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Intermediate arithmetic operations on ordinal numbers

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There are two well-known ways of doing arithmetic with ordinal numbers: the "ordinary" addition, multiplication, and exponentiation, which are defined by transfinite iteration; and the "natural" (or "Hessenberg") addition and multiplication (denoted $\oplus$ and $\otimes$), each satisfying its own set of algebraic laws. In 1909, Jacobsthal considered a third, intermediate way of multiplying ordinals (denoted $\times$), defined by transfinite iteration of natural addition, as well as the notion of exponentiation defined by transfinite iteration of his multiplication, which we denote $\alpha^{\times\beta}$. (Jacobsthal's multiplication was later rediscovered by Conway.) Jacobsthal showed these operations too obeyed algebraic laws. In this paper, we pick up where Jacobsthal left off by considering the notion of exponentiation obtained by transfinitely iterating natural multiplication instead; we shall denote this $\alpha^{\otimes\beta}$. We show that $\alpha^{\otimes(\beta\oplus\gamma)} = (\alpha^{\otimes\beta}) \otimes (\alpha^{\otimes\gamma})$ and that $\alpha^{\otimes(\beta\times\gamma)} = (\alpha^{\otimes\beta})^{\otimes\gamma}$; note the use of Jacobsthal's multiplication in the latter. We also demonstrate the impossibility of defining a "natural exponentiation" satisfying reasonable algebraic laws.

1 Introduction

In this paper, we introduce a new form of exponentiation of ordinal numbers, which we call *super-Jacobsthal exponentiation*, and study its properties. We show it satisfies two analogues of the usual laws of exponentiation. These laws relate super-Jacobsthal exponentiation to other previously studied operations on the ordinal numbers: natural addition, natural multiplication, and Jacobsthal's multiplication. We also show that there is no "natural exponentiation" analogous to natural addition and natural multiplication.

There are two well-known ways of doing arithmetic with ordinal numbers. Firstly, there are the "ordinary" addition, multiplication, and exponentiation. These are defined by starting with the successor operation S and transfinitely iterating; $\alpha + \beta$ is defined by applying to α the successor operation β-many times; $\alpha\beta$ is α added to itself β-many times; and α^β is α multiplied by itself β-many times. These also have order-theoretic definitions.

There are also infinitary versions of ordinary addition and ordinary multiplication, defined for families of operands with a well-ordered index set; using these, one can write

$$\alpha\beta = \sum_{i<\beta} \alpha; \qquad \alpha^\beta = \prod_{i<\beta} \alpha.$$

These can be defined either recursively or order-theoretically.

The ordinary operations obey some of the usual relations between arithmetic operations:

1. Associativity of addition: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
2. Left-distributivity of multiplication over addition: $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
3. Associativity of multiplication: $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
4. Exponentiation converts addition to multiplication: $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$.
5. Exponential of a product is iterated exponentiation: $\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$.

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Note that these operations are not commutative; e.g., $1 + \omega = \omega \neq \omega + 1$ and $2\omega = \omega \neq \omega 2$. Note further that distributivity does not work on the right; e.g.,

$$(1 + 1)\omega = \omega \neq \omega 2 = (1\omega) + (1\omega).$$

The infinitary versions of these operations also satisfy analogous laws, which we shall detail later.

Then there are the "natural" addition and multiplication, sometimes known as the Hessenberg operations [9, pp. 73–81], which we shall denote by $\alpha \oplus \beta$ and $\alpha \otimes \beta$, respectively. Natural addition and multiplication can be described as adding and multiplying ordinals as if they were "polynomials in ω"; cf. the next section for a more formal definition. These are the operations with which the ordinal numbers embed into the surreal numbers [5]. They also have order-theoretic definitions, due to Carruth [4]; cf. [6] for more on this.

The natural operations also have infinitary versions, but they are less well-behaved; cf. § 2.1.

Now, the operations in the ordinary family were formed by transfinite iteration; but we can transfinitely iterate the natural operations as well. Jacobsthal introduced a new sort of multiplication, which he denoted by $\alpha \times \beta$, by transfinitely iterating natural addition; we call it *Jacobsthal multiplication*. It is in a sense intermediate between ordinary multiplication and natural multiplication. In fact, one has the inequality

$$\alpha\beta \leq \alpha \times \beta \leq \alpha \otimes \beta$$

for all ordinals α and β. Jacobsthal then went on and defined a new form of exponentiation based on transfinitely iterating Jacobsthal multiplication. He denoted it by $\alpha^{\underline{\beta}}$, but we shall denote it by $\alpha^{\times\beta}$. One may consider infinitary Jacobsthal multiplication as well, so that

$$\alpha \times \beta = \bigoplus_{i<\beta} \alpha \quad \text{and} \quad \alpha^{\times\beta} = \bigtimes_{i<\beta} \alpha.$$

Jacobsthal's operations have been rediscovered several times. In the 1980s, Jacobsthal's multiplication was rediscovered by Conway and discussed by Gonshor and by Hickman [8, 10]; as such it has also been referred to as "Conway multiplication", though this name is used also of other operations. Both of Jacobsthal's operations were also later rediscovered by Abraham and Bonnet [1].

Just as we may transfinitely iterate natural addition, so may we transfinitely iterate natural multiplication. We call the resulting operation *super-Jacobsthal exponentiation*, and denote it $\alpha^{\otimes\beta}$. Another way of stating this, again, is that

$$\alpha^{\otimes\beta} = \bigotimes_{i<\beta} \alpha.$$

This type of exponentiation was previously considered briefly by de Jongh and Parikh [6], but has otherwise been mostly unexplored.

There are quite a few different notions of addition, multiplication, and exponentiation being considered here, so we shall summarize them with a table to help clarify the relations between them; cf. Table 1.

Note that there is no natural exponentiation to continue the "diagonal" family of natural operations. We shall prove this in § 4. (A version of this theorem was also proven independently by Asperó and Tsaprounis around the same time this paper was being written [2]. Their desiderata for natural exponentiation are slightly different, but the method of proof is essentially the same.)

One could continue any of these vertical families further, into higher hyper operations, as discussed in [3, pp. 66–79], but we shall not discuss that possibility here for several reasons, among them that higher hyper operations lack algebraic properties.

Our main interest here is in the algebraic laws sastisfied by these various operations, analogous to the algebraic laws satisfied by the ordinary operations discussed earlier. Such laws are already known for the natural and Jacobsthal operations; cf. § 2. The main result of this paper is that super-Jacobsthal exponentiation also satisfies such laws; cf. § 3 for the details.

Before we continue discussing these operations and their laws in more detail, let us conclude this section with Table 2 and Table 3, which list out all the relevant algebraic laws in a way that shows the relations between them. Table 2 includes the finitary versions, while Table 3 has the infinitary versions.

The new results of this paper, then, consist of the laws regarding super-Jacobsthal exponentiation shown in the tables, and the non-existence of natural exponentiation.

