Recovering Linear Controllability of an Underactuated Spacecraft by Exploiting Solar Radiation Pressure

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This paper describes a method for recovering linear controllability for the attitude of an underactuated spacecraft by accounting for the effects of solar radiation pressure in the spacecraft attitude model. The developments are based on a spacecraft model that has at least two functional reaction wheels. A solar radiation pressure torque model that is a function of spacecraft attitude is incorporated and, under suitable assumptions, can be simplified for spacecraft with body symmetry. Conditions are given under which a symmetric-body spacecraft will experience zero solar radiation pressure torque. The stability of the underactuated spacecraft model is discussed, and necessary and sufficient conditions are given for linear controllability to be regained when solar radiation pressure torques are included in the spacecraft attitude model. With linear controllability restored, conventional controllers can be designed for underactuated spacecraft. Controllability of a cuboid spacecraft under the influence of solar radiation pressure is then analyzed. Nonlinear simulations illustrate this novel approach to recover pointing by controlling two functioning reaction wheels while two other wheels undergo subsequent failures.

Nomenclature

A	=	dynamics matrix of an open-loop linear model
A_w	=	dynamics matrix of an open-loop linear model for angular velocities
A_j	=	area of panel j
B	=	input matrix of a linear model
$\mathcal{B}$	=	spacecraft bus-fixed frame
$C_{\text{diff},j}$	=	diffusion coefficient of panel j
c	=	speed of light
$d_{i,j}$	=	distance to point i on panel j from the center of the sun
d_0	=	1 AU
$\hat{g}_i,$ $i = 1,2,3,4$	=	physical spin axis vectors of reaction wheels 1, 2, 3, and 4, respectively
$\bar{g}_i,$ $i = 1,2,3,4$	=	mathematical vector corresponding to the expression $\hat{g}_i$ expressed in $\mathcal{B}$
$\boldsymbol{H}$	=	physical total angular momentum vector
$\mathcal{I}$	=	inertial frame
I_j	=	indicator function for determining if panel j contributes to solar radiation pressure torque
J	=	$\text{diag}(J_1, J_2, J_3)$
$\check{J}$	=	locked inertia matrix
$\bar{J}$	=	modified locked inertia matrix
J_{ind}	=	controllability index
J_w	=	moment of inertia of a reaction wheel about its spin axis
J_1, J_2, J_3	=	spacecraft principal moments of inertia
L_x, L_y, L_z	=	dimensions of the cuboid spacecraft
l_x, l_y, l_z	=	components of $\bar{r}_{C/O}$
$\boldsymbol{M}_{\text{ext}}$	=	physical external moment vector
$M(t_f, t_0)$	=	controllability gramian
n_1, n_2, n_3	=	components of $\bar{u}_s$
$\mathcal{O}_{\mathcal{B}/\mathcal{I}}$	=	orientation matrix of $\mathcal{B}$ relative to $\mathcal{I}$
$\boldsymbol{P}_j$	=	physical vector of solar radiation pressure of panel j
Q, R	=	weight matrices for linear quadratic controller
$\boldsymbol{r}_{C/O}$	=	physical position vector of the center of mass of the spacecraft C from a reference point O
$\bar{r}_{C/O}$	=	mathematical vector corresponding to $\boldsymbol{r}_{C/O}$ expressed in $\mathcal{B}$
$\bar{r}_j$	=	mathematical vector corresponding to $\boldsymbol{r}_{j/O}$ expressed in $\mathcal{B}$ when considering a pair of symmetric panels
$\boldsymbol{r}_{j/O}$	=	physical position vector of the center of the jth panel from a reference point O
T	=	linearized solar radiation pressure torques influence matrix
$\bar{u}_j$	=	mathematical vector corresponding to the expression of $\hat{u}_{n,j}$ expressed in $\mathcal{B}$ when considering a pair of symmetric panels
$\hat{u}_{n,j}$	=	outward physical normal vector to panels j
$\hat{u}_s$	=	sun direction physical vector
$\bar{u}_s$	=	mathematical vector corresponding to the expression of $\hat{u}_s$ expressed in $\mathcal{I}$
W_a	=	matrix of all operational reaction wheel spin axes
$\bar{z}_a$	=	mathematical vector of control inputs corresponding to the accelerations of operational reaction wheels
$\bar{z}_{a,0},$	=	control input values at the linearization point of operational reaction wheels

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$z_i, i = 1,2,3,4$	=	control inputs corresponding to the accelerations of reaction wheels 1, 2, 3, and 4, respectively
$\delta\bullet$	=	deviation from values at the linearization point
λ	=	eigenvalue of A
λ_w	=	eigenvalue of A_w
$\bar{\Theta}$	=	mathematical vector of Euler angles
$\bar{\nu}_a,$	=	mathematical vector of operational reaction wheels speeds
$\bar{\nu}_{a,0}$	=	reaction wheel speeds at the linearization point
$\nu_i, i = 1,2,3,4$	=	speeds (spin rates) of reaction wheels 1, 2, 3, and 4, respectively
$\boldsymbol{\tau}_p$	=	solar radiation pressure torque physical vector induced by a pair of symmetric panels
$\boldsymbol{\tau}_{\mathrm{srp}}$	=	physical solar radiation pressure torque vector
$\bar{\tau}_{\mathrm{srp}}$	=	mathematical vector corresponding to $\boldsymbol{\tau}_{\mathrm{srp}}$ expressed in $\mathcal{B}$
$\boldsymbol{\tau}_{\mathrm{srp},x},$ $\boldsymbol{\tau}_{\mathrm{srp},y}, \boldsymbol{\tau}_{\mathrm{srp},z}$	=	physical vectors of solar radiation pressure torques aligned with the principal axes of the cuboid spacecraft
$\bar{\tau}_{\mathrm{srp},x},$ $\bar{\tau}_{\mathrm{srp},y}, \bar{\tau}_{\mathrm{srp},z}$	=	mathematical vectors of $\boldsymbol{\tau}_{x,\mathrm{srp}}, \boldsymbol{\tau}_{y,\mathrm{srp}}$, and $\boldsymbol{\tau}_{z,\mathrm{srp}}$ expressed in $\mathcal{B}$
$\Phi_{\mathrm{sun,tot}}$	=	solar flux at 1 AU
ψ, θ, ϕ	=	3-2-1 Euler angles yaw, pitch, and roll
$\boldsymbol{\omega}$	=	physical angular velocity vector
$\bar{\omega}$	=	mathematical vector corresponding to $\rightarrow \omega$ expressed in $\mathcal{B}$
$\omega_1, \omega_2, \omega_3$	=	components of $\bar{\omega}$

I. Introduction

THE failure of reaction wheels (RWs) in an array can impair a spacecraft's ability to perform imaging missions, during which a prescribed inertial orientation has to be achieved and maintained with high accuracy and precision. The far ultraviolet spectroscopic explorer (FUSE) had three out of its four RWs fail within their expected lifetime and consequently had to perform attitude maneuvers using magnetic actuators and one RW [1]. The Kepler telescope lost two out of its four RWs before its mission was complete. Although its gas thruster can also be used to perform attitude maneuvers, this type of actuation expends fuel and was not designed to provide the precise orientation needed [2]. The Japanese Hayabusa satellite also lost two out its four RWs. Its mission profile had to be modified to conserve fuel to have enough propellant to return to Earth after performing the necessary attitude maneuvers with gas thrusters [3]. We note that RW failures cannot be addressed simply by increasing redundancy (i.e., adding more RWs) due to stringent volume, power, and weight constraints onboard the spacecraft. Control strategies to deal with underactuated spacecraft hence need to be developed.

In the case of an underactuated spacecraft with two internal torque actuators, the linearized dynamics are not controllable. In [4], the nonlinear dynamics are shown to be inaccessible with two or fewer reaction wheels due to the conservation of angular momentum. In [5], it is shown that if a spacecraft with two reaction wheels has zero angular momentum, the system becomes small-time locally controllable in a restricted (rest-to-rest) sense. However, the spacecraft cannot be smoothly or continuously stabilized by any time-invariant feedback law [6–9]. Stabilization is possible by time-periodic feedback laws, but exponential convergence rates cannot be achieved if the feedback law is smooth [10]. For a spacecraft with zero total angular momentum, open-loop reorientation methods are studied in [11–13], and discontinuous feedback laws are proposed in [5,14–17].

We note that the assumption of total zero angular momentum is limiting in practical spacecraft applications. It is difficult to achieve it in space environment, and it is undesirable for spacecraft with less than four RWs because the wheels must be spun down during inertial pointing and operated in the zero crossing region, where their accuracy is decreased, the friction coefficient is increased, and their operational life is reduced.

The case of an underactuated spacecraft with nonzero total angular momentum is less studied. Boyer and Alamir [18] define a set of feasible attitudes defined by the law of angular momentum conservation and give a procedure for constructing an open-loop control. A spin-axis stabilization is performed about the uncontrollable axis of a spacecraft with nonzero angular momentum in [19], but inertial pointing is not treated. Petersen et al. [20] exploit results in [21,22] to develop a hybrid feedback control scheme, which is, however, dependent on the assumption of angular momentum conservation and does not guarantee inertial pointing in the case where there is a nonzero projection of the angular momentum on the uncontrollable spacecraft axis. The techniques proposed in the aforementioned papers not only suffer from several limitations but are also unconventional and thus unfamiliar to many practicing aerospace engineers.

