



Fiber Path Optimization of a Symmetric Laminate with a Cutout for Thermal Buckling, using a Novel Finite Element Algorithm

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Automated fiber placement (AFP) technology has pushed for the need to explore non-conventional fiber paths in laminated composites. This paper investigates optimal spatially varying fiber paths in a symmetric linear orthotropic laminate which could increase the critical buckling temperature under uniform applied thermal loads. The key idea here is to achieve gains in buckling performance, yet focus on manufacturability of the obtained optimal fiber path. The subject of this study is a four layer symmetric orthotropic laminated plate, with a central circular cutout that is clamped on all the edges. A novel Finite Element algorithm is proposed which imposes a condition on the definition of discrete fiber angles within each element. The main objectives of the proposed finite element approach are to maintain continuity of the fiber paths, and to use a computationally efficient model by reducing the number of optimization variables.

Nomenclature

$T(m)$	Fiber angle within any finite element 'm'
$T_{i,p}^q$	Initial fiber angle for the ' p^{th} ' quadrant in the ' q^{th} ' layer
n	Number of elements in cycloidal direction
$d\theta$	Variation in the fiber angle across finite elements
z_q	Lamina thickness for the ' q^{th} ' layer
Q, \bar{Q}	Constitutive matrices in the global and fiber axes respectively
C	Upper-bound averages along thickness, of the constitutive matrices in the fiber axis
D	Finite element material matrix
h	Thickness of the lamina
u, v, w	Displacements in x, y, z directions
ΔT_{cr}	Critical buckling temperature
α	Coefficients of thermal expansion
B_1, B_2	1 st and 2 nd order derivatives of shape (interpolation) functions
K_b, K_g	Elastic and geometric stiffnesses

I. Introduction

Effects of cutouts on strength and buckling of thin plates and thin walled cylindrical shells have been presented by various researchers. Kirsch¹ in 1898, first proposed the exact closed-form analytical solutions for

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the stress concentrations around cutouts in an infinite isotropic lamina under uniaxial tension. Mansfield² proposed methods to restore the stress state of an isotropic plate with a cutout to that of one without, thereby reducing any stress concentrations that could cause fracture. Pipes *et. al*³ presented a study on notched strength of composite materials where as Tang⁴ proposed studies on interlaminar stresses in orthotropic plates with a cutout, under tension. Ahn and Waas⁵ provided experimental results and a novel mechanism based modeling approach to predict the compressive notched strength of laminates under uniaxial and biaxial loading. Senocak and Waas^{6,7} revisited the problem of a uniaxial tensile loaded plate with a cutout in the case of an orthotropic laminate and proposed methods to essentially nullify the stress concentrations around the cutouts, by duly considering the bending-stretching coupling and using stiffeners modeled as one dimensional 4-DOF beams. Acar *et.al*,⁸ presented an algorithm for optimal steered fiber paths to reduce stress concentrations around cutouts in a symmetric laminate. Lin *et.al*⁹ presented buckling models for laminated plates with holes and Topal *et.al*¹⁰ studied the maximization of buckling loads for laminated plates with cutouts. Ounis *et.al*¹¹ performed a finite element based study to analyze thermal buckling of laminated composites. A method to minimize the thermal expansion of laminates is presented by Rangarajan *et. al*.¹² Hyer and Lee¹³ studied a problem on the buckling resistance of composite plates with central circular holes in the case of curvilinear fibers and proposed a solution based on modeling individual fiber orientations within each finite element, where the fiber orientations are allowed to vary spatially within the lamina. Gurdal and Olmedo¹⁴ presented an optimization of spatially varying fiber paths for solution to a plane elasticity problem using variable stiffness concept.

The subject of interest in this paper is to derive optimal spatially varying fiber paths and layer thicknesses for a 4-layer symmetric orthotropic composite laminate with the central circular cutout, to improve the thermal buckling performance. The square plate is clamped on all the edges and is under uniform thermal loads. A novel finite element algorithm is proposed which ensures that the variation in fiber angles across adjacent finite elements pick discrete values, and thus maintain a constant fiber angle within each element, making it easy to obtain a smooth fiber path. This model can easily be reproduced for a laminate with as many layers following the same procedure. The problem formulation accounts for this fiber path definition to solve the critical buckling temperature. The multi-objective optimization is performed to identify the fiber path that maximizes the critical buckling temperature and minimizes the mass of the laminate. The organization of the paper is as follows: Section II introduces the composite plate thermal buckling problem. In Section III, the proposed meshing algorithm for fiber continuity and finite element methodology for the solution are discussed. The finite element solution is verified in Section IV . Section V provides information about the optimization problem and optimum results. Section VI concludes the paper with a discussion of future extensions of this study.

II. Problem Definition

The present work comprises thermal buckling of a clamped plate under the effect of thermal axial loads. The plate is modeled as a symmetric 4-layer composite laminate. The problem is separated into two parts - first part explains the algorithm and Finite Element procedure, where as the second explains the optimization problem setup. The loading and geometry are symmetric as the entire plate is under a uniform thermal load, inducing axial compressive and shear loads on the plate, described as N_x , N_y and N_{xy} in Figure 1. Even though the symmetry of the loading and geometry suggests the use of symmetry boundary conditions to obtain the pre-buckling stresses by analyzing only one quarter of the plate; it needs to be noted that the buckling modes need not be symmetric. By using symmetry boundary conditions, only the symmetric modes can be obtained. Therefore, the full plate model is modeled to obtain all the buckling modes. The critical buckling temperature is then solved for the 1st buckling mode.

