcal{M}}_{0,n}(X, \beta)^T$. We give a refinement of the definition that captures some of the "orbifold data:"

Definition 4.3.1. An *n-marked (X, T)-decorated tree* $\tilde{\Gamma} = (\Gamma, \mathrm{Mark}, \mathbb{f}, d, \mathrm{vdeg})$ is

- A tree Γ,
- A marking map $\mathrm{Mark} : \{1, \ldots, n\} \to V(\Gamma)$,
- A map $\mathbb{f}$ of graphs $\Gamma \to \Gamma_{X,T}$,
- An edge degree map $\beta_{edge} : E(\Gamma) \to H_2(X, \mathbb{Q})_e$, and
- A vertex degree map $\mathrm{vdeg} : V(\Gamma) \to H_2(\mathbb{f}(v), \mathbb{Q})$.
- A "monodromy map" Mon that assigns to each $i \in \{1, \ldots, n\}$ a component of the inertia stack of $\mathbb{f}(\mathrm{Mark}(i))$, and to each flag $(v, e) \in F(\Gamma)$ a conjugacy class in $G_{\xi(v,e)}$.

Remark 4.3.2. There are various conditions on these objects for the associated moduli space to be nonempty, such as: if $v \in V^2(\Gamma)$, then $\mathrm{Mon}(v, e_v^1)$ and $\mathrm{Mon}(v, e_v^2)$ must be in inverse conjugacy classes of $G_{\xi(v,e)}$.

In the nonorbifold case, there was a simple description of the T-fixed stable maps from an orbifold $\mathbb{P}^1$ to a specified 1-dimensional orbit closure; namely, a map is determined by its ramification degree at the two fixed points. In the orbifold case, it is much less clear what to do:

- When X is a toric stack, these maps may be computed exactly — this appears in [46], using methods from [36].

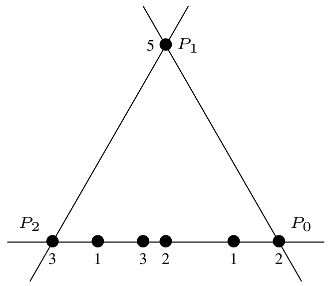


Figure 4.5: A point with one dimensional T-orbit in $\mathrm{Sym}^{17}\,\mathbb{P}^2$, with $i_1 = 1$ and $i_2 = 2$

- For $X = \mathrm{Sym}^d\,\mathbb{P}^r$, we are able to compute the maps, and we do so in the next two sections.
- More generally, if X is a quotient of a variety by a finite group G, then the maps may be described using principal G-bundles.
- If X (or more specifically the specified 1-dimensional orbit) is not toric, and is not a quotient of a variety by a finite group, then we do not know how to describe the maps.

4.4 T-Fixed points and 1-dimensional T-orbits in $\mathrm{Sym}^d\,\mathbb{P}^r$

As in Section 3.1.1, we write $(d(\bullet) \xrightarrow{P} \mathbb{P}^r)$ to denote a point $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Sym}^d\,\mathbb{P}^r$. We use notation from Section 3.2.1 for coordinate points and lines in $\mathbb{P}^r$.

Proposition 4.4.1. *1. A point* $(d(\bullet) \xrightarrow{P} \mathbb{P}^r) \in \mathrm{Sym}^d\,\mathbb{P}^r$ *is T-fixed if and only if* $\mathrm{Im}(P) \subseteq \{P_0, \ldots, P_r\}$.

2. $(d(\bullet) \xrightarrow{P} \mathbb{P}^r)$ *is in a 1-dimensional T-orbit if and only if (it is not T-fixed and)* $\mathrm{Im}(P) \subseteq \{P_0, \ldots, P_r\} \cup L_{\{i_1,i_2\}}$ *for some* $0 \leq i_1, i_2 \leq r$. *(See Figure 4.5.)*

Proof. 1 follows from the description of the T-action on $\widetilde{\mathrm{Sym}}^d\,\mathbb{P}^r$, and the fact that $\{P_0, \ldots, P_r\}$ is the T-fixed locus of $\mathbb{P}^r$.

The r-dimensional subtorus defined by $t_{i_1} = t_{i_2}$ acts trivially on $\{P_0, \ldots, P_r\} \cup L_{\{i_1,i_2\}}$, proving the backwards direction of 4.4.1. If $\mathrm{Im}(P) \not\subseteq \{P_0, \ldots, P_r\} \cup L_{\{i_1,i_2\}}$, the T-orbit is clearly at least 2-dimensional. $\square$

Remark 4.4.2. The T-fixed points of $\mathrm{Sym}^d\,\mathbb{P}^r$ are in natural bijection with $\mathrm{ZPart}(d, r+1)$, where the ith part is the number of points of $d(\bullet)$ mapping to P_i. (See Section 1.7.) We will use this identification from now on.

By the second part of Proposition 4.4.1, for each 1-dimensional T-orbit there is the associated data:

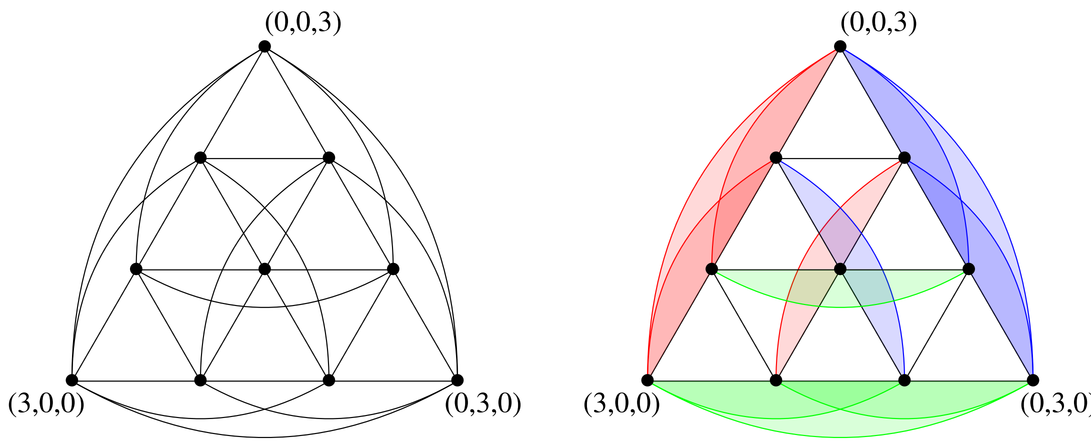


Figure 4.6: The T-graph, and a picture of 1-dimensional orbits, for $\operatorname{Sym}^3 \mathbb{P}^2$

1. Two (unordered) indices i_1 and i_2,
2. An element of $\operatorname{ZPart}(d')$, where $d' < d$ is the number of points of P mapping to $\{P_0, \ldots, P_r\}$, and
3. A $(d-d')$-tuple of points of $L_{\{i_1,i_2\}}$, up to scaling.

In Figure 4.5, 1 refers to the indices 0 and 2, and 2 refers to the nonnegative ordered partition $(2,5,3)$. (Of course, 3 depends on the relative coordinates of the 7 points on $L_{\{0,2\}}$.) Explicitly, this orbit is obtained from the point shown by simultaneously scaling the 7 points on the line $L_{\{0,2\}}$. As a consequence of Proposition 4.4.1, we obtain a description of $\Gamma_{\operatorname{Sym}^d \mathbb{P}^r, T}$, pictured in Figure 4.6 for $d = 3$ and $r = 2$. (The frontispiece to this thesis is $\Gamma_{\operatorname{Sym}^4 \mathbb{P}^2, T}$.)

4.5 The T-fixed locus in $\overline{\mathcal{M}}_{0,n}(\operatorname{Sym}^d \mathbb{P}^r, \beta)$

This section, which is in some sense the essential part of the proof of Theorem 6.1.2, draws heavily on the notations introduced in Sections 1.7, 3.1.1, 3.2.1, and 4.2. We will see that 1-dimensional orbits in $\operatorname{Sym}^d \mathbb{P}^r$ are reasonably well-behaved, in that the behaviors observed in Examples 4.2.14, 4.2.15, and 4.2.18 do not occur. As in Remark 4.2.12, we will define a partial ordering on decorated trees via chain contractions. The closure $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ of a *minimal* decorated tree $\mathcal{M}_{\tilde{\Gamma}}$ is an open and closed substack of $\overline{\mathcal{M}}_{0,n}(\operatorname{Sym}^d \mathbb{P}^r, \beta)^T$. Theorem 4.5.23 exhibits each of these substacks as a quotient of a specific variety by a finite group. The variety in question is a toric compactification of the moduli space $\mathcal{M}_{0,n}$, and parametrizes chains of rational marked curves glued at nodes. It was introduced by Losev and Manin [47].

4.5.1 $\mathrm{Sym}^d\,\mathbb{P}^r$-decorated trees

As above, for a T-fixed stable map $f : C \to \mathrm{Sym}^d\,\mathbb{P}^r$, each component C_ν of C maps into the fixed locus $(\mathrm{Sym}^d\,\mathbb{P}^r)^T$, or maps a 1-dimensional orbit closure, with special points (nodes and marks) and ramification points mapping to fixed points. Since T acts with isolated fixed points, we refer to the two types of components as *contracted* and *noncontracted*, since those of the first type map to a single point of $\mathrm{Sym}^d\,\mathbb{P}^r$.

Lemma 4.5.1. *Let $(f : C \to \mathrm{Sym}^d\,\mathbb{P}^r) \in \overline{\mathcal{M}}_{0,2}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$ be a T-fixed stable map of degree $\beta > 0$ with irreducible source curve. This is a map to a 1-dimensional orbit closure, which by Proposition 4.4.1 has associated indices i_1 and i_2. Denote by b_1 and b_2 the two marked points of C. Then:*

- *The associated étale cover $\rho : C' \to C$ from Section 3.1.1 is a disjoint union of rational connected components,*
- *Under the associated map $f' : C' \to \mathbb{P}^r$, each component of C' is either contracted to a T-fixed point of $\mathbb{P}^r$, or maps to the coordinate line $L_{\{i_1,i_2\}}$,*
- *On each component C'_η of the latter type, $\rho^{-1}(b_1)$ and $\rho^{-1}(b_2)$ are each a (single) fully ramified point, and*
- *If c_η is the degree of $\rho|_{C'_\eta} : C'_\eta \to C$ and β_η is the (coarse) degree of $f'|_{C'_\eta} : C'_\eta \to L_{\{i_1,i_2\}}$, then the ratio $q := \beta_\eta/c_\eta$ is independent of η (where η runs over* noncontracted *components of C').*

Proof. The first three statements follow from the fact that C has exactly two orbifold points, and from Proposition 4.4.1. It is straightforward to check that the last statement is equivalent to the fact that the T-action is compatible with the map ρ, i.e. that the action of $\lambda \in T$ corresponds to changing coordinates on C. $\square$

Remark 4.5.2. The same statement and proof apply to $\overline{\mathcal{M}}_{0,1}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$ and $\overline{\mathcal{M}}_{0,0}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$ and in these cases we have a slightly stronger statement: since C has at most one orbifold point, it has no nontrivial étale cover, i.e. $C' \cong C \times \{1, \ldots, d\}$.

For $X = \mathrm{Sym}^d\,\mathbb{P}^r$, we translate Definition 4.3.1 into the following equivalent, and more explicit, definition.

Definition 4.5.3. An *n-marked* $(\mathrm{Sym}^d\,\mathbb{P}^r, T)$*-decorated tree* $\tilde{\Gamma} = (\Gamma, \mathrm{Mark}, \mathrm{VEval}, q, \mathrm{Mon})$ is

- A tree Γ,

- A marking map $\mathrm{Mark} : \{1, \ldots, n\} \to V(\Gamma)$,
- A "vertex evaluation map" $\mathrm{VEval} = (\mathrm{VEval}_0, \ldots, \mathrm{VEval}_r) : V(\Gamma) \to \mathrm{ZPart}(d, r+1)$, where $\mathrm{ZPart}(d, r+1)$ is the set of nonnegative ordered partitions of d of length $r+1$,
- An "edge degree ratio" map $q : E(\Gamma) \to \mathbb{Q}_{>0}$,
- A "monodromy map" Mon that assigns to each $i \in \{1, \ldots, n\}$ an element of the set $\mathrm{MultiPart}(\mathrm{VEval}(\mathrm{Mark}(i)))$ of multipartitions (see Section 1.7) of $\mathrm{VEval}(\mathrm{Mark}(i))$, and assigns to a flag $(v, e) \in F(\Gamma)$ an element of $\mathrm{MultiPart}(\mathrm{VEval}(v))$,

subject to the conditions:

1. If e is an edge of Γ connecting vertices v and v', then $\mathrm{VEval}_k(v) = \mathrm{VEval}_k(v')$ for all but exactly two $0 \leq k \leq r$, denoted $i^{\mathrm{mov}}(v, e)$ and $i^{\mathrm{mov}}(v', e)$, which are defined so that $\mathrm{VEval}_{i^{\mathrm{mov}}(v,e)}(v) - \mathrm{VEval}_{i^{\mathrm{mov}}(v,e)}(v') > 0$,,
2. If e is an edge connecting vertices v and v', then for $i \neq i^{\mathrm{mov}}(v, e), i^{\mathrm{mov}}(v', e)$, $\mathrm{Mon}_i(v, e)$ is equal[2] (as a partition of $\mathrm{VEval}_i(v) = \mathrm{VEval}_i(v')$) to $\mathrm{Mon}_i(v', e)$. Furthermore, we have $\mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v', e) \subseteq \mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v, e)$, $\mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v, e) \subseteq \mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v, e)$, and the relation between complements holds:
$$\mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v, e) \setminus \mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v', e) = \mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v', e) \setminus \mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v, e) =: \mathrm{Mov}(e),$$
3. For $\eta \in \mathrm{Mov}(e)$, we have $q(e)\eta \in \mathbb{Z}$,
4. If $v \in V^1(\Gamma)$ then $\mathrm{Mon}(v, e_v)$ is the ones multipartition of $\mathrm{VEval}(v)$,
5. If $v \in V^{1,1}(\Gamma)$ with $v = \mathrm{Mark}(i)$, then $\mathrm{Mon}(v, e_v) = \mathrm{Mon}(i)$, and
6. If $v \in V^2(\Gamma)$, then $\mathrm{Mon}(v, e_v^1) = \mathrm{Mon}(v, e_v^2)$.

Definition 4.5.4. Let $\tilde{\Gamma}$ be an n-marked $(\mathrm{Sym}^d \mathbb{P}^r, T)$-decorated tree $\tilde{\Gamma}$. If $e \in E(\Gamma)_{v,v'}$, then the degree of e is $\beta_{edge}(e) := q(e)(\mathrm{VEval}_{i^{\mathrm{mov}}(v,e)}(v) - \mathrm{VEval}_{i^{\mathrm{mov}}(v,e)}(v'))$. The degree of $\tilde{\Gamma}$ is $\beta(\tilde{\Gamma}) = \sum_{e \in E(\Gamma)} \beta_{edge}(e)$.

Remark 4.5.5. We explain why this is equivalent to Definition 4.3.1.

[2] Here $\mathrm{Mon}(v, e)$ is an $r+1$-tuple of partitions, which we index by $\{0, \ldots, r\}$, so that $\mathrm{Mon}_i(v, e)$ is the $(i-1)$st partition.

- Because $\Gamma_{\mathrm{Sym}^d \mathbb{P}^r, T}$ has a unique edge between any two adjacent vertices, the map $\mathbb{f}$ is determined by the induced map of vertex sets $\mathbb{f}_v : V(\Gamma) \to V(\Gamma_{\mathrm{Sym}^d \mathbb{P}^r})$. The map VEval above is exactly this map, since we saw in Section 4.4 that fixed points of $\mathrm{Sym}^d \mathbb{P}^r$ are in natural bijection with $\mathrm{ZPart}(d, r+1)$. Condition 1 in the definition guarantees that VEval comes from a map of graphs.

- By Lemma 4.5.1, the edge degree map in Definition 4.5.4 agrees with the one in Definition 4.3.1.

- Since T acts with isolated fixed points, vdeg is trivial.

- The sets $\mathrm{MultiPart}(\mathrm{VEval}(\mathrm{Mark}(i)))$ and $\mathrm{MultiPart}(\mathrm{VEval}(v))$ are in bijection with the sets of appropriate conjugacy classes, so the monodromy map defined here is the same as that in Definition 4.3.1.

The analog of Proposition 4.2.7 holds:

Proposition 4.5.6. *There is a natural map*

$$\Psi : (\overline{\mathcal{M}}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T \to \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta).$$

The proof is identical to that of Proposition 4.2.7, with the obvious definition of the map Mon.

Remark 4.5.7. Conditions 2–6 on a decorated tree eliminate decorated trees for which $\mathcal{M}_{\tilde{\Gamma}}$ is trivially empty:

- Condition 2 is the condition that the monodromies at the two fixed points of a noncontracted component are compatible. (Unlike the others, this is specific to $\mathrm{Sym}^d \mathbb{P}^r$.)

- Condition 3 requires that the map $f'|_{C'_\eta}$ be a ramified cover with *integer* degree. (See Lemma 4.5.1)

- Condition 4 says that if a point of C mapping to a T-fixed point is not special, then it is not an orbifold point, hence has trivial monodromy.

- Condition 5 says that if a point of C mapping to a T-fixed point is marked, then the monodromy is the same whether this point is viewed as $\xi(v, e)$ or the ith marking.

- Condition 6 says that if the monodromy at the two branches of a node must be inverses, when both branches are noncontracted.

These conditions are still not enough to guarantee that $\mathcal{M}_{\tilde{\Gamma}}$ is nonempty. Another, more complicated, condition is needed on the set of monodromies at special points on a contracted component. Namely, there must exist a ramified degree d cover of $\mathbb{P}^1$ whose ramification profile is given by that set of monodromies. This is equivalent to the nonemptiness of a Hurwitz space $\mathcal{H}_v$ associated to each vertex v with $\left|\mathrm{Mark}^{-1}(v)\right| + \mathrm{val}(v) \geq 3$.

Alternatively, we could include in Definition 4.5.3 the additional data of a connected component of each Hurwitz space $\mathcal{H}_v$. (We could even define $\mathcal{H}_v$ by convention when $\left|\mathrm{Mark}^{-1}(v)\right| + \mathrm{val}(v) < 3$, in which case Conditions 4– 6 of Definition 4.5.3 would no longer be necessary.) In this case, Theorem 4.5.23 would imply that $\mathcal{M}_{\tilde{\Gamma}}$ is nonempty and *connected*. However, we do not adopt this definition, because the combinatorial description of the connected components of a Hurwitz space is quite complicated.

Notation 4.5.8. Note that the isotropy group at $\xi(v, e)$ (resp. b_i) has order $\mathrm{lcm}(\mathrm{Mon}(v, e))$ (resp. $\mathrm{lcm}(\mathrm{Mon}(i))$). For brevity we denote this by $r(v, e)$ (resp. r_i).

We now describe how the spaces $\mathcal{M}_{\tilde{\Gamma}}$ fit together via chain contractions (as defined in Remark 4.2.12). In particular, Theorem 4.5.15 tells us that any chain contraction gives rise to a limiting process of stable maps.

Definition 4.5.9. Let $\tilde{\Gamma} \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, and let $e_1, e_2 \in E(\Gamma)$. We say e_1 and e_2 are *combinable*, and write $e_1 \parallel e_2$, if there exists $v \in V^2(\tilde{\Gamma})$ with $\{e_1, e_2\} = \{e_v^1, e_v^2\}$ and the following hold:

- $q(e_1) = q(e_2)$,
- $i^{\mathrm{mov}}(v_1, e_1) = i^{\mathrm{mov}}(v, e_2)$ and
- $i^{\mathrm{mov}}(v, e_1) = i^{\mathrm{mov}}(v_2, e_2)$.

Denote by $\mathcal{P} \subseteq \binom{E(\Gamma)}{2}$ the set of pairs $\{\{e_1, e_2\} : e_1 \parallel e_2\}$.

Fix $(f : C \to \mathrm{Sym}^d \mathbb{P}^r) \in \mathcal{M}_{\tilde{\Gamma}}$. We compute that for a flag (v, e) with $e = \{v, v'\}$, the tangent weight at $\xi(v, e)$ along C_e is $w(v, e) = \frac{\alpha_{i^{\mathrm{mov}}(v,e)} - \alpha_{i^{\mathrm{mov}}(v',e)}}{q(e)}$. Thus two edges are combinable if and only if they correspond to components of C separated by a *nonbreaking* node.

Definition 4.5.10. Let $(v, e) \in F(\Gamma)$. We say (v, e) is a *steady flag* if either of the following holds:

1. $v \notin V^2(\Gamma)$, or
2. $v \in V^2(\Gamma)$ and $\{e_v^1, e_v^2\} \notin \mathcal{P}$.

Otherwise (v, e) is *unsteady*.

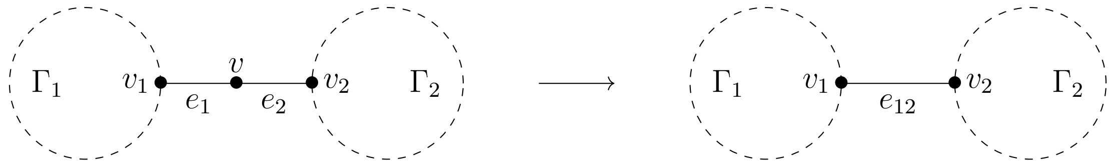


Figure 4.7: Combining edges

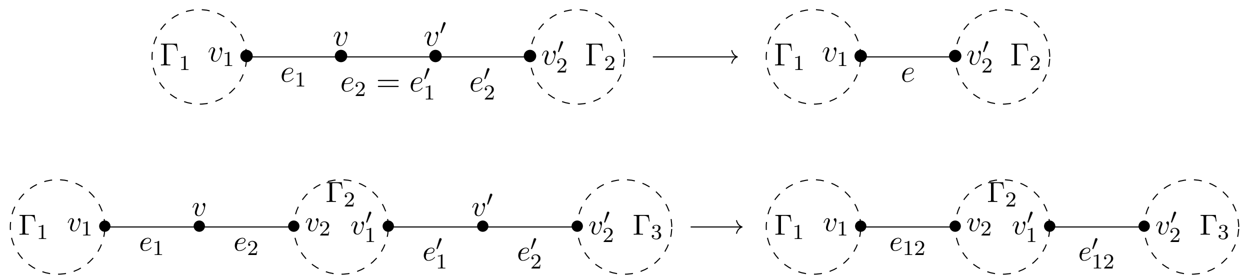


Figure 4.8: Combining two pairs of edges

Definition 4.5.11. Let $\tilde{\Gamma} \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ and let $e_1 \parallel e_2$ be a pair of combinable edges. We may define a new decorated tree $\mathrm{Comb}(\tilde{\Gamma}, e_1 \parallel e_2) \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ by **combining** e_1 **and** e_2. In other words, we delete the vertex v and the edges e_1 and e_2, and add an edge $e_{12} = \{v_1, v_2\}$ with $q(e_{12}) = q(e_1) = q(e_2)$, $\mathrm{Mon}(v_1, e_{12}) = \mathrm{Mon}(v_1, e_1)$, and $\mathrm{Mon}(v_2, e_{12}) = \mathrm{Mon}(v_2, e_2)$. (See Figure 4.7.) It is easy to check that $\tilde{\Gamma}(e_1, e_2)$ satisfies the two conditions of a decorated tree, and that $\mathrm{Mov}(e_{12}) = \mathrm{Mov}(e_1) \cup \mathrm{Mov}(e_2)$, and $\mathrm{Mon}(e_{12}) = \mathrm{Mon}(e_1) = \mathrm{Mon}(e_2)$. There is a natural map $\phi_{\{e_1,e_2\}} : E(\Gamma) \to E(\mathrm{Comb}(\tilde{\Gamma}, e_1 \parallel e_2))$ with $\phi_{\{e_1,e_2\}}(e_1) = \phi_{\{e_1,e_2\}}(e_2) = e_{12}$, and $\phi_{\{e_1,e_2\}}(e) = e$ for $e \in E(\Gamma) \smallsetminus \{e_1, e_2\}$.

