A "Re-Envisioned" Instruction Model:

Minimizing Students' Learning Difficulties in Mathematics

By

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Education
(Curriculum and Practice)
at the University of Michigan-Dearborn
2018

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Dedication

For the Camerons and Sams in our world.

For the learning theorists upon whose shoulders I stood.

For teachers who passionately strive to meet the needs of every child every day.

For God’s tender mercies along the way.

Acknowledgements

This work was made possible due to specific mentors in my life. Adele Sobania, Roger Verhey, Betty Garner, Rheta Rubenstein, Margaret Rathouz, Angela Krebs, Susan Deaner, Christopher Burke, and James Randall believed in me. Their counsel inspired and encouraged me to follow my dreams and passion for helping all students become mathematically powerful.

Thank you to Midwest district's director and the school's principal. They believed in this important work. Their support provided an excellent proving ground for the efficacy of the "re-envisioned" instruction model.

Thank you to the three teachers who participated in this study. The model's design imposed cognitive dissonance in their minds. Still, as warriors, they effected patience, steadfastness, and their own ingenuity for creating learning environments where students became reflective practitioners of their learning.

Thank you to all the students who participated in this study. Their minds and tender hearts revealed their innocence and trust in us as teachers.

Thank you to all the colleagues I have worked with throughout my career. Their work, reflections, wonderings, and questions caused me to reflect upon my own knowledge, beliefs, and practices. Through our associations and interactions, I have come to understand that this work is about inspiring others to accomplish things they did not believe possible.

A special thank you to my brother. His insightful counsel made a difference in the design of this study and our intense conversations instigated deep and thoughtful reflection.

A “forever” thank you to my dear, eternal companion. He was my fishing buddy when I needed to recalibrate. He inspired me to continue when I thought I could not write anymore. He was my advocate when I questioned if this work would make a difference for anyone.

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List of Abbreviations

CCSSM	Common Core State Standards for Mathematics
CCSSO	Council of Chief State School Officers
DBR	Design-Based Research
ELL	English Language Learners
LD	Learning Disability
MLE	Mediated Learning Experience
MTSS	Multi-Tiered Systems of Support
NCSM	National Council of Supervisors of Mathematics
NCTM	National Council of Teachers of Mathematics
NGA	National Governors Association
RTI	Response to Intervention
SES	Socio-Economic Status
ZPD	Zone of Proximal Development

Abstract

Persistent low levels of mathematics achievement are widely found throughout the United States. To reverse this trend, the researcher designed and tested the effects of a “re-envisioned” instruction model that incorporated a synthesis of six learning theories and current research from the fields of mathematics education, educational and cognitive psychologies, and neurosciences. General Tier I core instruction is the primary prevention component within a Multi-Tiered System of Support (MTSS). The “re-envisioned” instruction model and instructional tasks were used by two second-grade teachers to activate students’ cognitive structures and minimize students’ needs for Tier II and Tier III interventions. Cognitive structures are essential neurocognitive systems vital for student learning. This quasi-experimental research study was conducted in three second-grade classrooms. It took place in a midwestern part of the United States in a partial Title I Pre-K–5 elementary school. This study used mixed-research methods, including a variety of data collection instruments and statistical and qualitative measures, to explore four research questions. The findings suggest that effective implementation of the “Re-Envisioned” instruction model minimized the number of students needing Tier II and Tier III interventions and increased students’ mathematics achievement in statistically significant ways.

Keywords: cognitive structures, Tier I core instruction, inquiry-based instruction, mathematically “at risk” students, Multi-Tiered Systems of Support (MTSS), Elementary mathematics

Chapter 1: Introduction

Throughout the last decade, students’ mathematical proficiency levels have increased (National Center for Education Statistics [NCES], 2016). Students who increased their scores ranged from average to top performers in mathematics. Alternatively, the average to below-average students continue to underperform in this subject area and are “at-risk” for failure (Philipp et al., 2007). Data results from the Nation’s Report Card indicate that approximately 60% of our 4th-grade students and 65% of our 8th-grade students remain less than proficient in mathematics (NCES, 2015a).

Students’ lack of mathematics achievement holds long-term implications for society and for public education at large (Hill, Rowan, & Ball, 2005; Geary, 2011a). Low student achievement in this academic subject causes student failure in higher level math classes and adult life in general (Geary, 2013; Geary, Bailey, & Hoard, 2009; Jordan, Kaplan, Ramineni, & Locuniak, 2009). Future employability, rates of promotion, and annual incomes are in jeopardy as struggling mathematics students move into adulthood (Geary, 2011b). The mathematical skills and concepts learned in elementary school support the learning of advanced mathematics and are foundational for full participation in a technologically-advancing culture and society (Baroody & Ginsburg, 1990; Duval, 2006).

Noteworthy initiatives in mathematics education have been instituted to confront this trend (Ball & Cohen, 1999). These include delineated shifts in teachers’ instructional practices

(National Council of Teachers of Mathematics [NCTM], 2000), the creation of standards-based curricula, and the adoption of rigorous learning standards (National Governors Association & Council of Chief State School Officers [NGA & CCSSO], 2010). Unfortunately, these initiatives have not increased students’ levels of mathematical understanding and achievement at scale, especially for the average to below-average students (Ball, Hill, & Bass, 2005; Cohen & Ball, 2001). L.S. Fuchs and D. Fuchs (2001) claimed, “[The] prevention of mathematics difficulties in this country is generally ineffective not only for students with LD [learning disabilities], but also for nondisabled learners” (p. 85).

To potentially improve students’ mathematics achievement and learning on a greater scale, a review of the theoretical origins of students’ learning difficulties, accompanied by a re-examination of how learning occurs, was undertaken. An understanding of these key theoretical constructs and processes directed the design, implementation, and testing of the effects of an innovative instruction model used to implement general (Tier I) core mathematics instruction in two of three second-grade classrooms.

This research study was conducted at a partial Title I, K–5 school identified as Midwest Elementary School during the 2014–2015 school year. Two second-grade teachers used a “re-envisioned” instruction model to implement Tier I core mathematics instruction with their second-graders. The third second-grade teacher implemented mathematics instruction with her students using their district’s mathematics program and curricular resources. The time frame for applying treatment and collecting data was September 2014 through January 2015. Additional data collection occurred in May 2015—four months after treatment.

Results from this study illuminated important socio-cultural norms, essential pedagogical practices, and mathematical tasks that greatly influenced the experimental students’ increase in mean mathematics achievement scores beyond the scores of the control group. Quantitative results also indicated reduction in many students’ needs for Tier II and Tier III interventions relative to the school’s MTSS program. Analysis of qualitative results suggest ways teacher educators and mathematics coaches can support teachers and students shifting their beliefs and practices for teaching and learning mathematics.

A Perspective of the Problem

Seminal and contemporary learning theories were consulted for the literature review. Current research from the fields of mathematics education, educational and cognitive psychologies, and the neurosciences were also investigated. These studies highlighted cognitive processes essential for learning. They revealed key instructional practices, essential elements of productive learning environments, and important constructs for re-designing mathematical tasks to support students learning concepts at a deeper level.

Learning is cognitively and neurologically generated by the learner. Learning is the ability to create and strengthen one’s own neurological networks within the mind, creating new neuronal nodes that link to existing nodes (Devlin, 2010; Ifenthaler, Masduki, & Seel, 2011). Neurological connections and the strengthening of these connections occur through the mental processes of assimilation and accommodation. These mental processes require direct sensory exposure to environmental stimuli and mediated exposures to social stimuli such as language, signs, and symbols (Feuerstein, Feuerstein, Falik, & Rand, 2006; Vygotsky 1978/1930). Outward signs of learning include, but are not limited to, one’s ability to “make connections with prior knowledge and experiences, identify patterns [and relationships], identify predictable rules, and abstract generalizable principles” that can be applied to additional contexts and conditions (Garner, 2007, p. xiii).

The literature review also identified several theoretical origins for students' learning difficulties related to learning mathematics. These include, but are not limited to, historical, biological, cognitive, socio-cultural, and pedagogical (Baroody, 2011). Collectively, these categories exemplify the complex processes of teaching and learning this academic subject. For this study's purpose, only four of the five origins are specifically addressed through the "re-envisioned" instruction model, namely: biological, cognitive, socio-cultural, and pedagogical.

To introduce the need for this study, this next section briefly describes the nature of students' learning difficulties given a historical perspective. General descriptions of the other four origins are found in Chapter 2.

Historical Origin

A brief historical overview affords the reader an understanding that the problem of learning difficulties in mathematics is not a recent phenomenon, and neither are educators' and legislators' efforts to address them. Beginning in the early 1960s, federal initiatives attempted to target consistently low-performing students in significant ways. In 1963, the United States government issued the federal law Mental Retardation Facilities and Community Mental Health Centers Construction Act (P.L. 88-164). This law provided researchers monetary assistance to study and understand mental retardation and learning disabilities experienced by American students.

Additional public laws followed. In 1975, the federal government authorized the Education for All Handicapped Children Act (EAHCA), and then renamed and reauthorized this act as the Individuals with Disabilities Education Act of 1997 (IDEA). The 1997 law utilized a discrepancy process for identifying students with learning disabilities (Bradley, Danielson, & Doolittle, 2007; Fuchs & Fuchs, 2007; Riccomini & Smith, 2011). The 1997 law directed teachers and psychologists to document at least a two-year discrepancy between a student’s intellectual quotient and his or her academic achievement before providing and funding additional educational services (Fuchs & Fuchs, 2007). Under these federal requirements, a student “waited to fail” in mathematics, often until the 5th grade, before additional support or interventions were administered (Bradley et al., 2007; Fuchs, Fuchs, & Compton, 2012; Gresham & Little, 2012).

Fortunately for students and their learning, in 2004 the United States government amended the Individuals with Disabilities Education Improvement Act (United States Department of Education, IDEIA, 2004). It now read, “In determining whether a child has a specific learning disability, a local agency may use a process that determines if he [or she] responds to scientific, research-based intervention as part of the evaluation process” (20 U.S.C. §1414[b][6]). This legislation became the source waters for the phrase “Response to Intervention” (RTI), now titled “Multi-Tiered Systems of Support” (or MTSS).

The amended IDEIA (2004) law focused on improving student learning through a variety of means. One major alteration provided educators the ability to identify struggling students early in their formal schooling and permitted states to discontinue the use of the IQ discrepancy process. Thus, educators no longer waited for student failure before meeting their educational needs. Further amendments afforded more flexibility in the referral process, an increase in parental involvement, and included the use of evidence-based instructional materials and practices within the general education classroom. Fortunately for students and their learning, various options for early identification of students’ learning disabilities were finally endorsed by the federal government (Fuchs & Fuchs, 2007; Fuchs et al., 2012; Riccomini & Smith, 2011).

Once the 2004 IDEIA law was formally authorized and legally instituted, educational administrators quickly capitalized on these vital amendments. Central school administrations began implementing RTI (now known as MTSS) processes, including early identification and intervention. Naturally, educators wanted to reduce the number of students needing special education and/or tertiary programming, as well as to prevent students’ academic failure (Fuchs & Fuchs, 2007). Essentially, MTSS became a system for differentiating “between two explanations for low achievement: inadequate instruction versus disability” (Fuchs & Fuchs, 2007, p. 14).

Although IDEIA was a step in the right direction, the United States Department of Education and policy makers did not recommend, nor endorse a specific MTSS model for schools and teachers to systematically institute (Bradley et al., 2007; Fuchs, Fuchs, & Stecker, 2010). Nor did research identify a single model to be effective in all situations and in all cases (Mellard, McKnight, & Jordan, 2010). Left to interpret the implicit meaning within the law, many school staff interpreted the purposes and design of a MTSS model for themselves (D. Fuchs & L.S. Fuchs, 2005). School staffs readily instituted their own versions of MTSS frameworks to meet their students’ literacy and mathematics needs early on (Lembke, Hampton, & Beyers, 2012; Mellard et al., 2010).

To assist educators’ efforts, several national organizations offered their own interpretations of the law. For instance, leaders of the National Council of Supervisors of Mathematics (NCSM, 2013) published a position statement declaring that prevention and intervention measures are necessary and essential in *all* K–12 mathematics programs. They described RTI as a “systematic, data-based method for identifying, defining, and resolving students’ academic difficulties using collaborative, school-wide, problem-solving approaches”

(p. 1). The RTI Action Network (2015) defined RTI as a multi-tiered system for identifying and supporting students who have learning difficulties.

During a national conference in 2010, Francis (Skip) Fennell, former president of National Council of Teachers of Mathematics and professor of education at McDaniel College in Westminster, Maryland provided a brief description of the purposes for a RTI or MTSS model in mathematics. He described,

> Response to intervention (RTI) can be thought of as an early detection, prevention, and ongoing support system that identifies students and provides them with the support they need BEFORE they fall behind and before they are formally identified and designated for special education services. (Fennell, 2010, slide 2)

Key words and phrases in Fennell’s description stood out as critical themes: *early detection, prevention, ongoing support system, identifies students, provides them with support, before students fall behind,* and *before formal identification in special education services*.

To better understand how the teaching staff from Midwest Elementary School perceived the purpose of their RTI (or MTSS) model, Fennell’s (2010) terms and phrases were used to code staff’s perception data to the following question, “What is your working definition for Response to Intervention in mathematics?” Twenty-six teachers’ written responses were coded according to Fennell’s definition and themes. Results indicated that 19 of 26 teachers perceived the purpose of their school’s RTI program as *identifying students and providing support*. One-fifth of teachers’ responses suggested *early detection*. Three teachers mentioned the use of *prevention* strategies. Three included an *ongoing system of support* and one teacher suggested intervening *before students fall behind*. Still, the word “before” was never explicitly articulated in their 26 written definitions.

Teachers’ responses suggest that all 26 teachers held differing interpretations or definitions of their RTI (MTSS) program enacted within Midwest Elementary. Significantly, most teachers did not perceive their model as providing preventative measures for students’ learning difficulties. Instead, they perceived their RTI (MTSS) model as a system for remediation.

Challenging Expectations for Teaching Mathematics

Just as preventative medicine promotes good health, preventing students’ learning difficulties from occurring in the first place is an important philosophical stance and means for increasing students’ mathematics achievement if “educational improvement is to be a long-term, generative process” (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 9). The proponents of MTSS endorse effective Tier I core mathematics instruction as the *intended prevention mechanism* for all students in general education classrooms (Clements & Samara, 2007; Gresham & Little, 2012; NCSM, 2013). Furthermore, all students’ instructional needs can be met by knowledgeable teachers using evidence-based pedagogy and effective instructional design (NCTM, 2000).

Thus, the choices school districts, curriculum directors, general education teachers, and interventionists make for engaging students in learning mathematics hold explicit implications for improving student achievement. The mathematical tasks and instructional practices teachers use, the classroom environments teachers and students co-create, and the ways teacher and students engage with content are essential pathways for preventing students’ learning difficulties from occurring, as well as minimizing student difficulties when manifested (City, Elmore, Fiarman, & Teitel, 2009). Collectively, such processes for teaching and learning create challenging expectations for school districts, staff, and individual teachers to take on single-handedly.

Theoretical Framework

To understand the complexities of teaching and learning mathematics, multiple theoretical perspectives—both seminal and contemporary—were consulted. A variety of databases were accessed to identify and understand student learning and cognition. These databases included Psych INFO, Wiley Online Library, Open Access, Google Scholar, Google Search, JStor, ResearchGate, SAGE, ERIC, NCTM, and NCSM. Key terms such as cognitive structures, cognition, representations, spatial reasoning, learning theories, developmental stage theory, learning difficulties in mathematics, constructivism, socio-constructivism, and mediated learning theory were helpful in accessing prior research. Researchers' names such as Jerome Bruner, Jean Piaget, Lev Vygotsky, Reuven Feuerstein, Betty Garner, and David Geary were primary key terms used in the electronic search. Additional journal articles were selected based upon the researcher's knowledge and on the plethora of references existing within the extensive compilation of literature.

Subsequently, Design-Based Research (DBR), also known as Design Experiment, became the theoretical backdrop for designing and conducting this study (Barab & Squire, 2004; Cobb et al., 2003). Tracing back to the advanced and pragmatic research practices authored by Ann Brown (1992), DBR uses intervention to improve existing learning conditions as well as inform teacher practices. Cobb and his colleagues (2003) explained that DBR researchers labor to empirically fine tune what works while developing theories of intervention targeting specific domains.

DBR’s processes supported the design, implementation, and testing of the “re-envisioned” instruction model implemented in two of the three second-grade classrooms for this study. To advance existing theoretical constructs for learning and transcend the environmental contexts of these authentic classrooms, the instruction model utilized a synthesis of six learning theories and evidence-based practices (Barab & Squire, 2004). Then, as unexpected contextual conditions surfaced, as teachers’ and students’ differing needs arose, or when implementation of the treatment was deemed unsuccessful, DBR’s flexible design enabled teachers and researcher, as co-participants, to collaboratively analyze and revise the treatment. Although changing one aspect of the treatment created perturbations in other aspects of the treatment, attempts were always made to isolate dependent variables for testing purposes (Brown, 1992; City et al., 2009). Variables included the socio-cultural environment of each classroom and teachers’ varying levels of pedagogical and content knowledge for teaching mathematics (Ball et al., 2005). Students’ readiness to learn specific mathematical concepts was another variable impacting this study.

Research purists claim that DBR’s non-rigorous methodologies for establishing validity and reliability often lead to inaccurate results and reporting of data (Barab & Squire, 2004). To ensure valid and reliable results from this study, a quasi-experimental, concurrent mixed-methods approach (as described by Creswell, 2009) incorporating a pre- and post-test non-equivalent three-group and time series design was utilized (Cook & Campbell, 1979). The quasi-experimental aspect of the study enabled the detection of similarities and differences in student outcomes between students receiving intervention and those who did not. Concurrent mixed-methods data collection afforded the triangulation of multiple data sets. Merging thick descriptions of qualitative data with statistical results substantiated or challenged research results, thereby improving the validity of inferences made (Geertz, 1973). A pre- and post-test group design controlled threats to internal validity of the experiment, such as students' natural cognitive maturation (Johnson & Christensen, 2012). A time series design, whereby the same assessment measure was administered four months apart, helped determine the impact the model had upon students' learning mathematics and the development of two cognitive structures.

With roots firmly planted in an empirical base of learning theories, the "re-envisioned" instruction model was implemented in "average classrooms operated by and for average students and teachers" utilizing DBR processes (Brown, 1992, p. 143). Subsequently, this study became a "crucible for the generation and testing" (Cobb et al., 2003, p. 9) of the "re-envisioned" instruction model used to implement Tier I core instruction in two of three second-grade mathematics classrooms.

Cognitive Structures

Central to this study was the conceptual design and teachers' implementation of the "re-envisioned" instruction model and re-designed tasks to cognitively engage students. These elements were all aligned to learning theories predicated upon the mediation and development of students' cognitive structures. Activating and engaging students' cognitive structures afforded a natural and viable approach for increasing students' mathematics achievement and minimizing their learning difficulties in this academic subject.

Cognitive structures are humans' natural neuronal-mechanisms for learning (Geary, 1995). They are defined as neurocognitive networks within the mind. Different types of cognitive structures exist. Primitive structures appear to be inherent and existent at birth (Geary, 1995). Given appropriate stimuli and mediation, primitive structures develop into more complex neuronal systems supporting higher cognitive functioning (Feuerstein et al., 2006; Garner, 2007). For example, short- and long-term memories are mental structures integral to the assimilation and accommodation of new information. As students mentally accommodate new concepts or ideas, their minds generate new neuronal nodes, becoming linked to existing neuronal structures (Ifenthaler et al., 2011). The level of development of one’s cognitive structures influence “the way in which an individual arranges facts, concepts, propositions, theories, and raw data at any point in time” (Ifenthaler et al., 2011, p. 42). If an individual’s cognitive structures are highly organized, his or her structures support comprehension, integration, retention, and application of new information.

Cognitive structures such as memory, recognition, visualization, spatial orientation, and conservation of constancy enable students to make connections between and among concepts (Garner, 2007; Kamii, Lewis, & Kirkland, 2001b). Students’ abilities to make connections between and among mathematical representations and properties are vital cognitive processes for learning mathematics (Ball, 2001; Ball et al., 2005; Duval, 2006; Skemp, 1976/2006; Van de Walle, Karp, & Bay-Williams, 2012).

Purpose of the Study and Research Questions

The aims for conducting this research were multi-purposed. The first two goals were to understand students’ learning difficulties and the different learning theories relative to mediating the development of students’ cognitive structures. This information afforded insights into how a synthesis of these theories might increase students’ mathematical understanding and achievement. That knowledge led to the third goal, which was to align the theories to current research and best practices from the fields of mathematics education, educational and cognitive psychologies, and the neurosciences. The fourth and fifth goals were to design an instruction model used for Tier I core mathematics instruction that mediated the development of students’ cognitive structures, namely conservation of constancy and spatial orientation and then test the model’s effectiveness toward these ends. The final goals were to determine if the model was viable—meaning that teachers could effectively implement the instruction model using key processes and practices—and that students in the two experimental groups increased their understanding of mathematics in statistically significant ways when compared to the control group.

Utilizing a quasi-experimental mixed-methods approach, this study investigated the following questions:

1. To what extent did teacher implementation of the “re-envisioned” instruction model influence students’ mathematics achievement?

   $H1_0$: The change in students’ mathematics achievement scores between students who received treatment and students in the control group were not statistically different as determined by pre- to post- to end *i*-Ready Universal Screener assessments (Curriculum Associates, 2015).

   $H1_a$: The change in students’ mathematics achievement scores between students who received treatment and students in the control group were statistically different as determined by pre- to post- to end *i*-Ready Universal Screener assessments (Curriculum Associates, 2015).

2. Did teacher implementation of the “re-envisioned” instruction model minimize students' learning difficulties in mathematics? In other words, did teacher implementation of the model move students identified at Tier II and Tier III levels to Tier I and Tier II levels respectively as determined by the pre- to end tests from the *i*-Ready Universal Screening Assessment (Curriculum Associates, 2015)?

$H2_0$: When comparing students in the two treatment groups to students in the control group, there were no statistical differences in count patterns representing students' decrease (improvement) in Tier Levels from pre- to end according to the *i*-Ready Universal Screener assessment data.

$H2_a$: When comparing students in the two treatment groups to students in the control group, there were statistical differences in count patterns representing students' decrease (improvement) in Tier Levels from pre- to end according to the *i*-Ready Universal Screener assessment data.

3. To what extent did teacher implementation of the "re-envisioned" instruction model influence the development of students' cognitive structures, specifically spatial orientation and conservation of constancy?

$H3_0$: When comparing the treatment groups' development of their cognitive structures to the control group's development of their cognitive structures, there were no statistical differences as determined by pre- to post- test scores on the Adapted Cognitive Structure Assessment results.

$H3_a$: There were statistically significant differences in students' development of their cognitive structures between students who received treatment and students who did not as determined by students' pre- to post- test scores on the Adapted Cognitive Structure Assessment results.

4. By the end of this study, to what extent did teacher implementation of the "re-envisioned" instruction model influence students' beliefs and practices for learning mathematics?

$H4_0$: By the end of the study, qualitative differences in students’ beliefs and practices for learning mathematics did not exist between students who received treatment and students who did not as indicated by students’ and teachers’ qualitative data.

$H4_a$: By the end of the study, qualitative differences in students’ beliefs and practices for learning mathematics existed between students who received treatment and students who did not as indicated by students’ and teachers’ qualitative data.

Potential Limitations and Weaknesses

Limitations existed due to the complex nature of the research study and the design of the instruction model. At the beginning of the school year, the two teachers of the experimental groups were not ready to implement the instruction model as designed. Therefore, at the beginning of the study and periodically throughout, teachers requested implementation support. Support was provided in the form of modeling mathematics instruction—especially when instruction pertained to introducing novel mathematical representations. Furthermore, teachers requested lessons and tasks that supported their implementation of the “re-envisioned” instruction model. These forms of support may have influenced this study’s results thereby limiting generalization to other contexts (Barab & Squire, 2004).

Another limitation to this study involved using a small sample size of study subjects or participants. For the quantitative portion of this study, participation was limited to students across all three classrooms (n = 54). A small sample size limited the potential for generalizing results to a wider population (Johnson & Christensen, 2012).

A third limitation was that the participants were a convenience sample. All second-grade teachers were involved with elements of the “re-envisioned” instruction model the previous school year. As such, the teacher in the control group struggled in philosophically staying true to the design of the mathematical tasks and lessons found in her school district’s mathematics program.

A fourth limitation was that true experimental studies require random sampling (Creswell, 2009). Prior to this study, the school’s principal and first-grade teachers predetermined student placement with specific second-grade teachers. The principal also determined the classrooms of students who received treatment and those who functioned as the control. As such, this study was conducted using non-equivalent groupings.

A fifth limitation to this study involved the instruments used to collect, code, and quantify data. Some of the qualitative instruments were generated by the researcher and used for the first time in this study. Thus, these instruments were not tested for validity nor reliability.

Although other educators, researchers, and a statistician were consulted throughout the study, much of the qualitative data were analyzed and reported by one person. Rather than obtaining different perspectives and orientations to underlying phenomena, the data analysis and inferences may include biases relative to mathematics education. This presents the sixth foreseen limitation to this study.

Significance of the Study

Given these limitations, there were several factors motivating this research. First and foremost, the current data from the NCES (2016) suggest that the United States educational systems have not generated the level of mathematics achievement our students need to thrive in the 21st century (Stigler & Hiebert, 1999). Mathematics instruction that improves students’ mathematics achievement and prevents students' learning difficulties from occurring in the first place has been historically elusive and challenging to implement. No mathematics program is 100% effective for preventing students' mathematics learning difficulties (Fuchs et al., 2010). Accordingly, national concerns over the effectiveness of mathematics education and student learning are amplified (NCES, 2015a; National Research Council, 2001); and researchers continue to investigate ways to increase student achievement (National Research Council, 2001).

Secondly, it was the researcher's experience that districts adopting a MTSS model often hired untrained personnel (due to budget constraints) to enact Tier II interventions with their students. Tier II interventions require skilled personnel to implement explicit forms of supplemental interventions (Steedly, Dragoo, Arafeh, & Luke, 2008). The definitions for Tier II and Tier III levels of interventions are found in the glossary at the end of this chapter.

Thirdly, few research studies describe specific ways classroom teachers can implement general Tier I core mathematics instruction to mediate the development of students' cognitive structures. This includes the influence specific cognitive structures may have upon students' learning of mathematics. Furthermore, very few schools are implementing an instruction model like the one represented by the "re-envisioned" instruction model.

Hypothetically, designing and implementing an instruction model that utilizes a synthesis of learning theories, evidence-based instructional practices, and mathematical tasks that activate students' existing cognitive structures could invite and foster students' natural facility for learning. Thus, the results from this study may offer researchers and educators petite *generalizations* (Barab & Squire, 2004). One generalization is that this "re-envisioned" instruction model can be used to implement Tier I core mathematics instruction. Another generalization is the importance for mediating the development of students' cognitive structures to increase student learning.

Most importantly, obtaining higher levels of mathematics cognition and achievement affords equal opportunity for all students. Obtaining mathematical understanding and proficiency, as described by the Common Core State Standards of Mathematical Practices, prepares students for success in the 21st century workforce and within our global community (NGA & CCSSO, 2010). When all students can use mathematics to address our global community's needs, we will find solutions to the simple and complex problems we face today. Then, and only then, will everyone begin to "experience mathematics in ways that allow them to change the conditions of [all of our] lives" (Martin, Gholson, & Leonard, 2010, p. 17).

Definitions and Terms

The following terms and definitions provide the reader a foreshadowing of key concepts and vocabulary found throughout this study. These terms aid the reader's interpretation and comprehension of terminology found within various fields of education (e.g. educational psychology, mathematics education, special education, etc.).

- *Accommodation* occurs within the mind. When a new concept does not fit with existing schema, the result is a state of confusion, disequilibrium, or cognitive conflict for a student. To rectify the mental conflict, the human mind modifies existing structures or schema by generating new neuronal nodes and pathways (Ifenthaler et al., 2011). This physiological mental "rewiring" generates new neurological architecture or a re-design of students' schema, thus prompting stronger and more efficient capacities for connecting concepts, synthesizing information, and intellectualizing generalizations (Van de Walle et al., 2012).

- *Assimilation* occurs when concepts correspond with one's existing schema. Repeated practice of a known skill or concept results in the physiological mental "strengthening" of existing neuronal networks. As skills and practices are revisited, the myelin sheaths of axons within the human mind thicken, thus creating stronger and more efficient flows of electrical current (Devlin, 2010).
- *Cognitive structures* are innate interconnected sensory-motor and conceptual neurocognitive systems foundational for learning and for making meaning of one's external world (Geary, 1995; Mink, 1964). These mental structures are also known as knowledge structures or schema (Garner, 2007).
- *Conceptual understanding* is one's ability to make sense of and explain mathematical situations and underlying structures, and make connections between and among key mathematical ideas, procedures, and representations. Moreover, students use conceptual understanding to analyze various approaches and solutions to problems, apply knowledge to unfamiliar mathematical situations, and construct generalizations to solve problems (Ma, 1999; NCTM, 2014).
- *Constructivism* is a philosophical perspective that assumes human beings are already knowledgeable, thereby capable of creating, building, or constructing new understandings, giving new meaning to things or experiences. Constructivism occurs as students link what they already know (prior schema) to new learning. One's active construction modifies existing schema within the mind by assimilating or accommodating new information into existing mental networks (Cobb, 1994; Ifenthaler et al., 2011; Montague, 1997; Noddings, 1990; Van de Walle et al., 2012).

- *Disequilibration* is cognitive dissonance or cognitive tension generated within a student’s mind when his or her perceptions differ from reality. This tension is released by the person transitioning to higher levels of cognitive thought through the acts of accommodation and assimilation resulting in equilibration of cognitive thought (Fox & Riconscente, 2008).
- *Explicit instruction* involves highly organized, step-by-step processes whereby more knowledgeable persons make their thinking and decision-making processes visible and audible to learners. This form of instruction progresses from modeling and providing explanations of concepts and strategies to students practicing the action or concepts independently (Steedly et al., 2008; Van de Walle et al., 2014).
- *Figural units* are the different elements or attributes of an object or representation a student “quickly recognizes as significant or informative” (Duval, 2014, p. 160).
- *Instrumental understanding* involves learning mathematical rules and properties, basic math facts, procedures and algorithms without sense-making or reasoning processes. This is equivalent to procedural understanding (Skemp, 1976/2006).
- The *learning environment* includes learning conditions, instructional methods, and tasks found within the classroom impacting students’ intellectual, physical, social, and emotional well-being.
- *Logico-mathematical knowledge* is a network of inter-related mental relationships within one’s mind. These mental relationships support students making connections between parts and their respective wholes (Kato, Kamii, Ozaki, & Nagahiro, 2002). Logico-mathematical knowledge is developed by the individual.

- *Mathematical structure* is represented by numeric patterns (e.g. place value), numerical relationships (e.g. 6 is comprised of 5 and 1 more), classifications (all even numbers, squares are also rectangles), and mathematical properties (such as commutative, associative, identity, and inverse relationships).
- *Mediated Learning Experiences* (MLE), are catalysts for human experience and cognitive change. MLE involves the transmission of knowledge and information by a knowledgeable mediator who provides mediating interactions targeting the growth and development of a student’s cognitive structures (Feuerstein et al., 2006).
- *Mediated tasks* are activities intentionally designed to expose learners to stimuli that induce changes in the nature and structure of students’ neurocognitive networks or schema.
- *Metacognition* involves conscious awareness of one’s self as knower—as an awareness “of one’s own thoughts and thought processes” and the knowledge of one’s “capability of communicating one’s rationale” relative to perspective, reasoning and actions (Fox & Riconscente, 2008, p. 378). It involves intentionality, intelligence, logical and empirical thinking, and verbal communication.
- *Number sense* is defined as having a fluidity and flexibility for thinking about and working with number and operations. It includes “moving from initial development of counting techniques to more-sophisticated understandings of the size of numbers, number relationships, patterns, operations, and place value” (NCTM, 2000, p. 79).
- *Relational understanding* is the ability to see connections between and among concepts. It also involves one knowing and understanding the rules and properties of mathematics, along with procedures and algorithms, understanding why they exist,

how they work, appropriately applying them in different situations, and finding efficient and effective solutions to unfamiliar problems (Skemp, 1976/2006).

- *Representations* symbolize important features of mathematical constructs and actions that hold meaning. They illustrate ideas through visual, physical, contextual, verbal, and symbolic means. Words, story problems, symbols, diagrams, tables, graphs, equations, physical and pictorial models are all forms of mathematical representations (NCTM, 2014)
- *Scaffolding* is the effort on the part of a more knowledgeable person to control elements of a task or learning activity that are initially beyond the zone of a learner’s capabilities, thereby allowing the learner to focus on and complete elements that are within the range of aptitude (Wood, Bruner, & Ross, 1976).
- *Schemata (or structures)* are cognitive networks which hold orientation to a class of previous experiences, developed skills, and/or action sequences. Schemata enable students to relate and connect previous experiences and prior knowledge to new or current experiences and events by assimilating and accommodating information.
- *Self-regulation* is the “deliberate control of one’s thoughts and actions” relative to self, to knowledge of others and objects, and to use of language (Fox & Riconscente, 2008, p. 380).
- *Tier I core instruction* is defined as the primary prevention component where all students receive general core content through high-quality classroom instruction. This includes evidence-based classroom practices that provide instructional differentiation, accommodations, and strategies that address students’ learning, motivation, and behaviors (Fuchs et al., 2012; Fuchs et al., 2010). Tier I instruction should meet the

learning needs of at least 80% of the student population within a classroom and school (RTI Action Network, 2015).

- *Tier II instruction* is considered secondary prevention. Supplemental to Tier I core instruction, it involves small-group tutoring for students unresponsive to Tier I instruction. Mathematics instruction at this tier requires a specialist or interventionist who utilizes evidence-based pedagogy to accelerate students’ acquisition of new skills leading to mathematical proficiency (Fuchs & Fuchs, 2007; Fuchs et al., 2010). Tier II instruction should meet the learning needs of 10–15% of the student population within a classroom (RTI Action Network, 2015).
- *Tier III instruction* is considered tertiary intervention. It involves the most intensive forms of instruction and interventions, including multidisciplinary forms of evaluations to determine possible learning disabilities, individualized programming, and progress monitoring. This level of instruction is for students considered to be high-risk within the learning context because they fail to respond to both Tier I and Tier II forms of prevention (Fuchs & Fuchs, 2007). Tier III instruction should meet the learning needs of 3–5% of the student population within a classroom or school (RTI Action Network, 2015).
- *Understanding* is the “measure of the quality and quantity of connections that an idea has with existing ideas” (Van de Walle et al., 2012, p. 23). The greater the number and/or quality of connections students make, the greater their understanding.
- *Universal Screening Mechanism* is a brief assessment tool administered to all students to identify learning gaps. The screening tool contains a cut-point that reflects the likelihood of success or unsuccessful performance in the subject area being tested.

- *Visualization* is the spontaneous recognition of a concept relevant in any type of visual representation (Duval, 2014). Visualization is a tool used to remember, to construct meaning, and to generate knowledge and cognition.
- *Visual representations* include all types of iconic tools such as signs, symbols, numerals, tables and graphs, drawings, pictures, written words, and physical representations. Visual representations embody mathematical concepts and actions. As instructional tools, they are used to help students make meaning of mathematical ideas, structures, and procedures (Hiebert et al., 1997). As visible representations of student-cognition, they illuminate student understanding or lack thereof. They also provide a forum for discussion and assist students in making connections within and between mathematical concepts (Duval, 2014; NCTM, 2014).
- *Zone of Proximal Development* (ZPD) is the space between the actual intellectual development of a child during independent problem solving and the level of potential development for solving problems under the guidance of more knowledgeable persons (Vygotsky, 1978/1930). In other words, ZPD is at the "edge" of what students can do. With proper scaffolding, students can expand their mental regions and extend their behavioral capabilities.

Chapter 2: Literature Review

After a cool June morning of fly-fishing for trout, my husband and I rested upon the banks of Michigan’s pine-laden Pere Marquette River. Within five minutes, a kayak transporting two boys, about 12 years of age, landed at our feet. My husband began asking the boys if school had ended for the summer. The conversation that ensued will forever be etched in the researcher’s memory:

“So, are you boys done with school?” my husband inquired.

The boy in the front of the kayak responded excitedly, “School is out for me!”

With a forlorn face, the boy sitting in the back of the kayak replied softly, “It isn’t for me. I have to go to summer school.”

“Oh. So why do you have to go to summer school?” the researcher asked.

“I have to go for reading and for math,” the boy sadly replied.

Noting the negative tone in this young boy’s voice and trying to generate some positive energy regarding his predicament, the researcher emphatically responded, “You know, when you do well in reading and math, you can become anything and everything you want to be! Reading and math are important skills you need every day of your life.”

With a smile, the boy at the front of the kayak exclaimed, “You know, math is my best subject, but I can’t stand it!”

Taken aback, yet pondering his statement, the researcher questioned, “Perhaps it’s because your teacher doesn’t know how to teach it very well?”

> With a surprised look, the boy expressed, “That’s the same thing my parents said!” Then, with great emphasis and wonderment, the boy at the back of the kayak declared, “I just don’t get math. You know what our math teacher does? She starts in the middle and keeps on going!” (C.D. Zielinski, June 19, 2015)

This conversation was a powerful reminder of why the findings from this research study are important. The first boy’s comments suggested that, although math may be a student’s best subject, it does not guarantee one’s enjoyment. The second boy’s comments suggested that students who struggle often feel left behind and his teacher may not have had the skillset to help him “catch up.” The boy’s claim for starting in the middle also suggested that he may have cognitive structures needing further development!

There are varying perceptions for why students struggle learning mathematics. Some teachers, parents, and students believe that one’s ability to learn mathematics is genetically inherited (Dweck, 2006). If parents struggled learning mathematics, their children will likely struggle too. Others attribute students’ poor mathematical performance on teachers’ ineffective instruction and lack of mathematical knowledge (Hill et al., 2005). Some believe students’ inabilities to learn mathematics are caused by neurological dysfunctions in students’ minds (Baroody, 2011). Given these varying perceptions for low student achievement, it is astonishing that “inadequate attention has been given to developing effective instructional methods for implementing standards-based reforms with children having learning difficulties” (Baroody, 2011, p. 30). Understanding root causes for students’ learning difficulties and the ways students learn were integral in developing the conceptual and instructional framework represented by the “re-envisioned” instruction model.

The literature review within this chapter is multi-focused. This first section briefly describes general origins of students’ learning difficulties. The second and third sections present seminal and contemporary learning theories relative to cognitive structures, and identify key methods and processes for ways students learn. The fourth section strategically aligns learning theories to current research offered by educational and cognitive psychologists, neuroscientists, and mathematics education researchers. The last major section presents and describes the “re-envisioned” instruction model’s conceptual framework and implementation. A summary concludes this chapter.

Section 1: Possible Origins of Students’ Learning Difficulties in Mathematics

The historical origin for students’ learning difficulties was presented in Chapter 1. Chapter 2’s first section describes additional origins of students’ learning difficulties. These include brief explanations of biological, cognitive, socio-cultural, and pedagogical origins.

Biological origin. Some teachers and researchers believe students’ learning difficulties result from biological and genetic defects in brain functioning (Baroody, 2011; Geary, 2004; Gersten & Chard, 1999). It is true that a very nominal percentage of students possess neurological limitations (Baroody, 2011). According to the NCES (2015b), approximately 4.5% of students enrolled in public education are identified with a specific biological learning disability. Amongst this 4.5%, a small percentage of these students are affected by learning disabilities related to mathematics. For instance, students diagnosed with dyscalculia suffer from a specific mathematics learning disability. Dyscalculia can inhibit and impair one’s ability to discriminate amongst numbers, identify the magnitude of numbers, and perform simple calculations (Dehaene, 2010). Dyscalculia is one of many biological factors effecting students’ abilities towards learning mathematics.

Another biological factor impacting students’ abilities to learn mathematics is described by Baroody (2011) and Fox (2001). They suggested that students’ maturation and readiness to learn specific skills, concepts, and strategies influence their learning. For example, conservation of number is integral to a child’s ability to count and not all five-year-old students comprehend that the number of elements in a set is invariant when objects are stretched out, close together, stacked, or otherwise arranged. Children do not develop this concept within the same biological timeframe. One’s development of number conservation takes time and multiple experiences.

In a small portion of the 4.5% population of students identified with learning disabilities, their disabilities are attributed to biological factors. Among the 95% of the remaining students, some learners are influenced by other origins of learning difficulties.

Cognitive origin. A third origin of learning difficulties is initiated by students’ underdeveloped cognitive structures or schema (Feuerstein et al., 2006; Garner, 2007; Geary, 1995; Montague, 1997; Gruber & Voneche, 1995). Underdeveloped cognitive structures are created by impoverished stimulation within students’ learning environments (Feuerstein et al., 2006). Conservation of constancy and spatial orientation, as in identifying and comparing where locations of objects are relative to oneself or to a specific context, are two significant cognitive structures impacting students’ learning of mathematics (Kamii, Kirkland, & Lewis, 2001a).

Due to cognitive structures’ basic, yet significant nature, teachers, parents, and students frequently assume these mental structures are operational, when, in fact, they are not (Garner, 2007). Students who possess underdeveloped structures may not understand the reasons for their confusion when trying to learn a new concept. Alternatively, those who have well-developed cognitive structures find it challenging to comprehend why students struggle learning a concept that appears to be obvious. Garner (2007, 2013) and Feuerstein and his colleagues (2006), believed that students’ underdeveloped cognitive structures are causational factors for those who struggle learning academic subjects.

Another cognitive origin involves students’ lack of number sense. Students’ lack of early numeracy skills and their inabilities to abstract generalizable principles significantly contribute to students’ learning difficulties in mathematics (Bryant, 2005; Geary, 2011b; Gersten & Chard, 1999; Jordan et al., 2009; Montague, 1997). Van de Walle, Karp, Lovin, and Bay-Williams (2014) claimed that number sense involves conceptual understanding of the ways numbers are culturally used (e.g. number names, ordinality, notations etc.). Number sense includes students’ abilities to visualize quantities in a variety of contexts, understand numerical magnitude, and compose and decompose numbers in flexible ways. A student’s ability to subitize (quantify small groups of objects instantaneously), estimate, count, and solve basic arithmetic combinations and story problems denotes number sense. Such skills and concepts are integral to children’s mathematical cognition and understanding of number and operations (Bryant, 2005; Jordan, Kaplan, Olah, & Locuniak, 2006; Muldoon, Towse, Simms, Perra, & Menzies, 2012).

Socio-cultural origin. A fourth learning difficulty originates within students’ socio-cultural contexts. Interactions that occur between children and their environments are often defined by parents, teachers, children, and their respective communities (Vygotsky, 1978/1930). In fact, socio-cultural forces significantly shape students’ readiness to learn mathematics (Clements & Samara, 2009; Feuerstein et al., 2006; Gersten & Chard, 1999; Kozulin, 2002; Starkey & Klein, 2008). For instance, when compared to middle-class peer groups, children from low-income households often lack established logico-mathematical concepts of number (Kamii, Rummelsburg, & Kari, 2005). Logico-mathematical concepts enable children to connect concrete objects to abstract concepts, such as linking puzzle pieces together to create a full picture or identifying the quantity of "five" counters on a ten frame (Gruber & Voneche, 1995; Kamii & Rummelsburg, 2008). Students' mental constructions of mathematical relationships require logico-mathematical knowledge.

Logico-mathematical knowledge is vital as young students begin formal mathematics instruction in schools (Kamii, Rummelsburg, & Kari, 2005). Without it, students' lack of logico-mathematical knowledge contributes to learning gaps in mathematics. Learning gaps often transform into learning disabilities, thus creating learning delays (Baroody, 2011). When students' learning gaps are left unaddressed, learning delays result in students' inabilities to learn advanced mathematics (Starkey & Klein, 2008).

Additional socio-cultural factors contribute to the quality of students' learning environments. These include deficit-oriented perceptions and low expectations held by the broader social community. For instance, stereotyping low SES students as passive learners—capable of rote learning and incapable of self-directed learning—impacts student learning (Baroody, 2011; Haberman, 1991). The educational curricula assigned to students living within low socio-economic conditions are often restricted to memorization, skill, and drill of content (Haberman, 1991). Such socio-cultural beliefs, expectations, and outdated curricula and educational practices contribute to students becoming instructional casualties, preventing students from actualizing their true learning potentials (Baroody, 2011; Haberman, 1991; Silver & Stein, 1996).

Naturally, schools' and classrooms' socio-cultural learning environments impact student achievement. A school's curricula impact the content students learn and the rate at which they learn that content. Furthermore, teachers' perceptions of themselves as teachers of mathematics and their perceptions about specific students and students' capabilities enable or constrain students' opportunities and desires to learn mathematics (Haberman, 1991; Ma, 1999; Stigler & Hiebert, 1999).

Pedagogical (instructional) origin. Like the socio-cultural origin, this next and final origin for discussion commences in mathematics classrooms. Students' learning difficulties are influenced by teachers' lack of mathematical knowledge and by their outdated and inadequate instructional practices (Ball, 1990; Baroody, 2011; Hill et al., Ball, 2005; Stigler & Hiebert, 1999).

