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Engineering Notes

Norm-Constrained Consider Kalman Filtering

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I. Introduction

In many engineering applications, knowledge of a system's state is needed. Because states are rarely measured directly, the states are usually estimated using noise corrupted measurements. To improve upon the accuracy of the estimate, information from the measurements can be combined with knowledge of the system's assumed dynamics. This is the essence of state estimation.

In real-time applications, sequential state estimation methods are often employed. Sequential estimators operate by using the most recent measurements and state estimate. For linear systems, a popular method that yields an optimal solution is the Kalman filter [1]. Two nonlinear variants of the Kalman filter are the extended Kalman filter (EKF) ([2] pp. 400–409) and the unscented Kalman filter (UKF) [3].

State estimation in the presence of state constraints must be handled with care. Two popular approaches for addressing state constraints within the Kalman filter framework are the pseudomeasurement method and the projection method. The pseudomeasurement method, as discussed in [4–6], augments the Kalman filter by considering the constraint as a perfect measurement without noise. The drawback to this method is that it causes the measurement noise covariance to be singular, which in turn can lead to numerical problems when implemented [7]. The projection method presented in [8] projects the unconstrained solution to the Kalman filter onto a constraint surface for linear equality constraints. This work was extended in [9] to accommodate nonlinear equality constraints. In [10], the projection method is applied to the UKF. Further information concerning these methods and other algorithms for handling linear and nonlinear constraints applied to the discrete-time Kalman filter can be found in [7].

Estimating the state can be further complicated if additional parameters, such as measurement biases, are poorly known and left unaccounted for [11,12]. This issue can be addressed by either augmenting the filter's state vector to estimate parameters directly or accounting for the uncertainty by using the approach taken in the Schmidt–Kalman filter. The Schmidt–Kalman filter or consider Kalman filter (CKF) accounts for uncertainty in parameters used in the filter's process and measurement models by updating the state and covariance using an estimated parameter covariance [11,13]. This approach can yield computational and processing savings if the number of parameters is large. Derivation and further discussion of the CKF can be found in [11], ([14] pp. 281–286), and ([15] pp. 387–438). There has been some recent interest in consider analysis. Consider analysis has been applied to the problem of planetary entry [16] and orbit determination [12]. A UKF using the consider framework is derived in [17], consider square-root filters and smoothers are developed in [18], and a UDU formulation of the consider filter is presented in [19].

The contribution of this work is the presentation of a consider filter applicable to both discrete and hybrid systems subject to a norm-constrained state estimate. This is accomplished by adjusting the Kalman gain to solve the minimum variance estimation problem, while taking into account the constraint on the state estimate, as was done in [20]. Section II reviews the unconstrained discrete and hybrid forms of the minimum variance CKF. Section III outlines the discrete and hybrid norm-constrained CKF (NCCKF). Finally, in Sec. IV, the NCCKF is applied to a nonlinear attitude estimation problem through the EKF framework, and its performance is compared with the norm-constrained Kalman filter developed in [20] through a numerical simulation example.

II. Preliminaries

The linear discrete-time dynamic system to which the CKF is applied is given by [11]

$$\mathbf{x}_k = \mathbf{F}_{x,k-1}\mathbf{x}_{k-1} + \mathbf{F}_{p,k-1}\mathbf{p} + \mathbf{G}_{k-1}\mathbf{w}_{k-1}, \qquad \mathbf{w}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state at time t_k, $\mathbf{p}$ is a set of uncertain parameters on which the system depends, and $\mathbf{w}_{k-1}$ is the zero-mean Gaussian process noise with covariance $\mathbf{Q}_{k-1} = \mathbf{Q}_{k-1}^T \geq \mathbf{0}$. The corresponding continuous-time dynamic system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{F}_x(t)\mathbf{x}(t) + \mathbf{F}_p(t)\mathbf{p} + \mathbf{G}(t)\mathbf{w}(t), \quad \mathbf{w}(t) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}(t)\delta(t-\tau))$$

For discrete-time measurements at time t_k, the measurement model is given by

$$\mathbf{y}_k = \mathbf{H}_{x,k}\mathbf{x}_k + \mathbf{H}_{p,k}\mathbf{p} + \mathbf{M}_k\mathbf{v}_k, \qquad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$

where $\mathbf{y}_k$ is the measurement vector at time t_k, $\mathbf{M}_k$ has full row rank, and $\mathbf{v}_k$ is the measurement noise with covariance $\mathbf{R}_k = \mathbf{R}_k^T > \mathbf{0}$.

The derivation of the minimum variance CKF for both the discrete-time and hybrid systems is presented in [11]. The minimum variance CKF is obtained by including the consider states along with the original states in the state vector and partitioning the gain and the covariance matrices with respect to the original states and the consider states. By constraining the gain partition multiplying the consider state to zero, the minimum variance gain for the original states is found and the update equations for the consider states are discarded. These filters are summarized in [11].