Natural operations ↘	S-based ↓	⊕-based ↓	⊗-based ↓	
Successor	Successor $S\alpha$			
Addition	Ordinary $\alpha + \beta$	Natural $\alpha \oplus \beta$		
Multiplication	Ordinary $\alpha\beta$	Jacobsthal $\alpha \times \beta$	Natural $\alpha \otimes \beta$	
Exponentiation	Ordinary α^{β}	Jacobsthal $\alpha^{\times\beta}$	Super-J. $\alpha^{\otimes\beta}$	None —

Table 1 Each operation is the transfinite iteration of the one above it, yielding three vertical families of operations, in addition to the diagonal family of natural operations. Each operation not on the diagonal, being a transfinite iteration, is continuous in β. In addition, each operation is pointwise less-than-or-equal-to those on its right; cf. § 5

Successor-based	⊕-based	⊗-based
$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$	$\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$	Not applicable
$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$	$\alpha \times (\beta \oplus \gamma) = (\alpha \times \beta) \oplus (\alpha \times \gamma)$	$\alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma)$
$\alpha(\beta\gamma) = (\alpha\beta)\gamma$	$\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$	$\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$
$\alpha^{\beta+\gamma} = \alpha^{\beta}\alpha^{\gamma}$	$\alpha^{\times(\beta+\gamma)} = (\alpha^{\times\beta}) \times (\alpha^{\times\gamma})$	$\alpha^{\otimes(\beta\oplus\gamma)} = (\alpha^{\otimes\beta}) \otimes (\alpha^{\otimes\gamma})$
$\alpha^{\beta\gamma} = (\alpha^{\beta})^{\gamma}$	$\alpha^{\times(\beta\gamma)} = (\alpha^{\times\beta})^{\times\gamma}$	$\alpha^{\otimes(\beta\times\gamma)} = (\alpha^{\otimes\beta})^{\otimes\gamma}$

Table 2 A table of the (finitary) algebraic laws described in this paper. Each law has been placed into one of the three vertical families in Table 1 based on the "main" operation involved, i.e., whichever one is in the bottom-most row in Table 1—note that many of these laws relate operations in different vertical families, and so would go in more than one column without this choice of convention. In addition, the operations ⊕ and ⊗ are both commutative, but this is not listed here as it does not fit into any of the patterns displayed here

Successor-based	⊕-based	⊗-based
$\sum_i \sum_j \alpha_{i,j} = \sum_{(j,i)} \alpha_{i,j}$	Analogue is false	Not applicable
$\alpha \sum_i \beta_i = \sum_i \alpha\beta_i$	$\alpha \times \bigoplus_i \beta_i = \bigoplus_i (\alpha \times \beta_i)$	Analogue is false
$\prod_i \prod_j \alpha_{i,j} = \prod_{(j,i)} \alpha_{i,j}$	$\bigtimes_i \bigtimes_j \alpha_{i,j} = \bigtimes_{(j,i)} \alpha_{i,j}$	Analogue is false
$\alpha^{\sum_i \beta_i} = \prod_i \alpha^{\beta_i}$	$\alpha^{\times(\sum_i \beta_i)} = \bigtimes_i \alpha^{\times\beta_i}$	$\alpha^{\otimes(\bigoplus_i \beta_i)} = \bigotimes_i \alpha^{\otimes\beta_i}$

Table 3 The infinitary analogue of Table 2, organized the same way. The associativity laws are stated in an abbreviated form here for simplicity. The four rows here correspond to the first four rows of Table 2; the fifth row has no extension to the infinitary setting assuming we use only addition, multiplication, and exponentiation

2 Operations over the ordinals

Natural addition and natural multiplication have several equivalent definitions; the simplest definition is in terms of Cantor normal form. Recall that each ordinal number α can be written uniquely as $\omega^{\alpha_0} a_0 + \ldots + \omega^{\alpha_r} a_r$, where $\alpha_0 > \ldots > \alpha_r$ are ordinals and the a_i are positive integers (note that r may be 0); this is known as its Cantor normal form. (We shall also sometimes, when it is helpful, write $\alpha = \omega^{\alpha_0} a_0 + \ldots + \omega^{\alpha_r} a_r + a$ where a is a whole number and $\alpha_r > 0$—that is to say, we shall sometimes consider the finite part of α separately from the rest of the Cantor normal form.) Then natural addition and multiplication can roughly be described as adding and multiplying Cantor normal forms as if these were "polynomials in ω". More formally:

Definition 2.1 We define the *natural sum* of two ordinals α and β, here denoted $\alpha \oplus \beta$, as follows. Take ordinals $\gamma_0 > \ldots > \gamma_r$ and whole numbers $a_0, \ldots, a_r$ and $b_0, \ldots, b_r$ so that we may write $\alpha = \omega^{\gamma_0} a_0 + \ldots + \omega^{\gamma_r} a_r$ and $\beta = \omega^{\gamma_0} b_0 + \ldots + \omega^{\gamma_r} b_r$. Then

$$\alpha \oplus \beta = \omega^{\gamma_0}(a_0 + b_0) + \ldots + \omega^{\gamma_r}(a_r + b_r).$$

Definition 2.2 The *natural product* of α and β, here denoted $\alpha \otimes \beta$, is defined as follows. Write $\alpha = \omega^{\alpha_0} a_0 + \ldots + \omega^{\alpha_r} a_r$ and write $\beta = \omega^{\beta_0} b_0 + \ldots + \omega^{\beta_s} b_s$ with $\alpha_0 > \ldots > \alpha_r$ and $\beta_0 > \ldots > \beta_s$ ordinals and the a_i and b_i positive integers. Then

$$\alpha \otimes \beta = \bigoplus_{\substack{0 \le i \le r \\ 0 \le j \le s}} \omega^{\alpha_i \oplus \beta_j} a_i b_j.$$

The natural operations also have recursive definitions, due to Conway [5, pp. 3–14]. Let us use the following notation:

Notation 2.3 If T is a set of ordinals, $\sup' T$ will denote the smallest ordinal greater than all elements of T. (This is equal to $\sup\{S\alpha : \alpha \in T\}$; it is also equal to $\sup T$ unless T has a greatest element, in which case it is $S(\sup T)$.)

Then these operations may be characterized by:

Theorem 2.4 (Conway) *We have*

1. *for ordinals* α *and* β, $\alpha \oplus \beta = \sup'(\{\alpha \oplus \beta' : \beta' < \beta\} \cup \{\alpha' \oplus \beta : \alpha' < \alpha\})$.
2. *for ordinals* α *and* β, $\alpha \otimes \beta = \min\{x : x \oplus (\alpha' \otimes \beta') > (\alpha \otimes \beta') \oplus (\alpha' \otimes \beta)$ *for all* $\alpha' < \alpha$ and $\beta' < \beta\}$.

As was mentioned earlier, the natural operations also have order-theoretic interpretations [4, 6].

The natural operations have some better algebraic properties than the ordinary operations—they are commutative, and have appropriate cancellation properties; as mentioned earlier, these are the operations with which the ordinals embed in the field of surreal numbers. We list out explicitly the algebraic laws analogous to those satisfied by the ordinary operations:

Lemma 2.5 *The natural operations satisfy*

1. *associativity of addition:* $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$,
2. *distributivity of multiplication over addition:* $\alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma)$,
3. *associativity of multiplication:* $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$.

As these operations are commutative, $\otimes$ in fact distributes over $\oplus$ on both sides, but this will not be relevant.

The natural operations do not behave as well as the ordinary operations with regard to continuity; not being defined by transfinite iteration, these operations are not continuous in either operand, whereas the ordinary operations are continuous in the right operand.

As was mentioned earlier, there is no natural exponentiation, and we shall prove this in § 4.