III. Meshing Algorithm

The proposed finite element meshing method has multiple features. First, it always ensures continuity of the fiber paths across element boundaries. Second, it is computationally efficient since only four optimization variables per layer of the laminate are used to define the fiber path. Moreover, it could easily be scaled for any number of plies in any symmetric or asymmetric laminate. The algorithm explains how discrete fiber angles are computed for each finite element ensuring inter-element continuity of the fiber angle. This is essential to end up with a manufacturable design using AFP technology.

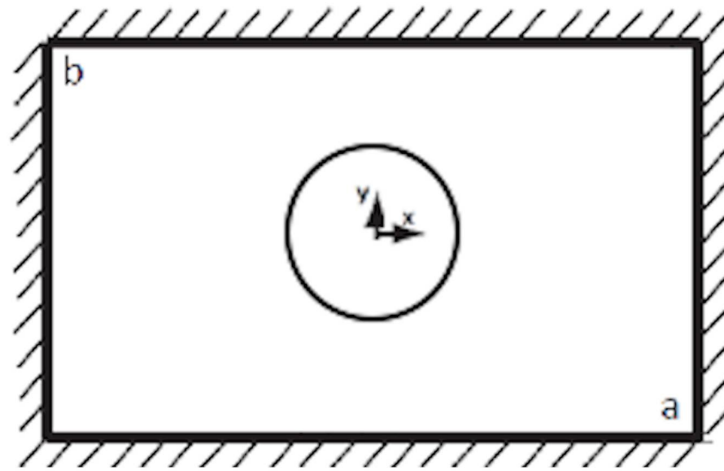


Figure 1. Clamped Plate with a Cutout under Uniform Heating

A. Algorithm for Discrete Fiber Angles

This section discusses the algorithm to compute a discrete fiber angle, $T(m)$, for each finite element under prescribed conditions of fiber path continuity. The direction of m (where $0 < m \leq n$, n being the total number of elements in cycloidal direction) represents the count of the finite element along each row. The plate is divided into 4 sub-domains (quadrants) to ensure fiber path continuity (Figure 2). In the context of spatially varying fiber paths, symmetry in the laminate is described by the same spatial variation of fiber paths in the layers symmetric with respect to the centerplane. In this specific case, Layer-1 is the same as Layer-4, and Layer-2 is the same as Layer-3. Detailed below is the formulation of the algorithm for the plate.

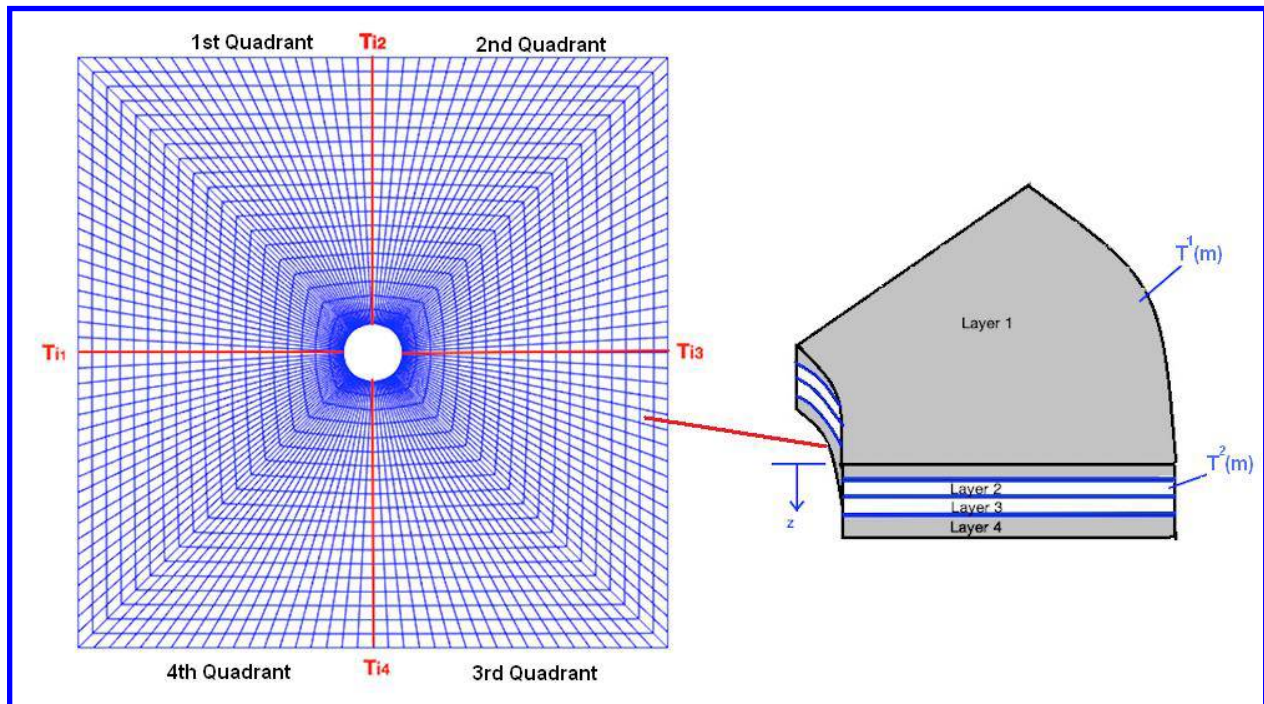


Figure 2. Finite Element Mesh for Quarter Plate (2888 Plate Elements)