Proposition 4.5.12. *Let $\tilde{\Gamma} \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, and let $e_1 \parallel e_2$ and $e_1' \parallel e_2'$ be two distinct pairs of combinable edges of Γ. Then $\phi_{\{e_1,e_2\}}(e_1') \parallel \phi_{\{e_1,e_2\}}(e_2')$ as edges of $\mathrm{Comb}(\tilde{\Gamma}, e_1 \parallel e_2)$ and $\phi_{\{e_1',e_2'\}}(e_1) \parallel \phi_{\{e_1',e_2'\}}(e_2)$ as edges of $\mathrm{Comb}(\tilde{\Gamma}, e_1' \parallel e_2')$. Also, combining pairs commutes, i.e.*

$$\mathrm{Comb}(\mathrm{Comb}(\tilde{\Gamma}, e_1 \parallel e_2), e_1' \parallel e_2') \cong \mathrm{Comb}(\mathrm{Comb}(\tilde{\Gamma}, e_1' \parallel e_2'), e_1 \parallel e_2),$$

and this isomorphism identifies the maps $\phi_{\{e_1,e_2\}} \circ \phi_{\{e_1',e_2'\}}$ and $\phi_{\{e_1',e_2'\}} \circ \phi_{\{e_1,e_2\}}$.

Proof. There are two cases, pictured in the left side of Figure 4.5.1; either the pairs $e_1 \parallel e_2$ and $e_1' \parallel e_2'$ share an edge, or they do not. Suppose we are in the first case, i.e. the top line of Figure 4.5.1. By definition of $\phi_{\{e_1,e_2\}}$, the edges $\phi_{\{e_1,e_2\}}(e_1')$ and $\phi_{\{e_1,e_2\}}(e_2')$ meet at v' (precisely, at the corresponding vertex in $\mathrm{Comb}(\tilde{\Gamma}, e_1 \parallel e_2)$), and satisfy the three conditions of Definition 4.5.9. Thus $\phi_{\{e_1,e_2\}}(e_1') \parallel \phi_{\{e_1,e_2\}}(e_2')$. Similarly $\phi_{\{e_1',e_2'\}}(e_1) \parallel \phi_{\{e_1',e_2'\}}(e_2)$. To see that $\mathrm{Comb}(\mathrm{Comb}(\tilde{\Gamma}, e_1 \parallel e_2), e_1' \parallel e_2') \cong \mathrm{Comb}(\mathrm{Comb}(\tilde{\Gamma}, e_1' \parallel e_2'), e_1 \parallel e_2)$, we note that both are obtained from the tree in Figure 4.5.1 by replacing the three edges shown with a single edge e connecting v_1 to v_2'. The decorations on this edge are:

- $q(e) := q(e_1) = q(e_2) = q(e_2')$,
- $\mathrm{Mon}(e) := \mathrm{Mon}(e_1) = \mathrm{Mon}(e_2) = \mathrm{Mon}(e_2')$,
- $i^{\mathrm{mov}}(v_1, e) := i^{\mathrm{mov}}(v_1, e_1) = i^{\mathrm{mov}}(v, e_2) = i^{\mathrm{mov}}(v', e_2')$, and
- $i^{\mathrm{mov}}(v_2', e) := i^{\mathrm{mov}}(v_2, e_2') = i^{\mathrm{mov}}(v', e_2) = i^{\mathrm{mov}}(v, e_1)$,

where the equalities follow from $e_1 \parallel e_2$ and $e_2 \parallel e_2'$. The maps $\phi_{\{e_1,e_2\}} \circ \phi_{\{e_1',e_2'\}}$ and $\phi_{\{e_1',e_2'\}} \circ \phi_{\{e_1,e_2\}}$ both send all of e_1, $e_2 = e_1'$, and e_2' to e.

The second case (the bottom line of 4.5.1) is a special case of this argument, so we omit it. $\square$

Corollary 4.5.13. *Let* $\tilde{\Gamma} \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$*, and let* $\mathcal{E}$ *be any subset of the set* $\mathcal{P}(\tilde{\Gamma})$ *of pairs of combinable edges in* Γ*. Then there is a well-defined tree* $\mathrm{Comb}(\tilde{\Gamma}, \mathcal{E}) \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ *obtained by combining all edge pairs in* $\mathcal{E}$*, in any order, and a well-defined associated map* $\phi_{\mathcal{E}} : E(\Gamma) \to E(\mathrm{Comb}(\tilde{\Gamma}, \mathcal{E}))$*. Furthermore,* $\mathcal{E}$ *is determined by the trees* $\tilde{\Gamma}$ *and* $\mathrm{Comb}(\tilde{\Gamma}, \mathcal{E})$*, and the map* $\phi_{\mathcal{E}}$*.*

Proof. The existence statement comes from repeatedly applying Proposition 4.5.12. The uniqueness statement amounts to the fact that if $e_1 \parallel e_2$ is a compatible pair of edges in $\tilde{\Gamma}$, then $\phi_{\mathcal{E}}(e_1) = \phi_{\mathcal{E}}(e_2)$ if and only if $(e_1, e_2) \in \mathcal{E}$. This follows from factoring $\phi_{\mathcal{E}}$ as a sequence of edge combination maps as in Definition 4.5.11. $\square$

Corollary 4.5.13 may be restated as follows. Definition 4.5.11 determines a partial order $\leq$ on $\mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, where $\tilde{\Gamma}' \leq \tilde{\Gamma}$ if $\tilde{\Gamma}'$ can be obtained from $\tilde{\Gamma}$ by combining edges. The Corollary then states that for $\tilde{\Gamma} \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, there is a natural order-reversing bijection between $\{\tilde{\Gamma}' : \tilde{\Gamma}' \leq \tilde{\Gamma}\}$ and $\{\text{subsets of } \mathcal{P}(\tilde{\Gamma})\}$, where the latter is partially ordered by inclusion. In particular, associated to $\tilde{\Gamma}$ is a unique *minimal* decorated tree $\mathrm{Comb}(\tilde{\Gamma}, \mathcal{P}(\tilde{\Gamma}))$.

Notation 4.5.14. Denote by $\mathrm{Trees}^{\min}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta) \subseteq \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ the set of minimal elements with respect to $\leq$.

Theorem 4.5.15. *Let* $\tilde{\Gamma}_0 \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$*. The closure of* $\Psi^{-1}(\tilde{\Gamma}_0)$ *is*

$$\bigcup_{\substack{\tilde{\Gamma} \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta) \\ \tilde{\Gamma}_0 \leq \tilde{\Gamma}}} \Psi^{-1}(\tilde{\Gamma}),$$

where Ψ *is the map from Proposition 4.2.7.*

Lemma 4.5.16. *Let* $\tilde{\Gamma}_0 = v_1 \bullet\!\xrightarrow{\quad e \quad}\!\bullet v_2$ *, where each of* v_1 *and* v_2 *contains a single marked point,* b_1 *and* b_2*. Let* $f : C \to \mathrm{Sym}^d \mathbb{P}^r$ *be in the closure of* $\Psi^{-1}(\tilde{\Gamma}_0)$*, and let* $\rho : C' \to C$ *and* $f' : C' \to \mathbb{P}^r$ *be the associated maps. Write* C'_η *for a noncontracted irreducible component of* C'*, corresponding to* $\eta \in \mathrm{Mov}(e) \subseteq \mathrm{Mon}(e)$*, as described in Lemma 4.5.1. Denote by* $L_e := L_{\{i^{\mathrm{mov}}(v_1,e), i^{\mathrm{mov}}(v_2,e)\}}$ *the line in* $\mathbb{P}^r$ *connecting* $P_{i^{\mathrm{mov}}(v_1,e)}$ *and* $P_{i^{\mathrm{mov}}(v_2,e)}$*. Then:*

1. *C and C'_η are nodal chains of rational curves,*
2. *$f'|_{C'_\eta}$ maps one irreducible component of C'_η to L_e with degree $\beta_\eta(e) = q(e) \cdot \eta$ (on coarse moduli spaces), and is fully ramified at the two special points of this component, and*
3. *$f'|_{C'_\eta}$ contracts all other irreducible components of C'_η to one of the endpoints of L_e.*

Proof of Lemma. Let $f : \mathcal{C} \to \mathbb{P}^r$ be a family over S of stable maps whose generic fiber is in $\Psi^{-1}(\tilde{\Gamma}_0)$, and let $s \in S$ such that the fiber over s is the stable map $f : C \to \mathrm{Sym}^d \mathbb{P}^r$. After an étale base change $\tilde{S} \to S$, $\mathcal{C}'$ is a union of connected components $\mathcal{C}'_\eta$ indexed by $\mathrm{Mon}(e)$, and the maps $\mathcal{C}'_\eta \to \mathcal{C}$ have degrees determined by $\mathrm{Mon}(e)$.

Consider the Stein factorization relative to S:

$$\mathcal{C}'_\eta \xrightarrow{\mathrm{sf}} \overline{\mathcal{C}'_\eta} \xrightarrow{\overline{f'}} \mathbb{P}^r.$$

The pullbacks along $\overline{f'}$ of the divisors $P_{i^{\mathrm{mov}}(v_1,e)}$ and $P_{i^{\mathrm{mov}}(v_2,e)}$ on L_e are divisors on $\overline{\mathcal{C}'_\eta}$, that by the definition of the Stein factorization do not contain a component of any fiber. On a generic fiber, these divisors are each supported on a single point, i.e. $\rho^{-1}(b_1)$ and $\rho^{-1}(b_2)$. Thus on the fiber $\overline{C'_\eta}$ over s, the divisors are still supported on single points, and $\rho^{-1}(b_1)$ and $\rho^{-1}(b_2)$ each lie above one of these points. (Also, the points are distinct since $\overline{f'}$ is well-defined.)

As any component of $\overline{C'_\eta}$ maps surjectively to L_e, the above implies that $\overline{C'_\eta}$ is irreducible. This proves claims (2) and (3).

Since f' is T-fixed, any irreducible components of C'_η that are contracted by f' map to either $P_{i^{\mathrm{mov}}(v_1,e)}$ or $P_{i^{\mathrm{mov}}(v_2,e)}$, i.e. they lie over either $(\overline{f'})^{-1}(P_{i^{\mathrm{mov}}(v_1,e)})$ or $(\overline{f'})^{-1}(P_{i^{\mathrm{mov}}(v_2,e)})$. Also, all nodes of C'_η lie over one of these two points. Since η was arbitrary, this shows that any irreducible component D of C' that is not contracted by f' has at most two special points, where a *special point* here means either a node or one of the points $\rho^{-1}(b_1)$ and $\rho^{-1}(b_2)$. Since $\rho^{-1}(b_1)$ and $\rho^{-1}(b_2)$ lie above distinct points of D, D has exactly two special points.

If C is not a chain, some component has only one special point. By stability there is a component D of $\rho^{-1}(D)$ that is not contracted by f', which contradicts the fact that D has two special points. Thus C is a chain, and it follows that each connected component C'_η is a chain. This proves claim (1). $\square$

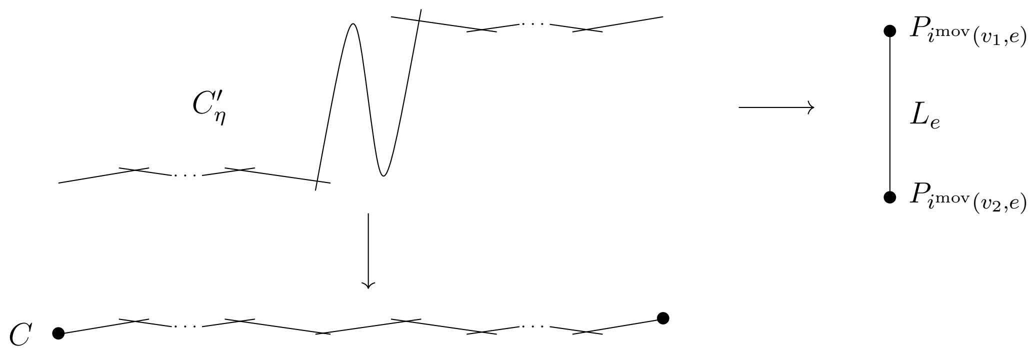


Figure 4.9: A portion of a map in $\overline{\Psi^{-1}(\tilde{\Gamma}_0)}$, with $\eta = 1$ and $q(e) = 3$

Remark 4.5.17. In summary, the restriction to C'_η of a point in $\overline{\Psi^{-1}(\tilde{\Gamma}_0)}$ may be represented as in Figure 4.5.1 (where despite appearances we mean for the map to L_e to have a single preimage point over each of $P_{i^{\mathrm{mov}}(v_1,e)}$ and $P_{i^{\mathrm{mov}}(v_2,e)}$).

Proof of Theorem 4.5.15. It is sufficient to consider the situation of Lemma 4.5.16. To see this, note that any $\tilde{\Gamma}_0 \in \mathrm{Trees}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ may be decomposed into subtrees of the form in the Lemma, together with single-vertex trees, glued at marked points. There is a corresponding decomposition of $\Psi^{-1}(\tilde{\Gamma}_0)$ as a product (up to a finite morphism), and this decomposition extends to the closure (see [2], Section 5.2, or [46], Section 9.2). Thus we may treat each factor of the product separately.

First, we show

$$\overline{\Psi^{-1}(\tilde{\Gamma}_0)} \subseteq \bigcup_{\tilde{\Gamma}, \tilde{\Gamma}_0 \leq \tilde{\Gamma}} \Psi^{-1}(\tilde{\Gamma}).$$

Let $(f : C \to \mathbb{P}^r) \in \overline{\Psi^{-1}(\tilde{\Gamma}_0)}$. It follows from Lemma 4.5.16 that $f^{-1}((\mathrm{Sym}^d \mathbb{P}^r)^T)$ is exactly the set of nodes of C, together with the two marked points. By stability, all irreducible components of C are noncontracted. Thus the tree $\Psi(f : C \to \mathrm{Sym}^d \mathbb{P}^r)$ is a chain with a vertex for each node and marked point, and an edge for each irreducible component.

Denote by v_1 and v_2 the leaves of C, such that $v_1 = \{b_1\}$ and $v_2 = \{b_2\}$. For $v \neq v_1, v_2$, we have $\mathrm{Mark}^{-1}(v) = \emptyset$. By claim 2 of Lemma 4.5.16, the degree ratios $q(e)$ are equal for all edges e. By the description of the connected components of C', the partitions $\mathrm{Mon}(e)$ are equal for all e. Finally, deleting an edge e breaks C into two connected components, one containing v_1 and one containing v_2. Let v be on the component with v_1, and v' on the component with v_2, such that $e = \{v, v'\}$. Then from the proof of Lemma 4.5.16, we have $i^{\mathrm{mov}}(v, e) = i^{\mathrm{mov}}(v_1, e_{12})$ and $i^{\mathrm{mov}}(v', e) = i^{\mathrm{mov}}(v_2, e_{12})$. Thus any pair of adjacent edges in $\Psi(f : C \to \mathrm{Sym}^d \mathbb{P}^r)$ is combinable. Combining them all yields $\tilde{\Gamma}_0$, i.e. $\tilde{\Gamma}_0 \leq \Psi(f : C \to \mathrm{Sym}^d \mathbb{P}^r)$.

For the converse, by induction on $\left|E(\tilde{\Gamma})\right| - \left|\tilde{\Gamma}_0\right| = \left|E(\tilde{\Gamma})\right| - 1$, it is sufficient to show that

$$\overline{\Psi^{-1}(\tilde{\Gamma}_0)} \supseteq \bigcup_{\substack{\tilde{\Gamma} \text{ such that} \\ \tilde{\Gamma}_0 = \mathrm{Comb}(\tilde{\Gamma}, e_1 \| e_2) \\ \text{for some } e_1 \,\|\, e_2 \in \mathcal{P}(\tilde{\Gamma})}} \Psi^{-1}(\tilde{\Gamma}).$$

Fix such a tree $\tilde{\Gamma} = v_1 \bullet\!\overset{e_1}{\text{——}}\!\underset{}{\overset{v}{\bullet}}\!\overset{e_2}{\text{——}}\!\bullet v_2$, and fix $(f : C \to \mathrm{Sym}^d \mathbb{P}^r) \in \Psi^{-1}(\tilde{\Gamma})$. We will construct a family $f : \mathcal{C} \to \mathrm{Sym}^d \mathbb{P}^r$ over $\mathbb{C}$ whose restriction to $0 \in \mathbb{C}$ is the map $f : C \to \mathrm{Sym}^d \mathbb{P}^r$.

By Lemma 4.5.1 and by representability of $f : C \to \mathrm{Sym}^d \mathbb{P}^r$, the orbifold points and nodes of C have order $\mathrm{lcm}(\mathrm{Mon}(e_1)) = \mathrm{lcm}(\mathrm{Mon}(e_2))$. Thus C is isomorphic to $V(xy) \subseteq [\mathbb{P}^2/\mu_{\mathrm{lcm}(\mathrm{Mon}(e_1))}]$, where $\mathbb{P}^2$ has coordinates x, y, z, and $\mathrm{lcm}(\mathrm{Mon}(e_1))$ acts by multiplication by inverse roots of unity on the first two coordinates. Define $\mathcal{C}$ so that $\mathcal{C}_t = V(xy - tz^2)$ for $t \in \mathbb{C}$. Precisely, $\mathcal{C}$ is an open subset of $\left[\mathcal{B}\ell_{[1:0:0],[0:1:0]}\mathbb{P}^2/\mu_{\mathrm{lcm}(\mathrm{Mon}(e_1))}\right]$.

For $\eta \in \mathrm{Mon}(e_1)$ a part, there is an étale quotient map $\tilde{\rho} : [\mathbb{P}^2/\mu_\eta] \to [\mathbb{P}^2/\,\mathrm{lcm}(\mathrm{Mon}(e_1))]$. As above, define $(\mathcal{C}'_\eta)_t = V(xy - tz^2) \subseteq [\mathbb{P}^2/\mu_\eta]$.

We must now define a map $\tilde{f}' : \mathcal{C}'_\eta \to \mathbb{P}^r$ for each $\eta \in \mathrm{Mon}(e_1)$. As $\mathbb{P}^r$ is a variety, it is enough to define this on coarse moduli spaces. We choose isomorphisms of the fibers $(\mathcal{C}'_\eta)_0$ and $\mathcal{C}_0$ with C'_η and C respectively, such that the maps $\tilde{\rho}$ and ρ are identified. Then f' defines a map $\tilde{f}'_0 : (\mathcal{C}'_\eta)_0 \to L_{e_1} = L_{e_2}$. (The case where C'_η is contracted is trivial, so we assume it is not contracted.) By Lemma 4.5.16, after equivariantly identifying $L_{e_1} \cong \mathbb{P}^1$, $\tilde{f}'_0$ is given (without loss of generality, on coarse moduli spaces) by

$$\begin{aligned}
[x : 0 : z] &\mapsto [0 : 1] \\
[0 : y : z] &\mapsto [y^{\beta_\eta(e_1)} : z^{\beta_\eta(e_1)}].
\end{aligned}$$

It remains to extend this to a map $\tilde{f}' : \mathcal{C}'_\eta \to L_{e_1}$ that is fixed with respect to the T-action, i.e. fully ramified over the endpoints of L_{e_1}. We observe that the rational map

$$[x : y : z] \mapsto [y^{\beta_\eta(e_1)} : z^{\beta_\eta(e_1)}]$$

is regular after blowing up the point $[1 : 0 : 0]$. This defines a map $\tilde{f}'$ as desired. Doing this for all η simultaneously shows that $f : C \to \mathrm{Sym}^d \mathbb{P}^r$ is in $\overline{\Psi^{-1}(\tilde{\Gamma}_0)}$ as desired. $\square$

Because $(\overline{\mathcal{M}}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T = \bigcup_{\tilde{\Gamma}} \Psi^{-1}(\tilde{\Gamma})$, we have:

Corollary 4.5.18. *Let* $\tilde{\Gamma} \in \mathrm{Trees}^{\min}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. *Then* $\overline{\Psi^{-1}(\tilde{\Gamma})}$ *is an open and closed substack of* $(\overline{\mathcal{M}}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T$. *We denote it by* $\overline{\mathcal{M}}_{\tilde{\Gamma}}$.

4.5.2 An explicit description of $\overline{\mathcal{M}}_{\tilde{\Gamma}}$

In order to describe $\overline{\mathcal{M}}_{\tilde{\Gamma}}$, we recall certain moduli spaces of marked curves, studied originally by Losev and Manin [47].

Definition 4.5.19. Let $k \geq 1$, and fix a 2-element set $\{0, \infty\}$. An $0|k|\infty$*-marked Losev-Manin curve* is a connected genus zero $k+2$-marked nodal curve $(C, b_0, b_1, \ldots, b_k, b_\infty)$, satisfying:

- The irreducible components of C form a chain, with two leaves C_0 and C_∞,
- The points $b_0, b_1, \ldots, b_k, b_\infty$ are smooth points of C, with $b_0 \in C_0$ and $b_\infty \in C_\infty$,
- $b_i \neq 0$ and $b_i \neq \infty$ for $i = 1, \ldots, k$ (though it is possible that $b_i = b_j$ for $i \neq j$), and
- Each irreducible component of C contains at least one point of $b_1, \ldots, b_k$.

Theorem 4.5.20 (Losev-Manin [47], Theorems 2.2 and 2.6.3)**.** *The moduli space of* $0|k|\infty$*-marked Losev-Manin curves* $\overline{\mathcal{M}}_{0|k|\infty}$ *is a smooth, proper variety, and the natural morphism* $\varphi : \overline{\mathcal{M}}_{0,k+2} \to \overline{\mathcal{M}}_{0|k|\infty}$ *is birational.*

Remark 4.5.21. The spaces $\overline{\mathcal{M}}_{0|k|\infty}$ are special cases of moduli spaces of weighted stable curves, developed by Hassett [31], and Theorem 4.5.20 is a special case of Theorems 2.1 and 4.1 of [31]. Specifically, there is a natural isomorphism $\overline{\mathcal{M}}_{0|k|\infty} \to \overline{\mathcal{M}}_{0,\mathcal{A}}$, where $\mathcal{A}$ is the weight datum $(1, \epsilon, \epsilon, \ldots, \epsilon, 1)$ of length $k+2$, for $\epsilon \leq 1/k$.