Schools are historically-situated and cultural institutions where teachers often use the same methods for instruction they experienced as K-12 students (Ball, 1990; Stigler & Hiebert, 1999). Teachers who have learned mathematics using rote memorization of rules, facts, and procedures tend to perpetuate this level of understanding with their own students (Hiebert, Morris, & Glass, 2003; Philipp et al., 2007; Skemp, 1976/2006). Teachers' perceptions of mathematics as a field of study, their level of conceptual understanding of mathematics, and their pedagogical knowledge for teaching mathematics, impact students' levels of achievement (Hill et al., 2005).

Hill, Rowan, and Ball (2005) described mathematical knowledge as the knowledge teachers use to "carry out the work of teaching mathematics" (p. 373). This knowledge encompasses effective use of instructional resources, including mathematical representations to represent concepts and processes, and involves teachers' accurate interpretations of students' ideas and solution strategies. Not only do teachers need to understand mathematics for teaching, they also need to understand mathematical content, including the use of precise, yet accessible, mathematical vocabulary.

Improving student achievement in mathematics requires a change in teachers' and students' perceptions that mathematics is not a system of facts and procedures to be memorized. It requires teachers embracing constructivist and socio-constructivist principles to facilitate learning. The types of mathematical tasks teachers engage their students with and ways students engage with those tasks impact students' learning in significant ways (City et al., 2009; Cohen & Ball, 2001; Henningsen & Stein, 1997; Smith & Stein, 1998).

Section summary. The origins of students' learning difficulties are varied, multi-faceted, and complex. Students' varying levels of biological maturation, cognitive development, and lack of number sense contribute to low student performance. Teachers' lack of mathematical and pedagogical knowledge also impacts student learning. Socio-cultural perceptions regarding specific students and their abilities to learn mathematics are additional contributors to students' mathematical learning difficulties.

The goals of this research study were to better understand how general classroom teachers might influence students' mathematics achievement using Tier I core instruction. As teachers and students attended to the cognitive, pedagogical, and socio-cultural origins simultaneously, it was hypothesized that the following make a difference in student learning and achievement:

- the ways teachers *stimulate and engage* students' existing cognitive structures
- the ways teachers *design, select, or modify mathematical tasks*
- the ways teachers and students *engage with those tasks*
- the ways teachers and students *structure their learning environment*

Unequivocally, Tier I core mathematics instruction can and should address students’ origins of learning difficulties (specifically cognitive, pedagogical, and socio-cultural, and in some cases, biological) by embracing theories for how students learn.

Section 2: Seminal Learning Theorists and Cognitive Structures

To improve students’ achievement in mathematics, it is critical to understand how young children make sense of and learn mathematics. Three seminal learning theorists’ conceptualizations for how students learn are offered in this next section. Piaget’s constructivism establishes a philosophical blueprint for designing and implementing the “re-envisioned” instruction model. Vygotsky’s socio-constructivism presents a framework for establishing productive socio-cultural learning environments (Cobb, 1994; Confrey, 1990; Noddings, 1990). Bruner’s (1966) representational learning theory describes the varied forms of mathematical representations used for instructional purposes, as well as the power and economy of those representations. His ideas illuminate key constructs for designing, selecting, and modifying the “re-designed” mathematical tasks used exclusively by the two experimental classrooms in this study. All three learning theorists’ definitions for cognitive structures, including their views for why and how these mental structures advance student learning, are presented in this next section.

Jean Piaget. Education predominantly relied on behaviorist learning theory from 1920 to 1950. This theory assumed children’s learning and behavior were shaped by punishments and rewards. Contrastingly, Jean Piaget, a scientist who studied the origins of knowledge, challenged the existing behaviorist theories of his day. Piaget (1964) believed student learning is based upon physical and neurological developments influenced by one’s genetics, biological maturation, and chronological age. He viewed children as knowledgeable and capable of developing new understandings by linking known concepts (existing schema) to new ideas. Thus, Piaget is considered the educational pioneer for advancing his theory of constructivism (Noddings, 1990; Steiner, 1974; von Glasersfeld, 1990).

Constructivism presupposed that children require and acquire physical, social, and logico-mathematical knowledge to support their construction of new ideas (Kamii, Kirkland, & Lewis, 2001a). As young children interact with objects in their environments, they notice the physical nature of those objects such as the shape, size, and color of a red ball. The color "red" and the name "ball" signify social knowledge. To understand the abstraction that not all spherical objects belong to the set of red balls requires logico-mathematical knowledge (Kamii, Kirkland, & Lewis, 2001a).

Logico-mathematical knowledge involves interconnected sets of sensory-motor, mental, and conceptual operations functioning simultaneously (Piaget, 1964). Piaget (1964) contended that these sense-making operations (or cognitive structures) are foundational for acquiring knowledge and for correcting one's underdeveloped or inaccurate perceptions of real-world abstractions (Gruber & Voneche, 1995).

Two fundamental mental processes students use to build and advance their own levels of cognition are *assimilation* and *accommodation* (Piaget, 1962, 1964). Assimilation occurs whenever new information strongly correlates with one's existing knowledge and understanding. It is through assimilation that students' mental structures function and expand without changing structurally (von Glasersfeld, 1990). Comparative thinking, pattern finding, and rule identification support the mental processes necessary for assimilating information (Mink, 1964; Sinclair & Kamii, 1970).

Piaget's second mental process is accommodation (Piaget, 1962, 1964). Accommodation occurs when students' perceptions of experiences or stimuli lead to unfamiliar or surprising results (von Glasersfeld, 1990). For instance, when young students first encounter letters of the alphabet to represent an unknown value (such as *a*, *x*, or *y*), they often experience mental disequilibrium. The representation of *a, x, or y*, triggers cognitive dissonance or tension because young students often associate letters with sounds and words, not mathematics. To accommodate the abstract concept that a letter can also represent an unknown value, students need to reestablish mental equilibrium within their minds.

Reestablishing mental equilibrium requires the reconfiguration of existing cognitive structures. Through the process of accommodation, new neurological nodes are constructed. These nodes support the construction of new neuronal pathways, thereby rewiring the electrical currents essential for processing information (Ifenthaler et al., 2011). Theoretically, these neuronal modifications “accommodate” the new concept of variable, creating and restoring a child’s cognitive balance or mental equilibration (Fox & Riconscente, 2008).

Both assimilation and accommodation are vital cognitive processes for developing and advancing students’ cognitive structures and levels of cognition. A cyclical movement between assimilation and accommodation strengthens one’s cognitive structures overall (Bruner, 1966). The mental process of accommodation enlarges students’ neurological networks within the mind (Gruber & Voneche, 1995).

Piaget believed that students’ mental cognition hinges upon their nervous system development, physical maturation, and the development of their sensory organs (Feuerstein et al., 2006; Wink & Putney, 2002). Piaget’s conclusions strongly influenced his four stages of cognitive development: *sensory-motor, pre-operational, concrete operational,* and *formal operational*. Piaget believed each subsequent stage represents a higher level of cognitive thought; and, a child’s cognitive structures functioning at one stage of development is determined by and becomes part of children’s cognitive structures during the next stage of development (Gruber & Voneche, 1995).

For instance, young children often rely upon visual and kinesthetic observations and other sensory data during the sensory-motor stage. Then, as children’s actions upon objects increase, their cognition levels rise. Additional physical and logico-mathematical experiences enable children to actively connect concrete objects to abstract concepts transitioning them from *sensory-motor* to *pre-operational* to *concrete operations* (Gruber & Voneche, 1995). According to Piaget (1964), concrete mental operations are essential to the development of students’ cognitive structures and their advancement to the highest cognitive stage of development, formal operations. Existing structures give rise to new structures, always evolving, with higher structures governing the lower structures.

Lev Vygotsky . As Piaget’s contemporary, Lev Vygotsky (1978/1930), a Russian psychologist, promulgated a growth and development model for learning. Like Piaget, Vygotsky believed human cognition is determined by an individual’s genetics and maturation, indicating biological or natural influencers (Wink & Putney, 2002). Where Piaget and Vygotsky differed was in their philosophical views for how learning occurs (Wink & Putney, 2002). Vygotsky believed human cognition is greatly influenced by the social interactions (nurture) children experience (Kozulin, 2002). He believed that when a child is inducted into a socio-cultural society, the level of that child’s cognition becomes qualitatively altered due to the historical properties of that child’s culture (Vygotsky, 1978/1930). Thus, Vygotsky is known as the father of socio-constructivist learning theory.

Vygotsky argued that one’s social (nurture) and mental activity (nature) form the bedrock of one’s cognitive advancement (van der Veer, 2007). Socially-mediated interactions involve more knowledgeable persons passing on their cultural knowledge to their children. According to Vygotsky, it is through these social interactions and use of cultural tools (i.e. language, signs and symbols) that children's natural intellects immediately move to higher planes (Kozulin, 1990; van der Veer, 2007; Vygotsky, 1978/1930; Wink & Putney, 2002).

Therefore, Vygotsky believed student learning occurs across two planes of intellectual thought. Learning first transpires on a social plane. Learning then occurs on a cognitive plane (Vygotsky, 1978/1930; Wink & Putney, 2002). For example, Vygotsky believed that a mediator or teacher passes on cultural information and knowledge causing the establishment of inter-psychological knowledge within the mind of a student (Kozulin, 1990). Next, the interplay of social-exchanges involving words and actions between teacher and student induces learning on an intra-psychological plane (Fox & Riconscente, 2008; Kozulin, 1990; Wink & Putney, 2002). Students then connect these two forms of knowledge, inter-psychological and intra-psychological, to make sense of concepts and regulate their own learning.

Collectively, these forms of mental actions generate new and complex neurological pathways, highly interconnected and unified within the human mind. Vygotsky identified these neurological pathways as cognitive structures (Kozulin, 1990; Vygotsky, 1965). For Vygotsky, biologically-derived cognitive structures such as perception, memory, attention, comparative thinking, and intelligence are organized, intricate cognitive systems vital for learning (Boettcher, 2007; Kozulin, 1990).

Vygotsky (1978/1930) believed that language, signs, and symbols are essential tools for mediating children's cognitive advancement in mathematics. He perceived language, signs, and symbols (including a child's inner speech) important microcosms of human consciousness (Kozulin, 1990). He was adamant that "cognitive functioning based on higher order symbolic tools associated with literacy and numeracy is superior to that based upon everyday experience and the oral transmission of culture" (Kozulin, 2002, p. 13).

Vygotsky's (1978/1930) Zone of Proximal Development (ZPD) embodied his theoretical views for social-constructivism. Unlike Piaget's sequential stages of maturational cognition, Vygotsky (1978/1930) claimed that a ZPD was the unique space between the child's intellectual development during independent problem solving and the child's potential development for problem-solving when guided by more knowledgeable persons. Teachers can advance student cognition beyond students' own natural cognitive endowments by aligning mathematical experiences to students' ZPD. Teachers need not wait for students' biological maturation to occur. Student cognition can be advance through challenging tasks. Teachers can pose novel representations and facilitate and sustain social communication between all those in the classroom. Social-constructivism suggests that learning through culturally- and socially-derived means (language, signs, and symbols) positions students for advancing to higher levels of cognitive thought.

Jerome Bruner. Largely influenced by Piaget's theory of constructivism and Vygotsky's theory of social-constructivism, Jerome Bruner, a research professor of psychology, claimed that children's cognitive development depends upon physical and biological maturation, as well as their natural and socio-cultural environments (Bruner, 1966, 1977, 1996, 1997; Takaya, 2008). Bruner (2008) believed that as a child entered a culture, that culture entered that child's mind.

For Bruner (1966), the classroom is an important communal space for furthering children's cognition. He argued that a socially-accommodating classroom promotes social acceptance, allowing students to obtain culturally-held knowledge (Bruner, 1996, 1997). For instance, as students develop and share mathematical representations reflecting the three distinct meanings of subtraction (take-away, comparison, and missing addend), mathematical language and interpretations for that operation become accessible knowledge (Fuson, 1984). Student participation, acceptance, and interactions within a socio-cultural environment are critical factors for developing student cognition, metacognition, and reflective thought processes.

Bruner (1977) agreed with Piaget that students use their prior knowledge to construct new insights and understandings *of* a concept. In fact, he believed that gaining knowledge *of* something was more valuable and applicable than acquiring knowledge *about* something (Takaya, 2008). Acquiring knowledge *of* a concept necessitates students transforming information into usable knowledge using their cognitive structures.

Bruner’s (1966) cognitive development model or representational learning theory epitomizes his own interpretation of the progression of one’s intellectual development. His model suggests that students need to engage with the following representational modes: enactive (doing or action), iconic (internal imagery development), and symbolic (symbolic-verbal encoding) to develop and generate their own cognitive structures.

Bruner’s (1966) first mode or phase of his developmental model is titled *enactive representation*. To physically encode information, students use their senses and physical bodies to explore their environment, manipulate objects, and practice skills. As students encounter mathematical concepts for the first time, students’ strategic use of physical mathematical tools such as multi-link cubes, counters, and fingers support them making sense of abstract concepts such as counting, mathematical properties and operations.

Bruner’s (1966) second mode of intellectual development is termed *iconic representation*. *Iconic representation* is generated as students represent their existing understanding using mental imagery. Mental iconic representations are derived from memories of past encounters and events. These image-based representations involve self-selected sequenced perceptions triggered by cognitive structures involving spatial, temporal, and quantitative orientations (Bruner, 1964). Mental images of numerical magnitudes, mathematical models, and other forms of representations often portray students' perceptions, interpretations, and reinterpretations of experiences existing in one's memory. Mental iconic images (or visualization) are important epistemological tools for all students, especially for those struggling to learn mathematics (Presmeg, 2014).

Bruner's (1966) third mode of intellectual development consists of *symbolic representations*. Symbolic representations involve diagrams, pictures, drawings, graphs, and the like. In fact, *symbolic memory* representations are associated with language in all forms, including musical and mathematical notations and abstract representations, all products of cultural innovations (Bruner, 1966). Expressing ideas and concepts using abstract representational forms require higher levels of cognition. For example, the internalization of language as a cognitive tool allows students to flexibly represent and transform previous experiences of "fiveness" into new symbolic forms. Five fingers on a hand or five petals on a flower can be depicted as five circles on a page or by the numeral 5. Bruner believed culturally-induced symbolic experiences increase students' sensory abilities; and, student use of language systems is indispensable for advancing student cognition (Bruner, 1964).

Bruner (1966) asserted that regardless of the levels of abstraction, all concepts can be represented by simple recognizable forms all students can interpret and understand. Teachers can effectively scaffold students' learning of new concepts by considering both the economy and power of mathematical representations relative to where students are within a mathematical learning trajectory. The *economy* of a representation is the amount of information students need to hold in their minds to process and make sense of that representation. The *power* of the representation pertains to students’ mental capacities to use the representation to make connections to what they already know about specific mathematical concepts.

Comparisons of seminal learning theorists’ ideas. Piaget’s, Vygotsky’s, and Bruner’s extensive work and invaluable insights are more numerous than these brief descriptions present. These theorists clearly indicated the existence of cognitive structures. Their work describes specific intellectual processes and instructional practices to advance student cognition.

Piaget contended that rich learning experiences support students constructing understanding of new concepts and ideas (Smith, 2000). To achieve understanding, students use their existing cognitive structures to mentally process information. Children’s mental construction of higher levels of cognition are achieved by assimilating familiar concepts and accommodating new and unfamiliar concepts. These ideas epitomize Piaget’s theory of constructivism.

Contrastingly, Vygotsky’s (1978/1930) socio-constructivist theory suggests that learning is cultivated through social means. Vygotsky believed students first construct ideas on a social plane, then on an individual plane. Mathematics lessons are naturally filled with language, signs, symbols, and tools that need to be socially introduced and communicated. As teachers and knowledgeable others pose meaningful mathematical representations and stimulating questions, children can individually construct their own ideas using existing cognitive structures involved with language, memory, attention, and comparative thinking. Then, children learn how to communicate mathematically using others and their own abstractions of mathematical ideas.

Bruner’s (1966) representational learning theory aligns to Vygotsky’s cultural signs and symbols. The use of enactive, iconic, and symbolic representations which align to students’ existing cognitive structures stimulate students’ assimilation and accommodation of new concepts and information (Bruner, 1977). Thus, Bruner (1966) and Vygotsky (1978/1930) contended that a teacher need not wait for students’ biological maturation as Piaget’s (1964) model suggests. Bruner’s ideas for considering the power and economy of a representation and Vygotsky’s ZPD reminds teachers that it is possible to effectively prepare and nurture students’ cognitive readiness for learning mathematics, as well as take them beyond their own natural cognitive endowments (Bruner, 1977).

Section 3: Contemporary Learning Theorists and Cognitive Structures

This next section presents three contemporary theorists’ views relative to meditating the development of students’ cognitive structures. Geary (1995) described two classifications of cognitive structures, biologically primary and biologically secondary, with the second form of structures developing through deliberate and sustained formal instruction. Feuerstein and his colleagues (2006) suggested that all students progress cognitively as they participate in strategically designed experiences targeting the development of their cognitive structures. Finally, Garner (2007) stressed that it is students who must develop their own cognitive structures and it takes strategic and reflective teachers to mediate such development.

David Geary. David Geary (1995, 2007), a developmental psychologist, claimed human cognition and development are influenced by inherited biological factors and deliberate socio-cultural experiences. Geary’s (1995) evolution-based learning theory draws upon Vygotsky’s and Bruner’s theories relative to the origins of students’ cognitive functioning (Sweller, 2008).

Geary’s (1995, 2007) learning theory identifies two classes of cognitive structures, each defined by the ease with which they are co-opted for different tasks. The first class of structures is considered biologically primary. Biologically primary structures are found across all human cultures and across many different animal species. Humans and animals both inherit common, yet specific cognitive structures that increase their chances for survival, allowing them to adapt to the contexts in which they live (Keil, 1981).

Biologically primary structures consist of highly-evolved specialized neurobiological systems which process domain-specific information. For example, perceptual and attentional cognitive structures, such as visual scanning and subitizing, enable youngsters to attend to the geometric and quantitative features of their environments (Geary, 1995). Human infants, as young as 18 months of age, demonstrate sensitivity to ordinal relationships up to three and four items (Geary, 1995; Sousa, 2008). Ordinality, subitizing, and simple arithmetic are essential functions for the young of a species to survive. These mental structures support processing and identification of the number of enemies approaching (fight or flee) or the quantity of berries for the purposes of gathering (Geary, 1995, 2011a; Sousa, 2008).

Other inherited biologically primary structures include one’s ability to navigate the environment, remember locations of objects, use objects as tools, and acquire language skills (Geary, 1995). As human fetuses are exposed to their mothers’ voices and voice patterns in utero, research suggests that the developing embryo becomes sensitized to the constructs of the mother’s cultural-language (Geary, 2007). Subsequently, biologically primary cognitive structures are essential and foundational for acquiring complex competencies involving language, symbols, and mathematics (Jordan et al., 2009). These same structures are involved in complex cognitive processing, allowing children to discover, acquire, process, assimilate, and learn vast amounts of information rapidly and effortlessly.

For higher levels of cognition, Geary’s (1995) second class of structures are classified biologically secondary. Unlike primary structures, biologically secondary structures vary across cultures and across generations of people. Secondary structures are developed through social-cultural means supporting their development (Geary, 1995, 2007). Aligning to Vygotsky’s socio-constructivism, secondary structures develop from the interplay between biologically evolved structures and the social interactions influencing their development. Geary (1995, 2007) hypothesized that students’ biologically secondary cognitive structures emerge through deliberate and sustained instruction and practice facilitated by those who already possess secondary structures.

For instance, young students’ higher forms of mathematical cognition are developed by observing others in the counting process and then counting physical objects themselves. Working- and long-term memory systems, which are considered biologically secondary cognitive structures, support students’ learning how to identify numbers by name, count large quantities, and use tools such as a number line to develop conceptual understanding of the Hindu-Arabic numeration system (Geary, 1995). According to Geary, if biologically primary or secondary structures—such as working and long-term memory—are underdeveloped, then student learning of advanced mathematics becomes a challenging feat for the learner (Feuerstein et al., 2006; Garner, 2007; Geary, 1995).

Reuven Feuerstein. Reuven Feuerstein, a developmental clinical cognitive psychologist and proponent of neuroscience, described the human mind as modifiable (Feuerstein et al., 2006; Tribus, 1996). He believed that intentional instructional experiences by more knowledgeable persons can modify a student’s neurological architecture (Byrnes & Fox, 1998; Feuerstein et al., 2006; Garner, 2007). Most importantly, Feuerstein’s *modifiability* refers to a learner’s capacity to change the course of his or her neurological development irrespective of the causes and conditions of his or her learning disabilities (Kozulin, 2002).

Feuerstein asserted that children’s level of intelligence increases through two modes of interactions between individuals and their environments: direct contact with stimuli and mediated experiences with stimuli. Direct contact with stimuli begins *in utero* (Feuerstein et al., 2006). The learner (or fetus) modifies his or her behavior to adapt to external stimuli. This change in the learner’s behavior generates new cognitive structures within the learner’s mind, which in turn, increases the learner’s intelligence.

Older children’s cognitive advancement relies heavily upon effective socio-cultural experiences or what Feuerstein (2006) described as mediated learning experiences (MLE). Consistent with Vygotsky’s (1978/1930) socio-constructivist learning theory, MLE is a proactive method to “change the cognitive structure of the learner and to transform him/her into an autonomous, independent thinker, capable of initiating and elaborating ideas” (Feuerstein et al., 2006, p. 124). Thus, the mediator’s intent is not to teach academic content, but to increase the learner’s and the mediator’s understanding of how the learner processes information and then seeks ways to improve the learner’s mental processing (Tribus, 1996).

Effective MLE includes: (a) an increase in the learner’s awareness for how he or she learns; (b) a change in the learner’s cognition and behavioral patterns; (c) the development of the learner’s abilities to recognize and identify relationships; and (d) an increase of strategies in the learner’s toolbox for problem solving. As such, MLE requires intentional and transcendent human interactions between a mediator and the learner. The mediator intentionally selects, frames, filters, schedules, and arranges a stimulus input. The mediator then acts as a filter, transforming the stimulus whereby the learner perceives the stimulus with a new or differing perception. These new perceptions involve the “temporal, spatial, and ordinal attributes of the stimulus and other such relationships such as the attribution of value and meaning” (Feuerstein et al., 2006, p. 88).

A fundamental feature of the MLE process is that there are three distinct movements or phases of mental processing: input, elaboration, and output (Feuerstein et al., 2006). *Input* is represented by various data forms or stimuli received through the learner’s sensory register. Once data is received, the learner *elaborates* upon the information by sorting, analyzing, classifying, and synthesizing the information for coding purposes (Bruner, 1957). Once the sensory information is coded, the learner makes decisions about the information and then presents his or her *output*. Student *output* is represented by the learner’s sense-making, coding processes, and decisions regarding the original sensory stimulus.

To describe the mental processes of input, elaboration and output, an example is presented here. To develop students’ understanding of numerical relationships, a teacher may pose the following number line represented in Figure 2.1 and ask students, “What do you notice?” Students use their sensory registers to observe the number line’s sensory data (*input*) and note the patterns and numerical structures inherent within the number line representation. Students use their cognitive structures to make sense of and *elaborate* upon the figural units inherent within the representation (Simon, 2001). Once students perceive and make sense on an individual mental plane, students engage in conversations on a socio-cultural plane (*output*).

Figure 1: Number line

To further expound upon students’ cognitive thought processes during a mediated learning experience in mathematics, this number line representation requires students to estimate the value of *n* given the spatial location of *n* relative to 20 and 35. Students’ understanding that number lines continue indefinitely in both directions and that number lines can begin with any value are pre-requisite knowledge for analysis and sense-making. Another pre-requisite for students is understanding that there is an underlying uniform scale proportional in the numerical values between 20 and 35. These concepts constitute the figural units within the representation above.

Students take in sensory input by noticing that *n* is situated between 20 and 35 and is about two-thirds the distance from 20 and closer to 35 (input). Here, students must mentally elaborate that the variable *n* represents an unknown value and that *n* can be determined by dividing the distance between 20 and 35 into equal segments. Student output is then represented by their expressions of ideas regarding the representation and providing an approximation or exact value for *n*, along with their justifications.

These fundamental mental processes of input, elaboration, and output require students to activate their existing knowledge and prior experiences and focus their attention on the figural units and inherent structure of the representation. Such cognitive processes reflect students’ abilities (or inabilities) to consider numerical relationships, find viable solutions, and articulate justifications for defining *n*’s value.

Consistent with Vygotsky’s (1978/1930) social-constructivist views, Feuerstein and his colleagues (2006) claimed that culturally-derived signs and symbols are important mediational tools that transmit cultural knowledge, as well as facilitate the growth of students’ cognitive structures (Kozulin, 2002). Additional tools that transmit socio-knowledge include language, gestures, and observation of behaviors. Still, language is the most efficient and economic tool for transmitting knowledge and skills and used for mediating meaning and understanding (Vygotsky, 1978/1930). As students engage in mediated learning experiences using culturally derived signs, symbols, and language, important neurological changes take place. These neurological changes indicate that learning is occurring within a child’s mind (Feuerstein et al., 2006).

Betty Garner. Betty Garner (2007, 2013), an education researcher and theorist, explained that learning occurs when students interact “creatively with information to construct meaning” (Garner, 2007, p. xv). Comparing her theory of metability to Feuerstein’s MLE, Garner proposed that students must actively alter their own neurological structures through “the ongoing, dynamic, interactive cycle of learning, creating, and changing” (Garner, 2007, p. xv). Neither mathematical tasks nor teachers alter or develop students’ cognitive structures, but students themselves generate these neurological changes within their own minds.

Garner’s (2007) theory of metability was generated through her personal observations of students as they learned. She noted students’ reflective awareness of sensory input supported their construction of meaning and understanding, leading to changes in students’ levels of cognition. Garner (2007) described cognitive structures as basic mental tools essential for making sense of information and for learning everything about the world and beyond.

Central to Garner’s work are the hierarchal classifications of cognitive structure systems. Classifications include comparative thinking structures, symbolic representation structures, and logical reasoning structures, all essential for learning mathematics (Garner, 2007). Each classification is categorized by type, by the kind of thinking it supports, how it affects student learning, and the applications necessary for developing student understanding.

Garner’s (2007) first classification system consists of *comparative thinking cognitive structures.* They include recognition, conservation of constancy, memory, classification, spatial and temporal orientation, and metaphorical thinking. All support the processing of information in distinguishing how differing stimuli are alike and different. These are also pre-requisites for developing higher-ordered cognitive structures and the understanding of mathematics (Bruner, 1957). For example, to conceptually understand mathematics, students must *recognize* numerals, signs and symbols, understand their relationships and significance, and identify and generate numerous equivalent representations. Students use *conservation of constancy* to perform operations or evaluate equations. With conservation of constancy, students notice what changes and what stays the same between representations. Students use *spatial orientation* to analyze the exponential structure of the Base Ten number system and to visualize a rectangular prism representing 1,000,000. *Temporal orientation* enables students to understand the sequential, step by step processes of algorithms and problem solving. Comparative thinking structures were foundational for learning everything else in mathematics (Garner, 2007).

Garner’s (2007) next classification of cognitive structures involve *symbolic representation cognitive structures.* This system of structures processes information by transforming comparative structure data into abstract coding systems. Thus, symbolic representations include *language* in all forms, written and spoken, from words to non-verbal representations. These structure types encompass music, rhythms, body expressions, and all types of physical movement that embody meaning. They also include graphics such as drawings, graphs, and other representations depicting mathematical phenomena.

Included within symbolic classifications is *quantification*. Quantification makes possible mathematical forms for computing and operating with number, for measuring in two- and threedimensional planes, and for expediting skills necessary for success in courses such as geometry, algebra, trigonometry and calculus. Hence, Garner’s symbolic structures embody all of Bruner’s (1957, 1966) enactive, iconic and symbolic representations, essential for working with and for developing understanding of abstract concepts within mathematics.

Garner’s last classification system of cognitive structures includes *logical reasoning structures*. These structures encompass higher levels of cognitive thought enabling students to “systematically process and generate information” (Garner, 2007, p. 2). These structures involve *deductive reasoning* and *inductive reasoning.* Deductive reasoning influences one’s ability to draw conclusions from existing generalizations. Inductive reasoning allows students to predict, forecast, and make conjectures or tentative generalizations from perceived patterns. Logical thinking structures support the identification of relationships related to *cause and effect*. *Analysis* structures support the discovery of an underlying nature or inner relationship between a whole and its parts. *Problem-framing* and *problem-solving* help students clarify and organize the relationships of elements and parameters within a problem for investigation, consideration, or solution finding. Such logical reasoning structures are vital cognitive processes involved in interpreting and understanding the abstractions in mathematics and for finding viable solutions.

Comparisons of contemporary learning theorists’ ideas. Geary, Feuerstein, and Garner addressed biological, cognitive, socio-cultural, and pedagogical origins of students’ learning difficulties suggesting the importance for mediating the development of students’ cognitive structures. Each theorist described cognitive structures as foundational neurological systems essential for learning. For example, Geary (1995; 2007) believed all children are born with genetically-inherited (biologically primary) cognitive structures enabling them to interpret, understand, and survive in their environments..

Feuerstein’s theory of MLE (2006) requires intentional planning of instructional tasks and careful selection of tools by a mediator to incite the development of students’ cognitive structures. The mediator models and verbalizes the cognitive processes students are to replicate. Focusing one’s attention, chunking information, asking questions, and sense-making are all important learning processes the mediator models for the learner.

Contrastingly, Garner’s (2007) theory of metability invites students to explore, observe, make sense, and problem solve for themselves. These ideas align to Piaget’s theory of constructivism. As the mediator poses representations, problems and questions, the mediator expects students to pause, notice, and reflect upon their existing knowledge, thereby focusing their attention on the details or figural units and structure of the sensory stimulus. Students’ reflective awareness supports them in making connections to prior knowledge. Unlike Feuerstein’s MLE, Garner’s (2007) theory of metability empowers students to create, learn, and change their own levels of cognition.

Fundamentally, all three theorists suggested that the development of students’ cognitive structures are necessary for learning. Geary (1995) believed that the stimulation and activation of genetically-inherited structures initiates the development of key biologically secondary structures that support the learning of advanced mathematics. Feuerstein et al. (2006) believed that a mediator transmits his or her knowledge and understanding to the student. The learner must first experience concepts and skills on a social plane before he or she can make sense of and use them on an individual plane. These ideas align to Vygotsky’s (1978/1930) notions of socio-constructivism.

Akin to Piaget’s constructivist ideas, Garner (2007) believed that the nature and design of mediational tasks and use of more open-ended question types allow students to reflect upon their existing knowledge, make comparisons and connections, and construct new insights. These mental actions on the part of the learners modify the neurological makeup of their cognitive structures and advance their cognition. Hence, metability represents the process for learning, changing, and growing on a mental plane. Even with the biological and developmental determinants for students’ cognitive growth, all three theorists’ learning theories suggested that mediating the development of students’ cognitive structures positively impact students’ learning in all areas of life, not only in mathematics (Garner, 2007).

Section 4: Articulation and Alignment of Current Research to Seminal and Contemporary Learning Theories

Section 1 described the origins of students’ learning difficulties. Section 2 and Section 3 presented seminal and contemporary theories addressing students’ cognitive structures and their importance for learning and advancing student cognition. Section 4 now presents research from the fields of mathematics education, cognitive and educational psychologies, and the neurosciences. Such fields provide a richness of current research, recommendations, and evidence-based practices for improving students’ learning of mathematics.

The synthesis of learning theories, combined with the recommendations for research-based processes and practices, influenced the design of the “re-envisioned” instruction model. A foreshadowing of these principles, elements, and processes for designing and implementing the model are described in this section. This includes a description of conducive environmental conditions, as well as an articulation of specific cognitive processes and instructional practices central to mediating the development of students’ cognitive structures.

Neuroscience and cognitive structures. Congruent with evolution-based learning theory, constructivism and MLE, today’s neuroscience research indicates that all children are born with neuronal networks, clusters, or assemblies of neurons waiting to activate, to communicate, and interconnect through synapse processes (Byrnes & Fox, 1998; Sedikides & Skowronski, 1991). Recent research indicates that cognitive structures are essential building blocks for cognitive processing (Ifenthaler et al., 2011) and many years of evolution have equipped humans with specific cognitive processes essential for survival. These include pattern recognition, subitizing nominal quantity of objects, and making “rapid judgements and inferences” (Devlin, 2010, p. 171).

Studies revealed that babies are born with core mental systems that support their deductions about the relative sizes of objects (Ansari, 2010; Devlin, 2010). As infants, babies can focus on the quantitative attributes of sets and can determine which of two sets has more objects or figures (Devlin, 2010; Fuson, 2009; Sousa, 2008). These findings are consistent with Geary’s (1995) biologically primary structures he identified through his work (e.g. visual scanning, simple quantification, and subitizing).

Piaget’s (1974) and Vygotsky’s (1978/1930) work determined that young children have evolutionary- and genetically- inherited language capabilities and core mental systems linked to mathematics. Neurological-imaging indicate that the left and right hemispheres of the brain are activated when students make inferences of semantics and syntax and when they attempt to comprehend narratives such as story problems and mathematical vocabulary and symbols (Devlin, 2010).

Brain research studies have also revealed that spatial orientation and reasoning occur in the two parietal lobes within the mind (Devlin, 2010). These lobes are found in the top posterior of both right and left hemispheres of the human brain. They support the integration of sensory input (primarily visual) and the construction of a spatial coordinate system. The parietal lobes of the brain "house number sense and support spatial reasoning" (Devlin, 2010, p. 172). As children mentally compare and operate with quantities and numbers, or when they consider numbers in written and verbal forms, the parietal lobes within the brain are stimulated. This region also lies near other neurological areas that engage students in spatial coding, subitizing, distinguishing the size of objects and their location, estimation, and determining the cardinality of a set of objects (Dehaene, 2010).

Mediating the development of students' spatial orientation and skills is integral to increasing students' mathematics achievement (Gunderson, Ramirez, Beilock, & Levine, 2012; Kell, Lubinski, Benbow, & Steiger, 2013). Incorporating tasks focused on subitizing, counting, estimation, and mapping numbers on a number line help mediate the development of students' cognitive structure for spatial orientation (Li & Geary, 2013). Students' ability to map numbers onto a linear number line are indicative of young students' later mathematical proficiencies (Booth & Siegler, 2006; Dehaene, 2010).

Children's cognitive structures and neuronal pathways within their minds are continually constructed, developed, and strengthened daily as children work to make sense of mathematics (Devlin, 2010). As children work to learn, the number of their neurological dendritic linkages increase, and the integration of cognitive structures strengthen. Theoretically, the number of linkages constitute effective and efficient (or ineffective and inefficient) information processing mechanisms, thereby impacting the ultimate quantity and patterns of neuronal connectivity within the human mind (Byrnes & Fox, 1998). The mental processes of assimilation and accommodation of mathematical information enable students to acquire more knowledge and skills through cognitive generation of mental networks (Byrnes & Fox, 1998).

Mathematics education research and cognitive structures. Today’s mathematics education researchers no longer debate between nature or nurture, genetics or environment (Dweck, 2006). Battista (2010), a current mathematics education researcher, claimed, “ALL students have pre-mathematical knowledge of mathematics topics that they are first learning” (p. 40). He further argued, “Research in mathematics education has repeatedly demonstrated that students build new mathematical understandings out of their current relevant mental structures” (p. 40). It is the nature and expanse of students’ pre-mathematical knowledge that varies amongst children.

Piaget (1964), Vygotsky (1978/1930), Bruner (1977), Garner (2007), and current researchers agree that young children are competent problem solvers; and, young children often understand more mathematics than adults assume (Dehaene, 2010). For instance, in Philipp’s and Schappelle’s study (2012), they observed children solving problems in novel ways without the “benefit” of first receiving explicit or direct instruction. They witnessed young children considering flexible solution strategies, thinking differently than what the researchers imagined. Other studies revealed young children setting their own mathematical challenges using building blocks during a classroom’s scheduled free play (Anghileri, 2006).

Corresponding to Piaget’s (1964) learning theory of constructivism, the above studies provide evidence that children use their existing mathematical knowledge, skills, and cognitive structures to construct their own understanding of mathematics via critical and creative thinking processes. These ideas align to the concepts of Garner’s (2007) metability and Vygotsky’s ideas of self-regulation (Fox & Riconscente, 2008). Young children often learn through trial and error (Anghileri, 2006).

In addition to spatial orientation, conservation of constancy is another comparative thinking structure vital for learning mathematics (Garner, 2007; Piaget, 1964). Conservation of constancy enables students to identify and distinguish an object’s characteristics (e.g. numeric values, quantities, a collection of 2-D shapes, etc.) and discern between abstract concepts that stay constant and those that change as in pattern finding (Garner, 2007).

Fundamentally, conservation of constancy is essential for students who are transitioning to more abstract mathematical thinking involving conservation of number. For instance, understanding that a pile of ten cubes will always equal 10 cubes regardless of the cubes’ arrangement, is one example of conservation of constancy. When working with number facts, students use conservation of constancy to perform operations and evaluate equations (as in 3 + 3 = 4 + 2). Conservation of constancy influences students’ creation of mathematical generalizations (Bruner, 1977; Garner, 2007). Teachers’ use of instructional tasks that activate students’ cognitive structures responsible for pattern recognition, visualization, spatial orientation, and conservation of constancy help students improve their mathematical understanding (Feuerstein et al., 2006; Kamii, Lewis, & Kirkland, 2001b).

Experts in the field of mathematics education embrace and promote constructivism and socio-constructivism as best philosophical approaches for teaching and learning mathematics (NCTM, 2000). Mathematics educators believe both theories, when effectively embraced by the school community, promote powerful learning experiences that facilitate students’ mathematical understanding (Baroody, 2011; Battista, 2010). Still, the concept and practice of strategically focusing teachers’ efforts on mediating the development of students’ cognitive structures is a concept all mathematics educators have yet to research and embrace.

Cognitive tools to mediate the development of cognitive structures. Teachers' carefully-selected instructional tasks (*input*), and the ways teachers engage students with those tasks, support students increasing and strengthening the neuronal connections within their minds. Just as Vygotsky (1978/1930) argued, it is through collaborative social interactions that students first gain important information about cultural symbolic systems (e.g., logic and language). This is because "language is a socially shared code representing concepts" and mathematics is a culturally-derived symbolic system (Jordaan & Moonsamy, 2015, p. 103).

Socially-shared language directs and develops an individual's thought processes and self-regulation of attention. For instance, children alter their own cognitive systems by becoming reflectively aware of a sensory input, identifying the cognitive strategies most useful to them, and then strengthening those interlinked neuronal systems by engaging in repeated, extended, and novel experiences (Dehaene, 2010). The ability to direct one's focus and mental processes using signs, words and symbols is fundamental for mathematics concept formation (Fox & Riconscente, 2008).

Garner (2007, 2013)—a strong proponent of constructivism, socio-constructivism, and mediated learning theory—understood that the information students retain is a function for how long and how deeply their attentions are allocated toward a stimulus. Thus, it is imperative teachers guide students in allocating their attention selectively, teaching them how to become self-reflective learners.

To support and engage students' selective attention and reflective awareness, teachers present a form of mathematical representation (*sensory input*) and then pose purposeful questions such as: "What do you notice?," "What sense can you make of this?," or "What do you know for sure?" These question-types support students' visual discrimination and mental processing of attributes and surface details of physical objects and of the figural units embedded within mathematical representations. Students make sense of the representation by making connections with their prior knowledge (schema or cognitive structures) and experiences. This is called *elaboration*. Students bring long- and short-term memory to the forefront to compare, classify, and connect the processed information to their existing knowledge (coding). This is the time when students make connections in their minds to creating mental iconic images.

Constructing mental images within one’s mind is essential for learning mathematics. Visualization is defined as a quick and often spontaneous recognition of what is mathematically relevant within a visual representation (Duval, 2014). As a mental activity, visualization is intentionally used for processing information, for remembering it, for planning, and for constructing meaning thereby generating knowledge and cognition (Garner, 2007). Current mathematics education researchers suggest that students’ depth of mathematical understanding highly correlates to the strength of connections among various forms of internalized mental representations (NCTM, 2014).

One’s ability to visualize correlates to Bruner’s (1966) iconic mode of representation. In mathematics, iconic images and symbols are mental images represented through words, numbers, pictures, diagrams, etc. (Garner, 2007). Students’ abilities to visualize iconic images within their minds involve the figural units or surface features of a visible representation (Duval, 2014; Garner, 2007). Mental images support students interpreting and making sense of quantities, number magnitudes, number and operations, and other forms of mathematical representations.

A student’s ability to transform one form of representation into another, from physical or symbolic representations, to internal visualizations, then to visible displays, while maintaining the same meaning and significance, is integral to developing mathematical proficiency (NCTM, 2014; Rubenstein & Thompson, 2012). Students’ external representations (*output*) become observable evidence of students’ capacities to “see” with their minds (Woleck, 2001). Thus, visualization and transforming iconic images into visible representations are essential cognitive processes for interpreting and learning mathematics.

Instructional tools to mediate the development of cognitive structures. Vygotsky (1978/1930) argued that the use of language creates the social (nurture) and mental (nature) constructs vital for students’ cognitive advancement (van der Veer, 2007). It is the classroom environment that supports culturally shared interpretations, meanings, and understandings whereby mathematical language, signs, and symbols are socially and culturally constructed, transmitted, and understood. These ideas strongly align to Vygotsky’s (1978/1930) work and to the other five learning theorists discussed earlier in the chapter.

Current mathematics education researchers have identified two pedagogical practices that align to Vygotsky’s claims. One practice is teacher’s artful facilitation of mathematical discourse (Herbel-Eisenmann, 2009). Another instructional practice is teachers’ questioning techniques (Boaler & Brodie, 2004; Rubenstein & Thompson, 2012).

Two main instructional intentions direct teachers’ mathematical discourse: creating productive social conditions for learning and the analysis of mathematical content (Herbel-Eisenmann, 2009). When discourse is socially focused, teachers and students work co-collaboratively to create safe learning spaces. Here, students understand that their mathematical ideas are expected, accepted, and respected. Further, teachers and students are responsible for ensuring discourse is respectful, open, inclusive, and learning-focused. Classroom expectations, routines, and even the organization of the furniture encourage or prohibit students sharing their mathematical thinking and work with each other (Herbel-Eisenmann, 2009).

When discourse is content focused, there are two approaches. There is the calculation approach and the conceptual approach (Herbel-Eisenmann, 2009). For the calculation approach, student discourse is focused on the operations and calculations used to solve mathematics problems. For a conceptual approach, student discourse is focused on why a specific strategy is selected. Student conversations may follow a specific line of thinking and address how students' ideas relate to the meanings of the problem. This last approach often leads to student explanations and justifications “grounded in the concepts and relationships that are central to the problem” (Herbel-Eisenmann, 2009, p. 31).

Additional content conversations may introduce mathematical vocabulary and contextual situations. Here, formal mathematical language is modeled, and vocabulary is often introduced and discussed using examples (Rubenstein, Beckmann, & Thompson, 2004). The introduction of formal vocabulary affords students opportunities to acquire more sophisticated language, glean new insights, and gain conceptual understandings of complex ideas such as mathematical operations, number relationships, and place value (Kozulin, 2002; Rubenstein et al., 2004). Ultimately, communication and language assist students in acquiring more complex behaviors and thought processes such as describing one's thinking using formal mathematical vocabulary and symbols, persevering through problem solving, and using mathematical tools strategically (Battista, 2010; NCTM, 2000, 2014).

Another key instructional tool is the set of questions teachers pose to students. Recent research findings demonstrate that posing authentic questions increases students' engagement and critical thinking skills (Boaler & Brodie, 2004). Authentic question types are questions without pre-determined answers. Additional question types focus students' attention on the following:

- gathering information
- exploring mathematical meanings and relationships
- linking concepts to representations
- extending student thinking
- establishing contexts for investigation

Current researchers claim that teachers’ strategic use of higher-level questions prompt students’ analysis, evaluation, and argumentation of mathematical ideas (Boaler & Brodie, 2004; Way, 2001/2011).

Asking good questions requires sufficient teacher pedagogical content knowledge. Pedagogical content knowledge includes knowing mathematics and knowing how students learn mathematics (e.g. specific representations, potential misconceptions, etc.) (Ball et al., 2005; Hill et al., 2005). Good questions often (a) reveal students’ conceptions and misconceptions about mathematical ideas; (b) afford students opportunities to generate substantive discourse; (c) invite all students to participate; and, (d) direct student focus, encourage reflective awareness, and increase their depth of thinking to develop conceptual understanding. Vygotsky (1978/1930) understood that language and communication are integral for student learning, metacognition, self-regulation, mathematical identity, and academic achievement (Hadjioannou, 2007; Kozulin, 2002; Mooney, 2013).

Objects, pictures, drawings, graphs, charts, tables, and symbolic notations are important cultural tools representing important mathematical phenomena, ideas, and concepts (Hiebert et al., 1997). Thus, mathematical representations are invaluable tools skillfully used by teachers in effective mathematics instruction (Duval, 2014; Rubenstein & Thompson, 2012). Mathematical representations assist students in exploring and grappling with conceptual complexities. They support problem solving, mathematical discourse, and afford students opportunities to recognize similarities and differences among related ideas (NCTM, 2000, 2014). The types of instructional tools teachers use to activate children’s cognitive structures are critical considerations at every juncture of classroom instruction (Sedikides & Skowronski, 1991).