The approach presented in this paper is based on including the effects of solar radiation pressure (SRP) torques, modeled following [23], into the spacecraft model. Our analysis shows that, under appropriate assumptions, linear controllability is regained, and hence spacecraft stabilization can be achieved with conventional control schemes. In particular, a linear quadratic (LQ) approach will be first applied. The LQ approach is chosen due to its robustness, its optimal control properties, and its familiarity to aerospace engineers. A pole placement scheme will also be used to improve convergence time. By taking advantage of the change in the dynamics induced by SRP torques, two RWs are able to slowly correct the attitude errors over time. This method is different from the previous approaches discussed previously. First, it exploits external disturbance torques, which many control techniques typically either neglect or reject. Second, this method is not restricted to zero total angular momentum or the constant angular momentum assumption. Third, using SRP torques allows designers to use conventional and familiar feedback control schemes that are guaranteed to locally stabilize the equilibrium.

As compared to a preliminary version of this work that has appeared in a conference paper [24], this paper contains several new results, extensions, and generalizations. These include 1) the characterization and discussion of the effects of SRP, specifically on symmetric-body spacecraft, 2) sufficient conditions for zero SRP torque acting on a symmetric spacecraft, 3) results on the fixed-attitude stability of an underactuated spacecraft with and without SRP, 4) necessary and sufficient conditions under which linear controllability is regained with SRP, and 5) extending a novel pathway to recover linear controllability and perform control of underactuated spacecraft by taking advantage of SRP torques to a broader class of spacecraft.

After [24] was submitted for publication, the press release [25] appeared, suggesting that SRP is actually used (in an unspecified control scheme) to restore Kepler's mission controllability. The controllability analysis and results in [24] and in this paper, obtained independently of [25], are thus indirectly corroborated by experimental evidence in [25].

The paper is organized as follows. The equations of spacecraft attitude kinematics and dynamics are derived in Secs. II.A and II.B. The SRP torque properties are characterized in Sec. II.C, and a linearized spacecraft model is presented in Sec. II.D. The SRP model is made specific to a symmetric spacecraft in Sec. III, and conditions under which SRP torque is zero for all attitudes for a symmetric spacecraft are subsequently derived in Sec. IV. The results on stability of an underactuated spacecraft are presented in Sec. V. Necessary and sufficient conditions for linear controllability of an underactuated spacecraft with two RWs exposed to SRP are given in Sec. VI.A, and a relative controllability metric is introduced in Sec. VI.B. The specific case of SRP torques acting on a cuboid shaped spacecraft is treated in Sec. VII and includes conditions for zero SRP torque for all attitudes as well as discussions on equilibria and controllability. The linear control design based on linear quadratic regulator theory and pole placement techniques is presented in Sec. VIII. Simulation results based on the nonlinear spacecraft model are reported in Sec. IX, where the controller is demonstrated to recover pointing after a sequence of two RW wheels failures. Concluding remarks are made in Sec. X.

Throughout this paper, the following notation is used. Frames are denoted by script, $\mathcal{S}$. A physical vector is designated by boldface italics, $\boldsymbol{r}$. Physical unit vectors are expressed with an overscript hat, $\hat{r}$. The notation for a mathematical vector obtained by resolving a physical vector in a given frame $\mathcal{S}$ is $\boldsymbol{r}|_{\mathcal{S}}$. Mathematical vectors are

denoted by an overscript bar, $\bar{r}$. The time derivative of a physical vector r with respect to a given frame $\mathcal{S}$ is $\overset{\mathcal{S}\cdot}{r}$. The orientation matrix (direction cosine matrix), representing the transformation from frame $\mathcal{S}$ to frame $\mathcal{R}$ is $\mathcal{O}_{\mathcal{R}/\mathcal{S}}$. An m by n zero matrix is denoted by $0_{m \times n}$, whereas an m by m identity matrix is denoted by $I_{m \times m}$. The nomenclature is summarized at the beginning of the paper.

II. Spacecraft Modeling

In this paper, a spacecraft configuration consisting of a bus and four RWs is considered. During the spacecraft's mission, two of the RWs fail and spin down to zero speed, leaving two operational RWs. Let the inertial frame be denoted by $\mathcal{I}$ and a spacecraft bus-fixed frame, with origin at the center of mass of the spacecraft bus, be denoted by $\mathcal{B}$. The axes of $\mathcal{B}$ are aligned with the spacecraft's principal axes.

All RWs rotors are assumed thin and identical, with moments of inertia about the spin axis being equal to J_w and moments of inertia about axes transversal to the spin axis being zero. The RWs are mounted such that the center of mass of the entire spacecraft assembly is the same as that of the spacecraft bus. The four RWs spin at angular rates ν_1, ν_2, ν_3, and ν_4 about the spin axes specified by unit vectors $\hat{g}_1$, $\hat{g}_2$, $\hat{g}_3$, and $\hat{g}_4$, which are fixed in $\mathcal{B}$. Let $\bar{g}_i = \hat{g}_i|_{\mathcal{B}}$, $i = 1$, 2, 3, 4. Define

$$\bar{\nu}_a = [\ldots \nu_b \ldots]^T, \qquad b \in \{1, 2, 3, 4\} \tag{1}$$

to be the vector of all operational RWs angular rates and

$$W_a = [\ldots \bar{g}_b \ldots], \qquad b \in \{1, 2, 3, 4\} \tag{2}$$

to be a matrix whose columns are the unit vectors of the spin axes of the operational RWs. For instance, if wheels 3 and 4 have failed, then $\bar{\nu}_a = [\nu_1 \quad \nu_2]^T$ and $W_a = [\bar{g}_1 \quad \bar{g}_2]$.

A. Kinematics of the Spacecraft

The orientation of $\mathcal{B}$ relative to $\mathcal{I}$ is characterized by three successive rotations, defined by 3-2-1 Euler angles ψ (yaw), θ (pitch), and ϕ (roll). Note that any three-parameter representation of orientation will contain singularities. It is assumed that the spacecraft's mission will not require large attitude transients and never approach the singularity. This assumption is reasonable, considering that we are interested in using RWs for accurate pointing near the target orientation. The kinematic equations, following the derivations of [26], are given by

$$\dot{\bar{\Theta}} = F(\bar{\Theta})\bar{\omega} \tag{3}$$

where $\bar{\Theta} = [\phi \quad \theta \quad \psi]^T$, $\bar{\omega} = \omega|_{\mathcal{B}} = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ is the angular velocity vector resolved in $\mathcal{B}$, and

$$F(\bar{\Theta}) = \frac{1}{\cos(\theta)} \begin{bmatrix} \cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi)\sin(\theta) \\ 0 & \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} \tag{4}$$

B. Dynamics of the Spacecraft

The evolution of the spacecraft's angular momentum in time with respect to the inertial frame $\mathcal{I}$ is given by

$$\overset{\mathcal{I}\cdot}{H} = \overset{\mathcal{B}\cdot}{H} + \omega \times H = M_{\text{ext}} \tag{5}$$

where H is the total angular momentum of the spacecraft (i.e., including the spacecraft bus and RWs), and M_{ext} is a vector of external moments about the center of mass of the spacecraft. It is assumed that the only external moments acting upon the spacecraft are torques induced by SRP (i.e., $M_{\text{ext}} = \tau_{\text{srp}}$). The total system's angular momentum expressed in $\mathcal{B}$ is

$$H|_{\mathcal{B}} = J\bar{\omega} + J_w W_c(\bar{\nu}_c + W_c^T\bar{\omega}) \tag{6}$$

where

$$J = \text{diag}(J_1, \quad J_2, \quad J_3),$$
$$\bar{\nu}_c = [\nu_1 \quad \nu_2 \quad \nu_3 \quad \nu_4]^T,$$
$$W_c = [\bar{g}_1 \quad \bar{g}_2 \quad \bar{g}_3 \quad \bar{g}_4] \tag{7}$$

All RWs are accounted for in Eq. (6) because any residual angular rate a failed RW has before it spins down to zero affects the angular momentum of the spacecraft. Taking the time derivative of Eq. (6) with respect to $\mathcal{B}$ and substituting this and Eq. (6) into Eq. (5) gives

$$J\dot{\bar{\omega}} + J_w W_c(\dot{\bar{\nu}}_c + W_c^T\dot{\bar{\omega}}) + \bar{\omega} \times (J\bar{\omega} + J_w W_c(\bar{\nu}_c + W_c^T\bar{\omega})) = \bar{\tau}_{\text{srp}} \tag{8}$$

where $\bar{\tau}_{\text{srp}} = \tau_{\text{srp}}|_{\mathcal{B}}$. Let

$$\tilde{J} = J + J_w W_c W_c^T \tag{9}$$

which is often referred to as the locked inertia matrix. Then, Eq. (8) simplifies to

$$\tilde{J}\dot{\bar{\omega}} = -\bar{\omega} \times (\tilde{J}\bar{\omega} + J_w W_c\bar{\nu}_c) - J_w W_c\dot{\bar{\nu}}_c + \bar{\tau}_{\text{srp}} \tag{10}$$

The contribution of the failed RWs to the dynamics in Eq. (10) vanishes once their speeds reach zero. Assuming that the failed RWs are at zero speed, the dynamics can be rewritten to account for the contributions of only operation RWs:

$$\bar{J}\dot{\bar{\omega}} = -\bar{\omega} \times (\tilde{J}\bar{\omega} + J_w W_a\bar{\nu}_a) - J_w W_a\dot{\bar{\nu}}_a + \bar{\tau}_{\text{srp}} \tag{11}$$

where

$$\bar{J} = J + J_w W_a W_a^T \tag{12}$$

is the modified locked inertia matrix.