III. Norm-Constrained Consider Kalman Filter

A. Fully Constrained State

Consider the same systems in Sec. II. It is desired that the posterior estimate $\hat{\mathbf{x}}_k$ satisfy the constraint $\|\hat{\mathbf{x}}_k\| = \sqrt{\ell}$. This can be expressed as

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$$\hat{\mathbf{x}}_k^T \hat{\mathbf{x}}_k = \ell \tag{1}$$

To enforce this constraint in the correction step of the CKF, the objective function for the minimum variance estimator $J_k = tr\{\mathbf{P}_{xx,k}\}$ is augmented with the constraint using a Lagrange multiplier:

$$\hat{J}_k = tr(\mathbf{P}_{xx,k}) + \lambda_k(\hat{\mathbf{x}}_k^T \hat{\mathbf{x}}_k - \ell)$$

Following [11], the updated covariance is given by

$$\begin{aligned} \mathbf{P}_{xx,k} &= \mathbf{P}_{xx,k}^- - \mathbf{K}_k(\mathbf{H}_{x,k}\mathbf{P}_{xx,k}^- + \mathbf{H}_{p,k}\mathbf{P}_{px,k}^-) \\ &\quad - (\mathbf{P}_{xx,k}^- \mathbf{H}_{x,k}^T + \mathbf{P}_{xp,k}\mathbf{H}_{p,k}^T)\mathbf{K}_k^T + \mathbf{K}_k \mathbf{W}_k \mathbf{K}_k^T \end{aligned} \tag{2}$$

where $\mathbf{W}_k = \mathbf{H}_{x,k}\mathbf{P}_{xx,k}^-\mathbf{H}_{x,k}^T + \mathbf{H}_{x,k}\mathbf{P}_{xp,k}^-\mathbf{H}_{p,k}^T + \mathbf{H}_{p,k}\mathbf{P}_{px,k}^-\mathbf{H}_{x,k}^T + \mathbf{H}_{p,k}\mathbf{P}_{pp,k}\mathbf{H}_{p,k}^T + \mathbf{M}_k\mathbf{R}_k\mathbf{M}_k^T$. This equation has a similar form to the update covariance for the regular Kalman filter

$$\mathbf{P}_k = \mathbf{P}_k^- - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^T\mathbf{K}_k + \mathbf{K}_k\mathbf{V}_k\mathbf{K}_k^T$$

where $\mathbf{V}_k = \mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{M}_k\mathbf{R}_k\mathbf{M}_k^T$. Thus, the derivation of the norm-constrained CKF can follow the same methodology as the derivation for the discrete-time norm-constrained Kalman filter presented in [20] by substituting $\mathbf{P}_{xx,k}$ for $\mathbf{P}_k$, $(\mathbf{H}_{x,k}\mathbf{P}_{xx,k}^- + \mathbf{H}_{p,k}\mathbf{P}_{px,k}^-)$ for $\mathbf{H}_k\mathbf{P}_k^-$, $(\mathbf{P}_{xx,k}^-\mathbf{H}_{x,k}^T + \mathbf{P}_{xp,k}\mathbf{H}_{p,k}^T)$ for $\mathbf{P}_k^-\mathbf{H}_k^T$, and $\mathbf{W}_k$ for $\mathbf{V}_k$.

The discrete-time and hybrid minimum variance norm-constrained CKFs are outlined in Table 1. In these equations, the prior estimate is denoted by a superscript $(\cdot)^-$ and the quantities $\hat{\mathbf{p}}_k$ and $\hat{\mathbf{p}}(t)$ denote the values of $\mathbf{p}$ that are assumed by the discrete-time and the hybrid filters. Because the CKF does not estimate these parameters, $\hat{\mathbf{p}}_k = \hat{\mathbf{p}}_{k-1}$ and $\dot{\hat{\mathbf{p}}} = \mathbf{0}$.

Both the discrete-time and the hybrid filters differ only in how the state dynamics are propagated. However, the structure of their gain and update steps displayed in Table 1 uses notation consistent with the discrete-time filter. When applying these equations to the hybrid case, the prior estimates are taken to be the propagated continuous-time values at t_k. For example, the prior estimate for the state covariance is $\mathbf{P}_{xx,k}^- = \mathbf{P}_{xx}(t_k)$.

