

Polyhedral Analysis of Plethysms and Kronecker Coefficients

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CHAPTER I

Introduction

Integer points in polyhedra show up regularly in combinatorial representation theory. Many combinatorial rules for computing multiplicities in representations can be expressed as counting the number of integer points in a polytope, and the weights of many representations end up corresponding to integer points in a polytope. In this thesis we will study three such polytopes which occur in the representation theory related to Kronecker coefficients and plethysms.

We will begin by looking at Kronecker coefficients. For λ a partition of size n , let I_λ denote the irreducible representation of S_n indexed by λ . Given a triple of partitions α, β , and γ of size n , the *Kronecker coefficient* is

$$g(\lambda, \mu, \nu) = \text{multiplicity of } I_\nu \text{ in } I_\lambda \otimes I_\mu.$$

In 1938, Murnaghan [11] published his well known result about a stability property of the Kronecker coefficients. If you grow the first row of the triple of partitions, then the Kronecker coefficient eventually stabilizes to a fixed value; i.e.

$$g(\lambda + (m), \mu + (m), \nu + (m))$$

is eventually constant for m large enough. More recently, Stembridge [14] generalized this result to show that a similar stability occurs for other triples of partitions. A

triple of partitions α, β, γ is *stable* if

$$g(\lambda + m \cdot \alpha, \mu + m \cdot \beta, \nu + m \cdot \gamma)$$

is eventually constant for any partitions λ, μ, ν . During the study of this generalized stability, Stembridge produced some sufficient conditions for a triple of partitions to be stable. He does this by finding a pair of polytopes whose integer points count the multiplicities of irreducible representations for representations related to $I_\alpha \otimes I_\beta$. He showed that a triple of partitions α, β, γ is stable if their Kronecker coefficient is nonzero and either of these related polytopes is 0-dimensional. In chapters II and III we investigate these polytopes. In both cases, we show how to find all triples for which this polytope is 0-dimensional. This is done by looking at the cone formed of all triples for which the polytope is nonempty, and then using the fact that the dimension of this polytope is constant on the interior of the faces of this cone.

Chapter II is dedicated to studying the simpler of these two polytopes for Kronecker coefficients. This polytope is related to the well known class of transportation polytopes. Vallejo [15] used this relation to transportation polytopes to show how to produce stable triples using ‘additive matrices’. Manivel [9], [10] takes a more geometric approach to produce the triples for which this polytope is 0-dimensional. Using an approach based on Schur-Weyl duality, he finds these stable triples by studying certain faces of the moment polytope for the Kronecker coefficients. We take a more combinatorial approach to finding these triples by looking at a cone formed from the polytopes.

In chapter III we study the ‘stronger’ of the two polytopes (this polytope can be used to produce any stable triple the other polytope can and more). This polytope is based on a sum of Littlewood-Richardson coefficients. We show how this polytope can be produced from a modification of the hives and honeycombs of Knutson and

Tao. Hives and honeycombs were first used by Knutson and Tao to prove the saturation conjecture for Littlewood-Richardson coefficients [5],[7]. They later used the honeycombs to show how to use certain puzzles to find the defining inequalities for the cone of triples for which the Littlewood-Richardson coefficients are nonzero [6]. We use a modification of their puzzles to produce the inequalities for the analogous cone for our polytope, and then show how to use these puzzles to find the dimension of our polytope for the points in the corresponding face of the cone.

In chapter IV we depart from Kronecker coefficients and instead look at plethysms. We will be interested in the plethystic representations of GL_n formed by composing Schur functors. While combinatorial rules are known for finding the decomposition into irreducibles for a few special cases, there is no general positive combinatorial formula for finding these multiplicities. In fact, there is no general rule for knowing when the multiplicities are nonzero. For any representation of GL_n , the weights of the representation which are maximal in the dominance order correspond to irreducible representations which have nonzero multiplicity. We investigate these maximal weights and find an algorithm for computing them. This is done by looking at the weight polytope for the representation. One surprising discovery made using this algorithm is that there are plethysms whose weight polytope is not saturated (i.e. there are integer points in the polytope which are not weights of the representations).

1.1 Symmetric Functions

1.1.1 Partitions

Definition I.1. A *composition* is a sequence (finite or infinite) of non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots)$ where only finitely many of the entries α_i are nonzero. We will consider two compositions to be equivalent if they only differ in the number of

0's at the end of the sequences.

Definition I.2. A *partition* is a composition λ whose entries are weakly decreasing, i.e. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$.

Remark I.3. Every composition has an associated partition formed by listing the entries of the composition in decreasing order.

Notation I.4. We call λ_i the *i-th part* of λ .

Definition I.5. The *length* of λ , denoted $l(\lambda)$, is the number of nonzero parts of λ .

Definition I.6. A partition λ is *strict* if its parts are strictly decreasing, i.e. $\lambda_1 > \lambda_2 > \dots > \lambda_l$.

Definition I.7. The *l-staircase partition* δ_l is the strict partition $(l-1, l-2, \dots, 0)$.

Definition I.8. The *size* of λ , denoted $|\lambda|$, is the sum of the parts of λ .

Definition I.9. The *dominance order* on the set of partitions of size n is the partial order defined by $\lambda \geq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for each $i \geq 1$, where we append 0's to the ends of λ and μ so that they are the same length.

Notation I.10. We will sometimes use $1^{n_1}2^{n_2}3^{n_3}\dots$ to denote the partition with n_i parts equal to i for $i = 1, 2, 3, \dots$.

1.1.2 Diagrams and Tableaux

Definition I.11. The *diagram*, $D(\lambda)$, of a partition λ is the set $\{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i, 1 \leq i\}$. We will use matrix notation for the diagrams, so the (i, j) position of the diagram is the cell i rows from the top and j columns from the left.

Notation I.12. We use $\mu \subset \lambda$ to denote that $D(\mu) \subset D(\lambda)$, i.e. $\lambda_i \geq \mu_i$ for all i .

Definition I.13. For any two partitions λ and μ with $\mu \subset \lambda$, we define the skew diagram to be $D(\lambda/\mu) = \{(i, j) \in \mathbb{Z}^2 : \mu_i < j \leq \lambda_i, 1 \leq i\}$. This diagram $D(\lambda/\mu)$ is equal to $D(\lambda) \setminus D(\mu)$ as sets.

Definition I.14. For any partitions λ and μ with $\mu \subset \lambda$, a *tableau of shape λ/μ* is an assignment of the cells of $D(\lambda/\mu)$ using entries in an ordered alphabet (usually the positive integers).