To ensure representations are most effective for mediating the development of students’ cognitive structures, it is essential that the figural units and structure of the representation are visible. Visibility offers students opportunities to make sense of the mathematical patterns and structures inherent within the representations. Thus, a teacher must be clear about the goals of the lesson, and then be strategic in the selection, implementation, and use of mathematical representation(s) (Kamii, Lewis, & Kirkland, 2001b).

To strategically select a mathematical representation for instructional purposes, teachers first consider the figural units, as well as the structure of that representation. Figural units are the different elements or attributes a student “quickly recognizes as significant or informative” within the representation itself (Duval, 2014, p. 160). Before presenting a representation to students for analysis, teachers analyze that representation for themselves, noting possible student responses. This intentional action supports teachers’ decisions for the best selection of representations, in that it meets students where they are on the learning continuum relative to the mathematical goals and concepts of the lesson.

For example, during students’ first encounter with place value, young students often struggle understanding that one unit can also represent the value of 10, 100, or 1,000 (Van de Walle et al., 2014). Therefore, rather than using base-ten blocks as students’ first manipulative for exploring place value concepts, a teacher may select multi-link cubes to represent these exponential denominations. Multi-link cubes allow students to build one stick of 10 at a time using 10 individual cubes. Whereas with base-ten blocks, 10 small cubes are already assembled into one stick and 10 sticks assembled into one flat representing 100 cubes. Students' considerations for composing one 10 from 10 cubes is done for them.

With multi-link cubes, students compose 10 individual cubes into a stick of 10 cubes by counting and connecting each cube. They visibly notice that one unit can represent a value of 10. Moving forward, students build 10 sets of “10” sticks and lay the sticks side-by-side constructing a 10 x 10 model or 100. These actions enable students to understand that one unit of 100 also has a value of 10 groups of 10, or 100 cubes. Using these enactive representations, students can visualize the magnitude of 100. Next, to visualize the magnitude of 1,000, students build a 10 x (10 x 10) cube or 10 groups of 100. Once students experience several compositions and decompositions of 10, 100, and 1,000 using multi-link cubes, students move toward representing one unit of 10, 100, and 1,000 using sets of base-ten blocks. While engaged with both sets of physical tools simultaneously, the teacher draws students' attention to comparing the sets of multi-link cubes with the base-ten block representations in physical and symbolic forms. Students use their comparative thinking structures (e.g. conservation of constancy, spatial orientation) to make connections between the various equivalent representations.

When introducing the concept of a number line up to 100, a teacher may first select a measuring tool such as a meter stick to convey the concept of a number line. During students' initial examination of a meter stick, students may notice a number as positional, as in the number 12 sits between the numbers 11 and 13. Students may also notice that, as numbers increase within a decade, the digit in the ten's place stays the same while the digit in the one's place increases by one. Students may notice that the distance between any two consecutive counting numbers on the meter stick is always equivalent regardless of the length of the “number line” (Charles, 2005).

Thus, the sets of multi-link cubes and meter sticks are physical objects representing the stimulus input for developing students’ numerical reasoning, a sense of magnitude, and conceptual understanding of place value. Such enactive models coincide with Bruner’s (1957, 1966) representational learning theory that students’ initial instruction is most effective using enactive representations. Studies show that children’s use of physical and visual representations of mathematical concepts support their mastering and maintaining mathematical competencies in later years (L.S. Fuchs & D. Fuchs, 2001).

Mathematical tasks to mediate the development of cognitive structures. Effective mathematics instruction engages students in cognitively demanding tasks where they experience productive struggle (Henningsen & Stein, 1997; Van de Walle et al., 2014). Cognitively-demanding tasks are often complex and non-routine. Bruner (1966) described such tasks as powerful and economic instructional tools because they compel students to become thinkers and creative users of mathematics, thereby influencing the quality of their learning (City et al., 2009; Paris & Paris, 2001; Silver & Stein, 1996). Furthermore, productive struggle indicates that such tasks fall within students’ ZPD, yet are still problematic for students to solve. Cognitively demanding task-types help mediate the development of students’ biologically secondary structures because they require cognitive effort (assimilation and accommodation), including sense-making and connection-making on the part of the learner.

One powerful and economic way teachers can meet varied student needs and readiness for learning within a classroom is by using open-ended tasks and prompts (Van de Walle et al., 2014). Open-ended tasks and prompts scaffold students’ access to solving problems. Such tasks afford students opportunities to reflectively access their current levels of understanding, use their existing knowledge, including familiar tools and representations, to find viable solutions. They provide access to a struggling student who may hold underdeveloped cognitive structures and engage students who have already developed conservation of constancy (as in $9 + 3 = 10 + 2$). Research demonstrates that students who participate in classroom environments where teachers posed open-ended tasks are found to be more cognitively engaged (Paris & Paris, 2001).

Another key feature of open-ended tasks is they often have more than one correct answer or solution path, inviting higher-levels of cognitive demand (Van de Walle et al., 2014). For instance, a teacher might pose an open-ended task such as the following, “The answer is 12. What might the mathematics context be?” The teacher asks students to reflect upon and create multiple problems that result in the answer 12. One student might record, “A friend has 15 pieces of candy and he gave me 3 pieces. So, my friend has 12 pieces left.” Another student might write, “Twelve equals a dozen eggs.” A different second-grade student might offer the following, “Well, if I owed my mom 13 dollars and I got 25 dollars from my grandma for my birthday present, I would pay my mom and then I would have 12 dollars left.” If the teacher were to record the symbolic representation of this student’s problem, it would look like this: $-(\$13.00) + \$25.00 = \$12.00$. Open-ended tasks engage students at their levels of difficulty. Students who engage in open-ended tasks often possess a variety of problem-solving strategies, persevere under challenging circumstances, and strive to construct meaning (Paris & Paris, 2001).

Subitizing tasks are another form of mathematical tasks teachers use to stimulate and engage students’ cognitive structures. Subitizing refers to one’s ability to recognize the numerosity of a set of objects efficiently without counting. Clements (1999), an early childhood mathematics education researcher, describes two major forms of subitizing: perceptual and conceptual. Perceptual subitizing involves the ability to immediately scan and recognize a small quantity of objects or pictures of dots without using additional mathematics or learned processes. Perceptual subitizing is considered a biologically primary cognitive structure because infants within the first two months of life and some animal species unitize or subitize up to three objects (Geary, 1995). This form of subitizing supports the development of concepts connected to cardinality.

Pattern recognition supports students moving toward conceptual subitizing (Clements, 1999). Conceptual subitizing is considered a more advanced form of perceptual subitizing and is an essential skill for developing students’ sense of number (Van de Walle et al., 2014). It also supports the development of students’ cognitive structures involving visualization, quantification, conservation of constancies, pattern finding, and logical reasoning. According to Geary’s (1995) definition, conceptual subitizing is considered a biologically secondary cognitive structure because it is developed over time and is mediated by others.

An example of conceptual subitizing is to present a domino containing four pips on one side and five pips on the other side. Flashing this domino in a mere four seconds, students may quickly “see” a double four and one more. Multiple experiences for conceptually subitizing quantities help students recognize that a double four equals eight and add one more, wherein $4 + 4 = 8$, $(4 + 4) + 1 = 8 + 1 = 9$.

Other foundational tasks in early elementary mathematics classrooms involve counting. Clements and Samara (2009) argued, “Without verbal counting, quantitative thinking does not develop” (p. 21). Geary, Hoard, Byrd-Craven, Nugent, and Numtee’s (2007) research findings supported this claim. Current research found that students engaging in counting tasks using physical tools minimizes their knowledge gaps, especially the gaps that low achieving students often hold (Bryant et al., 2008). Students who struggle with counting tasks often struggle with memory retrieval.

Studies suggest counting tasks support students developing more efficient and effective ways to count and learn how to keep track of those counts (Schwerdtfeger & Chan, 2007). Students’ development of counting abilities forecast their capacities for numerical learning and simple arithmetic (Jordan et al., 2009; Passolunghi, Vercelloni & Schadee, 2007). In fact, counting is a more powerful method than subitizing when quantifying groups of objects.

Counting collections of objects aligns to Piaget’s (1965/1952) concrete-representational stage of development, follows Bruner’s (1966) three modes of representations: enactive, iconic and symbolic, and his notions of power and economy. The development of students’ cognitive structures related to number sense and whole numbers and operations rely heavily upon students’ abilities to connect enactive quantitative representations to linguistic and symbolic representations representing numbers and space. As students engage with counting and subitizing tasks, their secondary cognitive structures such as memory-based retrieval and number name acquisition are mediated. Problem solving, quantification, conservation of constancy, and symbolic representation are also mediated and strengthened. Early detection of students’ underdeveloped counting abilities, followed by appropriate interventions, furthers students’ primary cognitive abilities and advances their biologically secondary cognitive structures (Geary, 1995, 2011a).

In addition to counting tasks, research studies also provide strong evidence linking students’ spatial skills to mathematics achievement (Gunderson et al., 2012; Kell et al., 2013). Number line tasks support the development of students’ spatial skills. For example, for making sense of the number line represented in Figure 1, students need to use their spatial reasoning skills to determine the value of *n*. Students’ “ability to use a number line is a key component of children’s understanding of number” (Geary et al., 2007, p. 1344).

Based upon these findings, a variety of number line activities underscoring one-to-one correspondence, counting, numerical magnitude, number relationships, and mathematical operations are essential for developing students’ cognitive structures relative to classification, seriation, and spatial and temporal orientations (Booth & Siegler, 2008; Gunderson et al., 2012; Jordan et al., 2009; Kamii et al., 2005). Current research studies prove that the use of a mental number line increases students’ number knowledge and understanding of simple addition and subtraction problems (Bryant, 2005).

Special education research and instructional practices. National mathematics organizations and mathematics educators believe that using constructivist and socio-constructivist principles and engage students with cognitively demanding tasks increase student achievement in mathematics (Baroody & Ginsburg, 1990; NCTM, 2014; National Research Council, 2012; Noddings, 1990; Van de Walle et al., 2012). However, the application of these philosophical stances and instructional practices with students who appear to have learning challenges remain the center of much controversy. Cognitive psychologists and researchers within special education conduct studies investigating the learning challenges exhibited by struggling students. Their findings suggest that explicit and systematic instruction are considered best pedagogical practices for students with learning challenges (Bryant et al., 2008; Geary, 1995; Gersten et al., 2009; Montague, 1997). Explicit and systematic instruction involve the use of teacher modeling and demonstration, intentional instructional scaffolding, student verbalization of thought processes, cumulative reviews of concepts, and corrective feedback (Gersten et al., 2009). In and of themselves, explicit and systematic practices are proven to offer valid and reliable learning outcomes (Gersten et al., 2009) and many of these practices align to Feuerstein’s (2006) MLE theory. This type of instruction is highly effective for improving students’ computation abilities; however, it is not applicable when the desired outcome is to develop higher-order thinking and problem solving (Steedly et al., 2008). Students who participate in step-by-step instruction or rote memorization without sense-making or developing conceptual understanding do not retain skills for long, nor are they able to transfer concepts to other contexts (Devlin, 2010; Woodward & Montague, 2002).

The continual challenge for teachers is finding a balance between developing students’ conceptual understanding and procedural knowledge. Engaging students in higher order thinking and problem solving supports the development of student’s biologically secondary cognitive structures which are vital for participating in advanced mathematics courses (Geary, 1995; Ifenthaler, 2011). It is important to remember that when considering instructional approaches constructivism, socio-constructivism, representational learning theory or explicit and systematic instruction all offer empirical evidence substantiating researchers’ claims.

Section 5: The “Re-Envisioned” Instruction Model

Teaching is a cultural and systemic activity (Stigler & Heibert, 1999). Until recently, the basic structure of mathematics lessons—especially at the secondary level—had remained unchanged for the past 100 years (Stigler & Heibert, 1999). Most lessons are still structured within a 45- to 60-minute time frame. At the beginning of a lesson, a teacher reviews or checks students’ homework assigned the previous school day. To introduce a new lesson, the teacher first demonstrates a skill or procedure, modeling it step-by-step while students observe. Once teachers perceive students can replicate that skill or procedure independently, the teacher assigns multiple problems for students to practice. See Figure 2 for the traditional framework and purpose for a mathematics lesson (Stigler & Hiebert, 1999).

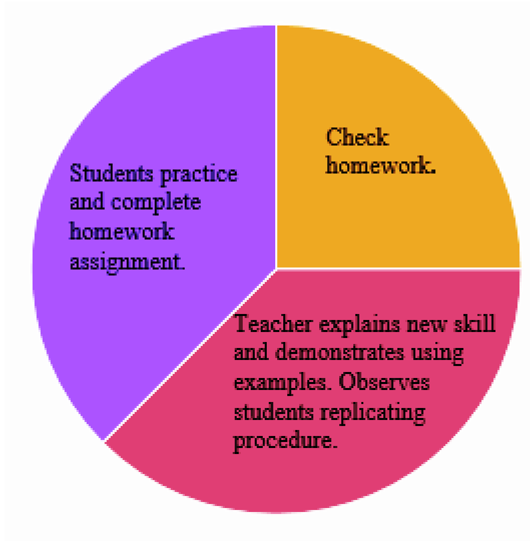

Figure 2. Traditional Framework and Purpose for Mathematics Lesson

Alternatively, this chapter describes a “Re-Envisioned” instructional model for implementing mathematics lessons. Six theorists’ views and current research attribute children’s cognitive advancement to neurological networks of cognitive structures. Current neuroscience research suggests there are highly interactive subsystems of structures within the human mind. These structures wait for stimulation and then work in concert to support students’ making connections neurologically, experientially, and abstractly (Byrnes & Fox, 1998; Ifenthaler, 2011; Sedikides & Skowronski, 1991). According to these theorists, cognitive structures clearly influence students’ cognition, cognitive growth, and learning (Duval, 2006; Feuerstein et al., 2006; Garner, 2007; Sweller, 2008).

Furthermore, all six theorists claimed that students’ cognitive structures change as they mature and learn. The stronger and more integrated students’ cognitive structures are, the more accessible mathematical concepts become (Wood et al., 1976). Current research is consistent with such claims. The human mind is malleable—not hard-wired as previously assumed—and all students can learn mathematics, irrespective of the origins of students’ learning difficulties (Dweck, 2006; Feuerstein et al., 2006). The premise for mediating the development of students’ cognitive structures to minimize their learning difficulties holds major implications. As Sweller (2008) claimed, instructional practices that fail to mediate the development of human cognition relative to students’ existing cognitive structures are likely to be haphazard in their effectiveness.

Two conceptual frameworks for the “re-envisioned” instruction model. The synthesis of learning theories and current research informed the conceptual framework identified as the “re-envisioned” instruction model. This model is used for teaching Tier I core mathematics instruction. The purpose of Tier I core instruction is to meet the learning needs of most students within a classroom and further their cognitive growth. Hence, the “re-envisioned” instruction model relies heavily upon the application of the six learning theories and current research detailed in this paper.

To advance learning and deepen students’ conceptual understanding, the “re-envisioned” instruction model embeds a synthesis of learning theories and current research within three distinct instructional segments: (a) *launch*, (b) *exploration*, and (c) *summary/reflection*. The purposes for each instructional segment resemble the intended constructs found in the Connected Mathematics Project’s framework for mathematics instruction (Michigan State University, 2017). The launch provides initial stimulus or sensory *input* to engage students in thinking about the mathematical focus of a lesson. The exploration segment encourages students to *elaborate* upon the mathematical concepts of the lesson using specific tasks. The summary/reflection affords time for whole-group sharing (*output*) of student-generated solution strategies, mathematical representations, justifications, critique, and argumentation. Outward signs of student learning include making connections to prior knowledge and experiences, identifying patterns [and relationships], identifying predictable rules, abstracting generalizable principles, and applying one’s learning to additional contexts and concepts (Garner, 2007). Each instructional segment is depicted in Figure 3.

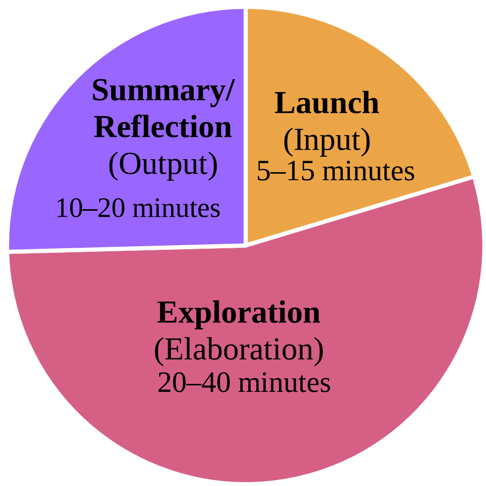

Figure 3. The Conceptual Framework for the “Re-Envisioned” Instruction Model

Within each segment, all three phases of mental actions: *input*, *elaboration*, and *output* also occur (see Figure 3). *Input*, *elaboration*, and *output* become critical mediated cognitive processes (thought and analysis) and externalized actions (e.g. intentional discourse, mathematical modeling, and written records) that assist students mentally engaging multiple times with the intended concepts of a lesson within a 60-90-minute timeframe (Feuerstein et al., 2006), vastly different from the traditional structure and purpose of mathematics instruction. See Figure 4 to understand these differences in processes and purpose of the “re-envisioned” instruction model.

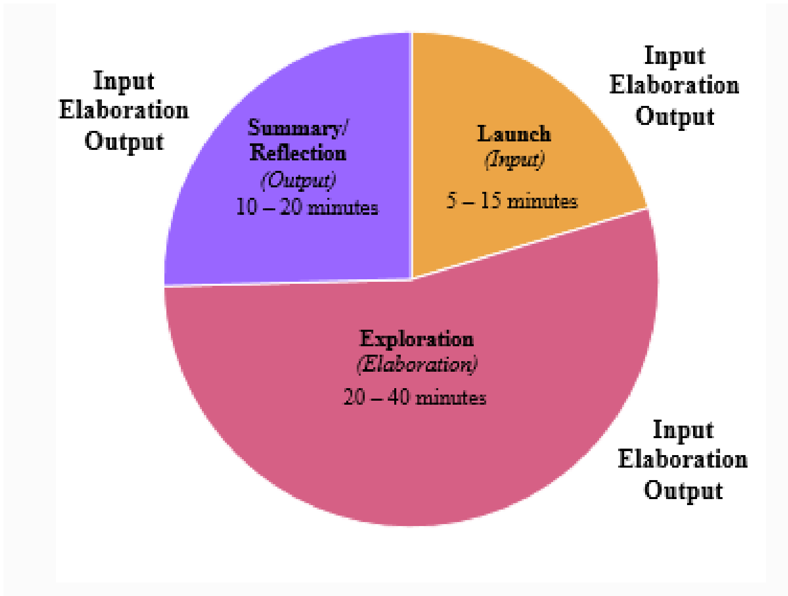

Figure 4. The Three Phases of Mental Actions for Lesson Implementation

To obtain a purview of specific teacher and student roles essential for engaging in the mental actions and processes of *input*, *elaboration*, and *output* during each instructional segment, see Table 2.1.

Table 2.1

The “Re-Envisioned” Instruction Model’s Framework for Three Mental Actions

Segment 1: Launch

Mental Actions	Teacher Role	Student Role
Input	Activate students’ cognitive structures by presenting a form of mathematical representation and asking open-ended questions such as “What do you notice?” and “What sense can you make of this?”	Use senses to take in sensory data and make observations of representations.
Elaboration	Move around the room. Ask clarifying questions such as, “What are you understanding the task to be?” “What do you know for sure?” Listen to students’ reasoning.	Connect observations to existing schema and prior experiences to make sense of information. Use memory, recognition, classification, logical and deductive reasoning to mentally code sensory data.
Output	Generate a public visual display (anchor chart) representing student thinking and sense-making, including students’ misconceptions. Ask clarifying questions to understand student thinking and perspectives. Present mathematical vocabulary as needed. This chart serves as a scaffolding tool and input for students as they begin the exploration segment.	Record observations and thoughts in a mathematician’s notebook. Use language, drawings, and mathematical representations to exhibit sense-making. Share own observations and sense-making ideas with whole group. Build upon each other’s ideas, ask questions, and critique one another’s mathematical thinking.

Segment 2: Exploration

Mental Actions	Teacher Role	Student Role
Input	Introduce selected task(s) to students. Ensure students understand the task(s), possibly referring to the anchor chart created in the launch of lesson.	Use the output from the launch and own logical reasoning to make sense of mathematical task(s).
Elaboration	Listen to students' explanations and justifications, and ask clarifying questions. Extend student thinking. Make mental or physical notes of students' conversations, solutions, and strategies to share during last segment of lesson. Encourage students to account for their thinking and strategies using recording sheets or mathematicians' notebooks.	Work toward comprehension and solution strategies using logical reasoning, pattern finding, problem solving, quantification, and discourse.
Output	Pre-select the order of students' sharing during Summary/Reflection of lesson.	Record thinking in various forms on a recording sheet or in a mathematician's notebook. Use concrete tools and manipulatives, draw pictorial representations, justify and explain thinking to peers and group members. Practice using precise mathematical language.

Segment 3: Summary/Reflection

Mental Actions	Teacher Role	Student Role
Input	Scaffold the order and presentation of student ideas, building from simplest to complex, concrete to abstract, or conceptual to procedural; or pose a new question for students to consider. Ensure that the lowest-performing students are heard, and their strategies shared.	Share thinking from exploration segment, including representations, solution strategies, findings, or connections, to demonstrate understanding of the concepts and goals of the lesson.
Elaboration	Use questions and student-to-student discourse to connect student explanations back to the launch of the lesson. Add onto the original anchor chart or create a new chart capturing students’ additional insights. Support students building upon each other’s ideas by promoting student-to-student discourse, making connections between ideas, strategies, solutions, procedures, and representations.	Make mental connections between the various mathematical representations, solution strategies, procedures, and ideas.
Output	Use public documents generated during Segment 1 and Segment 3 to compare, correct misconceptions, facilitate student-generation of new insights, meanings, connections, and revise student thinking. Pose a new question for students to consider and foreshadow upcoming lessons.	Process and summarize new insights, including generalizations. Reflect upon and revise thinking using analysis and synthesis. Demonstrate new understandings using words, pictures, numbers, equations, etc. Generate new questions for future investigations.

To support the reader gaining an in-depth understanding for how a teacher uses the launch of a lesson to stimulate students’ three mental actions of *input, elaboration, and output*, an in-depth description is provided below. Brief descriptions of the exploration and summary/reflection segments follow.

Preparing the launch. Teachers use their understanding of mathematics, their knowledge of students, and knowledge of their grade-level curricula to select, design, and modify varying forms of mathematical representations and tasks. Teachers’ selected representations and tasks fall within students’ zone of proximal development and are cognitively demanding. When considering the most effective representations and tasks to use, teachers consider the figural units, as well as the mathematical structures embedded within a graph, picture, mathematical model, a story problem, or student work. The teacher then identifies possible student responses to determine if the selected representation/task supports student understanding relative to the goals of the lesson. Once a representation or task is selected, the teacher launches the lesson by presenting the mathematical representation/task as students’ sensory input.

Launch implementation. The launch is the first instructional segment of the “re-envisioned” instruction model. It frames the initial *input* or beginning stimulus for the *entire mathematics lesson.* In addition to framing the *initial input* for a lesson, the teacher uses the launch to facilitate student progression through the three phases of mental actions: *input*, *elaboration*, and *output*.

The mathematical representation and the teacher’s prompts provide the sensory input necessary for stimulating and accessing students’ existing cognitive structures. The teacher’s mediational prompts, in the form of questions or instruction, invite students to focus and pay attention to the figural units, details, and mathematical structure of the representation (Learner & Johns, 2012). Teacher prompts may include, “What sense can you make of this?,” “What do you notice?,” “What mathematical connections might you make when you look at this representation?,” or “What patterns do you notice?” (Garner, 2007). Here, teachers do not tell students what to see, nor do teachers anticipate one specific response. For students to construct understanding, students need to make sense of the information for themselves. Students’ natural thought processes of memory, intelligence, and attention help bring their prior knowledge to the forefront.

All sensory stimuli received by the mind is coded, symbolized, and generalized according to students existing mental structures (Jensen, 2000). To make sense of the mathematical representation, students use their comparative thinking structures to compare, analyze, synthesize, and evaluate the representation’s figural units and mathematical structure. Figural units are the different elements or attributes a student “quickly recognizes as significant or informative” (Duval, 2014, p. 160).

Students’ sense-making processes of mental coding, symbolizing, and generalizing are considered *elaboration.* Elaboration is central to cognitive processing because it supports students’ transforming discreet bites of data into organized knowledge that is relational (Feuerstein et al., 2006). Elaboration involves retrieving information from memory and using existing cognitive structures to mentally represent, manipulate, and transform sensory data into visualized images within the mind. Hence, memory retrieval, reflective awareness, mental coding, and visualization are essential mental tools for creating meaning and for developing relational understanding.

Once mental images are in place, students transform their mental images into abstract and symbolic representations that are observable to others. This represents the third phase of mental actions: *output*. Observable representations include numbers, written and audible words, physical actions, drawings, graphs, etc. Student-generated mathematical representations are the visible manifestations of cognitive structures functioning within students’ minds (Duval, 2006; Geary, 1995).

Students share their own observations and insights in public ways during this third phase of the launch. Teachers solicit students’ visible and audible mathematical representations and records them verbatim onto an anchor chart or other recording device. Student *output* provides new data for the entire learning community to consider, reflect upon, and critique.

Student output can be used in formative ways. Students analyze and critique each other’s mathematical reasoning, revise and extend their learning, and engage in inter- and intra- personal exchanges necessary for mediating the development of their own cognitive structures (Kozulin, 1990; Vygotsky, 1965). Naturally, student output sheds light on their current understandings and misconceptions they may have.

Utilizing this on-going, formative, dynamic assessment, teachers gain insights to where students are on the learning continuum relative to the mathematical goals of the lesson. They can provide additional instructional scaffolds or build students’ background knowledge before continuing to segment two, the exploration segment of the lesson (Feuerstein et al., 2006; Garner, 2007; Kozulin, 1990).

Exploration implementation. The exploration segment of the lesson is the second instructional segment for mediating students’ cognitive structures relative to the goals of a lesson. During the exploration segment, stimulus *inputs* are additional representations or tasks offered by a teacher or classmates. These include instructions for a game or activity, or parts of an anchor chart generated during the launch. Mental *elaboration* occurs as students strategize to find a viable solution to a problem, while they investigate other students’ conjectures, or while playing a game. The *output* of the exploration segment may include students justifying a conjecture, identifying examples or counter-examples, recording steps in a solution strategy, or questioning opponents’ moves in a game.

Summary/reflection implementation. For the last segment of the lesson, the summary/reflection segment, *input*, *elaboration*, and *output* are again mental actions and processes that support student discourse and mathematical reasoning. Stimulus *input* to a whole group discussion might include the juxtaposition and comparison of two students’ solution strategies, a new or modified representation, a strategy in a game, or a student’s claim. Whole group discussion represents *elaboration* upon students’ ideas and claims, while a public record of students’ thinking, or modification of the initial anchor chart serves as *output*. To further understand how teachers and students engage with the mental actions of input, elaboration, and output, review Table 2.1.

Chapter Summary

Two conjectures were made at the beginning of this study. The first conjecture was that a single learning theory, by itself, could not counteract the historical trends of low student achievement in mathematics. A second conjecture was that an effective Tier I core instruction model could provide teachers effective ways for meditating the development of students’ biologically primary and secondary cognitive structures. These structures are essential for minimizing students learning difficulties and for increasing student achievement in mathematics. The number of research studies focused on mediating the development of students’ cognitive structures using Tier I core mathematics instruction is insufficient. Sweller (2008) claimed that instructional practices that fail to consider student’s existing cognitive structures relative to learning mathematics are likely to be haphazard in their effectiveness.

These conjectures and lack of research exposed the need for conducting this study: (a) to understand how a general education teacher might use Tier I core instruction to minimize students’ learning difficulties by addressing their origins, e.g. biological, cognitive, socio-cultural, and pedagogical; (b) reduce students’ need for Tier II and Tier III interventions by mediating the development of students’ cognitive structures; and, (c) build on existing research relative to increasing students’ mathematics achievement.

To conduct this study, the origins of students’ learning difficulties were reviewed. To understand the processes for learning, three seminal and three contemporary learning theories were consulted. Next, additional literature from the fields of mathematics education, cognitive and educational psychologies, and the neurosciences were reviewed and aligned to each learning theory. Each field revealed important cognitive processes and pedagogical practices for learning and for teaching mathematics. The extensive literature review supported a synthesis of theories, cognitive processes, and instructional practices used to create, refine, and test an innovative instruction model used for Tier I core instruction within a school’s MTSS program.

Chapter 3 presents the methods and procedures for conducting this study. It explains the research design. The chapter also offers an in-depth description of the study’s participants and location, as well as descriptions of instruments used to collect and analyze quantitative and qualitative data. Lastly, Chapter 3 provides details about specific conditions, methods, and processes necessary for replication.

Chapter 3: Methods and Procedures

Purposes of the Study

The main purpose for this study was to address the lack of student achievement in mathematics by creating an instruction model and using it to deliver Tier I core instruction. Tier I core instruction is the fundamental instructional mechanism for minimizing students’ learning difficulties in mathematics (Clements & Samara, 2007; Gresham & Little, 2012; NCSM, 2013). The development of the “re-envisioned” instruction model relied heavily upon six learning theorists’ conceptualizations for mediating the development of students’ cognitive structures. Cognitive structures are neurocognitive systems vital for student learning (Garner, 2007; Feuerstein et al., 2006). In addition to creating this model, the study tested the model’s effectiveness for improving students’ achievement and minimizing students’ learning difficulties in this academic subject.

The literature review briefly described five origins for students’ learning difficulties. For this study, the “re-envisioned” instruction model targets four origins of students learning difficulties: biological, cognitive, socio-cultural, and pedagogical. Collectively, these origins were addressed through teachers’ and students’ use of the instruction model.

The focused unpacking of each learning theory illuminated vital constructs and processes for ways children learn relative to the existence and influence of their cognitive structures. The literature review also supported the alignment between six theories and current research from multiple educational fields including the neurosciences. The model was then tested for four specific effects:

1. How effective was the model in increasing students’ mathematics achievement?
2. What was the model’s effectiveness in minimizing students’ learning difficulties in mathematics?
3. To what extent did teachers’ implementation of the model mediate the development of students’ cognitive structures, specifically spatial orientation and conservation of constancy?
4. How did teachers’ implementation of the model influence students’ beliefs about and practices for learning mathematics?

To answer these four questions, this study was conducted in three second-grade classrooms at a partial Title I, K-5 school—identified as Midwest Elementary School—during the 2014−2015 school year. The time frame for applying treatment and collecting data was September 2014 through January 2015. Additional data collection occurred in May 2015, four months after treatment.

Researcher’s Background

From 2005 to 2013 the researcher was employed by a county-wide Intermediate School District as a mathematics education consultant. The researcher worked with low-performing school districts relative to mathematics achievement. A main responsibility involved coaching K-8 general and special education teachers in improving their instructional practices. Additional responsibilities included designing and providing professional learning for teachers, writing mathematics curriculum aligned to CCSSM, and collaborating with other consultants to support and evaluate school districts’ implementation of MTSS. These experiences strikingly demonstrated to the researcher the need for strengthening Tier I core instruction as the means to improve student learning.

The responsibilities as a mathematics education consultant developed the researcher’s capacity to design and conduct this study. Curriculum work supported the selection and design of mathematical tasks. Coaching experiences prepared the researcher to support teacher implementation of the “re-envisioned” instruction model. The involvement with school districts’ MTSS implementation informed the processes for data collection. In short, many of the study’s processes were supported by the extensive work and experiences with different teachers in actual classrooms in a variety of educational settings.

Research Design

This study was quasi-experimental. Two classrooms received treatment, while the third classroom functioned as a control. The study’s theoretical framework was grounded in design-based research (Barab & Squire, 2004; Brown, 1992; Cobb et al., 2003). The premise of design-based research is that an intervention or treatment be conducted in authentic classrooms (Design-Based Research Collective, 2003). Authentic settings enable researchers to determine causational factors for how and why interventions or treatments work.

Through a variety of collection methods, both quantitative and qualitative data were concurrently collected throughout the study. The application of a mixed-methods approach increased the internal and external validity of the study. Merging teachers’ and students’ perceptions, explanations, and observations of behaviors and activities with quantifiable data supported cross-validation and data-triangulation (Creswell, 2009). The concurrent side-by-side design included quantitative data collected at the beginning and end of treatment, and four months after treatment. Thus, a time series design structured the gathering of students’ pre-, post- and end test scores (Creswell, 2009; Cook & Campbell, 1979).

Qualitative data were captured through a variety of means. Communication and interactions between the researcher and teachers, between the researcher and students, individually, in small groups, and in whole groups occurred throughout the study (DeWalt & DeWalt, 2011). Collection methods included surveys, questionnaires, face-to-face interviews, observations of teachers’ instruction and teacher/student dialogues, scripted field notes of verbal communications, and photos and photocopies of students’ work. Information- and image- rich transcriptions often originated in teacher and students’ handwritten texts, instructional artifacts, and photographs and records of students’ mathematical thinking. Much of this data reflected students’ words, phrases, sentences, and diagrams (Creswell, 2009). Collected with intentionality, the qualitative sources aided the researcher’s memory of the communication, activities, and learning experiences that transpired throughout the study.

A convenience sample generated the population for the study (Creswell, 2009). The researcher worked with all K-5 teachers at Midwest Elementary School the previous school year (2013–2014). Teachers were supported in improving their instructional practices for teaching mathematics. During that time, the researcher established a professional working relationship with all teachers and with the school’s principal.

The student subjects consisted of non-equivalent groupings. The school’s principal and first-grade teachers pre-determined first-grade students’ placement into non-equivalent second-grade groups for the school year, 2014–2015. Teachers’ placements of students were predicated on their knowledge of their first-graders and the strengths of second-grade teachers. With deliberation, the school’s principal determined the two teacher and student groups who were to receive treatment. In this study, these groups are referred to as CLB and CLC. The principal also determined the teacher and student group who functioned as the control. This teacher and her classroom is known as CLA. Having a control group supported the detection, isolation, and identification of possible causational factors and confounding variables impacting the overall effects of the treatment (Creswell, 2009).

Research Site

Midwest Elementary School housed approximately 419 general education students, kindergarten through fifth grade during the time of the study (September 2014–January 2015). In addition to the 419 general education students, approximately 40 pre-school students and two classes of cognitively impaired students in self-contained classrooms also attended this school. The general education students assigned to this school lived within the geographical boundaries defined by the school district’s administration and Board of Education.

Approximately 67% of the school’s student population were Caucasian. Hispanic, Asian, and African-American students were also members of the school’s demographics. Some families were transient, meaning they attended school for a short time period, moved, and then returned to Midwest Elementary. Approximately 40% of the student population were eligible for government-funded free or reduced lunch. Thus, this school was under a targeted assistance plan for Title I funds. Student attendance rate was 97%.

During the study, the employees at the school consisted of one principal, 47 staff members, and two secretaries. For K-5 general education teachers, each grade level was comprised of a three-member team. Each team member was near other team members to facilitate collaboration. Grade level teams collaborated two or more times each week to plan instruction and discuss student progress.

All core academic subjects were taught by highly qualified teachers as defined by Midwest State’s Department of Education and Midwest School District. Of the professional staff working at Midwest Elementary, eight staff were endorsed with bachelor’s degrees and 22 staff members were endorsed with master’s degrees. The principal was the instructional leader for 9 years with minimal staff turnover during the previous 5 years.

Parent enrollment in the school’s PTA organization was high, with 100% participation for male students and 99% participation for female students. Midwest Elementary school’s community took pride in their collaborative work and pledged to ensure quality learning experiences for all students attending this school (Midwest School’s Annual Report, 2013–2014).

All third-through fifth-grade students at Midwest Elementary School were tested using the 2014–2015 statewide assessment for mathematics.[1] More than half (58.1%) of all third-graders received a score of proficient, 24.2% were designated partially proficient, and 17.7% were deemed not proficient. For the 2015 fourth-grade NAEP assessment, 100% of their fourth-graders participated (NCES. 2015a). Assessment data suggested that only 18% of their students were considered proficient or advanced in mathematics. The remaining 82% of fourth-graders were considered basic or below proficient.

Study Subjects/Participants

Several variables justified the selection of Midwest Elementary School’s second-grade teachers as the teacher-subjects for this study. The second-grade teachers strengthened pedagogical skills for teaching mathematics through their participation in school-wide training the previous school year. Teachers learned how to facilitate mathematical discussions using

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[1] It was not possible to compare Midwest Elementary third-grade state test scores from 2013–2014 to third-grade students’ state test scores in 2014–2015 due to a change in the state assessment and the assessment’s parameters (i.e. spring- vs. fall-testing windows).

inquiry-based questions. They implemented the use of mathematician’s notebooks. They also learned about the launch, exploration, and summary/reflection segments of the “re-envisioned” instruction model.

A second motive for selecting this team was that there were many similarities amongst all three teachers. All three teachers held master degrees. All were considered highly effective for teaching literacy, while none specialized in teaching mathematics. All taught professionally for 10 to 11 years and taught second-grade students for at least 5 years. Each had served on a curriculum team either at their school or at the district level. Teachers’ similarities reduced the number of confounding variables impacting the results of this study (Creswell, 2009).

A third motive for this selection was that Midwest Elementary School’s principal considered the second-grade teachers a high-functioning team. She often observed them working together, discussing and planning instruction in all subject areas: literacy, mathematics, writing, social studies, and science. They each taught math in 60- to 75-minute time blocks per school day. They used their district’s Benchmark Assessments to guide their instructional planning and used their district’s mathematics program as their main resource for mathematics instruction. When one of the teachers accessed an additional resource, she shared it with the others (e.g. “Teachers Pay Teachers”). The final and most important variable pertained to teachers’ high expectations for students. Each teacher facilitated a classroom environment where students worked to make sense of mathematics. Teachers expected students to record their ideas in their mathematician’s notebooks and each teacher expected her students to make meaning using constructivist and socio-constructivist principles.

Like the larger school population, many of the second-grade students in these classrooms were Caucasian. Approximately 25% of the students, however, represented other ethnicities including Hispanic, African-American, Asian, and Russian. Two migrant families had second-graders attend school at the beginning of the school year. These families left mid-year and came back the following fall. Approximately one-fifth to one-third of the students in each classroom were eligible for federally-funded free-and-reduced lunch plans. At the beginning of the school year, each classroom had close to the same ratio of boys to girls: 13 males, 9 females; 12 males, 9 females; and 12 males, 10 females.

To control for regression and confounding variables (Creswell, 2009), exclusion criterion did exist in this study. Students who received alternative or additional mathematics support during the school day were excluded from the data set. Three second-grade students attended a third-grade classroom for mathematics instruction. Naturally, these students were excluded from the data set. One second-grade student required special education services and did not participate in general mathematics instruction. This student was excluded as well. Additionally, three second-grade students required English-Language services during mathematics instruction. Although these students participated in daily mathematics instruction along with their identified classmates, these students’ data were excluded from the study. Three students moved out of the school boundaries before the study ended. These students were not included in the data set. All student data analyzed and referenced in this study involved students who received Tier I core instruction from their assigned general education teacher during the school day and remained at the school site throughout the study. All students in this study, including Tier II and III students, did not receive additional educational services or support beyond the classroom. This decision reduced the threat of outside influences impacting the results of the study.

Overview of Methods and Procedures

Specific methods and procedures were used to conduct this study from September 2014 to January 2015 and one time in May 2015. As an active participant, all procedures were enacted or directed by the primary investigator/ researcher (DeWalt & DeWalt, 2011). Categories of procedures were classified under general headings. These headings included: recruitment, screening, data analysis, monitoring, teacher implementation support, and intervention. While most classifications are self-explanatory, “intervention” described methods and procedures unanticipated in the original proposal for this study.

An overview of enacted procedures is presented here. General procedures are sequenced as enacted. Following this brief overview, more specific details regarding methods and procedures are provided in subsequent sections in this chapter, as well as in Appendix A.

The first and most important procedure was to seek for and acquire IRB approval to conduct this study. Approval was granted in September 2014 by the University of Michigan’s IRB Committee. The remaining actions follow the sequence as described below:

1. *Recruitment* – Sought and gained consent from participating subjects: district administrator, school principal, teachers, parents, and students.
2. *Screening* – Conducted personal interviews with teachers and students using structured questionnaires.
3. *Screening* – Conducted formative assessment tasks and the Adapted Cognitive Structure Assessment (Tile Task, Number Line Assessment, and Cognitive Structure Protocol).
4. *Data analysis* – Analyzed student results from the formative assessment tasks to identify instructional tasks used in the two experimental classrooms.

5. *Teacher implementation support* – Modeled instruction using the “re-envisioned” instruction model, the cube task and arrow roads task with the two experimental groups.
6. *Monitoring* – Conducted first observations of classroom instruction in all classrooms. Created scripted field notes of observations and photographed instructional artifacts.
7. *Teacher implementation support* – Conferred with teachers relative to their implementation of “re-envisioned” instruction model and instructional tasks. Discussed student understanding of mathematical concepts.
8. *Teacher implementation support* – Provided the instructional resource *Number Talks* to the two experimental teachers. Modeled subitizing lessons for CLB and CLC Teachers with students.
9. *Screening* – CLA, CLB, and CLC Teachers administered the pre- *i*-Ready screening assessment to their students in the school’s computer lab, October 2014. Copies of student results were printed by each classroom teacher and shared with the researcher.
10. *Teacher implementation support* – Provided mathematics games to the two experimental teachers to implement with their students (see Table 3.1).
11. *Monitoring* – Observed student engagement with specific mathematical tasks and games. Generated scripted field notes of observations and photographed instructional artifacts.
12. *Intervention* – Provided support and resources to the two experimental teachers including a journal article for *Counting Collections*, student observation data, Benchmark Assessment I (BA) item analysis.
13. *Teacher Implementation Support* – Modeled counting task and pattern task.

14. *Monitoring* – Observed student engagement with specific mathematical tasks. Generated scripted field notes of observations and photographed instructional artifacts.
15. *Intervention* – Conducted an item analysis of BA II assessment items. Designed and created mathematics lessons aligned to the school district's Benchmark Assessment II, to the "re-envisioned" instruction model's conceptual framework, and to the district's mathematics program. These lessons were given to the teachers of the experimental groups.
16. *Monitoring* – Conducted second observation of all teachers' classroom instruction. Conferred with teachers to improve instruction relative to implementation of "re-envisioned" instruction model and students' current understanding of mathematics. Generated scripted field notes and photographed instructional artifacts.
17. *Intervention* – Supported CLA, CLB, and CLC Teachers by testing teacher-identified students relative to their understanding of numbers and operations. Provided teachers with assessment results and suggestions for instructing each student.
18. *Teacher implementation support* – Modeled novel tasks (open number line) for the two experimental teachers. Engaged teachers in reflective practice.
19. *Monitoring* – Conducted final classroom observations. Generated scripted field notes and photographed instructional artifacts.
20. *Screening* – CLA, CLB, CLC Teachers conducted the post- *i-Ready* screening assessment in the school's computer lab with their students. Copies of student results were printed by each classroom teacher and shared with the researcher.

21. *Screening* – Conducted post- Number Line Assessment to determine students' mathematical progress in spatial orientation. Conducted post- Adapted Cognitive Structure Assessment with each student. Assessed student-perceptions for learning mathematics with a questionnaire.
22. *Screening* – Conducted final data collection. Teachers shared copies of the end-of-year *i*-Ready assessment results with the researcher. Researcher conducted informal teacher interviews and students' self-analysis and reflections of their growth in mathematics.
23. *Data analysis* – Transcribed all qualitative data. Analyzed teachers' classroom instruction using the M-Scan rubric and questioning protocols. Analyzed students' qualitative data and coded for themes. Used specific instruments to conduct statistical tests to compare students' pre- to post- to end qualitative *i*-Ready assessment results.

The above information presents a general overview. For replication purposes, the following information offers further details about the procedures and methods used to conduct this study. The headings correlate to the classifications of procedures previously mentioned.

Recruitment. Midwest school district's Executive Director of Instruction, Technology, and Assessment, Midwest Elementary school's principal, second-grade teachers, respective parents and students were invited to participate in this study. The previous spring, the executive director gave his consent in electronic form. In a formal meeting conducted the first week of school, the principal and teachers signed forms documenting their consent for participation. These documents provided brief descriptions of the study, an outline of the work, expectations of teachers and researcher, and possible benefits to both teachers and students. The principal's and teachers' concerns and questions were discussed and addressed at that time.

Parents attended Midwest Elementary school’s curriculum night during the second week of school. Second-graders’ parents were informed about the purpose and design of the study at this meeting. After second-grade teachers explained second-grade curriculum and student expectations, the school’s principle invited parents to learn about the proposed research study. The study’s purpose, description, and expectations were shared by the researcher via a Power Point presentation and consent forms.