The operational RW accelerations are treated as the control inputs:

$$\dot{\bar{\nu}}_a = \bar{z}_a \tag{13}$$

where

$$\bar{z}_a = [\ldots z_b \ldots]^T, \qquad b \in \{1, 2, 3, 4\} \tag{14}$$

For instance, if wheels 1 and 2 are operational but wheels 3 and 4 have failed, $\bar{z}_a = [z_1 \quad z_2]^T$.

C. Solar Radiation Pressure Torque Model

Assume that the spacecraft is covered with ρ flat panels. The SRP torques induced by the panels are modeled based on the developments in [23]. Define

$$\alpha_{i,j} = \frac{\Phi_{\text{sun,tot}}}{c(d_{i,j}/d_0)^2} \tag{15}$$

$$\beta_j = \frac{4}{9}C_{\text{diff},j} \tag{16}$$

where c is the speed of light, d_0 is the nominal distance from the sun equal to 1 AU, $\Phi_{\text{sun,tot}}$ is the solar flux at d_0, $C_{\text{diff},j}$ is the diffusion coefficient for panel j, and $d_{i,j}$ is the distance to a point i on panel j from the center of the sun. The SRP at point i on panel j is given by

$$P_{i,j} = -\alpha_{i,j}(\hat{u}_{n,j} \cdot \hat{u}_s)(\hat{u}_{n,j} + \beta_j\hat{u}_s) \tag{17}$$
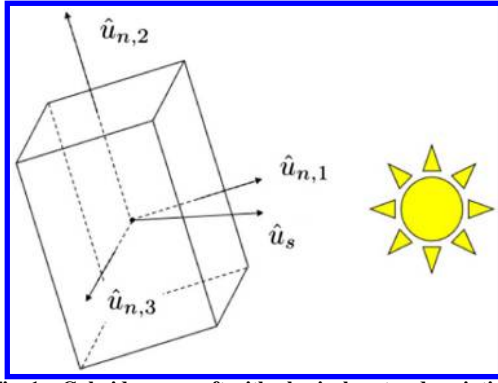
Fig. 1   Cuboid spacecraft with physical vector description.

where $\hat{u}_{n,j}$ is the normal to panel j pointing outward from the spacecraft, and $\hat{u}_s$ is the unit vector representing the sun direction. Figure 1 depicts the sun direction $\hat{u}_s$ and normal vectors $\hat{u}_{n,j}, j = 1, 2, 3$, as they apply to a cuboid spacecraft.

The distance between two points i and l on panels j and k (where j and k can represent the same panel) is very small when compared to $d_{i,j}$ and $d_{l,k}$. Therefore the difference between $d_{i,j}/d_0$ and $d_{l,k}/d_0$ becomes negligible, and $\alpha_{i,j}$ can be treated as a constant parameter α for all points across all panels. Thus, the SRP exerted at every point along the same panel is assumed to be identical. Note that there will be a variation in SRP between panels j and k if β_j and β_k are different. The SRP exerted at each point on panel j is then

$$P_j = -\alpha(\hat{u}_{n,j} \cdot \hat{u}_s)(\hat{u}_{n,j} + \beta_j \hat{u}_s) \tag{18}$$

Under the assumption that SRP acts identically across all points on the same panel, the total SRP torque about the spacecraft's center of mass due to the jth panel can be expressed as

$$\boldsymbol{\tau}_{j,\mathrm{srp}} = (\boldsymbol{r}_{j/O} - \boldsymbol{r}_{C/O}) \times A_j \boldsymbol{P}_j \tag{19}$$

where $\boldsymbol{r}_{j/O}$ is the vector from a reference point O to the center of the jth panel, $\boldsymbol{r}_{C/O}$ is the vector from O to the center of mass, and A_j is the area of the jth panel. Because SRP is additive across all panels,

$$\boldsymbol{\tau}_{\mathrm{srp}} = \sum_j^\rho \boldsymbol{\tau}_{j,\mathrm{srp}} I_j \tag{20}$$

where I_j is an indicator function used to identify which panels are facing the sun and are acted on by solar pressure. Assuming a regular, convex shape of the spacecraft, the indicator function is given by

$$I_j = \begin{cases} 1 & \text{if } (\hat{u}_{n,j} \cdot \hat{u}_s) > 0, \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Observe that the total SRP torque in Eq. (20) is only a function of $\bar{\Theta}$. In reality, SRP is a pressure-based torque and hence, strictly speaking, also depends on spacecraft velocities due to dynamic pressure effects. These effects are very small and are assumed to be negligible.

D.   Linearized Model

For analysis and controller design, the equations of motion [Eqs. (3) and (11)] are linearized. Without loss of generality, $\bar{\Theta} = 0$ is chosen as the desired attitude because $\mathcal{I}$ can be oriented to reflect desired pointing. Theorem 1 gives the requirement for the spacecraft to maintain $\bar{\Theta} = \bar{\omega} = 0$ in steady state (assuming no constraints on functioning RW speeds).

Theorem 1: $\bar{\Theta} = \bar{\omega} = 0$ can be maintained as a (relative) equilibrium by the closed-loop system if and only if $\bar{\tau}_{\mathrm{srp}}(0)$ is in the range of W_a.

Proof: For the spacecraft bus, if $\bar{\omega}(t) = 0$ for all t, then $\dot{\bar{\Theta}}(t) = 0$ and $\dot{\bar{\omega}}(t) = 0$. If $\dot{\bar{\omega}} = 0$, then Eq. (11) implies

$$0 = -J_w W_a \bar{z}_a + \bar{\tau}_{\mathrm{srp}}(0) \tag{22}$$

Equation (22) can be satisfied if and only if $\bar{\tau}_{\mathrm{srp}}(0)$ is in the range of W_a. □

Remark 1: Theorem 1 gives conditions under which $\bar{\Theta} = \bar{\omega} = 0$ can be maintained as a (relative) equilibrium for the spacecraft bus. The reaction wheels may be accelerating to compensate for SRP torque, in which case the (relative) equilibrium can be maintained until the RWs reach their saturation limits.

For three or more RWs, assuming that W_a is full rank, $\bar{\Theta} = \bar{\omega} = 0$ will always be a feasible equilibrium. With two or fewer RWs, Theorem 1 restricts the set of physical pointing orientations that can be maintained in steady state with functioning RWs.

Assuming that Theorem 1 holds, Eqs. (3) and (11) are linearized about $\bar{\Theta} = \bar{\omega} = 0$, $\bar{\nu}_a = \bar{\nu}_{a,0}$, and $\bar{z}_a = \bar{z}_{a,0} = \Xi$, where Ξ is the commanded RW accelerations to make $\bar{\Theta} = \bar{\omega} = 0$ a feasible equilibrium, i.e.,

$$\Xi = \begin{cases} (W_a^{\mathrm{T}} W_a)^{-1} W_a^{\mathrm{T}} \bar{\tau}_{\mathrm{srp}}(0) & \text{if } \mathrm{rank}(W_a) \le 3, \\ \text{an arbitrary } \bar{z}_a \text{ such that } W_a \bar{z}_a = \bar{\tau}_{\mathrm{srp}}(0) & \text{otherwise} \end{cases} \tag{23}$$

Assuming that there are m operational RWs and the failed RWs are at zero speed, the linearization yields the system of equations

$$\begin{bmatrix} \dot{\bar{\Theta}} \\ \dot{\bar{\omega}} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ \bar{J}^{-1} T & \bar{J}^{-1} J_w S[W_a \bar{\nu}_{a,0}] \end{bmatrix} \begin{bmatrix} \bar{\Theta} \\ \bar{\omega} \end{bmatrix} + \begin{bmatrix} 0_{3\times m} \\ -\bar{J}^{-1} J_w W_a \end{bmatrix} \delta\bar{z}_a \tag{24}$$

where $\delta\bar{z}_a = \bar{z}_a - \bar{z}_{a,0}$, T is the linearized matrix of $\bar{\tau}_{\mathrm{srp}}$ (i.e., $\bar{\tau}_{\mathrm{srp}} = \bar{\tau}_{\mathrm{srp}}(0) + T\bar{\Theta}$), and $S[*]$ is the 3×3 skew-symmetric matrix formed off a 3×1 mathematical vector. That is, if $\bar{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^{\mathrm{T}}$, then

$$S[\bar{a}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{25}$$

Letting $\bar{x} = [\bar{\Theta} \quad \bar{\omega}]^{\mathrm{T}}$, it follows that

$$\dot{\bar{x}} = A\bar{x} + B\delta\bar{z}_a \tag{26}$$

where

$$A = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ \bar{J}^{-1} T & \bar{J}^{-1} J_w S[W_a \quad \bar{\nu}_{a,0}] \end{bmatrix}, \qquad B = \begin{bmatrix} 0_{3\times m} \\ -\bar{J}^{-1} J_w W_a \end{bmatrix} \tag{27}$$