Definition 4.5.22. Let $s \geq 1$ be an integer. An *order* s *orbifold* $0|k|\infty$*-marked Losev-Manin curve* is a $k+2$-marked twisted curve $(C, b_0, b_1, \ldots, b_k, b_\infty)$ (in the sense of [51]) whose coarse moduli space is a k-marked Losev-Manin curve, such that C has orbifold structure only at b_0, b_∞, and the nodes of C, all of which have order s.

By standard arguments about twisted curves, the moduli space $\overline{\mathcal{M}}^s_{0|k|\infty}$ of order s orbifold k-marked Losev-Manin curves has a natural isomorphism $\overline{\mathcal{M}}^s_{0|k|\infty} \to \overline{\mathcal{M}}_{0|k|\infty}$ that comes from taking coarse moduli spaces. The rest of this section proves the following:

Theorem 4.5.23. *For a stable vertex* v *or edge* $e = \{v_1, v_2\}$ *of a minimal decorated tree* $\tilde{\Gamma} = (\Gamma, \mathrm{Mark}, \mathrm{VEval}, q, \mathrm{Mon}) \in \mathrm{Trees}^{\min}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$*, we define*

$$\overline{\mathcal{M}}_v := \overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}}(v)}(BS_{\mathrm{VEval}(v)}, 0)$$

$$\overline{\mathcal{M}}_e := \left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \Big/ \left(\prod_{\eta \in \mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \mathrm{\ wr\ } S_e\right)\right],$$

where:

- $\overrightarrow{\mathrm{Mon}}(v)$ *is the list of multipartitions* $\{\mathrm{Mon}(i)\}_{i\in\mathrm{Mark}^{-1}(v)} \cup \{\mathrm{Mon}(v,e)\}_{(v,e)\in F(\Gamma)}$,
- $\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$ *is the order* $\mathrm{lcm}(\mathrm{Mon}(e))$ *orbifold Losev-Manin space with* $\mathrm{mov}(e)$ *marked points* $b_1,\ldots,b_{\mathrm{mov}(e)}$ *and labeling set* $\{v_1,v_2\}$,
- S_e *is the group* $C_{\mathrm{Stat}}(e)\times S_{\mathrm{Mov}(e)}$, *where* $C_{\mathrm{Stat}}(e)$ *is the centralizer of any element of the conjugacy class* $\mathrm{Stat}(e)$ *in* $\prod_{i=0}^r S_{|\mathrm{Stat}(e)_i|}$, *and acts trivially on the Losev-Manin space,*
- *A generator of* $\mu_{\beta_\eta(e)}$ *acts by translating the marked point* b_η *by* $e^{2\pi i/q(e)}$, *and*
- wr *denotes the wreath product.*

Then the substack $\overline{\mathcal{M}}_{\tilde\Gamma}$ *associated to* $\tilde\Gamma$ *is isomorphic (see Remark 4.5.24) to*

$$\left[\left(\prod_{v\in V^S(\Gamma)}\overline{\mathcal{M}}_v\times\prod_{e\in E(\Gamma)}\overline{\mathcal{M}}_e\right)\Big/\mathrm{Aut}(\Gamma)\right]. \tag{4.4}$$

Remark 4.5.24. More precisely, $\overline{\mathcal{M}}_{\tilde\Gamma}$ has extra automorphisms coming from gluing at nodes, and is thus a gerbe over (4.4). Gluing of components is fibered over the *rigidified* inertia stack $\overline{I}\,\mathrm{Sym}^d\,\mathbb{P}^r$ (see [2] or [46]). In particular, for each steady flag (v,e) of $\tilde\Gamma$, we get an extra factor of $\left|C_{\mathrm{VEval}(v)}(\mathrm{Mon}(v,e))\right|/r(v,e)$ in the fundamental class of $\overline{\mathcal{M}}_{\tilde\Gamma}$, where $C_{\mathrm{VEval}}(\mathrm{Mon}(v,e))$ is the centralizer of any element of the conjugacy class $\mathrm{Mon}(v,e)$ of $G_{\mathrm{VEval}(v)}$. (We make the usual correction for double counting when $v\in V^2(\Gamma)$.)

Proof of 4.5.23. Using the gluing morphisms, we may write

$$\overline{\mathcal{M}}_{\tilde\Gamma}\cong\left[\left(\prod_{v\in V(\Gamma)}\overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}}(v)}(BS_{\mathrm{VEval}(v)},0)\times\prod_{e\in E(\Gamma)}\overline{\mathcal{M}}_{0,\{\mathrm{Mon}(e),\mathrm{Mon}(e)\}}(L_e,\beta(e))^T\right)\Big/\mathrm{Aut}(\Gamma)\right],$$

We need to show that, for all $e=\{v_1,v_2\}\in E(\Gamma)$, we have

$$\overline{\mathcal{M}}_{0,\{\mathrm{Mon}(e),\mathrm{Mon}(e)\}}(L_e,\beta(e))^T\cong\left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}\Big/\left(\prod_{\eta\in\mathrm{Mov}(e)}\mu_{\beta_\eta(e)}\,\mathrm{wr}\,S_e\right)\right].$$

(Note that the left hand side is isomorphic to $\overline{\mathcal{M}}_{\tilde\Gamma_e}$ for $\tilde\Gamma_e = v_1\bullet\overset{e}{\text{———}}\bullet v_2$, where the decorations on $\tilde\Gamma_e$ are induced from $\tilde\Gamma$. (Here the two vertices are labeled, i.e. $\mathrm{Aut}(\tilde\Gamma_e)=1$.) Write $P_e := P_{(i^{\mathrm{mov}}(v_1,e),i^{\mathrm{mov}}(v_2,e))}$ for the midpoint of L_e. For $(f:C\to\mathbb{P}^r)\in\overline{\mathcal{M}}_{\tilde\Gamma_e}$, consider the preimage of P_e under the associated map $f':C'\to\mathbb{P}^r$. By Lemma 4.5.16, C' is a union of connected components C'_η for $\eta\in\mathrm{Mon}(e)$, and if $\eta\in\mathrm{Mov}(e)$ then the preimage of P_e on C'_η consists of $\beta_\eta(e)$ points on the single noncontracted component of C'_η. These points are $\mu_{\beta_\eta(e)}$-translates of each other, under the natural action that fixes the two special points.

After a principal $(\prod_{\eta\in\mathrm{Mov}(e)}\mu_{\beta_\eta(e)} \mathrm{wr}\, S_e)$-cover $\widetilde{\overline{\mathcal{M}}}_{\tilde{\Gamma}_e} \to \overline{\mathcal{M}}_{\tilde{\Gamma}_e}$, we may fix a labeling of the connected components C'_η, and label a distinguished preimage of P_e on C'_η for $\eta \in \mathrm{Mov}(e)$. (The S_e-cover removes all automorphisms of stable maps induced by automorphisms of the image curve that commute with the monodromy at b_{v_1} and b_{v_2}.) Remembering the images of these distinguished points under ρ yields a nodal chain of rational curves with $\mathrm{mov}(e)$ labeled marked points, none of which coincides with b_{v_1} or b_{v_2}. The stability condition for $\overline{\mathcal{M}}_{0,\{\mathrm{Mon}(e),\mathrm{Mon}(e)\}}(L_e, \beta(e))$ implies that this is a Losev-Manin curve, with orbifold points of order $\mathrm{lcm}(\mathrm{Mon}(e))$ at marked points and nodes. This construction works in families, so it defines a map $\widetilde{\overline{\mathcal{M}}}_{\tilde{\Gamma}_e} \to \overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$, which is equivariant by definition with respect to the action of $\prod_{\eta\in\mathrm{Mov}(e)}\mu_{\beta_\eta(e)} \mathrm{wr}\, S_e$. This gives a map

$$\Phi : \overline{\mathcal{M}}_{\tilde{\Gamma}_e} \to \left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \Big/ \left(\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \mathrm{wr}\, S_e\right)\right].$$

We now construct an inverse to this map. Let $(C, b_{v_1}, b_1, \ldots, b_{\mathrm{mov}(e)}, b_{v_2}) \in \overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$ be a Losev-Manin curve whose points are indexed by the multiset $\mathrm{Mov}(e)$. Fix a curve $C' = \bigsqcup_{\eta\in\mathrm{Mon}(e)} C'_\eta$ with étale maps $\rho_\eta : C'_\eta \to C$ of degree η. This may be done uniquely up to isomorphism. Also, uniquely up to isomorphism (of C' commuting with $\rho : C' \to C$), for each $\eta \in \mathrm{Mov}(e) \subseteq \mathrm{Mon}(e)$ we may choose a preimage point $b'_\eta \in C'_\eta$ of the corresponding marked point $b_\eta \in C$. Finally, there is a unique map $f' : C' \to \mathbb{P}^r$ that sends:

- C'_η to a T-fixed point, for $\eta \notin \mathrm{Mov}(e)$,
- C'_η to L_e with degree $\beta_\eta(e)$, with b'_η mapping to P_e, $\rho^{-1}(b_{v_1})$ mapping to $P_{i^{\mathrm{mov}}(v_1,e)}$ and $\rho^{-1}(b_{v_2})$ mapping to $P_{i^{\mathrm{mov}}(v_2,e)}$, for $\eta \in \mathrm{Mov}(e)$.

Again, this works in families, and defines a map $\tilde{\Theta} : \overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \to \overline{\mathcal{M}}_{\tilde{\Gamma}_e}$, which we claim is invariant under the action of $\prod_{\eta\in\mathrm{Mov}(e)}\mu_{\beta_\eta(e)} \mathrm{wr}\, S_e$. Indeed, acting by $e^{2\pi i/q(e)}$ on b_η translates the preimage b'_η by some power of $e^{2\pi i/\beta_\eta(e)}$, and commutes with f'. Thus $\tilde{\Theta}$ descends to a map

$$\Theta : \left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \Big/ \left(\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \mathrm{wr}\, S_e\right)\right] \to \overline{\mathcal{M}}_{\tilde{\Gamma}_e},$$

which is by construction an inverse to Φ. □

Corollary 4.5.25. *The* $(\prod_{\eta\in\mathrm{Mov}(e)}\mu_{\beta_\eta(e)} \mathrm{wr}\, S_e)$*-action on* $\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$ *extends to the universal curve, so we have a universal curve on* $\overline{\mathcal{M}}_e$*, and by gluing, a universal curve on the left side of* (4.4)*. The isomorphism of 4.5.23 naturally identifies this with the universal curve on* $\overline{\mathcal{M}}_{\tilde{\Gamma}}$.

Proof. The first statement is by definition of the action, and the second is immediate from the proof of Theorem 4.5.23. $\square$

Remark 4.5.26. Theorem 4.5.23 shows in particular that $\overline{\mathcal{M}}_{\tilde{\Gamma}_e}$ is irreducible, so connected components of $(\overline{\mathcal{M}}_{0,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T$ are indexed by minimal decorated trees with the additional data of a connected component of $\overline{\mathcal{M}}_{\overrightarrow{\mathrm{Mon}}(v)}(BS_{\mathrm{VEval}(v)}, 0)$ for each v.

Notation 4.5.27. For a special flag $(v, e) \in F(\Gamma)$, we denote by $\psi_v^{\overline{\mathcal{M}}_e}$ the ψ-class on $\overline{\mathcal{M}}_e$ at the point labeled by v. If $v \in V^S(\tilde{\Gamma})$, we denote by $\psi_e^{\overline{\mathcal{M}}_v}$ the ψ-class on $\overline{\mathcal{M}}_v$ at the marked point $\xi(v, e)$. We use the same notation for the $\overline{\psi}$-classes.

4.6 The virtual normal bundle and virtual fundamental class of $\overline{\mathcal{M}}_{\tilde{\Gamma}}$

In this section we compute the Euler class of the virtual normal bundle to $\overline{\mathcal{M}}_{\tilde{\Gamma}}$, and show that the virtual fundamental class of $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ is equal to its fundamental class. Many of the arguments are "classical," and we refer the reader to [46] for these.

In this section we fix $\tilde{\Gamma} \in \mathrm{Trees}_{0,n}^{\min}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{\tilde{\Gamma}}$ and $\rho : \mathcal{C}' \to \mathcal{C}$ denote the universal curve and universal étale cover, respectively:

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "f'"] \arrow[d, "\rho"] & \mathbb{P}^r \\ \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & \mathrm{Sym}^d \mathbb{P}^r \\ \overline{\mathcal{M}}_{\tilde{\Gamma}} & \end{tikzcd}$$

By a standard argument (see [46], Section 9.3), we have an exact sequence of T-equivariant sheaves on $\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ giving the perfect obstruction theory[3]

$$\begin{aligned} 0 \to \mathrm{Aut}(\mathcal{C}) \to R^0\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \mathrm{Def}(\mathcal{C}, f) \to \\ \to \mathrm{Def}(\mathcal{C}) \to R^1\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \mathrm{Obs}(\mathcal{C}, f) \to 0, \end{aligned} \tag{4.5}$$

where $\mathrm{Aut}(\mathcal{C})$ (resp. $\mathrm{Def}(\mathcal{C})$) is the sheaf on $\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r)$ of infinitesimal automorphisms (resp. deformations) of the marked source curve $\mathcal{C}$. (See [46] for rigorous definitions.) For

[3] We will always use the notation in (4.5) for higher direct image sheaves, writing e.g. $R^i\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r)$ instead of $R^i\pi_* f^*T\,\mathrm{Sym}^d \mathbb{P}^r$. This is because we will restrict π to various substacks of $\mathcal{C}$, and wish to avoid renaming maps.

$(f : C \to \mathrm{Sym}^d \mathbb{P}^r) \in \overline{\mathcal{M}}_{\tilde{\Gamma}}$, we also have a normalization exact sequence computing the fibers of the middle terms:

$$\begin{aligned} 0 \to H^0(C, f^*T \,\mathrm{Sym}^d \mathbb{P}^r) \to \bigoplus_\nu H^0(C_\nu, f^*T \,\mathrm{Sym}^d \mathbb{P}^r) \to \bigoplus_\xi H^0(\xi, f^*T \,\mathrm{Sym}^d \mathbb{P}^r) \to \\ \to H^1(C, f^*T \,\mathrm{Sym}^d \mathbb{P}^r) \to \bigoplus_\nu H^1(C_\nu, f^*T \,\mathrm{Sym}^d \mathbb{P}^r) \to 0, \end{aligned} \tag{4.6}$$

where ν runs over the set of irreducible components of C, and ξ runs over nodes of C. The sequences (4.5) and (4.6) each split as direct sums of two exact sequences: the T-fixed part and the T-moving part. We use the notations $\mathrm{Aut}(\mathcal{C})^{\mathrm{fix}}$ and $\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}$ (and similar) to denote the T-fixed subsheaf or subspace and its T-invariant complement. By definition (see [29]), the *Euler class of the virtual normal bundle* $e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}})$ is

$$\frac{e_T(\mathrm{Def}(\mathcal{C}, f)^{\mathrm{mov}})}{e_T(\mathrm{Obs}(\mathcal{C}, f)^{\mathrm{mov}})} = \frac{e_T(\mathrm{Def}(\mathcal{C})^{\mathrm{mov}}) e_T(R^0\pi_*(\mathcal{C}, f^*T \,\mathrm{Sym}^d \mathbb{P}^r)^{\mathrm{mov}})}{e_T(\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}) e_T(R^1\pi_*(\mathcal{C}, f^*T \,\mathrm{Sym}^d \mathbb{P}^r)^{\mathrm{mov}})} \in H^*_T(\overline{\mathcal{M}}_{\tilde{\Gamma}}, \mathbb{Q}), \tag{4.7}$$

and the *virtual fundamental class* $[\overline{\mathcal{M}}_{\tilde{\Gamma}}]^{\mathrm{vir}}$ of $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ is $e_T(\mathrm{Obs}(\mathcal{C}, f)^{\mathrm{fix}})$. We compute the various terms of (4.5) and (4.6) one by one. It is convenient to compute by pulling back to the canonical $\mathrm{Aut}(\Gamma)$-cover $\overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}}$ of $\overline{\mathcal{M}}_{\tilde{\Gamma}}$, so that the correspondence between C and $\tilde{\Gamma}$ is more concrete.

The sheaves $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Def}(\mathcal{C})$. In the toric case, from [46] Section 9.3.1, we have

$$e_T(\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}) = \prod_{v \in V^1(\tilde{\Gamma})} e_T(T_{\xi(v,e_v)} C) = \prod_{v \in V^1(\tilde{\Gamma})} \psi_v^{\overline{\mathcal{M}}_{e_v}}. \tag{4.8}$$

The same argument and answer apply here, using (Theorem 4.5.15 and) the observation that combining edges gives a natural identification of $V^1(\tilde{\Gamma})$. Briefly, moving automorphisms come from noncontracted components with only one special point, and correspond to vector fields on such a component that are nonvanishing at the nonspecial T-fixed point.

Similarly, in the toric case [46] Section 9.3.2 gives

$$e_T(\mathrm{Def}(\mathcal{C})) = \left(\prod_{\substack{v \in V^2(\tilde{\Gamma}) \\ (v, e_v^1) \text{ steady}}} \left(-\psi_v^{\overline{\mathcal{M}}_{e_v^1}} - \psi_v^{\overline{\mathcal{M}}_{e_v^2}}\right) \right) \left(\prod_{\substack{(v,e) \in F(\Gamma) \\ v \in V^S(\tilde{\Gamma})}} \left(-\psi_e^{\overline{\mathcal{M}}_v} - \psi_v^{\overline{\mathcal{M}}_e}\right) \right). \tag{4.9}$$

This is again correct in our case. The factors in (4.9) come from smoothing nodes. (Classically, the deformation space of a node is the tensor product of the tangents spaces to the two branches.) Therefore the observation we need is that breaking nodes correspond to steady flags.

The bundles $R^0\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$ and $R^1\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$. We use the sequence (4.6). The computation is very similar to the original one by Kontsevich [41] (and the orbifold computations of Johnson [36] and Liu [46]), but requires some care due to the edge moduli spaces.

Because normalization does not commute with base change, (4.6) only computes fibers of $R^i\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$. However, normalization of breaking nodes does commute with base change on $\overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}}$, because the set of breaking nodes is canonically identified for any two points of $\overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}}$. Thus we have the sequence

$$\begin{aligned} 0 \to R^0\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \bigoplus_{\underline{\nu}} R^0\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \bigoplus_{\xi} R^0\pi_*(\xi, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \\ \to R^1\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \bigoplus_{\underline{\nu}} R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to 0, \end{aligned} \tag{4.10}$$

where $\underline{\nu}$ runs over maximal subcurves of $\mathcal{C}$ containing only nonbreaking nodes, and ξ runs over breaking nodes. ($\mathcal{C}_{\underline{\nu}}$ may contain a single branch of a breaking node, but not both branches.) Observe that either $\mathcal{C}_{\underline{\nu}}$ is contracted by f, or each fiber $C_{\underline{\nu}}$ of $\mathcal{C}_{\underline{\nu}}$ contains only noncontracted components.

By Section 3.1.2, we have

$$R^i\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r) = R^i\pi_*(\mathcal{C}_{\underline{\nu}}, \rho_*(f')^*T\mathbb{P}^r) = R^i(\pi\circ\rho)_*(\mathcal{C}_{\underline{\nu}}{}', (f')^*T\mathbb{P}^r).$$

(The second equality follows from the fact that ρ is étale, hence ρ_* is exact.) After an étale base change, we may distinguish the connected components of fibers of $\mathcal{C}'_{\underline{\nu}} \to \overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}}$. In other words, we may write

$$\mathcal{C}'_{\underline{\nu}} = \bigsqcup_{\eta} \mathcal{C}'_{\underline{\nu},\eta},$$

where $\mathcal{C}'_{\underline{\nu},\eta}$ has connected fibers. Then

$$R^i\pi_*(\mathcal{C}'_{\underline{\nu}}, (f')^*T\mathbb{P}^r) = \bigoplus_{\eta} R^i(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r). \tag{4.11}$$

If $\mathcal{C}_{\underline{\nu}} = \mathcal{C}_v$ is contracted, then $(f')^*T\mathbb{P}^r$ is trivial on $\mathcal{C}'_{\underline{\nu},\eta}$. Thus we have

$$R^i(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r) \cong R^i(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, \mathcal{O}_{\mathcal{C}'_{\underline{\nu},\eta}}) \otimes T_{P_{i(\eta)}}\mathbb{P}^r,$$

where as usual we write $P_{i(\eta)}$ for $f'(\mathcal{C}'_{\underline{\nu},\eta})$. In particular,

$$R^0\pi_*(\mathcal{C}_v, f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{fix}} = R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{fix}} = 0. \tag{4.12}$$

The bundle $R^1\pi_*(\mathcal{C}_v, f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{mov}}$ is nontrivial, and is isomorphic to a Hurwitz-Hodge bundle (see [46], Section 7.5). However, note that $e_T(R\pi_*(\mathcal{C}_v, f^*T\operatorname{Sym}^d\mathbb{P}^r))$ is the inverse of the twisting class from (2.6). We will use this fact in Section 5.1 in our characterization of $\mathcal{L}_{\operatorname{Sym}^d\mathbb{P}^r}$, and in Section 6.2.3 to apply the orbifold quantum Riemann-Roch theorem.

