JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 37, No. 4, July–August 2014

Engineering Notes

Identities for Deriving Equations of Motion Using Constrained Attitude Parameterizations

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DOI: 10.2514/1.G000221

I. Introduction

DERIVING the equations of motion of a flight vehicle modeled as a rigid body, a flexible body, or a system composed of many rigid and flexible bodies is necessary for simulation, estimation, and control. There is a wealth of literature devoted to the derivation of the equations of motion using Lagrange's equation ([1] p. 80), or the quasi-coordinate form of Lagrange's equation known and the Boltzmann–Hamel equation ([1] p. 227).

Although the attitude of one reference frame relative to another is globally and uniquely described by a rotation matrix ([2] p. 14), rotation matrix parameterizations are often used for derivations and computations. Euler angles and Rodrigues parameters are common unconstrained parameterizations, whereas axis/angle parameters and quaternions are well-known constrained parameterizations ([2] pp. 30–31).

The novel contribution of this note is the rigorous proof of three identities related to axis/angle and quaternion parameterizations of attitude. When using Lagrange's equation, the three identities realize a straightforward and concise derivation of the equations of motion of a rigid or flexible mechanical system in matrix form without the explicit use of the Boltzmann–Hamel equation or index notation. Highlighting the straightforward use of the identities is, although expected, also a contribution. In particular, the three identities are used to derive the equations of motion of a rigid spacecraft equipped with N thrusters being perturbed by a residual magnetic disturbance torque.

The remainder of this note is as follows. After reviewing preliminary material, the three identities are presented. Next, an example highlighting the utility of the three identities is discussed. Finally, the identities are rigorously proven to be true. This note closes with some final remarks. Relevant literature is discussed throughout the note as material is presented.

II. Preliminaries

Reference frames a and b are denoted $\mathcal{F}_a$ and $\mathcal{F}_b$. The components of the Gibbsian vector resolved in $\mathcal{F}_a$ are $\boldsymbol{v}_a = [v_{a,1} v_{a,2} v_{a,3}]^T$, and resolved in $\mathcal{F}_b$ are $\boldsymbol{v}_b = [v_{b,1} \quad v_{b,2} \quad v_{b,3}]^T$. The relationship between $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ is

$$\boldsymbol{v}_b = \boldsymbol{C}_{ba}\boldsymbol{v}_a \tag{1}$$

where $\boldsymbol{C}_{ba} = \boldsymbol{C}_{ab}^T$ is the 3×3 rotation matrix ([2] p. 528).

A. Axis/Angle Parameters

The rotation matrix $\boldsymbol{C}_{ba}$ can be written as ([2] p. 24)

$$\boldsymbol{C}_{ba}(\boldsymbol{a},\phi) = \cos\phi\mathbf{1} + (1-\cos\phi)\boldsymbol{a}\boldsymbol{a}^T - \sin\phi\boldsymbol{a}^\times = e^{-\phi\boldsymbol{a}^\times} \tag{2}$$

where $\boldsymbol{a}^T\boldsymbol{a} = 1$, $\boldsymbol{a} = [a_1 \quad a_2 \quad a_3]^T$ is the unit-length axis of rotation, and ϕ is the angle of rotation. When $\phi = 0, \pm 2\pi, \pm 4\pi, \ldots,$ the axis $\boldsymbol{a}$ is undefined ([2] p. 14), which can be considered a singularity. Throughout this note, derivations using axis/angle parameters assume this singularity is avoided. Let $\boldsymbol{\omega}_b^{ba}$ denote the angular velocity of $\mathcal{F}_b$ relative to $\mathcal{F}_a$ expressed in $\mathcal{F}_b$. The relationship between $\boldsymbol{a}$, ϕ, $\dot{\boldsymbol{a}}$, $\dot{\phi}$, and $\boldsymbol{\omega}_b^{ba}$ is ([2] p. 25)

$$\begin{bmatrix}\dot{\boldsymbol{a}}\\ \dot{\phi}\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{1}{2}\left(\boldsymbol{a}^\times - \cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times\right)\\ \boldsymbol{a}^T\end{bmatrix}}_{\boldsymbol{\Gamma}(\boldsymbol{a},\phi)}\boldsymbol{\omega}_b^{ba} \tag{3}$$

or

$$\boldsymbol{\omega}_b^{ba} = \underbrace{\begin{bmatrix}\sin\phi\mathbf{1} - (1-\cos\phi)\boldsymbol{a}^\times & \boldsymbol{a}\end{bmatrix}}_{\boldsymbol{S}(\boldsymbol{a},\phi)}\begin{bmatrix}\dot{\boldsymbol{a}}\\ \dot{\phi}\end{bmatrix} \tag{4}$$

Differentiating $\boldsymbol{a}^T\boldsymbol{a} - 1 = 0$ with respect to time gives

$$\underbrace{\begin{bmatrix}2\boldsymbol{a}^T & 0\end{bmatrix}}_{\boldsymbol{\Xi}(\boldsymbol{a},\phi)}\begin{bmatrix}\dot{\boldsymbol{a}}\\ \dot{\phi}\end{bmatrix} = 0 \tag{5}$$

From Eqs. (3) and (5), notice that

$$\boldsymbol{\Xi}(\boldsymbol{a},\phi)\boldsymbol{\Gamma}(\boldsymbol{a},\phi) = \mathbf{0} \tag{6}$$

As such, $\boldsymbol{\Gamma}(\boldsymbol{a},\phi)$ spans the null space of $\boldsymbol{\Xi}(\boldsymbol{a},\phi)$ [3]. Also, using Eqs. (3) and (4),

$$\boldsymbol{S}(\boldsymbol{a},\phi)\boldsymbol{\Gamma}(\boldsymbol{a},\phi) = \mathbf{1} \tag{7}$$

B. Quaternions

Quaternions are often used to parameterize $\boldsymbol{C}_{ba}$ and are defined as ([2] p. 17)

$$\begin{bmatrix}\boldsymbol{\epsilon}\\ \eta\end{bmatrix} = \begin{bmatrix}\boldsymbol{a}\sin\frac{\phi}{2}\\ \cos\frac{\phi}{2}\end{bmatrix}$$

where $\boldsymbol{\epsilon}^T\boldsymbol{\epsilon} + \eta^2 = 1$, and $\boldsymbol{\epsilon} = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3]^T$. In terms of quaternions, the rotation matrix $\boldsymbol{C}_{ba}$ is ([2] p. 18)

$$\boldsymbol{C}_{ba}(\boldsymbol{\epsilon},\eta) = (\eta^2 - \boldsymbol{\epsilon}^T\boldsymbol{\epsilon})\mathbf{1} + 2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T - 2\eta\boldsymbol{\epsilon}^\times = \mathbf{1} + 2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times - 2\eta\boldsymbol{\epsilon}^\times \tag{8}$$

The relationship between $\boldsymbol{\epsilon}$, η, $\dot{\boldsymbol{\epsilon}}$, $\dot{\eta}$, and $\boldsymbol{\omega}_b^{ba}$ is ([2] pp. 26–31)

$$\begin{bmatrix}\dot{\boldsymbol{\epsilon}}\\ \dot{\eta}\end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix}\eta\mathbf{1} + \boldsymbol{\epsilon}^\times\\ -\boldsymbol{\epsilon}^T\end{bmatrix}}_{\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)}\boldsymbol{\omega}_b^{ba} \tag{9}$$

or

Received 16 August 2013; revision received 30 October 2013; accepted for publication 8 November 2013; published online 8 April 2014.