Table 1 Minimum variance norm-constrained consider Kalman filter

CKF step	Equations
Discrete propagation	$\hat{\mathbf{x}}_k^- = \mathbf{F}_{x,k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{F}_{p,k-1}\hat{\mathbf{p}}_{k-1}$
	$\hat{\mathbf{p}}_k = \hat{\mathbf{p}}_{k-1}$
	$\mathbf{P}_{xx,k}^- = \mathbf{F}_{x,k-1}\mathbf{P}_{xx,k-1}\mathbf{F}_{x,k-1}^T + \mathbf{F}_{x,k-1}\mathbf{P}_{xp,k-1}\mathbf{F}_{p,k-1}^T + \mathbf{F}_{p,k-1}\mathbf{P}_{px,k-1}\mathbf{F}_{x,k-1}^T + \mathbf{F}_{p,k-1}\mathbf{P}_{pp,k-1}\mathbf{F}_{p,k-1}^T + \mathbf{G}_{k-1}\mathbf{Q}_{k-1}\mathbf{G}_{k-1}^T$
	$\mathbf{P}_{xp,k}^- = \mathbf{F}_{x,k-1}\mathbf{P}_{xp,k-1} + \mathbf{F}_{p,k-1}\mathbf{P}_{pp,k-1}$
	$\mathbf{P}_{px,k}^- = \mathbf{P}_{px,k-1}\mathbf{F}_{x,k-1}^T + \mathbf{P}_{pp,k-1}\mathbf{F}_{p,k-1} = \mathbf{P}_{xp,k}^{-T}$
	$\mathbf{P}_{pp,k} = \mathbf{P}_{pp,k-1}$
Continuous propagation	$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}_x(t)\hat{\mathbf{x}}(t) + \mathbf{F}_p(t)\hat{\mathbf{p}}(t)$
	$\dot{\hat{\mathbf{p}}}(t) = \mathbf{0}$
	$\dot{\mathbf{P}}_{xx}(t) = \mathbf{F}_x(t)\mathbf{P}_{xx}(t) + \mathbf{P}_{xx}(t)\mathbf{F}_x^T(t) + \mathbf{P}_{xp}(t)\mathbf{F}_p^T(t) + \mathbf{F}_p(t)\mathbf{P}_{px}(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t)$
	$\dot{\mathbf{P}}_{xp}(t) = \mathbf{F}_x(t)\mathbf{P}_{xp}(t) + \mathbf{F}_p(t)\mathbf{P}_{pp}(t)$
	$\dot{\mathbf{P}}_{pp}(t) = \mathbf{0}$
Gain	$\mathbf{W}_k = \mathbf{H}_{x,k}\mathbf{P}_{xx,k}^-\mathbf{H}_{x,k}^T + \mathbf{H}_{x,k}\mathbf{P}_{xp,k}^-\mathbf{H}_{p,k}^T + \mathbf{H}_{p,k}\mathbf{P}_{px,k}^-\mathbf{H}_{x,k}^T + \mathbf{H}_{p,k}\mathbf{P}_{pp,k}^-\mathbf{H}_{p,k}^T + \mathbf{M}_k\mathbf{R}_k\mathbf{M}_k^T$
	$\tilde{\mathbf{K}}_k = (\mathbf{P}_{xx,k}^-\mathbf{H}_{x,k}^T + \mathbf{P}_{xp,k}^-\mathbf{H}_{p,k})\mathbf{W}_k^{-1}$
	$\mathbf{r}_k = \mathbf{y}_k - \mathbf{H}_{x,k}\hat{\mathbf{x}}_k^- - \mathbf{H}_{p,k}\hat{\mathbf{p}}_k$
	$\tilde{r}_k = \mathbf{r}_k^T\mathbf{W}_k^{-1}\mathbf{r}_k$
	$\tilde{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \tilde{\mathbf{K}}_k\mathbf{r}_k$
	$\mathbf{K}_k = \tilde{\mathbf{K}}_k + (\frac{\sqrt{\ell}}{\lvert\lvert\tilde{\mathbf{x}}_k\rvert\rvert} - 1)\tilde{\mathbf{x}}_k\frac{\mathbf{r}_k^T\mathbf{W}_k^{-1}}{\tilde{r}_k}$
Update	$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k\mathbf{r}_k$
	$\mathbf{P}_{xx,k} = \mathbf{P}_{xx,k}^- - \mathbf{K}_k(\mathbf{H}_{x,k}\mathbf{P}_{xx,k}^- + \mathbf{H}_{p,k}\mathbf{P}_{px,k}^-) - (\mathbf{P}_{xx,k}^-\mathbf{H}_{x,k}^T + \mathbf{P}_{xp,k}^-\mathbf{H}_{p,k}^T)\mathbf{K}_k^T + \mathbf{K}_k\mathbf{W}_k\mathbf{K}_k^T$
	$\mathbf{P}_{xp,k} = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_{x,k})\mathbf{P}_{xp,k}^- - \mathbf{K}_k\mathbf{H}_{p,k}\mathbf{P}_{pp,k}$

B. Partially Constrained State

Suppose now that only part of the state estimate is constrained. Let $\hat{\mathbf{x}}_k = [\hat{\mathbf{q}}_k^T \quad \hat{\mathbf{z}}_k^T]^T$ where $\hat{\mathbf{q}}_k$ satisfies a norm constraint and $\hat{\mathbf{z}}_k$ is unconstrained. The Kalman gain and covariance matrices can be partitioned as

$$\mathbf{K}_k = \begin{bmatrix}\mathbf{K}_{q,k}\\ \mathbf{K}_{z,k}\end{bmatrix}, \qquad \mathbf{P}_{xx,k} = [\mathbf{P}_{1,k} \quad \mathbf{P}_{2,k}] = \begin{bmatrix}\mathbf{P}_{zz,k} & \mathbf{P}_{qz,k}\\ \mathbf{P}_{zq,k} & \mathbf{P}_{qq,k}\end{bmatrix},$$
$$\mathbf{P}_{zp,k} = \begin{bmatrix}\mathbf{P}_{qp,k}\\ \mathbf{P}_{zp,k}\end{bmatrix}$$