III.   Solar Radiation Pressure Torque on a Symmetric-Body Spacecraft

To provide insight into the effects of SRP, a general class of symmetric-body spacecraft that have pairs of panels with the same area and diffusion coefficients (i.e., same β_j), located on the opposite ends of the spacecraft, is now considered. The panels are equal distance away from the reference point O, and the normals to the panels are parallel but opposite in direction. Denote one panel in this pair by "$p+$" and another by "$p-$". Then,

$$\boldsymbol{r}_{p+/O} = -\boldsymbol{r}_{p-/O} = \boldsymbol{r}_p,$$
$$\hat{u}_{n,p+} = -\hat{u}_{n,p-} = \hat{u}_p,$$
$$\beta_{p+} = \beta_{p-} = \beta_p,$$
$$A_{p-} = A_{p+} = A_p \tag{28}$$

The SRP torque exerted by each individual panel, assuming they are exposed to the sun, is

$$\boldsymbol{\tau}_{p+} = (\boldsymbol{r}_{p+} - \boldsymbol{r}_{C/O}) \times A_{p+} \boldsymbol{P}_{p+},$$
$$= \gamma_p((\boldsymbol{r}_p \times \hat{u}_p) + \beta_p(\boldsymbol{r}_p \times \hat{u}_s) - (\boldsymbol{r}_{C/O} \times \hat{u}_p) - \beta_p(\boldsymbol{r}_{C/O} \times \hat{u}_s)) \tag{29}$$

$$\boldsymbol{\tau}_{p-} = (\boldsymbol{r}_{p-} - \boldsymbol{r}_{C/O}) \times A_{p-} \boldsymbol{P}_{p-},$$
$$= \gamma_p(-(\boldsymbol{r}_p \times \hat{u}_p) + \beta_p(\boldsymbol{r}_p \times \hat{u}_s) - (\boldsymbol{r}_{C/O} \times \hat{u}_p) + \beta_p(\boldsymbol{r}_{C/O} \times \hat{u}_s)) \tag{30}$$

where $\gamma_p = -\alpha A_p(\hat{u}_p \cdot \hat{u}_s)$. The total SRP torque induced by the two panels is

$$\boldsymbol{\tau}_p = \boldsymbol{\tau}_{p+} I_{p+} + \boldsymbol{\tau}_{p-} I_{p-} \tag{31}$$

By the assumption on directions of $\hat{u}_{p+}$ and $\hat{u}_{p-}$, I_{p+} takes on the opposite binary value of I_{p-} if $\gamma_p \neq 0$. If $\gamma_p = 0$, then the pair of panelsAU: The sentence that begins "If γp = 0, then . . . " is unclear, specifically the portion that reads "the pair of panels does to induce." Please review this sentence carefully, and edit/revise as necessary. does to induce any SRP torque. Knowing this, and noting that the second and third terms of Eqs. (29) and (30) are the same, whereas the first and fourth terms differ by a sign, Eq. (31) can be written as

$$\boldsymbol{\tau}_p = \gamma_p((-\boldsymbol{r}_{C/O} \times \hat{u}_p) + \beta_p(\boldsymbol{r}_p \times \hat{u}_s) + \text{sign}(\hat{u}_p \cdot \hat{u}_s)((\boldsymbol{r}_p \times \hat{u}_p)$$
$$- \beta_p(\boldsymbol{r}_{C/O} \times \hat{u}_s))) \tag{32}$$

IV. Conditions for Zero Solar Radiation Pressure Torque for All Orientations

There are cases in which the total SRP torque exerted on the spacecraft is zero for all possible orientations. Hence, utilization of SRP for control is not possible. Theorem 2 gives a condition for which a symmetric-body spacecraft covered in pairs of symmetric panels will experience zero SRP torque, regardless of $\bar{\Theta}$.

Theorem 2: Let a symmetric-body spacecraft be covered with ρ_s pairs of panels, having characteristics satisfying Eq. (28). Then, the SRP torque induced on the spacecraft is zero for all $\bar{\Theta}$ if the three relations

$$\boldsymbol{r}_i || \hat{u}_i, i = 1 \ldots \rho_s,$$
$$\boldsymbol{r}_{C/O} = 0,$$
$$\exists a \in \mathbb{R}: \sum_{i=1}^{\rho_s} \beta_i A_i \bar{r}_i \bar{u}_i^T = a I_{3\times3} \tag{33}$$

are all satisfied, where $\bar{r}_i = \boldsymbol{r}_i|_{\mathcal{B}}$ and $\bar{u}_i = \hat{u}_i|_{\mathcal{B}}$.

Proof: The total SRP torque exerted on the symmetric spacecraft can be written using Eq. (32):

$$\boldsymbol{\tau}_{\text{srp}} = \sum_{i=1}^{\rho_s} \gamma_i((-\boldsymbol{r}_{C/O} \times \hat{u}_i) + \beta_i(\boldsymbol{r}_i \times \hat{u}_i)$$
$$+ \text{sign}(\hat{u}_i \cdot \hat{u}_s)((\boldsymbol{r}_i \times \hat{u}_i) - \beta_i(\boldsymbol{r}_{C/O} \times \hat{u}_s))) \tag{34}$$

If the first two conditions of Eq. (33) hold, Eq. (34) becomes

$$\boldsymbol{\tau}_{\text{srp}} = \sum_{i=1}^{\rho_s} \gamma_i \beta_i(\boldsymbol{r}_i \times \hat{u}_i) \tag{35}$$

Let $\bar{u}_s = \hat{u}_s|_{\mathcal{I}}$. Resolving Eq. (35) in $\mathcal{B}$ yields

$$\bar{\boldsymbol{\tau}}_{\text{srp}} = -\alpha\left(\left(\sum_{i=1}^{\rho_s} \beta_i A_i \bar{r}_i \bar{u}_i^T\right) \mathcal{O}_{\mathcal{B}/\mathcal{I}} \bar{u}_s\right) \times \left(\mathcal{O}_{\mathcal{B}/\mathcal{I}} \bar{u}_s\right) \tag{36}$$

If the third condition of Eq. (33) holds, then there exists an $a \in \mathbb{R}$ such that

$$\bar{\boldsymbol{\tau}}_{\text{srp}} = -a\alpha \mathcal{O}_{\mathcal{B}/\mathcal{I}} \bar{u}_s \times \mathcal{O}_{\mathcal{B}/\mathcal{I}} \bar{u}_s = 0 \tag{37}$$

Therefore, regardless of orientation, the SRP will not exert any torques on the spacecraft.□

An example of when Theorem 2 holds is for a cuboid spacecraft where the reference point O coincides both with the center of mass and the center of solar radiation pressure of the spacecraft. See Sec. VII.A.

V. Stability of a Spacecraft

In general, the open-loop spacecraft attitude dynamics for a fixed-attitude equilibrium without SRP are unstable, though the dynamics of the angular velocities can be stable depending on the axis of rotation [27]. As we discuss in this section, a fixed-attitude equilibrium of an underactuated spacecraft with SRP has similar open-loop instability properties.

A. Linear Stability with Zero Solar Radiation Pressure

Consider the case of an underactuated spacecraft with two operational RWs, where effects of SRP torques are zero (i.e., $\boldsymbol{\tau}_{\text{srp}} = 0$). The state matrix A in Eq. (27) becomes

$$A = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ 0_{3\times3} & \bar{J}^{-1} J_w S[W_a \quad \bar{\nu}_{a,0}] \end{bmatrix} \tag{38}$$

The eigenvalues of matrix A, denoted as λ, are given by

$$\lambda = 0, 0, 0, 0, \pm J_w \sqrt{(W_a \bar{\nu}_{a,0})^T D(W_a \bar{\nu}_{a,0})} \tag{39}$$

where $D = -(\text{diag}(J_2 J_3, \quad J_1 J_3, \quad J_1 J_2))^{-1}$. The matrix D is negative definite, and therefore, A might have one pair of purely imaginary eigenvalues, provided that at least one of the RW speeds at the equilibrium is nonzero. Therefore, all eigenvalues lie on the imaginary axis.

The requirement for stability of a linear system with eigenvalues on the imaginary axis and in the open left half-plane is that each such eigenvalue with zero real part must be semisimple (i.e., the geometric multiplicity of the eigenvalue must be equal to its algebraic multiplicity). There is only one pair of purely imaginary eigenvalues, and so both are semisimple. Any instability may thus be caused only by the zero eigenvalue. It can be shown that the algebraic multiplicity of the zero eigenvalue of A is 4, whereas its geometric multiplicity is 3. Thus, the linearized system is open-loop unstable.

We note that the angular velocity variables are decoupled from the orientation variables for this linearized system. The state matrix for the angular velocities, denoted as A_ω, is given by

$$A_w = \bar{J}^{-1} J_w S[W_a \bar{\nu}_{a,0}] \tag{40}$$

whose eigenvalues λ_w are

$$\lambda_w = 0, \pm J_w \sqrt{(W_a \bar{\nu}_{a,0})^T D(W_a \bar{\nu}_{a,0})} \tag{41}$$

All the eigenvalues of A_w are semisimple, and hence it can be shown that the linearized angular velocity dynamics are stable but not asymptotically stable. Because of the double integrator structure of A, bounded angular velocities may cause unbounded drift in the attitude, resulting in instability.