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$$\boldsymbol{\omega}_b^{ba} = 2(\eta\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}\dot{\eta}) = \underbrace{[2(\eta\mathbf{1} - \boldsymbol{\epsilon}^\times) \quad -2\boldsymbol{\epsilon}]}_{\boldsymbol{S}(\boldsymbol{\epsilon},\eta)}\begin{bmatrix}\dot{\boldsymbol{\epsilon}}\\ \dot{\eta}\end{bmatrix} \tag{10}$$

Taking the time derivative of $\boldsymbol{\epsilon}^T\boldsymbol{\epsilon} + \eta^2 = 1$ yields

$$\underbrace{[2\boldsymbol{\epsilon}^T \quad 2\eta]}_{\boldsymbol{\Xi}(\boldsymbol{\epsilon},\eta)}\begin{bmatrix}\dot{\boldsymbol{\epsilon}}\\ \dot{\eta}\end{bmatrix} = 0 \tag{11}$$

Using Eqs. (9) and (11), it follows that

$$\boldsymbol{\Xi}(\boldsymbol{\epsilon},\eta)\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta) = \mathbf{0} \tag{12}$$

From Eq. (12), it is clear that $\boldsymbol{\Xi}(\boldsymbol{\epsilon},\eta)$ and $\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)$ are orthogonal complements [3]. Also, Eqs. (9) and (10) give

$$\boldsymbol{S}(\boldsymbol{\epsilon},\eta)\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta) = \mathbf{1} \tag{13}$$

III. Main Result: The Three Identities

When $\boldsymbol{C}_{ba}$, $\boldsymbol{\Gamma}$, $\boldsymbol{S}$, and $\boldsymbol{\Xi}$ are written without arguments of either $(\boldsymbol{a},\phi)$ or $(\boldsymbol{\epsilon},\eta)$, then either axis/angle parameters or quaternions may be assumed for each. In addition, $\boldsymbol{q} \in \mathbb{R}^4$ will be used to denote $\boldsymbol{q} = [\boldsymbol{a}^T \quad \phi]^T$ and $\boldsymbol{q} = [\boldsymbol{\epsilon}^T \quad \eta]^T$ when appropriate.

Recall $\boldsymbol{v}_b = \boldsymbol{C}_{ba}\boldsymbol{v}_a$ from Eq. (1). The main purpose of this note is rigorously proving that the following three identities hold when attitude is parameterized in terms of either $(\boldsymbol{a},\phi)$ or $(\boldsymbol{\epsilon},\eta)$.

Identity 1:

$$\left(\dot{\boldsymbol{S}} - \frac{\partial\boldsymbol{\omega}_b^{ba}}{\partial\boldsymbol{q}}\right)\boldsymbol{\Gamma} = -\boldsymbol{\omega}_b^{ba^\times}$$

Identity 2:

$$\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial\boldsymbol{q}}\boldsymbol{\Gamma} = (\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times$$

Identity 3:

$$\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{q}}\boldsymbol{\Gamma} = -\boldsymbol{C}_{ba}^T\boldsymbol{v}_b^\times$$

Two identities presented in [4] are similar but not identical to identities 2 and 3 when attitude is parameterized using quaternions; axis/angle parameters are not considered in [4]. When attitude is parameterized using Euler angles, equivalent identities to identities 1, 2, and 3 can be found in [5–9]. When any other unconstrained three-parameter attitude parameterization is used, the equivalent of identity 1 can be found in [7] and Appendix H of [10] in index form.

IV. An Example Highlighting the Utility of Identities One, Two, and Three

Before rigorously proving identities 1, 2, and 3, an example highlighting the utility of each identity will be presented. Specifically, a rigid spacecraft equipped with N thrusters being continually disturbed by a residual magnetic disturbance will be considered. Although a rigid body system is being considered, the identities can be used to derive the equations of motion of flexible systems as well.

A. Lagrange's Equation

For a system with nonholonomic constraints, Lagrange's equation takes the form ([1] p. 80)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial\dot{\boldsymbol{q}}}\right)^T - \frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})^T}{\partial\boldsymbol{q}} = \boldsymbol{\Xi}^T\boldsymbol{\lambda} + \sum_{i=1}^N \boldsymbol{g}^i \tag{14}$$

where $L = T - U$ is the Lagrangian, T and U are the kinetic and potential energy, $\boldsymbol{q}$ are the generalized coordinates, $\boldsymbol{\lambda}$ are the Lagrange multipliers associated with the constraints, and $\boldsymbol{g}^i$ are the generalized forces or torques. The generalized coordinates are either axis/angle parameters or quaternions (i.e., $\boldsymbol{q} = [\boldsymbol{a}^T \quad \phi]^T$ or $\boldsymbol{q} = [\boldsymbol{\epsilon}^T \quad \eta]^T$). Owing to the fact both $(\boldsymbol{a},\phi)$ and $(\boldsymbol{\epsilon},\eta)$ satisfy only one constraint, there is actually only one Lagrange multiplier, denoted λ.

B. Kinetic and Potential Energy

The kinetic energy of the rigid body is ([2] p. 60)

$$T = \frac{1}{2}\boldsymbol{\omega}_b^{ba^T}\boldsymbol{J}_b\boldsymbol{\omega}_b^{ba} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{S}^T\boldsymbol{J}_b\boldsymbol{S}\dot{\boldsymbol{q}}$$

where $\boldsymbol{J}_b = \boldsymbol{J}_b^T > 0$ is the inertia matrix (not necessarily diagonal) expressed in the body frame and the body frame is located at the mass center. The potential energy associated with a residual magnetic dipole is ([11] p. 281)

$$U = -\boldsymbol{m}_b^T\boldsymbol{b}_b$$

where $\boldsymbol{m}_b$ is the (constant) residual magnetic dipole and $\boldsymbol{b}_b = \boldsymbol{C}_{ba}\boldsymbol{b}_a$ is Earth's magnetic field vector ([12] pp. 779–786). Observe that U is a function of $\boldsymbol{q}$ (i.e., $U = U(\boldsymbol{q})$) because $\boldsymbol{b}_b = \boldsymbol{C}_{ba}(\boldsymbol{q})\boldsymbol{b}_a$. Combining the expressions for T and U, the Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{S}^T\boldsymbol{J}_b\boldsymbol{S}\dot{\boldsymbol{q}} + \boldsymbol{m}_b^T\boldsymbol{b}_b \end{aligned} \tag{15}$$

$$= \frac{1}{2}\boldsymbol{\omega}_b^{ba^T}\boldsymbol{J}_b\boldsymbol{\omega}_b^{ba} + \boldsymbol{m}_b^T\boldsymbol{b}_b \tag{16}$$

When the Lagrangian is written in the form shown in Eq. (15) (respectively, Eq. (16)), the notation $L(\boldsymbol{q},\dot{\boldsymbol{q}})$ (respectively, $\bar{L}(\boldsymbol{q},\boldsymbol{\omega}_b^{ba})$) will be used. This convention follows that of [5,7] and ([10] pp. 267–268).

C. Method of Virtual Work

The spacecraft in question is equipped with $i = 1, \ldots, N$ thrusters. The ith thruster produces a force $\mathbf{f}_b^i = \boldsymbol{C}_{ba}\mathbf{f}_a^i$ and is located at $\mathbf{r}_b^i = \boldsymbol{C}_{ba}\mathbf{r}_a^i$ from the origin of the body frame where $\mathbf{r}_b^i$ is constant, as seen in the body frame. The virtual work associated with the ith thruster is $\delta W^i = \mathbf{f}_a^{i^T}\delta\mathbf{r}_a^i$. The relationship between the virtual displacement $\delta\mathbf{r}_a^i$ and the (unconstrained) virtual displacement $\delta\boldsymbol{q}$ is

$$\delta\mathbf{r}_a^i = \frac{\partial(\boldsymbol{C}_{ba}^T\mathbf{r}_b^i)}{\partial\boldsymbol{q}}\delta\boldsymbol{q}$$

where $\mathbf{r}_a^i = \boldsymbol{C}_{ba}^T\mathbf{r}_b^i$. The virtual work associated with $\delta\boldsymbol{q}$ is

$$\delta W^i = \mathbf{f}_a^{i^T}\delta\mathbf{r}_a^i = \mathbf{f}_a^{i^T}\frac{\partial(\boldsymbol{C}_{ba}^T\mathbf{r}_b^i)}{\partial\boldsymbol{q}}\delta\boldsymbol{q} = \delta\boldsymbol{q}^T\underbrace{\frac{\partial(\boldsymbol{C}_{ba}^T\mathbf{r}_b^i)^T}{\partial\boldsymbol{q}}\mathbf{f}_a^i}_{\boldsymbol{g}^i} \tag{17}$$

where $\boldsymbol{g}^i$ is the generalized torque associated with the ith thruster.