Definition I.15. We say that a tableau is a *semistandard Young tableau* if in the assignment the entries are weakly increasing from left to right across each row and strictly increasing from top to bottom along each column.

Remark I.16. Any ordered alphabet can be used and this definition will still be valid.

Definition I.17. The *content* of a tableau T is the composition $(\alpha_1, \alpha_2, \dots)$ where α_i is the number of times that i occurs as an entry in T .

1.1.3 Symmetric functions

Let $\mathbb{C}[x_1, x_2, \dots]$ be the ring of polynomials in a countable set of indeterminates x_1, x_2, \dots .

Definition I.18. A *symmetric function* is a formal power series in $\mathbb{C}[x_1, x_2, \dots]$ that has bounded degree and is invariant under permutations of the variables.

Definition I.19. The set of symmetric functions forms a ring denoted $\Lambda = \Lambda_{\mathbb{C}}$.

Definition I.20. Let Λ^n denote the set of all homogeneous symmetric functions of degree n .

Definition I.21. For any integer $n \geq 1$, the *n -th power sum* symmetric function is $p_n = \sum_i x_i^n$. In addition, for any partition λ we set $p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots$.

Definition I.22. For any integer $n \geq 1$, the n -th complete homogeneous symmetric function is

$$h_n = \sum_{a_1 \leq a_2 \leq \dots \leq a_n} x_{a_1} x_{a_2} \cdots x_{a_n}.$$

In addition, for any partition λ we set $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots$.

Definition I.23. For any integer $n \geq 1$, the n -th elementary symmetric function is

$$e_n = \sum_{a_1 < a_2 < \dots < a_n} x_{a_1} x_{a_2} \cdots x_{a_n}.$$

In addition, for any partition λ we set $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$.

Definition I.24. For any partition λ , the monomial symmetric function m_λ is defined by

$$m_\lambda = \sum_{\alpha} x_{\alpha_1}^{\lambda_1} x_{\alpha_2}^{\lambda_2} \cdots$$

where the sum is over all compositions α whose associated partitions is λ .

Theorem I.25 (Theorem 7.4.4, Corollaries 7.6.2 and 7.7.2 of [13]). Λ is freely generated as a \mathbb{C} -algebra by the p_k , the h_k , or the e_k . i.e.

$$\Lambda = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[p_1, p_2, \dots]$$

Equivalently, for any n , the p_λ , the h_λ , and the e_λ over λ of size n , all form bases for Λ^n as a vector space.

Definition I.26. The Hall inner product is the inner product on Λ defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda, \mu}$.

Notation I.27. To each composition α we associate the monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, and to each semistandard Young tableau T we associate the monomial $x^T = x^\alpha$ where α is the content of T .

Definition I.28. For any partitions $\mu \subset \lambda$

- The *schur function* s_λ is

$$s_\lambda(x_1, x_2, \dots) = \sum_T x^T$$

where the sum is over all semistandard Young tableaux T of shape λ .

- The *skew-schur function* $s_{\lambda/\mu}$ is

$$s_{\lambda/\mu}(x_1, x_2, \dots) = \sum_T x^T$$

where the sum is over all semistandard Young tableaux T of shape λ/μ .

Theorem I.29 (Corollary 7.10.6 of [13]). *The s_λ form an orthonormal basis for Λ as a vector space.*

Definition I.30. For any partitions λ and μ , the *Kostka number* $K_{\lambda,\mu}$ is the coefficient of m_μ when s_λ is decomposed into the basis of monomial symmetric functions.

$$K_{\lambda,\mu} = \langle s_\lambda, h_\mu \rangle = \#\{\text{semistandard Young tableaux of size } \lambda \text{ and content } \mu\}.$$

Definition I.31. A *Gelfand-Tsetlin pattern* is a triangular array $(x_{i,j})_{1 \leq i \leq j \leq n}$ satisfying the inequalities:

$$x_{i,j} \geq 0, \text{ for } 1 \leq i \leq j \leq n \text{ and}$$

$$x_{i,j+1} \geq x_{i,j} \geq x_{i+1,j+1}, \text{ for } 1 \leq i \leq j \leq n-1.$$

Definition I.32. The *Gelfand-Tsetlin polytope* $GT(\lambda, \mu)$, $n \geq l(\lambda), l(\mu)$, is the polytope formed by Gelfand Tsetlin patterns satisfying the equalities:

$$\begin{aligned} x_{in} &= \lambda_i, \text{ for } 1 \leq i \leq n \text{ and} \\ \sum_{i=1}^j x_{ij} &= \mu_1 + \mu_2 + \dots + \mu_j. \end{aligned}$$

Proposition I.33. *The Kostka number $K_{\lambda,\mu}$ is equal to the number of integer points in the Gelfand-Tsetlin polytope $GT(\lambda, \mu)$.*

1.1.4 Plethysm

Definition I.34. Let $f, g \in \Lambda$ be symmetric functions, and suppose that f is expressible as a countable sum of monomials, say

$$f = x^\alpha + x^\beta + x^\gamma + \cdots .$$

The plethysm $g[f]$ is the symmetric function

$$g[f] = g(x^\alpha, x^\beta, x^\gamma, \cdots).$$

Remark I.35. The plethysms of the form $s_\lambda[s_\mu]$ will be of particular interest to us. In this symmetric function, we take s_λ and specialize the x_i to be the monomials we obtain from semistandard Young tableaux of shape μ . In essence, this means we are taking the sum of the monomials which occur from semistandard Young tableaux of shape λ where each of the entries in the tableau is itself a semistandard Young tableau of shape μ .

Theorem I.36 (Appendix A (7.4) of [8]). *For any two partitions λ and μ , the plethysm $s_\lambda[s_\mu]$ is a \mathbb{N} -linear combination of Schur functions.*

1.1.5 Kronecker product

Notation I.37. Given a partition $\lambda = 1^{n_1}2^{n_2}3^{n_3} \cdots$, let z_λ denote the product

$$z_\lambda := 1^{n_1}1!2^{n_2}2!3^{n_3}3! \cdots$$

Definition I.38. Define the *internal product*, $*$, on Λ by setting

$$p_\lambda * p_\mu = z_\lambda \delta_{\lambda, \mu} p_\mu$$

for any partitions λ, μ , and extending this to all of Λ using linearity.

Proposition I.39 (Exercise 7.78 of [13]). *There are nonnegative integers $g(\lambda\mu\nu)$ such that*

$$s_\lambda * s_\mu = \sum_\nu g(\lambda\mu\nu) s_\nu.$$

These $g(\lambda\mu\nu)$ are called Kronecker coefficients.