B. Stability Analysis of Linearized Underactuated Spacecraft Dynamics with Solar Radiation Pressure Torque

We now consider the case with nonzero SRP torque (i.e., $\boldsymbol{\tau}_{\text{srp}} \neq 0$) and two functioning RWs spinning about the first and second principal axes. We analyze the stability properties of the spacecraft

bus (relative) equilibrium corresponding to $\bar{\Theta} = \bar{\omega} = 0$. It is assumed that $\bar{\tau}_{\text{srp}}(0)$ is consistent with Theorem 1 so that the (relative) equilibrium can be maintained. Let the RW accelerations be set according to the following control law:

$$\delta\bar{z}_a = -(C^{\text{T}}C)^{-1}C^{\text{T}}\bar{J}^{-1}\tilde{T}\bar{\Theta} \qquad (42)$$

where $C = -(\bar{J}^{-1}J_wW_a)$. The feedback law in Eq. (42) cancels out the SRP torque components that are in the range of W_a. Then, the dynamics matrix has the following form:

$$A = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ \tilde{T} & \bar{J}^{-1}J_wS[W_a\bar{\nu}_{a,0}] \end{bmatrix} \qquad (43)$$

where

$$\tilde{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{t}_{31} & \tilde{t}_{32} & \tilde{t}_{33} \end{bmatrix} \qquad (44)$$

and $\tilde{t}_{3j} \in \mathbb{R}$, $j = 1, 2, 3$. Computing the eigenvalues of A_c yields three zero eigenvalues, a pair of complex eigenvalues, and an eigenvalue that is purely real (the equations for these eigenvalues are omitted due to lack of space). The zero eigenvalue has a geometric multiplicity of 2 and is, therefore, not semisimple. Hence, the equilibrium is unstable based on the linearized model. In addition, as apparent from Eq. (26), the SRP torque causes a coupling between the attitude and angular velocity dynamics, which does not allow the independent analysis of the stability of angular velocity dynamics.

C. Nonlinear Stability

The previous analysis is based on the linear model in Eq. (26). Because the linearized model has eigenvalues on the imaginary axis, the stability analysis is inconclusive because the stability/instability of an equilibrium in such a case may depend on nonlinear terms. Note that, without SRP, there do exist stable equilibria where the spacecraft can maintain directional pointing while rotating about an axis parallel to the pointing direction [27]. However, the previous analysis suggests that a fixed-attitude equilibrium either with or without SRP can be unstable. Intuitively, if one of the angular velocities is perturbed away from zero, a bounded drift in the Euler angles away from equilibrium can occur. This instability conclusion, supported by the linear analysis, has been verified through extensive simulations on a nonlinear model. The formal mathematical proof of instability (e.g., based on an application of Chetaev's theorem [28]) is left to future work.

VI. Regaining Linear Controllability Using Solar Radiation Pressure

A. Necessary and Sufficient Conditions for Regaining Linear Controllability

Without the effects of SRP included in Eq. (26), the spacecraft dynamics are linearly uncontrollable by RW accelerations if only two RWs are functioning. The following theorem gives necessary and sufficient conditions for restoring linear controllability to a spacecraft with two functioning RWs when SRP torques are included.

Theorem 3: Let the spacecraft have two operational RWs whose spin axis are nonparallel, i.e., rank$(W_a) = 2$. The system [Eq. (26)] is linearly controllable if and only if, for every eigenvalue λ of A and for any vector $\bar{\eta}_u$ in the null space of W_a^{T},

$$\bar{\eta}_u^{\text{T}}(\bar{J}\lambda^2 - S[W_a\bar{\nu}_{a,0}]\lambda - T) \neq 0 \qquad (45)$$

Proof: The Popov–Belevitch–Hautus test for controllability implies that Eq. (26) is linearly controllable if and only if

$$\text{rank}[(\lambda I_{6\times6} - A) \quad B] = \dim(A) \qquad (46)$$

for every eigenvalue λ of A; see [29]. By the converse, the system in Eq. (26) is uncontrollable if and only if there exists a nonzero vector $\bar{\eta} \in \mathbb{C}^6$ such that

$$\bar{\eta}^*[(\lambda I_{6\times6} - A) \quad B] = 0 \qquad (47)$$

for at least one eigenvalue, where $\bar{\eta}^*$ is the conjugate transpose of $\bar{\eta}$. Equation (47) can be reduced to two conditions:

$$\bar{\eta}^*(\lambda I_{6\times6} \quad - \quad A) = 0, \quad \bar{\eta}^*B = 0 \qquad (48)$$

Let $\bar{\eta} = [\bar{\eta}_1^* \quad \bar{\eta}_2^*]^*$, $\bar{\eta}_1, \bar{\eta}_2 \in \mathbb{C}^3$. Expanding the first condition of Eq. (48), it follows that

$$\bar{\eta}_1^*\lambda - \bar{\eta}_2^*\bar{J}^{-1}T = 0, \quad -\bar{\eta}_1^* + \bar{\eta}_2^*(I_{3\times3} - \bar{J}^{-1}J_wS[W_a\bar{\nu}_{a,0}]) = 0 \quad (49)$$

Multiplying the second part of Eq. (49) by λ and adding it together with the first part of Eq. (49) gives

$$\bar{\eta}_2^*(I_{3\times3}\lambda^2 - \bar{J}^{-1}J_wS[W_a\bar{\nu}_{a,0}] - \bar{J}^{-1}T) = 0 \qquad (50)$$

The second condition of Eq. (48) can be simplified to

$$\bar{\eta}_2^*\bar{J}^{-1}J_wW_a = 0 \qquad (51)$$

Taking the conjugate transpose of Eq. (51), and noting that $W_a^* = W_a^{\text{T}}$, $J_w^* = J_w$, and $(\bar{J}^{-1})^* = \bar{J}^{-1}$, yields

$$J_wW_a^{\text{T}}\bar{J}^{-1}\bar{\eta}_2 = 0 \qquad (52)$$

Let $\tilde{\eta}_2$ be the real part of $\bar{\eta}_2$. Then, $\bar{J}^{-1}\tilde{\eta}_2$ is in the null space of W_a^{T}. Denoting $\bar{\eta}_u = \bar{J}^{-1}\tilde{\eta}_2$, Eq. (50) can be written as

$$\bar{\eta}_u^{\text{T}}(\bar{J}\lambda^2 - S[W_a\bar{\nu}_{a,0}]\lambda - T) = 0 \qquad (53)$$

Thus, for Eq. (26) to be uncontrollable, there must exist an $\bar{\eta}_u$ in the null space of W_a^{T} and an eigenvalue λ such that Eq. (53) holds. Reversing the arguments, it can be similarly shown that if, for each eigenvalue of A, Eq. (53) cannot be satisfied with $\bar{\eta}_u$ in the null space of W_a^{T}, then the system is controllable. □

Corollary 3.1: Theorem 3 can be simplified for the case when there are two operational RWs about the first two principal moments of inertia, i.e.,

$$W_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (54)$$

Let $t_{3,i}, i = 1, 2, 3$ be the ith entry of the third row of T. Because $\tilde{\eta}_u$ is in the null of $W_a^{\text{T}}, \bar{\eta}_u = [0 \quad 0 \quad \eta]^{\text{T}}$, where $\eta \in \mathbb{R}$. Equation (53) can be rewritten as

$$\eta\lambda^2[0 \quad 0 \quad J_3] - \eta\lambda[-J_w\nu_{0,2} \quad J_w\nu_{0,1} \quad 0] = \eta[t_{31} \quad t_{32} \quad t_{33}] \qquad (55)$$

which gives the following three equations:

$$J_3\lambda^2 = t_{33}, \quad J_w\nu_{0,2}\lambda = t_{31}, \quad J_w\nu_{0,1}\lambda = -t_{32} \qquad (56)$$

For the system to be uncontrollable, the three parts of Eq. (56) must all hold for at least one eigenvalue of A. By the converse, if for each eigenvalue of A, one of the parts in Eq. (53) is not satisfied (it does not have to be the same equation for each eigenvalue), the system is controllable.

Theorem 4 is a consequence of Theorem 3 and provides a sufficient condition for an underactuated system to remain linearly uncontrollable even if SRP torques are added into the math model.

Theorem 4: Let the spacecraft be equipped with two operational RWs whose spin axes are not parallel. The system [Eq. (26)] is linearly uncontrollable if $\bar{\eta}_u^{\text{T}}T = 0$ for $\bar{\eta}_u$ in the null space of W_a^{T}.

Proof: Suppose that, for $\bar{\eta}_u$ in the null space of W_a^T,

$$\bar{\eta}_u^T T = 0 \tag{57}$$

By Theorem 3, if there exists an eigenvalue λ of A such that

$$\bar{\eta}_u^T (\bar{J}\lambda^2 - S[W_a\bar{\nu}_{a,0}]\lambda - T) = 0 \tag{58}$$

then the system is uncontrollable. Equation (57) simplifies Eq. (58) to

$$\bar{\eta}_u^T (\bar{J}\lambda^2 - S[W_a\bar{\nu}_{a,0}]\lambda) = 0 \tag{59}$$

If Eq. (57) holds, then T is not full rank, which implies that A is not full rank and contains a zero eigenvalue. The eigenvalue $\lambda = 0$ satisfies Eq. (59), and therefore the system is uncontrollable.□

Remark 2: The results in Theorem 3, Corollary 3.1, Theorem 4, and numerical examples suggest that the dynamics of the spacecraft with two RWs and SRP effects included are linearly controllable in a broad range of cases. Theorem 4 indicates that the linearized SRP torque must have nonzero projection on the uncontrolled direction for linear controllability to hold.