Substituting these partitioned matrices into the updated covariance matrix yields

$$\begin{aligned} \mathbf{P}_{xx,k} &= \begin{bmatrix}\mathbf{P}_{qq,k}^- & \mathbf{P}_{qz,k}^-\\ \mathbf{P}_{zq,k}^- & \mathbf{P}_{zz,k}^-\end{bmatrix} - \begin{bmatrix}\mathbf{K}_{q,k}\\ \mathbf{K}_{z,k}\end{bmatrix}(\mathbf{H}_{x,k}[\mathbf{P}_{1,k}^- \quad \mathbf{P}_{2,k}^-] \\ &\quad + \mathbf{H}_{p,k}[\mathbf{P}_{qp,k}^{-T} \quad \mathbf{P}_{zp,k}^{-T}]) \\ &\quad - \left(\begin{bmatrix}\mathbf{P}_{1,k}^{-T}\\ \mathbf{P}_{2,k}^{-T}\end{bmatrix}\mathbf{H}_{x,k}^T + \begin{bmatrix}\mathbf{P}_{qp,k}^-\\ \mathbf{P}_{zp,k}^-\end{bmatrix}\mathbf{H}_{p,k}^T\right)[\mathbf{K}_{q,k}^T \quad \mathbf{K}_{z,k}^T] \\ &\quad + \begin{bmatrix}\mathbf{K}_{q,k}\\ \mathbf{K}_{z,k}\end{bmatrix}\mathbf{W}_k[\mathbf{K}_{q,k}^T \quad \mathbf{K}_{z,k}^T] \end{aligned}$$

Recognizing that $\mathbf{P}_{qz} = \mathbf{P}_{zq}^T$, the partitions of the updated covariance are given by

$$\begin{aligned} \mathbf{P}_{qq,k} &= \mathbf{P}_{qq,k}^- - \mathbf{K}_{q,k}\mathbf{H}_{x,k}\mathbf{P}_{1,k}^- + \mathbf{K}_{q,k}\mathbf{H}_{p,k}\mathbf{P}_{qp,k}^{-T} - \mathbf{P}_{1,k}^{-T}\mathbf{H}_{x,k}^T\mathbf{K}_{q,k}^T \\ &\quad + \mathbf{P}_{qp,k}^-\mathbf{H}_{p,k}^T\mathbf{K}_{q,k}^T + \mathbf{K}_{q,k}\mathbf{W}_k\mathbf{K}_{q,k}^T, \\ \mathbf{P}_{qz,k} &= \mathbf{P}_{qz,k}^- - \mathbf{K}_{q,k}\mathbf{H}_{x,k}\mathbf{P}_{2,k}^- + \mathbf{K}_{q,k}\mathbf{H}_{p,k}\mathbf{P}_{zp,k}^{-T} - \mathbf{P}_{1,k}^{-T}\mathbf{H}_{x,k}^T\mathbf{K}_{z,k}^T \\ &\quad + \mathbf{P}_{qp,k}^-\mathbf{H}_{p,k}^T\mathbf{K}_{z,k}^T - \mathbf{K}_{q,k}\mathbf{W}_k\mathbf{K}_{z,k}^T, \\ \mathbf{P}_{zz,k} &= \mathbf{P}_{zz,k}^- - \mathbf{K}_{z,k}\mathbf{H}_{x,k}\mathbf{P}_{2,k}^- + \mathbf{K}_{z,k}\mathbf{H}_{p,k}\mathbf{P}_{zp,k}^{-T} - \mathbf{P}_{2,k}^{-T}\mathbf{H}_{x,k}^T\mathbf{K}_{q,k}^T \\ &\quad + \mathbf{P}_{zp,k}^-\mathbf{H}_{p,k}^T\mathbf{K}_{z,k}^T + \mathbf{K}_{z,k}\mathbf{W}_k\mathbf{K}_{z,k}^T \end{aligned}$$

Notice that $\mathbf{P}_{qq,k}$ and $\mathbf{P}_{zz,k}$ are only dependent on their respective partitions of the Kalman gain matrix, $\mathbf{K}_{q,k}$ and $\mathbf{K}_{z,k}$. Given that the objective function of the minimum covariance filter takes the trace of $\mathbf{P}_{xx,k}$, then

$$J_k = tr(\mathbf{P}_{xx,k}) = tr(\mathbf{P}_{qq,k}) + tr(\mathbf{P}_{zz,k})$$

Because $\mathbf{P}_{qq,k}$ is only dependent on $\mathbf{K}_{q,k}$ and $\mathbf{P}_{zz,k}$ is only dependent on $\mathbf{K}_{z,k}$, minimizing J_k with respect to $\mathbf{K}_k$ allows for each partition of $\mathbf{K}_k$ to be calculated independent of one another. Thus, $\mathbf{K}_{q,k}$ is given by the gain matrix in Table 1, substituting $\hat{\mathbf{q}}_k$ for $\hat{\mathbf{x}}_k$, and $\mathbf{K}_{z,k}$ can be found using the traditional formulation.