D. Application of Lagrange's Equation

Lagrange's equation given in Eq. (14) will now be employed to derive the motion equations of the system, starting with the computation of $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial\dot{\boldsymbol{q}}}\right)$. To this end,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{q}}^T \boldsymbol{S}^T \boldsymbol{J}_b \boldsymbol{S}) = \dot{\boldsymbol{\omega}}_b^{ba^T} \boldsymbol{J}_b \boldsymbol{S} + \boldsymbol{\omega}_b^{ba^T} \boldsymbol{J}_b \dot{\boldsymbol{S}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right)^T = \boldsymbol{S}^T \boldsymbol{J}_b \dot{\boldsymbol{\omega}}_b^{ba} + \dot{\boldsymbol{S}}^T \boldsymbol{J}_b \boldsymbol{\omega}_b^{ba} \tag{18}$$

Next, computation of $\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}$ will be considered. Because L can be written as $L(\boldsymbol{q},\dot{\boldsymbol{q}}) = \bar{L}(\boldsymbol{q},\boldsymbol{\omega}_b^{ba})$, it follows that

$$\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \frac{\partial \bar{L}(\boldsymbol{q},\boldsymbol{\omega}_b^{ba})}{\partial \boldsymbol{\omega}_b^{ba}}\frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}} + \frac{\partial \bar{L}(\boldsymbol{q},\boldsymbol{\omega}_b^{ba})}{\partial \boldsymbol{q}}$$

$$= \boldsymbol{\omega}_b^{ba^T} \boldsymbol{J}_b \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}} + \boldsymbol{m}_b^T \frac{\partial (\boldsymbol{C}_{ba}\boldsymbol{b}_a)}{\partial \boldsymbol{q}}$$

$$\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}})^T}{\partial \boldsymbol{q}} = \frac{\partial \boldsymbol{\omega}_b^{ba\,T}}{\partial \boldsymbol{q}} \boldsymbol{J}_b \boldsymbol{\omega}_b^{ba} + \frac{\partial (\boldsymbol{C}_{ba}\boldsymbol{b}_a)^T}{\partial \boldsymbol{q}} \boldsymbol{m}_b \tag{19}$$

Combining Eqs. (17), (18), and (19) with Lagrange's equation given in Eq. (14) and rearranging yields

$$\boldsymbol{S}^T \boldsymbol{J}_b \dot{\boldsymbol{\omega}}_b^{ba} + \left(\dot{\boldsymbol{S}}^T - \frac{\partial \boldsymbol{\omega}_b^{ba\,T}}{\partial \boldsymbol{q}}\right)\boldsymbol{J}_b \boldsymbol{\omega}_b^{ba}$$

$$= \boldsymbol{\Xi}^T \lambda + \frac{\partial (\boldsymbol{C}_{ba}\boldsymbol{b}_a)^T}{\partial \boldsymbol{q}} \boldsymbol{m}_b + \sum_{i=1}^{N} \frac{\partial (\boldsymbol{C}_{ba}^T \mathbf{r}_b^i)^T}{\partial \boldsymbol{q}} \mathbf{f}_a^i$$

Premultiplying both sides by $\boldsymbol{\Gamma}^T$ gives

$$\underbrace{\boldsymbol{\Gamma}^T \boldsymbol{S}^T}_{\mathbf{1}} \boldsymbol{J}_b \dot{\boldsymbol{\omega}}_b^{ba} + \boldsymbol{\Gamma}^T\left(\dot{\boldsymbol{S}}^T - \frac{\partial \boldsymbol{\omega}_b^{ba\,T}}{\partial \boldsymbol{q}}\right)\boldsymbol{J}_b \boldsymbol{\omega}_b^{ba}$$

$$= \underbrace{\boldsymbol{\Gamma}^T \boldsymbol{\Xi}^T}_{\mathbf{0}} \lambda + \boldsymbol{\Gamma}^T \frac{\partial (\boldsymbol{C}_{ba}\boldsymbol{b}_a)^T}{\partial \boldsymbol{q}} \boldsymbol{m}_b + \sum_{i=1}^{N} \boldsymbol{\Gamma}^T \frac{\partial (\boldsymbol{C}_{ba}^T \mathbf{r}_b^i)^T}{\partial \boldsymbol{q}} \mathbf{f}_a^i$$

where Eqs. (6) and (7), or (12) and (13), have been used to simplify. Next, using the transpose of identities 1, 2, and 3 and then simplifying the result leads to

$$\boldsymbol{J}_b \dot{\boldsymbol{\omega}}_b^{ba} + \boldsymbol{\omega}_b^{ba^\times} \boldsymbol{J}_b \boldsymbol{\omega}_b^{ba} = -(\boldsymbol{C}_{ba}\boldsymbol{b}_a)^\times \boldsymbol{m}_b + \sum_{i=1}^{N} \mathbf{r}_b^{i^\times} \boldsymbol{C}_{ba} \mathbf{f}_a^i$$

$$= -\boldsymbol{b}_b^\times \boldsymbol{m}_b + \sum_{i=1}^{N} \mathbf{r}_b^{i^\times} \mathbf{f}_b^i \tag{20}$$

where $\mathbf{f}_b^i = \boldsymbol{C}_{ba}\mathbf{f}_a^i$ and $\boldsymbol{b}_b = \boldsymbol{C}_{ba}\boldsymbol{b}_a$. Observe that the torque $-\boldsymbol{b}_b^\times \boldsymbol{m}_b = \boldsymbol{m}_b^\times \boldsymbol{b}_b$ stems from the potential energy associated with the residual magnetic dipole. Also, as expected, the generalized torques (which stem from the virtual work) lead to torques of the form $\boldsymbol{\tau}_b^i = \mathbf{r}_b^{i^\times} \mathbf{f}_b^i$ when identity 3 is employed. Augmenting the motion equations in Eq. (20) with the appropriate kinematics (e.g., Eqs. (3) or (9)) yields the rotational equations of motion that govern the motion of the rigid body.

E. Remarks on the Example

The intent of the example developed in Sec. IV.A–D is to highlight the utility of identities 1, 2, and 3, not to imply that the motion equations cannot be derived any other way. However, the derivation is quite straightforward and perhaps more attractive than other approaches for a variety of reasons. For instance, the introduction of quasi-coordinates and the Boltzmann–Hamel equation is avoided. Also, identities 1, 2, and 3 enable derivation of the equations of motion in a matrix form; scalar equations and index nation are not needed.

Observe that neither $\boldsymbol{C}_{ba}$ nor $\boldsymbol{q}$ appear explicitly in Eq. (20). This is consistent with the results obtained using a Newton–Euler approach ([2] pp. 55–61, [13] pp. 54–56, [14] pp. 272–278). Also, the derivation of the equations of motion using quaternions does not rely on any assumption of singularity avoidance. This is in contrast to the use of any three-parameter attitude representation where singularities are assumed to be avoided during the derivation.

The derivation presented herein is different than the existing literature in the following ways. First [4,7,15–18], consider the case where the Lagrangian is a function of the kinetic energy only; that is, $L = T$. Second, the form of the Lagrange multiplier is explicitly considered in [4,15,16], whereas in Sec. IV.D, the orthogonal complement method removes the Lagrange multiplier altogether. In [16,18] a 4×4 mass matrix is introduced where one element of the mass matrix is arbitrary; the formulation presented in Sec. IV.A–D avoids doing so. Fourth, in [15–18], the rotational equations of motion are written in terms of $\ddot{\boldsymbol{q}}$, $\dot{\boldsymbol{q}}$, and $\boldsymbol{q}$ (where $\boldsymbol{q}$ is the quaternion), masking the equality between results obtained using a Newton–Euler approach and a Lagrangian approach.

V. Proofs

In this section, identities 1, 2, and 3 will be rigorously shown to hold. In the forthcoming proofs, the various identities summarized in the Appendix will be used.

Theorem 1: Let $\boldsymbol{C}_{ba}$ be parameterized in terms of axis/angle parameters or quaternions, as in Eqs. (2) and (8), where $\boldsymbol{S}$ and $\boldsymbol{\Gamma}$ are given in Eqs. (3) and (4), as well as in Eqs. (9) and (10). Identity 1 holds.