Both forms of communications informed parents. Parents were notified that all second-grade students would participate in daily mathematics instruction. They were told that two classrooms would receive instruction using the “re-envisioned” instruction model and mathematical tasks, while the third classroom would receive instruction using the district’s mathematics program. Parents were informed that student work would be collected and analyzed and that their child’s data would be protected by school protocols in accordance with Federal Privacy Laws. This meant that all identifying information regarding each child would be transformed into numerical codes and stored in a secured physical and electronic location within the researcher’s home.

Nevertheless, parents’ main query was, “Which of the two classrooms will be the treatment and which classroom will be the control?” The school’s principal requested that this information be kept private. She believed this information was too sensitive for students and their families. Consent forms were provided at the end of the meeting to parents in attendance without divulging this information. Families who were not in attendance received personal letters describing the study, accompanied by corresponding consent forms. Teachers’ classroom rosters structured the gathering and documentation of parent consent forms in early September 2014.

Individual student assent was sought, once parent consent was achieved. Individual student meetings were conducted utilizing a protocol explaining the study in student-friendly language. Each child was informed that he/she would be included in the gathering of qualitative data (e.g. mathematician’s notebooks and interviews). After students attended to the brief description of the purpose and expectations, the researcher asked each child if he/she wanted to participate. If the child agreed to the conditions of the study, he or she signed an assent form. From all three classrooms, 34 students were granted permission by their parents to participate, while 33 students agreed to the conditions for their participation. Students’ assent and participation generated the collection of qualitative data analyzed in this study.

Screening and monitoring. Screening measures, assessment tasks, questionnaires, and personal interviews were vital methods for assessing and monitoring teachers’ and students’ mathematical content knowledge and skillsets. While the *i*-Ready screening data was not available until the second week of October, the researcher utilized two formative assessment tasks Mid-September 2014 to screen all students for number sense. NCTM (2000) defined number sense as having fluid and flexible ways of thinking about and working with number and operations, as well as “moving from initial development of counting techniques to more-sophisticated understandings of the size of numbers, number relationships, patterns, operations, and place value” (p. 79).

The following tasks were used to assess students’ development of number sense. The first formative assessment task, *How Many Squares?* was administered to all three student groups mid-September 2014. The task originated from the mathematics program *Investigations in Numbers, Data and Space,* Grade 1, Unit 8 (Russell, 2007, p. 76). This assessment task required students to identify the cardinality of a set of paper tiles using their understanding of counting and number combinations.

Material preparation included 25 sets of paper tiles that were precut into single squares, into sets of two-square tile strips, and five-square tile strips, with each set contained in individual plastic bags. To conduct the assessment, a plastic bag of paper tiles was distributed to each student. Students were directed to work individually to arrange their tile set in ways that made sense to them. To complete the task, students arranged the tiles and identified the cardinality of the set by counting tiles. They drew a pictorial representation of their tile arrangement onto a blank sheet of paper and recorded the way they calculated the total number of tiles. Students' resulting pictorial representations and equations illuminated their abilities to transfer enactive representations into iconic and symbolic representations. Data analysis also provided important information relative to students who correctly identified the cardinality of the set, recorded numerals accurately, recognized numeric relationships amongst tiles (2 two-tile sets + a tile = 5 tiles), and used symbols indicating addition and equality.

The second formative assessment task was an Open Number Line Task. This task was created by the researcher. Research studies indicated strong evidence linking students' visuospatial skills to their mathematical performance and achievement (Gunderson et al., 2012; Presmeg, 2014). Thus, the open number line task assessed students' cognitive structures involving visuospatial orientation, as well as students' mathematical understanding of one-to-one correspondence, counting, numerical magnitude, numerical relationships, and equal-distancing of consecutive multiples.

The Open Number Line Formative Assessment Task consisted of three open number lines parallel to each other on a single page (see Appendix F). Each number line positioned 0 on the far left and 100 on the far right. Students were expected to place the remaining numbers on each number line. The first number line required students to count, place, and record multiples of ten beginning at 0 and ending at 100. Parallel to the first number line was a second number line. The second number line required students to count, place, and record multiples of 20 from 0 to 100. A third number line was parallel to the first and second number lines. The third number line required students to count, place, and record multiples of 25 from 0 to 100.

Specific performance criteria were identified to analyze student data. A running record was used to document individual student performances. Students who completed the task correctly demonstrated their understanding of number sequences. They demonstrated their skills for skip counting by 10, 20, and 25 by placing multiples of these numbers using respective distances on corresponding number lines. Student data and subsequent analyses from the tile and number line task informed task selection for the two experimental groups involved in this study.

Multiple methods were used to screen and monitor teachers’ content knowledge and pedagogical skillsets for teaching mathematics. A semi-formal interview was conducted with each teacher at the beginning of the study. Teachers were asked about their strengths and challenges in teaching mathematics. In addition to the interviews, teachers’ mathematics instruction was observed several times throughout the study; and, the M-Scan Rubric supported the analysis of each teacher’s pedagogical skillset used to implement their lessons. A face-to-face debriefing occurred with each teacher after each observed lesson. Discussions revealed teachers’ knowledge of mathematical content and use of mathematical representations for instruction (or lack thereof). Teachers also shared their observations regarding student understanding. Teachers’ choices for instructional tasks also illuminated their level of understanding of Common Core State Standards for Mathematics (NGA & CCSSO, 2010)

Teacher implementation support. Implementation support was provided on an “as-needed” basis determined by the experimental teachers or by the researcher for the duration of the study. CLB and CLC Teachers immediately requested implementation support at the beginning of the study. They wanted to observe the researcher modeling lessons with their students using the mental and externalized processes and actions supported by the “re-envisioned” instruction model.

Modeling instruction provided teachers first-hand experiences in observing the cognitive processes and pedagogical actions required for effective implementation. Modeling lessons reduced variability in teacher implementation. For example, teachers observed the enactment of lessons utilizing mathematical representations in all forms: enactive, iconic, and symbolic. Teachers observed instructional practices that stimulated students’ cognitive structures (or schema) and reflective awareness. They witnessed the use of different question types for engaging students in higher-levels of cognitive thought. During the enactment of the launch and summary/reflection segments of lessons, teachers listened as students participated in mathematical discourse. Equally important, teachers observed their students using constructivist and socio-constructivist processes to record their mathematical observations and thinking in their mathematician’s notebooks. Teachers’ observations enabled them to implement the “re-envisioned” instruction model using the same practices and processes.

The experimental teachers also engaged in reflection and dialogue after observing a modeled lesson. Student learning and teachers’ implementation of mathematics instruction were discussed. The dialogue between researcher and teacher supported teachers’ decisions regarding next steps for their mathematics instruction.

Specific tasks were selected and provided to the experimental teachers to mediate the development of students' number sense and to develop their cognitive structures, essential for learning mathematics (See Table 3.1). The literature review also illuminated vital concepts and processes for developing students' sense of number. These processes included subitizing, estimating, and counting (Bryant, 2005; Jordan et al., 2006; Muldoon et al., 2012). It was essential students engaged with tasks that required them to visualize quantities in a variety of contexts (Duval, 2014), compose and decompose numbers in flexible ways, calculate using number relationships, and solve basic arithmetic combinations and story problems (Van de Walle et al., 2014).

One of the first mathematics tasks selected for CLB and CLC Teachers and students was the Cube Task. Stacks of multi-linked cubes introduced students to the concepts of number patterns, relationships, equalities, and counting. To focus students' attention on these concepts, pairs of students were given three individual stacks of cubes: a stack of 2 cubes, a stack of 4 cubes, and a stack of 6 cubes. They were then asked to analyze the representations through the teacher's prompt, "What do you notice?" (Garner, 2007). After a minute or so, students began to record their observations in their mathematician's notebooks. The facilitation of mathematical discourse supported the unpacking and sharing of students' initial perceptions and observations.

Another instructional task teachers used with their students was Arrow Roads. This task engaged students in considering the placement of numbers on a hundred chart. It also supported students using number relationships to add and subtract numeric values. An arrowhead directed students' move on the hundred chart. A sequence of arrowheads formed an "arrow road." For example, students started at 17 on the hundred chart and followed the directions of "arrows," one arrow at a time. To move on the hundred chart from 17 to another number, students needed to notice the direction of each arrowhead. The direction of the arrowhead told students to move one space, either up (-10), down (+10), to the right (+1) or to the left (-1). After completing the sequence of directional moves, students transformed an Arrow Road into symbolic equations, recording addition and subtraction of two-digit numbers.

Another task was called Counting Collections. Research demonstrated that counting tasks supported students in developing more efficient and effective ways to count objects and keep track of those counts (Schwerdtfeger & Chan, 2007). This task was enacted three-to-four times with CLB and CLC students during the study to mediate the development of students’ cognitive structures essential for conservation of constancy. From a mathematical perspective, a counting collections task engaged students in oral counting, organizing sets of objects, determining cardinality of a set, recording numbers, and writing equations. Once students organized and counted their collections, students drew pictorial representations depicting the ways they organized their collections. Students’ pictorial representations revealed their existing abilities and skills to transfer enactive representations into iconic and symbolic forms (Bruner, 1966; Bryant et al., 2008)

Other forms of implementation support that teachers were given were instructional resources. One important instructional resource was titled *Number Talks: Helping Children Build Mental Math and Computation Strategies* (Parrish, 2010). It included subitizing tasks and operation tasks. Subitizing tasks were provided to CLB and CLC Teachers to develop students’ number sense and flexibility using numbers and operations.

Although the book’s author designed the collection of tasks specifically to increase students’ abilities to perform mental calculations, the researcher of this study hypothesized that teachers’ use of Number Talks would also mediate the development of students’ cognitive structures: visualization, subitizing, quantification, and conservation of constancy. All were vital cognitive processes for mediating the development of students’ conceptual understanding and achievement in mathematics (Bryant, 2005; Geary, 2011b; Jordan et al., 2006).

To support teacher’s implementation of the Number Talks resource, three tasks were modeled with CLB and CLC students. One task involved cards with dots or circles. Another task involved the use of base-ten blocks. A third task involved students analyzing number patterns. The first task involved “flashing” for approximately 3 to 5 seconds, a handmade dot card containing different groupings of circles or dots. Once a dot card was flashed, students utilized their perceptual and conceptual subitizing skills to mentally calculate the number of dots on the card (Clements & Samara, 2007). After the two experimental teachers observed the processes for implementing subitizing tasks, they replicated the same processes with their students using additional dot cards, base-ten blocks, and number patterns.

In addition to the Number Talks book, other tasks and resources aided CLB and CLC Teachers’ alignment of their mathematics instruction to the conceptual framework of the instruction model. Table 3.1 provides the reader an overview of these resources. Garner’s work (2007) supported the correlation between specific tasks and the cognitive structures being mediated by those tasks. The unpacking of each task helped identify the associated mathematical concepts, skills, and Common Core Standards of Mathematical Practice (NGA & CCSSO, 2010). Both CLB and CLC Teachers were asked to use these games and tasks with their students; and, they often used them during the exploration segment of the lesson. An important caveat is that all lessons, tasks, and games used throughout the study were provided by the researcher or were found within Midwest School District’s standards-based (and research-based) mathematics program. Providing the same instructional resources to the two experimental teachers eliminated confounding variables between CLB and CLC Teachers’ implementation of the instructional tasks.

Table 3.1

Tasks Used to Mediate the Development of Students' Number Sense and Cognitive Structures

Mathematical Tasks	Cognitive Structures	Mathematics Skills and Concepts
Number Talks (subitizing)	Quantification/Subitizing	Subitize
	Visualization	Compose and decompose number
	Pattern recognition	Basic facts
	Conservation of Constancy	Addition
	Logical reasoning Memorization	Subtraction
		Mental computation strategies – making ten, doubles, counting on, counting all
		Use number relationships to solve problems
		Reason abstractly and quantitatively
		Attend to precision
		Look for and making use of structure
		Look for and express regularity in repeated reasoning
Counting Collections	Problem solving	One to One correspondence
	Symbolic representation	Cardinality
	Quantification	Quantification
	Number magnitude	Adding groups of 1, 2, 5, 10
	Conservation of constancy	Compose and decompose number
		Estimation
		Associative property
		Attend to precision

		Represent a quantity in multiple ways –concrete, pictorial, and symbolic Reason abstractly and quantitatively Model with mathematics Look for and make use of structure
Puzzles Number Lines	Quantification Spatial orientation Pattern recognition Visualization Verbal reasoning Logical reasoning	Spatial relationships Patterns Count Equivalence Part/whole relationships Make sense and persevere in problem solving Model with mathematics Use appropriate tools strategically Look for and make use of structure
Games On/Off Game (subitizing, missing addend) Making Complements of Ten or Twenty (Card game using numerical values from 0 – 9. Students had a choice to place cards face up or place cards face down as in a memory game). Place Value Game Compatible Pairs	Quantification/Subitizing Visualization Pattern recognition Conservation of Constancy Logical reasoning Memorization	Subitize Compose and decompose number Basic facts Addition Subtraction Mental computation strategies – making ten, doubles, counting on, counting all Place value Estimation Use number relationships to solve problems

	Commutative and identity properties of operations
	Pictorial and symbolic representations
	Attend to precision
	Reason abstractly and quantitatively
	Construct viable arguments and critique the reasoning of others
	Look for and make use of structure

All three teachers were given autonomy and decision-making power to meet their students’ instructional needs daily. CLA Teacher, however, was restricted to use and teach her mathematics lessons as designed by the authors of her district’s mathematics program.

Intervention. Teachers’ needs for instructional resources continued as gaps in students’ mathematical understanding became evident. When teacher’s content or pedagogical knowledge were challenged; and, as time drew near to administer Midwest school district’s Benchmark Assessments (BA), CLB and CLC Teachers requested additional consultation and implementation support.

For example, in November 2014—two weeks prior to the first Benchmark Assessment’s administration (BA I)—CLB and CLC Teachers stopped using the mathematics tasks provided for this study. Instead, they used lessons featured in their district’s standards-based mathematics program. The two experimental teachers expressed anxiety for their students’ success on the district’s BA I. The lessons within their district’s program directly correlated to the assessment items found on the BA and teachers had yet to explicitly teach the content of the BA. Thus, they attempted to explicitly teach those concepts two weeks prior to administering the assessment.

Attending to teachers’ instructional concerns and attempting to reduce their anxieties, the researcher analyzed the assessment items found on BA I and BA II. Each item was unpacked for mathematical concepts and for the level of cognitive demand. Each item was linked to tasks and lessons taught prior to the first test’s administration. This analysis was provided to the two experimental teachers.

To support teachers and students meeting the mathematical demands of the second quarter Benchmark Assessment, BA II, the researcher analyzed the next two units of study in their district’s mathematics program. The researcher identified specific lessons from their district’s mathematics program that aligned to BA I and BA II assessment items. These lessons were “re-designed” to meet the theoretical and conceptual frameworks of the instruction model. An example of a “re-designed” lesson is the Change-to-More lesson described later in this chapter.

The “redesigned” lessons included unit goals, teacher directions and questions, specific representations intended to stimulate students’ cognitive structures, and identified tasks for the exploration segment of the lesson. The launch of a lesson was designed to bring students’ prior mathematical knowledge to the forefront by stimulating their cognitive structures and developing conceptual understanding for whole number and operations.

These “re-designed” lessons engaged the experimental students in the analysis of the figural units embedded in the various mathematical representations. Figural units are the different elements or attributes of an object or representation a student “quickly recognizes as significant or informative” (Duval, 2014, p. 160). Students worked to construct and socially co-construct their own understanding of the mathematical representations via memory, prior experiences, visualization, and other sense-making tools, to give meaning to the figural units (Garner, 2007).

All three teachers requested another intervention approximately mid-way through the study. Three to four students struggled learning specific mathematical concepts in each classroom. They struggled identifying a digit’s place value correctly. They were challenged by composing numbers in the hundreds. Naturally, teachers asked for support for identifying their students’ understanding and knowledge gaps relative to the numbers and operations learning continuum. Teachers also requested suggestions for instructional tasks to fill and bridge students’ knowledge gaps because they were unfamiliar with this learning continuum.

To determine students’ number knowledge, an assessment protocol titled the Number Knowledge Test was conducted with students (Griffin, Clements, & Samara, 2015). This developmental test was designed to orally assess students’ conceptual understanding of foundational concepts for number and operations, as well as detect their sophistication for problem-solving. After analyzing student data, the researcher provided each teacher with student reports, including a personalized list of appropriate tasks and activities intended to “fill gaps” in students’ mathematical knowledge and understanding. Teachers did not use this information to support their students as reported by each teacher.

In summary, all methods and procedures for conducting this study were designed to do the following:

- expand experimental teacher’s pedagogical practices and conceptual understanding of mathematics
- align experimental teachers’ instructional practices to the conceptual framework of the “re-envisioned” instruction model

- provide experimental teachers with tasks that:
    - stimulated and mediated the development of students' cognitive structures
    - increased students' conceptual understanding for numbers and operations
- reduce teachers' anxieties for supporting student success on their district's Benchmark Assessment
- reduce experimental teachers' cognitive and instructional load due to their participation in this study
- reduce variability in the experimental teachers' implementation

Research Questions

Four research questions were posed in Chapter 1. The first three questions pertained to the effects the model had on students' mathematics achievement, mathematical learning difficulties, and the development of students' cognitive structures. The fourth research question pertained to students' beliefs and practices about learning mathematics. All four research questions, accompanied by their null and alternative hypotheses, are presented below.

1. To what extent did teacher implementation of the "re-envisioned" instruction model influence students' mathematics achievement?

    $H1_0$: The change in students' mathematics achievement scores between students who received treatment and students in the control group were not statistically different as determined by pre- to post to end *i*-Ready Universal Screener assessments (Curriculum Associates, 2015).

    $H1_a$: The change in students' mathematics achievement scores between students who received treatment and students in the control group were statistically different as determined by pre- to post to end *i*-Ready Universal Screener assessments (Curriculum Associates, 2015).

2. Did teacher implementation of the “re-envisioned” instruction model minimize students’ learning difficulties in mathematics? In other words, did teacher implementation of the model move students identified at Tier II and Tier III levels to Tier I and Tier II levels respectively as determined by the pre- to end tests from the *i*-Ready Universal Screening Assessment (Curriculum Associates, 2015)?

    $H2_0$: When comparing students in the two treatment groups to students in the control group, there were no statistical differences in count patterns representing students’ decrease (improvement) in Tier Levels from pre- to end according to the *i*-Ready Universal Screener assessment data.

    $H2_a$: When comparing students in the two treatment groups to students in the control group, there were statistical differences in count patterns representing students’ decrease (improvement) in Tier Levels from pre- to end according to the *i*-Ready Universal Screener assessment data.

3. To what extent did teacher implementation of the “re-envisioned” instruction model influence the development of students’ cognitive structures, specifically spatial orientation and conservation of constancy?

    $H3_0$: When comparing the treatment groups’ development of their cognitive structures to the control group’s development of their cognitive structures, there were no statistical differences as determined by pre- to post- test scores on the Adapted Cognitive Structure Assessment results.

$H3_a$: There were statistically significant differences in students' development of their cognitive structures between students who received treatment and students who did not as determined by students' pre- to post- test scores on the Adapted Cognitive Structure Assessment results.

4. By the end of this study, to what extent did teacher implementation of the "re-envisioned" instruction model influence students' beliefs and practices for learning mathematics?

   $H4_0$: By the end of the study, qualitative differences in students' beliefs and practices for learning mathematics did not exist between students who received treatment and students who did not as indicated by students' and teachers' qualitative data.

   $H4_a$: By the end of the study, qualitative differences in students' beliefs and practices for learning mathematics existed between students who received treatment and students who did not as indicated by students' and teachers' qualitative data.

Data Instruments, Methods of Collection, and Analyses

To establish validity and reliability of the study's findings, multiple assessment measures and concurrent methods for gathering quantitative and qualitative data were essential. Such methods and measures detected, isolated, and assessed confounding variables naturally occurring in authentic classrooms. Some of the measures, methods, and procedures used to reduce variability and validity threats were explained in previous sections in this chapter. Additional measures are detailed below.

Quantitative instruments: Methods of data collection and analyses. Students’ achievement levels pertaining to number and operations were assessed three times throughout the study, October 2014 (pre-), January 2015 (post-), and April 2015 (end-). The measurement instrument was Midwest School District’s *i*-Ready Screening Tool (Curriculum Associates, 2015).

The *i-Ready* Assessment is a criterion-referenced mathematics screening tool (Curriculum Associates, 2015) aligned to the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). It is an online computer adaptive program that monitors students’ levels of mathematical proficiencies across five domains: (1) Overall Math Levels; (2) Number and Operations; (3) Algebra and Algebraic Thinking; (4) Measurement and Data; and (5) Geometry.

The district’s main purpose for using the screener was to monitor students’ growth and achievement levels in reading and mathematics. The screener also identified students’ instructional Tier Levels and provided teachers with specific information prescribing instructional interventions. The principals and instructional support staff accessed this information to support their MTSS program. The researcher, however, used the *i*-Ready student results to provide statistical evidence of the effects the “re-envisioned” instruction model had upon students’ mathematics achievement and students’ needs for Tier II and Tier III interventions.

Pre-assessment data provided baseline data of students’ mathematics achievement. Post-assessment data occurred at the end of treatment. The end data occurred late April 2015 which provided specific information four months after treatment. Pre- to post- and end data allowed for comparisons within groups and between groups relative to students’ mathematics achievement and instructional Tier Levels.

After pre-, post-, and end-of-school year data were gathered, paired t-tests and two-sample t-tests were run to determine statistical correlations and variances within each group and between the treatment and control classrooms. The same tests were conducted to determine achievement growth amongst students (low-scorers) recommended for Tier II and Tier III interventions. Pearson Chi-Square Tests were conducted to compare patterns of students’ movement between Tier Levels, pre- versus end. This analysis helped determine if the “re-envisioned” instruction model was effective for reducing the number of students identified at Tier II and Tier III levels of instruction.

A modified version of Garner’s instrument (2007) was used to assess students’ development of their cognitive structures for spatial orientation and conservation of constancy (see Appendix B). This instrument was adapted from Garner’s (2007) Large Group Assessment of Basic Cognitive Structures and Square Search Assessment (see Appendix C). Three main sections comprised this assessment. The first section’s three test items assessed conservation of constancy. The second section’s three items again tested for conservation of constancy. The third section assessed students’ spatial orientation.

The same assessment instrument was used pre- and post- with all three student groups. The pre-assessment was administered in September 2014. It established baseline data depicting the initial stages of development relative to spatial orientation and conservation of constancy. The post-assessment data was administered in January 2015. There was a three-month interval between pre- and post- administrations. This action diminished the threat of testing and instrumentation (Creswell, 2009).

Students’ responses were hand-scored and quantified by the researcher to detect and analyze statistical changes in students’ growth or development of their cognitive structures pre- to post. Wilcoxon Signed Rank Tests and Z-tests for two proportions were conducted for statistical analysis. Students’ pre- and post-scores within each group and between groups were analyzed and compared. The use of these statistical analysis instruments increased the study’s validity and reduced error in the final data analysis and inferences pertaining to students’ development of these cognitive structures (Johnson & Christensen, 2012).

All three teachers administered two District Benchmark Assessments (BA) during the study. The district used student results to determine the effectiveness of teacher’s mathematics instruction, as well as measure and monitor students’ progress for understanding second-grade core content standards. Student results from the BAs were cross-referenced with student results from the *i-Ready* screening results. Cross-referencing students’ BA results with the i-Ready screening results indicated inconsistent student measures of mathematical understanding and achievement. Data from the BAs were not triangulated with the *i*-Ready data.

Benchmark Assessment data was used to support the monitoring of students’ mathematics achievement for numbers and operations relative to district and teacher expectations. Data also provided an important forum for conversing with teachers about individual student achievement. These conversations provided a more comprehensive understanding of teachers’ perceptions of the BAs’ prominence in influencing their day-to-day decisions regarding the content focus of their mathematics lessons. Refer to Appendix G to view students’ Benchmark Assessment results and their *i*-Ready results.

Qualitative instruments: Methods of data collection and analyses. All data were collected in efforts to create a full description of the processes and effects the “re-envisioned” instruction model had upon second-grade students’ beliefs and practices for learning mathematics. Thus, the instruments, methods, and sources for gathering qualitative data were invaluable to this study’s findings. These sources included classroom observations followed by conversations containing thick descriptions of each teachers’ reflections. Teachers’ perceptions and reflections of their own learning, of student learning, and the challenges incurred by their participation in this study were noted on questionnaires and in semi-structured interviews that invited open and closed responses. Rich dialogues and interactions between students, between teacher and students, between teacher and researcher, and student and researcher were recorded in the researcher’s private field notebook. Lesson observations were also recorded in the notebook. Pages from students’ mathematician’s notebooks were photocopied and analyzed. Photographs of instructional artifacts were captured on a locked private cell phone. All qualitative documents representing teachers’ and students’ thinking and work relative to this study were collected concurrently throughout the study. See Appendix D for teachers and students interview questions, questionnaire, and surveys.

Before data analysis and coding were conducted, transcriptions of the data were completed on the researcher’s locked and private computer. Teacher and student data were then transcribed using participants’ words, spelling, and punctuation. All qualitative data were manually coded by the researcher. Some data were analyzed for emerging themes. Initial readings and explorations of the transcriptions provided a general sense of the data and initial thoughts were recorded. Then, second and third readings necessitated the writing of memos, highlighting key words, and noting word repetitions. Patterns, similarities, and interrelated connections emerged and were used to reduce data into themes that were similar and diverse. Emerged themes and thick descriptions generated the narratives recorded in Chapters 4 and 5.

Other data were analyzed for specific content. For example, teachers’ mathematics lessons were coded for content using an analytic tool called the M-Scan Rubric (Berry III, Rimm-Kaufman, Ottmar, Walkowiak, & Merrit, 2012; Walkowiak, Berry III, Meyer, Rimm-Kaufman, & Ottmar, 2014). The M-Scan Rubric was designated as a valid and reliable instrument for lesson analysis (Walkowiak et al., 2014). The methods and processes for analyzing teacher data are described next.

Observations of teachers’ mathematics lessons were conducted multiple times throughout the study and used for analysis. This allowed for the detection of the confounding variables relative to the quality of teachers’ mathematics instruction and implementation of the “re-envisioned” instruction model. The researcher had been trained in lesson observation while an employee at Midwest County Intermediate School District. This helped identify and reduce the influence of confounding variables relative to teacher’s instructional effectiveness.

Lesson observations included many scripted details in the researcher’s field notebook. Once transcriptions of all lessons were complete, three lessons from each teacher were selected for analysis. These lessons correlated to the same mathematical concepts taught across all three classrooms. The reduction of lessons to those that addressed the same concepts was another attempt to control variability in interpretations of teachers’ quality of instruction.

Lessons’ transcriptions were analyzed using nine dimensions identified by the M-Scan Rubric (Berry III et al., 2012). Each dimension represented a single practice of high-quality mathematics instruction (see Table 3.2). Authors of the M-Scan provided statistical evidence that their rubric was a valid instrument for observing and rating mathematics instruction (Walkowiak et al., 2014). Teachers’ lessons were coded using all nine dimensions. Thus, across three lessons, a teacher received three scores for each dimension. Once analysis was completed, the three scores for each dimension were averaged to determine a mean score. All nine mean scores were used to describe and compare the quality of each teacher’s mathematics instruction (see Table 3.2).

Two protocols were used to determine and control for the variable of teacher’s practice relative to questioning. Once again, transcriptions of lesson observations provided important data regarding the types of questions teachers posed during mathematics instruction. Teacher’s question types constituted a vital part of mathematics instruction because they strongly influence the development of students’ mathematical understanding (Boaler & Brodie, 2004).

The first protocol was sourced from Boaler and Brodie (2004). Boaler and Brodie’s research produced observations and thick descriptions of mathematics lessons leading to nine classifications of question-types. Each question type constituted a different purpose for inviting student participation. The nine classifications consisted of (a) gathering information; (b) using mathematical vocabulary; (c) exploring relationships and meanings; (d) inviting students to explain thinking; (e) generating discourse; (f) linking and applying concepts; (g) extending student thinking; (h) orienting and focusing students’ attention; and (i) establishing a context for learning mathematics.

The second protocol for question-types was used to substantiate data findings from Boaler and Brodie’s (2004) protocol. This second protocol closely aligned to Bloom’s taxonomy (Way, 2001/2011). Question-types included the recalling of information, the transformation, interpretation and analysis of information. Descriptions also indicated questions that induced the cognitive processes of application, synthesis and evaluation. Identification and coding of teacher’s questions presented during instruction, as well as tracking frequencies of question-types, transformed qualitative data into quantitative data. Quantitative data triangulated with qualitative data provided plausible causational factors for student results.

Lesson observations often revealed mathematical content teachers understood and did not understand. Informal conversations conducted after each lesson clarified the concepts and mathematical models the experimental teachers struggled in implementing (e.g. open-number line, arrow roads, etc.). Questions posed during informal conversations, on questionnaires, and in semi-structured interviews targeted teachers' strengths and weaknesses for teaching mathematics. Teachers were also asked to provide demographic data, previous experience, and reflections on their learning due to their participation in this study.

Teachers' responses were triangulated with the numerically coded data derived from the M-Scan rubric (Berry III et al., 2012; Walkowiak et al., 2014) to substantiate inferences made. Teachers' observations and insights relative to student learning was triangulated with students' qualitative responses. The triangulation of data substantiated and augmented students' achievement results.

Additional Qualitative Measures and Analyses

Throughout the study, students' mathematical representations became windows into their mental processes, as well as personal mirrors for documenting their own mathematical growth (Woleck, 2001). In all three classrooms, students' words, pictures, and drawings captured and represented their perceptions of mathematical representations depicting relationships amongst numbers, numerical patterns, mathematical properties, and other mathematical structures. Student notebook pages documented evidence of specific connections they made between their existing schema and the mathematical representations presented to them. Pages from these notebooks were collected, photocopied, and reviewed throughout and after the study. Thus, students’ mathematician notebooks were integral resources for substantiating this study’s findings.

Teachers’ mathematics lessons provided the researcher, as a participant, opportunities to conduct observations and unstructured open-ended interviews with students. As students worked through specific tasks, questions were posed, observations made, and scripted field notes of students’ communication were recorded in the researcher’s field notebook.

Scripted conversations provided windows into the strategies students had for making sense and for constructing meaning throughout all three segments of the lesson: launch, exploration, and summary/reflection. Student data also provided key information for determining ways the mathematical tasks impacted student’s beliefs and practices for learning mathematics. This data was triangulated with teacher observation data and with student results from questionnaires and *i*-Ready Assessment Screening Tool (Curriculum Associates, 2015).

Debriefings with each teacher regarding student’s mathematical understanding of concepts were conducted after each observed lesson. Teachers’ perceptions supported the narrative interpretations and descriptions of their students’ learning. The transcriptions of each teacher’s communications provided opportunities to look for consistencies and themes per teacher and between teachers.

Classroom environments. Additional qualitative data collection pertained to the classroom environments teachers and students co-created. Classroom environments became important spaces for communicating the purposes for doing mathematics (Turner & Patrick, 2004). To determine the socio-cultural environment within each classroom, data were collected by a variety of means: a teacher survey, student questionnaires, and teacher observations.

The teachers’ survey consisted of 14 observable student attitudes and behaviors. Such attitudes and behaviors exemplified important elements and variables represented within the six learning theories, as well as, the eight Standards of Mathematical Practices (NGA & CCSSO, 2010). These included, but were not limited to, asking questions, using multiple representations, explaining and justifying mathematical thinking, making connections between and among concepts and representations, and self-selecting tools. Essentially, the indicators identified environmental conditions and behaviors necessary for implementing the “re-envisioned” instruction model (see Appendix E). Teachers were asked to complete the survey using a four-point Likert scale:

“1” indicated that a specific student behavior was observed all the time;

“2” indicated that the behavior was observed some of the time;

“3” indicated the behavior was observed occasionally; and,

“4” indicated that the student behavior was not observed at all.

Teachers recorded their observations and perceptions of their students’ behaviors three times throughout the study: September 2014, Mid-October 2014, and January 2015. Teacher perception data was then numerically recorded, totaled, and averaged. Numerical totals for each student behavior were then compared across all three student groups.

To understand students’ beliefs toward learning mathematics, multiple questions were posed to them in January 2015, at the end of treatment. All three classrooms of students were asked to record individual responses to the following prompts, “What is a mathematician?” and “What is the work they do?” Students recorded their descriptions in their mathematician’s notebooks. These questions allowed for accurate inferences regarding their beliefs and conceptions about who mathematicians were and the work they did. Their responses were collected, photocopied, and analyzed using an open-coding system. Within the first analytic pass, student responses revealed a list of common, yet specific words found in and across all three student groups. Key-words-in context and word frequencies generated the themes. Word frequencies were then calculated into percentages and used to compare the number of students expressing similar views across classrooms.

A questionnaire was also used to determine students' aspirations for learning mathematics. Students recorded all responses onto the questionnaire. For the first question, students had three choices to select from: "I don't like math"; "Math is O.K."; and "I love math." After selecting their choice, students were required to justify their response. The second question asked students to reflect upon a concept or behavior they had learned that they felt good about. The third question asked students, "What do you wish you understood better in math?" Student responses conveyed their attitudes and interests for learning this academic subject. Again, common themes arose within each group. Each group's responses were tabulated, calculated into percentages for comparisons and then transformed into narratives found in Chapters 4 and 5.

All three student groups were visited one final day in May 2015 four months after treatment ended. Students were asked to peruse their mathematician's notebooks and describe, in words, pictures, and numbers, ways they had grown as mathematicians. This survey was conducted to determine students' self-efficacy as mathematicians and identify their beliefs and practices for learning mathematics. Students scanned their notebooks noting their evidence onto paper and provided explanations.

Student responses were collected and electronically transcribed using students' exact words, spelling, and punctuation. Transcriptions were then manually coded for emerging and common themes. Once again, the frequencies of word repetitions within students' explanations were tabulated and calculated into percentage scores. Percentage scores and students’ exact descriptions provided explicit examples for ways teacher’s enactment of the instruction model, when compared to the control, impacted students’ beliefs and practices for learning mathematics.

Teachers’ Mathematics Instruction as Enacted in the Study

A total of 19 random and planned semi-structured classroom observations were conducted between October 2014 and January 2015 to obtain a clear understanding of the three teachers’ pedagogical skillsets and implementation of lessons and tasks. Lessons were scripted in the researcher’s private field notebook during each observation. Scripts included teacher’s enactment of the lesson, teacher questions, resultant student-teacher discourse, and mathematical representations used by teachers and by students. Descriptions of teachers’ pedagogical practices and representations used to launch lessons were captured in researcher’s field notes or through photographs taken on a private and locked cell phone. Photos of resulting student work were also obtained. Teachers’ levels of mathematical content knowledge were revealed through teacher’s classroom instruction and informal conversations and debriefings.

Once electronic transcriptions of all nineteen lessons were accomplished, common mathematical conceptual themes were identified. For instance, in each classroom, lessons pertaining to multiplicative arrays were observed. The three lessons (one from each classroom) were analyzed and coded. Other mathematical concepts held in common were selected for analysis. These included pattern-recognition on the hundreds chart, composing and decomposing number, and the use of diagrams. This cross-referencing resulted in three lessons from each teacher being coded as part of the study’s data. Thus, a total of nine lessons were numerically coded using the M-Scan Rubric (Berry III et al., 2012; Walkowiak et al., 2014).

The M-Scan Rubric measured teacher's use of nine pedagogical practices called Dimensions. Each Dimension represented a single practice of high-quality mathematics instruction. For example, the first Dimension or instructional practice focused on the structure of the lesson, specifically targeting a teacher's logical sequencing of a lesson's components, the coherency of those components, and the ways those components supported students developing a deeper understanding of mathematical concepts. Eight additional Dimensions were used to assess the quality and pedagogical skillsets of all three teachers' relative to teaching mathematics.

Each teacher's lessons were analyzed using the descriptors articulated within the M-Scan Rubric (Berry III et al., 2012; Walkowiak et al., 2014); and, each lesson was coded using all nine dimensions. As a result, each teacher received three scores for each dimension which were calculated to determine a mean score for each dimension. Teachers received nine mean scores in all. Table 3.2 details the results of this analyses.

Table 3.2

Mean Scores Representing Standards-Based Practices Exhibited in CLA, CLB, and CLC's Mathematics Instruction across Three Lessons

M-Scan Dimensions Scale is 1 to 7 Low (1,2); Med (3, 4, 5); High (6, 7)	Mean Score for three lessons for CLA (Control)	Mean Score for three lessons for CLB (Experimental)	Mean Score for three lessons for CLC (Experimental)
Structure of Lesson • Logical sequence • Mathematical coherence • Promotion of deeper understanding	4.33	6.33	5.67
Multiple representations • Teacher use of multiple representations • Student use of multiple representations • Translation/explanation among representations.	5.0	6.0	4.67
Use of mathematical tools • Opportunity to use tools • Depth of use	5.0	5.67	5.33
Cognitive Demand • Task selection • Teacher enactment	5.0	6.0	6.0
Mathematical discourse community • Teacher's use of discourse • Sense of mathematics community through student talk • Questions	4.67	6.0	5.0

Explanation and justification	5.33	6.33	4.67
• Presence of explanation/justification • Depth of explanation/justification			
Problem solving	6.0	6.33	5.33
• Students’ engagement with problems • Presence of multiple strategies • Student formulation of problems			
Connections and Applications	5.0	4.67	4.33
• Connections • Applications			
Mathematical Accuracy	4.33	6.0	6.0
• Accuracy in teacher presentation • Clarity of mathematical concepts • Responsiveness to students’ mathematical thinking			

All three teachers obtained close mean scores on seven of the nine dimensions. Teachers’ mean scores ranged from a medium scale score of 4.33 to a high scale score of 6.33. The two dimensions exhibiting the greatest difference in mean scale scores between the two experimental groups and the control group were the Structure of the Lesson and Mathematical Accuracy. Structure of the Lesson referred to the logical sequencing of a lesson’s concepts, mathematical coherence, and the promotion of deeper understanding. The National Research Council (2012) defines “deeper learning” as a process whereby individuals can apply and transfer knowledge to new contexts and situations. For this dimension, CLA Teacher obtained the lowest mean score 4.33. CLC Teacher’s mean score was 5.67 and CLB Teacher scored 6.33.

The dimension for Mathematical Accuracy consisted of three descriptors: accuracy in teacher presentation, clarity of concepts, and teacher’s responsiveness to students’ thinking. Specifically, mathematical accuracy pertained to ways mathematical concepts were presented throughout the lesson and the ways teachers addressed students’ understandings and misconceptions. It also involved whether the mathematical tasks enabled students to transfer mathematical concepts to future lessons, meaning students’ understanding embodied the creation of generalizations. CLA Teacher’s mean scale score was 4.33. CLB and CLC Teachers both obtained a mean scale score of 6.0. To illustrate teachers’ differences in their enactment of this dimension, all three teachers’ lessons pertaining to composition and decomposition of number are described over the next few pages.

A CLA lesson: Composing and decomposing number. As identified in Midwest School District’s mathematics program, a lesson that addressed basic fact calculations (composing and decomposing number) specifically asked students to consider what happens when 0 or 1 are added to a given number. To launch her lesson, CLA Teacher asked students to create a variety of representations for 10 + 4 and record them in their mathematician’s notebooks. When looking at their notebooks, students recorded a plethora of different equivalent representations for that expression. Students used number lines, ten frames, tally marks, counters, coins, and dominos.

Without asking students to share the representations they depicted, CLA Teacher asked students how they would use 9 + 1 to solve the sum for 9 + 2. Accepting one student’s accurate response of, “You can use the 9 + 1 = 10 and if you added one more then that equals eleven,” CLA Teacher attempted to extend students’ thinking by asking a second question, “If 6 + 1 = 7, how can this help you figure out 6 + 2?” To respond to the teacher’s question, a different student described hopping on a number grid, “You can make a hop. So, you can go to 6, then you can go to 7, then you hop 1, then you know to go one more.” CLA Teacher clarified student’s statement, “If you know 6 + 1 is 1 hop then 6 + 2 is 2 hops.” The teacher’s rephrasing focused on the number of hops or physical movements on the number grid.

CLA Teacher asked students to create a mental image of a number line in their minds to mentally calculate 20 + 1, 20 + 2, 20 + 3. While students were still processing these mental images, the teacher switched to a real-life context of a grocery store, which led a student to use place-value to find the sum of two addends rather than using a mental number line.

During the exploration segment of the lesson, the teacher introduced a mathematical game to support students practicing their basic-fact calculations using cards, a calculator, and students’ mental math capabilities. This game came from the district’s mathematics program and was part of the lesson. Students played the game in trios. While students played the game, the teacher moved about the classroom, observing and listening to students’ conversations. As she observed three students struggling with their basic-fact knowledge, she asked the trio, “What would a strategy be if the caller flipped a 3 and a 7?” The students appeared puzzled by the teacher’s question. Noticing the puzzled looks and pauses of the students, CLA Teacher immediately followed up with another question, “What’s the difference between 3 + 7 and 7 + 3?” At this point in the lesson, the teacher had not explicitly discussed the “counting on” strategy nor the commutative property of addition. To offer scaffolding support, the teacher then brought out a balance scale and asked the three students, “What can I do to figure this out? How can I use this balance, these dominos and these equations to figure it out?” The teacher and the three students unsuccessfully pursued these concepts.

After the lesson ended, CLA Teacher expressed her perplexity regarding students’ responses. Thus, she asked the researcher, “What lesson do you think I should follow up with?” After searching through the unit of study in the district’s mathematics program, the suggestion was to focus on patterns and rules for adding 0 and 1. She was advised to direct her students to complete a basic fact grid to notice the patterns when adding 0 and 1. CLA glanced at her teacher book and replied, "Students can write number stories for adding + 0 and + 1."

A CLB lesson: Composing and decomposing number. A corresponding lesson in CLB's classroom focused on composing and decomposing a numeric quantity in different ways. CLB launched her lesson by drawing an iconic representation for 25 using one unit (long) of 10 and 15 ones. She then focused students' attention whereby they were asked to determine the value of the representation and create as many equivalent representations using iconic symbols in different ways. Students responded, "I changed the ten ones and traded them for 1 ten and kept the five ones." "I traded in the ten for ten ones." As the teacher depicted different students' descriptions, one student noticed, "Hey, you have too many tens to make 25 now."

CLB Teacher used students' language to focus students' attention on the concept of trading. Precise language supported students understanding that the concept of trading meant that although the mathematical representation may change, the value remains the same. For example, regardless of how one represents 25 either by trading a ten for 10 cubes or 10 cubes for a ten, the value remains 25.

Students used base-ten blocks to decompose 63 in a variety of equivalent ways to continue the concept of trading during the exploration segment of the lesson. Students arrived at generalizable mathematical ideas using concrete models and equivalent symbolic representations. For example, one student claimed that there were more than 3 ways to represent 63. Another student added onto the first child's thinking, "The more tens you have in the tens place, there are more ways to show 63." A third student creatively represented 63 using benchmarks of 10 and turned three tens into three elevens, as in 10 + 10 + 10 + 11 + 11 + 11. As this student shared his representation with classmates, another student responded, “He built 6 tens and changed three tens to three elevens equaling 63. Because he needed three more.” Students made important connections and arrived at key informal generalizations.

A CLC lesson: Composing and decomposing number. CLC’s lesson for composing and decomposing 25 was launched in a similar fashion as CLB’s lesson. However, CLC focused students’ attention using an enactive representation of base-ten blocks. The teacher placed one stick of 10 and 15 ones under the document camera. Before asking students to calculate the total number of blocks, she questioned and clarified student understanding of each block’s value. Once each block’s value was accurately identified, CLC asked her students to identify the total number of blocks in the set. Students identified three possible sums: 20, 25, and 26.

Rather than correcting students’ errors or providing students with the correct answer, CLC asked a student to approach the document camera and make a trade using the blocks. She explained this instructional move supported students more readily identifying the correct total once all trades were made. After making the trades, one student commented about the different representations, “Just because they look different doesn’t mean that the number is different!”

During the exploration segment, students composed and decomposed numbers using Place-Value Mats and base-ten blocks. Many students struggled in making trades and recording the correct values represented by the blocks on their mats. For instance, one student had 212 represented on his place-value mat. When asked to read out loud his total value, he said, “Two hundred and two.”

To begin summary/reflection segment, CLC Teacher launched with the question, “Tell me some things you understand about a ten’s block and a one’s block. Anything.” Only a few students responded: “They help you build numbers”; “Ten and one more make eleven”; “I would take eleven ones”; “If you have ten cubes, you could trade for a long.” As noted, students’ interpretations of the teacher’s question were literal as in one 10-block and one 1-block equaled eleven. Receiving literal feedback from her students, CLC Teacher interpreted students’ bewildered facial expressions as their lack of understanding of the lesson’s goals. After her lesson, CLC Teacher shared that her students still needed focused and explicit experiences for composing and decomposing number, reading numbers correctly, and making trades. CLC expressed these concerns during the debriefing of her lesson.

Change-To-More lessons as enacted by each teacher. Another set of teachers’ lessons consisted of the same mathematical concepts, but were enacted very differently. In each of the three classrooms, teachers introduced a specific type of diagram, termed a Change-To-More diagram. This diagram was represented in their district’s mathematics program and on their district Benchmark Assessment II. It illustrated the relationships among quantities in addition and subtraction problems, as well as the meaning of the operations.