For the elements located in the p^{th} quadrant and in the q^{th} layer, $T_{i,p}^q$ and $T_{f,p}^q$ represent initial and final

fiber angles respectively (e.g. $T_{i,1}^1$ and $T_{f,1}^1$ are initial and final fiber angles for the 1st quadrant in Layer-1). Hence the change in fiber angle is defined across each finite element along the direction 'm' for Layer-1, $d\theta_1^1$, as follows:

$$d\theta_1^1 = \frac{T_{f,1}^1 - T_{i,1}^1}{n - 1} \quad (1)$$

To elaborate, for any q^{th} layer, $T_{i,1}^q, T_{i,2}^q, T_{i,3}^q$ and $T_{i,4}^q$ are the 'initial' fiber angles of quadrants 1, 2, 3, 4. Then the final fiber angles for these quadrants are noted as $T_{f,1}^q = T_{i,2}^q, T_{f,2}^q = T_{i,3}^q, T_{f,3}^q = T_{i,4}^q$ and $T_{f,4}^q = T_{i,1}^q$, which merely means that the final fiber angle of each quadrant becomes the initial fiber angle of the next quadrant to ensure continuity of fiber angles. Hence, the discrete variation in angle in each quadrant $d\theta_p^q$ for each p^{th} quadrant for the q^{th} layer is obtained as:

$$d\theta_p^q = \frac{T_{f,p}^q - T_{i,p}^q}{n - 1} \quad (2)$$

This would mean that the only condition imposed on each $T_{i,p}^q$ during optimization, on its range is ($0^\circ \leq T_{i,1}^q, T_{i,2}^q, T_{i,3}^q, T_{i,4}^q < 180^\circ$), which constrains the fiber angle inside each element, $T(m)$. By means of constraining the value of $d\theta_p^q$ and thereby imposing a specific $T(m)$ for each element, m , the essence of the solution is captured; *viz*: to enforce continuity of the fiber angle, making it easily adaptable for manufacturing. Fiber angle in each element in the p^{th} quadrant, for the q^{th} layer as it moves along direction 'm' in Figure 2 is given by the expression:

$$T_p(m)^q = T_{i,p}^q + (m - 1)d\theta_p^q \quad (3)$$

where m ($0 < m \leq n$) represents element number in the chosen cycloidal direction in Figure 2 and n being the maximum number of elements in this direction. As would be discussed in the next section, the fiber angle $T(m)$ specified for each element under restrictions arising from $d\theta$ would be used in the finite element formulation by effectively computing the finite element material matrix, D , using plane stress constitutive matrix, \bar{Q} , averaged over the thickness to obtain an upper bound matrix C . \bar{Q} is obtained by rotating the constitutive matrix Q by a fiber angle $T(m)$ for each element.

B. Finite Element Model

When heated uniformly, the thermal expansion of the restricted plates will result in uniformly distributed in-plane normal and shear forces (Figure 1). The constitutive matrix, Q , of each finite element depends on the particular fiber angle within that finite element. The regular finite element procedure starts with the computation of the constitutive matrix in the global frame, Q_{ij} (where $i, j = 1, 2, 3, 4, 5, 6$). However, the problem needs to be assembled and solved in the material frame. This means that material matrix in material frame, \bar{Q}_{ij} , can be computed for each finite element with the transformation through an angle, $T(m)$. The stress-strain relation in the global frame for a general three dimensional orthotropic material is:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zx} \\ \sigma_{yz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{zx} \\ \gamma_{yz} \\ \gamma_{xy} \end{bmatrix}$$

where the stiffness coefficients are functions of the orthotropic material properties $E_{11}, E_{22}, \nu_{12}, \nu_{23}$ and G_{23} . The material and thermal properties used in this work correspond to that of IM7/8551-7^{15, 16} and is detailed in Table 1. Note that extension and shear are uncoupled.

To assemble the global stiffness matrices $[K_b]$ and $[K_g]$, the stiffness of each finite element is to be calculated in the respective material coordinate systems and then assembled. For each finite element, a transformation matrix $[L] = f(T(m))$ is calculated as:

Parameter	Value
E_{xx}	162.0 GPa
E_{yy}	8.34 GPa
G_{xy}	2.07 GPa
G_{yz}	2.07 GPa
ν_{xy}	0.339
ν_{yz}	0.509
α_{xx}	$-0.48 \frac{\mu m}{m^{\circ}C}$
α_{yy}	$22.3 \frac{\mu m}{m^{\circ}C}$

Table 1. Material and Thermal Properties of IM7/8551-7

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_p^2 & s_p^2 & 2c_p s_p & 0 & 0 \\ 0 & s_p^2 & c_p^2 & -2c_p s_p & 0 & 0 \\ 0 & -c_p s_p & c_p s_p & c_p^2 - s_p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_p & -s_p \\ 0 & 0 & 0 & 0 & s_p & c_p \end{bmatrix}$$

where $c_p = \cos(T(m))$ and $s_p = \sin(T(m))$ for each element ' m '. Rotated constitutive matrix in the material frame for each element ' m ', $\bar{Q}_{ij}(m)$ is obtained as

$$\bar{Q}_{ij} = [L]^T [Q] [L]$$

Since the Finite element formulation is basically dealing with a multi-layer structure in this case and each Finite element was previously defined through the thickness of the whole structure, the material matrix should be averaged along the thickness of each particular element (Figure 3) to obtain an upperbound averaged material matrix C . This averaging along the thickness indicates the upper bound approach to compute Q_{ij} with the assumption of strain continuity through the thickness of the element. If z_1 and z_2 are the thicknesses of the 1st (and 4th) layers and the 2nd (and 3rd) layers respectively, then Q_{ij} elements of the multi-layer structure can be obtained as an upper bound average,¹⁷ $Q_{ij,avg}^- = \frac{z_1 Q_{ij}^- + z_2 Q_{ij}^-}{z_1 + z_2}$ (where $i, j = 1, 2, 3, 4, 5, 6$) for any element of averaged Q_{avg}^- matrix:

$$[C] = [Q_{avg}^-] = \begin{bmatrix} Q_{11,avg}^- & Q_{12,avg}^- & Q_{13,avg}^- & 0 & 0 & 0 \\ Q_{12,avg}^- & Q_{22,avg}^- & Q_{23,avg}^- & 0 & 0 & 0 \\ Q_{13,avg}^- & Q_{23,avg}^- & Q_{33,avg}^- & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44,avg}^- & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55,avg}^- & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66,avg}^- \end{bmatrix}$$

where C is the elastic stiffness matrix of each particular element. The flexural rigidities, D , can be obtained from the elastic stiffness matrix, using plane stress assumptions, by the relation:¹⁸

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} = \frac{h^3}{12} \times \left(\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{21} & C_{22} & C_{26} \\ C_{61} & C_{62} & C_{66} \end{bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{63} & C_{64} & C_{65} \end{bmatrix} \right) \times \left(\begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{43} & C_{44} & C_{45} \\ C_{53} & C_{54} & C_{55} \end{bmatrix} \right)^{-1} \times \begin{bmatrix} C_{31} & C_{32} & C_{36} \\ C_{41} & C_{42} & C_{46} \\ C_{51} & C_{52} & C_{56} \end{bmatrix} \quad (4)$$

The general classical buckling equations for a rectangular laminated plate with dimensions $a \times b \times h$ (length in x -direction \times length in y -direction \times height) subjected to uniform temperature rise are:

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} + 2A_{16} \frac{\partial^2 u_0}{\partial x \partial y} + A_{66} \frac{\partial^2 u_0}{\partial y^2} + A_{16} \frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} + A_{26} \frac{\partial^2 v_0}{\partial y^2} = 0 \quad (5)$$

$$A_{16} \frac{\partial^2 u_0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} + A_{26} \frac{\partial^2 u_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} + 2A_{26} \frac{\partial^2 v_0}{\partial x \partial y} + A_{22} \frac{\partial^2 v_0}{\partial y^2} = 0 \quad (6)$$

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \quad (7)$$

where, the first two equations govern the in-plane resultant loads (u_0 and v_0 are the in-plane displacements at mid plane), and the last equation governs the out-of-plane buckling deflection, $w = w(x, y)$. N_x , N_y are the in-plane resultant normal loads in the x and y directions respectively, and N_{xy} is the in-plane shear resultant. D_{11} , D_{12} , D_{16} , D_{22} , D_{26} , D_{66} are the flexural rigidities of the plate. The flexural rigidities are obtained from the material constitutive law for each particular finite element.

The boundary conditions for the case of a square plate (side lengths $a \times a$) with clamped sides are:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = -\frac{a}{2}, \frac{a}{2}$$

$$w = 0, \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = -\frac{a}{2}, \frac{a}{2} \quad (8)$$

$$u_0, v_0 = 0, \quad \text{at} \quad x, y = -\frac{a}{2}, \frac{a}{2} \quad (9)$$

The critical increase in temperature at buckling, ΔT_{cr} , is related to the in-plane resultant loads, and homogenized (thickness averaged) values of thermal expansion, $\bar{\alpha}_{xx}$, $\bar{\alpha}_{yy}$, $\bar{\alpha}_{xy}$, by the following equation:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \frac{12\Delta T_{cr}}{h^2} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \bar{\alpha}_{xx} \\ \bar{\alpha}_{yy} \\ \bar{\alpha}_{xy} \end{bmatrix} \quad (10)$$

A quarter plate was first analyzed using the Finite Element algorithm with the assumption of symmetric buckling modes. The plate was modeled using 2888 plate elements. Each plate element has 4 nodes and 5 degrees of freedom: u , v , w , $\frac{dw}{dx}$ and $\frac{dw}{dy}$. The general finite element formulation for this buckling problem is based on both the conventional element flexural stiffness matrix, $[k_b]$, and geometric element stiffness matrix, $[k_g]$, that takes account of the in-plane load resultants. The formulations for $[k_b]$ and $[k_g]$ are given in Eq. 11.

$$[k_b] = \int \int \int_V B_2^T D B_2 dV, \quad [k_g] = \int \int \int_V B_1^T P B_1 dV \quad (11)$$

where B_1 and B_2 are the matrices consisting of the first and second derivatives of the shape functions of the plate element respectively. The flexural rigidity, D , is defined in Eq. 4. The matrix, P , incorporates the in-plane load resultants, N_x , N_y and N_{xy} . Based on this formulation, the element stiffness matrix for the buckling problem is given below:

$$[k] = [k_b] \pm [k_g] \quad (12)$$

where the sign of the geometric stiffness matrix, $[k_g]$ depends on the type of in-plane loads. It is positive if the in-plane loads are tensile, and it is negative if the in-plane loads are compressive. At buckling, the condition given below is satisfied;

$$\det([K_b] \pm \lambda[K_g]) = 0 \quad (13)$$

where $[K_b]$ and $[K_g]$ are global flexural and geometric stiffness matrices respectively. Eq. 13 shows that the solution finally leads to an eigenvalue problem. The ΔT_{cr} value that satisfies the critical buckling condition is the critical increase in temperature.