IV. Application to Spacecraft Attitude Estimation

One application to which the norm-constrained CKF can be applied is the spacecraft attitude estimation problem using a unit-length quaternion. It is often the case that the spacecraft is endowed with a rate gyro that measures the spacecraft's angular velocity in the spacecraft's body frame. Unfortunately, gyro measurements are usually corrupted by both noise and a bias. Often this bias is directly estimated. However, modeling the bias is only useful insofar as it helps estimate the attitude. Thus, for this investigation, the quaternion $\mathbf{q}$ will be the state vector and the bias $\boldsymbol{\beta}$ will be considered as a parameter in the consider framework presented.

A. Process Model

For a principal angle of ϕ about an axis $\mathbf{a}$, the quaternion representation is given by

$$\mathbf{q} = \begin{bmatrix} \mathbf{a}\sin\left(\frac{\phi}{2}\right) \\ \cos\left(\frac{\phi}{2}\right) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\epsilon} \\ \eta \end{bmatrix}$$

where $\|\mathbf{a}\| = 1$ and $||\mathbf{q}|| = 1$. By using a power series approach ([21] pp. 457–460), the following discrete-time approximation of the quaternion kinematics can be made:

$$\mathbf{q}_k \approx \left[\mathbf{1}\cos\left(\frac{\|\boldsymbol{\omega}_{k-1}\|\Delta t}{2}\right) + \frac{\boldsymbol{\Omega}(\boldsymbol{\omega}_{k-1})}{\|\boldsymbol{\omega}_{k-1}\|}\sin\left(\frac{\|\boldsymbol{\omega}_{k-1}\|\Delta t}{2}\right)\right]\mathbf{q}_{k-1} = \mathbf{f}_{k-1}(\mathbf{q}_{k-1}, \boldsymbol{\omega}_{k-1}) \tag{3}$$

where Δt is the time elapsed from t_{k-1} and t_k, $\boldsymbol{\omega}_{k-1}$ is the angular velocity at t_{k-1},

$$\boldsymbol{\Omega}(\boldsymbol{\omega}_{k-1}) = \begin{bmatrix} -\boldsymbol{\omega}_{k-1}^{\times} & \boldsymbol{\omega}_{k-1} \\ -\boldsymbol{\omega}_{k-1}^{T} & 0 \end{bmatrix}$$

and the superscript $(\cdot)^{\times}$ denotes the a cross-product matrix as defined in ([21] p. 611). Equation (3) is in fact exact if the angular velocity is constant over Δt.

As previously mentioned, the gyro measurement of the spacecraft angular velocity $\mathbf{u}_k$ is corrupted by noise $\mathbf{w}_{\omega,k}$ and a static bias $\boldsymbol{\beta}$:

$$\mathbf{u}_k = \boldsymbol{\omega}_k + \mathbf{w}_{\omega,k} + \boldsymbol{\beta}$$

An estimate of the angular velocity is given by

$$\hat{\boldsymbol{\omega}}_k = \mathbf{u}_k - \hat{\mathbf{w}}_{\omega,k} - \hat{\boldsymbol{\beta}}_k \tag{4}$$

where $\hat{\mathbf{w}}_{\omega,k} = \mathbf{0}$. It should be noted that in the CKF, $\hat{\boldsymbol{\beta}}_k = \hat{\boldsymbol{\beta}}_{k-1}$.

The norm-constrained CKF presented in this paper is for linear systems. To apply it to the nonlinear attitude estimation problem, the filter can be modified using the EKF framework. This is accomplished by propagating the estimate using the nonlinear equations, assuming no noise and calculating the covariances using linearized Jacobians about the filter's estimate.

The propagation of the estimated state and covariance in discrete time can be found by substituting $\hat{\mathbf{q}}_k$ for $\hat{\mathbf{x}}_k$, $\hat{\boldsymbol{\beta}}_k$ for $\hat{\mathbf{p}}_k$, and $\mathbf{f}_{k-1}(\hat{\mathbf{q}}_{k-1}, \hat{\boldsymbol{\omega}}_{k-1})$ for $(\mathbf{F}_{x,k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{F}_{p,k-1}\hat{\mathbf{p}}_{k-1})$ in Table 1 where $\hat{\boldsymbol{\omega}}_{k-1}$ is given in Eq. (4). The discrete-time Jacobians for the discrete-time covariance propagation equations are

$$\mathbf{F}_{q,k-1} \approx \mathbf{1}\cos\left(\frac{\|\hat{\boldsymbol{\omega}}_{k-1}\|\Delta t}{2}\right) + \frac{\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}_{k-1})}{\|\hat{\boldsymbol{\omega}}_{k-1}\|}\sin\left(\frac{\|\hat{\boldsymbol{\omega}}_{k-1}\|\Delta t}{2}\right),$$