B. Relative Controllability

To assess the relative controllability of different spacecraft configurations, we consider the following controllability index:

$$J_{\text{ind}} = \lambda_{\max}(e^{A^T(t_f - t_0)} M(t_f, t_0)^{-1} e^{A(t_f - t_0)}) \tag{60}$$

where $\lambda_{\max}$ denotes the maximum eigenvalue of a matrix, t_0 is the initial time of the maneuver, t_f is the final time of the maneuver, and $M(t_f, t_0)$ denotes the controllability gramian. The controllability index J_{ind} corresponds to the maximum effort (where effort is defined as the minimum of the integral of the input squared) required to bring an initial state $x(t_0)$ of unit norm to zero. Note that the metric is defined over a finite time interval, given that the system is open-loop unstable. This metric will be used later to assess the controllability of the cuboid spacecraft.

VII. Solar Radiation Pressure Effects on a Cuboid Spacecraft

Suppose that the spacecraft of interest is a cuboid, which is a commonly used spacecraft shape. This spacecraft has a symmetric body and has dimensions L_x, L_y, and L_z. Let $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ be the normals to the panels covering the spacecraft sides, which, because the structure of the cuboid, are parallel to the principal axes. The reference point O is chosen as the geometric center of the cuboid spacecraft. Using the notation from Sec. III, the parameters to obtain the SRP torque are defined as

$$\bar{r}_1 = \begin{bmatrix} \frac{L_x}{2} \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{r}_2 = \begin{bmatrix} 0 \\ \frac{L_y}{2} \\ 0 \end{bmatrix}, \qquad \bar{r}_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{L_z}{2} \end{bmatrix} \tag{61}$$

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{62}$$

$$A_1 = L_y L_z, \qquad A_2 = L_x L_z, \qquad A_3 = L_x L_y \tag{63}$$

The center of mass of the spacecraft will also be offset from O, and

$$\bar{r}_{C/O} = r_{C/O}|_{\mathcal{B}} = [l_x \quad l_y \quad l_z]^T. \tag{64}$$

This offset is of particular importance because it influences the relative controllability of the system.

A. Conditions for Zero Solar Radiation Pressure

Based on Theorem 2, it may be shown that, under certain conditions, the cuboid spacecraft can experience zero SRP torque for all $\bar{\Theta}$. By the physical structure of the cuboid spacecraft, $r_i \| \hat{u}_i$, $i = 1, 2, 3$, and so the first condition of Eq. (33) in Theorem 2 is satisfied. If O is aligned with the center of mass (i.e., $r_{C/O} = 0$), the second condition of Eq. (33) is satisfied. Assume that the diffusion coefficient is the same for all panels ($\beta_1 = \beta_2 = \beta_3 = \beta$). Then, using the definitions in Eqs. (61–63),

$$\sum_{i=1}^{3} \beta_i A_i \bar{r}_i \bar{u}_i^T = \frac{\beta L_x L_y L_z}{2} I_{3 \times 3} \tag{65}$$

which satisfies the third condition of Eq. (33) with $a = L_x L_y L_z/2$. Therefore, by Theorem 2, if $r_{C/O} = 0$ and all panels have the same diffusion properties, a cuboid spacecraft will experience zero SRP torque. It should be noted that, if β_i, $i = 1, 2, 3$, were different, the conditions for zero SRP torque given by Theorem 2 may not hold, and it may be possible to take advantage of SRP torques to recover linear controllability.

B. Equilibrium Analysis

Let $\tau_{\text{srp},x}$, $\tau_{\text{srp},y}$, and $\tau_{\text{srp},z}$ be the SRP torques exerted on the cuboid spacecraft by panels whose normals are parallel with the principal axes. Suppose that $\bar{u}_s = [n_1 \quad n_2 \quad n_3]^T$, and let $\bar{\Theta} = 0$. Then the SRP torques exerted on the spacecraft, given by Eq. (32), resolved in $\mathcal{B}$, are

$$\tau_{\text{srp},x}\big|_{\mathcal{B}} = \bar{\tau}_{\text{srp},x}$$
$$= \frac{A_1 \alpha n_1}{2} \begin{bmatrix} 0 \\ 2l_z + L_x \beta n_3 \\ -2l_y - L_x \beta n_2 \end{bmatrix} + \text{sign}(n_1) A_1 \alpha n_1 \beta \begin{bmatrix} l_y n_3 - l_z n_2 \\ l_z n_1 - l_x n_3 \\ l_x n_2 - l_y n_1 \end{bmatrix} \tag{66}$$

$$\tau_{\text{srp},y}\big|_{\mathcal{B}} = \bar{\tau}_{\text{srp},y}$$
$$= \frac{A_2 \alpha n_2}{2} \begin{bmatrix} -2l_z - L_y \beta n_3 \\ 0 \\ 2l_x + L_y \beta n_1 \end{bmatrix} + \text{sign}(n_2) A_2 \alpha n_2 \beta \begin{bmatrix} l_y n_3 - l_z n_2 \\ l_z n_1 - l_x n_3 \\ l_x n_2 - l_y n_1 \end{bmatrix} \tag{67}$$

$$\tau_{\text{srp},z}\big|_{\mathcal{B}} = \bar{\tau}_{\text{srp},z}$$
$$= \frac{A_3 \alpha n_3}{2} \begin{bmatrix} 2l_y + L_z \beta n_2 \\ -2l_x - L_z \beta n_1 \\ 0 \end{bmatrix} + \text{sign}(n_3) A_3 \alpha n_3 \beta \begin{bmatrix} l_y n_3 - l_z n_2 \\ l_z n_1 - l_x n_3 \\ l_x n_2 - l_y n_1 \end{bmatrix} \tag{68}$$

The conditions for $\bar{\Theta} = \bar{\omega} = 0$ to be a feasible closed-loop (relative) equilibrium are given by Theorem 1 and supported by Eqs. (66–68). Suppose, for instance, $l_x = l_z = 0$ while l_y is nonzero, whereas the operational RWs have spin axes aligned with the first and second principal axes. If $n_1 = 0$, $n_2 = 1$, and $n_3 = 0$, then based on the aforementioned expressions, $\tau_{\text{srp}}(0) = 0$, and the spacecraft can remain at the relative equilibrium with $\bar{\Theta} = \bar{\omega} = 0$ without any RWs accelerating. If the possibility of RWs accelerating to maintain the (relative) equilibrium at $\bar{\Theta} = \bar{\omega} = 0$ is acceptable (e.g., to enable the spacecraft to obtain images while compensating for nonzero SRP with RWs), then the condition can be relaxed. In this case, as long as $n_1 = 0$ (i.e., the third principal axis is not pointed toward the sun), the SRP torque is of the form $\tau_{\text{srp}}(0) = [* \quad * \quad 0]^T$ and can be

Table 1   Model parameters

Parameter	Units	Value
J	kg · m^2	diag(430, 1210, 1300)
J_w	kg · m^2	0.043
L_x	m	2
L_y	m	2.5
L_z	m	5
l_x	m	0
l_y	m	0.1 and 0.5
l_z	m	0
W	— —	$\begin{bmatrix} 1 & 0 & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 & 1/\sqrt{3} \\ 0 & 0 & 1 & 1/\sqrt{3} \end{bmatrix}$
Φ_{sun}	W/m^2	1367
$C_{\text{diff},j}$	— —	0.2
c	m/s	299792458.0

compensated by the available RWs that are along the first and second principal axes.

C. Relative Controllability of the Cuboid Spacecraft

To demonstrate controllability of the cuboid spacecraft, it is assumed that the spacecraft has parameters listed in Table 1, the first and second RWs are operational, and $\bar{\nu}_{a,0} = [100 \quad 100]^T$ rad/s. The third and fourth RWs are assumed to have previously failed and are now at zero speed. The plot of J_{ind} in Eq. (60) versus l_y and the time of the maneuver t_f ($t_0 = 0$) is given in Fig. 2. The spacecraft is more controllable for larger l_y, and the control effort decreases if longer maneuver time is available. Intuitively, more SRP torque is produced when the distance between the center of pressure and the center of mass is greater. This torque can be exploited to enhance controllability.

VIII.   Controller Design

Because the linearized dynamics of a spacecraft body with certain asymmetry properties are linearly controllable with the addition of SRP torques to the spacecraft model, a controller based on conventional linear quadratic (LQ) theory [30] can be used to stabilize the system to equilibrium. The LQ controller minimizes the cost

$$J_{LQ} = \int_0^\infty x(\tau)^T Q x(\tau) + \delta\bar{z}_a(\tau)^T R \delta\bar{z}_a(\tau) \, d\tau \qquad (69)$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$ are weighting matrices. The controller has the following form:

$$\delta\bar{z}_a = K_{LQ} x \qquad (70)$$
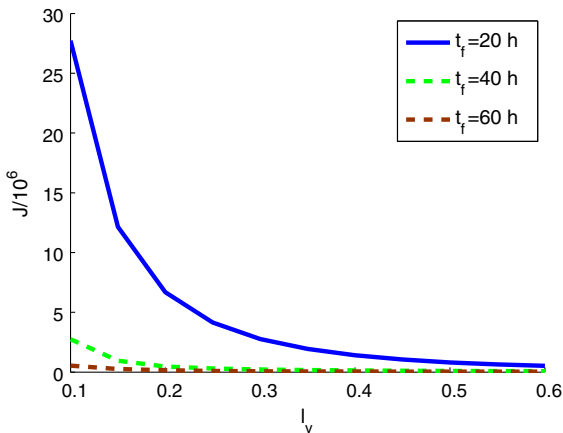
where K_{LQ} is the LQ gain.