Proof Using Axis/Angle Parameters: The matrix $\dot{\boldsymbol{S}}(\boldsymbol{a},\phi)$ will be computed first. Taking the derivative of $\boldsymbol{S}(\boldsymbol{a},\phi)$ shown in Eq. (4) gives

$$\dot{\boldsymbol{S}}(\boldsymbol{a},\phi) = [\,\dot{\phi}\cos\phi\mathbf{1} - (1-\cos\phi)\dot{\boldsymbol{a}}^\times - \dot{\phi}\sin\phi\boldsymbol{a}^\times \quad \dot{\boldsymbol{a}}\,]$$

Post multiplying $\dot{\boldsymbol{S}}(\boldsymbol{a},\phi)$ by $\boldsymbol{\Gamma}(\boldsymbol{a},\phi)$ given in Eq. (3) yields

$$\dot{\boldsymbol{S}}(\boldsymbol{a},\phi)\boldsymbol{\Gamma}(\boldsymbol{a},\phi) = [\,\dot{\phi}\cos\phi\mathbf{1} - (1-\cos\phi)\dot{\boldsymbol{a}}^\times - \dot{\phi}\sin\phi\boldsymbol{a}^\times \quad \dot{\boldsymbol{a}}\,] \times \begin{bmatrix} \frac{1}{2}(\boldsymbol{a}^\times - \cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times) \\ \boldsymbol{a}^T \end{bmatrix}$$

$$= \frac{1}{2}\Big(\dot{\phi}\cos\phi\boldsymbol{a}^\times - \dot{\phi}\cos\phi\cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times - (1-\cos\phi)\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times$$

$$+ \underbrace{(1-\cos\phi)\cot\frac{\phi}{2}}_{\sin\phi}\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times - \dot{\phi}\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times$$

$$+ \dot{\phi}\underbrace{\sin\phi\cot\frac{\phi}{2}}_{1+\cos\phi}\overbrace{\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times}^{-\boldsymbol{a}^\times} + 2\dot{\boldsymbol{a}}\boldsymbol{a}^T\Big)$$

$$= \frac{1}{2}\Big(\dot{\phi}\cos\phi\boldsymbol{a}^\times - \dot{\phi}\cos\phi\cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times - (1-\cos\phi)\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times$$

$$+ \sin\phi\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times - \dot{\phi}\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times - \dot{\phi}(1+\cos\phi)\boldsymbol{a}^\times + 2\dot{\boldsymbol{a}}\boldsymbol{a}^T\Big)$$

$$= \frac{1}{2}\Big(-\dot{\phi}\cos\phi\cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times - (1-\cos\phi)\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times$$

$$+ \sin\phi\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times - \dot{\phi}\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times - \dot{\phi}\boldsymbol{a}^\times + 2\dot{\boldsymbol{a}}\boldsymbol{a}^T\Big)$$

where various identities, such as $\tan\frac{\phi}{2} = \sin\phi/(1+\cos\phi)$, have been used to simplify. Using Eq. (4), it follows that

$$\frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}} = \begin{bmatrix} \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{a}} & \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \phi} \end{bmatrix}$$

$$= [\,\dot{\phi}\mathbf{1} + (1-\cos\phi)\dot{\boldsymbol{a}}^\times \quad (\cos\phi\mathbf{1} - \sin\phi\boldsymbol{a}^\times)\dot{\boldsymbol{a}}\,]$$

which, upon post multiplying by $\boldsymbol{\Gamma}(\boldsymbol{a},\phi)$, becomes

$$
\begin{aligned}
\frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}} \boldsymbol{\Gamma}(\boldsymbol{a},\phi) &= [\dot{\phi}\mathbf{1} + (1-\cos\phi)\dot{\boldsymbol{a}}^\times \quad (\cos\phi\mathbf{1} - \sin\phi\boldsymbol{a}^\times)\dot{\boldsymbol{a}}] \\
&\times \begin{bmatrix} \frac{1}{2}(\boldsymbol{a}^\times - \cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times) \\ \boldsymbol{a}^T \end{bmatrix} \\
&= \frac{1}{2}\left(\dot{\phi}\boldsymbol{a}^\times - \dot{\phi}\cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times + (1-\cos\phi)\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times - \sin\phi\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times\right) \\
&+ \cos\phi\dot{\boldsymbol{a}}\boldsymbol{a}^T - \sin\phi\boldsymbol{a}^\times\dot{\boldsymbol{a}}\boldsymbol{a}^T
\end{aligned}
$$

Subtracting $\frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{a},\phi)$ from $\dot{\boldsymbol{S}}(\boldsymbol{a},\phi)\boldsymbol{\Gamma}(\boldsymbol{a},\phi)$ gives

$$
\begin{aligned}
\dot{\boldsymbol{S}}(\boldsymbol{a},\phi)\boldsymbol{\Gamma}(\boldsymbol{a},\phi) - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{a},\phi) &= -\dot{\phi}\boldsymbol{a}^\times - (1-\cos\phi)\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times \\
&+ \sin\phi\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times \\
&+ \frac{1}{2}\left(-\dot{\phi}\cos\phi\cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times - \dot{\phi}\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times + \dot{\phi}\cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times\right) \\
&+ \dot{\boldsymbol{a}}\boldsymbol{a}^T - \cos\phi\dot{\boldsymbol{a}}\boldsymbol{a}^T + \sin\phi\boldsymbol{a}^\times\dot{\boldsymbol{a}}\boldsymbol{a}^T \\
&= -\dot{\phi}\boldsymbol{a}^\times - (1-\cos\phi)\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times + \sin\phi\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times \\
&+ (1-\cos\phi)\dot{\boldsymbol{a}}\boldsymbol{a}^T + \sin\phi\boldsymbol{a}^\times\dot{\boldsymbol{a}}\boldsymbol{a}^T \\
&+ \frac{\dot{\phi}}{2}\underbrace{\left(-\cos\phi\frac{(1+\cos\phi)}{\sin\phi} - \frac{\sin^2\phi}{\sin\phi} + \frac{(1+\cos\phi)}{\sin\phi}\right)}_{0}\boldsymbol{a}^\times\boldsymbol{a}^\times \\
&= -\dot{\phi}\boldsymbol{a}^\times - (1-\cos\phi)\boldsymbol{a}\dot{\boldsymbol{a}}^T + (1-\cos\phi)\dot{\boldsymbol{a}}\boldsymbol{a}^T \\
&+ \sin\phi(\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times + \boldsymbol{a}^\times\dot{\boldsymbol{a}}\boldsymbol{a}^T) \\
&= -\dot{\phi}\boldsymbol{a}^\times + (1-\cos\phi)\underbrace{(-\boldsymbol{a}\dot{\boldsymbol{a}}^T + \dot{\boldsymbol{a}}\boldsymbol{a}^T)}_{(\boldsymbol{a}^\times\dot{\boldsymbol{a}})^\times} \\
&+ \sin\phi(\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times - \dot{\boldsymbol{a}}^\times\boldsymbol{a}\boldsymbol{a}^T) \\
&= -\dot{\phi}\boldsymbol{a}^\times + (1-\cos\phi)(\boldsymbol{a}^\times\dot{\boldsymbol{a}})^\times + \sin\phi\dot{\boldsymbol{a}}^\times\underbrace{(\boldsymbol{a}^\times\boldsymbol{a}^\times - \boldsymbol{a}\boldsymbol{a}^T)}_{-\mathbf{1}} = -\boldsymbol{\omega}_b^{ba^\times}
\end{aligned}
$$

where $-\dot{\boldsymbol{a}}^\times\boldsymbol{a}^\times = \dot{\boldsymbol{a}}^T\boldsymbol{a}\mathbf{1} - \boldsymbol{a}\dot{\boldsymbol{a}}^T = \mathbf{0} - \boldsymbol{a}\dot{\boldsymbol{a}}^T$, $\boldsymbol{a}^\times\dot{\boldsymbol{a}} = -\dot{\boldsymbol{a}}^\times\boldsymbol{a}$, and $-\boldsymbol{a}^\times\boldsymbol{a}^\times = \mathbf{1} - \boldsymbol{a}\boldsymbol{a}^T$ have been used to simplify (see ([2] p. 25) for various identities related to $\boldsymbol{a}$ and $\dot{\boldsymbol{a}}$, and the identities in the Appendix). Therefore,

$$
\left(\dot{\boldsymbol{S}}(\boldsymbol{a},\phi) - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}}\right)\boldsymbol{\Gamma}(\boldsymbol{a},\phi) = -\boldsymbol{\omega}_b^{ba^\times}
$$

which is to say, identity 1 holds.