Similarly for a breaking node $\xi(v,e)$, we have

$$\begin{aligned} R^0\pi_*(\xi(v,e), f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{fix}} &= 0 \\ R^0\pi_*(\xi(v,e), f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{mov}} &= T_{(\mathrm{VEval}(v),\mathrm{Mon}(v,e))}I\operatorname{Sym}^d\mathbb{P}^r = \bigoplus_{\eta\in\mathrm{Mon}(v,e)} T_{P_{i(\eta)}}\mathbb{P}^r. \end{aligned} \tag{4.13}$$

Suppose $\mathcal{C}_{\underline{\nu}}$ is not contracted. The components $\mathcal{C}'_{\underline{\nu},\eta}$ are in bijection with $\mathrm{Mon}(e)$, where e is the edge of $\tilde{\Gamma}$ corresponding to $\mathcal{C}_{\underline{\nu}}$.) First, we argue that $R^1(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r)$ vanishes for all η. The normalization exact sequence for a fiber $C'_{\underline{\nu},\eta}$ reads:

$$\begin{aligned} 0 \to H^0(C'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r) &\to \bigoplus_{\nu\in\underline{\nu}} H^0(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to \bigoplus_{\xi} H^0(\xi, (f')^*T\mathbb{P}^r) \to \\ \to H^1(C'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r) &\to \bigoplus_{\nu\in\underline{\nu}} H^1(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to 0, \end{aligned}$$

where we also denote by $\underline{\nu}$ the set indexing irreducible components C_ν of $C_{\underline{\nu}}$ (equivalently, irreducible components $C'_{\nu,\eta}$ of $C_{\underline{\nu},\eta}$). For each $\nu\in\underline{\nu}$, we have

$$H^1(C_\nu, (f')^*T\mathbb{P}^r) = 0 \tag{4.14}$$

by convexity of $\mathbb{P}^r$. We claim that the map

$$\bigoplus_{\nu\in\underline{\nu}} H^0(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to \bigoplus_{\xi} H^0(\xi, (f')^*T\mathbb{P}^r)$$

is surjective, so that $H^1(C'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r) = 0$. (The map takes the difference of the sections on the two branches of a node.) If $C'_{\underline{\nu},\eta}$ has a component $C'_{\nu_0,\eta}$ not contracted by f', there is at most one, by Lemma 4.5.16. On any other component $C'_{\nu,\eta}$, we have $(f')^*T\mathbb{P}^r \cong \mathcal{O}_{C'_{\nu,\eta}}\otimes T\mathbb{P}^r$, i.e. $H^0(C'_{\nu,\eta}, \mathcal{O}_{C'_{\nu,\eta}}\otimes T\mathbb{P}^r) \cong T\mathbb{P}^r$. Fix an arbitrary section $s\in H^0(C'_{\nu_0,\eta}, (f')^*T\mathbb{P}^r)$. Then "working outward" from $C'_{\nu_0,\theta}$ shows that the map is surjective. The case where f' contracts $C'_{\underline{\nu},\eta}$ is similar and simpler.

Next, we compute $R^0(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r)$. If $\mathcal{C}'_{\underline{\nu},\eta}$ is contracted, $(f')^*T\mathbb{P}^r$ is trivial and we have

$$R^0(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta},(f')^*T\mathbb{P}^r)\cong T\mathbb{P}^r\otimes\mathcal{O}_{\overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}}}$$

by properness of $\pi\circ\rho$. Suppose $\mathcal{C}'_{\underline{\nu},\eta}$ is not contracted. Consider the Stein factorization of $f'|_{\mathcal{C}'_{\underline{\nu},\eta}}$ relative to $\pi\circ\rho$:

$$\begin{tikzcd}
\mathcal{C}'_{\underline{\nu},\eta} \arrow[r, "\mathrm{sf}"] \arrow[rr, bend left, "f'"] \arrow[d, "\pi\circ\rho"'] & \overline{\mathcal{C}'_{\underline{\nu},\eta}} \arrow[r, "f''"'] \arrow[dl, "\overline{\pi\circ\rho}"] & \mathbb{P}^r \\
\overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}} & &
\end{tikzcd}$$

If $(f:C\to\mathrm{Sym}^d\,\mathbb{P}^r)$ is in the dense open substack $\Psi^{-1}(\tilde{\Gamma})\subseteq\overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}}$, then $C_{\underline{\nu}}$ is irreducible, hence so is $C'_{\underline{\nu},\eta}$. This, with the fact that $\mathcal{C}'_{\underline{\nu},\eta}$ is not contracted, implies that sf is birational. By the projection formula for coherent sheaves,

$$\begin{aligned}
(\pi\circ\rho)_*(f')^*T\mathbb{P}^r &= (\pi\circ\rho)_*\mathrm{sf}^*(f'')^*T\mathbb{P}^r\\
&= (\overline{\pi\circ\rho})_*\mathrm{sf}_*\mathrm{sf}^*(f'')^*T\mathbb{P}^r\\
&= (\overline{\pi\circ\rho})_*((f'')^*T\mathbb{P}^r\otimes\mathrm{sf}_*\mathcal{O}_{C'_{\underline{\nu},\eta}})\\
&= (\overline{\pi\circ\rho})_*(f'')^*T\mathbb{P}^r.
\end{aligned}$$

After an étale base change on $\overline{\mathcal{M}}^{\mathrm{rig}}_{\tilde{\Gamma}}$, the map f'' trivializes the family $\overline{\mathcal{C}'_{\underline{\nu},\eta}}$. Thus the vector bundle $R^0(\overline{\pi\circ\rho})_*(\overline{\mathcal{C}'_{\underline{\nu},\eta}},(f'')^*T\mathbb{P}^r)$ is trivial.

Calculation of the T-weights of this vector bundle is identical to Kontsevich's calculation in Section 3.3.4 of [41], which uses the Euler sequence on $\mathbb{P}^r$. The weights are

$$\frac{A}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_1,e)}+\frac{B}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_2,e)}-\alpha_i,$$

where $0\leq A,B\leq\beta_\eta(e)$, $A+B=\beta_\eta(e)$, and $i\in\{0,\ldots,r\}$. Note that this is zero exactly when $A=0$ and $i=i^{\mathrm{mov}}(v_2,e)$, or $B=0$ and $i=i^{\mathrm{mov}}(v_1,e)$. (These factors contribute to $e_T(R^0(\overline{\pi\circ\rho})_*(\overline{\mathcal{C}'_{\underline{\nu},\eta}},(f'')^*T\mathbb{P}^r)^{\mathrm{fix}})$.) The Euler class $e_T(R^0(\overline{\pi\circ\rho})_*(\overline{\mathcal{C}'_{\underline{\nu}}},(f'')^*T\mathbb{P}^r)^{\mathrm{mov}})$ for $\underline{\nu}$ noncontracted is thus

$$\left(\prod_{\substack{\eta\in\mathrm{Stat}(e)\\ i\neq i(\eta)}}(\alpha_{i(\eta)}-\alpha_i)\right)\left(\prod_{\eta\in\mathrm{Mov}(e)}\prod_{\substack{A+B=\beta_\eta(e)\\ 0\leq i\leq r\\ (A,i)\neq(0,i^{\mathrm{mov}}(v_2,e))\\ (B,i)\neq(0,i^{\mathrm{mov}}(v_1,e))}}\left(\frac{A}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_1,e)}+\frac{B}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_2,e)}-\alpha_i\right)\right) \tag{4.15}$$

Summary. We collect the arguments of this section in the following two statements.

Proposition 4.6.1. *For any minimal decorated tree $\tilde{\Gamma}$, $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ is smooth, and the virtual fundamental class is equal to the fundamental class.*

Proposition 4.6.2. *The equivariant Euler class $e_T(N^{\mathrm{vir}}_{\overline{\mathcal{M}}_{\tilde{\Gamma}}})$ of the virtual normal bundle to $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ is*

$$\left(\frac{\prod_{v\in V^2(\tilde{\Gamma})}(-\psi_v^{\overline{\mathcal{M}}_{e_v^1}}-\psi_v^{\overline{\mathcal{M}}_{e_v^2}})\prod_{\substack{(v,e)\in F(\Gamma)\\ v\in V^S(\tilde{\Gamma})}}(-\psi_e^{\overline{\mathcal{M}}_v}-\psi_v^{\overline{\mathcal{M}}_e})}{\prod_{v\in V^1(\tilde{\Gamma})}\psi_v^{\overline{\mathcal{M}}_{e_v}}}\right)$$

$$\cdot\prod_{e\in E(\Gamma)}\left(\left(\prod_{\substack{\eta\in\mathrm{Stat}(e)\\ i\neq i(\eta)}}(\alpha_{i(\eta)}-\alpha_i)\right)\right.$$

$$\left.\cdot\left(\prod_{\eta\in\mathrm{Mov}(e)}\prod_{\substack{A+B=\beta_\eta(e)\\ 0\leq i\leq r\\ (A,i)\neq(0,i^{\mathrm{mov}}(v_2,e))\\ (B,i)\neq(0,i^{\mathrm{mov}}(v_1,e))}}\left(\frac{A}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_1,e)}+\frac{B}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_2,e)}-\alpha_i\right)\right)\right)$$

$$\cdot\frac{\prod_{v\in V^1(\Gamma)\cup V^{1,1}(\Gamma)\cup V^2(\Gamma)}e_T(T_{(\mathrm{VEval}(v),\mathrm{Mon}(v))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{\prod_{(v,e)\in F(\Gamma)}e_T(T_{(\mathrm{VEval}(v),\mathrm{Mon}(v))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}$$

$$\cdot\prod_{v\in V^S(\tilde{\Gamma})}e_T(R\pi_*(C_v,f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)).$$

Proof of Proposition 4.6.1. Recall from Theorem 4.1.2 that the virtual fundamental class of $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ is obtained from the fixed part of the perfect obstruction theory on $\overline{\mathcal{M}}_{0,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta)$. By (4.13), the fixed part of $\bigoplus_\xi R^0\pi_*(\xi, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)$ is zero. Thus by (4.10), we have

$$R^1\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r) \cong \bigoplus_{\underline{\nu}} R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r).$$

But we showed, in (4.12) and (4.14), that $\bigoplus_{\underline{\nu}} R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)$ has no fixed part. Thus $R^1\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)$ has no fixed part. By Proposition 5.5 of [6], the Proposition follows. (Smoothness already followed easily from Theorem 4.5.23.) $\square$

Proof of Proposition 4.6.2. The first line is the contribution from $\mathrm{Def}(\mathcal{C})^{\mathrm{mov}}$ and $\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}$, from (4.8) and (4.9). The second line is the contribution to $R\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)$ from noncontracted components $\mathcal{C}_{\underline{\nu}}$, as in (4.15) and (4.14). The third line is the contribution of breaking nodes to $R\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)$, from (4.13). (The numerator corrects for the fact that $F(\Gamma)$ overcounts the breaking nodes.) The last line is the contribution to $R\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)$ from contracted components $\mathcal{C}_{\underline{\nu}}$, by definition. $\square$

CHAPTER 5

The Givental cone is characterized by recursion relations

5.1 Characterization of the Givental cone $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$

In this section, we apply the results of Sections 4.5 and 4.6 to give a criterion (Theorem 5.1.4) that exactly determines whether a given power series lies on the Givental cone $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$.

Definition 5.1.1. Fix $(\mu, \sigma) \in (I \operatorname{Sym}^d \mathbb{P}^r)^T$. Let $\Upsilon(\mu, \sigma) \subseteq \mathrm{Trees}_{0,2}(\operatorname{Sym}^d \mathbb{P}^r, \beta)$ be the set of 1-edge decorated trees $\tilde{\kappa} = v_1 \bullet\!\xrightarrow{e}\!\bullet v_2$, with marking set $\{n+1, \bullet\}$, with $\mathrm{Mark}(n+1) = v_1$ and $\mathrm{Mark}(\bullet) = v_2$, such that $\mu = \mathrm{VEval}(v_1)$ and $\sigma = \mathrm{Mon}(v_1, e)$.

Notation 5.1.2. For $\tilde{\kappa} \in \Upsilon(\mu, \sigma)$ as in Definition 5.1.1, we write:

- $q(\tilde{\kappa}) := q(e)$,
- $\mathrm{Mov}(\tilde{\kappa}) := \mathrm{Mov}(e)$,
- $\mathrm{mov}(\tilde{\kappa}) := \mathrm{mov}(e)$,
- $\mathrm{Stat}(\tilde{\kappa}) := \mathrm{Stat}(e)$,
- $i_1^{\mathrm{mov}}(\tilde{\kappa}) := i^{\mathrm{mov}}(v_1, e)$,
- $i_2^{\mathrm{mov}}(\tilde{\kappa}) := i^{\mathrm{mov}}(v_2, e)$,
- $\mu'(\tilde{\kappa}) := \mathrm{VEval}(v_2)$,
- $\sigma'(\tilde{\kappa}) := \mathrm{Mon}(v_2, e)$, and
- $r(\tilde{\kappa}) := r(v_1, e) = r(v_2, e) = r_{n+1}$.
- $w(\tilde{\kappa}) := w(v_1, e)$.

We compute:

$$w(v_1, e) = \frac{\alpha_{i_1^{\mathrm{mov}}(\tilde{\kappa})} - \alpha_{i_2^{\mathrm{mov}}(\tilde{\kappa})}}{q(\tilde{\kappa})} \in H_T^2(\operatorname{Spec}\mathbb{C}, \mathbb{Q}).$$

Similarly to the notations ψ and $\overline{\psi}$, we write $\overline{w} = r(\tilde{\kappa})w$.

Definition 5.1.3. Let $\tilde{\kappa} \in \Upsilon(\mu, \sigma)$ and let $a \in \mathbb{Z}_{>0}$ We define the recursion coefficient as

$$\mathbf{RC}(\tilde{\kappa}, a) = \frac{(-1)^{\mathrm{mov}(\tilde{\kappa})-a}}{q(\tilde{\kappa})^{\mathrm{mov}(\tilde{\kappa})}} \binom{\sigma_{i_1^{\mathrm{mov}}(\tilde{\kappa})}}{\mathrm{Mov}(\tilde{\kappa})} \binom{\mathrm{mov}(\tilde{\kappa}) - 1}{a-1} \cdot$$
$$\cdot \frac{1}{\prod_{\eta \in \mathrm{Mov}(\tilde{\kappa})} \prod_{\substack{1 \le B \le \beta_\eta(e) \\ 0 \le i \le r \\ (B,i) \neq (\beta_\eta(e), i_2^{\mathrm{mov}}(\tilde{\kappa}))}} \left(\frac{\beta_\eta(e) - B}{\beta_\eta(e)} \alpha_{i_1^{\mathrm{mov}}(\tilde{\kappa})} + \frac{B}{\beta_\eta(e)} \alpha_{i_2^{\mathrm{mov}}(\tilde{\kappa})} - \alpha_i \right)},$$

where $\binom{\sigma_{i_1^{\mathrm{mov}}(\tilde{\kappa})}}{\mathrm{Mov}(\tilde{\kappa})}$ is the number of ways of choosing $\mathrm{Mov}(\tilde{\kappa})$ as a subpartition of $\sigma_{i_1^{\mathrm{mov}}(\tilde{\kappa})}$ with specified parts.

The following theorem and its proof are adapted from Theorem 41 of [18], which in turn is adapted from Theorem 2 of [8].

Theorem 5.1.4. *Let $\mathbf{f}$ be an element of $\mathcal{H}[[x]]$ such that $\mathbf{f}|_{Q=x=0} = -1z$, where 1 denotes the fundamental class of $\operatorname{Sym}^d \mathbb{P}^r \subseteq I \operatorname{Sym}^d \mathbb{P}^r$. Then $\mathbf{f}$ is a $\Lambda_{\mathrm{nov}}^T[[x]]$-valued point of $\mathcal{L}_{\operatorname{Sym}^d \mathbb{P}^r}$ if and only if for each T-fixed point $(\mu, \sigma) \in I \operatorname{Sym}^d \mathbb{P}^r$, the following three conditions hold:*

(I) *The restriction $\mathbf{f}_{(\mu,\sigma)}$ along $\iota_{(\mu,\sigma)} : (\mu, \sigma) \hookrightarrow I \operatorname{Sym}^d \mathbb{P}^r$ is a power series in Q and x, such that each coefficient of this power series is an element of $H^*_{T,\mathrm{loc}}(\bullet, \mathbb{Q})(z)$. Each coefficient is regular in z except for possible poles at $z = 0$, $z = \infty$, and*

$$z \in \{\overline{w}(\tilde{\kappa}) : \tilde{\kappa} \in \Upsilon(\mu, \sigma)\}.$$

(II) *The Laurent coefficients of $\mathbf{f}_{(\mu,\sigma)}$ at the poles (other than $z = 0$ and $z = \infty$) satisfy the recursion relation:*

$$\operatorname{Laur}(\mathbf{f}_{\mu,\sigma}, (\overline{w} - z)^{-a}) = \sum_{\substack{\tilde{\kappa} \in \Upsilon_{(\mu,\sigma)} \\ \overline{w}(\tilde{\kappa}) = \overline{w} \\ \mathrm{Mov}(\tilde{\kappa}) \ge a}} Q^{\beta(\tilde{\kappa})} \, \mathbf{RC}(\tilde{\kappa}, a) \operatorname{Laur}(\mathbf{f}_{(\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa}))}, (\overline{w} - z)^{\mathrm{mov}(\tilde{\kappa}) - a}) \tag{5.1}$$

for $a > 0$, and

(III) *The restriction $\mathbf{f}_\mu$ along $\iota_\mu : I\mu \hookrightarrow I \operatorname{Sym}^d \mathbb{P}^r$ is a $\Lambda_{\mathrm{nov}}^T[[x]]$-valued point of $\mathcal{L}_\mu^{T \operatorname{Sym}^d \mathbb{P}^r}$, the twisted Givental cone from Section 2.6.*

Remark 5.1.5. In (**III**), Λ^T_{nov} is the equivariant Novikov ring associated to $\mathrm{Sym}^d \mathbb{P}^r$, not μ. In other words, $\Lambda^T_{\mathrm{nov}}[[x]] = H^*_{CR,T,\mathrm{loc}}(\mu, \mathbb{Q})[[x, Q]]$.

Remark 5.1.6. The major difference between Theorem 5.1.4 and the corresponding theorems in [18] and [8] is that condition (**II**) gives a recursive relation for *all* negative-exponent Laurent coefficients at $z = w(\tilde{\kappa})$, in terms of positive-exponent ones. In [18] and [8], only stacks with isolated 1-dimensional T-orbits are considered. Thus the poles at $z = w(\tilde{\kappa})$ are simple, and a recursive relation is given for their residues.

Proof. For convenience, we abbreviate $\overline{\mathcal{M}}_{0,n+1,\beta} : \overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. Let $\mathbf{f}$ be a $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point of $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$. By definition, we can write

$$\begin{aligned}
\mathbf{f} &= -1z + \mathbf{t}(z) + \sum_{\substack{n,\beta \geq 0 \\ \beta > 0 \text{ or } n \geq 2}} \sum_{\phi} \frac{Q^\beta}{n!} \left\langle \mathbf{t}(\overline{\psi}), \ldots, \mathbf{t}(\overline{\psi}), \frac{\gamma_\phi}{-z - \overline{\psi}} \right\rangle^{\mathrm{Sym}^d \mathbb{P}^r, T}_{0,n+1,\beta} \gamma^\phi \\
&= -1z + \mathbf{t}(z) + \sum_{\substack{n,\beta \geq 0 \\ \beta > 0 \text{ or } n \geq 2}} \frac{Q^\beta}{n!} (\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}) \cup \frac{1}{-z - \overline{\psi}} \cap [\overline{\mathcal{M}}_{0,n+1,\beta}]^{\mathrm{vir}} \right)
\end{aligned}$$

for $\mathbf{t}(z) \in \mathcal{H}_+[[x]]$ with $\mathbf{t}|_{Q=x=0} = 0$. The restriction $\mathbf{f}_{(\mu,\sigma)}$ is then

$$\begin{aligned}
&-\delta_{\sigma=\{1,\ldots,1\}} z + \iota^*_{(\mu,\sigma)} \mathbf{t}(z) \\
&\qquad + \sum_{\substack{n,\beta \geq 0 \\ \beta > 0 \text{ or } n \geq 2}} \frac{Q^\beta}{n!} \iota^*_{(\mu,\sigma)} \left((\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}) \cup \frac{1}{-z - \overline{\psi}} \cap [\overline{\mathcal{M}}_{0,n+1,\beta}]^{\mathrm{vir}} \right) \right).
\end{aligned}$$

Using the projection formula, we write

$$\begin{aligned}
&\iota^*_{(\mu,\sigma)}\left((\mathrm{ev}_{n+1})_*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi}_j)\cup\frac{1}{-z-\overline{\psi}_{n+1}}\cap[\overline{\mathcal{M}}_{0,n+1,\beta}]^{\mathrm{vir}}\right)\right)\\
&=|C_\mu(\sigma)|\int_{\mathrm{Sym}^d\mathbb{P}^r}(\iota_{(\mu,\sigma)})_*\iota^*_{(\mu,\sigma)}\left((\mathrm{ev}_{n+1})_*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi}_j)\cup\frac{1}{-z-\overline{\psi}_{n+1}}\cap[\overline{\mathcal{M}}_{0,n+1,\beta}]^{\mathrm{vir}}\right)\right)\\
&=|C_\mu(\sigma)|\int_{\mathrm{Sym}^d\mathbb{P}^r}[(\mu,\sigma)]\cup\left((\mathrm{ev}_{n+1})_*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi}_j)\cup\frac{1}{-z-\overline{\psi}_{n+1}}\cap[\overline{\mathcal{M}}_{0,n+1,\beta}]^{\mathrm{vir}}\right)\right)\\
&=|C_\mu(\sigma)|\int_{\mathrm{Sym}^d\mathbb{P}^r}\left((\mathrm{ev}_{n+1})_*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi}_j)\cup\frac{\mathrm{ev}^*([(\mu,\sigma)])}{-z-\overline{\psi}_{n+1}}\cap[\overline{\mathcal{M}}_{0,n+1,\beta}]^{\mathrm{vir}}\right)\right)\\
&=|C_\mu(\sigma)|\int_{[\overline{\mathcal{M}}_{0,n+1,\beta}]^{\mathrm{vir}}}\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi}_j)\cup\frac{\mathrm{ev}^*([(\mu,\sigma)])}{-z-\overline{\psi}_{n+1}}\right)\\
&=|C_\mu(\sigma)|\left\langle\mathbf{t}(\overline{\psi}),\dots,\mathbf{t}(\overline{\psi}),\frac{[(\mu,\sigma)]}{-z-\overline{\psi}}\right\rangle^{\mathrm{Sym}^d\mathbb{P}^r,T}_{0,n+1,\beta}.
\end{aligned}$$

The first equality uses the identification of $\int_{\mathrm{Sym}^d\mathbb{P}^r}\circ\iota_{(\mu,\sigma)}$ with the identity map $\operatorname{Spec}\mathbb{C}\to\operatorname{Spec}\mathbb{C}$ on coarse moduli spaces, and the factor $|C_\mu(\sigma)|$ corrects for the isotropy at $(\mu,\sigma)\in I\,\mathrm{Sym}^d\mathbb{P}^r$. (Recall that $C_\mu(\sigma)$ denotes the centralizer of any element of σ in G_μ.) In summary,

$$\begin{aligned}
\mathbf{f}_{(\mu,\sigma)}&=-\delta_{\sigma=\{1,\dots,1\}}z+\mathbf{t}_{(\mu,\sigma)}(z)\\
&\quad+\sum_{\substack{n,\beta\geq 0\\ \beta>0\text{ or }n\geq 2}}\frac{|C_\mu(\sigma)|\,Q^\beta}{n!}\left\langle\mathbf{t}(\overline{\psi}),\dots,\mathbf{t}(\overline{\psi}),\frac{[(\mu,\sigma)]}{-z-\overline{\psi}}\right\rangle^{\mathrm{Sym}^d\mathbb{P}^r,T}_{0,n+1,\beta},
\end{aligned}\tag{5.2}$$

where $\mathbf{t}_{(\mu,\sigma)}(z):=\iota^*_{(\mu,\sigma)}\mathbf{t}(z)$. Now we calculate (5.2) by virtual torus localization (see Theorem 4.1.2). Namely, we may write

$$|C_\mu(\sigma)|\left\langle\mathbf{t}(\overline{\psi}),\dots,\mathbf{t}(\overline{\psi}),\frac{[(\mu,\sigma)]}{-z-\overline{\psi}}\right\rangle^{\mathrm{Sym}^d\mathbb{P}^r,T}_{0,n+1,\beta}=\sum_{\tilde{\Gamma}\in\mathrm{Trees}^{\min}_{0,n+1}(\mathrm{Sym}^d\mathbb{P}^r,\beta)}\mathrm{Contr}_{(\mu,\sigma)}(\tilde{\Gamma}).\tag{5.3}$$

We can partition $\mathrm{Trees}^{\min}_{0,n+1}(\mathrm{Sym}^d\mathbb{P}^r,\beta)$ into three subsets:

(i) $\tilde{\Gamma}$ such that $(\mathrm{VEval}(\mathrm{Mark}(b_{n+1})),\mathrm{Mon}(b_{n+1}))\neq(\mu,\sigma)$,

(ii) $\tilde{\Gamma}$ such that $(\mathrm{VEval}(\mathrm{Mark}(b_{n+1})),\mathrm{Mon}(b_{n+1}))=(\mu,\sigma)$ and $\mathrm{Mark}(b_{n+1})\in V^{1,1}(\Gamma)$, and

(iii) $\tilde{\Gamma}$ such that $(\mathrm{VEval}(\mathrm{Mark}(b_{n+1})),\mathrm{Mon}(b_{n+1}))=(\mu,\sigma)$ and $\mathrm{Mark}(b_{n+1})\in V^S(\Gamma)$.