Proposition I.40 (Exercise 7.78 of [13]). *The Kronecker coefficients are symmetric in λ , μ , and ν .*

Definition I.41. A *contingency table* of type (α, β) is a nonnegative integer matrix T whose row sum vector is α and column sum vector is β . The content $\text{co}(T)$ of the table is the partition formed by ordering all the entries in the matrix T .

Notation I.42. For any two partitions α and β , let $C(\alpha, \beta)$ denote the set of all contingency tables of type (α, β) .

Proposition I.43 (Exercise 7.84 of [13]). *Let λ and μ be partitions of size n with $l(\lambda) = l$.*

•

$$h_\lambda * s_\mu = \sum \prod_{i \geq 1} s_{\mu^i / \mu^{i-1}},$$

where the sum is over all sequences $(\mu^0, \mu^1, \dots, \mu^l)$ of partitions such that

$$\emptyset = \mu^0 \subset \mu^1 \subset \dots \subset \mu^{l(\lambda)} = \mu \text{ and } |\mu^i / \mu^{i-1}| = \lambda_i \text{ for all } i.$$

•

$$h_\lambda * h_\mu = \sum_T h_{\text{co}(T)},$$

where T ranges over all contingency tables of type (λ, μ) .

1.1.6 Littlewood-Richardson Rule

Definition I.44. Define the Littlewood-Richardson coefficients $c_{\mu,\nu}^\lambda$ by letting them be the coefficients in the expansion:

$$s_{\lambda/\mu} = \sum c_{\mu\nu}^\lambda s_\nu$$

Definition I.45. The *reverse row reading word* for a tableau T is the sequence $w(T) = w_1 w_2 \cdots$ formed by reading the rows of T from right to left, starting at the top row.

Definition I.46. A positive integer sequence $w = w_1 w_2 \cdots$ is said to be a *lattice word* if for any $i, j \geq 1$, the number of occurrences of i in the subword $w_1 w_2 \cdots w_j$ is greater than or equal to the number of occurrences of $i + 1$.

Theorem I.47 (The Littlewood-Richardson Rule, Theorem A1.3.3 of [13]). *The coefficient $c_{\mu\nu}^\lambda$ is equal to the number of semistandard Young tableaux of shape λ/μ with content ν whose reverse row reading word is a lattice word.*

1.2 Representation Theory

Notation I.48. For a finite dimensional vector space V , let $GL(V)$ be the group of linear automorphisms of V . We use the notation $GL_n = GL(V)$ if $V = \mathbb{C}^n$.

Notation I.49. For any integer n , let S_n denote the symmetric group on n letters.

Definition I.50. A *representation* of a group G on a finite dimensional complex vector space V is a homomorphism $\rho_V : G \rightarrow GL(V)$. Throughout this dissertation, there will be no ambiguity about the map ρ_V used for any V , so we will call V itself a representation of G and we will write $g \cdot v$ for $\rho_V(g)(v)$.

Remark I.51. For any two representations V and W of a group G , $V \oplus W$ and $V \otimes W$ are both also representations of G given by $g \cdot (v \oplus w) = (g \cdot v) \oplus (g \cdot w)$ and $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$.

Definition I.52. A *subrepresentation* of a representation V is a subspace W of V which is preserved under the action of G .

Definition I.53. A representation is called *irreducible* if it has no proper nonzero subrepresentation.

1.2.1 Representations of S_n

Definition I.54. Let $\mathbb{C}S_n$ be the vector space of formal \mathbb{C} -linear sums of permutations in S_n ; i.e. $\mathbb{C}S_n = \{c_1w_1 + c_2w_2 + \cdots + c_kw_k : c_i \in \mathbb{C}, w_i \in S_n\}$. We can define a product on $\mathbb{C}S_n$ by setting $(c_1w_1) \cdot (c_2w_2) = (c_1c_2)(w_1w_2)$ and extending this to all of $\mathbb{C}S_n$ by imposing the distributive property. This $\mathbb{C}S_n$ is a \mathbb{C} -algebra under this product which is called the *group algebra of S_n* .

Remark I.55. Any representation of S_n can be made into a $\mathbb{C}S_n$ -module by extending the action of S_n to the group ring using linearity. Similarly, any $\mathbb{C}S_n$ -module can be made into a representation of S_n by setting $\rho(w)$ to be the linear isomorphism $v \mapsto w \cdot v$. Hence, working with representations of S_n is equivalent to working with $\mathbb{C}S_n$ -modules.

Definition I.56. Fix a partition λ of size n and a tableau T of shape λ such that $1, \dots, n$ each occur once. We say that an element of S_n fixes a row (resp. column) of T if it preserves the set of values of T occurring in the row (resp. column). Set

$$P = \{w \in S_n : w \text{ preserves each row of } T\}$$

$$Q = \{w \in S_n : w \text{ preserves each column of } T\}.$$

We define two elements of $\mathbb{C}S_n$ as follows:

$$a_T = \sum_{w \in P} w$$

$$c_T = a_T \sum_{w \in Q} \text{sgn}(w)w.$$

We call this c_T a *Young symmetrizer*.

Theorem I.57 (§ 7.2 of [2]). *Given a partition λ of size n and tableau T such that $1, \dots, n$ each occur once, the following hold:*

- *The $\mathbb{C}S_n$ -module $I_T = \mathbb{C}S_n c_T$ is an irreducible module.*
- *If T' is any other tableau of λ then I_T and $I_{T'}$ are isomorphic. We will use I_λ to denote the irreducible representation of S_n given by these I_T .*
- *The irreducible representations of S_n are precisely the I_μ for μ of size n .*
- *The irreducible I_λ is a subrepresentation of the $\mathbb{C}S_n$ -module $A_\lambda = \mathbb{C}S_n a_T$.*

Theorem I.58 (Proposition 1.8 of [3]). *Every representation of S_n is a direct sum of irreducible representations of S_n . In addition, this decomposition is unique in the following sense: if $V_1 \oplus V_2 \oplus \dots \oplus V_k$ and $W_1 \oplus W_2 \oplus \dots \oplus W_l$ are two decompositions of a representation into irreducibles, then $k = l$ and by reordering the W_i we have $V_i \cong W_i$ for $i = 1, 2, \dots$.*

Definition I.59. The *character* of a representation V is the complex valued map $\chi_V : S_n \rightarrow \mathbb{C}$ defined by $\chi_V(w) = \text{Tr}(\rho_V(w))$, the trace of the linear endomorphism of V given by w .

Remark I.60. For any two representations V and W of S_n , $\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \chi_W$.