2.1 Infinitary ordinary and natural operations

One can, by taking limits, define infinitary versions of these operations as well. E.g., for the natural sum, one may define:

Definition 2.6 Given an indexed family of ordinals α_i indexed by the ordinals $i < \beta$ for some ordinal β, one can define the infinitary natural sum $\bigoplus_{i<\beta} \alpha_i$:

1. If $\beta = 0$, then $\bigoplus_{i<\beta} \alpha_i = 0$.

2. If $\beta = S\gamma$, then

$$\bigoplus_{i<\beta} \alpha_i = \left(\bigoplus_{i<\gamma} \alpha_i \right) \oplus \alpha_\gamma .$$

3. If β is a limit ordinal, then

$$\bigoplus_{i<\beta} \alpha_i = \lim_{\gamma<\beta} \bigoplus_{\alpha<\gamma} \alpha_i .$$

The definition for infinitary natural product is analogous; we shall not write it out explicitly.

Some care is warranted with the infinitary operations, though. E.g., as the natural operations are not continuous in the right operand, $1 \oplus (1 \oplus 1 \oplus \ldots)$ is not equal to $1 \oplus 1 \oplus \ldots$ (as $\omega + 1 \neq \omega$), and neither is $2 \otimes (2 \otimes 2 \otimes \ldots)$ equal to $2 \otimes 2 \otimes \ldots$ (as $\omega 2 \neq \omega$). Neither does natural multiplication distribute over infinitary natural addition; e.g., $2 \otimes (1 \oplus 1 \oplus \ldots)$ is not equal to $2 \oplus 2 \oplus \ldots$, as, again, $\omega 2 \neq \omega$.

This is in contrast to the ordinary operations, whose infinitary versions do satisfy laws extending those in the finitary case. One has:

1. Generalized associativity of addition:

$$\sum_{i<\gamma} \sum_{j<\beta_i} \alpha_{i,j} = \sum_{(j,i)\in\sum_{k<\gamma}\beta_k} \alpha_{i,j} .$$

2. Left-distributivity of multiplication over addition:

$$\alpha \sum_{i<\gamma} \beta_i = \sum_{i<\gamma} \alpha\beta_i .$$

3. Generalized associativity of multiplication:

$$\prod_{i<\gamma} \prod_{j<\beta_i} \alpha_{i,j} = \prod_{(j,i)\in\sum_{k<\gamma}\beta_k} \alpha_{i,j} .$$

4. Exponentiation converts addition to multiplication:

$$\alpha^{\sum_{i<\gamma}\beta_i} = \prod_{i<\gamma} \alpha^{\beta_i} .$$

Here, $\sum_{k<\gamma} \beta_k$ is the ordinary sum of the β_k, which is considered as a disjoint (tagged) union of the β_k; each element is an ordered pair (j, i) for some $i < \gamma$ and some $j < \beta_i$, and they are ordered lexicographically, first by i and then by j. This same convention will be used later as well.

It should also be pointed out that while the ordinary operations have a well-known order-theoretic meaning even when infinitary, the same cannot be said of the natural operations, whose order-theoretic definitions are not so easy to extend to the infinitary case. An order-theoretic characterization of the infinitary natural sum was recently discovered by Lipparini [13, 14], but none remains known for the infinitary natural product.

2.2 Jacobsthal's operations

In 1909, E. Jacobsthal introduced [11] the operation $\times$, which we refer to as "Jacobsthal multiplication", defined by transfinitely iterating natural addition; $\alpha \times \beta$ means α added to itself β-many times, using natural addition. More formally:

Definition 2.7 (Jacobsthal) We define the operation $\times$ by

1. For any α, $\alpha \times 0 := 0$.
2. For any α and β, $\alpha \times (S\beta) := (\alpha \times \beta) \oplus \alpha$.

3. If β is a limit ordinal, $\alpha \times \beta := \lim_{\gamma<\beta}(\alpha \times \gamma)$.

As noted earlier, this can be equivalently described as

$$\alpha \times \beta = \bigoplus_{i<\beta} \alpha.$$

This multiplication is not commutative; e.g., $2 \times \omega = \omega \neq \omega 2 = \omega \times 2$. We shall discuss other algebraic laws for it shortly.

Jacobsthal multiplication can be regarded as intermediate between ordinary and natural multiplication; like natural multiplication, it is related to natural addition, but like ordinary multiplication, it is based on transfinite iteration. Cf. also § 5.

Jacobsthal then went on to describe a notion of exponentiation obtained by transfinitely iterating $\times$, which we refer to as "Jacobsthal exponentiation". More formally:

Definition 2.8 (Jacobsthal) We define $\alpha^{\times\beta}$ by

1. For any α, $\alpha^{\times 0} := 1$.
2. For any α and β, $\alpha^{\times(S\beta)} := (\alpha^{\times\beta}) \times \alpha$.
3. If β is a limit ordinal, $\alpha^{\times\beta} := \lim_{\gamma<\beta}(\alpha^{\times\gamma})$.

Note that we can define infinitary Jacobsthal multiplication as well, analogous again to Definition 2.6 for the infinitary natural sum; we shall not write this out explicitly. With this definition, one then has, as noted earlier,

$$\alpha^{\times\beta} = \bigtimes_{i<\beta} \alpha.$$

Jacobsthal then proved [11] the algebraic law:

Theorem 2.9 (Jacobsthal) *For any ordinals α, β, and γ, one has*

$$\alpha \times (\beta \oplus \gamma) = (\alpha \times \beta) \oplus (\alpha \times \gamma).$$

That is to say, $\times$ left-distributes over $\oplus$.

This distributivity works only on the left and not on the right; e.g.,

$$(1 \oplus 1) \times \omega = \omega \neq \omega 2 = (1 \times \omega) \oplus (1 \times \omega).$$

Jacobsthal gave only a computational proof of Theorem 2.9, by computing the Cantor normal form of both sides and observing their equality. More specifically, he proved:

Theorem 2.10 (Jacobsthal) *Let α and β be ordinals. Write α in Cantor normal form as*

$$\alpha = \omega^{\alpha_0} a_0 + \ldots + \omega^{\alpha_r} a_r;$$

here $\alpha_0, \ldots, \alpha_r$ is a decreasing (possibly empty) sequence of ordinals and the a_i are positive integers. Write β in Cantor normal form as

$$\beta = \omega^{\beta_0} b_0 + \ldots + \omega^{\beta_s} b_s + b;$$

here $\beta_0, \ldots, \beta_s$ is a decreasing (possibly empty) sequences of nonzero ordinals, the b_i are positive integers, and b is a nonnegative integer. Then

$$\alpha \times \beta = \omega^{\alpha_0+\beta_0} b_0 + \ldots + \omega^{\alpha_0+\beta_s} b_s + \omega^{\alpha_0}(a_0 b) + \ldots + \omega^{\alpha_r}(a_r b).$$

In other words, if $\beta = \beta' + b$ where β' is either 0 or a limit ordinal and b is finite, then

$$\alpha \times \beta = \omega^{\alpha_0} \beta' + \alpha \times b.$$

With this in hand, Theorem 2.9 is straightforward, but as an explanation, it is not very satisfying. Here, we improve upon Jacobsthal's proof by presenting an inductive proof:

Inductive Proof of Theorem 2.9. We induct on β and γ. If $\beta = 0$ or $\gamma = 0$, the statement is obvious. If γ is a successor, say $\gamma = S\gamma'$, then we have

$$\alpha \times (\beta \oplus \gamma) = \alpha \times (\beta \oplus S\gamma') = \alpha \times S(\beta \oplus \gamma') = (\alpha \times (\beta \oplus \gamma')) \oplus \alpha$$
$$= (\alpha \times \beta) \oplus (\alpha \times \gamma') \oplus \alpha = (\alpha \times \beta) \oplus (\alpha \times \gamma),$$

as needed. If β is a successor, the proof is similar.