IV. Verification of Finite Element Methodology

Verification of the finite element approach of this work consists of three steps: fringe contour comparison for an isotropic plate problem, stress distribution verification for a unidirectional orthotropic composite laminate, and buckling solution for a benchmark orthotropic plate problem.

A. Fringe Contour Comparison

The developed finite element methodology is initially verified by comparing the fringe contours for a tension problem where an isotropic plate with a central cutout is subjected to remote tensile loading in the y-direction (Figure 3). This case has been selected since the fringe contours of the theoretical solution are available in literature¹⁹ and the problem geometry is similar to the buckling problem of interest in this work. The fringe contours plotted for the finite element solution match well with the theoretical result as shown in Figure 4, providing a degree of confidence in mesh selection.

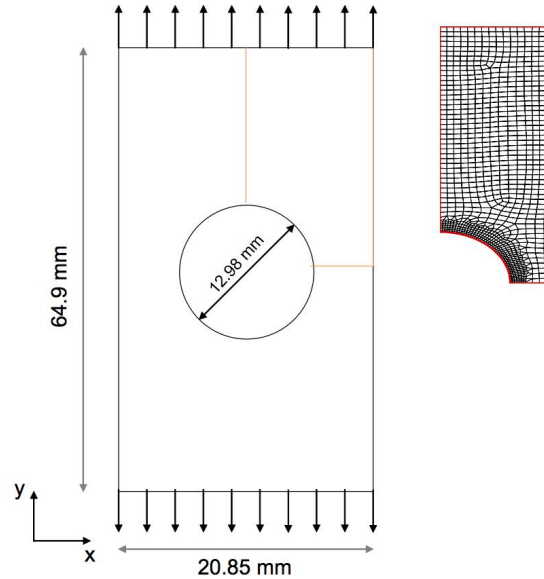


Figure 3. Problem Definition for Fringe Contour Comparison

B. Stress Distribution Verification for a Unidirectional Orthotropic Laminate

The second check on the finite element mesh quality is to verify the predicted stress concentration factor (SCF) of a 4-layer unidirectional orthotropic laminate, with a stacking sequence of $[45/0]_s$, to the theoretical solution. This problem has the same geometry with the problem of interest in this work, a plate with a central cutout, but in this case the plate is under the effect of tensile loads acting on the horizontal direction. A detailed explanation about the problem and solution methodology can be found in the earlier work of the authors.⁸ The maximum SCF value of the finite element solution was found to be at the cutout boundary, and it is 6.27. This value is close to the theoretical value, 5.96.²⁰

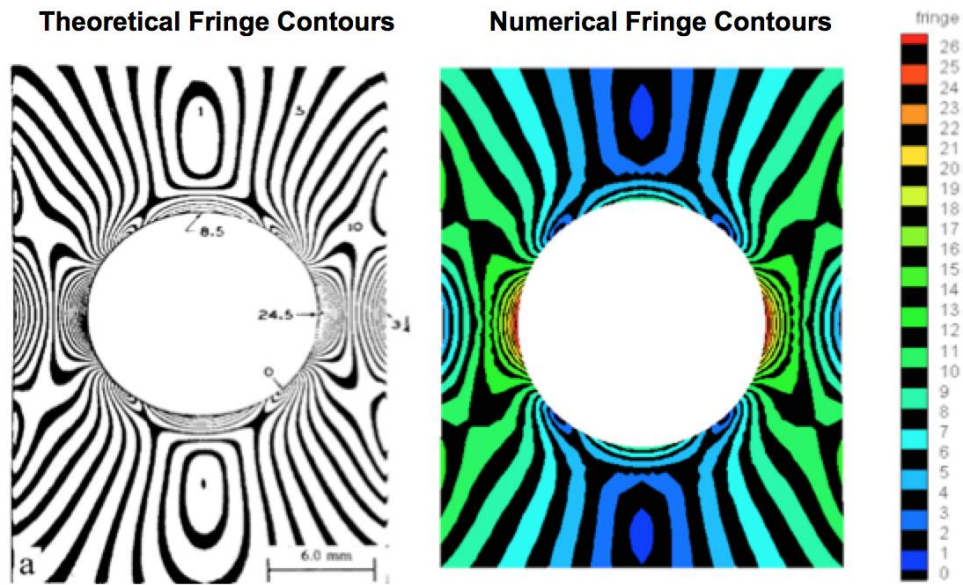


Figure 4. Fringe Contour Comparison

C. Benchmark Problem for Thermal Buckling of an Orthotropic Plate

Utilizing the finite element method, the critical increase in temperature was computed for symmetric buckling of a clamped symmetrically laminated plate under uniform heating. However, no analytical solution exists for the corresponding buckling problem. To verify the obtained buckling solution, a similar but simpler case of a specially orthotropic plate, which has a known analytical solution¹⁸ was employed. A rectangular orthotropic plate with principal directions parallel to the sides is compressed by uniformly distributed edge axial load, P_x (Figure 5).