Differences were noted between the control group and the two experimental groups when examining students’ representations and use of the Change-To-More diagrams within their mathematician’s notebooks. Photographs and Xeroxed copies of students’ mathematical representations and thinking were gathered for analysis. Thick descriptions emerged from students’ written work and from the analysis of students’ mathematical representations and recordings. Students’ use of mathematical language, models and representations, and the mathematical connections students made were evident within these invaluable documents. Such differences were reflections of the way each teacher launched her lesson of the diagram.

CLA teacher’s lesson. CLA Teacher (control) was asked to follow the lesson as written in her district’s mathematics program. This teacher, however, was given permission to alter a lesson per students’ needs. The following description and resulting student responses depicted the way CLA Teacher enacted her launch of the Change-To-More diagram with students and the ways students thought about the teacher’s representation.

To launch the Change-to-More diagram, CLA Teacher posed the following number story and Change-to-More diagram for students to consider: “I have three cats. I got one more. How many all-together?” Students were then asked to think about how they might represent the number story using the Change-to-More diagram within their mathematician’s notebooks. CLA teacher’s model is represented in Figure 5 below.

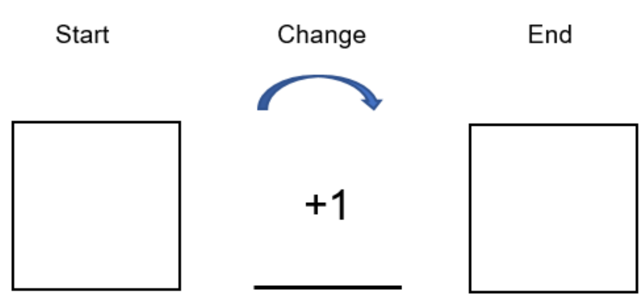

Figure 5. CLA Teacher’s mathematical representation to launch her Change-To-More lesson.

Table 3.3 contains CLA students’ responses to the teacher’s launch. Their writings and depictions within their mathematician’s notebooks capture their initial thinking relative to the story context and posted diagram. Students’ misspellings were corrected to make interpretation easier for the reader.

Table 3.3

CLA Students' Written Work and Mathematical Representations for Change-to-More Diagram

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A7: “The person now has four cats because she got one more. It is plus one so it is four!” [Student drew the diagram with the arrow beginning at 3 and ending at 4.]

A12: “I noticed at the top it has a number story. And at the bottom of the number story it has two boxes and in the middle, it says + 1.” [Student drew the diagram accurately and labeled “start” in her diagram.]

A8: “4 because 3 + 1 = 4” [Student drew the diagram and labeled the parts accurately.]

A5: “I noticed that she started out with 3 and she got 1 more and all together it equals 4 and how I figured this out was I know that 3 + 0 = 3 because 0 is not a number you can use it is nothing and so if 3 + 0 = 3 I add one more and it will give me 4 and the change of that is it starts out with 3 and the change is you add one more and in the ending you will have 4. Equation 3 + 1 = 4.” [Student drew diagram accurately, although student wrote only the word change.]

A6: “I have 3 cats. I got 1 more. How many all-together.” [Student drew the diagram accurately and included 3 + 1 = 4.]

A13: “She starts with 3 cats and she gets 1 more. Now she has 4 now 3 + 1 = 4”

A11: “I know that if you have 3 cats + 1 more cats = 4 cats because when you have 3 + 1 is.” [Student ended without completing the sentence.]

A3: “I notice that the number story sort of number story had a picture like this. I have 3 cats. I got 1 more. How many altogether? [Student drew the start/change/end model and labeled the parts of the model accurately: the beginning equation is 3, the change is the rule, and the equation ending is 4. Student recorded “full equation, but different.”]

A9: “I noticed that it's an equation. The equation is 3 + 1 = 4. And it's a diagram.”

A1: “I notice she has 3 cats and one gets one more. She has 4 cats in all.” [Student drew the start, change, end diagram accurately.]

A2: “I notice that he has 4 cats all together because he had 3 and then he got 1 more and that equals 4.” [Student drew the start, change, end diagram accurately, but wrote “cha” for change.]

CLA Teacher launched the Change-to-More diagram using a simple story problem about owning three cats and acquiring one more using the Change-to-More diagram as depicted in Figure 3. Students used words, numbers, equations, and Change-to-More diagrams to represent their understanding of their teacher’s story and model. All CLA students (100%) made mathematical sense of the story context. Approximately 75% of the students connected the numeric values within the story and applied these to personally-drawn Change-to-More diagrams. Most students focused on the story context. Few CLA students (27%) explained or correctly interpreted *the meaning* of the diagram. Students who correctly interpreted the meaning of the diagram claimed it represented an equation or that the change was the rule. When examining A7’s depiction of the diagram, it appeared A7 may have misapplied the meaning of the word “change.” A7’s arrows bridged the start box to the end box. Rather than seeing “the change” as the quantifiable difference between the two numbers, his drawing suggested that the start number “changed” to the end number.

As for making connections between the story context, the diagram, and the equation, almost half of CLA students (45%) made explicit connections. However, 9% of the students made connections to previously-learned mathematical diagrams or models that were different. Furthermore, 9% of students found, extended, or created patterns derived from the original representation.

CLB teacher’s lesson. To introduce the Change-to-More lesson and diagram to her students, CLB Teacher posed the following mathematical representation as represented in Figure 6.

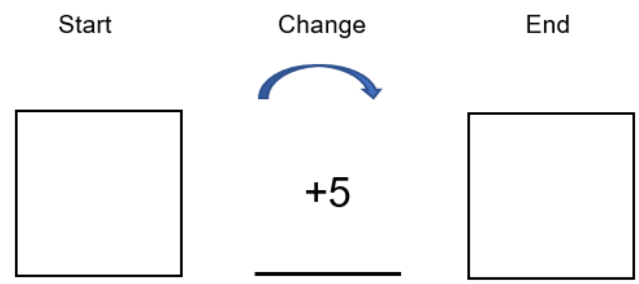


Figure 6. CLB Teacher’s mathematical representation to launch her Change-To-More lesson.

After posting the diagram, CLB Teacher asked her students, “What do you notice? What sense can you make of this?” After students shared their initial thoughts about the model, the teacher recorded 35 in the start box and asked students to determine the end number by noticing the structure of the diagram.

The information in Table 3.4 represent CLB students’ initial thinking found within their mathematicians’ notebooks. Students’ misspellings were corrected to make interpretation easier for the reader.

Table 3.4

CLB Students’ Written Work and Mathematical Representations for Change-to-More Diagram

B5: “You can put numbers in the start and end and the line and add Looks like an equation like a number rule. 35 + 5 = 40” [Student accurately drew the diagram using 35 + 5 = 40.]

B8: “I noticed that it says start, change and end. I think that you have an equation and you take the first the change is your – or +. The end is your sum.” [Student accurately drew the diagram using 35 + 5 = 40.]

B1: “What do you notice? It looks like an equation. Looks like a number rule.” [Student accurately drew the diagram using 35 + 5 =?]

B2: “I noticed…that it has a start change and end and it was on our test [pre-assessment] and I think if you were stuck on an equation in then you would draw it. And it would help you figure it out. It means that…it’s kind of like a number rule the change is a rule the end is the answer and the start is the number you start with. The starting number is 35 + 5 = 40” [Student accurately drew a different diagram using 22 + 5 = 27.]

B10: “What do you notice I think it’s 35 + 5 = 40. Jump + 5 or + 10” [Student accurately drew the diagram using 35 + 5 = 40.]

B7: “I noticed the start means the beginning number Change means plus more. The end means the finishing number for the equal number.” [Student accurately recorded the start−change−end model with an equation of her own: 13 + 6 = 19.] After the teacher presented 35 as the start number, the student said, “I notice that the end is 40 because the start is 35 the change is +5 and is 40 because the start change is an equation.” [Student then drew another start, change, end model using 22 + 5 = 27.]

B12: “I think you use it like this… [Student drew the start, change, end model incorporating his own values 5 – 1 = 4] and it’s the same as a minus equation. I seen it on my pre-test yesterday and that goes like this…and it’s a start, change, end.” [Student recorded 35 + 5 =? and rewrote the equation vertically 35 + 05 = 40.]

B3: [Student accurately drew the diagram using 35 + 5 = 40.] “I had seen this in my math or BA test. It looks like a in and out box and you write. It is an in and out box!!! And that was what I was thinking of. 35 + 5 = 40”

B8: “It has two boxes. It has one line. It has one arrow. It says start change end. Like an in and out machine or function machine. The equation is 35 + 5 = 40.” [Student accurately drew the diagram using 35 + 5 =? And drew a second one for 22 + 5 = 27.]

B13: [Student accurately drew the diagram using 35 + 5 = 40.] “I have seen this in our test.”

CLB Teacher launched the Change-to-More diagram by asking two questions, “What do you notice? What sense can you make of this?” Students used words, numbers, equations, and their own hand-drawn Change-to-More diagrams to represent their understanding of the diagram. Next, they discussed their ideas in whole group. Like students in the control group, all students (100%) made sense of the diagram relative to the problem at hand. Most students (86%) correctly interpreted the meaning of the diagram claiming it was “an equation,” “like a number rule,” or made explicit connections to other diagrams. For example, one CLB student described his initial thoughts,

> I noticed…that it has a start change and end and it was on our test [pre-assessment] and I think if you were stuck on an equation in then you would draw it. And it would help you figure it out. It means that…it’s kind of like a number rule the change is a rule the end is the answer and the start is the number you start with. The starting number is 35 + 5 = 40.

CLC teacher’s lesson. All three teachers were given permission to adapt or alter lessons per their students’ needs. To introduce the Change-to-More diagram to her students, CLC Teacher posted the following diagram as represented in Figure 7.

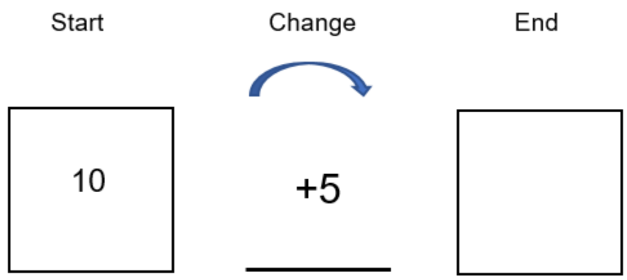


Figure 7. CLC Teacher’s mathematical representation to launch her Change-To-More lesson.

CLC Teacher asked her students, “What do you notice? What sense can you make of this?” Table 3.5 is presented below. It represents CLC students’ initial thinking found within their mathematician’s notebooks. As with other student entries, students’ misspellings were corrected to make interpretation easier for the reader.

Table 3.5

CLC Students’ Written Work and Mathematical Representations for Change-to-More Diagram

C6: “I notice that there are three words start change and end and has a 10 in a box. There is a middle line and says 5 + and the other box we need to answer. I think the answer is 15.” [Student accurately drew the diagram using 10 + 5 = 15.]
C3: “I think this is like part of equation adding and subtracting [[Student accurately drew the diagram without numbers] because it could be” [Student accurately drew the diagram using start 10 5+ end 15.]
C1: “10 + 5 = 15”
C5: “I think start with 10 part change + 5 end total 15.” [Student connected the first model with two more models of her own: 12 + 5 = 17 and start (also a part) 10 + 10 change (also a part) = total 20. She made explicit connections to the part/part/total model. Student accurately drew each model identifying start, change, end.]

C7: [Student extrapolated to three start, change, end diagrams. He labeled the end as "out."] including using subtraction, for start, change, "out" diagrams: "30 + 10 = 30; 100 + 10 = 110; 5000 – 5000 = 0"

CLC Teacher launched the Change-to-More diagram using a starting value of 10 and the change value of 5. Students used words, numbers, equations, and Change-to-More diagrams to represent their understanding of the diagram. Here, CLC students (80%) made sense of the model relative to the given problem. Two students (40%) extended their thinking using additional rules and equations.

Other instructional tasks implemented by CLB and CLC teachers. Vital resources for supporting student thinking in CLB and CLC classrooms were the "re-designed" tasks CLB and CLC Teachers and students engaged with throughout the study. The two experimental teachers appreciated the mathematical tasks provided to them because they afforded teachers and students opportunities to encounter novel mathematical representations focused on developing students' understanding of number and number relationships, as well as cognitive structures. The following tasks and representations were used in CLB and CLC classrooms but not used in CLA's classroom.

One novel mathematical representation used with the two experimental groups was called Arrow Roads. Arrow Roads depicted movement on a hundred chart. One began at an initial value and ended up at a final sum or difference by following the direction of arrowheads, one arrowhead at a time. The direction of an arrowhead directed students to move one space, either up (-10), down (+10), to the right (+1) or to the left (-1). After completing the sequence of directional moves, students transformed an arrow road into symbolic equations, thereby performing addition and subtraction of two-digit numbers. CLB Teacher explained, "This is the first year we have focused on the concept of number relationships. We had always assumed that first grade teachers had done that, but thinking about ten more and ten less, one more, one less, just the ideas of, we, as second-grade teachers, have never focused on.”

To deepen student understanding of number relationships, CLB and CLC Teachers and students used specific enactive representations. Multi-link cubes were one type of enactive representation used early in the school year. These cubes introduced the concepts of inequalities and equivalencies. To draw students’ attention to these concepts, pairs of students were given three stacks of cubes: a stack of 2 cubes, a stack of 4 cubes, and a stack of 6 cubes. They were then asked to analyze the representations and record things (figural units) they noticed about the stacks in their mathematician’s notebooks.

Many students first recorded that the stacks were different colors. However, students soon noticed that the stacks were different heights and claimed they looked like stairs. They noticed that each stack had two more or two less cubes than another stack and that one could begin with the shortest stack, count by twos to get to the next taller stack. A few students noticed that when they combined the stack of two cubes and the stack of four cubes, it equaled the number of cubes in the tallest stack (six). Other students noticed there were 12 cubes altogether.

Students’ notebook entries depicted the figural units embedded in the stacks of cubes. Student entries were in the form of words, drawings, numbers, and equations. Teachers next posed the question, “If the pattern of cubes was to continue, what would the next three stacks look like and why?” Students got to work developing their hypotheses using enactive models and their mathematician’s notebooks to explain and justify their thinking.

Another enactive model CLB and CLC Teachers used for the first time was a meter stick. A meter stick was an enactive representation used to introduce the concept of a number line up to 100. Pairs of students were each given a meter stick to notice the figural units embedded within the representation. Students noticed that the meter stick looked like a ruler that went up to 100. They noticed that there were 100 lines. One student noticed that he could go up to 1000. This student referred to the shortest increments representing millimeters. Other students predicted that they could count by fives and tens using the meter stick. They also predicted that the multiples of ten were bold because it takes ten tens to make 100. From prior knowledge, one student accurately described that the first centimeter meant 1 cm. Finally, students predicted that the meter stick started at 0 and ended at 100.

After students’ insights were recorded on to chart paper, students were asked to place translucent counters on multiples of ten and then describe what they noticed. Students noticed that the counters were placed equal distance from each other. When asked why this was so, many students needed to count the spaces in between the multiples of ten to verify that there were ten spaces between 10 and 20, and again between 20 and 30, and so forth. Visually seeing the equal distances between these consecutive multiples of ten was new phenomenon for students to notice. Students also discussed that as the numbers increased [within a decade], the digit in the ten’s place stayed the same, while the digit in the one’s place increased by one. Seeing the equal distances of space between the decades and noticing digits that stayed the same and digits that changed were easier to see on the meter stick than on a number grid, so students claimed.

As the lesson ended, one student, a very shy and low-performer (as identified by the *i-*Ready Screener, Curriculum Associates, 2015), approached the teacher and asked if she could do more work using the meter stick the following day. The teacher asked, “Did the meter stick help you think about numbers?” Without speaking, the student nodded her head in a positive manner. During the following two weeks of teachers’ instruction, CLB and CLC students made connections between the meter stick, a number line, and a thermometer.

Teachers’ question-types. Another potential variable between teachers pertained to the types of questions teachers asked during mathematics instruction. Two different classification systems for coding were used to identify differences in teacher question-types. The first classification system was Boaler and Brodie’s (2004) nine question-types. The second classification system was derived from the University of Cambridge (Way, 2001/2011). It consisted of seven question-types aligned to Bloom’s taxonomy.

Differences between CLB and CLC Teachers’ questioning and CLA’s questioning was found under three categories: (a) exploring mathematical meanings and/or relationships, (b) synthesis, and (c) extending thinking. The classification of exploring mathematical meanings and/or relationships was identified under Boaler and Brodie’s (2004) system of nine question-types. These researchers defined this classification as linking mathematical ideas to respective representations. During CLA’s lessons, 13% of her questions fell under this category. Whereas, 21% of CLB’s questions and 24% of CLC’s questions fell under this category.

Questions pertaining to the classification of synthesis fell under the University of Cambridge’s system (Way, 2001/2011). Synthesis questions asked students to solve a problem necessitating students’ original and creative thinking. For example, “Who has a different solution?” Ten percent of CLA Teacher’s questions pertained to this category. Whereas, 20% of CLB’s questions and 18% of CLC’s question-types fell under this category. In both categories, the two experimental teachers scored at least eight percentage points higher than the control teacher.

Extending thinking was a classification found under Boaler and Brodie’s (2004) system for question-types. Extending thinking questions supported students making connections between mathematical concepts and real-world contexts. CLA Teacher asked twice as many questions under this classification than CLB and CLC Teachers did. About 22% of CLA’s question-types were identified as extending thinking, whereas, only 8% of CLB and 11% of CLC’s question-types pertained to this category. CLA Teacher believed her questioning techniques had improved over the course of the year because they supported student inquiry. However, she also claimed that her district’s curriculum didn’t support her asking the types of questions that engaged students in thinking more deeply about mathematics.

Additional Qualitative Analyses

Time allotted for mathematics instruction. Data collected by a questionnaire and semi-structured interviews revealed similarities between teachers and their instructional practices not captured by the M-Scan Rubric (Berry III et al., 2012). Each teacher reported they taught mathematics for 60 to 75 minutes per school day. All teachers reported they consistently used games to support the development of students’ mathematical understanding and skills. Likewise, students were expected to first make sense of mathematical tasks before whole group discussions took place; and, the same types of mathematics tools were used across all three classrooms on a regular basis (e.g. cards, counters, number grid, base-ten blocks, etc.). One tool used exclusively in the two experimental classrooms was the number line.

One noted difference between teachers consisted of the amount of time it took for teachers to plan their mathematics lessons. CLA reported she spent 1.5−2.0 hours per week preparing for her weekly lessons. For the two experimental teachers, they spent 3.0–5.0 hours per week preparing for their weekly lessons. This was double the amount of preparation time they indicated for preparing literacy instruction.

Teacher use of students’ mathematician’s notebooks. To detect additional variances in teacher’s instructional practices and processes not explicitly screened for in the M-Scan Rubric

(Berry III et al., 2012; Walkowiak et al., 2014), yet vital for implementing the “re-envisioned” instruction model with fidelity, this researcher used three to four students’ mathematician’s notebooks from each classroom as data sources.

From September 8, 2014 to January 21, 2015, each teacher’s instruction incorporated the use of students’ notebooks. Students’ entries revealed the mathematical concepts teachers addressed, as well as the number of times students recorded their mathematical thinking. Upon analysis, every notebook entry involved some form of symbolic representation using words, numbers, or signs; however, not every depiction revealed students’ engagement with enactive representations. Also detected were variances in students’ record of dates and the number of summary/reflections. Table 3.6 presents the data generated by analyzing student representations (artifacts) recorded in their entries.

Table 3.6

Mathematical Concepts and Instructional Processes in CLA, CLB, and CLC’s Classrooms as Represented in Students’ Mathematician’s Notebooks, September 9, 2014 – January 21,2015

Group	Total Note-book Entries by Students	Percentage of Entries Involving Reflection	Lessons Involving Enactive Representation	Lessons Focused on Patterns and Relationships	Number of Lessons Modeled by this Researcher
CLA	38	66%	26%	50%	3
CLB	54	Undetermined	57%	63%	13
CLC	51	39%	63%	65%	13

To determine the number of summary/reflection entries in CLB’s classroom, additional mathematician’s notebooks were reviewed. The variances continued to exist across additional students’ notebooks. Lacking consistency between notebooks, the number of reflection entries remained undetermined for CLB.

Furthermore, each teacher described the purpose and their perceptions of students’ mathematician’s notebooks. CLA Teacher reported that her students “used their notebook every single day. They also work in them.”

CLB Teacher explained,

> It’s a place to share their [students] ideas, get their thinking in writing. I launch using correct vocabulary. I also use them [notebooks] as formative assessment. I look at what they wrote and use it for instruction the next day.

CLC Teacher described, “I use them to record [student] thinking, beginning of a lesson, middle of a lesson, and end of the lesson.”

Methodological Assumptions and Reduction of Threats

Whereas all three teachers had an existing professional relationship with the researcher; and, the teachers in the two experimental groups utilized a synthesis of learning theories and evidence-based practices, while the teacher in the control group utilized an evidence-based mathematics program, several methodological assumptions were made. These assumptions included the following:

- The application of a synthesis of learning theories could counteract the historical trends of low-student achievement in mathematics

- A control group was necessary to detect and isolate the confounding variables, and compare the effects the instruction model had upon students’ mathematics achievement to students’ achievement in the control group
- The survey and questionnaire questions reflected existing research regarding best instructional practices, and thus, provided validity to inferences made
- The study’s participants provided honest responses to all questions
- Data triangulation between qualitative and quantitative data provided reliability and validity to inferences made
- The study’s findings provided petite generalizations for ways to improve student achievement and minimize students’ learning difficulties in mathematics
- This research would invite further investigations for how students’ cognitive structures influenced their learning of mathematics

Naturally, external and internal threats to validity compromised the reliability of inferences made (Creswell, 2009). External threats existed relative to replication and generalization of the “re-envisioned” instruction model to a larger population. All three teachers experienced elements of the instruction model the previous school year before the study began. Additionally, all three teachers possessed similar and defining characteristics. They had experience in teaching elementary students. They were thoughtful, reflective practitioners and used constructivist and socio-constructivist principles to promote student learning. Due to these similar traits, conducting additional studies under different conditions *before* generalizing the instruction model to a larger K-12 population is advised. Key variables found in this study need to be considered before replication and generalization is considered.

Four internal threats to validity also existed. Internal validity threats impacted the ability to draw accurate and reliable inferences regarding students’ mathematics achievement (Creswell, 2009). Such threats included student maturation, diffusion of treatment, resentful demoralization, testing, and instrumentation. However, these threats were mitigated through intentional decisions and actions.

To mitigate the threat of differences in student maturation, all three student groups were considered second-graders and, therefore, received second-grade mathematics instruction. Exceptions to this condition were students who attended third grade mathematics instruction, one student who received special education services, and three students who needed ELL support. To reduce this threat, these students were not included in this study’s data collection.

The two experimental teachers (CLB and CLC) were asked to stop planning mathematics instruction with their colleague who was the teacher of the control group (CLA). This reduced the threat of diffusion of treatment. The two experimental teachers were also directed not to share or discuss the instructional tasks, or anything related to mathematics instruction with CLA Teacher during treatment. All three teachers agreed to these conditions.

Midwest Elementary School’s principal and CLA Teacher were promised that she and her students would receive instructional support upon completion of treatment with CLB and CLC. Remember that for CLA Teacher’s daily mathematics instruction, she used the district’s mathematics program, along with other resources prepared by their district’s mathematics curriculum team. This helped reduce the threat of resentful demoralization.

Finally, to reduce threats to validity due to testing and instrumentation, the same measurement instruments were used to test all students’ mathematics achievement and development of their cognitive structures, pre- and post. Classroom instruction did not explicitly inculcate the items of the Adapted Cognitive Structure Assessment during the study. Accordingly, at least three months passed between the administration of the pre-and post-assessments. The longer time interval reduced the threat of testing.

Limitations to the Study

Several limitations existed in this study. Although all three teachers worked with this researcher the previous school year, the two experimental teachers were not prepared to implement the model. Full implementation of the instruction model integrated complex theories and design. Successful implementation necessitated teachers’ mathematical and pedagogical content knowledge. The two experimental teachers needed time and requested implementation support to learn specific content and pedagogy to enact the instruction model with fidelity.

Another limitation involved using a small convenience sample of teachers with whom this researcher already had existing relationships. CLA Teacher of the control group knew enough to taint or invalidate the study’s findings. She was challenged to stay true to her district’s mathematics program and resources.

Another limitation to the study was the lack of random assignment of student subjects to both experimental and control groups. Students were placed into classrooms by the principal and staff which may have introduced confounding variables that were uncontrolled. Additionally, this study was conducted with second-grade students only. Therefore, the inferences derived from this study cannot be generalized to a larger population of K-12 students and teachers. The conditions necessary for replication must be further explored to consider application to a wider population.

A fourth limitation to this study involved the data instruments used to collect, code and quantify data. Some of the data collection instruments were generated and used for the first time by the researcher. Although data instrumentation aligned to the research articulated in the literature review, these instruments had not been tested for validity nor reliability prior to this study.

A fifth limitation was that most of the data results were analyzed solely by one person, the researcher. Although a learning theorist, educational colleagues, committee co-chair, a statistician, and participating teachers were consulted, most data analysis and inferences incorporated one perspective. Such inferences may result in biased interpretations of the data.

Chapter Summary

Two main hypotheses were prominent throughout this study. The first hypothesis was that a single learning theory could not counteract the historical trends for low student achievement in mathematics. Thus, a synthesis of learning theories permeated the conceptual framework of the instruction model. The instruction model provided teachers and students with a design for Tier I core instruction, as well as instructional tasks for stimulating students' cognitive structures, an essential cognitive process for learning abstract concepts like mathematics. The second hypothesis proposed that teachers' and students' effective implementation of the "re-envisioned" instruction model could mediate the development of students' cognitive structures, thereby minimizing students' learning difficulties in mathematics.

Design-based research provided the theoretical backdrop for conducting this study (Barab & Squire, 2004; Brown, 1992; Cobb et al., 2003). This study included and focused on teachers and students. The study occurred in three authentic settings in second-grade classrooms. The instruction model was implemented by two of the second-grade teachers and their students. The third second-grade teacher and her students functioned as the control.

A quasi-experimental concurrent mixed-methods study, with pre- and post-test, non-equivalent, three-group time series design was used to answer the four research questions presented by this study. Selected measures, methods, and processes helped identify causational factors for how and why teacher’s implementation of the “re-envisioned” instruction model impacted students’ mathematics achievement and their beliefs and practices for learning mathematics (Cook & Campbell, 1979).

Multiple measures, methods, and processes were used to gather quantitative and qualitative data to determine the model’s impact upon student achievement when comparing the two treatment groups to the control group. Surveys, questionnaires, personal interviews, semi-structured observations, informal conversations, photocopies of student work, and instructional artifacts became the primary sources for qualitative data. The primary sources for quantitative data were the *i*-Ready Screening Measures (Curriculum Associates, 2015) and the adapted Cognitive Structure Assessments.

The “re-designed” tasks and lessons students in the two treatment groups engaged with supported them in making sense of the figural units within a vast array of physical, iconic, and symbolic representations. CLB and CLC students continuously created, transposed, and transformed their visualized perceptions of mathematical concepts using words, signs, and symbols. Students’ transformations from physical representations into iconic and symbolic forms supported their development of visual and mathematical literacy.

Limitations were encountered due to the convenience and non-equivalent groupings of the sample population, as well as to the authenticity of the study’s context. However, attempts were made to control specified threats to internal validity. Time constraints limited the ability to generalize the study’s results to other populations and academic subjects.

Chapter 4 provides statistical details that answer the research questions. These questions help to determine the effects the instruction model had upon students’ mathematics achievement, beliefs, and practices. Thick descriptions of narratives, statistical analyses, tables and figures describe this study’s results and findings in the next chapter.

Chapter 4: Results of Findings

Within recent years, mathematics education researchers have identified pedagogical practices that, when used effectively, increase student learning and achievement in mathematics (Ball, 2001). This study proposed four research questions to support these efforts. The research – questions led to the design, implementation, testing, and analysis of an innovative instruction model used with two of three second-grade teachers and their students at Midwest Elementary School during the school year 2014–2015. One second-grade teacher and her classroom served as the control group.

Teachers and students provided quantitative and qualitative data that was collected concurrently over a four and a half month period, and one day at the end of the school year, 2015. Standardized instruments, as well as surveys, interviews, instructional artifacts, and researcher’s field notes documented vital information detailing the efficacy of the “re-envisioned” instruction model. Statistical analysis and thematic coding of the data, including data triangulation, answered the four research questions.

Chapter 4 presents the study’s statistical and qualitative findings. It is organized into five main sections that describe the effects the instruction model had upon student learning. The first section consists of descriptive comparisons of the study’s participants. The second section describes the methodological approaches and procedures applied to testing the research questions. The third section presents each research question followed by data and data analyses to respond to that question. The fourth section reports additional qualitative analyses. The fifth section, the summary, concludes this chapter.

Descriptive Statistics

As an active participant, the researcher collaborated with an elementary principal, three second-grade teachers, and fifty-four second-graders for the duration of the study (DeWalt & DeWalt, 2011). Two teachers (identified as CLB and CLC) implemented the “re-envisioned” instruction model and respective tasks designed to mediate the development of students’ cognitive structures. A third teacher (identified as CLA) and her students served as the control group. This teacher implemented Midwest Elementary School’s district mathematics program with her students.

A non-equivalent group design was used to conduct the study. Midwest Elementary School’s principal identified the classroom of students who acted as the control and the two classrooms of students who received treatment. This designation was based upon the principal’s knowledge of teachers and students in each classroom. Furthermore, during the previous spring of 2014, first-grade teachers generated student placement lists for the following 2014–2015 school year. Student placements were based upon students’ needs and teachers’ strengths.

The school site and the study’s participants formed a convenience sample. The researcher worked with all K-5 teachers at Midwest Elementary the previous school year, 2013–2014. The second-grade teachers were strategically selected and given approval to participate in this study by their school principal and their district’s Executive Director of Instruction, Technology, and Assessment.

All three teachers were considered highly qualified by Midwest State and tenured by Midwest School District. All three teachers held Bachelor’s and Master’s degrees in education. Each teacher taught approximately the same number of years and taught second-graders for at least five years. Table 4.1 provides specific self-reported data per teacher.

Table 4.1

Descriptive Data of Teacher’s Educational Background and Instructional Experiences

Teacher	Bachelor Degree and Endorsement	Master Degree Focus	Number of Years Teaching	Number of Years Teaching Second-Grade	Self-Reported Strengths for Teaching Mathematics
CLA Control	Elementary Education: Early Childhood	Reading Specialist	11	5	Questioning that encourages students to think deeply.
CLB Experimental	Elementary Education: Social Studies and Science	Reading Specialist	10	9	Desire to learn math for self and for students. Growth mindset.
CLC Experimental	Elementary Education: Language Arts	Curriculum and Practice	10	9	Presenting information using a constructivist approach.

Teacher’s descriptive similarities proved to be important controls for the many variables in this study. Teachers’ prior education and teaching experiences were similar. All specialized in literacy instruction, while none held degrees or specific endorsements for teaching mathematics.

Demographic composition of second-grade students consisted of student group, gender, socio-economic status as measured by free/reduced lunch eligibility, students receiving special education services, and English Language Learners (ELL). The total population for all three classrooms were 63 students at the beginning of the study. Table 4.2 provides specific demographic data regarding students who populated each classroom.

Table 4.2.

Descriptive and Demographic Data of Students in Control and Experimental Groups

Student Group	Frequency	Male	Female	Free/Reduced Lunch	Special Education	English Language Learners
CLA Control	21	13	8	5	0	3
CLB Treatment	21	12	9	6	0	0
CLC Treatment	21	12	9	7	1	0
Totals	63	36	25	18	1	3

Approximately 44% of Midwest Elementary Schools’ entire student population were eligible for government-funded free and reduced lunch program during the school year 2014–2015. This school was under a targeted assistance plan for Federal Title I. As evident across the three second-grade classrooms, approximately 22%–33% of students were eligible for free and reduced lunch.

An exclusion criterion consisted of students who received alternative or additional instructional supports for mathematics instruction during the school day. This criterion controlled for regression and confounding variables regarding student participants. Students who received instructional services such as special education services, resource-room teacher support, ELL support, or who received mathematics instruction beyond their grade level were excluded from this study.

Special cases, as described, existed in all three classrooms. Three CLA students were eliminated from the study due to needing ELL support during mathematics instruction. Three

CLB students were eliminated due to attending third grade mathematics instruction. One CLC student was eliminated because he received mathematics instruction through special education services. Two CLC students moved prior to the end of the study. Thus, the final composition of this study's second-grade participants is depicted in Table 4.3 below. Due to participation criteria, total number of student participants in this study decreased from 61 to 54.

Table 4.3

Final Count of Student Participants at Midwest Elementary School, 2014–2015

Student Group	**Frequency**	**Male**	**Female**	**Free/Reduced Lunch**
CLA (Control)	18	11	7	4
CLB (Treatment)	18	9	9	6
CLC (Treatment)	18	10	8	6
Totals	54	30	24	16

Methodological Approaches

A concurrent collection of quantitative and qualitative data and triangulation of results strengthened the validity of inferences made. Electronic reports of students' mathematics achievement scores were generated by Midwest School District's *i*-Ready Screening Tool (Curriculum Associates, 2015). The *i*-Ready criterion-referenced mathematics screening tool was designated as a valid and reliable online computer adaptive program. The National Center on Intensive Intervention established this tool's validity and reliability by following the established guidelines outlined in the Standards for Education and Psychological Testing, as well as Rasch and Item Response Theory (Curriculum Associates, 2015). This tool is used by schools and districts to monitor students’ levels of mathematics achievement and growth across five domains: (1) Overall Math Levels; (2) Number and Operations; (3) Algebra and Algebraic Thinking; (4) Measurement and Data; and (5) Geometry.

Students’ *i*-Ready’s quantitative dataset was generated by this online tool and spanned 6.5-month time interval from pre- to end of study. Students’ mathematics achievement scores from each classroom were gathered three times throughout the study: pre-test, October 2014; post-test, January 2015; and end test, April 2015. Quantitative assessment measures established students’ mathematics achievement levels, as well as MTSS instructional Tiers levels before, during, and after treatment. The end-of-year assessment measured the extent to which the results of treatment persisted three months after treatment.

Printed copies of each classroom’s *i*-Ready computer-generated reports were made available to determine and compare students’ overall development of mathematics proficiencies encompassing four domains (Curriculum Associates, 2015). Paired t-tests were performed to determine statistical differences between students’ pre-assessment to post-assessment measures, and then again from post to end of study measures. Two-sample t-tests were conducted to compare students’ mathematics achievement results between those who received treatment and those who did not. For each group, Pearson Chi-Square tests were conducted to compare count patterns of movement in students’ Tier Levels, pre- versus end.

Other statistical analyses involved a pre- (Fall, 2014) and post-assessment (Winter, 2015) to determine the developmental functioning of students’ cognitive structures for spatial orientation and conservation of constancy. A modified version of Garner’s Large Group Assessment of Basic Cognitive Structures and Square Search Assessment was administered to detect and analyze students’ cognitive growth relative to two classes of cognitive structures:

spatial orientation and conservation of constancy (see Appendix B). Student responses were hand-scored. Data results were analyzed using Wilcoxon signed rank tests and Z-tests for two proportions. The Wilcoxon signed rank test was the statistical analysis tool used to detect growth differences in the development of students’ cognitive structures for conservation of constancy. The same assessment was used pre- and post in each classroom. The two proportion Z-test was used to compare pre- to post student data for spatial orientation because more than 30 data points existed for each classroom of students.

To substantiate quantitative findings, qualitative data was collected, transcribed, and coded for emerging themes. Data collection instruments consisted of teacher and student interviews, participant questionnaires of quantifiable and open-ended response questions, and Xeroxed copies and photographs of student work. Photographs were captured by the researcher’s locked and private cell phone. Xeroxed copies of students’ drawings and writings depicted their thinking and work and their notions for what a mathematician is. Student writings also captured their impressions of the work mathematicians do. Students were prompted to reflect upon their writings and drawings and the ways they grew as mathematicians throughout the school year.

Both teacher and student qualitative data were transcribed and coded using an open coding system (Miles & Huberman, 1994). For both teachers and students, the unit of qualitative analysis was an individual’s responses, as well as themed responses from each classroom. This system illuminated common and differing themes that emerged amongst participants. Relative to teacher participants, themes specific to each teacher, as well as comparable themes were identified. Patterned responses captured teachers’ perceptions, conceptions, and values influencing the results in this study (DeWalt & DeWalt, 2011). For students, common themes and different themes emerged across all three student-groups and were compared.

Key-words-in-contexts and word repetitions or frequencies were transformed into numerical data while other qualitative data was kept thematic. Themes were then analyzed to establish valid inferences. Qualitative data results were triangulated with multiple data sources to substantiate or challenge quantitative results thereby reducing errors in inferences.

Testing the Four Research Questions

A synthesis of six learning theories and evidence-based pedagogy (See Section 5 in Chapter 2) framed the two experimental teachers’ implementation of the “re-envisioned” instruction model to teach mathematics to their second-grade students. The quantitative and qualitative data was collected concurrently throughout the study. The use of statistical analysis measures, thematic coding, and researcher’s analysis of results provide insights for how the “re-envisioned” instruction model effected students’ mathematics achievement levels, instructional Tier Levels, and perceptions for learning mathematics. Four research questions, including their respective null and alternative hypotheses, subsequent data, and data analysis offer a structure for presenting the results from this study. To guide the reader, each question is presented and followed by corresponding data and data analysis. The first research question is presented here.

Quantitative analyses and comparisons: Research question 1. To what extent did teacher implementation of the “re-envisioned” instruction model influence students’ mathematics achievement?

$H1_0$: The change in students’ mathematics achievement scores between students who received treatment and students in the control group were not statistically different as determined by pre- to post to end *i*-Ready Universal Screener assessments (Curriculum Associates, 2015).

$H1_a$: The change in students’ mathematics achievement scores between students who received treatment and students in the control group were statistically different as determined by pre- to post to end *i*-Ready Universal Screener assessments (Curriculum Associates, 2015).

To identify and monitor students’ levels of mathematics achievement and instructional needs in mathematics, Midwest School District established teacher and student use of the *i-Ready* criterion-referenced computer-adaptive mathematics screening tool during the 2014–2015 school year (Curriculum Associates, 2015). Teachers were trained on its administration late September and early October 2014.

Student data generated by this screening tool consisted of quantitative assessment measures for pre-, post-, and end-of-school-year. The data was used to identify students’ mathematics achievement and instructional Tier Levels throughout the study, as well as three months after treatment. The pretest measure was administered early October 2014. The posttest measure was administered January 2015. The end-of-school-year measure was administered late April 2015. The end-of-school-year data determined whether students maintained, increased, or decreased their rate of mathematics growth and levels of proficiencies (Tier Levels) after treatment.

Students’ overall mathematics achievement scores were determined using the following sub-categories as defined by Curriculum Associates (2015): (1) Number and Operations; (2) Algebra and Algebraic Thinking; (3) Measurement and Data; and (4) Geometry. Only students’ overall mathematics achievement scores were used to calculate and compare individual and aggregated student results. This allowed for variances between the scope and sequence of mathematics concepts presented by Midwest School District’s mathematics program and the scope and sequence of mathematics concepts presented by the two experimental teachers.

Statistical analyses were conducted to establish, verify, and compare students’ aggregate mean scores for mathematics achievement before and after treatment. A paired t-test was used to detect statistical similarities and differences between the pre-, post- and end- aggregate mean scores from each student group. Using an alpha level of 0.05 to argue for significance, test results indicated that all student groups experienced statistically significant increases in mean scores from pre- to post- and from post- to end. Paired t-test analysis identified each student group’s aggregate mean scores before treatment, at the end of treatment, and after treatment. Student group’s scores are listed in Table 4.4.

Table 4.4

i-Ready Mathematics Achievement Aggregate Mean Scores for CLA, CLB, and CLC from Pre- to Post- to End

Group	Pre-Assessment (Fall)	Post-Assessment (Winter)	After Treatment (Spring or End)
CLA (n=18) Control	414.78	424.22	437.89
CLB (n=18) Experimental	420.89	440.39	454.67
CLC (n=18) Experimental	398.00	417.00	430.39

As evident by the repeated measures of the *i*-Ready Assessment results, CLA students (n=18) remained the mid-achieving group throughout the study. They were also designated as the control group. Paired t-test analysis suggested that from pre- to post-, students’ aggregate mean scores increased from 414.78 to 424.22, with an overall increase of 9.44, significant with t =

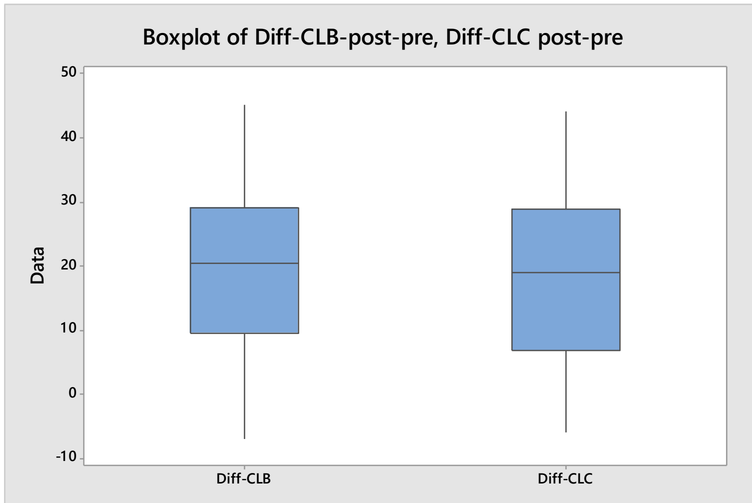
2.16, SD = 18.52, p-value < 0.05. Students’ post- to end aggregate mean scores increased from 424.22 to 437.89, significant with t = 3.54, SD = 16.37, p-value < 0.05.

CLB students (n=18) were one of the two treatment (experimental) groups. The *i*-Ready Assessment results indicated that this group remained the highest-achieving group throughout the study. Paired t-test data results for this group demonstrated that from pre- to post-, students’ aggregate mean scores increased from 420.89 to 440.39, with an overall increase of 19.50, significant with t = 6.26, SD = 13.21, p-value < 0.05. Students’ post- to end aggregate mean scores increased from 440.39 to 454.67, significant with t = 4.63, SD = 13.08, p-value < 0.05.

CLC students (n=18) were the second of the two treatment (experimental) groups; and, relative to *i*-Ready Assessment results (Curriculum Associates, 2015), remained the lowest-achieving group throughout the study. Paired t-test data results indicated, pre- to post-, students’ aggregate mean scores increased from 398.00 to 417.00, with an overall increase of 19.00, significant with t = 5.90, SD = 13.66, p-value < 0.05. Students’ post- to end aggregate mean scores increased from 417.00 to 430.39, significant with t = 3.47, SD = 16.39, p-value < 0.05.

Two sample t-tests were conducted to compare the differences of mean scores between the two treatment groups and between the control group and treatment groups. Note that the mean increase from pre- to post- Assessment for CLA was 9.44, while CLB and CLC student groups mean increases were 19.50 and 19.00 respectively. Although CLB was the highest performing group and CLC was the lowest performing group of all three groups, CLB and CLC groups experienced comparable statistical increases in mean scores pre- to post.

Another two-sample t-test tested for differences relative to students’ mean increases in mathematics achievement scores between the two experimental groups pre- to post. As shown in Figure 4.1, there appeared to be no real teacher effect. The sample t-test, with a t-value = 0.11, SD = 13.44, p-value > 0.05, indicated no evidence of differences in students' mean increases in achievement scores between CLB and CLC groups. Figure 8 is presented on the next page.



	N	Mean	St Dev	SE Mean
Diff-CLB	18	19.5	13.2	3.1
Diff-CLC	18	19.0	13.7	3.2

T-Test of difference = 0; (vs ≠): T-Value = 0.11; p-value > 0.05
DF = 34

Figure 8. Comparing *i*-Ready mean difference scores between CLB and CLC, post to pre-. Statistical evidence proving that the two experimental groups made comparable increases in mean scores. This justifies combining the two experimental groups into one large group (n=36).

These statistical findings justify the appropriateness for combining the experimental groups when looking at their mean increases in achievement scores. Comparing the combined experimental groups' pre- to post differences in mean scores to the control groups' differences in mean scores, two-sample t-test analysis indicated that the experimental groups experienced significantly higher increases in their mathematics achievement scores than the control group. When combined, experimental groups' (n=36) mean increase in scores was 19.3, while CLA group’s (n=18) mean increase in scores was 9.4, with t = 2.00, SD = 13. 2 and 18.5 respectively, p-value < 0.05.

From post to end, a different statistical outcome occurred (see Figure 9). Two sample t-tests indicated no real differences in students’ differences in the increases in mean scores between the two experimental groups and between the two experimental groups and the control group. The slopes indicating students’ growth in mathematics achievement were comparable across all three classrooms. When combined, experimental groups’ (n=36) mean increase in scores was 13.8, while the mean increase in scores for CLA students was 13.7, with t = 0.04, SD = 14.6 and 16.4 respectively, p-value > 0.05.