This leaves the case where β and γ are both limit ordinals. Note that in this case, $\beta \oplus \gamma$ is a limit ordinal as well, and that

$$\beta \oplus \gamma = \sup(\{\beta \oplus \gamma' : \gamma' < \gamma\} \cup \{\beta' \oplus \gamma : \beta' < \beta\}).$$

So

$$\begin{aligned} &\alpha \times (\beta \oplus \gamma) = \sup\{\alpha \times \delta : \delta < \beta \oplus \gamma\} = \\ &\sup(\{\alpha \times (\beta' \oplus \gamma) : \beta' < \beta\} \cup \{\alpha \times (\beta \oplus \gamma') : \gamma' < \gamma\}) = \\ &\qquad \sup(\{(\alpha \times \beta') \oplus (\alpha \times \gamma) : \beta' < \beta\} \cup \{(\alpha \times \beta) \oplus (\alpha \times \gamma') : \gamma' < \gamma\}). \end{aligned} \tag{1}$$

Since $\alpha \times \beta$, $\alpha \times \gamma$, and their natural sum are all limit ordinals as well, we have

$$(\alpha \times \beta) \oplus (\alpha \times \gamma) = \sup(\{\delta \oplus (\alpha \times \gamma) : \delta < \alpha \times \beta\} \cup \{(\alpha \times \beta) \oplus \varepsilon : \varepsilon < \alpha \times \beta\}). \tag{2}$$

So we want to show that these two sets we are taking the suprema of (in the final expressions in Equations (1) and (2)) are cofinal, and thus have equal suprema. The first of these is actually a subset of the second, so it suffices to check that it is cofinal in it. So if $\delta < \alpha \times \beta$, then $\delta \le \alpha \times \beta'$ for some $\beta' < \beta$, so $\delta \oplus (\alpha \times \gamma) \le (\alpha \times \beta') \oplus (\alpha \times \gamma)$; similarly with $\varepsilon < \alpha \times \gamma$.

So our two suprema are equal and $\alpha \times (\beta \oplus \gamma) = (\alpha \times \beta) \oplus (\alpha \times \gamma)$; this proves the theorem. □

Once one has Theorem 2.9 in hand, it is straightforward to prove by transfinite induction, as Jacobsthal did, that

Theorem 2.11 (Jacobsthal) *The following algebraic relations hold:*

1. *Jacobsthal multiplication is associative: For any α, β, and γ, one has*
$$\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma.$$
2. *Jacobsthal exponentiation converts ordinary addition to Jacobsthal multiplication: For any α, β, and γ, one has*
$$\alpha^{\times(\beta+\gamma)} = (\alpha^{\times\beta}) \times (\alpha^{\times\gamma}).$$
3. *The Jacobsthal exponential of an ordinary product is an iterated Jacobsthal exponentiation: For any α, β, and γ, one has*
$$\alpha^{\times(\beta\gamma)} = (\alpha^{\times\beta})^{\times\gamma}.$$

The same methods easily show infinitary versions of these.

Theorem 2.12 *The following algebraic relations hold:*

1. *Jacobsthal multiplication distributes over infinitary natural sum:*
$$\alpha \times \bigoplus_{i<\gamma} \beta_i = \bigoplus_{i<\gamma} (\alpha \times \beta_i).$$
2. *Infinitary Jacobsthal multiplication satisfies "generalized associativity":*
$$\bigtimes_{i<\gamma} \bigtimes_{j<\beta_i} \alpha_{i,j} = \bigtimes_{(j,i)\in\sum_{k<\gamma}\beta_k} \alpha_{i,j}$$

3. *Jacobsthal exponentiation converts infinitary addition to Jacobsthal multiplication:*

$$\alpha^{\times(\sum_{i<\gamma}\beta_i)} = \bigtimes_{i<\gamma} \alpha^{\times\beta_i}.$$

2.3 Jacobsthal's laws: discussion

We have just given an inductive proof of Theorem 2.9. However, one obvious question remains: Is there an order-theoretic proof? We can ask the same for Theorems 2.11 and 2.12 as well. Of course, to write an order-theoretic proof of any of these, one would first need an order-theoretic interpretation of Jacobsthal multiplication.

As mentioned earlier, however, an order-theoretic characterization of the infinitary natural sum was recently found by Lipparini [13, 14], which in particular yields an order-theoretic characterization of Jacobsthal multiplication. This charaterization does not make Theorem 2.9 or part (1) of Theorem 2.11 obvious, so there is still work to do there, but an answer may be close at hand. As for parts (2) and (3), no order-theoretic intepretation has yet been found for Jacobsthal exponentiation, or for infinitary Jacobsthal multiplication more generally.

There is an additional mystery to part (1) of Theorem 2.11. While the proof is a simple transfinite induction using Theorem 2.9, the statement itself still looks strange; why should the operation of $\times$ be associative? Typically, when we prove that an operation $*$ is associative, we are not just proving that $a*(b*c) = (a*b)*c$; rather, we usually do it by proving that $a*(b*c)$ and $(a*b)*c$ are both equal to some object $a*b*c$, and that indeed $a_1 * \ldots * a_r$ makes sense for any finite r—not just proving that this makes sense *because* $*$ happens to be associative, so that $a*b*c$ is may be written as a notational shortcut; but that $a*b*c$ makes sense as an object on its own, and that this relation is *why* $*$ must be associative. The same question applies, perhaps even more so, to part (2) of Theorem 2.12. (Note that the generalized associativity laws satisfied by ordinary sum and ordinary product have both been stated in this relation-between-arities form, because this is the simplest way to do so.)

Consider, e.g., multiplication of cardinal numbers; the simplest way to show associativity of the binary version is to first define it for any number of operands. One would define the product $\kappa\lambda\mu$ to be the cardinality of the Cartesian product $\kappa \times \lambda \times \mu$, a set of ordered triples, and then observe that $\kappa(\lambda\mu) = \kappa\lambda\mu = (\kappa\lambda)\mu$. Multiplication of cardinal numbers actually provides an especially clear illustration of this tendency, if one considers the infinitary version. Whereas a finitary product of cardinals, though it may be taken all at once as described, may also be broken down in terms of iterated binary products, an infinitary product of cardinals cannot be written as a limit of finitary products in the obvious fashion; it must be taken all at once. But with Jacobsthal multiplication—unlike, say, with ordinary multiplication of ordinals, where the infinitary product has a clear order-theoretic meaning—it's not clear what it would mean to take the product all at once, how one would define it other than as a limit of iterated binary products. Even though the infinitary version was stated in the form of relation between arities, for now those higher arities remain simply a notational convention. (Infinitary natural multiplication has a lesser version of the same problem, of course, since there is still no known interpretation of the infinitary natural product other than as a limit; but there at least finite products make sense taken all at once, without recourse to iteration.)