In some literature, e.g. [13], decorated trees of type (ii) are called *recursion type* and those of type (iii) are called *initial type*. Let $v_1 := \mathrm{Mark}(b_{n+1})$ be the vertex containing the point b_{n+1}. (We will see below, however, that in our setup both types are used recursively.)

For a tree $\tilde{\Gamma}$ of type (i), the restriction $\mathrm{ev}_{n+1}^*([(\mu,\sigma)])$ vanishes, hence $\mathrm{Contr}_{(\mu,\sigma)}(\tilde{\Gamma}) = 0$. For this reason, we may simplify our notation, and write $\mathrm{Contr}(\tilde{\Gamma}) := \mathrm{Contr}_{(\mu,\sigma)}(\tilde{\Gamma})$, where $\mu = \mathrm{VEval}(\mathrm{Mark}(b_{n+1}))$ and $\sigma = \mathrm{Mon}(\mathrm{Mark}(b_{n+1}))$.

If $\tilde{\Gamma}$ is a tree of type (iii), then by Theorem 4.5.23 and Corollary 4.5.25, $\overline{\psi}_{n+1}$ is pulled back from $\overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}}(v_1)}(BG_\mu, 0)$, where G_μ is the isotropy group of μ. Since this stack parametrizes maps that factor through the fixed point μ, the action of T is trivial, hence

$$H^*_{T,\mathrm{loc}}(\overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}}(v_1)}(BG_\mu, 0), \mathbb{Q}) \cong H^*(\overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}}(v_1)}(BG_\mu, 0), \mathbb{Q}) \otimes H^*_{T,\mathrm{loc}}(\bullet, \mathbb{Q}).$$

In particular, $\overline{\psi}_{n+1}$ is nilpotent. It follows that $\mathrm{Contr}(\tilde{\Gamma})$ is a polynomial in z^{-1}, hence has a pole only at $z = 0$.

Finally, let $\tilde{\Gamma}$ be a tree of type (ii). By (4.1), we have

$$\mathrm{Contr}(\tilde{\Gamma}) = |C_\mu(\sigma)| \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}}]'} \frac{1}{e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}})} \iota^*_{\tilde{\Gamma}} \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}) \cup \frac{\mathrm{ev}^*_{n+1}[(\mu,\sigma)]}{-z - \overline{\psi}_{n+1}} \right), \tag{5.4}$$

where $\iota_{\tilde{\Gamma}}$ is the inclusion $\overline{\mathcal{M}}_{\tilde{\Gamma}} \hookrightarrow \overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, and $[\overline{\mathcal{M}}_{\tilde{\Gamma}}]'$ denotes the fundamental class, weighted by factors from Remark 4.5.24. Note that $\mathrm{ev}_{n+1} \circ \iota_{\tilde{\Gamma}}$ factors through (μ, σ), hence $\iota^*_{\tilde{\Gamma}} \mathrm{ev}^*_{n+1}[(\mu,\sigma)]$ is the weight $e_T(T_{(\mu,\sigma)} I \, \mathrm{Sym}^d \mathbb{P}^r)$.

Then $\tilde{\Gamma}$ has a decorated subtree $\tilde{\kappa} \in \Upsilon(\mu,\sigma)$, obtained by removing all edges except for $e := e_{v_1}$ (and necessary vertices), and all marked points except b_{n+1}. Let $\tilde{\Gamma} \smallsetminus \tilde{\kappa}$ denote the tree obtained by *pruning* $\tilde{\kappa}$. That is, $\tilde{\Gamma} \smallsetminus \tilde{\kappa} \in \mathrm{Trees}^{\min}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta - \beta(\tilde{\kappa}))$ is defined by $V(\Gamma \smallsetminus \kappa) = V(\Gamma) \smallsetminus \{v_1\}$, $E(\Gamma \smallsetminus \kappa) = E(\Gamma) \smallsetminus e$, and decorations Mark, VEval, q, and Mon are unchanged, except $\mathrm{Mark}(b_{n+1}) := v_2$, where v_2 is the common vertex of $\tilde{\kappa}$ and $\tilde{\Gamma} \smallsetminus \tilde{\kappa}$. Observe that an automorphism of Γ fixes b_{n+1}, and therefore fixes e, so we have $\mathrm{Aut}(\tilde{\Gamma}) = \mathrm{Aut}(\tilde{\Gamma} \smallsetminus \tilde{\kappa})$ and may write

$$\overline{\mathcal{M}}_{\tilde{\Gamma}} \cong \overline{\mathcal{M}}_e \times \overline{\mathcal{M}}_{\tilde{\Gamma} \smallsetminus \tilde{\kappa}}$$

by Theorem 4.5.23, up to a gerbe from Remark 4.5.24. We factor the T-equivariant map $\overline{\mathcal{M}}_{\tilde{\Gamma}} \to$

$\operatorname{Spec}\mathbb{C}$ through the second projection, i.e. we integrate over $\overline{\mathcal{M}}_e$:

$$\begin{aligned}\operatorname{Contr}(\tilde{\Gamma}) = & \frac{|C_\mu(\sigma)|\left|C_{\mu'(\tilde{\kappa})}(\sigma'(\tilde{\kappa}))\right|}{r(\tilde{\kappa})} \\ & \cdot \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\setminus\tilde{\kappa}}]'} \left(\int_{\overline{\mathcal{M}}_e} \frac{e_T(T_{(\mu,\sigma)} I \operatorname{Sym}^d \mathbb{P}^r)}{e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}})} \iota^*_{\tilde{\Gamma}} \left(\prod_{j=1}^n \operatorname{ev}_j^* \mathbf{t}(\overline{\psi}) \cup \frac{1}{-z-\overline{\psi}_{n+1}} \right) \right).\end{aligned}$$

The factor $\left|C_{\mu'(\tilde{\kappa})}(\sigma'(\tilde{\kappa}))\right| / r(\tilde{\kappa})$ is the correction from Remark 4.5.24. From Proposition 4.6.2, we may write

$$\frac{e_T(T_{(\mu,\sigma)} I \operatorname{Sym}^d \mathbb{P}^r)}{e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}})} = \frac{1}{W} \cdot \frac{e_T(T_{(\mu'(\tilde{\kappa}),\sigma'(\tilde{\kappa}))} I \operatorname{Sym}^d \mathbb{P}^r)}{e(N^{\mathrm{vir}}_{\tilde{\Gamma}\setminus\tilde{\kappa}})(-\psi_e^{\overline{\mathcal{M}}_{v_2}} - \psi_{v_2}^{\overline{\mathcal{M}}_e})}, \tag{5.5}$$

where

$$\begin{aligned} W &= \frac{\prod_{\substack{\eta\in\operatorname{Stat}(e)\\ i\neq i(\eta)}} (\alpha_{i(\eta)} - \alpha_i)}{e_T(T_{(\mu,\sigma)} I \operatorname{Sym}^d \mathbb{P}^r)} \prod_{\eta\in\operatorname{Mov}(\tilde{\kappa})} \prod_{\substack{A+B=\beta_\eta(\tilde{\kappa})\\ 0\le i\le r\\ (A,i)\neq(0,i^{\mathrm{mov}}(v_2,e))\\ (B,i)\neq(0,i^{\mathrm{mov}}(v_1,e))}} \left(\frac{A}{\beta_\eta(\tilde{\kappa})} \alpha_{i^{\mathrm{mov}}(v_1,e)} + \frac{B}{\beta_\eta(\tilde{\kappa})} \alpha_{i^{\mathrm{mov}}(v_2,e)} - \alpha_i \right) \\ &= \prod_{\substack{\eta\in\operatorname{Mov}(\tilde{\kappa})\\ 1\le B\le\beta_\eta(\tilde{\kappa})\\ 0\le i\le r\\ (B,i)\neq(\beta_\eta(\tilde{\kappa}),i^{\mathrm{mov}}(v_2,e))}} \left(\frac{\beta_\eta(\tilde{\kappa}) - B}{\beta_\eta(\tilde{\kappa})} \alpha_{i^{\mathrm{mov}}(v_1,e)} + \frac{B}{\beta_\eta(\tilde{\kappa})} \alpha_{i^{\mathrm{mov}}(v_2,e)} - \alpha_i \right) \in H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C},\mathbb{Q}) \end{aligned}$$

Note that the cancellation in the last step removes the terms with $B \neq 0$ in the product, and that $1/W$ is the product appearing $\mathbf{RC}(\tilde{\kappa}, a)$.

To avoid confusion, we write $\overline{\psi}^{\tilde{\Gamma}}_{n+1}$ (resp. $\overline{\psi}^{\tilde{\Gamma}\setminus\tilde{\kappa}}_{n+1}$) for the $\overline{\psi}$-class at the $(n+1)$st marked point on $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ (resp. $\overline{\mathcal{M}}_{\tilde{\Gamma}\setminus\tilde{\kappa}}$), recalling that on $\tilde{\Gamma}\setminus\tilde{\kappa}$ we defined $\operatorname{Mark}(b_{n+1}) = v_2$. We also have $\iota^*_{\tilde{\Gamma}}\overline{\psi}^{\tilde{\Gamma}}_{n+1} = \overline{\psi}^{\overline{\mathcal{M}}_e}_{v_1}$. The T-weight on $\overline{\psi}^{\overline{\mathcal{M}}_e}_{v_1}$ is $-\overline{w}(\tilde{\kappa})$ (see Notation 5.1.2), so we have

$$\overline{\psi}^{\overline{\mathcal{M}}_e}_{v_1} = \overline{\psi}^{\mathrm{ne}}_{v_1} - \overline{w}(\tilde{\kappa}) \in H^*_T(\overline{\mathcal{M}}_{\tilde{\Gamma}},\mathbb{Q}) \cong H^*(\overline{\mathcal{M}}_{\tilde{\Gamma}},\mathbb{Q}) \otimes H^*_T(\operatorname{Spec}\mathbb{C},\mathbb{Q}), \tag{5.6}$$

where $\overline{\psi}^{\mathrm{ne}}_{v_1}$ denotes the nonequivariant $\overline{\psi}$-class. Similarly $\overline{\psi}^{\overline{\mathcal{M}}_e}_{v_2} = \overline{\psi}^{\mathrm{ne}}_{v_2} + \overline{w}(\tilde{\kappa})$. Then since $\iota^*_{\tilde{\Gamma}} \operatorname{ev}_j^* \mathbf{t}(\overline{\psi})$

is pulled back from $\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}$,

$$\mathrm{Contr}(\tilde{\Gamma}) = \frac{|C_\mu(\sigma)|\left|C_{\mu'(\tilde{\kappa})}(\sigma'(\tilde{\kappa}))\right|}{r(\tilde{\kappa})}\frac{e_T(T_{(\mu'(\tilde{\kappa}),\sigma'(\tilde{\kappa}))}I\operatorname{Sym}^d\mathbb{P}^r)}{W}$$
$$\cdot\int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}]'}\left(\frac{\iota_{\tilde{\Gamma}}^*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi})\right)}{e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}})}\int_{\overline{\mathcal{M}}_e}\frac{1}{(-\psi_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\psi_{v_2}^{\mathrm{ne}}-w(\tilde{\kappa}))}\frac{1}{(-z-\overline{\psi}_{v_1}^{\mathrm{ne}}+\overline{w}(\tilde{\kappa}))}\right).$$

We compute the last integral using the fact that $w(\tilde{\kappa})$ is invertible, the following lemma, which follows immediately from Lemma 2.3 of [48]:

Lemma 5.1.7. *Let $\psi_{0,LM}$ and $\psi_{\infty,LM}$ denote the tautological cotangent classes at b_0 and b_∞ on $\overline{\mathcal{M}}_{0|k|\infty}$. The pullbacks $\varphi^*\psi_{0,LM}$ and $\varphi^*\psi_{\infty,LM}$ along the reduction morphism $\overline{\mathcal{M}}_{0,k+2}\to\overline{\mathcal{M}}_{0|k|\infty}$ are the cotangent classes ψ_0 and ψ_∞, respectively.*

This says that we may integrate on $\overline{\mathcal{M}}_{k+2}$ (with an appropriate constant factor) instead of $\overline{\mathcal{M}}_e$. We use $r(\tilde{\kappa})(-\psi_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\psi_{v_2}^{\mathrm{ne}}-w(\tilde{\kappa}))=\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{\psi}_{v_2}^{\mathrm{ne}}-\overline{w}(\tilde{\kappa})$. It is well-known (see e.g. [40], Lemma 1.5.1) that

$$\int_{\overline{\mathcal{M}}_{0,k}}\psi_1^m\psi_2^{k-3-m}=\binom{k-3}{m}.$$

By Lemma 5.1.7, this identity holds on $\overline{\mathcal{M}}_{0|k|\infty}$ also. Thus:

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_e}\frac{1}{(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{\psi}_{v_2}^{\mathrm{ne}}-\overline{w}(\tilde{\kappa}))}\frac{1}{(-z-\overline{\psi}_{v_1}^{\mathrm{ne}}+\overline{w}(\tilde{\kappa}))}\\
&\quad=\frac{1}{|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde{\kappa})}\beta_\eta(\tilde{\kappa})}\int_{\overline{\mathcal{M}}_{v_1|\mathrm{mov}(\tilde{\kappa})|v_2}}\left(\sum_{m_1,m_2=0}^{\infty}\frac{(\overline{\psi}_{v_2})^{m_1}}{(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{w}(\tilde{\kappa}))^{m_1+1}}\frac{(\overline{\psi}_{v_1})^{m_2}}{(-z+\overline{w}(\tilde{\kappa}))^{m_2+1}}\right)\\
&\quad=\frac{1}{|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde{\kappa})}\beta_\eta(\tilde{\kappa})}\sum_{m_1+m_2=\mathrm{mov}(\tilde{\kappa})-1}\frac{\binom{\mathrm{mov}(\tilde{\kappa})-1}{m_1}}{(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{w}(\tilde{\kappa}))^{m_1+1}(-z+\overline{w}(\tilde{\kappa}))^{m_2+1}} \qquad (5.7)\\
&\quad=\frac{1}{|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde{\kappa})}\beta_\eta(\tilde{\kappa})}\frac{(-z-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}})^{\mathrm{mov}(\tilde{\kappa})-1}}{(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{w}(\tilde{\kappa}))^{\mathrm{mov}(\tilde{\kappa})}(-z+\overline{w}(\tilde{\kappa}))^{\mathrm{mov}(\tilde{\kappa})}}.
\end{aligned}$$

(The last inequality is gotten in the backwards direction by writing the numerator as $((-z+\overline{w}(\tilde{\kappa}))+$

$(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}} - \overline{w}(\tilde{\kappa})))^{\mathrm{mov}(\tilde{\kappa})-1}$ and expanding.) We have

$$\mathrm{Contr}(\tilde{\Gamma}) = \frac{|C_\mu(\sigma)|\,\left|C_{\mu'(\tilde{\kappa})}(\sigma'(\tilde{\kappa}))\right|}{|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde{\kappa})}\beta_\eta(\tilde{\kappa})}\frac{e_T(T_{(\mu'(\tilde{\kappa}),\sigma'(\tilde{\kappa}))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{W(-z+\overline{w}(\tilde{\kappa}))^{\mathrm{mov}(\tilde{\kappa})}} \cdot \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}]'}\left(\frac{\iota_{\tilde{\Gamma}}^*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi})\right)}{e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}})}\frac{(-z-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}})^{\mathrm{mov}(\tilde{\kappa})-1}}{(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{w}(\tilde{\kappa}))^{\mathrm{mov}(\tilde{\kappa})}}\right). \tag{5.8}$$

For fixed β_0, and n_0, from (5.3), the coefficient of $Q^{\beta_0}x^{n_0}$ in $\mathbf{f}_{(\mu,\sigma)}$ only has contributions from $\tilde{\Gamma}\in\mathrm{Trees}_{0,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta)$ for $\beta+n\le\beta_0+n_0$. This is because $\mathbf{t}(z)\in\langle x,Q\rangle$, so if $\mathcal{H}[[x]]$ is graded by giving Q and x degree 1, then the (n,β) term in (5.2) has degree at least $n+\beta$. In particular, $\bigcup_{\beta+n\le\beta_0+n_0}\mathrm{Trees}_{0,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta)$ is a finite set. Thus (5.3) and (5.8) realize the contribution to such a coefficient from trees of type (ii) as a finite sum of rational functions with poles at the weights $\tilde{\kappa}$. Together with the analysis above for types (i) and (iii), this proves that $\mathbf{f}_{(\mu,\sigma)}$ satisfies condition (**I**) of the Theorem.

We consider the Laurent coefficient $\mathrm{Laur}(\mathrm{Contr}(\tilde{\Gamma}),(\overline{w}-z)^{-a})$. By (5.8), $\mathrm{Laur}(\mathrm{Contr}(\tilde{\Gamma}),(\overline{w}-z)^{-a})$ is zero if $\overline{w}\neq\overline{w}(\tilde{\kappa})$, or if $\mathrm{mov}(\tilde{\kappa})<a$. Otherwise,

$$\begin{aligned}
&\mathrm{Laur}(\mathrm{Contr}(\tilde{\Gamma}),(\overline{w}-z)^{-a})\\
&= \frac{1}{(\mathrm{mov}(\tilde{\kappa})-a)!}\left(\frac{d^{\mathrm{mov}(\tilde{\kappa})-a}}{d(\overline{w}(\tilde{\kappa})-z)^{\mathrm{mov}(\tilde{\kappa})-a}}(\overline{w}(\tilde{\kappa})-z)^{\mathrm{mov}(\tilde{\kappa})}\,\mathrm{Contr}(\tilde{\Gamma})\right)\Bigg|_{z\mapsto\overline{w}(\tilde{\kappa})}\\
&= \frac{(-1)^{\mathrm{mov}(\tilde{\kappa})-a}\,|C_\mu(\sigma)|\,\left|C_{\mu'(\tilde{\kappa})}(\sigma'(\tilde{\kappa}))\right|\binom{\mathrm{mov}(\tilde{\kappa})-1}{a-1}}{W\,|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde{\kappa})}\beta_\eta(\tilde{\kappa})}\\
&\qquad\cdot\int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}]'}\left(\frac{\iota_{\tilde{\Gamma}}^*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi})\right)}{e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}})}\frac{e_T(T_{(\mu'(\tilde{\kappa}),\sigma'(\tilde{\kappa}))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{w}(\tilde{\kappa}))^{\mathrm{mov}(\tilde{\kappa})-a+1}}\right).
\end{aligned}$$

Now, summing over all $\tilde{\Gamma}$ of type (ii) with associated subtree $\tilde{\kappa}$ yields

$$\frac{(-1)^{\mathrm{mov}(\tilde{\kappa})-a}\,|C_\mu(\sigma)|\,\left|C_{\mu'(\tilde{\kappa})}(\sigma'(\tilde{\kappa}))\right|\binom{\mathrm{mov}(\tilde{\kappa})-1}{a-1}}{W\,|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde{\kappa})}\beta_\eta(\tilde{\kappa})} \left\langle \mathbf{t}(\overline{\psi}),\ldots,\mathbf{t}(\overline{\psi}),\frac{[(\mu'(\tilde{\kappa}),\sigma'(\tilde{\kappa}))]}{(-\overline{\psi}_{n+1}^{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}-\overline{w}(\tilde{\kappa}))^{\mathrm{mov}(\tilde{\kappa})-a+1}}\right\rangle^{\mathrm{Sym}^d\,\mathbb{P}^r,T}_{0,n+1,\beta-\beta(\tilde{\kappa})}. \tag{5.9}$$

On the other hand, the coefficient $\mathrm{Laur}(\mathbf{f}_{(\mu'(\tilde\kappa),\sigma'(\tilde\kappa))}, (\overline{w}(\tilde\kappa)-z)^{\mathrm{mov}(\tilde\kappa)-a})$ is

$$\sum_{\substack{n,\beta\geq 0\\ \beta>0 \text{ or } n\geq 2}} \frac{\left|C_{\mu'(\tilde\kappa)}(\sigma'(\tilde\kappa))\right| Q^\beta}{n!} \left\langle \mathbf{t}(\overline{\psi}),\dots,\mathbf{t}(\overline{\psi}), \frac{[(\mu'(\tilde\kappa),\sigma'(\tilde\kappa))]}{(-\overline{\psi}_{n+1}^{\tilde\Gamma\smallsetminus\tilde\kappa}-\overline{w}(\tilde\kappa))^{\mathrm{mov}(\tilde\kappa)-a+1}} \right\rangle_{0,n+1,\beta}^{\mathrm{Sym}^d\mathbb{P}^r,T} \tag{5.10}$$

We compute $\frac{|C_\mu(\sigma)|}{|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde\kappa)}\beta_\eta(\tilde\kappa)}$ explicitly:

$$|C_\mu(\sigma)| = |S_\sigma|\prod_{\eta\in\sigma}\eta$$

$$|S_e| = |C_{\mathrm{Stat}}(\tilde\kappa)|\left|S_{\mathrm{Mov}(\tilde\kappa)}\right| = \left|S_{\mathrm{Stat}(\tilde\kappa)}\right|\left|S_{\mathrm{Mov}(\tilde\kappa)}\right|\prod_{\eta\in\mathrm{Stat}(\tilde\kappa)}\eta$$

$$\frac{|C_\mu(\sigma)|}{|S_e|\prod_{\eta\in\mathrm{Mov}(\tilde\kappa)}\beta_\eta(\tilde\kappa)} = \frac{|S_\sigma|\prod_{\eta\in\mathrm{Mov}(\tilde\kappa)}\eta}{\left|S_{\mathrm{Stat}(\tilde\kappa)}\right|\left|S_{\mathrm{Mov}(\tilde\kappa)}\right|\prod_{\eta\in\mathrm{Mov}(\tilde\kappa)}\beta_\eta(\tilde\kappa)} = \frac{1}{q(\tilde\kappa)^{\mathrm{mov}(\tilde\kappa)}}\binom{\sigma}{\mathrm{Mov}(\tilde\kappa)}$$

With (5.9) and (5.10), this proves (**II**). Note that the contribution from all graphs of type (ii) (and the term $\mathbf{t}_{(\mu,\sigma)}(z)$) is

$$\boldsymbol{\tau}_{(\mu,\sigma)}(z) := \mathbf{t}_{(\mu,\sigma)}(z) + \sum_{\substack{\tilde\kappa\in\Upsilon(\mu,\sigma)\\ a\leq\mathrm{mov}(\tilde\kappa)}} \frac{Q^{\beta(\tilde\kappa)}\,\mathbf{RC}(\tilde\kappa,a)}{(\overline{w}(\tilde\kappa)-z)^a}\,\mathrm{Laur}(\mathbf{f}_{(\mu'(\tilde\kappa),\sigma'(\tilde\kappa))}, (\overline{w}(\tilde\kappa)-z)^{\mathrm{mov}(\tilde\kappa)-a}). \tag{5.11}$$

The proof of condition (**III**) is identical to that of condition (C3) in [18], and we reproduce the argument here for convenience.