Proof Using Quaternions: First, the matrix $\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta)$ will be computed. From Eq. (10) where $\boldsymbol{S}(\boldsymbol{\epsilon},\eta)$ is given, it follows that

$$
\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta) = [2(\dot{\eta}\mathbf{1} - \dot{\boldsymbol{\epsilon}}^\times) \quad -2\dot{\boldsymbol{\epsilon}}]
$$

Using Eq. (10) again, we can compute

$$
\frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}} = \left[\frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{\epsilon}} \quad \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \eta}\right] = [2(-\dot{\eta}\mathbf{1} + \dot{\boldsymbol{\epsilon}}^\times) \quad 2\dot{\boldsymbol{\epsilon}}] = -\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta)
$$

Thus, the left-hand side of identity 1 can be written

$$
\left(\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta) - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}}\right)\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta) = 2\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta)\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)
$$

Evaluating $2\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta)\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)$ explicitly yields

$$
\begin{aligned}
2\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta)\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta) &= 2[2(\dot{\eta}\mathbf{1} - \dot{\boldsymbol{\epsilon}}^\times) - 2\dot{\boldsymbol{\epsilon}}]\begin{bmatrix} \frac{1}{2}(\eta\mathbf{1} + \boldsymbol{\epsilon}^\times) \\ -\frac{1}{2}\boldsymbol{\epsilon}^T \end{bmatrix} \\
&= 2(\dot{\eta}\mathbf{1} - \dot{\boldsymbol{\epsilon}}^\times)(\eta\mathbf{1} + \boldsymbol{\epsilon}^\times) + 2\dot{\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^T \\
&= 2(\dot{\eta}\eta\mathbf{1} + \dot{\eta}\boldsymbol{\epsilon}^\times - \eta\dot{\boldsymbol{\epsilon}}^\times - \dot{\boldsymbol{\epsilon}}^\times\boldsymbol{\epsilon}^\times + \dot{\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^T)
\end{aligned}
$$

Recall the identity $-\boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}}^\times = \boldsymbol{\epsilon}^T\dot{\boldsymbol{\epsilon}}\mathbf{1} - \dot{\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^T$ (see the Appendix); substituting $\dot{\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^T = \boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}}^\times + \boldsymbol{\epsilon}^T\dot{\boldsymbol{\epsilon}}\mathbf{1}$ into the last line gives

$$
\begin{aligned}
&2(\underbrace{(\dot{\eta}\eta + \boldsymbol{\epsilon}^T\dot{\boldsymbol{\epsilon}})}_{0}\mathbf{1} + \dot{\eta}\boldsymbol{\epsilon}^\times - \eta\dot{\boldsymbol{\epsilon}}^\times - \dot{\boldsymbol{\epsilon}}^\times\boldsymbol{\epsilon}^\times + \boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}}^\times) \\
&= -2(\eta\dot{\boldsymbol{\epsilon}}^\times - \dot{\eta}\boldsymbol{\epsilon}^\times + \dot{\boldsymbol{\epsilon}}^\times\boldsymbol{\epsilon}^\times - \boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}}^\times) \\
&= -2(\eta\dot{\boldsymbol{\epsilon}}^\times - \dot{\eta}\boldsymbol{\epsilon}^\times - (\boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}})^\times) \quad = -\boldsymbol{\omega}_b^{ba^\times}
\end{aligned}
$$

where the identities $(\dot{\boldsymbol{\epsilon}}^\times\boldsymbol{\epsilon})^\times = \dot{\boldsymbol{\epsilon}}^\times\boldsymbol{\epsilon}^\times - \boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}}^\times$ and $\dot{\boldsymbol{\epsilon}}^\times\boldsymbol{\epsilon} = -\boldsymbol{\epsilon}^\times\dot{\boldsymbol{\epsilon}}$ have been used to simplify. It follows that

$$
\left(\dot{\boldsymbol{S}}(\boldsymbol{\epsilon},\eta) - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \boldsymbol{q}}\right)\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta) = -\boldsymbol{\omega}_b^{ba^\times}
$$

which is to say, identity 1 holds.

Theorem 2: Let $\boldsymbol{C}_{ba}$ be parameterized in terms of axis/angle parameters or quaternions, as in Eqs. (2) and (8), where $\boldsymbol{\Gamma}$ and $\boldsymbol{S}$ are given in Eqs. (3) and (4), as well as in Eqs. (9) and (10). Identity 2 holds for any $\boldsymbol{v}_a \in \mathbb{R}^3$.

Proof Using Axis/Angle Parameters: To be concise, $\boldsymbol{C}_{ba}(\boldsymbol{a},\phi)$ will be written $\boldsymbol{C}_{ba}$. To begin, $\boldsymbol{C}_{ba}\boldsymbol{v}_a$ and $(\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times$ will be computed. Using Eq. (2), it follows that

$$
\boldsymbol{C}_{ba}\boldsymbol{v}_a = \cos\phi\boldsymbol{v}_a + (1-\cos\phi)\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v}_a - \sin\phi\boldsymbol{a}^\times\boldsymbol{v}_a \tag{21}
$$

$$
\begin{aligned}
(\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times &= (\cos\phi\boldsymbol{v}_a + (1-\cos\phi)\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v}_a - \sin\phi\boldsymbol{a}^\times\boldsymbol{v}_a)^\times \\
&= \cos\phi\boldsymbol{v}_a^\times + (1-\cos\phi)\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_a - \sin\phi(\boldsymbol{a}^\times\boldsymbol{v}_a)^\times \\
&= \cos\phi\boldsymbol{v}_a^\times + (1-\cos\phi)\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_a - \sin\phi\boldsymbol{a}^\times\boldsymbol{v}_a^\times + \sin\phi\boldsymbol{v}_a^\times\boldsymbol{a}^\times
\end{aligned} \tag{22}
$$

where the identity $(\boldsymbol{a}^\times\boldsymbol{v}_a)^\times = \boldsymbol{a}^\times\boldsymbol{v}_a^\times - \boldsymbol{v}_a^\times\boldsymbol{a}^\times$ has been used. Next,

$$
\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{q}} = \left[\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{a}} \quad \frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \phi}\right]
$$

will be computed, starting with $\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{a}}$. Using Eq. (21),

$$
\begin{aligned}
\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{a}} &= (1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_a\mathbf{1} + (1-\cos\phi)\boldsymbol{a}\boldsymbol{v}_a^T + \sin\phi\boldsymbol{v}_a^\times \\
&= 2(1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_a\mathbf{1} + (1-\cos\phi)(-\boldsymbol{v}_a^T\boldsymbol{a}\mathbf{1} + \boldsymbol{a}\boldsymbol{v}_a^T) + \sin\phi\boldsymbol{v}_a^\times \\
&= 2(1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_a\mathbf{1} + (1-\cos\phi)\boldsymbol{v}_a^\times\boldsymbol{a}^\times + \sin\phi\boldsymbol{v}_a^\times
\end{aligned}
$$

where $\boldsymbol{v}_a^\times\boldsymbol{a}^\times = -\boldsymbol{v}_a^T\boldsymbol{a}\mathbf{1} + \boldsymbol{a}\boldsymbol{v}_a^T$ has been used. Using Eq. (21) once again,

$$
\begin{aligned}
\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \phi} &= -\sin\phi\boldsymbol{v}_a + \sin\phi\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v}_a - \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_a \\
&= \sin\phi(-\mathbf{1} + \boldsymbol{a}\boldsymbol{a}^T)\boldsymbol{v}_a - \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_a \\
&= \sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{v}_a - \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_a
\end{aligned}
$$

where $\boldsymbol{a}^\times\boldsymbol{a}^\times = -\mathbf{1} + \boldsymbol{a}\boldsymbol{a}^T$ has been used to simplify.