So we ask the questions:

Question 2.13 Can Theorem 2.9 be proven by giving an order-theoretic interpretation to both sides? Can the same be done for the various parts of Theorem 2.11 and Theorem 2.12?

Question 2.14 Can the associativity of Jacobsthal multiplication be proven by finding a natural way of interpreting $\alpha \times \beta \times \gamma$ without first inserting parentheses? Can the same be done for the infinitary version, finding a way of interpreting $\bigtimes_i \alpha_i$ other than as a limit?

To go in a different direction, rather than restricting surreal operations to the ordinals, or trying to define a natural exponentiation on the ordinals analogous to surreal exponentiation, one could also attempt to extend the ordinary ordinal operations, or these intermediate ones, to the surreal numbers. This was accomplished for ordinary addition by Conway [5, Chapter 15]; indeed, he extended it to all games, not just numbers. For ordinary multiplication, there is a definition of Norton which was proven by Keddie [12] to work for surreal numbers written in a particular form, namely, those written with no reversible options; cf. his paper for more. It remains to be seen whether this can be done for Jacobsthal multiplication, or for any of the exponentiation operations considered here; Keddie [12] gives reasons why this may be difficult for exponentiation.

3 Super-Jacobsthal exponentiation

Having discussed Jacobsthal's operations, there is still one spot missing from Table 1: The transfinite iteration of natural multiplication, or "super-Jacobsthal exponentiation", as we call it here. (Rather, it is the one spot still missing that actually exists.) As mentioned earlier, it was considered briefly by de Jongh and Parikh [6], but has otherwise remained mostly unexplored.

Definition 3.1 We define $\alpha^{\otimes\beta}$ by

1. For any α, $\alpha^{\otimes 0} := 1$.
2. For any α and β, $\alpha^{\otimes(S\beta)} := (\alpha^{\otimes\beta}) \otimes \alpha$.
3. If β is a limit ordinal, $\alpha^{\otimes\beta} := \lim_{\gamma<\beta}(\alpha^{\otimes\gamma})$.

An equivalent way of stating this, as mentioned earlier, is that

$$\alpha^{\otimes\beta} = \bigotimes_{i<\beta} \alpha.$$

Before we continue, it is worth noting that all the notions of multiplication and exponentiation considered here are in fact different. An example is provided by considering $(\omega+2)(\omega+2)$, or $(\omega+2)^2$, since one has the equations

$$(\omega+2)^2 = \omega^2 + \omega 2 + 2,$$
$$(\omega+2)^{\times 2} = \omega^2 + \omega 2 + 4,$$
$$(\omega+2)^{\otimes 2} = \omega^2 + \omega 4 + 4.$$

With Definition 3.1 in hand, we can now state:

Theorem 3.2 *For any ordinals α, β, and γ, one has*

$$\alpha^{\otimes(\beta\oplus\gamma)} = (\alpha^{\otimes\beta}) \otimes (\alpha^{\otimes\gamma}).$$

That is to say, super-Jacobsthal exponentiation converts natural addition to natural multiplication.

Before we prove this theorem, let us make some further notes. Once it is proven, it will be straightforward to prove by transfinite induction that

Theorem 3.3 *For any ordinals α, β, and γ, one has*

$$\alpha^{\otimes(\beta\times\gamma)} = (\alpha^{\otimes\beta})^{\otimes\gamma}.$$

That is to say, the super-Jacobsthal exponential of a Jacobsthal product is an iterated super-Jacobsthal exponential.

More generally, given ordinals α and γ and a family of ordinals β_i indexed by γ, one has

$$\alpha^{\otimes(\bigoplus_{i<\gamma}\beta_i)} = \bigotimes_{i<\gamma} \alpha^{\otimes\beta_i}.$$

Once this is proven, it will complete Tables 2 and 3.

Note the appearance of Jacobsthal multiplication—not ordinary or natural multiplication—on the left-hand side of the first equation. This occurs because Theorem 3.3 comes from transfinitely iterating Theorem 3.2, and when one transfinitely iterates natural addition, one gets Jacobsthal multiplication.

Now we prove Theorem 3.2. This will require a tiny bit more setup. First, some notation and two lemmas:

Notation 3.4 For an ordinal α which is either 0 or a limit ordinal, $\omega^{-1}\alpha$ will denote the unique ordinal β such that $\alpha = \omega\beta$.

Notation 3.5 For an ordinal $\alpha > 0$, $\deg\alpha$ will denote the largest exponent appearing in the Cantor normal form of α.

Lemma 3.6 *Suppose $a > 1$ is finite and and let β be an ordinal. Write $\beta = \beta' + b$, where β' is 0 or a limit ordinal and b is finite. Then*

$$a^{\otimes\beta} = \omega^{\omega^{-1}\beta'} a^b.$$

Proof. We induct on β. If $\beta = 0$, then both sides are equal to 1. If β is a successor ordinal, say $\beta = S\gamma$, then by the inductive hypothesis,

$$a^{\otimes\gamma} = \omega^{\omega^{-1}\gamma'} a^c,$$

where we write $\gamma = \gamma' + c$ analogously to $\beta = \beta' + b$. As $\beta = S\gamma$, we have $\beta' = \gamma'$ and $b = c + 1$. Thus

$$a^{\otimes\beta} = a^{\otimes\gamma} \otimes a = (\omega^{\omega^{-1}\gamma'} a^c) \otimes a = \omega^{\omega^{-1}\beta'} a^b.$$

If β is a limit ordinal, we have two further cases, depending on whether or not β is of the form $\omega^2\gamma$ for some ordinal γ. If not, then β is of the form $\gamma' + \omega$, where γ' is either 0 or a limit ordinal. This means that β is the limit of $\gamma' + c$ for finite c. So then by the inductive hypothesis,

$$a^{\otimes\beta} = \lim_{c<\omega}(\omega^{\omega^{-1}\gamma'} a^c) = \omega^{S(\omega^{-1}\gamma')} = \omega^{\omega^{-1}\beta'},$$

as required.

If so, then we once again consider $\deg a^{\otimes\beta}$. Since β is of the form $\omega^2\gamma$, β is the limit of all ordinals less than it of the form $\omega\gamma$, i.e., it is the limit of all limit ordinals less than it. And for $\gamma < \beta$ a limit ordinal, by the inductive hypothesis, $\deg a^{\gamma} = \omega^{-1}\gamma$. So again applying the fact that the deg function is increasing, we have that $\deg a^{\otimes\beta} \geq \omega^{-1}\beta$, i.e., that $a^{\otimes\beta} \geq \omega^{\omega^{-1}\beta}$. (Here we also use the continuity of "division by ω", which follows from the continuity of left-multiplication by ω.) Conversely, for $\gamma < \beta$ with γ a limit ordinal, one has $\omega^{-1}\gamma < \omega^{-1}\beta$, and so $a^{\otimes\gamma} < \omega^{\omega^{-1}\beta}$; thus one has $a^{\otimes\beta} \leq \omega^{\omega^{-1}\beta}$. So we conclude, as needed, that $a^{\otimes\beta} = \omega^{\omega^{-1}\beta}$. This proves the lemma. □

Notation 3.7 For ordinals α and β, $\alpha \ominus \beta$ will denote the smallest γ such that $\beta \oplus \gamma \geq \alpha$. For convenience, we shall also define

$$f_{\alpha,\beta}(\alpha', \beta') = ((\alpha \otimes \beta') \oplus (\alpha' \otimes \beta)) \ominus (\alpha' \otimes \beta').$$