Additional two-sample t-tests were conducted to establish statistical comparisons between all three student groups’ mathematics achievement at pre-, at post-, and at end, with the group’s mean score being the unit of analysis. With alpha levels set at 0.05 for significance, results indicated there were no statistical differences between CLA’s (control) pre- mean score of 414.78 and CLB’s pre-mean score (the highest-performing experimental group) of 420.89, with t-value = 0.90, SD = 20.43, p-value > 0.05.

When comparing groups’ mean scores at end of treatment (post), close significant differences between CLA’s and CLB’s mathematics achievement scores were manifested. CLA’s mean score of 424.2 compared to CLB’s mean score of 440.4 revealed differences with t = 2.01, SD = 24.13, p-value = 0.052 at the end of treatment. By the end of the study, however, CLA’s mean score of 437.9 and CLB’s mean score of 454.7 indicated significant differences three months after treatment, with t = 2.23, SD = 22.55, p < 0.05. View Figure 9 on the following page.

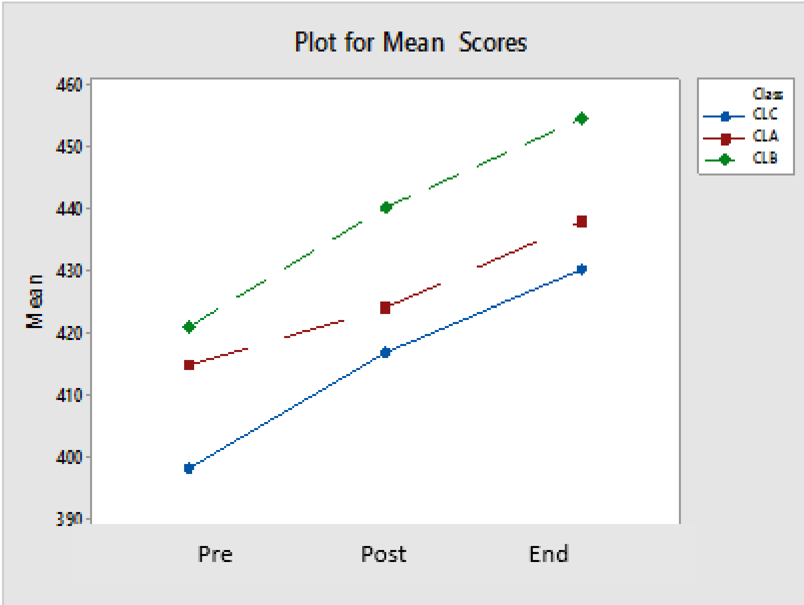


Figure 9. CLA, CLB, CLC *i*-Ready mean scores at pre-, post, end; school year 2014–2015. Statistical analysis of each classroom’s mean scores indicating mathematics achievement.

Whereas with CLA’s (control) pre- mean score of 414.8 and CLC’s pre- mean score of 398.0 (the lowest-performing experimental group), there were statistical differences at the onset of the study. The control group’s mean score was statistically significantly higher than CLC’s mean score, with t = 2.12, SD = 23.78, p-value < 0.05. While there were statistical mean differences at the beginning of the school year between CLA and CLC, these statistical differences ceased to exist by January’s post-assessment (see Figure 9). Comparing CLA’s post mean score of 424.2 to CLC’s post mean score of 417.0, a two-sample t-test analysis indicated no significant differences, with t = 0.83, SD = 25.97, p-value > 0.05. The differences or gap between CLC students’ and CLA students’ mean scores decreased from 16.8 (Pre) to 7.2 (Post). Two-sample t-tests indicated that CLA’s mean score of 437.9 was not significantly different from CLC’s mean score of 430.4, with t-value = 0.81, SD = 27.9, p-value > 0.05 post to end.

Students in all three groups experienced statistically significant increases in their respective mathematics achievement from pre- to post and post to end. Students in the two experimental groups experienced greater statistical differences from pre- to post than students in the control group during that timeframe. The increase in the experimental groups' mean scores doubled that of the control group. The null hypothesis for Research Question 1 was rejected.

Quantitative analyses and comparisons: Research question 2. In Chapter 1's glossary of terms and definitions, proponents of MTSS claimed that Tier I general core instruction should meet the instructional needs of 80% of the students in the classroom. To determine the effects teacher implementation of the "re-envisioned" instruction model had upon students' Tier Levels of instruction, the following question was posed:

Research question 2. Did teacher implementation of the "re-envisioned" instruction model minimize students learning difficulties in mathematics? In other words, did implementation of the model move students identified at Tier II and Tier III levels to Tier I and Tier II levels respectively as identified by the pre- to end tests from the *i*-Ready Screening Assessment (Curriculum Associates, 2015)?

$H2_0$: When comparing students in the two treatment groups to students in the control group, there were no statistical differences in count patterns representing students' decrease (improvement) in Tier Levels from pre- to end according to the *i*-Ready Universal Screener assessment data.

$H2_a$: When comparing students in the two treatment groups to students in the control group, there were statistical differences in count patterns representing students' decrease (improvement) in Tier Levels from pre- to end according to the *i*-Ready Universal Screener assessment data.

The *i*-Ready Screening results was the data source (Curriculum Associates, 2015) to determine how CLA, CLB, and CLC Teachers’ mathematics instruction (“re-envisioned instruction model” compared to Midwest School District’s mathematics program) affected students’ instructional Tier Levels. Utilizing teachers’ *i*-Ready reports, students’ instructional Tier Levels were tabulated into a table indicating Tier I, II, or III. These tabulations informed aggregated percentages of students in each class at each instructional Tier pre-, post-, and end of study. Table 4.5 identifies the percentage of students at each instructional Tier level in each classroom throughout the school year, 2014–2015.

Table 4.5

i-Ready Instructional MTSS Tier- Levels for Mathematics Instruction for CLA, CLB, and CLC Students, Pre- to Post- to End

Group (N=18)	Pre-Assessment (Fall)	Post-Assessment (Winter)	After Treatment (Spring or End)
CLA (Control)			
Tier Level I	28%	56%	61%
Tier Level II	61%	33%	39%
Tier Level III	11%	11%	0%
CLB (Experimental)			
Tier Level I	28%	72%	89%
Tier Level II	72%	28%	11%
Tier Level III	0%	0%	0%
CLC (Experimental)			
Tier Level I	17%	39%	50%
Tier Level II	44%	44%	44%
Tier Level III	39%	17%	6%

Over the course of treatment, from pre- to post-, the number of CLA students who improved in instructional Tier Levels was 7 (i.e. moved from a Tier II or Tier III to a Tier II or Tier I); while the number of students who regressed in instructional Tier Levels were 2. One

CLA student moved from a Tier I to a Tier II instructional level. The second student moved from a Tier II to a Tier III instructional level.

From pre- to post-, the number of CLB students who improved in Tier Levels were 8. No students regressed in Tier Levels during treatment. For CLC students, the number of students who improved in Tier Levels was 7. Again, for this treatment group, no students regressed in Tier Levels during treatment.

To statistically test for significant differences in count patterns for each student group, Pearson’s Chi-Square tests were used to compare patterns of movement in Tier Levels, pre- versus end. Pre-assessment measures established baseline data of all students’ Tier Levels for instruction. End assessment measures established the effects teacher instruction had upon students’ instructional Tier Levels. Students included in this study did not experience additional instructional supports during the school day.

All three groups demonstrated significant differences in count patterns pre- vs. end. Most students moved from a Tier II to a Tier I or from a Tier III to a Tier II or I instructional level. Due to low counts, the number of CLA students in Tiers II and III had to be combined for testing purposes. CLA group’s data results suggested significant differences in count patterns with a Chi-Square = 4.050, p-value < 0.05.

For CLB and CLC, combining Tier Levels was not necessary. CLB’s count patterns for pre- versus end were also significantly different, with a Chi-Square = 13.829, p-value < 0.05. CLC’s count patterns, pre- versus end, were also significantly different, with a Chi-Square = 7.500, p-value < 0.05.

When comparing CLA students to CLB students’ mathematics achievement scores at the end of the study, CLB’s end was significantly higher. Yet, relative to students’ instructional Tier

Levels, the Chi-Square test for differences in patterns, Pearson Chi-Square = 3.704, DF = 1, p-value = 0.054. These measures demonstrated only mild significance. The p-value of 0.054 was just outside the range of significance. This was possibly due to the small sample sizes for each classroom.

The number of students who achieved at least one year's worth of growth in mathematics achievement in each classroom during the 6.5 months of the study was vastly different. The authors of the *i*-Ready Assessment tool (Curriculum Associates, 2015) identified students' individual Scale Scores relative to chronological grade levels and independent of instructional levels. According to Curriculum Associates (2015) established measures, four students in CLA group achieved one year's worth of growth in mathematics achievement. Ten students in CLB group and nine students in CLC group achieved one year's worth of growth in mathematics achievement. Both experimental groups experienced more than twice the number of students achieving at least one year's growth relative to this measure when compared to students in the control group. Appendix G provides individual classroom lists of students' *i*-Ready Mean Scores for 2014-2015 and District Benchmark Assessment I & II Results.

Analysis of low-scorers' mean scores. A main goal for this study was to design an instruction model that reduced the number of students needing Tier II and Tier III interventions. Consistent with *i*-Ready reports, any student who scored ≤ 429 was designated at Tier II or Tier III instructional levels. To determine how low-scoring students faired in each classroom during and after treatment, paired t-tests with an alpha level of 0.05 to argue for significance were conducted. When following all original low scorers across the three classrooms, all three student groups demonstrated statistically significant improvements from pre- to post- and again, from post- to end.

In CLA's classroom (n=18), thirteen students were considered low-scorers needing instructional interventions at the beginning of the school year. By post, eight students were identified as low scorers; and by the end of the school year, seven students were considered low scorers. Paired t-test results indicated that CLA students' post- mean score of 416.46 was significantly higher than their pre- mean score of 405.08, with $t = 2.17$, $SD = 18.89$, and p-value < 0.05. From post- to end, their mean score of 428.92 was significantly higher than their post- mean score of 416.46, having $t = 2.45$, $SD = 18.36$, p-value < 0.05.

In CLB's classroom (n=18), thirteen students were identified as low-scorers needing instructional interventions at the beginning of the school year. By post, five students were identified as low scorers; and, by the end of the school year, two students were identified as low scorers. Low-scorer's post mean score of 433.00 was significantly higher than their pre- mean score of 412.38, with $t = 5.40$, $SD = 13.76$, and p-value < 0.05. End mean score was 445.77, which was significantly higher than the post mean score of 433.00, $t = 3.16$, $SD = 14.59$, p-value < 0.05.

In CLC's classroom ($n = 18$), fifteen students were identified as low-scorers needing interventions at the beginning of the school year. By post, eleven students were identified as low scorers; and, by the end of the school year, nine students were identified as low scorers. Post mean score 410.40 was significantly higher than their pre- mean score of 389.47, with $t = 5.88$, $SD = 13.79$, and p-value < 0.05. End mean score was 421.27, which was statistically significantly higher than the post mean score of 410.40, $t = 2.54$, $SD = 15.56$, p-value < 0.05.

Similarly, all three classrooms of low scorers experienced improvements in instructional Tier Levels. Between pre- and post-assessments, however, two students in CLA classroom experienced regression in instructional Tier Levels, whereas no students in the experimental groups regressed. Patterns were clearly different. To answer Research Question 2, however, the statistical analyses of count patterns in students’ movement of instructional Tier Levels justify the acceptance of the null hypothesis. Proof indicating statistical differences in count patterns between the experimental groups and the control group was insufficient. Results may be due to small sample sizes.

Quantitative analysis and comparisons: Research question 3. In Chapter 2, all six learning theorists described students’ cognitive structures as essential neurocognitive systems that support students’ learning of mathematics. The two experimental teachers used the “re-envisioned” instruction model, along with “re-designed” mathematics tasks, to activate and mediate the development of students’ cognitive structures. The following data and data analysis answered Research Question 3.

Research question 3. To what extent did teacher implementation of the “re-envisioned” instruction model influence the development of students’ cognitive structures, specifically spatial orientation and conservation of constancy?

$H3_0$: When comparing the treatment groups’ development of their cognitive structures to the control group’s development of their cognitive structures, there were no statistical differences as determined by pre- to post- test scores on the Adapted Cognitive Structure Assessment results.

$H3_a$: There were statistically significant differences in students’ development of their cognitive structures between students who received treatment and students who did not as determined by students’ pre- to post- test scores on the Adapted Cognitive Structure Assessment results.

To assess the level of development of students’ cognitive structures relative to conservation of constancy and spatial orientation, a modified version of Garner’s Large Group Assessment of Basic Cognitive Structures and Square Search Assessment was utilized. The same assessment was administered to all three student groups for pre- and post. This assessment comprised three sub-tests and was conducted in whole group settings (refer to Appendix B).

Student responses on the assessments were hand-scored and quantified. The first section contained three test items assessing for conservation of constancy. The first question required students to look at two markers that were the same brand, type, width, and length. The markers were placed side by side so that students perceived they were equivalent in length. When the second marker was moved to the left of the first marker, students were asked to determine if one marker was now longer than the other or if the two markers were still the same length (conservation of length). The other tasks in the first section were similar in nature, meaning that equivalency was first established between two balls of clay having the same mass and two plastic bottles holding the same amount of liquid. When one of two clay balls were flattened, and one of the two bottles of water was tipped upside down, students were asked to determine if the ball’s mass was still the same or if it had changed, and if the amount of liquid in the bottle remained the same or if it had changed.

When conducting Wilcoxon Signed Rank Tests using a test of median = 0.000 versus median > 0.000, only the control group, CLA, provided statistically convincing evidence that they improved on this set of test items. In CLA’s classroom, eleven students improved over the treatment period with a statistical value at 56.0, p-value <0.05, and estimated median set at 1.000. The two experimental groups’ results did not indicate statistical differences pre- to Post.

For the second set of tasks testing for conservation of constancy, students drew three pictorial representations of half-filled glasses of water. The first glass was positioned on its base, the second half-filled glass of water tipped to the right, and the third half-filled glass of water tipped to the left. Analyzing the total correct responses from each classroom pre- to post-, slight changes were evident in CLA’s students, but not enough to test for significant differences. No changes were evident from pre- versus post- in both treatment groups. See Table 4.6 on the next page for the results from all three classrooms relative to students’ development for conservation of constancy.

Table 4.6

Adapted Cognitive Structure Assessment Results with Wilcoxon Signed Rank Test for Conservation of Constancy

Class	N	N for Test	Wilcoxon Statistic	p-Value	Estimated Median
CLA	17	11	56.0	0.023	1.000
CLB	18	11	46.0	0.133	0.000
CLC	16	9	30.0	0.203	0.000

For the final portion of the cognitive structure assessment, students used their spatial orientation skills to depict 12 different configurations for squares and 12 different configurations for triangles by connecting sets of small dots. Z-Tests were conducted to determine statistical changes in students’ responses pre- to post. With statistical analysis of student results, each student group provided statistically significant evidence that they developed their cognitive structures involving spatial orientation.

CLA students increased by 14 percentage points, with a confidence level of at least 8 percentage points, $Z = 3.77$, $p\text{-value} < 0.05$. CLB students increased by 38 percentage points, with a confidence level of at least 31 percentage points, Z = 8.79, p-value < 0.05. CLC students increased by at least 10 percentage points, with a confidence level of 4 percentage points, Z = 2.70, p-value < 0.05.

Students’ development of their cognitive structures varied. CLA student group demonstrated statistical differences in their development of their cognitive structures for Conservation of Constancy. The two experimental groups did not. All student groups demonstrated differences for their cognitive structures for spatial orientation. Therefore, the null hypothesis for Research Question 3 was rejected and the alternative hypothesis was accepted. There were statistical differences between the control group and the two experimental groups relative to their development of their cognitive structures.

Qualitative analysis and comparisons: Research question 4. The fourth and final question pertained to teacher implementation of the model and the effects this had upon students’ beliefs and practices for learning mathematics. Multiple sources of qualitative data were gathered. Qualitative data were triangulated with other data sources to establish validity of quantitative results and reliability of inferences made. Data from each classroom of students were collected, transcribed, analyzed for content, coded for emerging themes, and reduced for data manageability. Interviews, questionnaires, surveys, researcher’s observations, and scripted field notes of teacher’s mathematics lessons generated thick descriptions of teachers’ and students’ beliefs and practices for teaching and for learning mathematics. Thick descriptions originated in students’ mathematics work recorded in their mathematicians’ notebooks. Student-to-student and student-to- teacher dialogues provided rich contexts offering insights into ways students made sense of the “re-designed” tasks teachers presented. Students’ pictures and written and oral descriptions illustrated their beliefs and practices for learning mathematics

Research question 4. By the end of this study, to what extent did teacher implementation of the “re- envisioned” instruction model influence students’ beliefs and practices for learning mathematics?

$H4_0$: By the end of the study, qualitative differences in students’ beliefs and practices for learning mathematics did not exist between students who received treatment and students who did not as indicated by students’ and teachers’ qualitative data.

$H4_a$: By the end of the study, qualitative differences in students’ beliefs and practices for learning mathematics existed between students who received treatment and students who did not as indicated by students’ and teachers’ qualitative data.

Classroom Environments

To determine the quality of each classroom’s environment, all three teachers were asked to complete surveys indicating fourteen specific beliefs and practices they observed in their students. Beliefs included students’ self-perceptions as mathematicians and understanding their roles in the learning process. Learning practices included: asking questions, using multiple representations, explaining and justifying mathematics thinking, making connections between and among concepts and representations, and self-selecting tools.

The survey utilized a four-point Likert scale rating:

“1” indicated that a specific student behavior was observed all the time;

“2” indicated that the behavior was observed some of the time;

“3” indicated the behavior was observed occasionally; and,

“4” indicated that the student behavior was observed not at all.

Utilizing classroom observations of their students, each teacher scored the fourteen student beliefs and practices three times during treatment: September 2014, Mid-October 2014, and in January 2015.

All three teachers claimed their students perceived themselves as mathematicians. Throughout the study, each teacher put forth intentional efforts for creating environments where all students’ self-efficacy for learning mathematics was enhanced. Each teacher questioned her students about their roles and their work as mathematicians and created hallway bulletin boards posting each student’s photo accompanied by students’ corresponding responses. Each teacher also charted students’ collective responses onto chart paper and posted it in their classrooms. Time after time, teachers consistently referred to their students as mathematicians. By post-study, each teacher indicated that their students consistently perceived themselves as mathematicians. In fact, CLA Teacher described that when comparing her current students to students from previous years, she claimed that this year’s group believed they were mathematicians more intensely than previous groups.

City and colleagues (2009) argued that to increase student achievement, the student’s role must change in the learning process. When comparing CLB and CLC Teacher responses to CLA Teacher responses on the survey, one student belief was markedly different. This indicator referred to students understanding that their roles were to think deeply about mathematics during instruction. At the beginning of the study, all three teachers indicated that their students did not see their roles as thinking deeply about mathematics. By the end of the study, both experimental teachers indicated their students explicitly understood their role was to think about math, and think about math deeply. CLA Teacher recorded that her students understood this role “some of the time.” To verify teacher’s perceptions, we turn to data indicating students’ beliefs.

Students’ Beliefs

To accurately infer students’ perceptions about mathematicians and the work they do, all three classrooms of students were asked to record individual responses to the following prompts, “What is a mathematician?” and “What is the work they do?” Students’ replies to these open-response questions were recorded at the end of treatment in January 2015. Students recorded their perceptions in their mathematician’s notebooks using words, drawings, and symbols. Their responses were collected, photocopied, and analyzed using an open-coding system. Within the first analytic pass, student responses revealed a list of common, yet specific words found in and across all three student groups.

Next, student responses were reduced to fourteen different codes. Codes included “thinking,” “smart,” “learner,” “knows math,” “a reader,” “a writer,” “uses tools,” etc. Repetition of these codes amongst student groups generated frequency patterns. These codes were reduced one more time to specific themes to calculate the percentages of student responses within each class. These percentages were used to create a common definition or theme representing each student population.

CLA students’ written descriptions were wide and varied. No single code or common theme emerged in at least 50% of students’ responses. In other words, no common description emerged from at least half the respondents. When 40% of respondents were needed to generate a common description, then CLA students described mathematicians as “someone who is smart and they solve problems.” When 30% of students’ responses were needed to create their common definition of a mathematician, CLA group’s description was, “Mathematicians use their brains to think. They are smart and they solve problems. They are readers and writers. I am a mathematician!”

Following the same themes and percentage criteria for developing a common definition from CLB’s class, more than half (> 50%) of CLB respondents claimed, “Mathematicians use their brains to think. They share and explain their thinking. They use tools and make connections.” Between 40 to 49% of CLB students’ responses remained the same, meaning there was no change in student descriptions. At least 30% of student respondents claimed, “Mathematicians use their brains to think and to learn. They know math so they share and explain their thinking. They write to solve problems and operate with numbers. They use tools and make connections.”

According to CLC student respondents, at least 50% of CLC students described mathematicians as, “Mathematicians use their brains to think. They are readers and writers which help them solve problems. They also use tools and make connections. I am a mathematician!” Like CLB, there was no additional change in student descriptions when 40%–49% was the range for defining mathematicians and their work. When 30% or more students’ responses generated a common definition, then CLC students reported, “Mathematicians use their brains to think and to learn. They are readers and writers which helps them solve problems. They also use tools and make connections. I am a mathematician!” The difference between 40% and 30% responses involved the word *learn*.

Students’ Self-Reflections of their Learning

Each teacher believed her actions and words in the classroom supported students developing self-efficacy as mathematicians. Experimental teacher’s implementation of the “re-envisioned” instruction model supported students becoming consciously aware of one’s self as knower relative to learning, reasoning, and specific practices. Thus, near the end of the school year in May 2015, each classroom was visited a final time. Students were invited to look through their mathematician’s notebooks and describe, using words, pictures, and numbers, the ways they grew as mathematicians throughout the school year.

In whole group, yet working individually, students responded to the following prompts, “Looking at your mathematician’s notebook, how have you grown as a mathematician? What are things you notice? You can use words, pictures, numbers, whatever you need to describe how you’ve changed as a mathematician.” As students leafed through their notebooks, giggles, smiles, and expressions of wonderment erupted. Some students verbalized it was hard to read their initial entries from the beginning of the year, or that their writing at the beginning of the year didn’t make sense to them now. All three student groups noticed changes in their handwriting and spelling abilities from the beginning of the year to the end of the school year.

Most CLA students focused on certain aspects of their handwriting, including neatness, spelling, punctuation, and their ability to read and make sense of their writing. Some students also focused on their changes in learning specific mathematics content. For example, one CLA student compared what he did at the beginning of the year to the ways he changed over the course of the year (spelling has been corrected).

This CLA student recorded,

> Before: It was about Base 10 Blocks and games. Doesn’t make sense my writing. Missing stuff in my writing; Reflect a lot more. After: Now it is about subtraction; now it makes sense; now I don’t leave stuff out of my writing; now I don’t have time to reflect.

Another CLA student wrote, “I noticed none of the words are spelled correctly. I learned how to tell [time]. I was smart. I learned how to count money.” A third example from a CLA student included,

> At the beginning I could not read my writing. I notice that we glued number stories in our note book. We wrote about mathematicians. I notice at the beginning I spelled words wrong and added more punctuation at the end.

Finally, another CLA student’s response focused on specific mathematics models represented in her mathematician’s notebook. She wrote, “I learned Start and Change End. I know a long time ago my writing was funny now it is good. I learned Partial Sums. I learned Part, Part, Total.”

While both CLB and CLC students noted changes in their reading and writing capabilities, they also noted increases in their mathematics learning and understanding, in the amount of writing needed to explain their thinking, and their inclusion of evidence. One CLB student observed,

> I noticed that I used to not have as much words as now. I do more math than I used to do. I didn’t really get math really good but now I do. I don’t get what I used to write. I changed as a math mathematician because I do more things and in different ways and more ways than later. I used to call tens blocks. I trade now and I never did that, Wow. I work more with money. I think at the beginning of the year I was a 1, but now I changed to 5s and 3s. I used to think ¼ was three parts shaded.

This student’s 5’s and 3’s referred to the teacher’s system of engaging students in self-reflection for understanding. At the end of a lesson, CLB Teacher asked her students to reflect upon their understanding of the specific mathematics concepts taught that day. A fist of five indicated that the student could teach another student about the mathematics concepts they learned.

Another CLB student noted,

> I noticed that I am using more harder [words] than before. I noticed that we have changed the subject of math a lot. I noticed that in the past I didn’t use too much precise math

> words. I noticed that I used to just work to find the answer and now I just show evidence. I noticed that I don’t remember the lesson until I actually see it with my own eyes. I noticed that my now work in the past it doesn’t make sense but the present does make sense. I noticed that I barely had any thinking and now I have a butload of thinking on my page.

Likewise, a CLC student recorded, “At the beginning I had only 1 or 2 ideas. But now I have all most 5 or 6 ideas. At the beginning I didn’t understand.” Similarly, another student wrote, “I did not have much on my paper before. I use strategies more now. My stuff did not make sense before. I can do a lot more stuff and strategies. Better handwriting. I understand things more. We did reflections.” Still, another CLC student observed, “I use more evidence. My writing makes more sense. I did not understand a lot of my words in the beginning of the year. I used pictures. My writing has increased.” Finally, one CLC student noticed she made fewer mistakes as the year progressed. She recorded, “I do more words now. Now I do more evidence. I do more explaining now than back then. I had to do more crossing out back then. I didn’t solve one of my equations. I made a wrong answer.”

Looking beyond students’ handwriting capabilities, students’ responses were analyzed from a mathematics perspective. Seven common themes emerged pertaining to learning mathematics across all three groups of students. Table 4.7 represents the percentage of student responses from each classroom aligning with these seven themes.

Table 4.7

Common Themes for How Students Grew as Mathematicians

Class	Understanding Sense Making Thinking	Learning	Explaining	Showing Evidence	Using Strategies/ Properties of Mathematics	Using Tools	Specific Math Content
CLA (Control)	50%	35%	15%	25%	35%	10%	35%
CLB (Exp.)	63%	81%	63%	63%	54%	25%	50%
CLC (Exp.)	67%	67%	33%	56%	22%	65%	30%

When comparing student responses from the control group to student responses from the two experimental groups, the percentages that were double the control group were ways students grew as mathematicians. These involved an increase in student learning, in the amount and depth of their written explanations for explaining their thinking, in the amount of evidence they provided, and their use of tools.

Utilizing an open-response survey, students were also asked about the concepts they wanted to better understand in mathematics. All three student groups responded. Student responses were coded for key-words-in-contexts and word repetitions or frequencies.

The majority of CLA students reported they wanted to understand more about the operations of multiplication and division. One CLA student recorded, “Devison 6 ÷ 2 = 3.” Another student reported, “Divsen and really hard problems and using base-ten block.” A third student claimed, “Do times because there yougly hard for me in math.” Others wrote, “That I can know every math eqagun”; “Paper and pesor”; “Wene there’s a adish pamol I don’t get it”; and “Math game’s. Because I do not understand the deirecttions.” Finally, four CLA students reported that they had learned all they needed to learn. When asked what they wished they understood better in math, three of the four students responded, “Nothing” or “Really nothing,” while the fourth student responded, “I think I get it all?”

CLB students’ responses focused upon a wide variety of concepts. Students reported they wanted to learn more about measurement, quantity, arrays, benchmark numbers, odds and evens, and using the number line to solve problems. Others reported that they wanted to improve their understanding of meanings for multiplication and division. One CLB student recorded, “Maltiplacashon because I can’t understand Maltaplacashon is all about I mean what’s the point if you don’t know.” Another student wrote, “÷ because I do not really understand it. I do not really now the mening of it.” Two students expressed their desires to better understand how to explain their thinking because it was challenging for them. Two more students responded, “Megerment becase I don’t think I know most of the megemet words. But I want to know most of them”; and, “I do not no all the ansers yet.” None of the CLB students expressed there was nothing more to learn in this classroom. Overall, CLB students sought to gain a better understanding of a variety of mathematics concepts.

Similarly, CLC students’ responses focused upon a wide range of concepts as well. Students reported that they wanted to better understand numbers and number relationships, equations, jumping on the number line, addition, multiplication, division, and fractions. One student recorded, “Fractions and devitoin because I want to cathch up with other kids and I want to be smart.” Another student wrote, “Number relashonship’s because it is hard to explan.” A third student reported that he wanted to understand division because his grandmother taught him division, but he still didn’t understand it. A student had high expectations for herself. She reported, “Tines colleage math.” Only one CLC student claimed there was nothing more for her to learn in this classroom.

Teachers’ Reflections of Students’ Practices for Learning Mathematics

Teachers’ qualitative data was triangulated with student data to confirm the above inferences relative to students’ beliefs and practices for learning mathematics.

CLA Teacher reported,

> I don’t think I got the digging deeper questions like I got from you and our [school’s math coach]. These were too challenging to do. The lessons don’t dig deep, don’t have the same types of questions. The district has some good lessons, good parts of lessons, but they do not have the same quality for when we do the new workshop.

Teachers’ implementation of the “re-envisioned” instruction model and “re-designed” mathematics tasks enabled the two experimental teachers to observe and compare students’ mathematics skills, mathematics thinking, beliefs, and practices relative to student groups from prior years.

CLB Teacher recorded,

> Students are truly thinking like mathematicians. They are talking about math more during the lesson and while playing games. They are more engaged. They are trying things at home on their own. Questioning more. Finding more connections. Beginning to understand the relationship between numbers. Better understanding that math learning is thinking. More cooperative learning. Writing! The amount and quality of their math writing is far more substantial than past years.

CLC Teacher claimed,

> I see students thinking more deeply, making more connections between math ideas than before. Students are linking equations to games, writing more about math, explaining their mathematics thinking and drawing visual representations to explain thinking. Students add onto other students’ thinking, making sense of each other’s ideas and thoughts.

Given the triangulation between teachers’ and students’ qualitative data, differences were evident in students’ beliefs and practices for learning mathematics between students in the control group and students who received mathematics instruction via the “re-envisioned” instruction model. Students’ differences in their responses for what it means to be a mathematician, the work mathematicians do, and students’ beliefs for future learning of mathematics provide justification for rejecting the null hypothesis relative to Question 4.

Chapter Summary

The results from this study addressed four critical questions pertaining to increasing students’ mathematics achievement and for minimizing their learning difficulties in mathematics. Quantitative results from Midwest School District’s *i*-Ready Universal Screening Measure (Curriculum Associates, 2015) indicated greater significant differences in mean increases in the experimental students’ mathematics achievement scores when compared to the control students’ mean increases in achievement scores. Students who experienced the instruction model and “re-designed” tasks demonstrated increases in mean scores doubling the increase of students’ mean scores in the control group. The experimental students also experienced comparable mean increases pre- to post even though the two experimental groups were widely divergent in their mathematics achievement levels. Statistical analyses provided justification for rejecting the null hypothesis related to Research Question 1.

CLA Teacher reunited with her two grade-level colleagues to plan mathematics instruction after treatment. From post- to end of school year, *i*-Ready results revealed a different outcome than the one experienced during treatment (Curriculum Associates, 2015). There were no statistical differences in students’ increase of mean scores when comparing the two experimental groups to the control group. Students’ mean increases in mathematics achievement scores were comparable in all three classrooms.

Screener results at the beginning of the school year indicated there were no statistical differences when comparing CLA group’s mean scores to CLB group’s mean scores. By the end of the study, however, there were statistically significant differences between CLA’s and CLB’s mean scores in mathematics achievement. The achievement gap between CLA and CLB students statistically widened from post to end of study.

Opposite results occurred between CLA group and the lowest-performing experimental group, CLC. Where there were statistical differences in students’ mean scores between these two student populations at the beginning of the study, test measures revealed that the achievement gap between CLC students’ mathematics achievement and CLA students’ achievement narrowed by the end of treatment. When compared to CLA students, CLC students’ learning gap or achievement gap was no longer significantly different by post-assessment and remained non-existent till the end of the study.

The above results corresponded to the number of students achieving at least one year’s worth of growth in mathematics achievement. As indicated by students’ individual Scale Scores relative to chronological grade levels and independent of instructional levels, the *i*-Ready measures indicated four CLA students achieved one year’s worth of growth in mathematics achievement from pre- to end of study. Ten students in CLB group and nine students in CLC group achieved one year’s worth of growth in their mathematics achievement during the same timeframe when applying the same statistical criteria. This data suggested that teacher’s enactment of the “re-envisioned” instruction model and “re-designed” tasks more than doubled the number of students experiencing at least one year’s worth of growth in their mathematics achievement scores when comparing the experimental groups to the control group.

Further analysis was conducted to determine whether the “re-envisioned” instruction model was effective in reducing the number of students identified at Tier II and Tier III levels of instruction. This analysis addressed Research Question 2. Pearson Chi-Square Tests were conducted to compare count patterns of students’ movement between Tier Levels, pre- versus end. Although pre- to post data from *i*-Ready Assessment indicated two CLA students regressed in instructional Tier Levels, students’ count patterns of movement from pre- to end in all three classrooms demonstrated convincing statistical evidence that students improved in instructional Tier Levels.

The low-scorers’ analysis for any student who achieved a mean score ≤ 429 at the beginning of the school year on their overall mathematics achievement was designated as a Tier II or Tier III student. Statistical analysis indicated that low-scoring students in each classroom demonstrated statistically significant differences when comparing their increases in mean scores pre- to Post. Furthermore, when comparing students in the two treatment groups to students in the control group, no statistical differences in students’ count patterns representing students’ improvement in Tier Levels pre- to end were displayed. The null hypothesis for Research Question 2 was accepted.

A Wilcoxon Signed Rank Test and Z-Test for Two Proportions were applied for statistical analysis to assess whether teacher’s enactment of the “re-envisioned” instruction model mediated the development of students’ cognitive structures for conservation of constancy and spatial orientation. Only CLA students provided statistically convincing evidence that they furthered their development of their cognitive structures for conservation of constancy. Eleven students improved over the treatment period with a statistical value at 56.0, $p\text{-value} < 0.05$, and estimated median set at 1.00. The two experimental groups’ findings were not statistically different pre-to post. When analyzing the second set of tasks for conservation of constancy, slight changes were evident in CLA’s students, but not enough to show significant differences. No changes were evident from pre- versus post in both treatment groups with the second set of tasks for conservation of constancy. This variance in data between task types suggests more data should be collected.

To test for students’ development of the cognitive structures for spatial orientation, statistical changes in students’ responses pre- to post were manifested for each student group. Each student group made statistically significant increases pre- to post. For Research Question 3, the null hypothesis was rejected. CLA students did mediate the development for their cognitive structures for conservation of constancy. Thus, there were statistical differences between groups.

Multiple sources of qualitative data were collected, analyzed, and triangulated with these results to substantiate or challenge students’ quantitative results from the *i*-Ready Assessment Screener (Curriculum Associates, 2015). Differences were manifested when comparing student beliefs and practices in CLA’s classroom to the two experimental groups. CLA students were often expected to create equivalent representations. CLB and CLC students engaged with novel tasks that focused their attention on the figural units within varied representations. Students in the experimental classrooms were expected to interpret the meaning of the representations, make connections to other mathematics concepts previously learned; or, extend the original problem.

Differences in CLB and CLC student’s definitions for mathematicians and the work they do were demonstrated when compared to CLA students. CLA students (40%) believed “Mathematicians are smart and they solve problems.” More than 50% of CLB students described, “Mathematicians use their brains to think. They share and explain their thinking. They use tools and make connections.” At least half of CLC students claimed, “Mathematicians use their brains to think. They are readers and writers which help them solve problems. They also use tools and make connections. I am a mathematician!”

Qualitative differences in student’s descriptions and perceptions for ways they grew mathematically during the school year were found amongst students who experienced the “re-envisioned” instruction model when compared to students who did not. A greater number of students in CLB and CLC’s classrooms recorded they observed increases in their learning, in the amount and depth of their written explanations, in the amount of evidence they provided, and in their abilities to use tools. This data provides justification to reject the null hypothesis for Research Question 4.

To conclude this chapter, all three student groups statistically increased their quantitative mean scores from pre- to post- to end. However, there were stark statistical differences in students’ measures across groups relative to students’ mathematics achievement. Students in both experimental groups outperformed students in the control group. The number of students in CLB and CLC classrooms who achieved one year’s worth of growth in mathematics achievement were more than twice the number of students in the control group.

All three student groups statistically improved in instructional Tier Levels from pre- to end of study. Regarding cognitive structure development for conservation of constancy, only

CLA students demonstrated a statistical difference. All three student groups demonstrated statistical increases for their cognitive structures related to spatial orientation.

Qualitative differences were noted in students’ beliefs and practices for learning mathematics. Students in the experimental groups recognized that their roles for learning mathematics was to think deeply about mathematics. CLB and CLC students worked to provide evidence of their thinking, share their thinking with others, make connections, and use tools to learn mathematics.

Contrastingly, the control classroom of students explained that mathematicians were smart and solved problems. Four students from CLA classroom believed they knew everything there was to know about mathematics. If students were to learn a new concept, it related to multiplication and division. Students in the two experimental groups expressed they wanted to learn a wide variety of mathematics concepts, including understanding how to explain their ideas more effectively, apply mathematics concepts, and understand the meanings of the operations.

The triangulation between the quantitative data and the multiple sources and analyses of teachers’ and students’ qualitative data supported justification of these differences between CLA, CLB, and CLC student groups. The null hypotheses for Research Questions 1, 3, and 4 were rejected. The null hypotheses for Research Question 2 regarding students’ movement in instructional Tier Levels was accepted. Interpretations of the data, conclusions, and recommendations for future research are presented in the next and final chapter, Chapter 5.

Chapter 5: Interpretations, Conclusions, and Recommendations

The first four chapters culminate in Chapter 5 where the “re-envisioned” instruction model’s effectiveness for increasing students’ mathematics achievement is discussed. This instruction model is grounded in a synthesis of six learning theories and current research from mathematics education, cognitive and educational psychologies, and the neurosciences. During the 2014–2015 school year, two second-grade teachers and their students attending Midwest Elementary School implemented the instruction model in their classrooms. The model structured second-grade teachers’ implementation of Tier I core instruction—the primary prevention component within a Multi-Tiered System of Support (MTSS; Fuchs et al., 2012; Fuchs et al., 2010).

Students in the two experimental groups (or classrooms) received Tier I core instruction using “re-designed” tasks. These tasks focused on mediating the development of students’ mental cognitive structures and numeracy which are vital constructs for learning mathematics. Another second-grade teacher and her students functioned as the control group. Students in the control group were instructed by their teacher using Midwest School District’s mathematics program and curricular resources.

Chapter 5 encompasses a short summary of the study, a discussion of the findings, conclusions, implications, limitations, recommendations for future research, and suggestions for implementing this model at scale. Closing remarks conclude this chapter and dissertation.

Summary of the Study

Researchers claim that low student achievement in mathematics contribute to students' failures as they reach adulthood. Students' future employability, rates of promotion, and their annual income levels are at stake (Geary, 2013; Geary et al., 2009; Jordan et al., 2009). The mathematical skills and competencies learned in elementary school form a critical foundation for students' full adult- participation in a technologically-oriented and information-rich society (Baroody & Ginsburg, 1990). To improve average to below-average students' mathematics achievement, design-based research became the theoretical backdrop for designing an effective Tier I model of instruction and intervention that minimized students' learning difficulties in mathematics (Barab & Squire, 2004; Cobb et al., 2003).

To conduct this study, origins of learning difficulties were reviewed. This was followed by a reexamination of the cognitive mechanisms that support learning. Next, key theoretical constructs and evidence-based processes for teaching and for learning mathematics were identified. All informed the design, implementation, and testing of the conceptual framework known as the “re-envisioned” instruction model. Data analysis revealed that implementing the model in second-grade classrooms statistically improved students' mathematics achievement scores above the mean scores achieved by the control group.

Qualitative and quantitative data were concurrently collected over a four and a half month period (from September 2014–January 2015), as well as one day in May 2015. Quantitative measures consisted of students' pre-, post, and end *i*-Ready Universal Screening Assessment data (Curriculum Associates, 2015). Data results originated from the three non-equivalent student groups involved in the study.

The control teacher and her classroom of students (n=18) are referenced as CLA. The two experimental teachers and their classrooms of students (n=18 each) are referenced as CLB and CLC. Quantitative data and statistical analyses determined each group’s mean increase in mathematics achievement scores and enabled the comparison of mean scores across classrooms. Additionally, Pearson Chi-Square tests were used to detect statistical differences in count patterns representing students’ improvement or regression in instructional Tier Levels from pre- to end of school year. Other statistical analysis measures (Wilcoxon Signed Rank Test; Z tests; CI for Two Proportions) identified changes in students’ development for two specific classes of cognitive structures: conservation of constancy and spatial orientation.

Qualitative instruments for gathering data included semi-structured observations of teacher’s mathematics instruction, teacher and student interviews, questionnaires, and surveys utilizing quantifiable and open-ended response questions. Xeroxed copies and photographs of instructional artifacts were also collected. A researcher’s personal field notebook captured observations and interactions between teacher and students, between researcher and students, and between researcher and teachers. Qualitative measures were used for analysis. Thick descriptions emerged from teachers’ and students’ qualitative data which were coded for themes and transformed into narratives. Some qualitative data were quantified and transformed into numerical codes and percentages.

Data triangulation between multiple data sources established validity of empirical evidence when the following research questions were applied:

1. To what extent did teacher implementation of the “Re-Envisioned” Instruction Model influence students’ mathematics achievement?

2. Did teacher implementation of the “re-envisioned” instruction model minimize students’ learning difficulties in mathematics? Specifically, did implementation of the model move students identified at Tier II and Tier III levels to Tier I and Tier II levels respectively as identified by the pre- to end *i*-Ready Universal Screening Assessments (Curriculum Associates, 2015)?
3. To what extent did teacher implementation of the “re-envisioned” instruction model influence the development of students’ cognitive structures, namely spatial orientation and conservation of constancy?
4. To what extent did teacher implementation of the “re-envisioned” instruction model influence students’ beliefs and practices for learning mathematics?

According to quantitative data results, all three classrooms experienced increases in students’ aggregate mean scores from pre- to post, and from post- to end of study. Statistical comparisons between the two experimental groups and the control group revealed statistically greater increases in students’ mean achievement scores. Teacher implementation of the “re-envisioned” instruction model proved successful for increasing students’ mean mathematics achievement scores, doubling the increase in mean scores experienced by the control group. Qualitative data also revealed differences in student thinking, beliefs, and practices when comparing the control group to the two experimental groups.

Discussion of Quantitative Findings

A diverse collection of teachers’ and students’ data were collected concurrently throughout the study. Students’ quantitative results from the *i*-Ready Universal Screening Assessment (Curriculum Associates, 2015) were analyzed using multiple statistical measures. Results from these analyses were classified as significant, interesting, and puzzling.

Significant findings resulted from the analytical comparisons of pre- to post assessment results from the control group and the two experimental groups. To argue for significance, alpha levels were set at 0.05 for the following test measures. Two-sample t-tests revealed that the two experimental groups' mean increases from pre- to post doubled that of the control group.

Another significant finding revealed that the two experimental groups' increases in mathematics achievement scores were comparable. Even though CLB remained the highest achieving group throughout the study and CLC remained the lowest achieving group throughout the study, students in both experimental groups experienced similar increases in mean scores pre- to post. Data suggested that it did not matter whether students were high or low achievers in mathematics. The number of students in each experimental group who demonstrated at least one-year's growth from pre- to end of study were more than twice the number of CLA students as determined by the *i*-Ready Diagnostic Assessment student growth measures (Curriculum Associates, 2015).

One more significant finding pertained to the statistical data representing achievement gaps between groups. At the beginning of the school year, statistical analyses of students' mathematical achievement data revealed an achievement gap between the control group (CLA) and the lowest-performing experimental group of students (CLC). By post-assessment, this gap statistically closed and remained non-existent till the study's end. While there were no statistical differences in achievement at the beginning of the school year between the control group of students (CLA) and the highest-preforming group of students (CLB), CLB students' mean scores moved ahead of CLA students' mean scores demonstrating a statistically significant difference by post-assessment. This achievement gap expanded from post- till end of study.

An interesting finding resulted from students’ movement in instructional Tier Levels. The school district’s *i*-Ready Assessment data (Curriculum Associates, 2015) and the analysis from Pearson’s Chi-Square Tests indicated students in all three groups experienced statistical differences in count patterns from pre-to-end of school year. Upon closer analysis of *i*-Ready pre- to post- results, two CLA students regressed in instructional Tier Levels, whereas no students from either experimental classroom regressed in instructional Tier Levels pre- to post.

Other interesting and unexpected findings resulted after treatment. As previously stated, the two experimental groups’ increases in mean achievement scores were twice the mean increase of the control group. Three months later, from post- to end of study, two-sample t-test analyses of students’ achievement data indicated that statistical increases in CLA, CLB, and CLC’s mean achievement scores were comparable. From post- to end, students’ similarities in their growth rate may have been influenced by teachers’ collaborative planning for mathematics instruction after the study was completed.

Another interesting finding occurred when CLB and CLC students’ pre-to post-achievement results were compared to their post- to end achievement results. CLB and CLC students did not maintain the same growth rate in mathematics achievement as experienced during the first half of the school year during treatment. This phenomenon may have resulted from lack of coaching support, lack of “re-designed” tasks as instructional tools, or teachers’ lack of mathematical understanding of concepts taught during the second half of the school year.