Theorem I.61 (Corollary 2.14 of [3]). *A representation of S_n is uniquely determined up to isomorphism by its character.*

Definition I.62. The *cycle type* of a permutation $w \in S_n$ is the partition $\rho(w) = (\rho_1, \rho_2, \dots)$ such that ρ_1, ρ_2, \dots are the lengths of the cycles of w .

Definition I.63. A *class function* of S_n is a map $f : S_n \rightarrow \mathbb{C}$ that is constant on conjugacy classes of S_n (i.e. $f(\nu^{-1}w\nu) = f(w) \forall w, \nu \in S_n$). The set of class functions of S_n , denoted CF^n , is a vector space over \mathbb{C} .

Definition I.64. We can define a hermitian inner product on CF^n by setting

$$\langle f, g \rangle = \frac{1}{n!} \sum_{w \in S_n} f(w) \overline{g(w)}$$

Proposition I.65 (Proposition 2.30 of [3]). *The characters of S_n are class functions, and the irreducible characters form an orthonormal basis.*

Definition I.66. The *Frobenius characteristic map* ch_n is the map $CF^n \rightarrow \Lambda^n$ defined by

$$ch_n(f) = \frac{1}{n!} \sum_{w \in S_n} f(w) p_{\rho(w)}.$$

Proposition I.67 (§ 7.18 and Exercise 7.78 of [13]). *The characteristic map has the following properties:*

- *It is a ring homomorphism; i.e. it is \mathbb{C} -linear and $ch_n(fg) = ch_n(f) * ch_n(g)$ where $*$ is the internal product.*
- *It preserves the scalar product, i.e. $\langle f, g \rangle = \langle ch_n(f), ch_n(g) \rangle$ where the we use the Hall inner product on Λ .*
- *For λ a partition of size n , $ch_n(\chi_{I_\lambda}) = s_\lambda$.*
- *For λ a partition of size n , $ch_n(\chi_{A_\lambda}) = h_\lambda$ where $A_\lambda = \mathbb{C}S_n a_T$ for any T of shape λ .*

Remark I.68. For any partition λ of size n and representation V , the multiplicity of W_λ in any decomposition of V into irreducibles is equal to $\langle \chi_V, \chi_{I_\lambda} \rangle = \langle ch_n(\chi_V), s_\lambda \rangle$. Hence, the Kronecker coefficient

$$g(\lambda\mu\nu) = \langle s_\lambda * s_\mu, s_\nu \rangle = \langle \chi_{I_\lambda \otimes I_\mu}, \chi_{I_\nu} \rangle$$

is the multiplicity of I_ν in the tensor product $I_\lambda \otimes I_\mu$.

1.2.2 Representations of GL_n

Definition I.69. A representation V of GL_n is called *polynomial* if each of the coordinate functions in the map $GL_n \rightarrow GL(V)$ is a polynomial in the coordinates of GL_n . We will restrict ourselves to only working with polynomial representations of GL_n , and will omit the word polynomial from now on.

Notation I.70. Given a vector space V and partition λ , we define a $GL(V)$ representation $\mathbb{S}^\lambda V = V^{\otimes d} \otimes_{\mathbb{C}S_d} I_\lambda$ where I_λ is the irreducible representation of S_d and $GL(V)$ acts on this space by acting on the first factor. This construction is functorial and we call \mathbb{S}^λ a *Schur functor*. The representation $\mathbb{S}^\lambda V$ is nonzero if and only if $l(\lambda) \leq n$.

Theorem I.71 (Theorem 6.4(4) of [3]). *The irreducible representations of GL_n are the representations $\mathbb{S}^\lambda(\mathbb{C}^n)$ for λ of length at most n .*

Theorem I.72 (§9.3 of [3]). *Every representation of GL_n is a direct sum of irreducible representations and the decomposition is unique in the same sense as for S_n (see Theorem I.58).*

Notation I.73. Let D_n denote the set of diagonal matrices in GL_n , and let $\text{diag}(x_1, \dots, x_n)$ denote the diagonal matrix with values x_1, \dots, x_n going southeast along the diagonal.

Definition I.74. A vector v in a representation is said to be a *weight vector with weight* (a_1, \dots, a_n) if for any $D = \text{diag}(x_1, \dots, x_n) \in D_n$,

$$D \cdot v = x_1^{a_1} \cdots x_n^{a_n} v.$$

Definition I.75. Given a representation V and $(a_1, \dots, a_n) \in \mathbb{Z}_+^n$, the (a_1, \dots, a_n) *weight space* of V is $V(\lambda) =$ the span of all weight vectors with weight $a = (a_1, \dots, a_n)$.

Proposition I.76 ((14.4) of [3]). *Every representation V is a direct sum of its weight spaces, i.e. $V = \bigoplus_a V(a)$.*

Definition I.77. The character of a GL_n representation V is the complex valued map $\chi_V : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ defined by $\chi_V(x_1, \dots, x_n) = \text{Tr}(\rho_V(\text{diag}(x_1, \dots, x_n)))$.

Remark I.78. For a GL_n representation V , the character χ_v is a symmetric polynomial. Moreover, by Proposition I.76, χ_V is equal to the sum $\sum_a \dim(V_a) x^a$ where the sum is over all weights a of V .

Remark I.79. For any two representations V and W of GL_n , $\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \chi_W$.

Proposition I.80 (pg. 78 of [3]). *A representation of GL_n is uniquely determined up to isomorphism by its character.*

Theorem I.81 (Theorem 6.4(3) of [3]). *The character for the irreducible representation $\mathbb{S}^\lambda(\mathbb{C}^n)$ is the schur function $s_\lambda(x_1, \dots, x_n)$ (where we set all the coordinates x_{n+1}, x_{n+2}, \dots to be 0).*

Remark I.82. This theorem gives us that for any GL_n representation V , the multiplicity of $\mathbb{S}^\lambda(\mathbb{C}^n)$ in any decomposition of V into irreducibles is equal to $\langle \chi_V, s_\lambda \rangle$.

Remark I.83. We can apply a Schur functor to any representation of GL_n to get another representation of GL_n . For example, given any two partitions λ and μ we

can construct the GL_n representation $\mathbb{S}^\lambda(\mathbb{S}^\mu(\mathbb{C}^n))$. This representation has character equal to the plethysm $s_\lambda[s_\mu]$.