Note that with this definition, we can rewrite Conway's definition of $\alpha \otimes \beta$ as

$$\alpha \otimes \beta = \sup{}'\{f_{\alpha,\beta}(\alpha', \beta') : \alpha' < \alpha, \beta' < \beta\}.$$

Lemma 3.8 *For fixed α and β, $f_{\alpha,\beta}(\alpha', \beta')$ is increasing in α' and β'.*

Proof. Observe that $f_{\alpha,\beta}(\alpha', \beta')$ is the smallest ordinal greater than the surreal number $\alpha'\beta + \alpha\beta' - \alpha'\beta'$ (where these operations are performed in the surreal numbers, and are therefore natural operations on the corresponding ordinals). This expression is increasing in α' and β', since it can be written as $\alpha\beta - (\alpha - \alpha')(\beta - \beta')$. Therefore so is $f_{\alpha,\beta}(\alpha', \beta')$, the smallest ordinal greater than it. □

Now, the proof:

Proof of Theorem 3.2. We split this into several cases depending on the value of α. If $\alpha \in \{0, 1\}$ the theorem is obvious.

Now we have the case where $\alpha > 1$ is finite; in this case, we shall use Lemma 3.6 to give a computational proof. Let us rename α to a to make it clear that it is finite. Let $\beta = \beta' + b$ and $\gamma = \gamma' + c$ where β' and γ' are limit ordinals or 0, and b and c are finite.

So observe first that

$$\omega^{-1}(\beta' \oplus \gamma') = \omega^{-1}\beta' \oplus \omega^{-1}\gamma'.$$

This can be seen as, if $\beta' = \omega\beta''$ and $\gamma' = \omega\gamma''$, then

$$\omega(\beta'' \oplus \gamma'') = \omega\beta'' \oplus \omega\gamma'',$$

which can be seen by comparing Cantor normal forms. (This can also be seen by noting that for any ordinal δ, $\omega\delta = \omega \times \delta$, since if ε is a limit ordinal then $\varepsilon \oplus \omega = \varepsilon + \omega$, and by induction this quantity will always be a limit ordinal.)

Now, $\beta \oplus \gamma$ can be written as $(\beta' \oplus \gamma') + (b + c)$; here, $\beta' \oplus \gamma'$ is either 0 or a limit ordinal, and $b + c$ is finite. Thus,

$$a^{\otimes(\beta\oplus\gamma)} = \omega^{\omega^{-1}(\beta'\oplus\gamma')} a^{b+c} = \omega^{(\omega^{-1}\beta')\oplus(\omega^{-1}\gamma')} a^b a^c = (\omega^{\omega^{-1}\beta'} a^b) \otimes (\omega^{\omega^{-1}\gamma'} a^c) = a^{\otimes\beta} \otimes a^{\otimes\gamma},$$

as required.

This leaves the case where α is infinite. In this case we give an inductive proof, inducting on β and γ. If $\beta = 0$ or $\gamma = 0$ the theorem is obvious. If γ is a successor ordinal, say $\gamma = S\gamma'$, then

$$\alpha^{\otimes(\beta\oplus\gamma)} = \alpha^{\otimes(\beta\oplus S\gamma')} = \alpha^{\otimes S(\beta\oplus\gamma')} = \alpha^{\otimes(\beta\oplus\gamma')} \otimes \alpha = \alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma'} \otimes \alpha = \alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma},$$

as needed. If β is a successor, the proof is similar.

This leaves the case where β and γ are both limit ordinals. As before, not only are β and γ limit ordinals but so is $\beta \oplus \gamma$. So

$$\alpha^{\otimes(\beta\oplus\gamma)} = \sup\{\alpha^{\otimes\delta} : \delta < \beta \oplus \gamma\} = \sup(\{\alpha^{\otimes(\beta'\oplus\gamma)} : \beta' < \beta\} \cup \{\alpha^{\otimes(\beta\oplus\gamma')} : \gamma' < \gamma\}) \tag{3}$$

On the other hand,

$$\begin{aligned}
\alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma} &= \sup\left\{ f_{\alpha^{\otimes\beta},\alpha^{\otimes\gamma}}(\delta, \varepsilon) : \delta < \alpha^{\otimes\beta}, \varepsilon < \alpha^{\otimes\gamma} \right\} \\
&= \sup\left\{ f_{\alpha^{\otimes\beta},\alpha^{\otimes\gamma}}(\alpha^{\otimes\beta'}, \alpha^{\otimes\gamma'}) : \beta' < \beta, \gamma < \gamma' \right\} \\
&= \sup\left\{ ((\alpha^{\otimes\beta'} \otimes \alpha^{\otimes\gamma}) \oplus (\alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma'})) \ominus (\alpha^{\otimes\beta'} \otimes \alpha^{\otimes\gamma'}) : \beta' < \beta, \gamma' < \gamma \right\} \\
&= \sup\left\{ (\alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes(\beta\oplus\gamma')}) \ominus \alpha^{\otimes(\beta'\oplus\gamma')} : \beta' < \beta, \gamma' < \gamma \right\}.
\end{aligned} \tag{4}$$

Note that here we have used not only the inductive hypothesis, but have also used Lemma 3.8 and the fact that $\alpha^{\otimes\gamma}$, $\alpha^{\otimes\beta}$, and their natural product are all limit ordinals.

So now once again we must show that the two sets we are taking the suprema of in the final expressions of Equations (3) and (4) are cofinal with each other. Let us call these sets S and T, respectively.

So let us take an element of S; say it is $\alpha^{\otimes(\beta'\oplus\gamma)}$ for $\beta' < \beta$. We want to show it is bounded above by some element of T. (If instead it is of the form $\alpha^{\otimes(\beta\oplus\gamma')}$ for $\gamma' < \gamma$, the proof is similar.) But certainly, choosing $\gamma' = 0$,

$$\alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes\beta'} < \alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes\beta}$$

and so

$$\alpha^{\otimes(\beta'\oplus\gamma)} < (\alpha^{\otimes(\beta'\oplus\gamma)} \oplus \alpha^{\otimes\beta}) \ominus \alpha^{\otimes\beta'}.$$

Conversely, say we take an element δ of T. Since we assumed α infinite, and in general we have $\deg(\alpha \otimes \beta) = (\deg \alpha) \oplus (\deg \beta)$, it follows that the sequence $\deg \alpha^{\otimes\beta}$ is strictly increasing in β. So here, we have an element δ of T given by $(\alpha^{\otimes(\beta\oplus\gamma')} \oplus \alpha^{\otimes(\beta'\oplus\gamma)}) \ominus \alpha^{\otimes(\beta'\oplus\gamma')}$ for some $\beta' < \beta$ and $\gamma' < \gamma$, and we want to determine its degree. Now, in general, if we have ordinals α and β, then $\deg(\alpha \oplus \beta) = \max\{\deg \alpha, \deg \beta\}$, and so it follows that if $\deg \alpha > \deg \beta$ then $\deg(\alpha \ominus \beta) = \deg \alpha$. So here it follows that

$$\deg \delta = \max\left\{ \deg \alpha^{\otimes(\beta'\oplus\gamma)}, \deg \alpha^{\otimes(\beta\oplus\gamma')} \right\}.$$

But this means we can find an element of S with degree at least $\deg \delta$; and since β and γ are limit ordinals, we can find an element with degree even larger than $\deg \delta$, which in particular means that δ is less than some element of S.