More interesting findings came from the control group CLA. Students increased their mean scores by a greater growth rate during the second half of the school year when comparing their pre- to post- to post- to end results. These findings suggest that, again, teachers’ collaborative planning of mathematics instruction may have influenced CLA Teacher’s practice, leading to the rate increase in her students’ mean scores.

Finally, puzzling findings originated from the statistical analyses of CLA, CLB, and CLC students’ pre-to- post- assessment results relative to the development of two classes of cognitive structures: spatial orientation and conservation of constancy. Teachers’ implementation of the “re-envisioned” instruction model was intentionally designed to mediate the development of these two mental structures. Wilcoxon Signed Rank Tests and Z tests indicated that only CLA students demonstrated convincing evidence for furthering their development of conservation of constancy. CLB and CLC student data provided no statistical evidence for the same claim. However, all three student groups demonstrated statistical evidence for developing their cognitive structure for spatial orientation.

What were the defining factors influencing statistical differences in CLA, CLB, and CLC students’ quantitative results? Qualitative data analyses detected variations in teachers’ strengths and their practices for implementing mathematics instruction. Implementation support, especially the “re-designed” tasks and the content of those tasks used by the experimental groups, appeared to be contributing factors. Data pertaining to each classroom’s learning environment also presented notable differences. Thus, pedagogical, cognitive, biological, and socio-cultural factors influenced CLB and CLC students’ learning and achievement in significant ways when compared to CLA students.

The upcoming sections discuss plausible causational factors for why these significant, interesting, and puzzling findings may have occurred. Interpretations and inferences target the design of the “re-envisioned” instruction model, teachers’ implementation of mathematics instruction, the “re-designed” instructional tasks, and the learning environments of each classroom. Data triangulation between qualitative and quantitative sources were essential for establishing reliability of inferences made.

Discussion of Qualitative Findings

Comparing teachers’ implementation of mathematics instruction. Qualitative data revealed that all three teachers had been teaching for approximately the same number of years and were familiar with teaching second-grade students. In fact, each teacher had taught at this grade level for a minimum of 5 years. All teachers in this study held equitable education degrees and certifications for teaching literacy, but did not possess strong backgrounds for teaching mathematics. During the previous school year, all three collaborated in lesson design to engage students in sense-making of mathematical ideas. Similarities existed amongst teachers at the onset of the study. Teachers also had defining differences.

Differences in teachers’ strengths. Teachers’ reflections regarding personal strengths for teaching mathematics highlighted differences amongst teachers. CLA Teacher believed her personal strength for teaching mathematics was in posing thought-provoking questions. CLA reported that her professional growth plan focused on questioning her students. She claimed her questions made students think beyond the classroom. The question-type analyses verified CLA’s claim. CLA Teacher asked more extending-thinking questions than either CLB or CLC Teacher. CLA Teacher’s questions inspired her students to make real-world connections beyond their school walls.

CLB Teacher described her strength as wanting to know more about mathematics. She explained that she was willing to commit time and effort to learning mathematics, as well as work to increase student learning by integrating mathematics with other subject areas. When describing her participation in this study, CLB Teacher recorded, “This whole process has been a learning curve. I am learning in a conceptual way. I need to adjust my schema so much it is a struggle. Not in a bad way. I’m proud I get to do this.” Despite her apparent struggles in learning mathematics, CLB Teacher’s strength was embracing a growth mindset to learn mathematics for herself and improve her instruction to benefit her students (Dweck, 2006).

CLC Teacher professed her instructional strength was through the various ways she presented mathematical concepts, allowing students to discover concepts and talk about their discoveries. She also encouraged students to use multiple solution strategies. Classroom observations of CLC Teacher’s instruction suggested that her educational experiences in curriculum and practice enabled her to embrace constructivist and socio-constructivist principles to enhance student learning. Her education enabled her to reconsider the structure of her lessons, implement lessons in ways that met students’ needs, particularly those who struggled learning mathematics. As evidence, at the beginning of the school year, only three students in CLC Teacher’s classroom were identified for Tier I core instruction, while fifteen students were identified at Tiers II and III. By the end of the school year, half her students were identified for Tier I core instruction.

Differences in teachers’ implementation of instruction. Experimental teachers’ implementation of the “re-envisioned” instruction model drew heavily upon six learning theories. Jean Piaget, Lev Vygotsky, Jerome Bruner, David Geary, Reuven Feuerstein, and Betty Garner’s theories explicated neurological networks of cognitive structures strongly influencing student cognition, cognitive growth, and their learning of mathematics. Thus, theorists’ and researchers’ contributions became important constructs in the design of the model, in teacher implementation of mathematics instruction, and for student engagement with mathematical concepts.

Feuerstein et al.’s (2006) Mediated Learning Theory (MLE) influenced the overall design of the “re-envisioned” instruction model. Feuerstein’s mental actions of *input, elaboration,* and *output* provided the iterative processes for students’ mental processing of information during the three instructional segments of the model: launch, exploration, and summary/reflection. Input, elaboration, and output supported students’ internal thoughts and analysis and externalized actions (e.g. intentional discourse, mathematical modeling, and written records) whereby the experimental students mentally engaged with the same mathematical concepts multiple times throughout a lesson.

Piaget’s (1964) constructivist theory proposed philosophical and scientific justifications for utilizing an inquiry-based approach for teaching mathematics. Teachers’ instruction was not about telling students how to think about a concept. Nor did teachers model a mathematical procedure. Instead, teachers worked to activate students’ existing structures (or schema) and invite their mental reflections and cognizing of mathematical ideas.

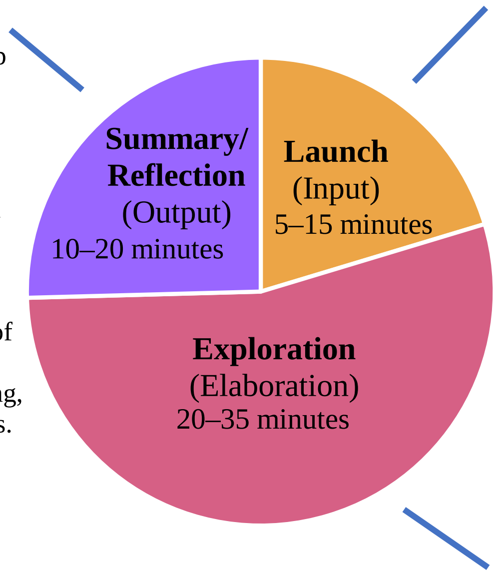
To accomplish this, the experimental teachers utilized specific mathematical representations as sensory *input*, activating their students’ cognitive structures for memory, visual scanning, perception, comparative thinking, and pattern finding. Students were then given time to mentally code these concepts by making sense of them and *elaborating* upon the sensory data. Following elaboration, students constructed iconic and symbolic mathematical representations of their own *(output)* and recorded them in their mathematicians’ notebooks. Students’ choice of words, pictures, numbers, and symbols became personal transformations and externalizations of the original sensory *input*. For the two experimental groups, the figural units within teachers’ and students’ representations exemplified mathematical structure, numeric patterns and relationships, the meanings of operations, mathematical properties, and connections between concepts.

The students in the treatment groups consistently worked to make sense of the vast array of physical, iconic, and symbolic representations CLB and CLC Teachers used in their lessons. CLB and CLC students created, transposed, and transformed their visualized perceptions of mathematical concepts using words, signs, and symbols. Students’ actions for transforming physical representations into iconic and symbolic forms supported their development of visual literacy. Diezmann and English (2001) described visual literacy as one’s ability to read, write, use, think, and learn relative to iconic images. This assertion held true for all three student groups, but especially for students in the two experimental groups. Bruner’s (1966) representational learning theory, including his notions of power and economy, were actualized in both experimental classrooms.

Garner (2007) termed students’ transformations of mathematical representations “metability,” or “the ongoing, dynamic, interactive cycle of learning, creating, and changing” (p. xv). CLB and CLC students often needed to mentally accommodate the information within novel representations to make sense of them and produce their own external representations. Unfamiliar or novel representations incited students’ mental disequilibrium (Fox & Riconscente, 2008). To re-establish mental equilibrium, students used principles from Vygotsky’s socio-constructivist theory to make sense of information. Through teacher facilitation of mathematical discourse, students worked to mentally accommodate their thoughts and mathematical interpretations. Theoretically, accommodation requires students’ construction of new neurological networks, thereby generating new neuronal nodes and linking existing mental pathways (Ifenthaler et al., 2011; Piaget, 1964). The stronger and more integrated CLB and CLC students' mental networks became, the more accessible mathematical concepts became (Wood et al., 1976).

Thus, Garner's theory of metability strongly corresponds to Piaget's constructivism. To readily perceive how learning theorists' suppositions influenced the experimental teachers' philosophical stances and instructional practices, see Figure 10.

- Use student representations, solution strategies, or post a question to mediate the development of students' secondary cognitive structures.
- Facilitate whole group discussion around student ideas.
- Expect students to make connections between concepts and all forms of representations.
- Engage students in reflective awareness of their mathematical thinking, sense-making, and solution strategies.



- Promote inquiry-based stance.
- Use a mathematical representation to activate students' primary schema and prior experiences.
- Use open-ended questions to prompt students to take in and make sense of sensory data and focus students' attention on the figural units and structure within the representation.
- Encourage students to record initial thinking in notebook or on paper.
- Facilitate whole group discussion.

- Select open-ended mathematical task(s) that connect to goals of lesson and extend student learning.
- Support students making connections between enactive, iconic, and symbolic representations.
- Mediate students biologically secondary structures by asking scaffolding questions and engaging students in mathematical discourse.

Figure 10. The "re-envisioned" instruction model's essential instructional practices. A description of essential elements and instructional practices necessary for activating students' cognitive structures.

Differences between teachers’ mathematics instruction were further substantiated through lesson observations. A fine-grained analysis of M-Scan results (Berry III et al., 2012; Walkowiak et al., 2014) suggested that the two experimental teachers’ implementation of the “re-envisioned” instruction model strongly affected two dimensions of their instruction: Mathematical Accuracy and the Structure of the Lesson. CLB and CLC Teachers scored higher than CLA Teacher on both dimensions.

According to the authors of the M-Scan rubric (Berry III et al., 2012), the structure of a lesson relates to a teacher’s propensity to lead students to deeper understandings of mathematical concepts within a lesson. The National Research Council (2012) defined “deeper learning” as a process whereby individuals can apply and transfer knowledge to new contexts and situations. The transcribed and coded analyses of all three teachers’ implementation of lessons pertaining to Composing and Decomposing Numbers and the Change-to-More Diagram (as described in Chapter 4) indicates that CLB and CLC Teachers engaged their students in deeper learning during those lessons, more so than CLA Teacher.

CLB and CLC students recorded mathematical connections to previously-learned models, and their extensions of models provided evidence of their deeper, transferrable learning to other mathematical problems and concepts. Mathematical connections were represented in students’ writing and drawings of visual representations. Connections were also evidenced in their classroom discussions when they explained and elaborated upon their own mathematical thinking and the thinking of their classmates.

Triangulating these claims to students’ qualitative responses, supporting evidence was found in students’ mathematicians notebooks. Analyzing their own work, CLA, CLB, and CLC students noted evidence of specific ways they grew as mathematicians throughout the school year. Words, numbers, drawings, depictions of mathematical models, and symbolic representations within students’ notebooks became windows into their own minds (Woleck, 2001).

When comparing CLA students’ analyses to CLB and CLC students’ analyses, greater percentages of CLB and CLC students noted increases in their learning and understanding, in the amount and depth of their written explanations, and in the amount of evidence they provided.

One CLB student reported,

> I noticed that I am using more harder [words] than before. I noticed that we have changed the subject of math a lot. I noticed that in the past I didn’t use too much precise math words. I noticed that I used to just work to find the answer and now I just show evidence…I noticed that I barely had any thinking and now I have a butload of thinking on my page.

A CLC student described, “At the beginning I had only 1 or 2 ideas. But now I have all most 5 or 6 ideas [on a page].”

Data triangulation substantiates students’ analyses. CLC Teacher witnessed her students making more connections between mathematical ideas than students from previous years. She claimed that these students linked equations to games and added on to another students’ thinking. CLC Teacher reported, “This is the first year I have focused on the concept of visualization in math. Getting students to talk about what they see in their heads helps them talk about mathematics.”

CLB Teacher described that her students were now making connections and perceiving relationships between mathematical models. For example, she witnessed students making connections between the number grid, a number line, and arrow roads. CLB Teacher stated, “It is like students are connecting the dots. The visual of the number line is helpful. It makes you see that everything is connected.”

CLB and CLC Teachers’ launches and summary/reflections of their lessons “made serious use of students’ thinking” (Ball, 2001, p. 11). The experimental teachers first expected their students to make sense of the mathematical representations. Students focused their attention upon the figural units embedded in *novel* mathematical representations (enactive, iconic, or symbolic forms), in student-generated solutions, and in the problem-solving strategies students employed. Figural units are defined as the different elements or attributes a student “quickly recognizes as significant or informative” (Duval, 2014, p. 160).

CLB and CLC Teachers provided students time to construct meaning by attending to the informative “figural units” using their perceptions, memory, and comparative thinking structures. Students then recorded their “noticings” in their mathematician’s notebooks. Afterwards, teachers encouraged students to publicly discuss their noticings and interpretations with each another and collectively with the entire class. In both CLB and CLC classrooms, mathematical discussions increased in substance over time and often highlighted similarities and differences between students’ interpretations and representations of their thinking (Diezmann & English, 2001).

During summary/reflections of lessons, the two experimental teachers expected students to re-engage with the lesson’s concepts. They encouraged students to reflect upon their learning by comparing different solution strategies, addressing students’ misconceptions, and clarifying mathematical vocabulary. These observations corroborated with the analyses of teachers’ question-types. CLB and CLC Teachers asked more “exploring mathematical meanings and relationships” and “synthesis” type questions than CLA Teacher. CLB and CLC Teachers’ questions prompted students to make sense of underlying concepts, connect meanings of operations to diagrams, and perceive numeric relationships. Teacher questions and novel tasks provided students’ more opportunities to reflect and solve problems requiring students’ original and creative thinking.

Contrastingly, during CLA Teacher’s launches for her lessons, she often asked students to focus on making sense of *familiar* mathematical representations in the form of story contexts and real-world examples. Students then constructed drawings and used symbols representing *equivalent variations* of the original stimulus. CLA’s representations of real-world contexts aligned to current brain-based research which asserts that the more background or familiarity students have with the subject matter, the quicker they cognitively process new concepts (Jensen, 2000).

During lesson observations of CLA Teacher’s summary/reflections, students often reflected upon their learning by recording their personal perceptions and representations in their mathematicians’ notebooks, but they did not share their insights publicly. Although student data indicated that CLA Teacher conducted a greater percentage of summary/reflection segments than CLC Teacher, lesson analyses revealed that very little time was dedicated to support and sustain student-generated generalizations in CLA’s classroom. CLA Teacher explained,

> I have looked at the lessons and now realize that the lessons in this program are not engaging enough. I don’t think I got the ‘digging deeper’ questions like I got from you and [our math coach]. These were too challenging to do.

Facilitating student discourse relative to mathematical concepts is a vital process for students learning mathematics (Vygotsky, 1930/1978). These experiences afford students opportunities to acquire more sophisticated language, glean new insights, and gain conceptual understandings of complex ideas such as mathematical operations, numbers, and place value (Kozulin, 2002; Rubenstein et al., 2004).

Differences in instructional practices influencing cognitive structure development. The review of learning theories and origins of students' learning difficulties suggested that students' underdeveloped cognitive structures were a significant reason for students' low achievement in mathematics (Feuerstein et al., 2006; Garner, 2007, 2013, Geary, 1995). Pre- to post- data analyses from all three classrooms suggested that many students statistically increased their development for spatial orientation. Statistical results indicating students' development for conservation of constancy were different. Only CLA student data indicated statistical differences for conservation of constancy, pre- to post. For CLB and CLC student groups, pre- to post- statistical increases were not evident.

Possible factors for CLA students making gains relative to conservation of constancy, when both experimental groups did not, may be attributed to CLA teacher's emphasis on constructing equivalent representations. Frequent constructions of representations that are equivalent, yet different in structure or appearance, may have supported CLA students' understanding that iconic representations can look different, yet still depict the same concept and maintain the same meaning. Another probable cause for these differences may be attributed to CLA Teacher's continual practice of introducing mathematical concepts using simple story contexts. This practice afforded students' familiarity when learning new concepts. Theoretically, familiarity with concepts incites mental assimilation (von Glasersfeld, 1990). Mental assimilation involves students' repeated experiences with familiar representations, strengthening existing mental networks, and creating more efficient flows of electrical current within their minds (Devlin, 2010; Ifenthaler et al., 2011).

As for their mathematical tasks, CLB and CLC students consistently engaged with a variety of novel representations. Novel representations necessitate mental work for accommodating new information (Dehaene, 2010; von Glasersfeld, 1990). Theoretically, accommodation is a more challenging mental activity than assimilation. New neuronal nodes and networks are created during accommodation to re-establish mental equilibrium (Ifenthaler et al., 2011). To accommodate concepts, students needed time to reflect, and work, and reflect again to gain a deeper, transferable, and lasting understanding of mathematical concepts.

Another reason for this difference in students’ outcomes may be attributed to students developing their cognitive structures at different rates. However, one may conjecture that the students who obtained the highest mathematics achievement scores would have increased their development of conservation of constancy. This was not the outcome in this study. Perhaps student development is not necessarily related to maturation, but to the types of experiences students engage in (Bruner, 1966; City et al., 2009; Henningsen & Stein, 1997). This hypothesis would require a fine-grained analysis of the “re-designed” tasks. An analysis may reveal that the experimental students needed additional opportunities to engage with representations by noting the figural units that remained constant and compare them to the figural units that changed, which is the essence for conservation of constancy (Garner, 2007). Mathematical tasks and the ways students engage with those tasks do make a difference in student learning and in students’ cognitive development.

Differences in instructional tasks and mathematical content. The instructional tasks used by the two experimental groups during the four-and-a-half months of treatment appeared to influence the statistical differences in student achievement when compared to the control group. Instructional materials such as “re-designed” tasks and Number Talks (Parrish, 2010) provided the experimental teachers and students opportunities to focus on developing students’ sense of number and operations. “Re-designed” tasks were used exclusively by CLB and CLC Teachers and students.

NCTM (2000) defined number sense as having a fluidity and flexibility for thinking about and working with number and operations, including “moving from initial development of counting techniques to more-sophisticated understandings of the size of numbers, number relationships, patterns, operations, and place value” (p. 79). Developing number sense in young students requires numerous opportunities to visualize quantities in a variety of contexts, compose and decompose numbers flexibly, subitize, estimate, count, and solve basic arithmetic combinations and story problems. (Bryant, 2005; Jordan et al., 2006; Muldoon et al., 2012). Number sense concepts and skills defined the content of the “re-designed” tasks’ that CLB and CLC Teachers engaged their students with.

“Re-designed” tasks were selected or designed based upon Geary’s (1995) theoretical notions for mediating the development of students’ biologically secondary structures. This was accomplished by considering Vygotsky’s (1978/1930) zone of proximal development and Bruner’s (1966) three modes of representations. For example, some mathematical representations involved task cards comprised of specific quantities of dots, as well as specific patterns of dots signifying numeric relationships. These cards were used to activate students’ existing biologically primary structures for memory, visual scanning, subitizing, spatial orientation, and language. Activating and engaging primary structures furthered students’ development for number sense while inducing higher-levels of cognitive thought processes requisite for learning advanced mathematics (Clements & Samara, 2009; Devlin, 2010; Geary, 1995).

Bruner’s (1966) representational learning theory also influenced the design of tasks implemented in the two experimental classrooms. CLB and CLC students used enactive representations such as meter sticks and translucent counters, multi-link cubes, and base-ten blocks to visualize mathematical concepts. Base ten blocks supported students composing and decomposing a quantity in equivalent ways. Many students identified numeric relationships and “saw” mathematical structure when engaging with these materials. After multiple decompositions of a specified quantity, one student generalized, “The more tens you have in the tens place, there are more ways to show 63 [that number].”

Bruner’s (1966) ideas for power and economy of a representation, regardless of the level of abstraction, was represented in enactive recognizable forms students interpreted, understood, and generalized. Meter sticks and counters facilitated students visualizing equal-distance (or differences) between multiples of ten and twenty. They identified numeric relationships and developed a deeper understanding of the patterns within the base-ten number system. Students also made explicit connections between a meter stick, a thermometer, and an open number line. Multi-link cubes supported students visualizing and exploring concepts of equalities and inequalities, addition and subtraction, repeating and growing patterns, and enabled students to generate new patterns by identifying algebraic rules.

According to Vygotsky’s (1978/1930) zone of proximal development, teachers’ artful scaffolding from enactive, to iconic, to symbolic representations enabled CLB and CLC students to go beyond their own natural cognitive endowments (Wink & Putney, 2002). For example, reflective engagement with a hundred-chart game and an arrow road task supported students adding and subtracting two- and three- digit numbers even before teachers formally introduced these concepts. The meter stick and open number line models assisted students rounding to the nearest benchmark number and extrapolating to adding greater numbers. This example is further illustrated when a student used a hundred chart to think about adding 5 + 3. The student then mentally calculated the sum for 50 + 30. As a final connection, she successfully extrapolated that pattern for adding 150 and 130. As research studies demonstrate, young students’ use of enactive and iconic (visual) representations of mathematical concepts enable them to transfer concepts to other problems, and master and maintain mathematical competencies in later years (L.S. Fuchs & D. Fuchs, 2001).

CLB Teacher reported,

> This is the first year we focused on the concept of number relationships. We had always assumed that first grade teachers had done that, but thinking about ten more and ten less, one more, one less, just the ideas of more or less, we, as second-grade teachers, have never focused on.

Number sense concepts and skills supported CLB and CLC students acquiring a deeper understanding of numbers and operations. The experimental groups’ qualitative data suggested they assimilated and accommodated concepts and transferred their understandings to concepts not yet formally introduced.

Contrastingly, CLA Teacher engaged her students with instructional tasks derived from the school district’s mathematics program, their district’s Benchmark Assessments, and Midwest School District’s second-grade mathematics units of study. Students’ daily tasks involved participating in mathematics games using decks of cards, reading information from their student’s reference manuals, using enactive tools such as base ten blocks, counters, and hundred charts. Mathematical concepts consisted of composing and decomposing number, computing basic facts, solving problems using money, measuring temperature, and using different models and strategies to add, subtract, multiply, and divide, including the use of traditional algorithms.

Ultimately, when comparing tasks, CLA Teacher's explanation accurately justifies the statistical differences in students' mean achievement scores between the control group and the two experimental groups:

> The lessons don't dig deep, don't have the same types of questions. The district has some good lessons, good parts of lessons, but they do not have the same quality for when we do the new workshop ["re-envisioned" instruction model].

Henningsen and Stein (1997) claimed, "the nature of tasks can potentially influence and structure the way students think" (p. 525). Doyle (1988) argued that the tasks students engage with are the "proximal causes" of their learning. Meaningful mathematical tasks cause students to think more deeply about mathematical concepts and structure. They support students in developing conceptual understanding and procedural knowledge and assist them in making connections between concepts and models. Quality tasks fall within students' zone of proximal development and, therefore, are central to deepening and extending students' conceptual understanding.

Differences in the socio-cultural environments of classrooms. A third variable influencing student achievement between classrooms were the classrooms' socio-cultural environments (Cobb, 1994; Fuson, 2009; Mooney, 2013; Nelson et al., 2001). All three teachers promoted constructivist and socio-constructivist principles within their classrooms to some degree. Importantly, they considered all students as mathematicians; and, all three teachers maintained that their students perceived themselves as mathematicians.

CLA Teacher's analysis of student behaviors within her classroom, however, indicated her students held differing beliefs for learning mathematics than the students in the two experimental classrooms. Student data substantiated CLA Teacher's inferential analyses. One significant difference between CLA students and CLB and CLC students involved their definitions of mathematician's and the work they do. When coding and comparing students' qualitative responses, the majority of CLA students described, "Mathematicians are smart and they solve problems." Dweck (2006) explained that students who hold fixed mindsets attribute their success with smartness, not effort.

Alternatively, when describing mathematicians and the work they do, over 50% of CLB and CLC students used comments like, "Mathematicians use their brains to think. They share and explain their thinking. They use tools and make connections." CLB and CLC students' collective responses suggested that their environments fostered growth mindsets. A growth mindset is not about smartness, but about one's effort at thinking (Dweck, 2006).

Correspondingly, differences in teacher's perceptions about the purposes for students' mathematician's notebooks also existed. For example, the two experimental teachers viewed students' mathematician's notebooks as a record of student thinking and as a resource to share students' thinking with others. In fact, CLB and CLC Teachers' statements suggested that learning mathematics was all about thinking! The teacher of the control classroom (CLA) viewed the notebook as a daily repository of student work.

Another distinguishing difference between socio-cultural environments in each classroom revealed that the types of instructional tasks teachers engaged their students with significantly influenced students' perceptions for what it means to know and do mathematics. For instance, when asked what they wanted to better understand relative to learning mathematics, most CLA students responded that they wanted to learn more about operations, specifically multiplication and division. Four CLA students reported that they already knew everything and that there was nothing more to learn. Scardamalia and Bereiter (2006) argued that “students in regular classrooms tend to say that the more they learn and understand, the less there remains to be learned and understood” (p. 104).

Students in knowledge-generating classrooms lean toward the opposite view. Students in knowledge-generating classrooms realize that the more they learn, the more information they do not know. Accordingly, CLB students recorded they wanted to learn more about measurement, quantity, arrays, benchmark numbers, odds and evens, and how to use the number line to solve problems. Not only did CLB students want to learn more, but they expressed they were seeking understanding and meaning of concepts. Similarly, CLC students claimed they wanted to learn how to explain numbers and number relationships, learn equations, use an open number line, tell time, learn addition and multiplication, understand division and fractions. Only one student in CLC’s classroom reported there was nothing more to learn.

Thus, with only one exception between the two experimental groups, the students who experienced the “re-envisioned” instruction model, exemplified growth mindsets. Students desired to gain a deeper understanding of mathematical concepts. Their responses revealed their personal awareness (or metacognition) of where they were on the learning continuum: “Number relationship’s because it is hard to explan”; “Like expaning your thinking becase sometimes it is hard for me”; “Maltaplacashon because I can’t understand what Maltaplacashon is all about I mean what’s the point if you don’t know”; and “÷ because I do not really understand it. I do not really now the mening of it.”

As the school year ended, teachers provided a final reflection. CLB Teacher reported, We feel we are stronger mathematicians as adults going through this process; and, when you delve deeper into number sense and Number Talks books, you understand more. Looking at the distance between numbers on the number line for subtraction, making students think about what they know, talking through the process out loud, listening to what each other is saying. How limitless it can be!

CLC Teacher testified,

The data you shared with us shows us that no matter where kids are in math, they can make gains and close the gap. I've never had a class of seven outliers before that were really low. The connections we've built as mathematicians like making connections with the number grid, number relationships, inverse operations, and place value. They [low-performers] are now more willing to try something because they have that schema to fall back on. I really see them making connections now.

CLA Teacher claimed,

I think I would have gotten similar results [CLB and CLC] had my class been part of the study. I have kids who performed between the two classrooms. I did do the math workshop technique, but I did not do Number Talks [2010] and no Van de Walle [2014]. I did follow the games but not any that were not from our [mathematics program]. I would find myself resorting to the [Re-Envisioned Instruction Model], then I would have to think again. Our district's lessons do not have the same quality as the new workshop.

Addressing Four Origins of Students' Learning Difficulties

At the beginning of this study, it was hypothesized that the following make a difference in students' achievement outcomes:

- the ways teachers *activate and engage* students' existing cognitive structures

- the ways teachers’ *design, select, or modify mathematical tasks*
- the ways teachers’ and students *engage with those tasks*
- the ways teachers and students *structure their learning environment*

When comparing qualitative and quantitative data results between the two experimental groups and the control group, stark differences were revealed in students’ mean increases in mathematics achievement scores from pre- to Post, which were then validated by teachers and students’ qualitative coded responses. The qualitative analyses indicated that CLB and CLC students’ beliefs and practices for learning mathematics were greatly influenced by:

- teacher’s implementation of the “re-envisioned” instruction model;
- the mathematical tasks they used (Henningsen & Stein, 1997);
- student use of higher-level cognitive processes such as visualization, analysis, evaluation, and generalization;
- teachers’ question-types (Boaler & Brodie, 2004);
- students’ expectations for seeking meaning and justifying one’s thinking in visible and public ways;
- the quantity and focus of mathematical discourse (Herbel-Eisenmann, 2009; Vygotsky 1978/1930); and
- the classrooms’ socio-cultural learning environments that supported a growth mindset (Dweck, 2006; Haberman, 1991).

The above practices and cultural beliefs address four of the five origins of students learning difficulties. These include biological, cognitive, socio-cultural, and pedagogical. Tasks that fell within students’ zone of proximal development were used to increase students’ readiness to learn mathematics. ZPD addresses the biological origin of students’ learning difficulties. “Re-designed” tasks often took students beyond their current level of understanding as in adding and subtracting multi-digit numbers before formal introduction of the algorithms.

The cognitive origin of students’ learning difficulties was targeted through tasks that focused on developing students’ sense of number. Number Talks were used to increase students’ abilities to visualize quantities in a variety of contexts. The use of enactive tools engaged students in discussing numerical magnitude and composing and decomposing numbers in flexible ways.

The socio-cultural origin was addressed by providing all students time to make sense of novel representations. Students made connections between models and concepts. They also recorded, explained, and justified their thinking in their notebooks and discussed their insights, understandings, and misconceptions with classmates. As teachers recorded students’ ideas onto chart paper, students’ ideas were honored, accepted, and acknowledged.

Coaching support addressed the pedagogical origin of students’ learning difficulties. The two experimental teachers requested implementation support for the instruction model and mathematical tasks. Most often, teachers were unfamiliar with the novel representations and requested implementation support for the first lesson. Teachers then implemented subsequent lessons using the model. In coaching meetings, discussions often pertained to teacher’s mathematical understanding of specific concepts. Lesson plans were also provided to assist teachers in making important shifts in instructional practices.

Conclusions: Answering the Research Questions

The first research question was, “To what extent did teacher implementation of the ‘re-envisioned’ instruction model influence students’ mathematics achievement?” Statistical differences in students’ increases in mean achievement scores pre- to post between the students in the two experimental groups and students in the control group were evident. One needs to first return to the definition for learning established back in Chapter 1 to support rejecting the null hypothesis.

Direct sensory exposure to environmental stimuli and mediated exposures to social knowledge support students creating, learning, and altering the neurological structures within their own minds (Garner, 2007). Theoretically and neuro-scientifically, creating and strengthening one’s own mental neurological structures signifies learning has occurred (Devlin, 2010; Ifenthaler et al., 2011). Outward signs of learning include, but are not limited to, one’s ability to “make connections with prior knowledge and experiences, identify patterns [and relationships], identify predictable rules, and abstract generalizable principles” that are then applied to additional contexts and conditions (Garner, 2007, p. xiii).

When compared to the students in the control classroom, the triangulation of quantitative and qualitative data suggests that the experimental teachers’ implementation of the “re-envisioned” instruction model and “re-designed” tasks advanced students’ mathematics achievement in statistically significant ways pre- to post-. The analysis of teacher and student qualitative data indicated that the experimental groups were expected to make connections with prior knowledge, visualize the figural units within novel representations, identify patterns and mathematical structures, record their own perceptions using multiple representations, make connections to other concepts, and abstract generalizable principles.

Kamii, Kirkland, and Lewis (2001a) explained that when teachers and textbooks focus and foster children’s higher levels of abstraction, “high levels of representation will follow” (p. 32). Thus, CLB and CLC students’ external abstractions often exemplified their perceptions, connections, and level of understanding and exhibited their high levels of abstraction (Kamii, Kirkland, & Lewis, 2001a). Theoretically, the mathematical connections the experimental students made imitated the neuronal connections within their minds (Van de Walle et al., 2012). These webs of connected concepts became essential building blocks for developing CLB and CLC students' cognition and cognitive growth relative to number and operations (Clements & Samara, 2009).

The *i*-Ready Assessment measures (Curriculum Associates, 2015) also indicated that the number of CLB and CLC students in each classroom who experienced at least one year's worth of growth in mathematical understanding were more than twice the number of students in the control group. Theoretically, the "re-designed" tasks and mathematical content met CLB and CLC students' instructional needs for numbers and operations. Through the intentional selection of enactive, iconic, and symbolic representations representing abstract concepts, CLB and CLC Teachers prepared and nurtured their students' readiness for conceptually understanding abstract concepts before they were formally introduced (Bruner, 1977; Wink & Putney, 2002). Employing Bruner's (1966) power and economy of representations and Vygotsky's ZPD, prompts, open-ended questions, and mathematical representations supported students going beyond their own natural abilities (Wink & Putney, 2002). For instance, various enactive and iconic representations enabled students to successfully add and subtract multi-digit numbers before teachers taught these formal algorithmic procedures.

Furthermore, the fact that each segment of the instruction model facilitated the cognitive processes of input, elaboration, and output, CLB and CLC students engaged and re-engaged in cognizing abstract concepts multiple times within the same lesson. During the summary/reflection of the lesson, the higher-level of mathematical discourse that the experimental students participated in supported students becoming self-regulators of their own learning. It is plausible that these strategic elements of design supported CLC’s students closing the achievement gap between themselves and the control group and CLB’s students widening the achievement gap between themselves and the control group.

By the study’s end, each teacher was grateful for her participation in the study and was ready to independently apply the practices she learned. CLA Teacher looked forward to rejoining her colleagues to collaboratively plan mathematics instruction. CLB and CLC Teachers were ready to collaborate with her. Due to their co-planning of mathematics instruction from post- to end, CLB and CLC’s experiences with the model appeared to have influenced CLA Teacher’s instructional practices. Collaborative planning may have contributed to the increase in CLA students’ mean scores for mathematics achievement during the second half of the school year. Alternatively, CLB and CLC students’ mean increases post to end were less than previously experienced. To maintain a similar increase or growth rate in mathematics achievement during the second half of the school year, the data suggests that the two experimental teachers may have benefitted from additional implementation support, including “re-designed” tasks.

Research Question 2:

> “Did teacher implementation of the “re-envisioned” instruction model minimize students’ learning difficulties in mathematics? Specifically, did implementation of the model move students identified at Tier II and Tier III levels to Tier I and Tier II levels respectively as identified by the pre- to end *i*-Ready Universal Screening Assessments (Curriculum Associates, 2015)?

According to the statistical analyses of the *i*-Ready Assessment data for comparing the control classroom and the two experimental classrooms’ count patterns relative to reducing students’ needs for Tier II and Tier III interventions, individual classroom count patterns were not statistically different from each other. Count patterns in all three classrooms demonstrated positive movement in instructional Tier Levels. From pre- to post-, however, the data analyses indicated two students in the control group regressed to Tier II and III instruction levels. No students in the two experimental groups experienced this same regression. This data answers the second question in the research study. The null hypothesis is accepted because, although each classroom’s count patterns changed significantly pre- to post-, the count patterns were not significantly different in the experimental classrooms vs. the control classroom.

Research Question 3 was, “To what extent did teacher implementation of the “re-envisioned” instruction model influence the development of students’ cognitive structures, namely spatial orientation and conservation of constancy? The statistical analysis of students’ cognitive structure assessment results supports rejecting the null hypothesis for this question. Data analyses suggests that only CLA students statistically advanced their development of cognitive structure for conservation of constancy. This was most likely due to CLA students’ continuous engagement with creating equivalent mathematical representations. It is plausible that students mentally noted the elements that stayed constant and those that changed while representing multiple representations of the same concept. For the cognitive structure of spatial orientation, students in all three classrooms experienced statistical increases in their advancement for this cognitive structure. These findings present a puzzling aspect of this study.

To respond to Research Question 4, “To what extent did teacher implementation of the “re-envisioned” instruction model influence students’ beliefs and practices for learning mathematics?,” we turn to qualitative results and analyses. Although CLA Teacher claimed her students saw themselves as mathematicians, her students did not necessarily perceive their responsibilities as thinking deeply about mathematics. CLA Teacher attributed this phenomena to her district's mathematics program, to the lessons she was required to teach, and the questions she developed autonomously because she did not receive coaching support nor support from her colleagues.

To verify CLA Teacher's claims, three CLA students claimed they needed harder problems or paper and pencil problems, while four CLA students claimed there was nothing more to learn. These students believed they knew everything they needed to know about the mathematics they were learning in CLA's classroom. Perhaps CLA Teacher's instructional practice for generating equivalent representations and simple story problems may have undermined students' higher levels of thinking, representation, and abstraction.

Essentially, teachers and students who implemented the "Re-Envisioned" Instruction model and "redesigned" tasks experienced key pedagogical practices and processes necessary for

- developing conceptual understanding of number and operations;
- cognitively engaging students beyond their natural abilities;
- supporting students' sense-making of novel representations;
- visualizing and making mathematical sense of the figural units within representations;
- analyzing and abstracting generalizable principles;
- engaging young students in high-levels of discourse;
- fostering students' growth mindsets;
- influencing students' beliefs and practices for thinking deeply about mathematics; and
- supporting students becoming self-regulators of their own learning.

The null hypothesis for Research Question 4 was rejected. Data analysis revealed qualitative differences in students' beliefs and practices between students who experienced the "re-envisioned" instruction model and students who did not.

Implications

Barab and Squire (2004) explained, “Design-based research that advances theory but does not demonstrate the value of the design in creating an impact on learning in the local context of the study has not adequately justified the value of the theory” (p. 6). The “re-envisioned” instruction model was implemented in authentic classrooms to provide insights to school districts and staffs implementing MTSS programs. Authentic settings assist researchers in determining causational factors for why and how an intervention or treatment works (Barab & Squire, 2004; Brown, 1992; Cobb et al., 2003).

Implementing this innovative instruction model to minimize students’ learning difficulties displayed the complex and intricate nature for teaching and learning mathematics. Identifying, isolating, and controlling the many variables impacting this study’s findings was challenging. It is not surprising that researchers study only one aspect relative to effective instruction or student learning.

The synthesis and coordination of all six theories and the evidence-based instructional practices led to the creation of several conceptual frameworks represented by the “re-envisioned” instruction model and its implementation. Two conceptual frameworks were depicted and explained near the end of Chapter 2. For example, MLE’s constructs of input, elaboration, and output are represented within each learning segment, respectively: the launch, exploration, and summary/reflection. Through iterative engagements with input, elaboration, and output, teachers and students mentally engage and reengage with a lesson’s mathematical concepts multiple times throughout a lesson.

The literature review also highlighted specific design elements that guided the “re-design” of instructional tasks used by teachers to support Tier I core instruction. Bruner’s (1966) theoretical concept of power and economy guided the strategic selection of enactive, iconic, and symbolic representations that CLB and CLC Teachers used with their students. Several of the “re-designed” tasks embodied key figural units pertaining to mathematical structures, patterns, and meanings of operations and equalities. Dependent upon where students were on the learning trajectory for number sense and number and operations, students’ attention was focused on vital concepts and skills such as subitizing, visualizing, counting and comparing quantities, identifying number magnitudes, composing and decomposing number, observing numeric patterns and relationships, and making sense of operations and mathematical properties. CLB and CLC Teachers reported that the quality of the tasks they used with their students made a difference in their students’ mathematical thinking and understanding.

The efficacy of teachers’ open-ended questions, such as: “What do you notice?”; “What sense can you make of this?”; “What information do you know for sure?” cannot be undervalued. Not only did these questions invite all students to activate their cognitive structures, they were also an effective scaffolding tool. These questions invited students to “notice,” think, reflect, and make sense of the figural units and mathematical structures within the representations. All students noticed something. Thus, open-ended questions became powerful and economic tools for meeting each student at his or her individual cognitive level in CLB and CLC classrooms (Bruner, 1966).

Constructivism and socio-constructivism were also vital philosophical stances that supported CLB and CLC Teachers and students constructing understanding *of* concepts, as compared to *about* concepts. Novel mathematical representations necessitated students’ mental accommodation of mathematical concepts, resulting in the creation of more powerful and memorable ideas (Nelson, Warfield, & Wood, 2001). Theoretically, the mental processes for accommodating information supported students expanding their existing mental networks or schema, not just filling their minds with rote-memorized procedures. More research is needed to identify mathematical tasks that mediate students’ cognitive structures for conservation of constancy.

Implementing the “Re-Envisioned” Instruction Model at Scale

American schools are under extreme scrutiny and pressure for increasing student achievement (City et al., 2009; Cobb & Jackson, 2011). Researchers have intensified their efforts to identify ways students learn, provide evidence-based practices, document learning progressions, and create research-based instructional materials (Cobb & Jackson, 2011). According to Stigler and Hiebert (1999), these initiatives have had limited effects upon teachers’ classroom instruction.

City and her colleagues (2009) argue that to improve student achievement at scale, improvement can only occur by focusing on the following:

- expand teacher’s knowledge and skills for teaching mathematics;
- increase the rigor of the content; and
- change the role of the student when engaging with mathematical content.

These three elements comprise the instructional core. If one attempts to “change any single element of the instructional core,” the other two must also change if student achievement is to improve (City et al., 2009, p. 25).

Teachers’ implementation of the “re-envisioned” instruction model and tasks required attention to all three elements found in the instructional core. CLB and CLC Teachers consistently worked to increase their content and pedagogical knowledge to implement the instruction model and “re-designed” tasks with fidelity. Novel tasks often created cognitive dissonance for CLB and CLC Teachers and challenged and expanded the limits of their existing content and pedagogical knowledge. Therefore, the instructional modeling of lessons scaffolded the experimental teachers’ implementation of the open number line, arrow roads, etc. Informal interviews and collaborative meetings provided time for teachers to reflect upon the effectiveness of the model, their implementation and instructional practices, as well as student understanding. Coaching support addressed teacher questions and feedback. Teachers and students were given instructional resources that focused on developing students’ conceptual understanding for number and operations and for mediating the development of their cognitive structures.

Students who experienced the “re-envisioned” instruction model and “re-designed” tasks were engaged in learning mathematics differently than the control group. As one student recorded, “I noticed that we have changed the subject of math a lot.” Novel representations pressed students to accommodate abstract information. To support students’ mental accommodation of these concepts, time was essential. Students needed time to become reflectively aware of the sensory data and use their existing schema to make sense of that data. Students were encouraged to focus upon the figural units embedded within enactive, iconic, and symbolic representations through open-ended prompts. They worked to visualize these concepts within their minds. Students’ entries in their mathematicians’ notebooks presented evidence of students’ observations, perceptions, transformations, and reflective thinking. Students’ verbalizations and depictions of solution strategies and mathematical representations enabled students to make connections between their own and their classmates’ ideas. Collectively, these practices enhanced CLB and CLC students’ abilities to connect to previously-learned models, diagrams, symbols, and generalize abstract concepts.

Cobb and colleagues (2003) explained that educational improvement is a generative process. For the two experimental groups at Midwest Elementary, teacher participation in this study helped teachers develop pedagogical toolkits of essential practices they still use today. The mathematical experiences students encountered help them generate growth mindsets for learning mathematics and minimize their learning difficulties.

Cobb and Jackson (2011) suggest five components that delineate a theory of action when implementing a program, treatment, or intervention on a grander scale:

- provide a blend of support for teachers, including professional development and job-embedded experiences;
- create teacher cohorts or networks that offer collegial support;
- provide job-embedded coaching support that is timely;
- train school leaders to become instructional leaders; and
- build school-level capacities to support instructional improvement.

Time is essential for theories and pedagogical practices represented by the “re-envisioned” instruction model to become the norm for practice in teachers’ classrooms. Teachers need ongoing support to become effective and efficient implementers of the model. Coaching can support teachers becoming reflective practitioners. Students need time and support to become deep thinkers of mathematics to realize the lasting effects of the model. It is likely that successful implementation of this instruction model may require a one- to two-year implementation period. This implies that the greatest influence of teachers’ and students’ implementation of the model relative to student achievement has yet to be realized at Midwest Elementary School.

Limitations

There were several limitations which may have influenced this study’s findings. The first limitation was the complex theoretical design of the instruction model. The instruction model’s synthesis of six learning theories and evidence-based research demanded that the theoretical constructs, processes, and instructional practices be implemented simultaneously, subsequently, or iteratively. It was challenging to isolate and detect if teacher’s enactment of specific processes from one theory strongly impacted students’ learning relative to processes from a different theory.

A second limitation was that all three teachers learned about the instruction model the previous school year. Due to CLA Teacher’s responsibility as the control group, she was instructed to use her district’s instructional resources to teach her students mathematics. This led to CLA Teacher comparing the efficacy of her district’s program to the efficacy of the “re-envisioned” instruction model. Through her comparisons, this teacher realized her district’s mathematics program did not contain the cognitively-challenging tasks that supported students’ development of deeper mathematical thinking.