Fig. 2   Controllability metric (scaled by 10^6) vs l_y and maneuver time, t_f.

In the subsequent simulations, $R = 1000 I_{m \times m}$, where m is the number of operational RWs. If $m \geq 3$, $Q = \text{diag}(10, 10, 10, 0.01, 0.01, 0.01)$ is used. If $m = 2$ and the only operational RWs spin about the first and second principal axes, $Q = \text{diag}(40, 10, 10, 0.04, 0.01, 0.01)$ is used to emphasize faster regulation of ϕ and ω_1, which improves the transient response.

IX.   Results

In this section, nonlinear simulations are presented where four wheels are initially operational. Two separate wheel failures occur during these simulations: one wheel fails at 5 h and the other at 20 h. The speed response of a failed wheel spinning down is modeled by a first-order lag with a 10 min settling time. The simulations are run on the full nonlinear model of the spacecraft kinematics and dynamics, including the nonlinear model of SRP torques. All spacecraft parameters are given by Table 1. Nominally, $\bar{u}_s = [0 \quad 1 \quad 0]^T$, and simulation results are presented for two different sequences of wheel failures. Additional simulation results are then reported for the case when $\bar{u}_s = [0 \, 1/\sqrt{2} \, 1\sqrt{2}]^T$, creating a situation where the RWs must accelerate to maintain spacecraft pointing. Finally, convergence times from different initial conditions are quantified.

A. Wheel 3 Fails First

The first case considered is when RW 3 fails first, followed by RW 4. The offset of the center of pressure from the center of mass is given by $l_y = 0.5$ m and $l_x = l_z = 0$ m. The responses are shown in Fig. 3. There is a much larger disturbance to orientation when the second wheel fails. The controller manages the first wheel failure quickly, then reconfigures and handles the second wheel failure over a longer period of time, coordinating two operational RWs in presence of SRP.

B. Wheel 4 Fails First, Reduced l_y

Now, the case is considered when l_y is reduced to 0.1 m so that the spacecraft is less controllable (see Sec. VI.B). In this simulation, RW 4 fails first, followed by RW 3. The responses are shown in Fig. 4. The controller is able to manage the failures and reduce the spacecraft orientation error over time. However, the responses are slower due to worse spacecraft controllability.

C. Wheel 4 Fails First, Skewed Pointing

In this case study, which is referred to as the skewed pointing, $l_y = 0.1$ m while $\bar{u}_s = [0 \, 1/\sqrt{2} \, 1\sqrt{2}]^T$. The controller is modified with an additional feedforward term, which cancels out $\bar{\tau}_{srp}(0)$. Note that the third component of $\bar{\tau}_{srp}(0)$ is zero; hence, canceling the steady-state value of SRP torque by the acceleration of RWs 1 and 2 is feasible. The response is shown in Fig. 5. Observe that the controller handles RW 3 and 4 failures and that RWs 1 and 2 continue to accelerate to be able to maintain the spacecraft orientation in steady state with desired pointing.

D. Achieving Faster Closed-Loop Response Time

A faster linear controller may be designed using the pole placement method. As confirmed by simulations, more aggressive controllers, however, have smaller regions of attraction (ROAs) for the nonlinear system and are not able to always recover after the simulated sequence of wheel failures. If spacecraft thrusters (which are not as precise as reaction wheels) are employed to first reduce the attitude and angular velocity errors before the two functional RWs are used, the implementation of more aggressive controllers for RWs becomes feasible. Toward this end, a pole-placement-based linear controller was designed with the closed-loop poles of $-0.0137 \pm 0.0068i$, $-0.0208 \pm .0021i$, -0.0001, and -0.0075 (versus $-0.0012 \pm 0.0068i$, $-0.0019 \pm .0021i$, -6.4906×10^{-6}, and -0.007 for the nominal LQ controller), and Monte–Carlo simulations were performed to characterize the closed-loop response from various initial conditions. For each simulation, the initial Euler angles were selected from a uniform distribution in the interval [−2, 2 deg], and the angular velocity was initially zero. Only two RWs (those aligned with the first two principal axes) were assumed to be
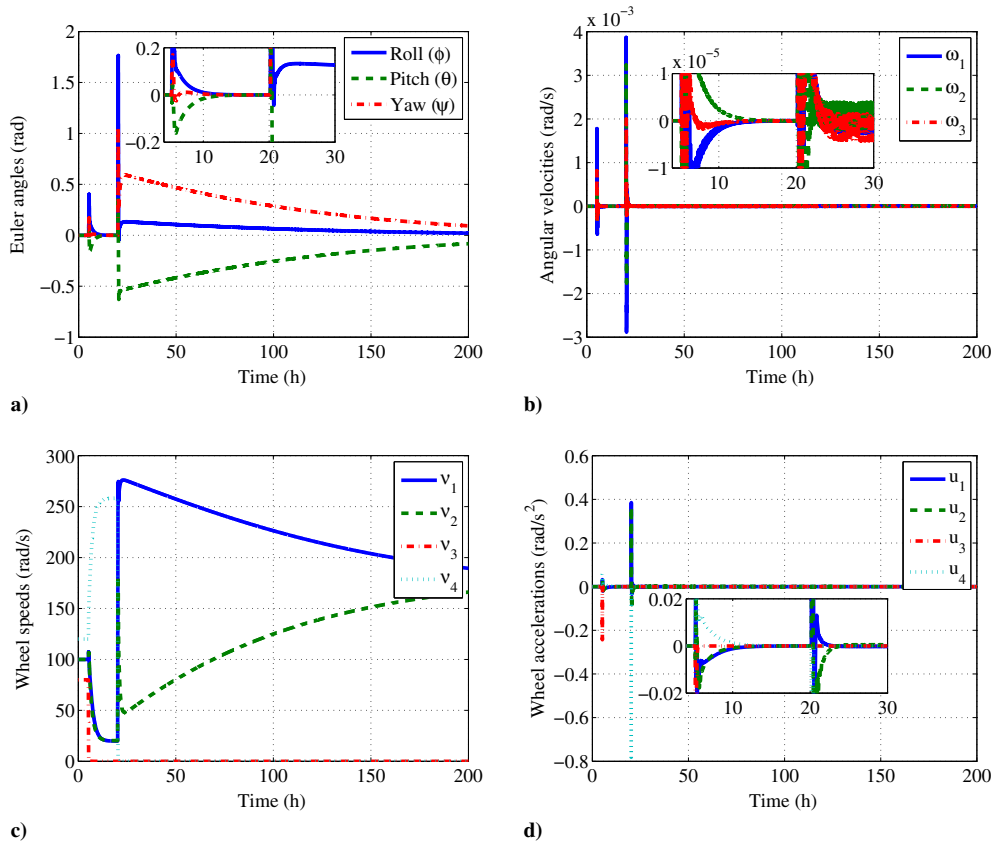
Fig. 3 Representations of a) Euler angles, b) angular velocities, c) RW speeds, and d) RW accelerations in the nonlinear simulation when wheel 3 fails first, $l_y = 0.5$ m.
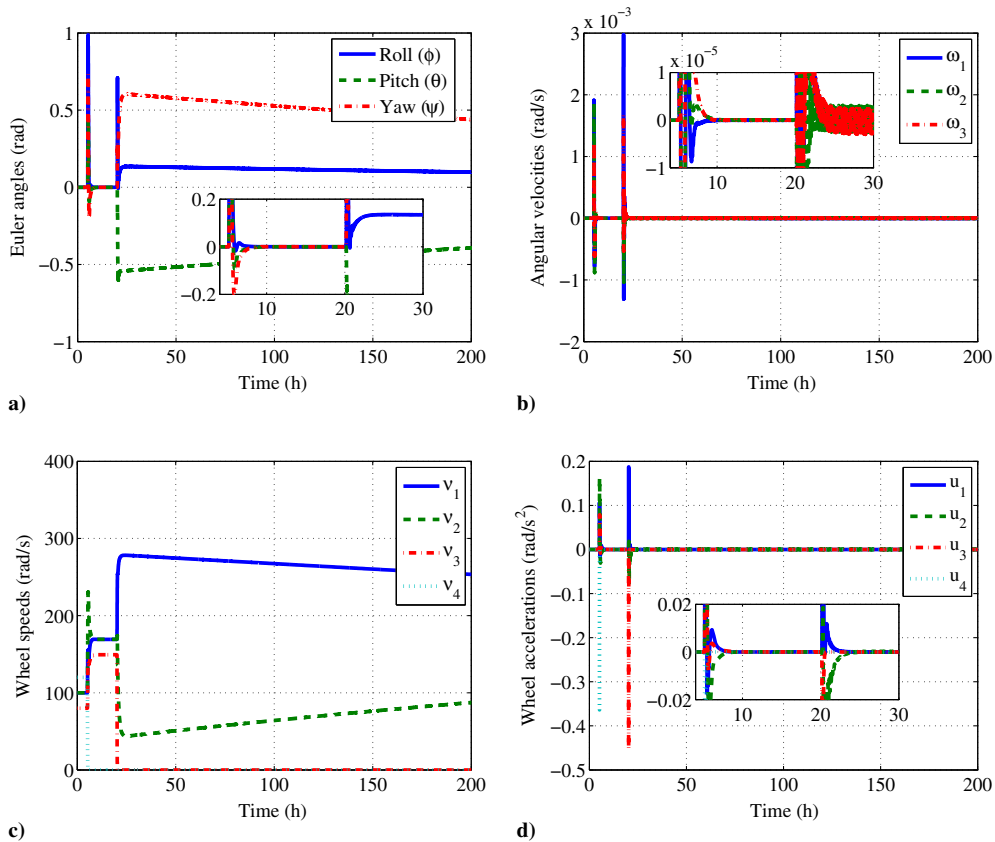