Using the expression for $\boldsymbol{\Gamma}(\boldsymbol{a},\phi)$ given in Eq. (3), along with $\cot\frac{\phi}{2} = \sin\phi/(1-\cos\phi) = (1+\cos\phi)/\sin\phi$, the left-hand side of identity 2 will be evaluated:

$$\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{a},\phi) = \left[\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{a}}\ \frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \phi}\right]\underbrace{\begin{bmatrix}\frac{1}{2}\left(\boldsymbol{a}^\times - \cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times\right)\\ \boldsymbol{a}^T\end{bmatrix}}_{\boldsymbol{\Gamma}(\boldsymbol{a},\phi)}$$

$$\begin{aligned}
&= (1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_a\boldsymbol{a}^\times - \sin\phi\boldsymbol{a}^T\boldsymbol{v}_a\boldsymbol{a}^\times\boldsymbol{a}^\times\\
&+\frac{1}{2}(1-\cos\phi)\boldsymbol{v}_a^\times\boldsymbol{a}^\times\boldsymbol{a}^\times - \frac{1}{2}\sin\phi\boldsymbol{v}_a^\times\underbrace{\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times}_{-\boldsymbol{a}^\times}\\
&+\frac{1}{2}\sin\phi\boldsymbol{v}_a^\times\boldsymbol{a}^\times - \frac{1}{2}(1+\cos\phi)\boldsymbol{v}_a^\times\boldsymbol{a}^\times\boldsymbol{a}^\times\\
&+\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{v}_a\boldsymbol{a}^T - \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_a\boldsymbol{a}^T\\
&= (1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_a\boldsymbol{a}^\times + \sin\phi\boldsymbol{v}_a^\times\boldsymbol{a}^\times - \cos\phi\boldsymbol{v}_a^\times\underbrace{\boldsymbol{a}^\times\boldsymbol{a}^\times}_{-\mathbf{1}+\boldsymbol{a}\boldsymbol{a}^T}\\
&+\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times\underbrace{(-\boldsymbol{a}^T\boldsymbol{v}_a\mathbf{1}+\boldsymbol{v}_a\boldsymbol{a}^T)}_{\boldsymbol{a}^\times\boldsymbol{v}_a^\times} - \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_a\boldsymbol{a}^T\\
&= (1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_a\boldsymbol{a}^\times + \sin\phi\boldsymbol{v}_a^\times\boldsymbol{a}^\times + \cos\phi\boldsymbol{v}_a^\times - \cos\phi\boldsymbol{v}_a^\times\boldsymbol{a}\boldsymbol{a}^T\\
&+\sin\phi\underbrace{\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times}_{-\boldsymbol{a}^\times}\boldsymbol{v}_a^\times - \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_a\boldsymbol{a}^T\\
&= \underbrace{\cos\phi\boldsymbol{v}_a^\times + (1-\cos\phi)\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_a - \sin\phi\boldsymbol{a}^\times\boldsymbol{v}_a^\times + \sin\phi\boldsymbol{v}_a^\times\boldsymbol{a}^\times}_{(\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times \text{ via (22)}}\\
&- \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_a\boldsymbol{a}^T - \cos\phi\boldsymbol{v}_a^\times\boldsymbol{a}\boldsymbol{a}^T\\
&= (\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times - \cos\phi(\boldsymbol{a}^\times\boldsymbol{v}_a + \boldsymbol{v}_a^\times\boldsymbol{a})\boldsymbol{a}^T\\
&= (\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times - \cos\phi(-\boldsymbol{v}_a^\times\boldsymbol{a} + \boldsymbol{v}_a^\times\boldsymbol{a})\boldsymbol{a}^T = (\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times
\end{aligned}$$

Thus,

$$\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{a},\phi) = (\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times$$

Proof Using Quaternions: As shown earlier, $\boldsymbol{C}_{ba}(\boldsymbol{\epsilon},\eta)$ will be written $\boldsymbol{C}_{ba}$. To start, the terms $\boldsymbol{C}_{ba}\boldsymbol{v}_a$ and $(\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times$ will be computed first. Using Eq. (8), it follows that

$$\begin{aligned}
\boldsymbol{C}_{ba}\boldsymbol{v}_a &= \boldsymbol{v}_a + 2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times\boldsymbol{v}_a - 2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_a\\
&= \boldsymbol{v}_a - 2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times\boldsymbol{\epsilon} + 2\eta\boldsymbol{v}_a^\times\boldsymbol{\epsilon} \qquad (23)
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times &= \boldsymbol{v}_a^\times + 2(\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times\boldsymbol{v}_a)^\times - 2\eta(\boldsymbol{\epsilon}^\times\boldsymbol{v}_a)^\times\\
&= \boldsymbol{v}_a^\times + 2(-\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\boldsymbol{v}_a + \boldsymbol{\epsilon}\boldsymbol{\epsilon}^T\boldsymbol{v}_a)^\times - 2\eta(\boldsymbol{\epsilon}^\times\boldsymbol{v}_a)^\times\\
&= \boldsymbol{v}_a^\times - 2\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\boldsymbol{v}_a^\times + 2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^T\boldsymbol{v}_a - 2\eta(\boldsymbol{\epsilon}^\times\boldsymbol{v}_a)^\times \qquad (24)
\end{aligned}$$

Next,

$$\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{q}} = \left[\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{\epsilon}}\ \frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \eta}\right]$$

will be computed. Starting with $\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial\boldsymbol{\epsilon}}$, and using Eq. (23),

$$\begin{aligned}
\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{\epsilon}} &= -2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times + 2(\boldsymbol{v}_a^\times\boldsymbol{\epsilon})^\times + 2\eta\boldsymbol{v}_a^\times\\
&= -2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times + 2\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times - 2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times + 2\eta\boldsymbol{v}_a^\times\\
&= 2\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times + 2\eta\boldsymbol{v}_a^\times - 4\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times
\end{aligned}$$

Using Eq. (23) once again,

$$\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \eta} = 2\boldsymbol{v}_a^\times\boldsymbol{\epsilon}$$

The left-hand side of identity 2 can now be computed. To do so, $\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)$ given in Eq. (9) will be used. Then, using Eq. (9), the left-hand side of identity 2 is

$$\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta) = \left[\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{\epsilon}}\ \frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \eta}\right]\underbrace{\begin{bmatrix}\frac{1}{2}(\eta\mathbf{1}+\boldsymbol{\epsilon}^\times)\\ -\frac{1}{2}\boldsymbol{\epsilon}^T\end{bmatrix}}_{\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)}$$

$$\begin{aligned}
&= \eta\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times + \eta^2\boldsymbol{v}_a^\times - 2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times + \boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times + \eta\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times\\
&- 2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times - \boldsymbol{v}_a^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T\\
&= 2\eta\underbrace{(\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times - \boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times)}_{(\boldsymbol{v}_a^\times\boldsymbol{\epsilon})^\times} + \eta^2\boldsymbol{v}_a^\times - \boldsymbol{v}_a^\times\underbrace{(-\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times + \boldsymbol{\epsilon}\boldsymbol{\epsilon}^T)}_{\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\mathbf{1}} - 2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times\\
&= 2\eta(\boldsymbol{v}_a^\times\boldsymbol{\epsilon})^\times + (\eta^2 - \boldsymbol{\epsilon}^T\boldsymbol{\epsilon})\boldsymbol{v}_a^\times - 2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times\\
&= \boldsymbol{v}_a^\times - 2\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\boldsymbol{v}_a^\times - 2\boldsymbol{\epsilon}^\times\boldsymbol{v}_a^\times\boldsymbol{\epsilon}^\times + 2\eta(\boldsymbol{v}_a^\times\boldsymbol{\epsilon})^\times\\
&= \boldsymbol{v}_a^\times - 2\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\boldsymbol{v}_a^\times + 2(\boldsymbol{\epsilon}^T\boldsymbol{v}_a\mathbf{1} - \boldsymbol{v}_a\boldsymbol{\epsilon}^T)\boldsymbol{\epsilon}^\times + 2\eta(\boldsymbol{v}_a^\times\boldsymbol{\epsilon})^\times\\
&= \boldsymbol{v}_a^\times - 2\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\boldsymbol{v}_a^\times + 2\boldsymbol{\epsilon}^T\boldsymbol{v}_a\boldsymbol{\epsilon}^\times - 2\eta(\boldsymbol{\epsilon}^\times\boldsymbol{v}_a)^\times = (\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times
\end{aligned}$$

As such,

$$\frac{\partial(\boldsymbol{C}_{ba}\boldsymbol{v}_a)}{\partial \boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta) = (\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times$$

which is identity 2.

Theorem 3: Let $\boldsymbol{C}_{ba}$ be parameterized in terms of axis/angle parameters or quaternions, as in Eqs. (2) and (8), where $\boldsymbol{\Gamma}$ and $\boldsymbol{S}$ are given in Eqs. (3) and (4), as well as in Eqs. (9) and (10). Identity 3 holds for any $\boldsymbol{v}_b \in \mathbb{R}^3$.