$$\mathbf{F}_{\beta,k-1} \approx \frac{\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}_{k-1})\Xi(\hat{\mathbf{q}}_{k-1})}{\|\hat{\boldsymbol{\omega}}_{k-1}\|^2}\cos\left(\frac{\|\hat{\boldsymbol{\omega}}_{k-1}\|\Delta t}{2}\right) - \frac{\Xi(\hat{\mathbf{q}}_{k-1})}{\|\hat{\boldsymbol{\omega}}_{k-1}\|}\sin\left(\frac{\|\hat{\boldsymbol{\omega}}_{k-1}\|\Delta t}{2}\right) - \frac{\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}_{k-1})\Xi(\hat{\mathbf{q}}_{k-1})}{\|\hat{\boldsymbol{\omega}}_{k-1}\|^2},$$

$$\mathbf{G}_{k-1} = \mathbf{F}_{\beta,k-1}$$

where $\Xi(\hat{\mathbf{q}}_k) = [\,(\hat{\eta}_k\mathbf{1} + \hat{\boldsymbol{\epsilon}}_k^{\times})^T \quad -\hat{\boldsymbol{\epsilon}}_k\,]^T$.

B. Measurement Model

The exteroceptive measurements, measurements of external features relative to the spacecraft, are assumed to be taken by sensors that each provide a vector. The measurement vector has the form

$$\mathbf{y}_k = \begin{bmatrix} \mathbf{y}_{b,k}^{m,1} \\ \vdots \\ \mathbf{y}_{b,k}^{m,n} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{bi}(\mathbf{q}_k)\mathbf{y}_{i,k}^1 \\ \vdots \\ \mathbf{C}_{bi}(\mathbf{q}_k)\mathbf{y}_{i,k}^n \end{bmatrix} + \begin{bmatrix} \mathbf{v}_k^1 \\ \vdots \\ \mathbf{v}_k^n \end{bmatrix} = \begin{bmatrix} \mathbf{Y}(\mathbf{y}_{i,k}^1, \mathbf{q}_k) \\ \vdots \\ \mathbf{Y}(\mathbf{y}_{i,k}^n, \mathbf{q}_k) \end{bmatrix}\mathbf{q}_k + \mathbf{v}_k = \mathbf{h}(\mathbf{q}_k, \mathbf{v}_k)$$

where n is the number of sensors, $\mathbf{y}_{b,k}^{m,j}$ is the jth vector measurement in the body frame, $\mathbf{C}_{bi}(\mathbf{q}_k)$ is the rotation matrix from the inertial frame to the body frame corresponding to $\mathbf{q}_k$, $\mathbf{y}_{i,k}^j$ is the corresponding reference vector expressed in the inertial frame, $\mathbf{v}_k^j$ is the zero-mean white noise with covariance $\mathbf{R}_k^j = \mathbf{R}_k^{jT} > \mathbf{0}$, and [22]

$$\mathbf{Y}(\mathbf{y}_{i,k}^j, \mathbf{q}_k) = [\,\eta_k\mathbf{1} - \boldsymbol{\epsilon}_k^{\times} \quad -\boldsymbol{\epsilon}_k\,]\begin{bmatrix} \mathbf{y}_{i,k}^{j\times} & \mathbf{y}_{i,k}^j \\ -\mathbf{y}_{i,k}^{jT} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\epsilon}_k \\ \eta_k \end{bmatrix}$$

As with the propagation step, the measurement update also needs to be modified to work with this nonlinear system. Using the EKF framework, the innovation is computed using the nonlinear measurement equation, and the Kalman gain and the covariance update are calculated using Jacobians linearized about the estimated state. The correction of the estimated state and covariance can be found by substituting $\hat{\mathbf{q}}_k$ for $\hat{\mathbf{x}}_k$, $\hat{\boldsymbol{\beta}}_k$ for $\hat{\mathbf{p}}_k$, and $\mathbf{h}(\hat{\mathbf{q}}_k, \mathbf{0})$ for $\mathbf{H}_{x,k}\hat{\mathbf{x}}_k^- + \mathbf{H}_{p,k}\hat{\mathbf{p}}_k$ in Table 1. The measurement Jacobians are given by

$$\mathbf{H}_{q,k} = \left.\frac{\partial\mathbf{h}(\mathbf{q}_k, \mathbf{v}_k)}{\partial\mathbf{q}_k}\right|_{\hat{\mathbf{q}}_k^-,\mathbf{0}} = \begin{bmatrix} \bar{\mathbf{Y}}(\mathbf{y}_{i,k}^1, \hat{\mathbf{q}}_k^-) \\ \vdots \\ \bar{\mathbf{Y}}(\mathbf{y}_{i,k}^n, \hat{\mathbf{q}}_k^-) \end{bmatrix},$$

$$\mathbf{H}_{\beta,k} = \left.\frac{\partial\mathbf{h}(\mathbf{q}_k, \mathbf{v}_k)}{\partial\beta}\right|_{\hat{\mathbf{q}}_k^-,\mathbf{0}} = \mathbf{0}, \qquad \mathbf{M}_k = \left.\frac{\partial\mathbf{h}(\mathbf{q}_k, \mathbf{v}_k)}{\partial\mathbf{v}_k}\right|_{\hat{\mathbf{q}}_k^-,\mathbf{0}} = \mathbf{1}$$