Consider a decorated tree $\tilde\Gamma$ of type (iii). We write $v := \mathrm{Mark}(b_{n+1}) \in V^S(\Gamma)$. The marked points of $\overline{\mathcal{M}}_v$ correspond to (1) elements of $\mathrm{Mark}^{-1}(v)$, and (2) edges $e \in E(\Gamma)_v$. To e is associated a maximal subtree Γ_e containing v, with $E(\Gamma_e, v) = e$. We decorate Γ_e so that $\mathrm{Mark}^{-1}(v) = b$, and the rest of the decorations inherited from $\tilde\Gamma$. We will then write $\mathrm{Contr}(\tilde\Gamma)$ in terms of $\mathrm{Contr}(\tilde\Gamma_e)$ for $e \in E(\Gamma)_v$, and integrals over the vertex moduli space $\overline{\mathcal{M}}_v$.

We apply (5.4) again. After an étale base change $\widetilde{\overline{\mathcal{M}}}_{\tilde\Gamma} \to \overline{\mathcal{M}}_{\tilde\Gamma}$, we may label the subtrees $\tilde\Gamma_e$. (Write M for the degree of this base change.) We then write $\widetilde{\overline{\mathcal{M}}}_{\tilde\Gamma} \cong \overline{\mathcal{M}}_v \times \prod_{e\in E(\Gamma)_v}\overline{\mathcal{M}}_{\tilde\Gamma_e}$. Now we again apply Proposition 4.6.2, to see that

$$\frac{1}{e_T(N^{\mathrm{vir}}_{\tilde\Gamma})} = e_T^{-1}(R\pi_*(C_v, f^*T\,\mathrm{Sym}^d\mathbb{P}^r)) \prod_{e\in E(\Gamma)_v} \frac{r(v,e)e_T(T_{(\mu,\mathrm{Mon}(v,e))}I\,\mathrm{Sym}^d\mathbb{P}^r)}{(-\overline{\psi}_e^{\overline{\mathcal{M}}_v}-\overline{\psi}_v^{\overline{\mathcal{M}}_e})e_T(N^{\mathrm{vir}}_{\tilde\Gamma_e})}$$

Observe that $\frac{e_T(T_{(\mu,\mathrm{Mon}(v,e))} I \operatorname{Sym}^d \mathbb{P}^r)}{(-\overline{\psi}_e^{\overline{\mathcal{M}}_v} - \overline{\psi}_v^{\overline{\mathcal{M}}_e})}$ is the insertion at b in $\mathrm{Contr}(\tilde{\Gamma}_e)|_{z\mapsto \overline{\psi}_e^{\overline{\mathcal{M}}_v}}$. Thus

$$\mathrm{Contr}(\tilde{\Gamma}) = \frac{1}{M}\int_{\overline{\mathcal{M}}_v} \left(\prod_{e\in E(\Gamma)_v} |C_\mu(\sigma)|\, Q^{\beta(\tilde{\Gamma}_e)} \,\mathrm{Contr}(\tilde{\Gamma}_e)|_{z\mapsto \overline{\psi}_e^{\overline{\mathcal{M}}_v}} \right) \cup \left(\prod_{b_i \in \mathrm{Mark}^{-1}(v)} \mathbf{t}(\overline{\psi}) \right)$$
$$\cup \frac{e_T(T_{(\mu,\sigma)} I \operatorname{Sym}^d \mathbb{P}^r)}{-z-\overline{\psi}_{n+1}} \cup e_T^{-1}(R\pi_*(C_v, f^* T \operatorname{Sym}^d \mathbb{P}^r)).$$

This is almost a twisted Gromov-Witten invariant of $\mathrm{VEval}(v)$, but not quite, since there are restrictions on the monodromies at the marked points. Summing over $\tilde{\Gamma}_e$ for a single e, with everything else fixed, gives the insertion $\boldsymbol{\tau}_{(\mu,\mathrm{Mon}(v,e))}(\psi)$, where the initial term comes from replacing $\tilde{\Gamma}_e$ with a marked point. Thus summing over all σ, and over all $\tilde{\Gamma}$ of type (iii), gives

$$\sum_{m=2}^{\infty}\sum_{\sigma} \frac{1}{m!} \left\langle \boldsymbol{\tau}_\mu(\overline{\psi}), \ldots, \boldsymbol{\tau}_\mu(\overline{\psi}), \frac{[(\mu,\sigma)]}{-z-\overline{\psi}_{n+1}} \right\rangle_{0,m+1,0}^{\mathrm{VEval}(v),T,T \operatorname{Sym}^d \mathbb{P}^r} 1_{(\mu,\sigma)} \in H^*_{T,\mathrm{loc}}(I\mu, \mathbb{Q}),$$

where $1_{(\mu,\sigma)}$ is the fundamental class of $(\mu,\sigma) \in I\mu$, and $\boldsymbol{\tau}_\mu(z) = \sum_{\sigma' \in \mathrm{MultiPart}(\mu)} \boldsymbol{\tau}_{(\mu,\sigma')}(z) 1_{(\mu,\sigma)}$. Adding in the contributions from type (ii) graphs, summing (5.2) over σ yields:

$$\begin{aligned}
\mathbf{f}_\mu &= \sum_\sigma \mathbf{f}_{(\mu,\sigma)} 1_{\mu,\sigma} \\
&= -1_\mu z + \boldsymbol{\tau}_\mu(z) + \sum_{m=2}^{\infty}\sum_{\sigma} \frac{1}{m!} \left\langle \boldsymbol{\tau}_\mu(\overline{\psi}), \ldots, \boldsymbol{\tau}_\mu(\overline{\psi}), \frac{[(\mu,\sigma)]}{-z-\overline{\psi}_{n+1}} \right\rangle_{0,m+1,0}^{\mathrm{VEval}(v),T,T \operatorname{Sym}^d \mathbb{P}^r} 1_{(\mu,\sigma)},
\end{aligned}$$

where 1_μ is the untwisted fundamental class on $I\mu$. This shows that $\mathbf{f}_\mu$ is a $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point of $\mathcal{L}_\mu^{T \operatorname{Sym}^d \mathbb{P}^r}$. (Here we are using both comments in Remark 2.6.6.)

The converse also requires no modification from [18]. Suppose $\mathbf{f}$ satisfies the conditions of theorem. By conditions (**I**) and (**II**), we may uniquely write

$$\mathbf{f}_\mu = -1_\mu z + \sum_{\sigma \in \mathrm{MultiPart}(\mu)} \boldsymbol{\tau}_{(\mu,\sigma)}(z) 1_{(\mu,\sigma)} + O(z^{-1}),$$

where $\boldsymbol{\tau}_{(\mu,\sigma)}(z)$ is the expression in (5.11), for some $\mathbf{t}_{(\mu,\sigma)}(z) \in \iota_\mu^*(\mathcal{H}_+)[[x]]$. We claim that the set $\{\mathbf{t}_{(\mu,\sigma)}(z)\}$ for all fixed points (μ,σ) determines $\mathbf{f}$. By the localization isomorphism, if suffices to show that it determines $\mathbf{f}_{(\mu,\sigma)}$ for all (μ,σ). We induct on the degree $\beta+k$, where k is the exponent of x. The base case $\beta = k = 0$ is taken care of by the assumption $\mathbf{f}|_{Q=x=0} = -1z$. Assume the coefficients of $\mathbf{f}_{(\mu,\sigma)}$ up to degree $\beta+k$ are determined by $\{\mathbf{t}_{(\mu,\sigma)}\}$. Consider the coefficients of degree $\beta+k+1$. Some of these appear in $\mathbf{t}(z)$, but these are given. Some of them appear in $\boldsymbol{\tau}_{(\mu,\sigma)}(z)$, but these are determined since they are of the form: $Q^{\beta(\tilde{\kappa})}$ multiplied by a factor determined by the inductive hypothesis. The sum of all of these terms is in $H^*_{CR,T,\mathrm{loc}}(\mu,\mathbb{Q})[[x,Q]][[z]]$.

Finally, some of them appear in $O(z^{-1})$. However, condition (**III**) and (2.6) show that these are determined by terms of $-1z+\boldsymbol{\tau}_{(\mu,\sigma)}(z)$ of degree at most $\beta+k+1$. Since all such terms are determined by $\mathbf{t}_{(\mu,\sigma)}$ and induction, the degree $\beta+k+1$ coefficients of $\mathbf{f}_{(\mu,\sigma)}$ are determined. Thus in fact $\mathbf{f}$ is determined by $\{\mathbf{t}_{(\mu,\sigma)}(z)\}$.

Again by the localization isomorphism, the set $\{\mathbf{t}_{(\mu,\sigma)}(z)\}$ corresponds uniquely to an element $\mathbf{t}(z)\in\mathcal{H}_+[[x]]$ that restricts to each $\mathbf{t}_{(\mu,\sigma)}(z)$. This in turn corresponds uniquely to a $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point $\mathbf{f}_{\mathrm{GW}}$ of $\mathcal{L}_x$. By the uniqueness argument above we have $\mathbf{f}=\mathbf{f}_{\mathrm{GW}}$. $\square$

Remark 5.1.8. No modifications are required to replace x with a tuple $(x_1,\ldots,x_m)$.

5.2 Characterization of the Givental cone $\mathcal{L}_X$, general (nonorbifold) case

In this section, we discuss the extent to which Theorem 5.1.4 can be extended to a general variety X with action of a torus T. Here are a few of the major points:

1. In Condition (**II**) of Theorem 5.1.4, the terms of the recursion are elements of $H^*((\mu,\sigma),\mathbb{Q})$ for **various** (μ,σ), and the equality is via the canonical isomorphism between these groups. For more general (X,T), these terms a priori lie in the homology groups of **various** components of X^T. Thus (5.1) must have a somewhat different form when X does not have isolated fixed points.

2. As in the proof of Theorem 5.1.4, for each component μ of X^T, one may compute $\mathbf{f}_\mu$ by localization, and decompose $\mathbf{f}_\mu$ as a sum of two terms, the contributions from trees of initial and recursion type. As before, these correspond to poles at $z=0$ or ∞, or poles at $z=w(\tilde{\kappa})$, respectively.

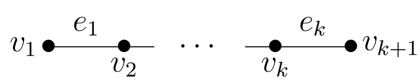
3. Essentially the same argument as in the last section shows that Condition (**III**) of Theorem 5.1.4 holds in general, with $\mathcal{L}_\mu^{T\,\mathrm{Sym}^d\,\mathbb{P}^r}$ replaced with the Givental cone $\mathcal{L}_\mu^{N_{\mu|X}}$ of μ twisted by the normal bundle to μ in X. (This coincides with TX when μ is a point.)

4. In [24] Theorem 3.5, Fan and Lee show the existence of a general recursion for points on the Givental cone, via a similar method. Our recursion (5.1) is, in appearance, much simpler than theirs. In particular, the left side of ours is a single Laurent coefficient of $\mathbf{f}_\mu$, and the right side is expressed in terms of Laurent coefficients of $\mathbf{f}_{\mu'}$ for various μ'. On the other hand, the left side of their recursion is the entire principal part $\mathrm{Prin}_{z=w}\,\mathbf{f}_\mu$, and the right side is expressed in

terms of *nonprincipal* parts of certain derivatives of $\mathbf{f}_{\mu'}$ for various μ'. Theorem 5.1.4 makes us hope that a simpler form such as this is possible for arbitrary X, and we suspect that it is. For $X = \mathrm{Sym}^d\,\mathbb{P}^r$, the refined form was enabled by the "miracle" computation (5.7).

Though we are unable to prove a sufficiently refined recursion, we go through some of the steps to demonstrate what is needed and what is difficult.

Definition 5.2.1. (CHANGE THIS) Analogously to 5.1.1, for $\mu \in V(\Gamma_{X,T})$ a connected component of X^T, let $\Upsilon(\mu) \subseteq \mathrm{Trees}_{0,2}(X, \beta)$ be the set of X-decorated trees

$\tilde{\kappa} =$ 

with $\mathrm{Mark}(1) = v_1$, $\mathrm{Mark}(2) = v_2$, $\mu = \mathbb{f}(v_1)$, $\mathrm{vdeg}(v_1) = \cdots = \mathrm{vdeg}(v_{k+1}) = 0$, and (v_i, e_{i-1}) *unsteady* for $2 \leq i \leq k$.

Notation 5.2.2. For $\tilde{\kappa} \in \Upsilon(\mu)$ as in Definition 5.1.1, we write $\mu'(\tilde{\kappa}) := \mathbb{f}(v_{k+1})$ and $w(\tilde{\kappa}) := w(v_1, e_1) = w(v_i, e_i)$.

Let $\mathbf{f}$ be a $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point of $\mathcal{L}_X$.Using the projection formula, we can write

$$\mathbf{f}_\mu = -1_\mu z + \mathbf{t}_\mu(z) + \sum_{\substack{n,\beta \geq 0 \\ \beta > 0 \text{ or } n \geq 2}} \frac{Q^\beta}{n!} \sum_\varphi \left\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), \frac{(\iota_\mu)_*(\gamma^\mu_\varphi)}{-z - \psi} \right\rangle^{X,T}_{0,n+1,\beta} \gamma^\varphi_\mu. \tag{5.12}$$

where $\mathbf{t}_\mu(z) := \iota^*_\mu \mathbf{t}(z)$. Ignoring the subtleties of Section 4.2, we define a partial order on decorated trees using (not necessarily realizable) chain contractions, as in Section 4.5.1. We consider minimal trees $\tilde{\Gamma}$, and write $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ for the locus of T-fixed stable maps $(f : C \to X)$ satisfying $\Psi(f) \geq \tilde{\Gamma}$. Then we may write

$$\sum_{\beta,n} \sum_\varphi \left\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), \frac{(\iota_\mu)_*(\gamma^\mu_\varphi)}{-z - \psi} \right\rangle^{X,T}_{0,n+1,\beta} \gamma^\varphi_\mu = \sum_{\tilde{\Gamma} \in \mathrm{Trees}^{\min}_{0,n+1}(X,\beta)} \mathrm{Contr}_\mu(\tilde{\Gamma}), \tag{5.13}$$

and we may decompose the right side as two sums, one over trees of initial type, and the other over trees of recursion type.

Let $\tilde{\Gamma}$ be a tree of recursion type. By (4.1), we have

$$\mathrm{Contr}(\tilde{\Gamma}) = \sum_\varphi \gamma^\varphi_\mu \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}}]^{vir}} \frac{1}{e_T(N^{\mathrm{vir}}_{\tilde{\Gamma}})} \iota^*_{\tilde{\Gamma}} \left(\prod_{j=1}^n \mathrm{ev}^*_j\, \mathbf{t}(\psi) \cup \frac{\mathrm{ev}^*_{n+1}(\iota_\mu)_*(\gamma^\mu_\varphi)}{-z - \psi_{n+1}} \right). \tag{5.14}$$

Since $\mathrm{ev}_{n+1} \circ \iota_{\tilde{\Gamma}}$ factors through μ, we have

$$\iota_{\tilde{\Gamma}}^* \mathrm{ev}_{n+1}^* (\iota_\mu)_* (\gamma_\varphi^\mu) = \iota_{\tilde{\Gamma}}^* \mathrm{ev}_{n+1}^* (\gamma_\varphi^\mu \cup e_T(N_{\mu|X})).$$

From $\tilde{\Gamma}$, via "edge pruning" we again obtain two decorated trees $\tilde{\kappa} \in \Upsilon(\sigma)$ and $\tilde{\Gamma} \setminus \tilde{\kappa} \in \mathrm{Trees}_{0,n+1}(X, \beta - \beta(\tilde{\kappa}))$. This realizes $\overline{\mathcal{M}}_{\tilde{\Gamma}}$ as the fiber product of moduli spaces $\overline{\mathcal{M}}_e$ and $\overline{\mathcal{M}}_{\tilde{\Gamma}\setminus\tilde{\kappa}}$. Note that $\overline{\mathcal{M}}_e$ may be quite complicated — for example, the generic source curve may be nodal. (These are the moduli spaces $M_{\omega,2}(X)$ in [49], which works in the case $T = \mathbb{C}^*$, and in the additional generality of weighted stable maps.)

We factor the T-equivariant map $\overline{\mathcal{M}}_{\tilde{\Gamma}} \to \operatorname{Spec} \mathbb{C}$ through the second projection, i.e. we integrate over $\overline{\mathcal{M}}_e$:

$$\mathrm{Contr}(\tilde{\Gamma}) = \sum_\varphi \gamma_\mu^\varphi \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\setminus\tilde{\kappa}}]^{vir}} \int_{[\overline{\mathcal{M}}_e]^{\mathrm{vir}}} \frac{\iota_{\tilde{\Gamma}}^* \left(\mathrm{ev}_{n+1}^* (\gamma_\varphi^\mu \cup e_T(N_{\mu|X}) \cup \prod_{j=1}^n \mathrm{ev}_j^* \mathbf{t}(\psi) \cup \frac{1}{-z-\psi_{n+1}} \right)}{e_T(N_{\tilde{\Gamma}}^{\mathrm{vir}})}. \tag{5.15}$$

By the proof of Lemma 7.5.2 of [13] (pp. 441–442), we may generalize (5.5) to:

$$\frac{1}{e_T(N_{\tilde{\Gamma}}^{\mathrm{vir}})} = \frac{e_T(\mathrm{ev}_\bullet^*(N_{\mu'(\tilde{\kappa})|X}))}{e_T(N_{\overline{\mathcal{M}}_e}^{\mathrm{vir}}) e_T(N_{\tilde{\Gamma}\setminus\tilde{\kappa}}^{\mathrm{vir}})(-\psi_e^{\overline{\mathcal{M}}_{v_2}} - \psi_{v_2}^{\overline{\mathcal{M}}_e})}.$$

Then (5.15) becomes

$$\mathrm{Contr}(\tilde{\Gamma}) = \sum_\varphi \gamma_\mu^\varphi \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\setminus\tilde{\kappa}}]^{vir}} \frac{\iota_{\tilde{\Gamma}}^* \left(\prod_{j=1}^n \mathrm{ev}_j^* \mathbf{t}(\psi) \right)}{e_T(N_{\tilde{\Gamma}\setminus\tilde{\kappa}}^{\mathrm{vir}})} \left(\int_{[\overline{\mathcal{M}}_e]^{\mathrm{vir}}} \frac{e_T(\mathrm{ev}_\bullet^*(N_{\mu'(\tilde{\kappa})|X})) \cup \mathrm{ev}_{n+1}^*(\gamma_\varphi^\mu \cup e_T(N_{\mu|X}))}{e_T(N_{\overline{\mathcal{M}}_e}^{\mathrm{vir}})(-\psi_e^{\overline{\mathcal{M}}_{v_2}} - \psi_{v_2}^{\overline{\mathcal{M}}_e})(-z-\psi_{n+1})} \right). \tag{5.16}$$

As in (5.6), we write $\psi_{v_1}^{\overline{\mathcal{M}}_e} = \psi_{v_1}^{\mathrm{ne}} - \overline{w}(\tilde{\kappa})$ and $\psi_{v_2}^{\overline{\mathcal{M}}_e} = \psi_{v_2}^{\mathrm{ne}} + \overline{w}(\tilde{\kappa})$. Expanding the denominator in

(5.16),

$$\mathrm{Contr}(\tilde{\Gamma}) = \sum_{\varphi} \gamma_{\mu}^{\varphi} \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}]^{vir}} \left(\frac{e_T(\mathrm{ev}_{\bullet}^*(N_{\mu'(\tilde{\kappa})})) \cup \iota_{\tilde{\Gamma}}^* \left(\prod_{j=1}^n \mathrm{ev}_j^* \mathbf{t}(\psi) \right)}{e_T(N_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}^{\mathrm{vir}})} \right.$$
$$\left. \cdot \left(\int_{[\overline{\mathcal{M}}_e]^{\mathrm{vir}}} \frac{\mathrm{ev}_{n+1}^*(\gamma_{\varphi}^{\mu} \cup e_T(N_{\mu|X}))}{e_T(N_{\overline{\mathcal{M}}_e}^{\mathrm{vir}})} \sum_{m_1,m_2\geq 0} \frac{(\psi_{v_2}^{\mathrm{ne}})^{m_1}(\psi_{v_1}^{\mathrm{ne}})^{m_2}}{(-\psi_{n+1}^{\tilde{\Gamma\smallsetminus\tilde{\kappa}}} - w(\tilde{\kappa}))^{m_1+1}(-z+w(\tilde{\kappa}))^{m_2+1}} \right) \right)$$

$$= \sum_{\varphi} \gamma_{\mu}^{\varphi} \sum_{m_1,m_2\geq 0} \int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}]^{vir}} \left(\frac{e_T(\mathrm{ev}_{\bullet}^*(N_{\mu'(\tilde{\kappa})})) \cup \iota_{\tilde{\Gamma}}^* \left(\prod_{j=1}^n \mathrm{ev}_j^* \mathbf{t}(\psi) \right)}{e_T(N_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}^{\mathrm{vir}})(-\psi_{n+1}^{\tilde{\Gamma\smallsetminus\tilde{\kappa}}} - w(\tilde{\kappa}))^{m_1+1}(-z+w(\tilde{\kappa}))^{m_2+1}} \right)$$
$$\cdot \left(\int_{[\overline{\mathcal{M}}_e]^{\mathrm{vir}}} \frac{\mathrm{ev}_{n+1}^*(\gamma_{\varphi}^{\mu} \cup e_T(N_{\mu|X}))(\psi_{v_2}^{\mathrm{ne}})^{m_1}(\psi_{v_1}^{\mathrm{ne}})^{m_2}}{e_T(N_{\overline{\mathcal{M}}_e}^{\mathrm{vir}})} \right) \tag{5.17}$$