Proof Using Axis/Angle Parameters: To be concise, $\boldsymbol{C}_{ba}(\boldsymbol{a},\phi)$ will be written $\boldsymbol{C}_{ba}$ once again. Using Eq. (2), the right-hand side of identity 3 is

$$-\boldsymbol{C}_{ba}^T\boldsymbol{v}_b^\times = -\cos\phi\boldsymbol{v}_b^\times - (1-\cos\phi)\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v}_b^\times - \sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b^\times \qquad (25)$$

Next, the computation of

$$\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial \boldsymbol{q}} = \left[\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial \boldsymbol{a}}\ \frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial \phi}\right]$$

will be considered. Observe that

$$\boldsymbol{C}_{ba}^T\boldsymbol{v}_b = \cos\phi\boldsymbol{v}_b + (1-\cos\phi)\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v}_b + \sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b \qquad (26)$$

Using Eq. (26), the term $\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{a}}$ is

$$\begin{aligned}
\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial \boldsymbol{a}} &= (1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_b\mathbf{1} + (1-\cos\phi)\boldsymbol{a}\boldsymbol{v}_b^T - \sin\phi\boldsymbol{v}_b^\times\\
&= 2(1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_b\mathbf{1} + (1-\cos\phi)\underbrace{(-\boldsymbol{v}_b^T\boldsymbol{a}\mathbf{1} + \boldsymbol{a}\boldsymbol{v}_b^T)}_{\boldsymbol{v}_b^\times\boldsymbol{a}^\times} - \sin\phi\boldsymbol{v}_b^\times\\
&= 2(1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_b\mathbf{1} + (1-\cos\phi)\boldsymbol{v}_b^\times\boldsymbol{a}^\times - \sin\phi\boldsymbol{v}_b^\times
\end{aligned}$$

Similarly, using Eq. (26), the $\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\phi}$ term is

$$\begin{aligned}
\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial \phi} &= -\sin\phi\boldsymbol{v}_b + \sin\phi\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v}_b + \cos\phi\boldsymbol{a}^\times\boldsymbol{v}_b\\
&= \sin\phi\underbrace{(-\mathbf{1} + \boldsymbol{a}\boldsymbol{a}^T)}_{\boldsymbol{a}^\times\boldsymbol{a}^\times}\boldsymbol{v}_b - \cos\phi\boldsymbol{v}_b^\times\boldsymbol{a}\\
&= \sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{v}_b - \cos\phi\boldsymbol{v}_b^\times\boldsymbol{a}
\end{aligned}$$

It follows that, when various identities such as $\cot\frac{\phi}{2} = \sin\phi/(1-\cos\phi) = (1+\cos\phi)/\sin\phi$ are used, the left-hand side of identity 3 is

$$\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{a},\phi)=\left[\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\mathbf{a}}\;\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\phi}\right]\underbrace{\begin{bmatrix}\frac{1}{2}\left(\boldsymbol{a}^\times-\cot\frac{\phi}{2}\boldsymbol{a}^\times\boldsymbol{a}^\times\right)\\ \boldsymbol{a}^T\end{bmatrix}}_{\boldsymbol{\Gamma}(\boldsymbol{a},\phi)}$$

$$=(1-\cos\phi)\boldsymbol{a}^T\boldsymbol{v}_b\boldsymbol{a}^\times-\sin\phi\boldsymbol{a}^T\boldsymbol{v}_b\boldsymbol{a}^\times\boldsymbol{a}^\times$$
$$+\frac{1}{2}(1-\cos\phi)\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times-\frac{1}{2}\sin\phi\boldsymbol{v}_b^\times\underbrace{\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times}_{-\boldsymbol{a}^\times}$$
$$-\frac{1}{2}\sin\phi\boldsymbol{v}_b^\times\boldsymbol{a}^\times+\frac{1}{2}(1+\cos\phi)\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times$$
$$+\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{v}_b\boldsymbol{a}^T-\cos\phi\boldsymbol{v}_b^\times\boldsymbol{a}\boldsymbol{a}^T$$
$$=(1-\cos\phi)\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_b+\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times$$
$$+\sin\phi\boldsymbol{a}^\times\boldsymbol{a}^\times\underbrace{(-\boldsymbol{a}^T\boldsymbol{v}_b\mathbf{1}+\boldsymbol{v}_b\boldsymbol{a}^T)}_{\boldsymbol{a}^\times\boldsymbol{v}_b^\times}-\cos\phi\boldsymbol{v}_b^\times\underbrace{\boldsymbol{a}\boldsymbol{a}^T}_{\boldsymbol{a}^\times\boldsymbol{a}^\times+\mathbf{1}}$$
$$=(1-\cos\phi)\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_b+\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times+\sin\phi\underbrace{\boldsymbol{a}^\times\boldsymbol{a}^\times\boldsymbol{a}^\times}_{-\boldsymbol{a}^\times}\boldsymbol{v}_b^\times$$
$$+\cos\phi\boldsymbol{v}_b^\times(-\boldsymbol{a}^\times\boldsymbol{a}^\times-\mathbf{1})$$
$$=(1-\cos\phi)\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_b+\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times-\sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b^\times$$
$$-\cos\phi\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times-\cos\phi\boldsymbol{v}_b^\times$$
$$=(1-\cos\phi)\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_b+(1-\cos\phi)\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times-\sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b^\times$$
$$-\cos\phi\boldsymbol{v}_b^\times$$
$$=(1-\cos\phi)(\boldsymbol{a}^\times\boldsymbol{a}^T\boldsymbol{v}_b+\boldsymbol{v}_b^\times\boldsymbol{a}^\times\boldsymbol{a}^\times)-\sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b^\times-\cos\phi\boldsymbol{v}_b^\times$$
$$=(1-\cos\phi)\underbrace{(\boldsymbol{a}^T\boldsymbol{v}_b\mathbf{1}+\boldsymbol{v}_b^\times\boldsymbol{a}^\times)}_{\boldsymbol{a}\boldsymbol{v}_b^T}\boldsymbol{a}^\times-\sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b^\times-\cos\phi\boldsymbol{v}_b^\times$$
$$=(1-\cos\phi)\boldsymbol{a}\boldsymbol{v}_b^T\boldsymbol{a}^\times-\sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b^\times-\cos\phi\boldsymbol{v}_b^\times$$
$$=-(1-\cos\phi)\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{v}_b^\times-\sin\phi\boldsymbol{a}^\times\boldsymbol{v}_b^\times-\cos\phi\boldsymbol{v}_b^\times$$
$$=-\boldsymbol{C}_{ba}^T\boldsymbol{v}_b^\times$$

via Eq. (25), thus proving identity 3.

Proof Using Quaternions: Again, $\boldsymbol{C}_{ba}(\boldsymbol{\epsilon},\eta)$ will be written $\boldsymbol{C}_{ba}$ in order to be concise. Using Eq. (8), the right-hand side of identity 3 is

$$-\boldsymbol{C}_{ba}^T\boldsymbol{v}_b^\times=-\boldsymbol{v}_b^\times-2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times-2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times \tag{27}$$

Next, the computation of

$$\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{q}}=\left[\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{\epsilon}}\;\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\eta}\right]$$

will be considered. Observe that

$$\boldsymbol{C}_{ba}^T\boldsymbol{v}_b=\boldsymbol{v}_b+2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times\boldsymbol{v}_b+2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b=\boldsymbol{v}_b-2\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times\boldsymbol{\epsilon}-2\eta\boldsymbol{v}_b^\times\boldsymbol{\epsilon} \tag{28}$$

Next, using Eq. (28), $\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{\epsilon}}$ and $\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\eta}$ are

$$\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{\epsilon}}=-2\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times+2(\boldsymbol{v}_b^\times\boldsymbol{\epsilon})^\times-2\eta\boldsymbol{v}_b^\times\qquad\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\eta}=-2\boldsymbol{v}_b^\times\boldsymbol{\epsilon}$$