Therefore S and T are cofinal and so have the same supremum. This completes the proof. □

As mentioned above, this then implies Theorem 3.3:

Proof of Theorem 3.3. We prove the more general version by induction on γ. If $\gamma = 0$, then

$$\alpha^{\otimes(\bigoplus_{i<0}\beta_i)} = \alpha^{\otimes 0} = 1 = \bigotimes_{i<0} \alpha^{\otimes\beta_i},$$

as needed.

If γ is a successor ordinal, say $\gamma = S\delta$, then

$$\alpha^{\otimes(\bigoplus_{i<S\delta}\beta_i)} = \alpha^{\otimes((\bigoplus_{i<\delta}\beta_i)\oplus\beta_\delta)} = \alpha^{\otimes(\bigoplus_{i<\delta}\beta_i)} \otimes \alpha^{\otimes\beta_\delta} = \bigotimes_{i<\delta}(\alpha^{\otimes\beta_i}) \otimes \alpha^{\otimes\beta_\delta} = \bigotimes_{i<S\delta}\alpha^{\otimes\beta_i},$$

again as needed, where we have applied both Theorem 3.2 and the inductive hypothesis.

Finally, if γ is a limit ordinal, so $\gamma = \lim_{\delta<\gamma}\delta$, then

$$\alpha^{\otimes(\bigoplus_{i<\gamma}\beta_i)} = \alpha^{\otimes(\lim_{\delta<\gamma}\bigoplus_{i<\delta}\beta_i)} = \lim_{\delta<\gamma}\alpha^{\otimes(\bigoplus_{i<\delta}\beta_i)} = \lim_{\delta<\gamma}\bigotimes_{i<\delta}\alpha^{\otimes\beta_i} = \bigotimes_{i<\gamma}\alpha^{\otimes\beta_i},$$

where here we have used both the inductive hypothesis and the fact that $\alpha^{\otimes\beta}$ is continuous in β (a fact which follows immediately from the definition).

The restricted version then follows by letting $\beta_i = \beta$ for all i. □

Thus we see that super-Jacobsthal exponentiation admits algebraic laws similar to those followed by ordinary exponentiation and Jacobsthal exponentiation, thus completing Table 2.

3.1 Super-Jacobsthal exponentiation: discussion

The theorems above raise some questions, analogous to those discussed in § 2.3. Specifically:

Question 3.9 Can Theorem 3.2 be proven by giving an order-theoretic interpretation to both sides? Can the same be done for Theorem 3.3?

Of course, proving the theorem in this way would require first finding an order-theoretic interpretation for super-Jacobsthal exponentiation; none is currently known.

Even if one cannot do that, there is still the question of improving on the proof of Theorem 3.2 given here by giving a more unified proof. The proof given here requires separating out the case where the base α is finite and handling that case “computationally”. A unified proof, if one could be found, would be preferable.

4 Natural exponentiation

In this section we discuss the question of “natural exponentiation” and show that there is no such thing, that Table 1 is complete as-is. Table 1 has several vertical families of operations, defined by transfinite iteration. This raises the question: Can we continue further the diagonal family in Table 1, the sequence of natural operations, and get a natural exponentiation?

Let us denote such an operation by $e(\alpha, \beta)$, where α is the base and β is the exponent. In this section, we shall show that such an operation cannot exist, unless one is willing to abandon basic properties it ought to possess. Now, one could produce a whole list of conditions that such an operation might be expected to satisfy. E.g., one might desire:

1. $e(\alpha, 0) = 1$.
2. $e(\alpha, 1) = \alpha$.
3. $e(0, \alpha) = 0$ for $\alpha > 0$.
4. $e(1, \alpha) = 1$.
5. For $\alpha > 1$, $e(\alpha, \beta)$ is strictly increasing in β.
6. For $\beta > 0$, $e(\alpha, \beta)$ is strictly increasing in α.
7. $e(\alpha, \beta \oplus \gamma) = e(\alpha, \beta) \otimes e(\alpha, \gamma)$.
8. $e(\alpha, \beta \otimes \gamma) = e(e(\alpha, \beta), \gamma)$.
9. $e(\alpha \otimes \beta, \gamma) = e(\alpha, \gamma) \otimes e(\beta, \gamma)$.
10. $e(2, \alpha) > \alpha$.

But even only a small number of these is enough to cause a contradiction. In this section we prove:

Theorem 4.1 *There is no natural exponentiation $e(\alpha, \beta)$ on the ordinals satisfying the following conditions:*

1. $e(\alpha, 1) = \alpha$.
2. *For* $\alpha > 0$, $e(\alpha, \beta)$ *is weakly increasing in* β.
3. $e(\alpha, \beta)$ *is weakly increasing in* α.
4. $e(\alpha, \beta \oplus \gamma) = e(\alpha, \beta) \otimes e(\alpha, \gamma)$.
5. $e(\alpha, \beta \otimes \gamma) = e(e(\alpha, \beta), \gamma)$.

The same holds if hypothesis (5) is replaced with the following hypothesis (5'): $e(\alpha \otimes \beta, \gamma) = e(\alpha, \gamma) \otimes e(\alpha, \gamma)$.

Remark 4.2 The version of this theorem where hypothesis (5') is used was also proven independently, in a slightly stronger form, by Asperó and Tsaprounis [2], using essentially the same means.

Before we go on and prove this, let us make a note about one way that one could attempt to define $e(\alpha, \beta)$, even though we know it will not work. Since addition and multiplication in the surreals agree with natural addition and natural multiplication on the ordinals, one might attempt to define a natural exponentiation based on the theory of the surreal exponential (developed by Gonshor [7, pp. 143–190]). One could define $e(\alpha, \beta) = \exp(\beta \log \alpha)$ for $\alpha > 0$, and then define $e(0, 0) = 1$ and $e(0, \beta) = 0$ for $\beta > 0$. And indeed, the operation on the surreals defined this way will satisfy all of the desiderata in the long list above, so long as all terms involved are defined. But there is one fatal problem: the ordinals are not closed under this operation. E.g., it turns out that, using the usual notation for surreal numbers, one has

$$\exp(\omega \log \omega) = \omega^{\omega^{1+1/\omega}},$$

which is not an ordinal. One could attempt to remedy this by rounding up to the next ordinal, but unsurprisingly the resulting operation is lacking in algebraic laws.

Now, the proof:

Proof of Theorem 4.1. Suppose we had such an operation $e(\alpha, \beta)$. Note that hypotheses (1) and (4) together mean that if k is finite and positive, then $e(\alpha, k) = \alpha^{\otimes k}$, and in particular, if n is also finite, then $e(n, k) = n^k$. By hypothesis (2), this means that for $n \geq 2$ we have $e(n, \omega) \geq \omega$. Let us define $\delta = \deg \deg e(2, \omega)$; since $e(2, \omega)$ is infinite, this is well-defined.

Observe also that by hypothesis (5), we have for n and k as above,

$$e(n^k, \alpha) = e(e(n, k), \alpha) = e(n, k \otimes \alpha) = e(n, \alpha \otimes k) = e(e(n, \alpha), k) = e(n, \alpha)^{\otimes k}.$$

(If we had used instead the alternate hypothesis (5'), this too would prove that $e(n^k, \alpha) = e(n, \alpha)^{\otimes k}$.)