where

$$\bar{\mathbf{Y}}(\mathbf{y}_{i,k}^j, \mathbf{q}_k) = \mathbf{Y}(\mathbf{y}_{i,k}^j, \mathbf{q}_k) + \left[\,((\mathbf{y}_{i,k}^{j\times}\boldsymbol{\epsilon}_k + \mathbf{y}_{i,k}^j\eta_k)^{\times} + \mathbf{y}_{i,k}^{jT}\boldsymbol{\epsilon}_k\mathbf{1}) \quad (\mathbf{y}_{i,k}^{j\times}\boldsymbol{\epsilon}_k + \mathbf{y}_{i,k}^j\eta_k)\,\right]$$

C. Numerical Simulation Results

To evaluate the performance of the norm-constrained consider filter, the filter is used to estimate the attitude of a rigid-body spacecraft in a circular orbit. The orbit inclination and altitude are 97.6 deg and 600 km, respectively. The spacecraft's inertia matrix is $\mathbf{I} = \mathrm{diag}\{17, 25, 27\}\ \mathrm{kg}\cdot\mathrm{m}^2$, its initial attitude is $\mathbf{q}(0) = [0 \quad 0 \quad 0 \quad 1]^T$, and it is tumbling with an initial angular velocity of $\boldsymbol{\omega}_0 = [0.001 \quad 0.001 \quad 0.002]^T$ rad/s.

The spacecraft is outfitted with a rate gyro and a magnetometer. Earth's magnetic field is modeled as a magnetic dipole. All measurements are corrupted by zero-mean Gaussian white noise. The covariance of the gyroscope measurement noise is $\mathbf{Q}_k = \sigma_\omega^2\mathbf{1}$ with $\sigma_\omega = 3.1623\times10^{-7}$ rad/s and the standard deviation of the magnetometer noise is given by $\mathbf{R}_k = \sigma_m^2\mathbf{1}$ with $\sigma_m = 150$ nT. Gyro measurements are taken every 0.1 s, and magnetometer measurements are taken once every 2 s.

The performance of this norm-constrained consider EKF (NCCEKF-q) is compared with the norm-constrained EKF developed in [20] in the presence of a relatively small bias and a relatively large bias. In the first case, the bias is $\boldsymbol{\beta} = 0.04$ deg/h, and in the second case, $\boldsymbol{\beta} = 0.5$ deg/h. The EKF from [20] is implemented by both directly estimating the bias (NCEKF-qb) and ignoring the bias (NCEKF-q).

Because $\hat{\boldsymbol{\beta}}_k$ is treated differently in the NCEKF-qb and the NCCEKF-q, the role of its covariance $\mathbf{P}_{\beta\beta,k}$ differs as well. When estimating $\boldsymbol{\beta}$, the filter will minimize $tr\{\mathbf{P}_{\beta\beta,k}\}$. When considering $\boldsymbol{\beta}$, $\mathbf{P}_{\beta\beta,k}$ is used by the filter to account for the uncertainty of $\boldsymbol{\beta}$ when estimating $\mathbf{q}_k$.

Because the purpose of the simulations is to compare the ability of these filters to cope with a bias, only a modest 3 deg initial attitude estimate error about each of the principal body axes is applied to the filters. At time t_0 the bias is assumed to be $\hat{\boldsymbol{\beta}}_0 = \mathbf{0}$ by all the filters. The initial quaternion covariance and quaternion-bias cross covariance are $\mathbf{P}_{qq,0} = (0.2)\mathbf{1}$ and $\mathbf{P}_{q\beta,0} = \mathbf{0}$ for both cases. The initial bias covariances are $\mathbf{P}_{\beta\beta,0} = (0.08)^2\mathbf{1}$ (deg /h)^2 for case 1 and $\mathbf{P}_{\beta\beta,0} = \mathbf{1}$ (deg /h)^2 for case 2.

1. Case 1

Figure 1 shows the attitude error in terms of an angle $\delta\phi_k$ for each of the filters over one orbit, where $\delta\phi_k$ is the principal angle obtained from the error quaternion $\delta\mathbf{q}_k = \mathbf{q}_{k,\text{true}} \otimes \hat{\mathbf{q}}_k^{-1}$. The quaternion multiplication operator $\otimes$ is defined in ([21] p. 614) and $\hat{\mathbf{q}}_k^{-1} = [-\hat{\boldsymbol{\epsilon}}_k^T \quad \hat{\eta}_k]^T$. From Fig. 1, it can be seen that the performance of the NCEKF-qb and NCCEKF-q are relatively close, whereas the NCEKF-q starts to diverge.

Although the attitude error plots and bias plots are omitted for brevity, it was found that both the NCEKF-qb and the NCCEKF-q are consistent with respect to attitude, where consistency implies that the state estimate remains within the $\pm 3\sigma$ bounds at all times. The bias estimates for the NCEKF-qb are also consistent, as is the treatment of the bias as a parameter in the NCCEKF-q. The relatively close performance of the NCEKF-qb and the NCCEKF-q can be attributed to the initial bias having small enough values that the process noise causes the NCEKF-qb to make only a modest improvement on the bias estimate. However, the NCEKF-q becomes inconsistent because it does not consider the bias.