The left-hand side of identity 3 can now be computed. Using the expression for $\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)$ in Eq. (9), it follows that

$$\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{q}}\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)=\left[\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\boldsymbol{\epsilon}}\;\frac{\partial(\boldsymbol{C}_{ba}^T\boldsymbol{v}_b)}{\partial\eta}\right]\underbrace{\begin{bmatrix}\frac{1}{2}(\eta\mathbf{1}+\boldsymbol{\epsilon}^\times)\\ -\frac{1}{2}\boldsymbol{\epsilon}^T\end{bmatrix}}_{\boldsymbol{\Gamma}(\boldsymbol{\epsilon},\eta)}$$

$$=-\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times+\eta(\boldsymbol{v}_b^\times\boldsymbol{\epsilon})^\times-\eta^2\boldsymbol{v}_b^\times-\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times+(\boldsymbol{v}_b^\times\boldsymbol{\epsilon})^\times\boldsymbol{\epsilon}^\times-\eta\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times$$
$$+\boldsymbol{v}_b^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$
$$=-\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times+\eta(\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times-\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times)-\eta^2\boldsymbol{v}_b^\times-\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times$$
$$+(\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times-\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times)\boldsymbol{\epsilon}^\times-\eta\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times+\boldsymbol{v}_b^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$
$$=-\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times+\eta\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times-\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times-(1-\boldsymbol{\epsilon}^T\boldsymbol{\epsilon})\boldsymbol{v}_b^\times-\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times$$
$$+\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times-\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times-\eta\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times+\boldsymbol{v}_b^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$
$$=-2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times-2\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times-\boldsymbol{v}_b^\times+\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\boldsymbol{v}_b^\times+\boldsymbol{v}_b^\times\underbrace{\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times}_{-\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\mathbf{1}+\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T}+\boldsymbol{v}_b^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$
$$=-\boldsymbol{v}_b^\times-2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times-2\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^\times+\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\boldsymbol{v}_b^\times-\boldsymbol{v}_b^\times\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}+\boldsymbol{v}_b^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$
$$+\boldsymbol{v}_b^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$
$$=-\boldsymbol{v}_b^\times-2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times+2(\boldsymbol{\epsilon}^T\boldsymbol{v}_b\mathbf{1}-\boldsymbol{v}_b\boldsymbol{\epsilon}^T)\boldsymbol{\epsilon}^\times+2\boldsymbol{v}_b^\times\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$
$$=-\boldsymbol{v}_b^\times-2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times+2\boldsymbol{\epsilon}^T\boldsymbol{v}_b\boldsymbol{\epsilon}^\times-2\boldsymbol{\epsilon}^\times\boldsymbol{v}_b\boldsymbol{\epsilon}^T$$
$$=-\boldsymbol{v}_b^\times-2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times+2\boldsymbol{\epsilon}^\times(\boldsymbol{\epsilon}^T\boldsymbol{v}_b\mathbf{1}-\boldsymbol{v}_b\boldsymbol{\epsilon}^T)$$
$$=-\boldsymbol{v}_b^\times-2\eta\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times-2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times\boldsymbol{v}_b^\times=-\boldsymbol{C}_{ba}^T\boldsymbol{v}_b^\times$$

where the last line follows from Eq. (27), proving identity 3.

VI. Conclusions

This note rigorously proves three identities that are useful for deriving the equations of motion of mechanical systems in matrix form using Lagrange's equation when attitude is parameterized using axis/angle parameters or quaternions. The utility of the three identities has been highlighted through an example. Specifically, the equations of motion of a rigid-body spacecraft perturbed by a residual magnetic disturbance torque and actuated by N thrusters has been derived. In the future, the utility of the three identities in other engineering fields such as estimation and control will be considered.

Appendix A: Various Identities

The following identities have been employed throughout this note:

$$(\boldsymbol{u}+\boldsymbol{v})^\times=\boldsymbol{u}^\times+\boldsymbol{v}^\times,$$
$$\boldsymbol{u}^\times\boldsymbol{v}=-\boldsymbol{v}^\times\boldsymbol{u},$$
$$-\boldsymbol{u}^\times\boldsymbol{v}^\times=(\boldsymbol{u}^T\boldsymbol{v})\mathbf{1}-\boldsymbol{v}\boldsymbol{u}^T,$$
$$(\boldsymbol{u}^\times\boldsymbol{v})^\times=\boldsymbol{u}^\times\boldsymbol{v}^\times-\boldsymbol{v}^\times\boldsymbol{u}^\times=-\boldsymbol{u}\boldsymbol{v}^T+\boldsymbol{v}\boldsymbol{u}^T,$$
$$(\boldsymbol{C}_{ba}\boldsymbol{v}_a)^\times=\boldsymbol{C}_{ba}\boldsymbol{v}_a^\times\boldsymbol{C}_{ba}^T$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are arbitrary 3×1 column matrices, $\mathbf{1}$ is the identity matrix (of appropriate dimension), and

$$\boldsymbol{v}^\times=\begin{bmatrix}0 & -v_3 & v_2\\ v_3 & 0 & -v_1\\ -v_2 & v_1 & 0\end{bmatrix}$$

for any arbitrary 3×1 column matrix $\boldsymbol{v}=[v_1\quad v_2\quad v_3]^T$. These identities can be found throughout [2,13].

References

[1] Greenwood, D. T., *Advanced Dynamics*, Cambridge Univ. Press, New York, 2003.

[2] Hughes, P. C., *Spacecraft Attitude Dynamics*, 2nd ed., Dover, New York, 2004.

[3] Laulusa, A., and Bauchau, O. A., "Review of Classical Approaches for Constraint Enforcement in Multibody Systems," *ASME Journal of*

Computational and Nonlinear Dynamics, Vol. 3, No. 1, Jan. 2008, Paper 011004-1–011004-8.
doi:10.1115/1.2803257

[4] Nikravesh, P. E., Wehage, R. A., and Kwon, O. K., "Euler Parameters in Computational Kinematics and Dynamics: Parts 1 and 2," *ASME Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 107, No. 3, Sept. 1985, pp. 358–369.

[5] Quinn, R. D., "Equations of Motion for Structures in Terms of Quasicoordinates," *ASME Journal of Applied Mechanics*, Vol. 57, No. 3, 1990, pp. 745–749.
doi:10.1115/1.2897086

[6] Meirovitch, L., and Stemple, T., "Hybrid Equations of Motion for Flexible Multibody Systems Using Quasicoordinates," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 4, 1995, pp. 678–688.
doi:10.2514/3.21447

[7] Hurtado, J. E., "Hamel Coefficients for the Rotational Motion of a Rigid Body," *Journal of the Astronautical Sciences*, Vol. 52, No. 1–2, 2004, pp. 129–147.

[8] D'Eleuterio, G. M. T., and Barfoot, T. D., "A Discrete Quasicoordinate Formulation for the Dynamics of Elastic Bodies," *ASME Journal of Applied Mechanics*, Vol. 74, No. 2, 2007, pp. 231–239.
doi:10.1115/1.2189873

[9] Barfoot, T., Forbes, J. R., and Furgale, P. T., "Pose Estimation Using Linearized Rotations and Quaternion Algebra," *Acta Astronautica*, Vol. 68, Nos. 1–2, 2011, pp. 101–112.
doi:10.1016/j.actaastro.2010.06.049

[10] Schaub, H., and Junkins, J. L., *Analytical Mechanics of Space Systems*, 2nd ed., AIAA, Reston, VA, 2009.

[11] Griffiths, D. J., *Introduction to Electrodynamics*, 3rd ed., Prentice–Hall, Upper Saddle River, NJ, 1999.

[12] Wertz, J. R., *Spacecraft Attitude Determination and Control*, Reidel, Dordrecht, The Netherlands, 1978.

[13] de Ruiter, A. H., Damaren, C. J., and Forbes, J. R., *Spacecraft Dynamics and Control: An Introduction*, Wiley, New York, 2013.

[14] O'Reilly, O., *Intermediate Dynamics for Engineers*, Cambridge Univ. Press, New York, 2008.

[15] Vadali, S., "On the Euler Parameter Constraint," *Journal of the Astronautical Sciences*, Vol. 36, No. 3, 1988, pp. 259–265.

[16] Morton, H. S., Jr., "Hamiltonian and Lagrangian Formulations of Rigid-Body Rotational Dynamics Using Euler Parameters," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 569–591.

[17] Hurtado, J. E., and Sinclair, A. J., "Lagrangian Mechanics of Overparameterized Systems," *Nonlinear Dynamics*, Vol. 66, Nos. 1–2, 2011, pp. 201–212.
doi:10.1007/s11071-010-9921-1

[18] Udwadia, F. E., and Schutte, A. D., "An Alternative Derivation of the Quaternion Equations of Motion for Rigid-Body Rotational Dynamics," *ASME Journal of Applied Mechanics*, Vol. 77, No. 4, July 2010, Paper 044505-1–044505-4.
doi:10.1115/1.4000917

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