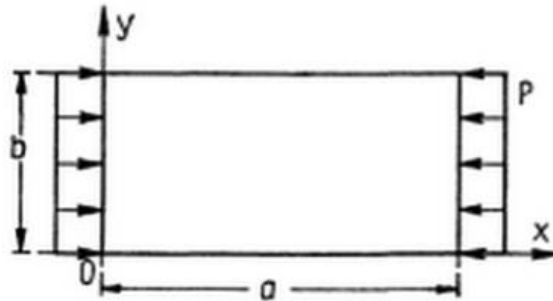


Figure 5. Benchmark Buckling Problem Definition for an Orthotropic Plate

The buckling problem for such a plate with four simply supported sides is given by Lekhnistkii.¹⁸ The buckling equation is:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2D_{33} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \quad (14)$$

The solution of the form:

$$w = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (15)$$

is sought, where A_{mn} are constant coefficients, m and n are integers. The following boundary conditions are utilized:

$$\begin{aligned} w = 0, \quad \frac{\partial w^2}{\partial x^2} + \nu_y \frac{\partial w^2}{\partial y^2} &= 0 \quad \text{at } x = 0, a \\ w = 0, \quad \frac{\partial w^2}{\partial y^2} + \nu_x \frac{\partial w^2}{\partial x^2} &= 0 \quad \text{at } y = 0, b \end{aligned} \quad (16)$$

where ν_x and ν_y are the Poisson's ratios. By requiring that Eq. 15 be a solution of Eq. 14, the following solution is obtained:

$$N_x \left(\frac{m}{a}\right)^4 + N_y \left(\frac{n}{b}\right)^4 = \pi^2 \left[D_{11} \frac{m}{a} + 2D_{33} \frac{mn}{ab} + D_{22} \frac{n}{b} \right] \quad (17)$$

The formulation is performed for the case of proportional loading where forces N_x and N_y may vary (general case), but must maintain a constant ratio β .

$$N_x = \lambda, \quad N_y = \lambda\beta \quad (18)$$

The critical value of λ is found from the formula:

$$\lambda = \frac{\pi^2 \sqrt{D_{11} D_{22}}}{b^2} \cdot \frac{\sqrt{\frac{D_{11}}{D_{22}} \left(\frac{m}{c}\right)^2 + \frac{2D_{33}}{\sqrt{D_{11} D_{22}}} n^2 + \frac{D_{22}}{D_{11}} \left(\frac{c}{m}\right)^2} n^4}{1 + \beta \left(\frac{c}{m}\right)^2 n^2} \quad (19)$$

where c is the ratio between the lengths of the sides of the plate ($c = \frac{a}{b}$). The problem then consists of seeking the values of m and n which give the smallest λ and hence the critical buckling load, λ_{cr} . Based on the solution of the critical buckling load, the expression can be formulated for the critical increase in temperature using the coefficient of thermal expansion tensor (α):

$$\Delta T_{cr} = \frac{\lambda}{\frac{12}{h^2} (D_{11} \cdot \alpha_x + D_{12} \cdot \alpha_y + D_{13} \cdot \alpha_{xy})} \quad (20)$$

By modifying the boundary conditions in the original calculations (from clamped to simply supported for an orthotropic plate), the results are compared to the analytical solution and found to be a very close match. Sample comparisons are shown below. The default orthotropic rectangular plate has dimensions ($a \times b \times h$) of $0.1 \text{ m} \times 0.4 \text{ m} \times 0.004 \text{ m}$. Its flexural rigidities are $D_{11} = 8.0 \times 10^{10}$, $D_{12} = D_{21} = 2.0 \times 10^9$, $D_{22} = 4.0 \times 10^{10}$, $D_{33} = 1.2 \times 10^{11}$, and its coefficients of thermal expansion are $\alpha_x = 9.0 \times 10^{-6}$ and $\alpha_y = 1.2 \times 10^{-6}$. Only one property is varied at a time, and the average percentage error between the results is 0.0314%. The analytical and computational critical increases in temperature are presented below.

V. Optimization of Fiber Path

The optimization problem deals with multiple objectives since the goals are defined to find the fiber path and layer thicknesses that maximize the critical increase in buckling temperature, ΔT_{cr} , and minimize the mass of the structure as well. The fiber path depends on 4 independent variables for each layer: $T_{i,1}^q$, $T_{i,2}^q$, $T_{i,3}^q$ and $T_{i,4}^q$. All $T_{f,p}^q$ values have $T_{i,p}^q$ equivalents, and $d\theta_p^q$ can also be computed in terms of them. Considering the fact that the problem of interest includes a 4-layer symmetric laminate - 2 independent layers, the number of optimization parameters relevant to fiber path distribution is 8. The remaining optimization variables are the thicknesses of these independent 2 layers (z_1 and z_2) since the area is constant for every layer, and therefore the total mass varies with respect to the thicknesses. A multi-objective optimization problem is defined since proposing only one objective which is related to buckling may cause designs which are not cost efficient. In order to eliminate this possibility, minimization of the plate mass is defined as the second design objective. Since the distribution constraints were already imposed to the problem through meshing and methodology, no additional design constraints are defined for the optimization problem. The mathematical formulation of the optimization problem is given in Equation 21.

$$\max f_1 = \Delta T_{cr}, \quad \min f_2 = m(z_1, z_2)$$

D_{11} (Nm)	Analytical ΔT_{cr} (K)	Computed ΔT_{cr} (K)
7.0×10^{10}	192.9	193.2
8.0×10^{10}	187.4	187.4
9.0×10^{10}	183.2	183.2
D_{12} (Nm)	Analytical ΔT_{cr} (K)	Computed ΔT_{cr} (K)
1.5×10^9	188.9	188.9
2.0×10^9	187.4	187.4
2.5×10^9	186.0	185.9
α_x (K^{-1})	Analytical ΔT_{cr} (K)	Computed ΔT_{cr} (K)
8.0×10^{-6}	209.0	209.0
9.0×10^{-6}	187.4	187.4
10×10^{-6}	169.9	169.8
b (m)	Analytical ΔT_{cr} (K)	Computed ΔT_{cr} (K)
0.3	211.1	211.2
0.4	187.4	187.4
0.5	171.3	171.2