Theorem I.84 (§8.3 Corollary 2(c) of [2]). *The Littlewood-Richardson coefficient $c_{\mu\nu}^\lambda$ is the multiplicity of $\mathbb{S}^\lambda(\mathbb{C}^n)$ in $\mathbb{S}^\mu(\mathbb{C}^n) \otimes \mathbb{S}^\nu(\mathbb{C}^n)$.*

CHAPTER II

Kronecker Stability and the $\bar{\mathcal{H}}_{abc}$ Polytope

2.1 Stability in the Kronecker Coefficients

One interesting property of the Kronecker coefficients is that they display a stability property. Namely, there are triples of partitions $\alpha\beta\gamma$ such that for any triple $\lambda\mu\nu$ the sequence

$$g(\lambda + n\alpha, \mu + n\beta, \nu + n\gamma) = g(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$$

converges to a fixed positive value as $n \rightarrow \infty$ (Note: In order for this limit to be nonzero we need $\alpha\beta\gamma$ to be of the same size and similarly for $\lambda\mu\nu$).

Definition II.1. We say that a triple $\alpha\beta\gamma$ is *stable* if $g(\alpha\beta\gamma) > 0$ and it satisfies this condition.

Stembridge studied this generalized notion of stability in [14]. In his paper, Stembridge showed that $g(n \cdot \alpha\beta\gamma)$ has to equal 1 for all n in order for the triple to be stable and conjectured that this necessary condition for stability was also sufficient. Sam and Snowden later proved that this is in fact sufficient [12].

Theorem II.2 (Sam-Snowden). *A triple $\alpha\beta\gamma$ is stable if and only if*

$$(2.1) \quad g(n \cdot \alpha\beta\gamma) = 1 \quad \forall n \geq 1.$$

This provides us with a means of not only testing if a triple is stable but of also producing stable triples. The difficulty is that there currently is no positive combinatorial formula known for computing Kronecker coefficients, so it is not easy to directly produce triples which satisfy (2.1). Stembridge suggests a method of getting around this by working with coefficients which are in some cases larger than the Kronecker coefficients, but which have a nice combinatorial description that can be used to show that (2.1) is satisfied.

Definition II.3. For any triple $\alpha\beta\gamma$ of partitions, define coefficients $f(\alpha\beta\gamma)$ and $h(\alpha\beta\gamma)$ as follows:

$$(2.2) \quad f(\alpha\beta\gamma) = \langle h_\alpha * s_\beta, s_\gamma \rangle$$

$$(2.3) \quad h(\alpha\beta\gamma) = \langle h_\alpha * h_\beta, s_\gamma \rangle$$

Recall that $h_\alpha = \sum K_{\lambda,\alpha} s_\lambda$. Plugging this into the definitions for $f(\alpha\beta\gamma)$ and $h(\alpha\beta\gamma)$ we obtain following expression for these new coefficients in terms of the Kronecker coefficients:

$$(2.4) \quad f(\alpha\beta\gamma) = \sum_{\lambda} K_{\lambda,\alpha} g(\lambda\beta\gamma)$$

$$(2.5) \quad h(\alpha\beta\gamma) = \sum_{\lambda,\mu} K_{\lambda,\alpha} K_{\mu,\beta} g(\lambda\mu\gamma)$$

These coefficients are therefore positive sums of Kronecker coefficients, and the coefficient of $g(\alpha\beta\gamma)$ in both sums is 1. Stembridge uses this to show that if an analogous notion of stability occurs for these coefficients, then there must be a related triple which is stable for the Kronecker coefficients.

Theorem II.4 (Theorem 6.1 of [14]). *If $\alpha\beta\gamma$ is a triple of partitions such that*

$$h(n * \alpha\beta\gamma) = 1 \quad \forall n \geq 1$$

then there exists a unique Kronecker stable triple $\alpha^+\beta^+\gamma$ with $\alpha^+ \geq \alpha$ and $\beta^+ \geq \beta$ in the dominance order.

Theorem II.5 (Theorem 7.1 of [14]). *If $\alpha\beta\gamma$ is a triple of partitions such that*

$$f(n * \alpha\beta\gamma) = 1 \quad \forall n \geq 1$$

then there exists a unique Kronecker stable triple $\alpha^+\beta\gamma$ with $\alpha^+ \geq \alpha$ in the dominance order.

For any partition λ , the symmetric function h_λ is a positive sum of schur functions in which s_λ occurs with coefficient 1. Applying this to our definitions for the f and h coefficients, we obtain the following inequalities:

$$g(\alpha\beta\gamma) \leq f(\alpha\beta\gamma) \leq h(\alpha\beta\gamma).$$

This means that any stable triple we can detect using Theorem II.5 can also be found using Theorem II.4. Despite this, looking into producing h -stable triples will still be fruitful since finding them will be computationally less intensive than finding f -stable triples.

We introduced these coefficients since, unlike the Kronecker coefficients, we do have nice combinatorial equations for computing them. Namely, both the f and h coefficients are equal to the number of integer points inside certain rational polytopes. In both cases, the polytope (call it $P(\alpha\beta\gamma)$ for now) will have the following 2 key properties:

- $P(n \cdot \alpha\beta\gamma) = nP(\alpha\beta\gamma) \quad \forall n \geq 0$.
- The vertices of $P(\alpha\beta\gamma)$ are rational points.

These two properties are useful since they allow us to use $P(\alpha\beta\gamma)$ to narrow down when the f or h coefficients will be stable.

Proposition II.6. *If a triple of partitions $\alpha\beta\gamma$ is f or h stable, then $P(\alpha\beta\gamma)$ is 0-dimensional.*

Proof. Suppose $P(\alpha\beta\gamma)$ is not 0-dimensional. Then $P(\alpha\beta\gamma)$ must have at least 2 vertices. Let n be the least common multiple of all the denominators of two of the vertices of $P(\alpha\beta\gamma)$. Then $nP(\alpha\beta\gamma)$ must contain at least 2 integer points, and hence the coefficient at $n \cdot \alpha\beta\gamma$ is not equal to 1. \square

The converse of this theorem doesn't necessarily hold. The polytope being 0-dimensional does not guarantee that the triple is stable since the polytope could be a rational point that is not integer causing the coefficient to be 0. However, the polytope being 0-dimensional does guarantee that some multiple of the triple will be stable; namely if k is the least common multiple of all the denominators of the vertex of $P(\alpha\beta\gamma)$, then $P(k \cdot \alpha\beta\gamma)$ is an integer point and so is $P(n \cdot k \cdot \alpha\beta\gamma) \forall n \geq 1$, and so $k \cdot \alpha\beta\gamma$ is stable. Finding the triples where the polytope is 0-dimensional asymptotically solves the problem of finding f or h stable triples. We can therefore reduce our search for f and h stable triples to searching for when the corresponding polytope $P(\alpha\beta\gamma)$ is 0-dimensional.