Naturally, CLA Teacher struggled in philosophically staying true to the design of her district’s mathematics program. She implemented the three instructional segments (launch, explore, summary/reflection) even though these were not part of her written curriculum. She used mathematical representations from her district’s program to activate student thinking and their reflective awareness, and facilitated student dialogue in her classroom during the launch and exploration segments of her lessons. Subsequently, CLA Teacher was not a pure control classroom relative to *all* aspects of the study. She was a pure control classroom with respect to the “re-designed” tasks and she did not receive coaching support for teaching her curriculum.

Had all three teachers not experienced implementation support the previous school year, it is logical to predict the possibility of two scenarios: (a) a starker contrast between the control population and the experimental populations regarding students’ mean increases in their mathematics achievement scores; (b) the two experimental teachers may have experienced an even greater challenge implementing the instruction model and “re-designed” tasks during the study.

A third limitation relates to implementation support provided to the two experimental teachers. Although these teachers received some implementation support the previous school year, they were not ready to independently implement the model and “re-designed” tasks at the beginning of the study. Lesson plans and resources that guided their implementation had not been established. Coaching, instructional tasks, and modeling of lessons were provided to teachers as they requested. As a result, the researcher’s presence at the school site and the implementation support provided to teachers may have influenced students’ outcomes thereby limiting *Grande generalizations* to other contexts and age-groups (Barab & Squire, 2004).

A fourth limitation to this study pertained to the small sample size of teachers and students. At the beginning of the school year, each classroom consisted of 21 students. Due to specific needs of some students, not all students participated in the study. For the quantitative portion of this study, n=54 students across all three classrooms. For the qualitative portion of the study, n = 33 across all three classrooms. A small sample size limited the potential for making inferences and generalizations that could be attributed to a larger population (Johnson & Christensen, 2012).

A fifth limitation was that true experimental studies require random sampling (Creswell, 2009). Since this study was conducted in authentic classroom settings, the study used non-equivalent groupings. Prior to this study’s commencement, the school’s principal and first grade teachers predetermined student placement with specific second-grade teachers. The principal independently selected the classrooms of students who received treatment and the classroom who served as the control. For these reasons, this study is considered a “*quasi-experimental*” study.

A sixth limitation pertained to the quantitative and qualitative instruments used to gather data. The only valid and reliable quantitative measure was the i-Ready Assessment measure (Curriculum Associates, 2015). The district’s Benchmark Assessments aligned to the district’s mathematics program, not to Common Core State Standards. This limited the ability to triangulate and validate students’ growth data pertaining to mathematics achievement.

Regarding qualitative data, additional samples of student work across all three classrooms would have supported additional comparisons of students’ mathematical thinking between the control group and the two experimental groups. Conducting more student interviews as follow-ups to their work would have provided additional insights to students’ sense-making processes.

Highly-sensitive and reliable instruments to detect students’ development of conservation of constancy and spatial orientation would have improved the triangulation of data relative to cognitive structure analysis. Additionally, the instruments used to measure each classroom’s learning environment were created by the researcher and utilized for the first time. These were not tested for validity nor reliability prior to this study.

A seventh and final limitation pertains to data analyses. Although a learning theorist, a committee co-chair, a statistician, educators, and participating teachers were consulted during the study, most of the qualitative data were analyzed by one person. Rather than containing differing perspectives and orientations to underlying phenomena, some results may contain specific biases towards mathematics education, teaching, and learning.

Recommendations for Future Research

Learning theorists and neuro-scientists emphasized that the development of students' cognitive structures enhance their abilities to learn. CLB and CLC student groups demonstrated the greatest increase in mathematics achievement from pre- to post- assessments. As such, it is expected that these two groups would have demonstrated an increase in their cognitive structure development for conservation of constancy. This was not the case. Only CLA students improved their development of conservation of constancy.

The first recommendation for future research is to determine the type of mathematical tasks and instructional practices that mediate students' development of conservation of constancy. Although a plausible causational factor was proposed, additional research is still recommended. According to learning theorists and neuro-scientists, these insights will benefit all students in their learning (Duval, 2006; Ifenthaler, 2011; Sweller, 2008).

A second recommendation involves Tier II and III interventions. By the end of the school year, the *i*-Ready Universal Screening Assessment (Curriculum Associates, 2015) results indicated that some students (in all three classrooms) still required Tier II interventions and one CLC student required Tier III interventions. The *i*-Ready Assessment data provided teachers important information regarding students' positions on the learning continuum and identified targeted interventions. Teachers did not utilize the program's data. Nor did they use the suggested tasks identified by the researcher. Teachers expressed their uncertainty for how to best implement the suggested interventions to students during Tier I core instruction.

Implementing interventions during Tier I core instruction is challenging work for teachers to do. Teachers' limited content and pedagogical knowledge also constrains their abilities to determine and implement appropriate interventions. Teachers expressed they need specific training on how to use the data to implement interventions during Tier I core instruction. Further research can assist teachers in this implementation process.

A fourth suggestion for future research focuses on the “re-designed” tasks used to focus students’ attention and engage them in mathematical thinking. CLB and CLC Teachers struggled in creating tasks like the ones provided. One recommendation for future research is to conduct a fine-grain analysis of the “re-designed” mathematical representations and tasks. This analysis would identify key features and concepts. The tasks’ features can then be generalized to create a wider range of mathematical tasks for K-12 students.

The fifth recommendation aligns to the observed changes in teachers’ pedagogical practices. The two experimental teachers became more student-responsive during the study. Rather than following a mathematics program page by page, these teachers began attending to students’ mathematical thinking during the three instructional segments. Teachers listened to students and attended to student reasoning, including their misconceptions. Students’ external representations provided important data teachers used to plan their next steps of instruction.

The sources for change and the processes that supported teacher’s shifting their practices can be identified through additional interviews and focused conversations. Mathematics education researchers, mathematics teacher educators, and professional development providers will benefit from teachers’ insights (Ball, 1990; Stigler & Hiebert, 1999).

A sixth recommendation relates to a limitation in this study. All three teachers experienced implementation support the previous school year. Still, the two experimental teachers required additional support to implement the model and tasks with fidelity. Additional research is needed to identify teachers’ specific needs relative to implementation process. This data informs schools and districts when designing their own theories of action for increasing student achievement.

The final recommendation extends to all academic contexts and populations. Implementing the “re-envisioned” instruction model across a variety of contexts and subjects with students of all ages can illuminate the full efficacy of the “re-envisioned” instruction model. The findings from this study can be realized by a larger population.

Closing Remarks

The “re-envisioned” instruction model was designed to mediate the development of students’ cognitive structures because, as Sweller (2008) claimed, teachers’ instructional practices that fail to consider student’s existing cognitive structures and their development are likely to be haphazard in their effectiveness. Learning theorists, mathematics education researchers, neuroscientists, and cognitive psychologists all agree with Sweller. However, when comparing students’ results from the cognitive structure assessment, all three student groups in this study increased their cognitive development for spatial orientation. Whereas only CLA students (control group) demonstrated statistical evidence for increasing their cognition for conservation of constancy.

Suggestions were offered as to why this occurred. These suggestions included CLA students’ frequent constructions of representations that were equivalent and CLA teacher’s use of simple story contexts to introduce mathematical concepts. Furthermore, students, at this age, may develop conservation of constancy at different rates, even in “normal” children at this age. Another possible reason is that the “re-designed” tasks needed to offer students more opportunities to create equivalent representations, as well as identify the figural units that stayed the same and compare them to those that changed.

Two other factors may have attributed to these surprising results. These included the small sample sizes of study subjects and utilizing less sensitive tools for assessing students’ developmental levels of cognitive structures. Larger sample sizes increase the likelihood of drawing correct conclusions from the data (Johnson & Christensen, 2012); and, the use of highly sensitive instruments that measure students’ levels of development may have provided different results. Clearly, further research is needed to investigate and identify the causational factors for why teachers’ implementation of the “re-envisioned” instruction model did not mediate the development of students’ cognitive structure for conservation of constancy, whereas CLA Teacher’s mathematics instruction appeared to accomplish this.

Still, teachers’ and students’ implementation of the “re-envisioned” instruction model prompted systems of change for the experimental teachers and their students. Their implementation affected distinct positive effects upon students’ learning and understanding of mathematics when compared to the control classroom of students. Students in the two experimental groups increased their mean mathematics achievement scores, doubling that of the control group’s mean increase. When compared to the control group, a significant number of students in the two experimental groups embraced growth mindsets for learning and understanding mathematics. These students perceived mathematics as concepts to be understood and identified a variety of concepts they wanted to continue to learn.

Consistently, students in the two experimental groups understood that learning mathematics was about deep thinking. Thus, a greater number of CLB and CLC students proclaimed that they grew in their abilities to make sense of mathematics, explain their thinking, and provide evidence of that thinking using tools and representations. Analysis of qualitative data indicated that these two student groups made important connections to other models, to classmates’ solution strategies, and arrived at informal generalizations through student-to-student groups’ mathematical discussions.

Experimental teachers’ implementation of the “re-envisioned” instruction model supported their shifting from traditional and historical practices of teacher-led lectures and demonstrations to practices that strategically targeted more authentic ways students learned. Students in the two experimental groups became active participants in their learning - able to create, learn, and change their thinking of mathematical concepts individually, and importantly, as a learning community (Bruner, 1966; Garner, 2007; Piaget, 1964; Vygotsky, 1978/1930). CLB and CLC teachers’ questions required students to explore mathematical meanings and relationships and synthesize their learning. Teachers *and* students collectively shared responsibilities for applying constructivist and socio-constructivist principles to generate knowledge and cultivate productive environments that led to sense-making and mathematical generalizations.

Teachers also learned how to implement “re-designed” novel tasks, transforming student and teacher engagement with mathematical concepts. The novel tasks and teacher’s open-ended questions focused students’ attention, encouraging them to analyze, visualize, and synthesize the figural units and mathematical structures embedded within enactive, iconic, and symbolic representations. The novel features of the representations generated cognitive opportunities to mentally engage with and make sense of abstract concepts. The use of the “re-designed” tasks enhanced teachers’ content knowledge for teaching mathematics.

Accordingly, teacher and student implementation of the “re-envisioned” instruction model addresses four of the five origins for students’ learning difficulties (biological, cognitive, socio-cultural, and pedagogical) because it supports generative processes. These processes include socio-cultural environments that inspire *growth mindsets* for teachers and for students. It delineates a superior and effective framework for implementing mathematics lessons and tasks that increase *students' sense-making abilities*. It strongly supports teachers engaging students in thinking deeply about mathematical concepts *enabling them to move beyond their natural cognitive endowments*. The model structures teachers' *strategic pedagogical actions* that facilitate students' conceptual understanding of mathematical content, thereby changing traditional purposes for instruction. Teachers can influence these four origins by infusing the synthesis of learning theories and evidence-based practices into their Tier I core instruction.

Decades of low student performance and achievement in mathematics continue to challenge our nation, businesses, communities, schools, families and their students (National Research Council, 2001; NCES, 2015a; NCES, 2016). Researchers and teachers must strive to improve our students' numerical literacy in the United States. Our students' active participation in an ever-changing, information and technologically-dependent world is critical for the environmental, economic, and political health of our country. Mathematical principles underlie everything we do.

The synthesis of viable learning theories and evidence-based practices, as represented by the "re-envisioned" instruction model, increases students' achievement. The model inspires the necessary and crucial shifts in beliefs and practices for teaching and for learning mathematics. Teachers engage in pedagogical practices that minimize students' learning difficulties in mathematics. Students become the confident, reflective, and dynamic change-agents of their own intelligence. Together, researchers and teachers can become effective stewards of our students' futures by "re-envisioning" ways children authentically learn mathematics.

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Appendix A: Detailed Work Schedule Throughout Research Study

Dates and Days of Implementation	Description of Work and Researcher Contacts	Researcher Observations, Notes, and Communications
5/13/2014 (none)	Gained approval to conduct study from school district. Executive Director of Instruction, Technology and Assessment for WLCSD) via email approved this research project at the school site for school year 2014-2015.	
Week 1 9/1/14-9/6/15 (W)	- Met with Principal and teachers to describe study. - IRB Board at University of Michigan gave permission to conduct research study - Submitted electronic versions of teacher, parent, student and administration consent forms to principal.	Relative to Principal’s selection of the teachers (CLB and CLC Teachers) who received the intervention and the teacher who was the control (CLA Teacher), Principal shared that CLC Teacher has the greatest number of struggling students in all three classrooms. She wants this classroom to receive the intervention. As for the other two classrooms, she mentioned that CLA Teacher was the strongest pedagogically of the three, so she believed she would be the best to function as the control. The CLA Teacher has worked at the district level for supporting second grade math. Thus, CLB and CLC will serve as the intervention groups as designated by the principal.

Week 2 9/7/14-9/14/14 (M, T, W)	- All three teachers and principal signed consent forms for participation. - Presented research study to parents at curriculum night. Answered parent questions and provided parent consent forms. - Observed teachers' math lessons in CLA and CLC classrooms. - Received 17 parent consent forms across three classrooms. - Interviewed 16 students to gain their consent for participation.	When interviewing individual students, the protocol was followed. During the interview, most students were very trusting. All but one agreed to participate in this study. However, rather than looking at the researcher, five students' eyes darted around, and their hands rubbed their legs or other body parts.
Week 3 9/15/14-9/21/14 (M, T, W)	- Modeled a Number Talk using Ten Frames (Parrish, 2010, p. 92–93). - Administered How Many Squares Task to all students, (see Russell, 2007). - Each teacher rated their students using the Student-Centered Mathematics Classroom Indicators. - Received remaining parent consent forms for a total of 14 students in CLA; 11 students in CLB; and 10 students in CLC gave permission for their child to be part of the study. - Completed student assent interviews with remaining students to gain their consent for participation.	Analyzed student results from How Many Squares task. Majority of students did not sort, nor recognize number relationships with the Square Task. They also struggled in the number relationships Number Talk tasks. Teachers commented that these were different types of tasks that the teachers nor students had encountered before.
Week 4 9/22/14-9/28/14 (M, T)	- Met with CLB and CLC Teachers to discuss the types of tasks students needed to be engaged with given the data. - Modeled Cube Task and Arrow Road Tasks in CLC classroom (see Chapter 3 in Dissertation.) - Modeled Cube Task and Arrow Road Tasks in CLB classroom. - Observed the teacher teaching the lesson in CLA's classroom.	Discussed with CLC Teacher how to take student's misconceptions and use them as a launch for the next lesson. Teachers explained this was the first time this year students asked for more time to record their ideas during the launch of a lesson. I wonder if it was because they had the physical cubes right in front of them and they could touch them?

Week 5 9/29/14- 10/5/14 (M, W)	- Administered Adapted Cognitive Structure Assessment to students in all three classrooms. - Conversed with all three teachers and asked how they would feel to be the opposite of what they are doing in the study (i.e. If you were the control rather than the treatment and if you were the treatment rather than the control.) - Observed math lessons in CLB and in CLC classrooms (Arrow Roads and Number Grids).	Most students struggled with finding the triangles and square within the task. Spatial reasoning is a cognitive structure we will need to develop. After discussions with the two experimental teachers and hearing their feelings regarding the work load of preparing lessons that meet the cognitive demand for the "Re-Envisioned" Instruction Model, some Base Ten Block activities were provided, (see Steward, Walker, & Reak, 1995.)
Week 6 10/6/14- 10/12/14 (W)	- Observed I-Ready Assessment Measure administered to 4th grade students. - Administered Pre-assessment of the Number line task in all three second grade classrooms. - Wrote 12 lesson plans for CLB and CLC Teachers to use with their students for this study. Used District's Mathematics Program and Van de Walle, Karp, Lovin, & Bay-Williams (2014) to create lesson plans. Articulated the following Big Ideas for all 12 lessons.	Due to the two teachers in the experimental groups feeling overwhelmed and not being able to see where their work was headed, 12 lessons for CLB and CLC Teachers were created to use for instructional purposes. Some lessons were in EDM and were modified or "re-designed" to align to the "Re-Envisioned" Instruction Model. These lessons were designed to activate students' cognitive structures and develop their sense of number and number relationships (i.e. Counting Collections which teachers did not implement until I modeled it for them with their students).
Week 7	- I-Ready screener administered to all second-graders.	CLB and CLC Teachers asked students to take their time and use blank sheets of

10/13/14-10/19/14 (T, Th)	- Conducted a face-to-face interview with each teacher. - Each teacher rated their students using the Student-Centered Mathematics Classroom Indicators - Met with both CLB and CLC Teachers in the morning for a math planning meeting to discuss the cognitive structures the 12 lessons attended to and discussed a possible place value lesson. - Modeled "Quick Images" using Base Ten Blocks in CLB and in CLC classrooms. Task was created by the researcher. - Observed the I-Ready Screener being administered to 3rd graders.	paper to think about the questions on the I-Ready screener. Teachers did this because they wanted their students to think about the knowledge they knew to solve problems. They also wanted students to slow down their thought processes.
Week 8 10/20/14-10/26/14 (M, T)	- Observed lessons in all three classrooms	
Week 9 10/27/14- 11/2/14 (M)	- Checked with CLB and CLC Teachers to ensure they had sufficient lesson materials. Asked all three teachers for questions they might have. All teachers were notified that the researcher would be out of town but would return on November 10th. Teachers could contact the researcher by email.	Teachers shared they needed to stop teaching the lessons given them and focus on using lessons that prepared students to take the district's first Quarter Benchmark assessment. What was interesting is that CLB and CLC Teachers used two weeks to prepare students to take this test. They did not make the connection that the lessons provided them would prepare students for the assessment. Students' Benchmark Assessment performance scores were reported on students' report cards and used for teacher evaluation purposes.
Week 10	- No visits made	

11/2/14 – 11/9/14 (none)	- All three teachers administered District's Benchmark Assessment I to students.	
Week 11 11/10/14-11/16/14 (M, T, W)	- Modeled lesson looking for Patterns using puzzle pieces in CLB and CLC classrooms. - Taught second lesson this week re: looking for patterns. 9 + 9 + 2 =; 8 + 8 + 4 =; 7 + 7 + 6 =; etc.… - Taught third lesson this week modeling for teachers the use of the double ten frame, Number Talks, (Parrish, 2010, p. 104) - Met with CLB and CLC Teachers for 30 minutes to discuss EDM Unit 4 Addition and Subtraction and Mental Arithmetic. - Conducted an item analysis of District's Benchmark Assessment II for math content before writing next unit's lesson plans for teachers CLB and CLA. - Used EDM resource and Van de Walle, Karp, Lovin, & Bay-Williams (2014) to create 15 more lessons to activate students' cognitive structures for Addition/subtraction and Mental Arithmetic. Big Ideas included: o Addition and subtraction involves composing and decomposing numbers and quantities o Addition and subtraction are inverse operations, one "undoes" the other. o Addition and subtraction can be represented in different ways.	Used EDM as a resource to create additional lessons because teachers are comfortable with that program and format. Asked teachers if they had ever done an item analysis on their benchmark assessments to inform their instruction before giving assessments to their students in their classroom. They replied no.
Week 12 11/17/14- 11/23/14 (W, F)	- Analyzed student results from the first Benchmark Assessment. - Conducted mid-study interviews with participating students in all three classrooms. - Modeled Counting Collections for CLB and CLC Teachers with students (see Schwerdtfeger & Chan, 2007).	

Week 13 11/24/14- 11/30/14 (M, T)	- Observed lessons in all three classrooms. - All three teachers had great concerns for a small group of students in each class. Met with teacher-selected students from all three classrooms and assessed each student using the Number Knowledge Assessment. Generated specific directions for each teacher regarding additional math supports in the form of tasks and interventions they could provide each student they were concerned about.	
Week 14 12/1/14- 12/7/14 (W)	- Modeled how to teach and use an open number line representation with CLB and CLC Teachers and students. - Met with CLB and CLC Teachers to discuss their concerns about the study. - Asked CLB and CLC Teachers to reflect upon the following: o Changes they made in their instruction o Changes in students' thinking and/or skills/Behaviors from Previous Years of teaching.	The open number line representation was new to both CLB and CLC Teachers.
Week 15 12/8/14- 12/14/14 (M, T)	- In both CLB and CLC classrooms, modeled a lesson using a meter stick to focus on equal intervals between whole numbers and multiples of ten preparatory to using the open number line as a representation to add and subtract. - Conducted second lesson by asking students to compare their meter stick experience to an open number line and to the thermometer. - Prepared all three teachers for the site visit in a different school district.	CLC Teacher asked how the meter stick, the open number line and the thermometer could be the same. She had never thought about all three representations being a type of number line.
Week 16 12/15/14- 12/21/14 (T)	- All three teachers and researcher visited another district to observe two second-grade teachers' mathematics instruction.	Teachers observed two teachers implementing the "Re-Envisioned" Instruction Model with their students. Teachers appreciated visiting two other second-grade teachers' classrooms.

		Teachers commented they were amazed at the types of higher-cognitive tasks students engaged in. They
Week 17 12/22/14- 12/28/14	- Holiday Break	
Week 18 12/29/14- 1/4/15	- Holiday Break	
Week 19 1/5/15- 1/11/15 (M, T)	- Observed lessons in all three classrooms to determine if student’s mathematical understanding needed additional review due to the two-week holiday break.	Teacher were surprised at how much students retained regarding math concepts and classroom routines. Both CLB and CLC Teachers stopped using the “re-designed” lessons provided and began using their district’s mathematics program to prepare students for success on Benchmark Assessment II.
Week 20 1/12/15- 1/18/15 (M, T)	- Met with CLB and CLC Teachers to answer their questions and provide feedback of their instruction and student learning. - Observed teachers’ lessons in all three classrooms - Observed student choices during indoor recess in all three classrooms.	During recess, CLA and CLC students were engaged in conversations, drawing or coloring, playing school, playing games. Students in CLB classroom were engaged with puzzles, mind-benders, Legos, conversations, and math manipulatives.
Week 21 1/19/15- 1/25/15 (W)	- Each teacher administered District Benchmark Assessment II. - Copied all students’ district benchmark assessment to analyze data results. - Administered the Post-assessment for the Number Line Task	

	- Collected data from three to four students regarding the math topics recorded in their mathematician's notebooks.	
Week 22 1/26/15-1/28/15 (M, T, W)	- Post *i*-Ready Screening Measure was administered to majority of second grade students. Some students in CLA and CLB classrooms did not complete the screener until a later date. - Each teacher rated their students using the Student-Centered Mathematics Classroom Indicators - Continued to copy student data from student's mathematicians' notebooks - Administered Cognitive Structure Assessment to students in all three classrooms - Taught lesson in CLA, CLB, and CLC classrooms using Tangrams to support spatial reasoning and introduced the Game "Capture 5" to students. - Provided Capture 5 game to all three teachers for students to play to continue to develop mathematical proficiencies with number and operations. - Conducted debriefing with CLB Teacher.	CLA and CLB Teachers did not notice they had a few students who did not finish their *i*-Ready screening assessment. This was brought to teachers' attention. Teachers provided additional time for these students to finish their assessment.
4/2014	- End *i*-Ready Screening Measure was administered to all students.	
5/19/14 (T)	- Teachers provided students' End data results from the *i*-Ready Screening Measure with researcher. - Conducted face-to-face interviews with each teacher to understand their perceptions re: assessment results	

Teacher and Student Interactions with the Researcher

22 weeks for the research study spanning September 4, 2014 until January 28, 2015

This researcher was at the school site for 36 days during that time frame.

Researcher’s direct interactions with students consisted of the following:

- solicitation of students’ consent to participate in research study: a 5-6-minute interview per student at the beginning of the school year;
- modeled (or taught) 13 lessons with CLB Teacher’s students;
- modeled (or taught) 13 lessons with CLC Teacher’s students;
- modeled (or taught) 3 lessons with CLA Teacher’s students;
- assessed students for Cognitive Structures (no explicit instruction was done relative to this assessment (2 days in each classroom);
- assessed students for number sense using the Number Line task and Tile Task (2 days each in each classroom); and
- conducted conversations with participating students regarding their understanding of mathematical concepts. After teachers launched their math lessons, students discussed their ideas with the researcher. (3 days in each classroom.)

Teacher and Researcher Interactions

Observed Teacher’s implementation of lessons

- CLA: 6 times
- CLB: 5 times
- CLC: 7 times

Number of times this researcher met with each teacher throughout the course of the study

- All three teachers together: 4 times approximately 30-45 minutes in length

In addition to the above meetings, the following indicate additional meetings lasting from 5 minutes to 25 minutes in length depending upon the purpose and content of the meeting.

- CLB and CLC Teacher together: 5 times
- CLA Teacher individually: 4 times
- CLB Teacher individually: 4 times
- CLC Teacher individually: 4 times

Infrequent brief discussions occurred with teachers in the hallway as students lined up to use the restrooms. Most discussions occurred with CLB Teacher when she wanted to share an observation she made regarding her students or inquire about an upcoming lesson.

Appendix B: Adapted Cognitive Structure Assessment

Student Name________________________ Date______________________

Please circle **only one answer:**

1. Which piece of clay has more, or do they have the same amount?

Clay Ball Clay Pancake Same

2. Which bottle has more water or are they the same amount?

Right side up Upside down Same

3. Which is longer? The one on the top? The one on the bottom? Or are they the same?

Top Bottom Same

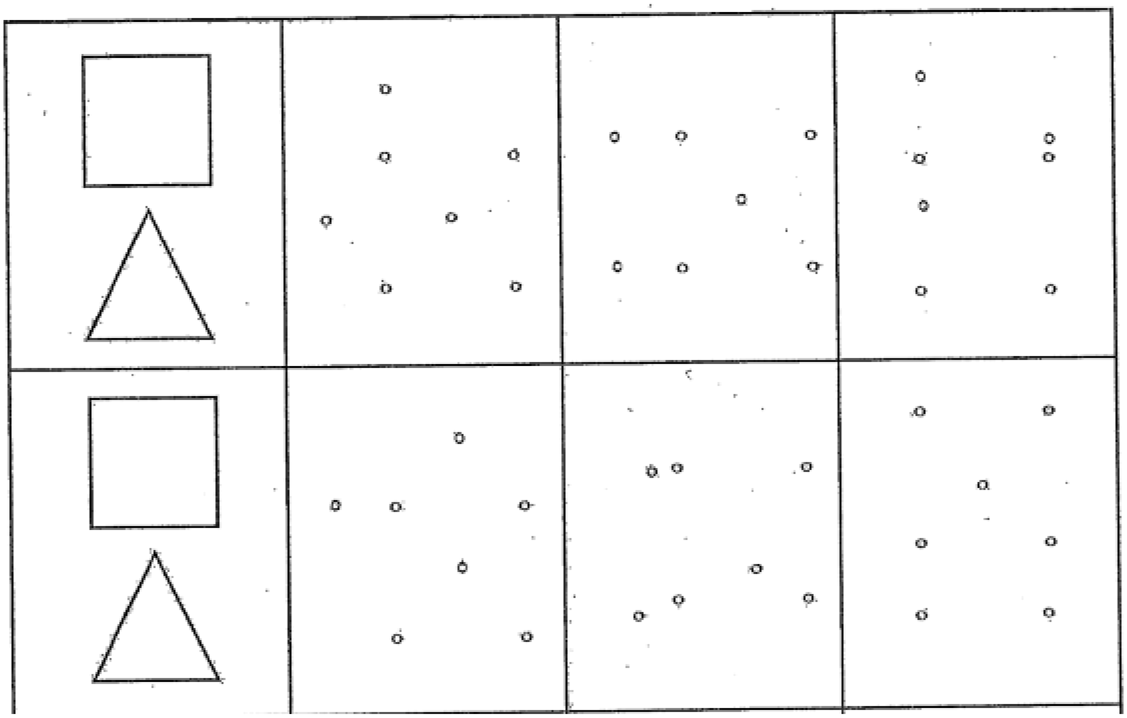
Drawing a Glass of Water

4. Draw a glass of water standing straight up half full.

5. Draw a glass of water tipped to the right half full.

6. Draw a glass of water tipped to the left half full.

Look at the model on the left. Decide which dots can be connected to match the model. The new shapes must be the exact size and shape as the model.



Excerpted from Putting the Pieces Together by Kim D. Ellis. Published by aha! Process, Inc. www.ahaprocess.com

Appendix C: Original Cognitive Structure Assessment

Aesthetic of Lifelong Learning Betty K. Garner, Ed.D. bettygarner@yahoo.com 19

LARGE GROUP ASSESSMENT of BASIC COGNITIVE STRUCTURES

PROTOCOL

Student Name: ______________________ Date: __________
School: ______________ Grade:____ Teacher :______________

Please circle the correct answer:

1. Which piece of clay has more or do they have the same amount? Ball Pancake Same

2. Which jar has more water or are they the same amount of water? Right side up Upside down Same

3. Which is longer the one on the top or the one on the bottom, or are they the same length? Top Bottom Same

4. Draw a glass of water **standing straight up** half full.

5. Draw a glass of water **tipped to the right** half full.

6. Draw a glass of water **tipped to the left** half full.

7. Write a couple sentences about this room ______________________

8. What do you wish was easier in school?______________________

9. Draw a floor plan of your house. (On back of paper or on another sheet of paper)

10. Draw a self portrait. (On back of this paper or on another sheet of paper)

Appendix D: Teacher and Student Semi-Structured Interviews, Questionnaires, and Surveys

Teacher Semi-Structured Interview Questions Conducted at the Beginning of Study

1. How many years of teaching?
2. Number of years teaching in second grade?
3. Degree(s)?
4. On average, how much time per week do you spend preparing for reading lessons?
5. On average, how much time per week do you spend preparing for math lessons?
6. What is challenging for you while preparing math lessons?
7. On average, how much time per day is spent teaching math?
8. What are some of your strengths for teaching math?
9. What is challenging for you when teaching math?
10. What normally drives your mathematics instruction of concepts: based on a calendar, based on student needs, based on a program, based on ___________________________
11. What was a significant change for you teaching math last year?

11. What do you owe that change to?

12. In what ways has your math instruction changed already this year?

13. What do you owe that change to?

14. In what ways do you utilize students’ mathematicians’ notebooks?

15. What else would you like to share with me?

Teacher Reflection at the End of Study (Teacher completed this questionnaire individually)

1. Write a brief definition for the following:

    a. Teaching is…

    b. Learning is…

    c. Cognitive structures are…

2. How do students learn mathematics?

3. Describe changes you’ve made in your math instruction this year. Use a ☆ to indicate changes that were caused by your participation in this study.

4. Describe differences in students’ mathematical thinking, skills and/or behaviors this year from previous years.

5. What were the most challenging math concepts and/or tasks for your students to understand during this study? Please explain.

6. What were the most challenging math concepts and/or tasks for you, as a teacher, to understand during this study? Please explain.

Indicate the level of implementation of the following by circling the corresponding value.

7. On average, the days per week a math lesson was taught:

   0 < 1 1 2 3 4 5

8. On average, the number of instructional minutes per day devoted to math instruction:

   0 Less than 30 30-45 46-60 61-75 76-90 91 +

9. On average, the days per week a math game was used to support student learning:

   0 < 1 1 2 3 4 5

10. On average, the days per week a form of Number Talks was implemented?

    0 < 1 1 2 3 4 5

11. On average, the days per week students were required to first make sense of a math task before whole group discussion occurred:

    0 < 1 1 2 3 4 5

12. On average, the days per week students used their Student Reference Book as a resource to make sense of mathematical ideas:

    0 < 1 1 2 3 4 5

13. On average, the days per week students used math tools to make sense of mathematical ideas:

    0 < 1 1 2 3 4 5

14. List the math tools students used on a ***regular basis*** in your classroom:

15. On average, the days per week the SIPP information/instructional strategies provided by the researcher were used as intervention support for struggling students: Please explain.

    0 <1 1 2 3 4 5

16. Please list other forms of math instruction you provided your students and indicate how often they were used (i.e. morning work packets, homework packets, drills in the hallways, etc.…)

18. The number of times your students did Counting Collections for their math lesson:

19. On average, how many minutes per day did you spend planning a math lesson?

20. What were the most challenging aspects for you while participating in this study? Please explain.

21. Please provide suggestions to the researcher for how to improve the overall design of the research study.

Student Questionnaire Conducted at Post Treatment

Name________________________________ Date____________________________

Circle the way you feel about learning math.
I don’t like math. Math is O.K. I love math.
Explain your thinking.

____

____

____

What is something you’ve learned in math you feel really good about?

____

____

____

What do you wish you understood better in math?

____

____

____

Appendix E: Student-Centered Classroom Indicators

Focusing on student outcomes, indicate the level in each of the following behaviors you have observed in your students in math class. Use the following scale: (1) All the time; (2) Some of the time; (3) Once-in-a-while; (4) Not at all.

______ a. Students see themselves as mathematicians.

______ b. Students incorporate the five talk moves within their classroom dialogue.

______ c. Students ask one another questions to support each other's learning about mathematics.

______ d. Students understand their role is to think about math.

______ e. Students understand they are expected to make sense of complex mathematical ideas.

______ f. Students demonstrate their understanding of ideas using multiple representations.

______ g. Students use self-talk to support themselves solving a math problem.

______ h. Students generate their own questions about mathematical ideas.

______ i. Students notice details embedded within various mathematical representations.

______ j. Students make connections between and among mathematical ideas from prior lessons.

______ k. Students explain and justify their mathematical reasoning and thinking.

______ l. Students critique the mathematical reasoning of their peers.

______ m. Students self-select math tools to support their own mathematical reasoning and thinking.

______ n. Students attempt to construct meaning of multiple solution strategies by making connections between them.

Indicator	Learning Theories Connected to Indicator	Standards of Mathematical Practices Connected to Indicator
a. Students see themselves as mathematicians.	• Self-efficacy	# 1 – Make sense of problems and persevere in solving them.
b. Students incorporate the five talk moves within their classroom dialogue.	• Social Constructivist • Mediated	# 1 – Make sense of problems and persevere in solving them. # 3 – Construct Viable arguments and critique the reasoning of others # 6 – Attend to precision
c. Students ask one another questions to support each other's learning mathematics.	• Constructivist • Social Constructivist • Mediated	# 1 – Make sense of problems and persevere in solving them. # 2 – Reason abstractly and quantitatively # 3 – Construct Viable arguments and critique the reasoning of others # 6 – Attend to precision
d. Students understand their role is to think about math.	• Self-efficacy	# 1 – Make sense of problems and persevere in solving them.
e. Students understand they are expected to make sense of complex mathematical ideas.	• Constructivism	# 1 – make sense of problems and persevere in solving them. # 2 – Reason abstractly and quantitatively # 6 – attend to precision
f. Students demonstrate their understanding of ideas using multiple representations.	• Representational • Constructivism • Evolution-based • Metability	# 1 – make sense of problems and persevere in solving them

		# 2 – Reason abstractly and quantitatively # 4 – Model with mathematics # 5 – Use appropriate tools strategically # 6 – Attend to precision
g. Students use self-talk to support themselves solving a math problem.	• Constructivism • Evolution-based	# 1 – Make sense of problems and persevere in solving them. # 6 – Attend to precision # 7 – Look for and make use of structures # 8 – Look for and express regularity in repeated reasoning
h. Students generate their own questions about mathematical ideas.	• Constructivism • Metability • Mediated	# 1 – Make sense of problems and persevere in solving them # 3 – Construct Viable arguments and critique the reasoning of others # 6 – Attend to precision
i. Students notice details embedded within various mathematical representations.	• Evolution-based • Representational • Mediated • Metability	# 1 – Make sense of problems and persevere in solving them. # 6 – Attend to precision # 7 – Look for and make use of structures # 8 – Look for and express regularity in repeated reasoning

j. Students make connections between and among mathematical ideas from prior lessons.	• Constructivism • Social Constructivism • Representational • Evolution-based • Mediated • Metability	# 1 – Make sense of problems and persevere in solving them. # 2 – Reason abstractly and quantitatively # 4 – Model with mathematics # 6 – Attend to precision # 7 – Look for and make use of structures # 8 – Look for and express regularity in repeated reasoning
k. Students explain and justify their mathematical reasoning and thinking.	• Constructivism • Social Constructivism • Representational • Evolution-based • Mediated • Metability	# 1 – Make sense of problems and persevere in solving them. # 2 – Reason abstractly and quantitatively # 3 – Construct Viable arguments and critique the reasoning of others # 4 – Model with mathematics # 5 – Use appropriate tools strategically # 6 – Attend to precision # 7 – Look for and make use of structures #8 – Look for and express regularity in repeated reasoning
l. Students critique the mathematical reasoning of their peers.	• Constructivism • Social Constructivism • Representational • Evolution-based	# 1 – Make sense of problems and persevere in solving them.

	• Mediated • Metability	# 2 – Reason abstractly and quantitatively # 3 – Construct Viable arguments and critique the reasoning of others # 4 – Model with mathematics # 5 – Use appropriate tools strategically # 6 – Attend to precision # 7 – Look for and make use of structures #8 – Look for and express regularity in repeated reasoning
m. Students self-select math tools to support their own mathematical reasoning and thinking.	• Constructivism • Social Constructivism • Representational • Evolution-based • Mediated • Metability	# 1 – Make sense of problems and persevere in solving them. # 2 – Reason abstractly and quantitatively # 4 – Model with mathematics # 5 – Use appropriate tools strategically
n. Students attempt to construct meaning of multiple solution strategies by making connections between them.	• Constructivism • Social Constructivism • Representational • Evolution-based • Mediated • Metability	# 1 – Make sense of problems and persevere in solving them. # 2 – Reason abstractly and quantitatively # 3 – Construct Viable arguments and critique the reasoning of others

4 – Model with mathematics

6 – Attend to precision

7 – Look for and make use of structures

#8 – Look for and express regularity in repeated reasoning

- All fourteen indicators require making sense of problems.
- Those that require all eight Standards are:
    - Students explain and justify their mathematical reasoning and thinking.
    - Students critique the mathematical reasoning of their peers.
    - Students attempt to construct meaning of multiple solution strategies by making connections between them.

Appendix F: Open Number Line Formative Assessment Task

Name ______________________________ Date_________________

Count from 0 to 100 by tens. Place the numbers on the number line below.

Count from 0 to 100 by twenties. Place the numbers on the number line below.

Count from 0 to 100 by twenty-fives. Place the numbers on the number line below.

Appendix G: CLA, CLB, CLC, Students’ i-Ready Results and District Benchmark Test Results

CLA Results

CLA Students’ *i*-Ready Results and District Benchmark Test Results (Control Group; n=18)

Student ID	i-Ready Pre-Assessment Score Administered the week of 10/13/2014	Tier Designation as Determined by i-Ready Screener Pre-Assessment.	District Benchmark Quarter 1 Administered the week of 11/2/2014	I-Ready Post-Assessment Score Administered the week of 1/26/2015	Tier Designation as Determined by i-Ready Screener Post-Assessment.	District Benchmark Quarter 2 Administered the week of 1/19/2015	I-Ready End-Assessment Score Administered the week of 4/27/2015	Tier Designation as Determined by i-Ready Screener End-Asst.	Difference in i-Ready Scores from Pre to End
A1	391	II	78%	370	III	64%	416	II	+24
A6	415	II	88%	436	I	71%	416	II	+1
A15	409	II	94%	413	II	82%	435	I	+26
A17	428	II	91%	396	II	96%	422	II	-6
A10	356	III	63%	379	III	71%	414	II	+58
A14	415	II	94%	433	I	89%	445	I	+30
A5	441	I	91%	427	II	100%	461	I	+20
A9	446	I	97%	479	I	100%	494	I	+48
A2	440	I	97%	452	I	100%	458	I	+18
A3	437	I	91%	432	I	89%	444	I	+7
A19	383	III	53%	401	II	71%	402	II	+19
A20	405	II	81%	433	I	89%	450	I	+45
A12	418	II	94%	451	I	89%	449	I	+31
A7	436	I	100%	432	I	100%	449	I	+13
A4	410	II	91%	415	II	86%	426	II	+16
A11	405	II	91%	414	II	100%	431	I	+26
A8	420	II	100%	439	I	93%	425	II	+5
A13	411	II	91%	434	I	82%	445	I	+34

Key

- Students who increased in Tier Levels during the study (e.g. was at Tier II for Pre-Assessment and went to Tier III at Post-Assessment)
- Students who decreased in Tier Levels during the study (e.g. was at Tier II for Pre-Assessment and went to Tier I at Post-Assessment)
- Students who made at least 1.0 year of growth by end of school year for 2014-2015 (For Grade 2, 1.0 Year Ranges between 32 – 41) [1]

[1] *i-Ready Diagnostic Scale Score Increases to Achieve Specified Years of Growth in Mathematics*: Growth targets are for all students in a chronological grade, independent of level in i-Ready. Retrieved from http://www.kent.k12.wa.us/cms/lib/WA01001454/Centricity/Domain/728/Using-i-Ready-as-a-Student-Growth-Measure_2014-2105.pdf

CLB Results

CLB Students' i-Ready Results and District Benchmark Test Results (Experimental Group; n=18)

Student ID	i-Ready Pre-Assessment Score Administered the week of 10/13/2014	Tier Designation as Determined by i-Ready Screener Pre-Assessment	District Benchmark Quarter 1 Administered the week of 11/2/2014	i-Ready Post-Assessment Score Administered the week of 1/26/2015	Tier Designation as Determined by i-Ready Screener Post-Assessment	District Benchmark Quarter 2 Administered the week of 1/19/2015	i-Ready End-Assessment Score Administered the week of 4/27/2015	Tier Designation as Determined by i-Ready Screener End-Assessment	Difference in i-Ready Scores from Pre to End
B5	409	II	84%	431	I	86%	439	I	+30
B14	417	II	88%	446	I	89%	453	I	+36
B4	407	II	75%	410	II	71%	413	II	+5
B10	419	II	88%	430	I	93%	460	I	+41
B1	425	II	91%	451	I	89%	475	I	+50
B6	460	I	97%	462	I	100%	486	I	+26
B2	430	I	94%	435	I	96%	460	I	+30
B11	432	I	100%	460	I	75%	467	I	+35
B19	399	II	59%	417	II	50%	423	II	+24
B12	421	II	88%	458	I	79%	434	I	+13
B16	396	II	69%	425	II	68%	443	I	+47
B13	412	II	91%	405	II	86%	432	I	+20
B8	436	I	97%	455	I	86%	468	I	+32
B7	422	II	97%	441	I	82%	461	I	+39
B3	418	II	94%	431	I	75%	442	I	+24
B9	457	I	100%	486	I	89%	508	I	+49
B15	420	II	97%	465	I	100%	472	I	+52
B17	396	II	84%	419	II	43%	448	I	+52

Key

- Students who increased in Tier Levels during the study (e.g. was at Tier II for Pre-Assessment and went to Tier III at Post-Assessment)
- Students who decreased in Tier Levels during the study (e.g. was at Tier II for Pre-Assessment and went to Tier I at Post-Assessment)
- Students who made at least 1.0 year of growth by end of school year for 2014-2015 (For Grade 2, 1.0 Year Ranges between 32 – 41) [1]

[1] *i-Ready Diagnostic Scale Score Increases to Achieve Specified Years of Growth in Mathematics*: Growth targets are for all students in a chronological grade, independent of level in i-Ready. Retrieved from http://www.kent.k12.wa.us/cms/lib/WA01001454/Centricity/Domain/728/Using-i-Ready-as-a-Student-Growth-Measure_2014-2105.pdf

CLC Results

CLC Students' i-Ready Results and District Benchmark Test Results (Experimental Group; n=18)

Student ID	i-Ready Pre-Assessment Score Administered the week of 10/13/2014	Tier Designation as Determined by i-Ready Screener Pre-Assessment	District Benchmark Quarter 1 Administered the week of 11/2/2014	i-Ready Post-Assessment Score Administered the week of 1/26/2015	Tier Designation as Determined by i-Ready Screener Post-Assessment	District Benchmark Quarter 2 Administered the week of the 1/19/2015	i-Ready End-Assessment Score Administered the week of 3/27/2015	Tier Designation as Determined by i-Ready Screener End-Assessment	Difference in i-Ready Scores from Pre to End
C17	387	II	50%	406	II	32%	399	II	+12
C16	378	III	84%	384	III	75%	417	II	+39
C12	394	II	78%	436	I	89%	429	II	+35
C4	408	II	75%	415	II	71%	407	II	-1
C2	382	III	72%	395	II	36%	393	II	+11
C18	444	I	94%	447	I	100%	463	I	+19
C7	404	II	97%	435	I	93%	435	I	+31
C3	415	II	94%	435	I	96%	461	I	+46
C19	369	III	59%	363	III	68%	389	II	+20
C5	441	I	97%	461	I	96%	490	I	+49
C20	352	III	56%	388	III	39%	380	III	+28
C15	395	II	75%	414	II	61%	430	I	+35
C8	388	III	91%	432	I	86%	453	I	+65
C11	409	II	94%	428	II	100%	470	I	+61
C1	384	III	50%	396	II	39%	404	II	+20
C6	399	II	100%	427	II	86%	440	I	+41
C14	437	I	100%	442	I	93%	475	I	+38
C13	378	III	53%	402	II	29%	406	II	+28

Key

- Students who increased in Tier Levels during the study (e.g. was at Tier II for Pre-Assessment and went to Tier III at Post-Assessment)
- Students who decreased in Tier Levels during the study (e.g. was at Tier II for Pre-Assessment and went to Tier I at Post-Assessment)
- Students who made at least 1.0 year of growth by end of school year for 2014-2015 (For Grade 2, 1.0 Year Ranges between 32 – 41) [1]

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