Note that the sums over m_1 and m_2 are finite, since $\psi_{v_1}^{\mathrm{ne}}$ and $\psi_{v_2}^{\mathrm{ne}}$ are nonequivariant, hence nilpotent. Also, the second line of (5.17) is now valued in $H_T^*(\mathrm{Spec}\,\mathbb{C}, \mathbb{Q})$ instead of $H_T^*(\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}, \mathbb{Q})$. Applying the projection formula, (5.17) is equal to

$$\sum_{m_1,m_2\geq 0} \frac{1}{(-z+w(\tilde{\kappa}))^{m_2+1}} \left(\int_{[\overline{\mathcal{M}}_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}]^{vir}} \frac{e_T(\mathrm{ev}_{\bullet}^*(N_{\mu'(\tilde{\kappa})})) \cup \iota_{\tilde{\Gamma}}^* \left(\prod_{j=1}^n \mathrm{ev}_j^* \mathbf{t}(\psi) \right)}{e_T(N_{\tilde{\Gamma}\smallsetminus\tilde{\kappa}}^{\mathrm{vir}})(-\psi_{n+1}^{\tilde{\Gamma\smallsetminus\tilde{\kappa}}} - w(\tilde{\kappa}))^{m_1+1}} \right)$$
$$\cdot (\mathrm{ev}_{n+1})_* \left((\mathrm{ev}_{n+1}^* e_T(N_{\mu|X})) \cup \frac{(\psi_{v_2}^{\mathrm{ne}})^{m_1}(\psi_{v_1}^{\mathrm{ne}})^{m_2}}{e_T(N_{\overline{\mathcal{M}}_e}^{\mathrm{vir}})} \cap [\overline{\mathcal{M}}_e]^{\mathrm{vir}} \right) \tag{5.18}$$

The integral in this expression is, formally, the contribution of $\tilde{\Gamma} \smallsetminus \tilde{\kappa}$ to

$$\frac{1}{m!} \int_{\mu'(\tilde{\kappa})} \left. \frac{d^{m+1}\mathbf{f}_{\mu'(\tilde{\kappa})}}{dz^{m+1}} \right|_{z\mapsto w(\tilde{\kappa})}.$$

However, this expression is not well-defined, since $\frac{d^{m+1}\mathbf{f}_{\mu'(\tilde{\kappa})}}{dz^{m+1}}$ has, in general, a pole at $z = w(\tilde{\kappa})$. Summing over all trees of recursion type gives instead

$$\sum_{\tilde{\kappa}} \sum_{m_1,m_2\geq 0} \frac{1}{(-z+w(\tilde{\kappa}))^{m_2+1}} \frac{1}{m!} \int_{\mu'(\tilde{\kappa})} \left. \left(\frac{d^{m+1}\mathbf{f}_{\mu'(\tilde{\kappa})}}{dz^{m+1}} - \mathrm{Prin}_{z=w(\tilde{\kappa})} \frac{d^{m+1}\mathbf{f}_{\mu'(\tilde{\kappa})}}{dz^{m+1}} \right) \right|_{z\mapsto w(\tilde{\kappa})}$$
$$\cdot (\mathrm{ev}_{n+1})_* \left((\mathrm{ev}_{n+1}^* e_T(N_{\mu|X})) \cup \frac{(\psi_{v_2}^{\mathrm{ne}})^{m_1}(\psi_{v_1}^{\mathrm{ne}})^{m_2}}{e_T(N_{\overline{\mathcal{M}}_e}^{\mathrm{vir}})} \cap [\overline{\mathcal{M}}_e]^{\mathrm{vir}} \right), \tag{5.19}$$

where Prin denotes the principal part of the Laurent expansion.

Remark 5.2.3. As mentioned, this recursion (which is the one appearing in [24]) does not contain individual Laurent coefficients of the other restrictions $\mathbf{f}_{\mu'(\tilde{\kappa})}$. We have hope that summing over the index m_1 should give a more elegant expression similar to that in Theorem 5.1.4. If true, this would give us information about the geometry of the mysterious spaces $\overline{\mathcal{M}}_e$.

CHAPTER 6

The I-function and mirror theorem for $\mathrm{Sym}^d \mathbb{P}^r$

6.1 Statement of the main theorem

In this section we introduce the function $I_{\mathrm{Sym}^d \mathbb{P}^r}(t, \mathbf{x}, Q, -z)$. We will then show that it is a $\Lambda^T_{\mathrm{nov}}[[t, \mathbf{x}]]$-valued point of $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$, where $\mathbf{x} = \{x_\varpi\}_{\varpi \in \mathrm{Part}(d)}$ are formal variables.

Definition 6.1.1. The (extended) I-function is

$$I_{\mathrm{Sym}^d \mathbb{P}^r}(t,\mathbf{x}, Q, z) = z \sum_{\sigma \in \mathrm{Part}(d)} 1_\sigma \sum_{\beta \geq 0} \exp\left(\sum_{i=0}^{r} t_i([H_{\sigma,i}]/z + \beta)\right) Q^\beta \sum_{\substack{\mathbb{Z}_{>0}\text{-labels } L = (L_\eta) \\ \text{of } \sigma \text{ with sum } \beta}} \cdot \left(\sum_{\substack{\mathbf{k}=(k_\varpi)_{\varpi \in \mathrm{Part}(d)} \\ k_\varpi \geq 0}} \frac{\mathbf{x}^{\mathbf{k}} H(\sigma \prod_\varpi \varpi^{k_\varpi})}{\mathbf{k}! z^{\mathbf{k}}} \right) \left(\frac{|S_\sigma|}{|S_{\sigma,L}|} \right) \left(\prod_{\eta \in \sigma} \frac{1}{\prod_{\gamma=1}^{L_\eta} \prod_{i=0}^{r} \left(H_{\sigma,\eta,i} + \frac{\gamma}{\eta} z \right)} \right) \tag{6.1}$$

where:

- $1_\sigma \in H^*_{CR,T}(\mathrm{Sym}^d \mathbb{P}^r, \mathbb{Q})$ is the fundamental class of the twisted sector corresponding to σ,
- $[H_{\sigma,i}]$ and $[H_{\sigma,\eta,i}]$ are defined in Section 3.1.3,
- $\mathbf{x}^{\mathbf{k}} := \prod_\varpi x_\varpi^{k_\varpi}$,
- $\mathbf{k}! := \prod_\varpi k_\varpi!$,
- $z^{\mathbf{k}} := \prod_\varpi z^{k_\varpi}$, and
- $H(\sigma \prod_\varpi \varpi^{k_\varpi})$ is the number of ways of factoring $1 \in S_d$ as a product $a_1 \cdots a_{1+\sum k_\varpi}$, where the conjugacy classes (i.e. partitions) of the permutations a_j are given by the list $(\sigma \prod_\varpi \varpi^{k_\varpi})$.

Note that (6.1) uses the normal cup product on $H^*(I\operatorname{Sym}^d\mathbb{P}^r,\mathbb{Q})$, not the Chen-Ruan product. As mentioned, we prove:

Theorem 6.1.2. *$I_{\operatorname{Sym}^d\mathbb{P}^r}(t,\mathbf{x},Q,-z)$ is a $\Lambda^T_{\mathrm{nov}}[[t,\mathbf{x}]]$-valued point of $\mathcal{L}_{\operatorname{Sym}^d\mathbb{P}^r}$.*

6.2 Proof of Theorem 6.1.2

Instead, we will show that a different series $\underline{I}_{\operatorname{Sym}^d\mathbb{P}^r}(\mathbf{x},Q,-z)$ is a $\Lambda^T_{\mathrm{nov}}[[\mathbf{x}]]$-valued point of $\mathcal{L}_{\operatorname{Sym}^d\mathbb{P}^r}$, where $\underline{I}_{\operatorname{Sym}^d\mathbb{P}^r}(\mathbf{x},Q,-z)$ is obtained from $I_{\operatorname{Sym}^d\mathbb{P}^r}(\mathbf{x},Q,-z)$ by removing the exponential factor. The divisor equation in Gromov-Witten theory then implies Theorem 6.1.2.

It is immediate that $\underline{I}_{\operatorname{Sym}^d\mathbb{P}^r}(0,0,0,-z) = -1z$. Per Theorem 5.1.4, it now suffices to prove conditions (**I**), (**II**), and (**III**), which we do in Sections 6.2.1, 6.2.2, and 6.2.3. We write $\underline{I}_{(\mu,\sigma)}$ for the restriction of $\underline{I}_{\operatorname{Sym}^d\mathbb{P}^r}(\mathbf{x},Q,-z)$ to a T-fixed point $(\mu,\sigma)\in \underline{I}\operatorname{Sym}^d\mathbb{P}^r$. We write $r_\sigma := \operatorname{lcm}(\sigma)$.

6.2.1 $\underline{I}_{\operatorname{Sym}^d\mathbb{P}^r}(\mathbf{x},Q,-z)$ satisfies condition (I)

From (6.1),

$$\underline{I}_{(\mu,\sigma)} = -z\sum_{\beta\geq 0} Q^\beta \sum_{\substack{\text{labels } L=(L_\eta)\\ \text{of } \sigma \text{ with sum } \beta}} \left(\sum_{\substack{\mathbf{k}=(k_\varpi)_{\varpi\in\mathrm{Part}(d)}\\ k_\varpi\geq 0}} \frac{\mathbf{x}^{\mathbf{k}} H(\sigma\prod_\varpi \varpi^{k_\varpi})}{\mathbf{k}!(-z)^{\mathbf{k}}}\right)\left(\frac{|S_\sigma|}{|S_{\sigma,L}|}\right)\left(\frac{1}{\prod_{\eta\in\sigma}\prod_{\gamma=1}^{L_\eta}\prod_{i=0}^{r}\left(r_\sigma(\alpha_{i(\eta)}-\alpha_i)-\frac{\gamma}{\eta}z\right)}\right). \tag{6.2}$$

It is clear that the coefficient of a single power $\mathbf{x}$ and Q is a rational function in z. The poles of such a coefficient are (at worst) $z=0$, $z=\infty$, and $z=\frac{r_\sigma(\alpha_{i_1}-\alpha_{i_2})}{q}$, where $i_1 = i(\eta)$ for some $\eta\in\sigma$, and $q\in\frac{1}{\eta}\mathbb{Z}$. This is exactly the set of values arising as $\overline{w}(\tilde{\kappa})$ for $\tilde{\kappa}\in\Upsilon(\mu,\sigma)$. This proves (**I**).

6.2.2 $\underline{I}_{\operatorname{Sym}^d\mathbb{P}^r}(\mathbf{x},Q,-z)$ satisfies condition (II)

We work with the left and right sides of (5.1). Fix a and $\overline{w} = \frac{r_\sigma(\alpha_{i_1}-\alpha_{i_2})}{q}$. When applied to $\mathbf{f} = \underline{I}_{\operatorname{Sym}^d\mathbb{P}^r}(\mathbf{x},Q,-z)$ the factor

$$\Omega := -z\left(\sum_{\substack{\mathbf{k}=(k_\varpi)_{\varpi\in\mathrm{Part}(d)}\\ k_\varpi\geq 0}} \frac{\mathbf{x}^{\mathbf{k}} H_\sigma(\sigma\prod_\varpi \varpi^{k_\varpi})}{\mathbf{k}!(-z)^{\mathbf{k}}}\right)\left(\frac{1}{\prod_{\substack{\eta\in\sigma\\ i(\eta)\neq i_1 \text{ or}\\ L_\eta<q\eta}}\prod_{\gamma=1}^{L_\eta}\prod_{i=0}^{r}\left(r_\sigma(\alpha_{i(\eta)}-\alpha_i)-\frac{\gamma}{\eta}z\right)}\right)$$

appears identically on both sides, so we may prove (5.1) instead for $\underline{I}_{(\mu,\sigma)}/\Omega$. We break up $\underline{I}_{(\mu,\sigma)}/\Omega$, the left-hand side of (5.1), into terms by the label L (with sum β):

$$T_{(\mu,\sigma),L}(Q,z) := \frac{|S_\sigma|}{|S_{\sigma,L}|}\left(\frac{Q^\beta}{\prod_{\substack{\eta\in\sigma\\ i(\eta)=i_1\\ L_\eta\geq q\eta}}\prod_{\gamma=1}^{L_\eta}\prod_{i=0}^{r}\left(r_\sigma(\alpha_{i(\eta)}-\alpha_i)-\frac{\gamma}{\eta}z\right)}\right). \tag{6.3}$$

Write σ_T for the multiset $\{\eta\in\sigma | i(\eta)=i_1, L_\eta\geq q\eta\}$. This consists of parts of σ that are in $\operatorname{Mov}(\tilde{\kappa})$ for some $\tilde{\kappa}$ with weight $\overline{w}$ based at (μ,σ). We compute

$$\begin{aligned}\operatorname{Laur}(T_{(\mu,\sigma),L}(Q,z),(\overline{w}-z)^{-a}) &= \frac{1}{(|\sigma_T|-a)!}\left(\frac{d^{|\sigma_T|-a}}{d(\overline{w}-z)^{|\sigma_T|-a}}(\overline{w}-z)^{|\sigma_T|}T_{(\mu,\sigma),L}(Q,z)\right)_{z\mapsto\overline{w}}\\ &= \frac{|S_\sigma|/|S_{\sigma,L}|}{(|\sigma_T|-a)!}\left(\sum_A\prod_{(\eta,\gamma,i)\in A}\frac{\gamma/\eta}{\left(r_\sigma(\alpha_{i_1}-\alpha_i)-\frac{\gamma}{\eta}\overline{w}\right)}\right)\\ &\qquad\left(\frac{Q^\beta}{q^{|\sigma_T|}\prod_{\eta\in\sigma_T}\prod_{i=0}^{r}\prod_{\substack{1\leq\gamma\leq L_\eta\\ (\gamma,i)\neq(q\eta,i_2)}}\left(r_\sigma(\alpha_{i_1}-\alpha_i)-\frac{\gamma}{\eta}\overline{w}\right)}\right),\end{aligned} \tag{6.4}$$

where A ranges over $(|\sigma_T|-a)$-tuples of factors in the denominator of $T_{(\mu,\sigma),L}(Q,z)$, i.e. over unordered tuples of triples (η,γ,i), with $\eta\in\sigma_T$, $1\leq\gamma\leq L_\eta$, $0\leq i\leq r$, and $(\gamma,i)\neq(q\eta,i_2)$. Observe that

$$r_\sigma(\alpha_{i_1}-\alpha_i)-\frac{\gamma}{\eta}\overline{w} = r_\sigma\left(\frac{q\eta-\gamma}{q\eta}\alpha_{i_1}+\frac{\gamma}{q\eta}\alpha_{i_2}-\alpha_i\right).$$

For the right-hand side of (5.1), let $\tilde{\kappa}\in\Upsilon(\mu,\sigma)$ with $\overline{w}(\tilde{\kappa})=\overline{w}$. Such an edge corresponds to a subset $\operatorname{Mov}(\tilde{\kappa})\subseteq\sigma_T$. Let $L'(\tilde{\kappa})$ be the label of $\sigma'(\tilde{\kappa})$ obtained by decreasing L_η by $q\eta$ for $\eta\in\operatorname{Mov}(\tilde{\kappa})$, using an identification of σ and $\sigma'(\tilde{\kappa})$ as partitions. (There is a factor of $\left|S_{\sigma,L'(\tilde{\kappa})}\right|/|S_{\sigma,L}|$ from different choices of identification.) As above we write $T_{(\mu'(\tilde{\kappa}),\sigma'(\tilde{\kappa})),L'(\tilde{\kappa})}(Q,z)$ for the factors of $\underline{I}_{(\mu'(\tilde{\kappa}),\sigma'(\tilde{\kappa}))}$ coming from σ_T. (The meaning of the multiset $\sigma_T\subseteq\sigma$ has not changed.) Then as before

$$\begin{aligned}
&\mathrm{Laur}(T_{(\mu'(\tilde\kappa),\sigma'(\tilde\kappa)),L'(\tilde\kappa)}(Q,z),(\overline{w}-z)^{\mathrm{mov}(\tilde\kappa)-a})\\
&\qquad\qquad=\frac{1}{(|\sigma_T|-a)!}\left(\frac{d^{|\sigma_T|-a}}{d(\overline{w}-z)^{|\sigma_T|-a}}(\overline{w}-z)^{|\sigma_T|-\mathrm{mov}(\tilde\kappa)}T_{(\mu'(\tilde\kappa),\sigma'(\tilde\kappa)),L'(\tilde\kappa)}\right)\\
&\qquad\qquad=\frac{|S_\sigma|\,/\,|S_{\sigma,L}|\,Q^{\beta-\beta(\tilde\kappa)}}{(|\sigma_T|-a)!q^{|\sigma_T|-\mathrm{mov}(\tilde\kappa)}}\left(\sum_{B_{\tilde\kappa}}\prod_{(\eta,\gamma,i)\in B_{\tilde\kappa}}\frac{\gamma/\eta}{\left(r_\sigma(\alpha_{i(\eta)}-\alpha_i)-\frac{\gamma}{\eta}\overline{w}\right)}\right) \qquad (6.5)\\
&\qquad\qquad\qquad\cdot\left(\frac{1}{\prod_{\eta\in\mathrm{Mov}(\tilde\kappa)}\prod_{i=0}^{r}\prod_{\gamma=1}^{L_\eta-q\eta}\left(r_\sigma(\alpha_{i_2}-\alpha_i)-\frac{\gamma}{\eta}\overline{w}\right)}\right)\\
&\qquad\qquad\qquad\quad\cdot\left(\frac{1}{\prod_{\eta\in\sigma_T\smallsetminus\mathrm{Mov}(\tilde\kappa)}\prod_{i=0}^{r}\prod_{\substack{1\le\gamma\le L_\eta\\(\gamma,i)\ne(q\eta,i_2)}}\left(r_\sigma(\alpha_{i_1}-\alpha_i)-\frac{\gamma}{\eta}\overline{w}\right)}\right),
\end{aligned}$$

where $B_{\tilde\kappa}$ runs over $(|\sigma_T|-a)$-tuples of factors in the denominator. The product in the denominator over $\eta\in\sigma_T\smallsetminus\mathrm{Mov}(\tilde\kappa)$ appears identically in (6.4), and the product over $\eta\in\mathrm{Mov}(\tilde\kappa)$ of the factors

$$\left(r_\sigma(\alpha_{i_2}-\alpha_i)-\frac{\gamma}{\eta}\overline{w}\right)=\left(r_\sigma(\alpha_{i_1}-\alpha_i)-\left(q+\frac{\gamma}{\eta}\right)\overline{w}\right)$$

appears in (6.4) via the substitution $\gamma\mapsto\gamma-q\eta$. Together with the denominator of $\mathbf{RC}(\tilde\kappa,a)$, this makes up entire denominator of (6.4), excluding the sum over A. The factor Q^β also appears on both sides, so it remains to prove:

$$\sum_A\prod_{(\eta,\gamma,i)\in A}\gamma/\eta=\sum_{\tilde\kappa}\sum_{B_{\tilde\kappa}}(-1)^{\mathrm{mov}(\tilde\kappa)-a}\binom{\sigma_{i_1^{\mathrm{mov}(\tilde\kappa)}}}{\mathrm{Mov}(\tilde\kappa)}\binom{\mathrm{mov}(\tilde\kappa)-1}{a-1}\prod_{(\eta,\gamma,i)\in B_{\tilde\kappa}}\gamma/\eta. \qquad (6.6)$$

We switch the order of summation on the right-hand side, and identify each tuple $B_{\tilde\kappa}$ with one of the tuples A via the substitution $\gamma/\eta\mapsto\gamma/\eta-q$ for $\eta\in\mathrm{Mov}(\tilde\kappa)$. We now want to prove:

$$\sum_A\prod_{(\eta,\gamma,i)\in A}\gamma/\eta=\sum_A\sum_{\substack{\mathrm{Mov}\subseteq\sigma_T\\|\mathrm{Mov}|\ge a}}(-1)^{|\mathrm{Mov}|-a}\binom{\sigma_{i_1}}{\mathrm{Mov}}\binom{|\mathrm{Mov}|-1}{a-1}\prod_{\substack{(\eta,\gamma,i)\in A\\ \eta\in\mathrm{Mov}}}(\gamma/\eta-q)\prod_{\substack{(\eta,\gamma,i)\in A\\ \eta\notin\mathrm{Mov}}}\gamma/\eta.$$

We break up the right side further by fixing the set $A' := \{(\eta,\gamma,i) \in A | \eta \in \mathrm{Mov}\}$:

$$\sum_{A}\sum_{A'\subseteq A}\prod_{(\eta,\gamma,i)\in A'}(\gamma/\eta - q)\prod_{(\eta,\gamma,i)\in A\smallsetminus A'}\gamma/\eta \sum_{\substack{\mathrm{Mov}\subseteq\sigma_T\\ |\mathrm{Mov}|\geq a\\ \eta\in\mathrm{Mov}\ \text{for}\ (\eta,\gamma,i)\in A'\\ \eta\notin\mathrm{Mov}\ \text{for}\ (\eta,\gamma,i)\in A\smallsetminus A'}}(-1)^{|\mathrm{Mov}|-a}\binom{\sigma_{i_1}}{\mathrm{Mov}}\binom{|\mathrm{Mov}|-1}{a-1}. \tag{6.7}$$

The factor $\binom{\sigma_{i_1}}{\mathrm{Mov}}$ turns the second summation on the right into a sum over *labeled* submultisets $\mathrm{Mov} \subseteq \sigma_T$. We then use the straightforward combinatorial identity:

$$\sum_{\substack{\text{labeled multisets }\ \mathrm{Mov}\subseteq\sigma_T\\ |\mathrm{Mov}|\geq a\\ \eta\in\mathrm{Mov}\ \text{for}\ (\eta,\gamma,i)\in A'\\ \eta\notin\mathrm{Mov}\ \text{for}\ (\eta,\gamma,i)\in A\smallsetminus A'}}(-1)^{|\mathrm{Mov}|-a}\binom{|\mathrm{Mov}|-1}{a-1} = \begin{cases}0 & A'\neq\emptyset\\ 1 & A'=\emptyset.\end{cases}$$

Thus (6.7) is equal to $\sum_A \prod_{(\eta,\gamma,i)\in A}\gamma/\eta$, proving (6.6) and (**II**).