Table 2. Comparison of results for a simply supported rectangular orthotropic plate

$$\begin{aligned}
 s &= (T_{i_1}^1, T_{i_2}^1, T_{i_3}^1, T_{i_4}^1, T_{i_1}^2, T_{i_2}^2, T_{i_3}^2, T_{i_4}^2, z_1, z_2) \\
 0^\circ &\leq T_{i_1}^1, T_{i_2}^1, T_{i_3}^1, T_{i_4}^1, T_{i_1}^2, T_{i_2}^2, T_{i_3}^2, T_{i_4}^2 < 180^\circ \\
 0.1 \text{ mm} &\leq z_1, z_2 \leq 1 \text{ mm}
 \end{aligned} \tag{21}$$

In this formulation, the first line shows the objective functions where m shows the mass of the plate. The set of optimization variables is defined in the second line. The upper and lower limits of the optimization variables are given in the last two lines. Since the problem of interest here is a non-linear multi-objective optimization problem, a genetic algorithm that is also capable of working with multiple objectives is implemented. Considering the large computational time requirement for a finite element buckling solution, an efficient optimization algorithm needs to be chosen. For this purpose, Non-Dominated Sorting Genetic Algorithm (NSGA-II), which is known to be one of the fastest genetic algorithms, was implemented. The sampling was performed with Incremental Space Filter (ISF) with 10 Design of Experiments (DoE). The optimization algorithm provided 150 total designs since it used 15 function evaluations for each DoE. The summary of the optimization workflow is illustrated in Figure 6.

The parameters of the optimum design are shown in Table 3. The optimum design performance is compared to selected random designs in Table 4. These random designs have the same layer thickness values with the optimum design. However, their layers are defined as unidirectional laminae. The comparison indicates that the optimum design has provided a significant increase in the thermal buckling performance of the composite structure having the same mass. The fiber path distributions of the optimum design for Layer-1 and Layer-2 are illustrated in Figure 7 and Figure 8 respectively.

VI. Conclusion

A computationally efficient and novel finite element approach to identify optimum steered fiber paths that can be manufactured using AFP technology is presented in this work. The meshing algorithm using discrete fiber angle values within each element as control variables for each independent layer is developed so as to obtain continuous fiber paths, as well as to reduce the computational time. The specific problem discussed here addresses investigating optimal steered fiber paths for an orthotropic symmetric laminate under uniform thermal loads, in order to maximize the critical buckling temperature. The plate has a cutout at its center and is clamped on all sides. A full plate model is developed to include symmetric and