For the case considered, there are four states of interest associated with the quaternion and three considered parameters associated with the bias. By considering the number of operations required for the calculations in the prediction and correction steps as outlined in ([23], pp. 661–664), it was found that the NCCEKF-q performs approximately 1798 less floating point operations compared with the NCEKF-qb for the correction step, and 1069 less operations for the propagation step. For a contemporary desktop computer, this difference may be imperceptible. The savings would likely be more significant in the case where the hardware was more restrictive or if there were a larger number of considered parameters compared with the number of variables of interest.

2. Case 2

Figure 2 shows the attitude error for each of the filters over one orbit. From Fig. 2 it is apparent that the NCCEKF-q no longer performs as well as the NCEKF-qb. In this case, the larger bias causes the process noise to have a lesser effect on the bias estimate and the NCEKF-qb can make a more dramatic improvement in the bias estimate. Nevertheless, both the NCEKF-qb and the NCCEKF-q are consistent. For brevity, plots of the quaternion error are omitted.

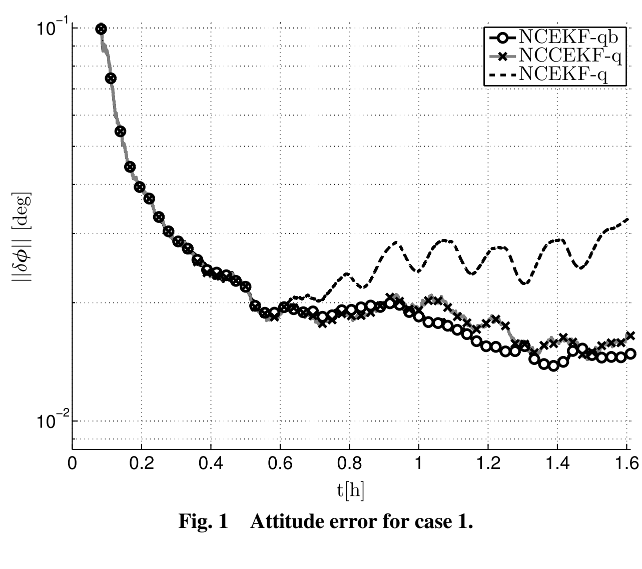

Fig. 1 Attitude error for case 1.

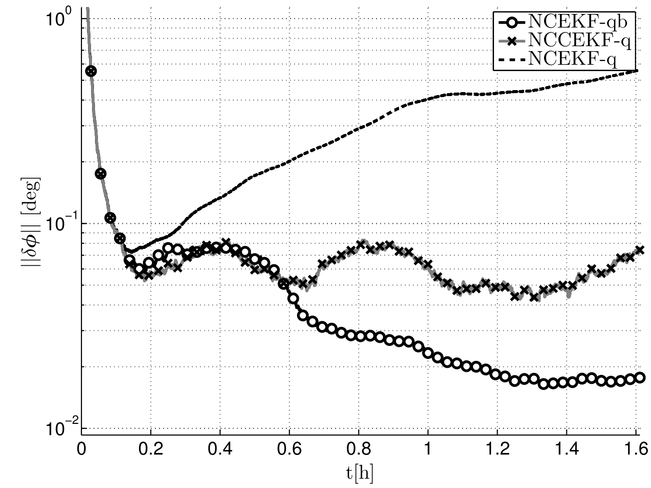

Fig. 2 Attitude error for case 2.

3. Remarks on Results

These two cases develop a profile of the usefulness of the NCCEKF-q versus the NCEKF-qb. For the first case, it is seen that, if the uncertainty in the unknown parameters are small, the performance of the NCCEKF-q is close to the NCEKF-qb. Thus, one may consider the parameters rather than estimate them, because the smaller state results in less computational load. From case 2, it is seen that the NCCEKF-q does not perform as well as the NCEKF-qb for larger uncertainties in the unknown parameters. As such, if a better state estimate is needed and the extra computational load associated with estimating parameters is of no concern, one should estimate the parameters with larger uncertainties. That said, the NCCEKF-q is still able to cope with the larger uncertainties and remain consistent while considering the uncertain parameters, rather than directly estimating them. Finally, neglecting the bias altogether leads to inconsistent state estimates.

V. Conclusions

In this study, a consider Kalman filter that directly accounts for a norm-constrained state estimate has been derived for discrete-time and hybrid systems. This is done by solving the minimum variance optimization problem while accommodating for the norm constraint using a Lagrangian multiplier. The norm-constrained consider Kalman filter was then applied to a nonlinear spacecraft attitude estimation problem by employing the EKF framework. Numerical simulation results were presented showing that, depending on the size of the gyro bias, the bias can be considered rather than estimated directly, yielding consistent results with comparable performance to a filter that directly estimates the bias.

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