6.2.3 $\underline{I}_{\mathrm{Sym}^d\,\mathbb{P}^r}(\mathbf{x}, Q, -z)$ satisfies condition (III)

Finally, we prove (**III**) using Tseng's orbifold quantum Riemann-Roch operator. From Proposition 3.4 of [34], Proposition 3.4, we compute that $J_\mu(\mathbf{x}, Q, -z)$ is a $\Lambda^T_{\mathrm{nov}}[[\mathbf{x}]]$-valued point of the *untwisted* Givental cone $\mathcal{L}_\mu$, where[1]

$$J_\mu(\mathbf{x}, Q, z) := z\sum_{\sigma}\sum_{\mathbf{k},\beta}\frac{1_\sigma Q^\beta \mathbf{x}^{\mathbf{k}} H_\sigma(\sigma,\mathbf{k})}{\beta! z^\beta \mathbf{k}! z^{\mathbf{k}}}.$$

Here σ runs over conjugacy classes in G_μ, and $\mathbf{t}(\mathbf{x}, Q, z) = Q\cdot 1 + \sum_\varpi x_\varpi 1_\varpi$. We introduce variables $Q_{\sigma,\eta}$ indexed by a multipartition σ and part η, and define

$$J^Q_\mu(\mathbf{x}, Q, -z) := -z\sum_{\sigma}\sum_{\mathbf{k},\beta,L}\frac{1_\sigma \mathbf{x}^{\mathbf{k}} H_\sigma(\sigma,\mathbf{k})}{\mathbf{k}!(-z)^{\mathbf{k}}}\left(\frac{|S_\sigma|}{|S_{\sigma,L}|}\prod_{\eta\in\sigma}\frac{Q^{L_\eta}_{\sigma,\eta}\eta^{L_\eta}}{(-z)^{L_\eta}L_\eta!}\right)$$

$$\underline{I}^Q_\mu := -z\sum_{\sigma,\beta,\mathbf{k},L}\frac{1_\sigma \mathbf{x}^{\mathbf{k}} H_\sigma(\sigma\prod_\varpi \varpi^{k_\varpi})}{\mathbf{k}!(-z)^{\mathbf{k}}}\left(\frac{|S_\sigma|}{|S_{\sigma,L}|}\prod_{\eta\in\sigma}\frac{Q^{L_\eta}_{\sigma,\eta}\eta^{L_\eta}}{(-z)^{L_\eta}L_\eta!\prod_{\gamma=1}^{L_\eta}\prod_{i=0}^{r}\left(\alpha_{i(\eta)}-\alpha_i-\frac{\gamma}{\eta}z\right)}\right).$$

Using combinatorics we may check that specializing $Q_{\sigma,\eta} = Q$ for all σ, η recovers $J_\mu(\mathbf{x}, Q, z)$ and $\underline{I}_\mu$. This is the analog of the change of variables after Equation 36 of [18].

[1] Note that the coefficients of $J_\mu(\mathbf{x}, Q, z)$ differ from the Gromov-Witten invariants of μ (as calculated in [34]) by a factor of $|G|$; this is because on an orbifold point, the Poincaré dual to the fundamental class 1 is $|G|\cdot 1$.

From here, (**III**) essentially follows from the proof in Section 7.3 of [18], despite the fact that $T_\mu \operatorname{Sym}^d \mathbb{P}^r$ is not a direct sum of line bundles. We give an outline.

As in [18], we work with a general multiplicative characteristic class $c_{\mathbf{s}}$. Denote by $\Delta_{\mathbf{s}}$ the orbifold quantum Riemann-Roch operator, which by [54] maps $\mathcal{L}_\mu$ to $\mathcal{L}_\mu^{T\operatorname{Sym}^d \mathbb{P}^r}$. Explicitly,

$$\Delta_{\mathbf{s}} := \bigoplus_{\sigma} \exp\left(\sum_{\substack{\eta\in\sigma \\ 0\le i\le r \\ i\neq i(\eta)}} \sum_{0\le \ell\le \eta-1} \sum_{m\ge 0} s_{m-1} R(\sigma,\ell) \frac{B_m(\ell/\eta)}{m!} z^{m-1} \right),$$

where B_m is the mth Bernoulli polynomial, and $R(\sigma,\ell)$ is the rank of the eigenbundle of $T_\mu \operatorname{Sym}^d \mathbb{P}^r$ on which elements of $\sigma \subseteq G_\mu$ act with eigenvalue $e^{2\pi i\ell/\eta}$. Note that the values

$$s_k = \begin{cases} -\log(\alpha_{i(\eta)} - \alpha_i) & k = 0 \\ (-1)^k \frac{(k-1)!}{(\alpha_{i(\eta)}-\alpha_i)^k} & k > 0 \end{cases}$$

recover the T-equivariant Euler class. Using the functional equation of the Bernoulli polynomials, we may check that $\underline{I}^Q_{\mu,\sigma} = \Delta_{\mathbf{s}}(J^{\mathrm{mod}}_\mu(\mathbf{x}, Q, -z))$, where

$$\begin{aligned} J^{\mathrm{mod}}_\mu(\mathbf{x}, Q, -z) :&= -z \sum_{\sigma} \sum_{\mathbf{k},\beta,L} \frac{1_\sigma \mathbf{x}^{\mathbf{k}} H_\sigma(\sigma \prod_{\varpi} \varpi^{k_\varpi})}{\mathbf{k}!(-z)^{\mathbf{k}}} \frac{|S_\sigma|}{|S_{\sigma,L}|} \prod_{\eta\in\sigma} \left(\frac{Q_{\sigma,\eta}^{L_\eta} \eta^{L_\eta}}{(-z)^{L_\eta} L_\eta!} \right) \\ &\exp\left(\sum_{i\neq i(\eta)} \sum_{\ell=1}^{\eta-1} \sum_{n,m\ge 0} s_{n+m-1} B_m(0) R(\sigma,\ell) \frac{\left(z\ell/\eta - z\left\lfloor \frac{L_\eta+\ell}{\eta}\right\rfloor\right)^n}{n!} \frac{z^{m-1}}{m!} \right). \end{aligned} \tag{6.8}$$

Analyzing the floor function, we have

$$\sum_{\ell=1}^{\eta-1} \frac{\left(z\ell/\eta - z\left\lfloor \frac{L_\eta+\ell}{\eta}\right\rfloor\right)^n}{n!} = \sum_{\ell=1}^{\eta-1} \frac{(z(-L_\eta+\ell)/\eta)^n}{n!}.$$

Now (6.8) is equal to $P(Q_{\sigma,\eta}\frac{\partial}{\partial Q_{\sigma,\eta}})(J^Q_\mu)$, for

$$P(a) = \exp\left(\sum_{i\neq i(\eta)} \sum_{\ell=1}^{\eta-1} \sum_{n,m\ge 0} s_{n+m-1} B_m(0) R(\sigma,\ell) \frac{(z(-a+\ell)/\eta)^n}{n!} \frac{z^{m-1}}{m!} \right).$$

The inductive argument at the end of the proof of Theorem 4.6 of [16] shows that (after specializing $Q_{\sigma,\eta} = Q$) the series $J^{\mathrm{mod}}_\mu(\mathbf{x}, Q, -z)$ is a $H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C}, \mathbb{Q})[[Q_{\sigma,\eta}, \mathbf{x}]]$-valued point of $\mathcal{L}_\mu$, and orbifold quantum Riemann-Roch then shows that $\underline{I}_\mu(\mathbf{x}, Q, -z)$ is a $H^*_{T,\mathrm{loc}}(\operatorname{Spec}\mathbb{C}, \mathbb{Q})[[Q_{\sigma,\eta}, \mathbf{x}]]$-valued point of $\mathcal{L}^{T\,\mathrm{Sym}^d\,\mathbb{P}^r}_\mu$.

6.3 $I_{\mathrm{Sym}^d\,\mathbb{P}^r}(t, \mathbf{x}, Q, z)$ is equal to the J-function of $\mathrm{Sym}^d\,\mathbb{P}^r$

The beginning terms of $I_{\mathrm{Sym}^d\,\mathbb{P}^r}(t, \mathbf{x}, Q, z)$ are

$$I_{\mathrm{Sym}^d\,\mathbb{P}^r}(t, \mathbf{x}, Q, z) = 1z + \sum_\sigma \sum_{i=0}^r t_i[H_{\sigma,i}] + \sum_{\varpi\in\mathrm{Part}(d)} x_\varpi 1_\varpi + O(z^{-1}).$$

On the other hand, by definition, there is a unique $\Lambda^T_{\mathrm{nov}}[[t, \mathbf{x}]]$-valued point of $\mathcal{L}_{\mathrm{Sym}^d\,\mathbb{P}^r}$ of this form, namely the *J-function*

$$J_{\mathrm{Sym}^d\,\mathbb{P}^r}(Q, \theta, z) = 1z + \theta + \sum_{\beta,n} \frac{Q^\beta}{n!} \left\langle \theta, \ldots, \theta, \frac{\gamma_\phi}{z - \overline{\psi}} \right\rangle^{\mathrm{Sym}^d\,\mathbb{P}^r, T}_{0,n+1,\beta} \gamma^\phi, \tag{6.9}$$

where $\theta = \sum_\sigma \sum_{i=0}^r t_i[H_{\sigma,i}] + \sum_{\varpi\in\mathrm{Part}(d)} x_\varpi 1_\varpi$. Thus:

Corollary 6.3.1. $I_{\mathrm{Sym}^d\,\mathbb{P}^r}(t, \mathbf{x}, Q, z) = J_{\mathrm{Sym}^d\,\mathbb{P}^r}(Q, \theta, z)$.

6.4 The origins of the I-function

6.4.1 Quasimaps

One of the main challenges of proving a mirror theorem such as Theorem 6.1.2 is that the series $I_{\mathrm{Sym}^d\,\mathbb{P}^r}(t, \mathbf{x}, Q, z)$ must be guessed in some way (or, in the original story, predicted by physicists).

Recently, there has been major progress on this front, due to work of Ciocan-Fontanine and Kim [13] on *quasimaps*. For $X = W /\!\!/ G$ a GIT quotient of an affine variety, they produce a collection of moduli spaces $\overline{\mathcal{Q}}^{\epsilon,\delta}_{0,n}(X, \beta)$ of (ϵ, δ)-*stable quasimaps*, such that when $\epsilon, \delta \gg 0$, we have $\overline{\mathcal{Q}}^{\epsilon,\delta}_{0,n}(X, \beta) \cong \overline{\mathcal{M}}_{0,n}(X, \beta)$. (When $\epsilon \gg 0$ they are spaces of weighted stable maps.)

Precisely, these spaces parametrize maps to $[W/G]$, with weighted markings, where points of C may map into the G-unstable locus of W. The parameters $\epsilon > 0$ and $0 < \delta \leq 1$ specify a stability condition which governs the number of marked points that may collide, and the maximum order of contact the curve may have with the unstable locus at a point.

The authors then define *quasimap invariants*, a generalization of Gromov-Witten invariants, via a perfect obstruction theory on $\overline{\mathcal{Q}}^{\epsilon,\delta}_{0,n}(X, \beta)$. As in Gromov-Witten theory, twisted invariants and equivariant invariants can also be defined.

6.4.2 Graph spaces and I-functions

By the stability condition on stable maps, the space $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is empty when $\beta = 0$ and $n < 3$. Indeed, in the definition of a point of the Givental cone, the sum is over $n, \beta \geq 0$ with $(n+1, \beta) \notin \{(1,0),(2,0)\}$. The first two terms $-1z$ and $\mathbf{t}(z)$ of such a function $\mathbf{f}$ may be understood as corrections, corresponding to these two "missing" cases (respectively).

This can be made precise. There is a moduli space, called a *graph moduli space* and denoted $\overline{\mathcal{M}}G_{0,n}(X,\beta)$, that parametrizes stable maps with a single parametrized component (on which the stability condition is not required). This space has a $\mathbb{C}^*$-action, and one may exhibit a Gromov-Witten invariant of X appearing in $\mathbf{f}$ as a localization contribution to a certain integral on $\overline{\mathcal{M}}G_{0,n}(X,\beta)$. The advantage of this point of view is that $\overline{\mathcal{M}}G_{0,n}(X,\beta)$ exists for any n and β. Applying the same computation when $\beta = 0$ and $n < 3$ yields a conventional way of defining "extra" Gromov-Witten invariants. These exactly give the terms $-1z$ and $\mathbf{t}(z)$ in $\mathbf{f}$. This technique was used in Givental's proof of a mirror theorem for quintic threefolds [28].

The same technique can be adapted to quasimaps, as follows. The moduli space $\overline{\mathcal{Q}}^{\epsilon,\delta}_{0,n}(X,\beta)$ is instead empty whenever

$$-2 + \delta n + \epsilon\beta \leq 0. \tag{6.10}$$

There is a *quasimap graph space* $\overline{\mathcal{Q}}G^{\epsilon,\delta}_{0,n}(X,\beta)$ that is defined for all n and β, even in the limit $\epsilon, \delta \to 0$. The same operation as above of defining conventional extra invariants works here. Note that in the limit $\epsilon, \delta \to 0$, condition (6.10) is *always* satisfied, i.e. *all* quasimap invariants are defined via this convention. The I-function $\mathbb{I}_X$ of X is a generating function analogous to $\mathbf{f}$ (with the simple value $\mathbf{t}(z) = \mathbf{t}(\{t_\phi\}, Q, z) = \sum_\phi t_\phi \gamma_\phi$), but defined using *quasimap* invariants in the limit $\epsilon, \delta \to 0$.

In many special cases, $\mathbb{I}_X$ may be computed explicitly. The authors conjecture that the I-function defined this way is on the Givental cone $\mathcal{L}_X$, and prove it in a special case:

Theorem 3.3 of [14]. *If X has a T-action with isolated fixed points and 1-dimensional orbits, then $\mathbb{I}_X$ is on the (equivariant) Givental cone $\mathcal{L}_X$.*

6.4.3 I-functions of orbifolds

In the case that X does not admit such a torus action, we may still hope that the I-function produced this way would lie on the Lagrangian cone, and that this could be proven by other means. In general this is a very promising method, but there is an obstruction when X is an orbifold.[2] To explain this

[2] An orbifold GIT quotient is the stack quotient $[W^{ss}/G]$ of the semistable locus by G. For example, $X = \mathrm{Sym}^d\,\mathbb{P}^r$ is a GIT quotient via the presentation $\mathbb{C}^{(r+1)d} /\!\!/ ((\mathbb{C}^*) \operatorname{wr} S_d)$.

obstruction, we need to be more explicit about how to express a quasimap invariant as a localization contribution on $\overline{Q}G^{\epsilon,\delta}_{0,n}(X,\beta)$.

The group $\mathbb{C}^*$ acts on $\overline{Q}G^{\epsilon,\delta}_{0,n}(X,\beta)$ by scaling on the parametrized component of each curve C. If a point $(f : C \to [W/G])$ is to be $\mathbb{C}^*$-fixed, then this component, identified with $\mathbb{P}^1$, must map to a single point of $X = [W^{ss}/G]$, except for 0 and ∞, which may map into the unstable locus $[W^{uns}/G]$. Similarly marked points and nodes must be at 0 or ∞. If there are nodes at 0 and ∞, then chopping C at 0 and ∞ yields two quasimaps to X, of degrees adding up to β.

Consider the case where there is a node at 0, but just the marked point b_{n+1} at ∞. This locus $F_{n,\beta}$ is an open and closed substack of the $\mathbb{C}^*$-fixed locus of $\overline{Q}G^{\epsilon,\delta}_{0,n+1}(X,\beta)$. It is isomorphic to $\overline{\mathcal{Q}}^{\epsilon,\delta}_{0,n+1}(X,\beta)$. The normal bundle $N_{F_{n,\beta}}$ parametrizes smoothings of the node 0, as well as infinitesimal movements of the special points at 0 and ∞ on $\mathbb{P}^1$. It has equivariant Euler class

$$e(N_{F_{n,\beta}}) = -z^2(-z-\psi),$$

where ψ is the cotangent class of the $(n+1)$st marked point under the identification with $\overline{\mathcal{Q}}^{\epsilon,\delta}_{0,n+1}(X,\beta)$, and z is the equivariant parameter. Thus (using the fact that ev_{n+1} is identified with itself under this isomorphism) the contribution of $F_{n,\beta}$ to an integral $\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi),\phi\rangle^{X,G}_{0,n+1,\beta}$ over the graph space is precisely the *non-graph* quasimap invariant $\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi),\frac{\phi}{-z^2(-z-\psi)}\rangle^{X}_{0,n+1,\beta}$. Up to the factor $-z^2$, this is the invariant appearing in $\mathbf{f}$.

The previous paragraph only makes sense when condition (6.10) does not hold, since in that case there can be no node at zero. When it does hold, however, the locus $F_{n,\beta}$ still makes sense; it parametrizes maps $\mathbb{P}^1 \to [W/G]$, where n marked points are at 0, one marked point b_{n+1} is at ∞, and where the point 0 maps into the unstable locus. This moduli space is now, at least in many cases, isomorphic to X via ev_{n+1}. Computing the virtual normal bundle $N^{\mathrm{vir}}_{F_{n,\beta}}$ is now more difficult, but by doing so one essentially obtains the I-function.

We can now see why this strategy fails when X is an orbifold. For one thing, above we had all n marked points of C concentrated at the point $0 \in \mathbb{P}^1$. But there is no known way to make sense of multiple orbifold points colliding in a family. Second, there are additional complications if one would like orbifold points to map into the unstable locus, which generally has an infinite stabilizer.

There is a modification presented in [12] for dealing with these problems. The authors simply restrict to the case $n = 0$. This gives an I-function with significantly less data than hoped (it is a Λ^T_{nov}-valued point of $\mathcal{L}_X$, rather than a $\Lambda^T_{\mathrm{nov}}[[x_1,\dots,x_n]]$-valued point), but the authors show that in the case where X is a toric orbifold, by carefully picking a GIT presentation all of the data may be recovered.

Let us see why this modification is not available in the case $X = \mathrm{Sym}^d\,\mathbb{P}^r$. When $n = 0$, the source curve C has a single orbifold point, at ∞. A quasimap $f : C \to \mathrm{Sym}^d\,\mathbb{P}^r$ consists of a principal S_d-bundle over C, and an S_d-equivariant quasimap to $(\mathbb{P}^r)^d$. However, there are no nontrivial S_d-bundles over a $\mathbb{P}^1$ with one orbifold point. Therefore in some sense, the I-function obtained via this modification will not see the fact that $\mathrm{Sym}^d \mathbb{P}^r$ is an orbifold at all! Precisely, the I-function will take values in the usual cohomology of $\mathrm{Sym}^d \mathbb{P}^r$, rather than the Chen-Ruan cohomology. In general, we only expect the recovery process mentioned above to work in rare cases, possibly only for toric orbifolds.

6.4.4 I-functions of orbifolds, attempt 2

We propose a method that in some simpler sense allows orbifold points to collide. We will show how this method gives rise to the I-function of Theorem 6.1.2, which gives strong evidence that the method is good.

Let $f : C \to X$ be a map from an orbifold curve into an orbifold X. By pulling back a cover of orbifold charts of X, we may cover C with open sets $\{U_i\}$ such that $f|_{U_i}$ is a map to a quotient $[M_i/G_i] \subseteq X$, i.e. $f|_{U_i}$ is given locally by a principal G_i-bundle $\tilde{U}_i \to U_i$ and a G_i-equivariant map $\tilde{U}_i \to M_i$, where G_i is finite. This still works if $f : C \to X$ is a family of maps over a base scheme S.

Suppose $S = \mathbb{C}^*$, and as $s \to 0$, two orbifold points of C approach each other. (It is easy to make rigorous sense of this, as in the case of usual moduli of weighted stable maps.) From above, we have the data of a principal G_i-bundle on C. Is it possible to find a flat limit of C over $\mathbb{C}$ such that the map and principal G_i-bundle extend? First, we deal with the question of how to extend C.

Example 6.4.1. Consider the family $[\{xy = tz^2\}/S_2] \subseteq [\mathbb{P}^2/S_2] \times \mathbb{C}^*$, where μ_r acts by permuting x and y. This is a family over $\mathbb{C}^*$ of orbifold curves, each of which has two orbifold points of order 2. When we fill it in the natural way, the limit is $[\{xy = 0\}/\mu_2]$, a *singular* Deligne-Mumford curve with coarse moduli space isomorphic to $\mathbb{P}^1$.

This construction generalizes as follows. We replace the group S_2 with the dihedral group D_{2n}, where a reflection acts by permuting x and y, and a rotation acts by scaling x and y by an nth root of unity. Then one checks that this is also a family of orbifold curves, each with two orbifold points of order 2. The limit is again singular with coarse moduli space $\mathbb{P}^1$, but it is not isomorphic to the previous one (e.g. it has isotropy group D_{2n} at the singular point). This shows that in general there are many possible limiting curves resulting from the collision of two orbifold points. (In fact, they may come in a positive-dimensional family, if the base S has higher dimension.) These particular singularities were studied by Ekedahl [23].

In any example like this, properness of BG_i allows one to extend the principal G_i-bundle to the filled-in family. However, because the behavior of degenerating orbifold curves is extremely complicated, we do not know how to define a quasimap moduli space where orbifold points may collide. Even if we could do so, we would not know how to compute with it.

Instead we propose the following solution. Again suppose we are given a family of orbifold stable maps over $S = \mathbb{C}^*$. There is a well-defined monodromy, a conjugacy class in G_i, around a marked section. In the limit, there is also a well-defined monodromy around the collection of colliding marked sections, given by taking a small loop that encloses those sections. Rather than taking the limiting curve as $s \to 0$, we will puncture C at the marked sections (which does not lose any data) and then fill in a limiting punctured curve. This curve has a natural principal G_i-bundle with monodromy around the collided marked points equal to the one just described. We may use this monodromy to fill in a curve and a map to X, but the resulting curve does **not** fit in a flat family with C. Therefore our moduli space will parametrize quasimaps from punctured curves, with specified monodromy data at the punctures.

Here is how we use this to compute, when $X = [M/G]$ is a global quotient by a finite group. The part of the fixed locus $F_{n,\beta}$ in $\overline{Q}G^{\epsilon,\delta}_{0,n+1}(X, \beta)$ parametrizes maps where all marked points have collided at 0. A point of $F_{n,\beta}$ now carries not only the information of the target point in X, but also the monodromy data of all n collided marked points. Thus it contains an open and closed substack for each possible assignment of monodromy data; these are indexed by factorizations of the identity in G. In particular, for $X = \operatorname{Sym}^d \mathbb{P}^r$, we have $G = S_d$. This is where the Hurwitz number appearing in (6.1) comes from.

The other interesting part of (6.1) is the product at the end, which comes from the normal bundle to $F_{n,\beta}$. For $(f : C \to \operatorname{Sym}^d \mathbb{P}^r) \in F_{n,\beta}$, the étale cover $C' \to C$ is simply a union of covers, fully ramified over 0 and ∞. The monodromy is given by the partition $\sigma = \operatorname{Mon}(n + 1)$. The degrees of the map $C' \to \mathbb{P}^r$ on each rational connected component give a $\mathbb{Z}_{\geq 0}$-labeling of σ with sum β. The deformations of the maps $C' \to \mathbb{P}^r$ are a product over these connected components, and the weights are computed using Kontsevich's argument. The result is the product in (6.1) (when done equivariantly).

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