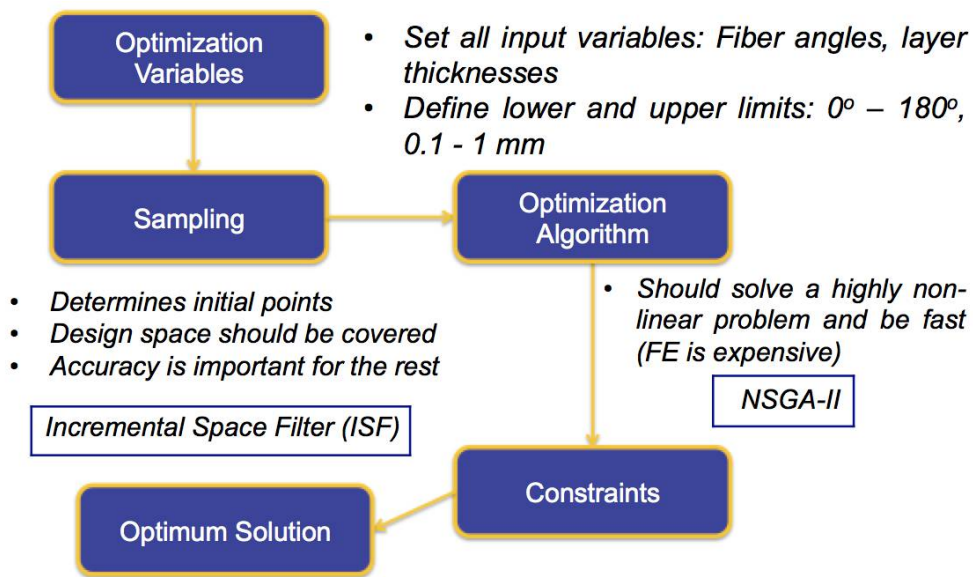


Figure 6. Optimization Workflow

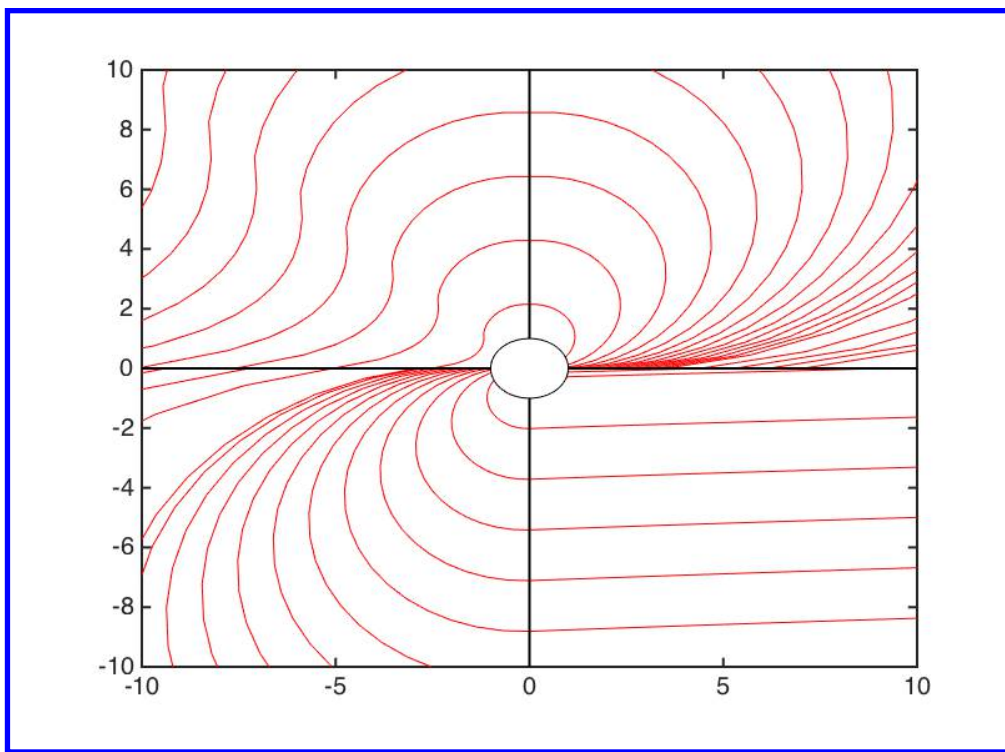


Figure 7. Optimum Fiber Path (Layer-1 and Layer-4)

asymmetric buckling modes. The laminate is modeled in four separate quadrants, using finite elements, ensuring continuity of displacements and fiber angles across the quadrants. A verification of the algorithm is provided by three specific cases- a fringe contour comparison for an isotropic plate that is available in literature, an orthotropic plate tension problem using unidirectional plies and a bench mark buckling problem for orthotropic laminates with unidirectional plies. After the verification of the solution methodology, the

Parameter	Optimum Value
$T_{i_1}^1$	5.5571°
$T_{i_2}^1$	179.99°
$T_{i_3}^1$	2.7640°
$T_{i_4}^1$	1.9010°
$T_{i_1}^2$	2.2260°
$T_{i_2}^2$	6.0188°
$T_{i_3}^2$	89.9840°
$T_{i_4}^2$	1.9855°
z_1	0.10085 mm
z_2	0.10120 mm
ΔT_{cr}	$205.75^\circ C$

Table 3. Optimum Design Parameters

Design	$T_{i_1}^1$	$T_{i_2}^1$	$T_{i_3}^1$	$T_{i_4}^1$	$T_{i_1}^2$	$T_{i_2}^2$	$T_{i_3}^2$	$T_{i_4}^2$	ΔT_{cr}
Random-1	45°	45°	45°	45°	0°	0°	0°	0°	$151.68^\circ C$
Random-2	90°	90°	90°	90°	135°	135°	135°	135°	$121.42^\circ C$
Optimum	5.5571°	179.99°	2.7640°	1.9010°	2.2260°	6.0188°	89.9840°	1.9855°	$205.75^\circ C$

Table 4. Performance Comparison for Optimum Design

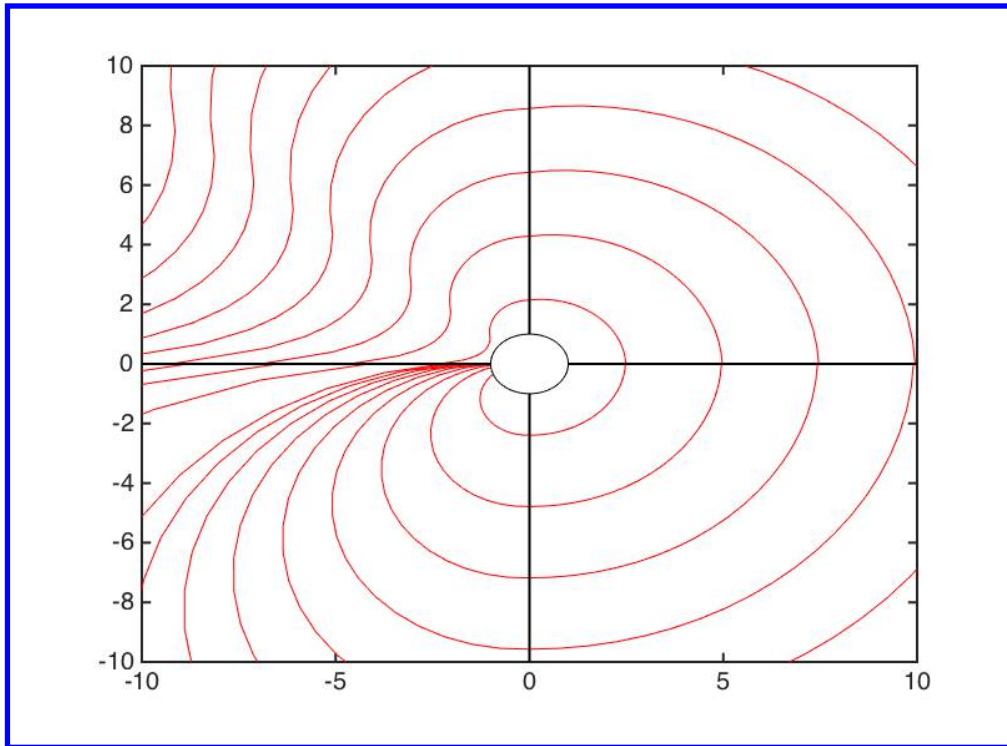


Figure 8. Optimum Fiber Path (Layer-2 and Layer-3)

global optimization is performed using a genetic algorithm using the defined objective functions and the optimization parameters. The optimum design is selected among all feasible solutions and it provides a very significant increase in the thermal buckling capacity.

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