Given any finite $n \geq 2$, choose some k such that $n \leq 2^k$; then by the above and hypothesis (3),

$$e(2, \omega) \leq e(n, \omega) \leq e(2, \omega)^{\otimes k},$$

and so

$$\deg e(2, \omega) \leq \deg e(n, \omega) \leq (\deg e(2, \omega)) \otimes k,$$

and so

$$\deg \deg e(2, \omega) \leq \deg \deg e(n, \omega) \leq \deg \deg e(2, \omega),$$

i.e., $\deg \deg e(n, \omega) = \delta$.

Thus we may define a function $f : \mathbb{N} \to \mathbb{N}$ by defining $f(n)$ to be the coefficient of ω^δ in the Cantor normal form of $\deg e(n, \omega)$. Then since $e(n^k, \omega) = e(n, \omega)^{\otimes k}$, we have $f(n^k) = kf(n)$. And by the above and hypothesis (3) we have that f is weakly increasing, since $\deg e(n, \omega)$ is weakly increasing and no term of size $\omega^{S\delta}$ or higher ever appears in any $\deg e(n, \omega)$. Finally, we have that $f(2) \geq 1$.

But no such function can exist; given natural numbers n and m, it follows from the above that

$$\lfloor \log_m n \rfloor f(m) \leq f(n) \leq \lceil \log_m n \rceil f(m)$$

or in other words that

$$\left\lfloor \frac{\log n}{\log m} \right\rfloor \le \frac{f(n)}{f(m)} \le \left\lceil \frac{\log n}{\log m} \right\rceil.$$

If one takes the above and substitutes in n^k for n, one obtains

$$\left\lfloor k\frac{\log n}{\log m} \right\rfloor \le k\frac{f(n)}{f(m)} \le \left\lceil k\frac{\log n}{\log m} \right\rceil.$$

But in particular, this means that

$$k\frac{\log n}{\log m} - 1 \le k\frac{f(n)}{f(m)} \le k\frac{\log n}{\log m} + 1,$$

or in other words, that

$$\frac{\log n}{\log m} - \frac{1}{k} \le \frac{f(n)}{f(m)} \le \frac{\log n}{\log m} + \frac{1}{k};$$

since this holds for any choice of k, we conclude that

$$\frac{f(n)}{f(m)} = \frac{\log n}{\log m}.$$

But the right hand side may be chosen to be irrational, e.g., if $m = 2$ and $n = 3$; thus, the function f cannot exist, and thus neither can our natural exponentiation e. □

Remark 4.3 Note that the only use of hypotheses (1) and (4) was to show that for k a positive integer, $e(\alpha, k) = \alpha^{\otimes k}$, so strictly speaking the the theorem could be stated with (1) and (4) replaced by this single hypothesis.

5 Comparison between the operations

In Table 1 it was asserted that each operation appearing in the table is pointwise less-than-or-equal-to those appearing to the right of it in the table. In this section we justify that assertion. Let us state this formally:

Proposition 5.1 *For any ordinals α and β, one has:*

1. $\alpha + \beta \le \alpha \oplus \beta$.
2. $\alpha\beta \le \alpha \times \beta \le \alpha \otimes \beta$.
3. $\alpha^\beta \le \alpha^{\times\beta} \le \alpha^{\otimes\beta}$.

The inequalities $\alpha + \beta \le \alpha \oplus \beta$ and $\alpha\beta \le \alpha \otimes \beta$ are well known; the inequalities $\alpha\beta \le \alpha \times \beta$ and $\alpha^\beta \le \alpha^{\times\beta}$ are due to Jacobsthal [11]. We shall give proofs of all of the above nonetheless.

Proof. First we prove that $\alpha + \beta \le \alpha \oplus \beta$, by induction on β. If $\beta = 0$, both sums are equal to α. If $\beta = S\gamma$, then by the inductive hypothesis,

$$\alpha + \beta = S(\alpha + \gamma) \le S(\alpha \oplus \gamma) = \alpha \oplus \beta.$$

Finally, if β is a limit ordinal, then since $\alpha \oplus \beta$ is increasing in β, we have that

$$\alpha \oplus \beta \ge \sup_{\gamma<\beta}(\alpha \oplus \gamma) \ge \sup_{\gamma<\beta}(\alpha + \gamma) = \alpha + \beta.$$

So $\alpha + \beta \le \alpha \oplus \beta$. It then immediately follows from transfinite induction and the definitions of each that $\alpha\beta \le \alpha \times \beta$, and $\alpha^\beta \le \alpha^{\otimes\beta}$.

Next we prove that $\alpha \times \beta \le \alpha \otimes \beta$, again by induction on β. If $\beta = 0$, both products are equal to 0. If $\beta = S\gamma$, then by the inductive hypothesis,

$$\alpha \times \beta = (\alpha \times \gamma) \oplus \alpha \le (\alpha \otimes \gamma) \oplus \alpha = \alpha \otimes \beta.$$

Finally, if β is a limit ordinal, then since $\alpha \otimes \beta$ is (possibly weakly) increasing in β, we have that

$$\alpha \otimes \beta \geq \sup_{\gamma<\beta}(\alpha \otimes \gamma) \geq \sup_{\gamma<\beta}(\alpha \times \gamma) = \alpha \times \beta.$$

So $\alpha \times \beta \leq \alpha \otimes \beta$. It then immediately follows from transfinite induction and the definitions of each that $\alpha^{\times\beta} \leq \alpha^{\otimes\beta}$. This completes the proof. □

Of course, this is not the only possible proof. E.g., all the above inequalities could also be proven by comparing Cantor normal forms. Perhaps more meaningfully, the inequalities $\alpha + \beta \leq \alpha \oplus \beta$ and $\alpha\beta \leq \alpha \otimes \beta$ also both follow immediately from the order-theoretic interpretation of these operations. This leaves the question of order-theoretic proofs of the other inequalities. Lipparini's order-theoretic interpretation [14] of $\alpha \times \beta$ does immediately make it clear that $\alpha\beta \leq \alpha \times \beta$—indeed, it shows more generally that $\sum_i \alpha_i \leq \bigoplus_i \alpha_i$. However, it does not seem to immediately prove that $\alpha \times \beta \leq \alpha \otimes \beta$, so finding an order-theoretic proof there remains a problem.

Question 5.2 Can the inequality $\alpha \times \beta \leq \alpha \otimes \beta$, and part (3) of Proposition 5.1, be proven by giving order-theoretic interpretations to all the quantities involved? What about the infinitary analogue of part (2)?

All these inequalities hold equally well, of course, for the infinitary versions of these operations. Also, note that if we had a natural exponentiation $e(\alpha, \beta)$, the same the same style of argument used above to prove $\alpha + \beta \leq \alpha \oplus \beta$ and $\alpha \times \beta \leq \alpha \otimes \beta$ could also be used to prove $\alpha^{\otimes\beta} \leq e(\alpha, \beta)$, in accordance with Table 1. But, as we showed in the previous section, there is no natural exponentiation. However, if one is willing to look a little bit outside of the ordinals, this line of reasoning could be used to prove that $\alpha^{\otimes\beta}$ is pointwise at most the surreal exponential discussed in § 4.

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