The rest of this chapter will be devoted to showing how to find the triples for which the polytope for h is 0-dimensional, while Chapter III will be devoted to doing the same thing for the polytope for the f coefficients. In both cases, we will see that the set of such triples correspond to certain faces of a cone. By finding points in these faces of the cones, it is possible to produce all stable triples whose stability can be shown using Theorems II.4 and II.5.

2.2 The $H(\alpha\beta\gamma)$ polytope

Recall from Proposition I.43 that we have the following expression for the internal product:

$$h_\alpha * h_\beta = \sum_{T \in C(\alpha, \beta)} h_{\text{co}(T)}$$

where the sum is over all contingency tables of content (α, β) . Applying this to our definition of $h(\alpha\beta\gamma)$ we find the following formula for the h coefficients:

$$(2.6) \quad h(\alpha\beta\gamma) = \sum_{T \in C(\alpha, \beta)} K_{\gamma, \text{co}(T)}.$$

Our goal now is to find a polytope $H(\alpha\beta\gamma)$ such that $h(\alpha\beta\gamma)$ is equal to the number of integer points in $H(\alpha\beta\gamma)$. We will do this using the same polytope and reasoning of Stembridge (Lemma 7.2 and Remark 7.3 of [14]).

Fix α and β , and let a and b be integers such that $l(\alpha) \leq a$ and $l(\beta) \leq b$. Given a contingency table T , let $\text{co}'(T)$ be the sequence formed by reading the entries of the rows of T from left to right, going from the top row to the bottom row. We have $K_{\gamma, \text{co}(T)} = K_{\gamma, \text{co}'(T)}$ since $\text{co}(T)$ and $\text{co}'(T)$ have the same entries but in a different order and the Kostka numbers are constant under permuting the entries of the composition of the second index. Hence, $h(\alpha\beta\gamma) = \sum_{T \in C(\alpha, \beta)} K_{\gamma, \text{co}'(T)}$. Similarly, when working with semistandard Young tableaux, the alphabet used does not change the values of the Kostka number.

Notation II.7. Let \leq_L denote lexicographical order on \mathbb{Z}^2 :

$$(i, j) <_L (k, l) \text{ if either } i < k \text{ or } i = k \text{ and } j < l.$$

Instead of using \mathbb{N} as the alphabet, consider tableaux using entries in $\{1, \dots, a\} \times \{1, \dots, b\}$ ordered using \leq_L . The Kostka number $K_{\gamma, \text{co}'(T)}$ is then equal to the number of semistandard Young tableau of shape γ such that (i, j) occurs with multiplicity

$T(i, j)$ for $1 \leq i \leq a, 1 \leq j \leq b$. Define the *bicontent* of a tableau S to be the pair of compositions (λ, μ) where λ_i (resp. μ_i) is the number of times that i occurs in the first (resp. second) coordinate of an entry in S . A tableau of shape γ contributes to the sum in (2.6) if and only if it has bicontent (α, β) . We therefore have that (2.6) is equivalent to the formula

$$(2.7) \quad h(\alpha\beta\gamma) = \#\{\text{semistandard Young tableau of shape } \gamma \text{ with bicontent } (\alpha, \beta)\}.$$

If we set $c := ab$, then the partition γ must have length less than or equal to c in order for the right hand side of (2.7) to be nonzero. Suppose S is a semistandard Young tableau of shape γ . Since the rows of a semistandard Young tableau must be weakly increasing, a semistandard Young tableau is uniquely determined by how many of each letter of the alphabet there are in each row. Let $x(i, j, k)$ be the number of times (i, j) occurs as an entry in row k of S . Since S has bicontent (α, β) and shape γ , for any i, j, k we must have that

$$(2.8) \quad \sum_{j^*, k^*} x(i, j^*, k^*) = \alpha_i, \quad \sum_{i^*, k^*} x(i^*, j, k^*) = \beta_j, \quad \sum_{i^*, j^*} x(i^*, j^*, k) = \gamma_k.$$

Given a set of coordinates $x(i, j, k)$ satisfying (2.8), we can use them to produce a semistandard Young tableau of shape γ and bicontent (α, β) if and only if when we fill in the rows in weakly increasing order, we never have an entry whose value above it is less than or equal to it, i.e. for any row k and entry (i, j) , we never have the boxes in row $k + 1$ with entries $\leq_L (i, j)$ extend further than the boxes in row k with entries $<_L (i, j)$. Expressing this criterion in terms of the $x(i, j, k)$ we obtain the following inequality:

$$(2.9) \quad \sum_{(i^*, j^*) <_L (i, j)} x(i^*, j^*, k) \geq \sum_{(i^*, j^*) \leq_L (i, j)} x(i^*, j^*, k + 1).$$

We therefore have that a set of nonnegative integers $x(i, j, k)$ for $1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c$ produces a semistandard Young tableau of shape γ and bicontent (α, β) if and only if they satisfy (2.8) and (2.9).

Definition II.8. Fix positive integers a, b and set $c := ab$. For any triple of partitions $\alpha\beta\gamma$ of lengths at most a, b, c respectively, let $H(\alpha\beta\gamma)$ be the polytope defined by the following set of inequalities:

$$\begin{aligned} x(i, j, k) &\geq 0 \quad (1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c), \\ \sum_{j^*, k^*} x(i, j^*, k^*) &= \alpha_i, \quad \sum_{i^*, k^*} x(i^*, j, k^*) = \beta_j, \quad \sum_{i^*, j^*} x(i^*, j^*, k) = \gamma_k, \\ \sum_{(i^*, j^*) <_L (i, j)} x(i^*, j^*, k) &\geq \sum_{(i^*, j^*) \leq_L (i, j)} x(i^*, j^*, k+1). \end{aligned}$$

Note: In these equations, i^*, j^*, k^* are summation variables whereas i, j, k parameterize the inequalities.

Proposition II.9. $h(\alpha\beta\gamma)$ is equal to the number of integer points in the polytope $H(\alpha\beta\gamma)$.

Proof. By what we have shown, any set of coordinates $x(i, j, k)$ corresponding to a semistandard Young tableau of (2.8) must occur as an integer point in $H(\alpha\beta\gamma)$. In addition, every integer point inside of $H(\alpha\beta\gamma)$ has a unique tableau producing that point. We therefore have a bijection between the integer points of $H(\alpha\beta\gamma)$ and the set on the right hand side of (2.8), hence $H(\alpha\beta\gamma)$ is a polytope with the claimed features. \square

Remark II.10. The definition of $H(\alpha\beta\gamma)$ did not require any of α, β , or γ to be partitions, they simply have to be points in $\mathbb{R}^a, \mathbb{R}^b$, and \mathbb{R}^c respectively. We will therefore relax our restrictions so that $H(\alpha\beta\gamma)$ is defined in this more general setting. When we do this, the inequalities of (2.9) force the entries of γ to be nonincreasing

in order for $H(\alpha\beta\gamma)$ to be nonempty, but there are no such constraints on α and β . This is related to the fact that the definition of h_α does not depend on the ordering of the parts of α where as s_γ requires γ to be a partition.