Fig. 4 Representations of a) Euler angles, b) angular velocities, c) RW speeds, and d) RW accelerations in the nonlinear simulation when wheel 4 fails first, $l_y = 0.1$ m.
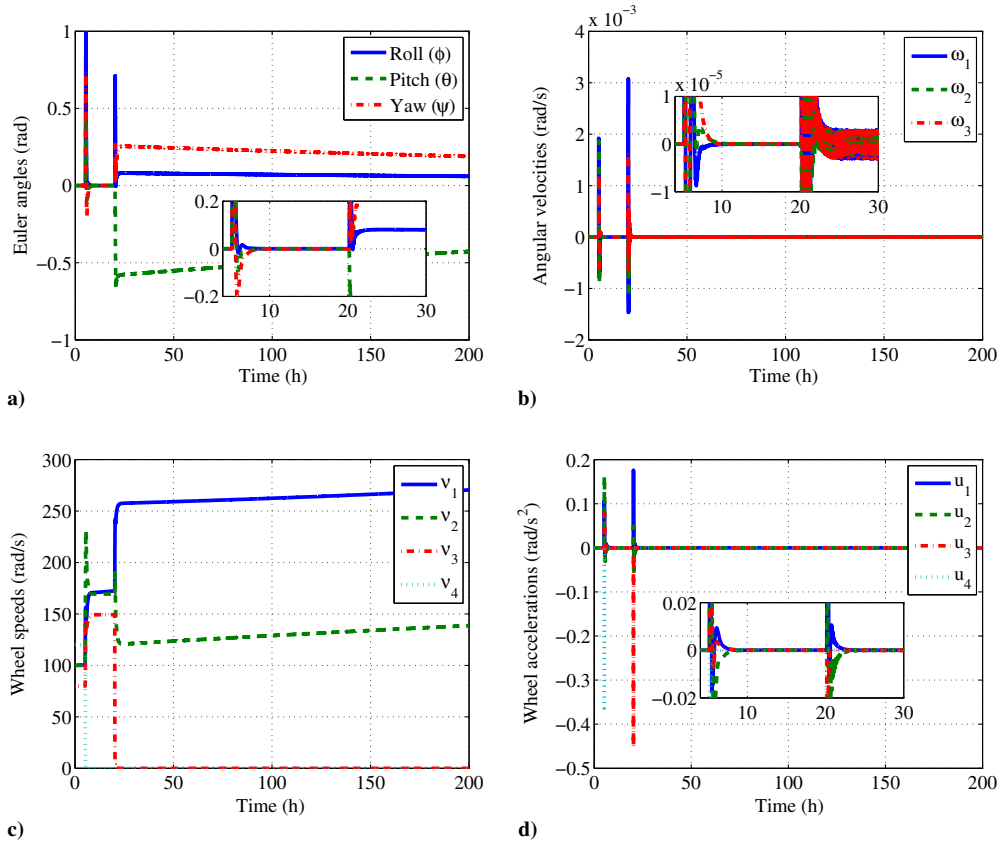
Fig. 5 Representations of a) Euler angles, b) angular velocities, c) RW speeds, and d) RW accelerations in the nonlinear simulation when wheel 4 fails first, skewed pointing.

functional, each initially rotating at a speed of 100 rad/s. 1900 simulation runs were performed. The average time it took for the spacecraft orientation to enter a 0.001 deg box around the equilibrium was 35.1554 h, with a standard deviation of 4.2532 h. In all cases, the controller was convergent and able to bring the Euler angles to the target box. The maximum angular speed and accelerations of the wheels in these simulations were 247.5929 rad/s and 9.7833 rad/s^2, respectively, which are within actuator capability limits.

E. Regions of Attractions

Each controller has associated with it a unique ROA such that if the spacecraft starts within this region, the controller will stabilize the system to equilibrium. Such regions can be estimated by using scaled sublevel sets of the Lyapunov function for the linearized system, but these estimates may be quite conservative [31]. The ROA in our case cannot be easily or analytically described for an arbitrary spacecraft because the dynamics depend on the shape of spacecraft, the desired
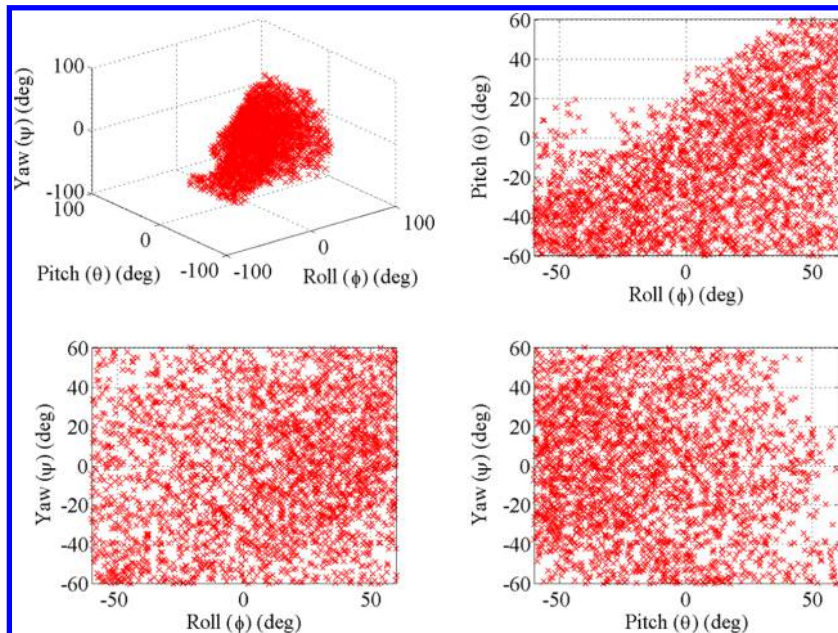

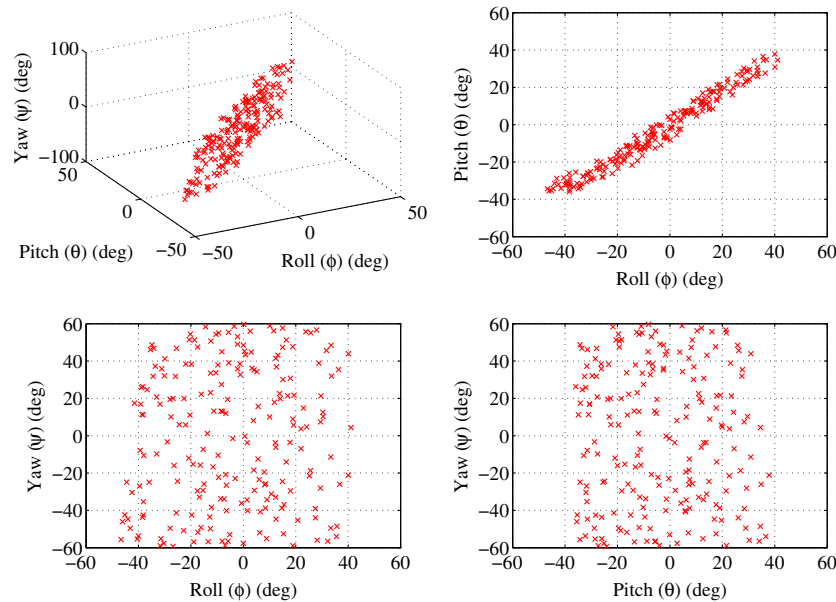**Fig. 6 Numeric ROA calculation for the LQ controller.**

Fig. 7 Numeric ROA calculation for the pole placement controller.

inertial pointing direction, the spin axis directions of operational RWs, and the chosen control scheme. Therefore, nonlinear simulations are used to approximate the ROA for a cuboid spacecraft using the LQ and the pole placement controllers in this work. 5000 random test simulations were run using each controller with initial Euler angles belonging to the interval of [−60, 60 deg], initial zero angular velocity, and RW speeds initially at 100 rad/s. The operational RWs are aligned with the first and second principal axes. The results for both controllers are shown in Figs. 6 and 7, in which the X's designate all the initial attitudes that the controllers can stabilize to the desired pointing equilibrium. As is to be expected, the LQ controller has a larger ROA than the more aggressive pole placement controller.

X. Conclusions

This paper treated an inertial pointing attitude control problem for an underactuated spacecraft with two reaction wheels. An unconventional pathway to recover linear controllability was demonstrated through incorporating solar radiation pressure torques into the spacecraft model. For certain fixed inertial pointing directions and realistic spacecraft configurations, linear controllability is recovered, and spacecraft control becomes feasible with conventional linear quadratic and pole placement techniques. Although the maneuvers take time, the approach presented is not restricted to the assumption of zero angular momentum; in fact, because of the presence of solar radiation pressure torques, the total angular momentum may not be zero and is not conserved. The results open up the possibility of applying a variety of conventional control techniques to the underactuated spacecraft control problem.

Acknowledgments

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