2.3 When is $H(\alpha\beta\gamma)$ nonempty?

Before finding when $H(\alpha\beta\gamma)$ is 0-dimensional, we will need to get a grasp of when the polytope is nonempty.

Definition II.11. Fix positive integers a, b, c where $c = ab$. Let $\bar{\mathcal{H}}_{abc}$ be the polyhedron consisting of all triples (α, β, γ) , $\alpha \in \mathbb{R}^a$, $\beta \in \mathbb{R}^b$, $\gamma \in \mathbb{R}^c$ such that $H(\alpha\beta\gamma)$ is nonempty.

In order to study this cone, we will need to use some intermediary polyhedra between $\bar{\mathcal{H}}_{abc}$ and the $H(\alpha\beta\gamma)$.

Definition II.12. Let \mathcal{H}_{abc} be the rational cone given by the inequalities

$$\begin{aligned} x(i, j, k) &\geq 0 \quad (1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c), \\ \sum_{(i^*, j^*) <_L (i, j)} x(i^*, j^*, k) &\geq \sum_{(i^*, j^*) \leq_L (i, j)} x(i^*, j^*, k + 1). \end{aligned}$$

These are the same inequalities as those for $H(\alpha\beta\gamma)$, except we have now removed the restriction that the points correspond to tableau of shape γ and bicontent (α, β) . The integer points in \mathcal{H}_{abc} correspond to all semistandard Young tableaux using the alphabet $\{1, \dots, a\} \times \{1, \dots, b\}$.

Notation II.13. Set

$$S_1(x) := \left(\sum_{j^*, k^*} x(1, j^*, k^*), \sum_{j^*, k^*} x(2, j^*, k^*), \dots, \sum_{j^*, k^*} x(a, j^*, k^*) \right),$$

the vector whose n -th component is the sum of the entries of x whose first coordinate is n . Define $S_2(x)$ and $S_3(x)$ similarly by letting them be the vectors whose entires

are these sums when the second and third components respectively are the fixed coordinate.

Then $\bar{\mathcal{H}}_{abc}$ is the image of \mathcal{H}_{abc} under the linear map

$$(2.10) \quad x \mapsto (S_1(x), S_2(x), S_3(x)).$$

Note: $\bar{\mathcal{H}}_{abc}$ is the image of a rational cone under a rational linear map so it is also a rational cone.

Definition II.14. Let \mathcal{I}_{abc} be the image of \mathcal{H}_{abc} under the linear map

$$(2.11) \quad x \mapsto (S_{12}(x), S_3(x))$$

where $S_{12}(x)$ is the matrix whose (i, j) entry is $\sum_{k^*} x(i, j, k^*)$.

The points in \mathcal{I}_{abc} are of the form (T, γ) where T is a ‘generalized’ contingency table of content (α, β) where the entries in the matrix only have to be nonnegative and don’t have to be integers (see Definition I.41).

Remark II.15. The preimage of an integer point (T, γ) under this map is a polytope which is isomorphic to the Gelfand-Tsetlin polytope for $K_{\gamma, co'(T)}$ (Definition I.32). This implies that the preimage of (T, γ) is nonempty precisely when γ is larger than $co(T)$ ($= co'(T)$ with the entries reordered to be nonincreasing) in the dominance order.

Our first linear map (2.10) : $\mathcal{H}_{abc} \rightarrow \bar{\mathcal{H}}_{abc}$ factors through \mathcal{I}_{abc} using (2.11): it is $\mathcal{H}_{abc} \rightarrow \mathcal{I}_{abc}$ composed with the linear map

$$(2.12) \quad (T, \gamma) \mapsto (S_1(T), S_2(T), \gamma),$$

where $S_1(T)$ and $S_2(T)$ are similarly defined for T . Hence, $\bar{\mathcal{H}}_{abc}$ is also the image of \mathcal{I}_{abc} under a linear map.

From here, we will continue by finding the inequalities defining the cone $\bar{\mathcal{H}}_{abc}$. We will compute the inequalities by first finding the extreme rays for the cone $\bar{\mathcal{H}}_{abc}$ and then finding the inequalities these rays all satisfy. To do this, we will find the rays of \mathcal{H}_{abc} , which gives us the rays of \mathcal{I}_{abc} and $\bar{\mathcal{H}}_{abc}$ by taking the image of these rays under maps (2.10) and (2.12). We will then use the properties of the rays for \mathcal{I}_{abc} to put constraints on the inequalities.

Definition II.16. Let $s = (s_1, \dots, s_n)$ be a sequence of pairs of integers $s_l = (i_l, j_l)$ such that s_1, s_2, \dots, s_n are strictly increasing in $<_L$. For any such sequence, let x_s be the point in \mathcal{H}_{abc} defined by

$$x_s(i, j, k) = \begin{cases} 1 & \text{if } (i, j, k) = (i_l, j_l, l) \text{ for some } l \\ 0 & \text{otherwise} \end{cases}$$

Lemma II.17. For any $x \in \mathcal{H}_{abc}$,

$$x = t_1 x_{s_1} + t_2 x_{s_2} + \dots + t_m x_{s_m}$$

for some $t_i \in \mathbb{R}_{>0}$ and sequences s_1, \dots, s_m .

Proof. Let x be a point in \mathcal{H}_{abc} . We will prove this using induction on the number of nonzero entries of $x(i, j, k)$. Let (i', j', k') be such that $x(i', j', k')$ is the smallest nonzero $x(i, j, k)$. Set $s_{k'} := (i', j')$. The inequality of (2.9) indexed by (i', j', k') implies that there is an entry $(i'', j'', k' - 1)$ such that $x(i'', j'', k' - 1) > 0$ and $(i'', j'') <_L (i', j')$ (in terms of tableaux, this corresponds to the columns being strictly increasing, so if row k' has a box with (i', j') then the box above it in row $k' - 1$ must have a smaller value). Set $s_{k'-1} := (i'', j'')$. Repeatedly applying the inequality of (2.9) in this way we get a sequence $s = (s_1, s_2, \dots, s_{k'})$ such that $s_l <_L s_{l+1}$ and $x(i_l, j_l, l) > 0$. Consider the point $x - x(i', j', k') \cdot x_s$. All of its coordinates are

nonnegative since we have subtracted the value of the the smallest positive entry from coordinates with positive values. It still satisfies (2.9) since $x(i, j, k)$ is only subtracted from one of the terms on the left hand side if it is also subtracted from one of the terms on the right hand side. Therefore, the point $x - x(i', j', k') \cdot x_s$ is in \mathcal{H}_{abc} , and it has at least one more zero entry than x . By induction,

$$x - x(i', j', k') \cdot x_s = t_1 x_{s'_1} + t_2 x_{s'_2} + \cdots + t_{m-1} x_{s'_{m-1}}$$

for some t_i, s'_i . Setting $t_m = x(i', j', k')$ and $s'_m = s$ we therefore have that

$$x = t_1 x_{s'_1} + t_2 x_{s'_2} + \cdots + t_m x_{s'_m}.$$

□

Corollary II.18. *The extreme rays of \mathcal{H}_{abc} are a subset of the rays given by the x_s .*

Definition II.19. For any sequence s of pairs of integers, let T_s be the contingency table defined by

$$T_s(i, j) = \begin{cases} 1 & \text{if } (i, j) = (i_l, j_l) \text{ for some } l \\ 0 & \text{otherwise} \end{cases}$$

Proposition II.20. *The cone \mathcal{I}_{abc} is the convex hull of the rays in the directions given by $(T, 1^n)$ for T any $\{0, 1\}$ -matrix consisting of n entries equal to 1.*

Proof. Consider the image of one of these x_s in \mathcal{I}_{abc} under the map (2.11). Under this map, $x_s \mapsto (T_s, \gamma)$ where γ is the partition 1^n , $n = \text{length}(s)$. We know that (2.11) maps \mathcal{H}_{abc} onto \mathcal{I}_{abc} , so the rays of \mathcal{I}_{abc} must be a subset of the images of the rays of \mathcal{H}_{abc} , which are in turn a subset of the rays given by pairs $(T_s, 1^n)$. □

Applying the map (2.12) to Proposition II.20, we find that $\bar{\mathcal{H}}_{abc}$ is the convex hull of the rays of the images of these $(T, 1^n)$. Now that we know the rays of $\bar{\mathcal{H}}_{abc}$, we

can plug them into a generic linear inequality to find what restrictions there are on the inequalities defining $\bar{\mathcal{H}}_{abc}$.

Definition II.21. An *additive inequality* for the partition triple $\alpha\beta\gamma$ is an inequality of the form

$$\rho_1\alpha_1 + \rho_2\alpha_2 + \cdots + \rho_a\alpha_a + \sigma_1\beta_1 + \sigma_2\beta_2 + \cdots + \sigma_b\beta_b \geq \mu_1\gamma_1 + \mu_2\gamma_2 + \cdots + \mu_c\gamma_c$$

where the ρ_i and σ_j are nonnegative integers and $\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_c$ are the values $\rho_i + \sigma_j$, $1 \leq i \leq a$, $1 \leq j \leq b$, ordered from smallest to largest.

Definition II.22. The *chamber inequalities* are the inequalities

$$\gamma_{i+1} \leq \gamma_i$$

for $i = 1, \dots, c$, where we set $\gamma_{c+1} = 0$.

Theorem II.23. $\bar{\mathcal{H}}_{abc}$ is the cone defined by the set of all additive inequalities and the chamber inequalities.

To prove this theorem we will need the following lemma:

Lemma II.24. Suppose $\bar{\mathcal{H}}_{abc}$ satisfies a linear inequality

$$\rho_1\alpha_1 + \rho_2\alpha_2 + \cdots + \rho_a\alpha_a + \sigma_1\beta_1 + \sigma_2\beta_2 + \cdots + \sigma_b\beta_b \geq \tau_1\gamma_1 + \tau_2\gamma_2 + \cdots + \tau_c\gamma_c.$$

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_c$ be the ordering of the values $\rho_i + \sigma_j$, $1 \leq i \leq a$, $1 \leq j \leq b$.

Then $\mu_1 + \cdots + \mu_l \geq \tau_1 + \cdots + \tau_l$ for $l = 1, 2, \dots, c$.

Proof. Fix $1 \leq l \leq c$ and let $i_1, j_1, \dots, i_l, j_l$ be the indices such that $\mu_k = \rho_{i_k} + \sigma_{j_k}$ for $k = 1, \dots, l$. Then the image under (2.12) of the point (T, γ) defined by $T(i_k, j_k) = 1$ for $1 \leq k \leq l$ with every other entry zero and $\gamma = 1^l$ is in $\bar{\mathcal{H}}_{abc}$ (it is itself the image of x_s where $s = ((i_1, j_1), \dots, (i_l, j_l))$). Plugging this point into the inequality for $\bar{\mathcal{H}}_{abc}$ gives the desired result. \square

Proof of Theorem II.23. We will begin by showing that $\bar{\mathcal{H}}_{abc}$ satisfies any additive inequality. From Proposition II.20, we know that any extreme ray of $\bar{\mathcal{H}}_{abc}$ is the image of some pair $(T, 1^n)$ where T is a $\{0, 1\}$ -matrix with n entries equal to 1. Let $(T, 1^n)$ be such a pair and consider plugging in its image into an additive inequality. If the (i, j) -entry of T is 1, then it will contribute a value of $\rho_i + \sigma_j$ to the left hand side of the additive inequality. Hence, the left hand side of the additive inequality will have a value greater than or equal to the sum of the n smallest values of the $\rho_i + \sigma_j$, i.e. $\mu_1 + \cdots + \mu_n$. This is the value of the right hand side of the inequality, so the image of $(T, 1^n)$ satisfies the inequality. We therefore have that $\bar{\mathcal{H}}_{abc}$ satisfies the additive inequality since each of its extreme rays does.

Next, we will show that any linear inequality which $\bar{\mathcal{H}}_{abc}$ satisfies is a sum of an additive inequality with some chamber inequalities. Suppose $\bar{\mathcal{H}}_{abc}$ satisfies a linear inequality as in the setup of the lemma. We know that $\bar{\mathcal{H}}_{abc}$ is a rational cone, so we may assume that all the coefficients in the inequality are integers. By adding multiples of the equalities

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_k \gamma_k$$

we may assume that the ρ_i and σ_j are nonnegative.

The lemma tells us that $\mu_1 + \cdots + \mu_l \geq \tau_1 + \cdots + \tau_l$ for $l = 1, \dots, c$. Set δ_l to be the difference between the two sums in the inequality indexed by l :

$$\delta_l := (\mu_1 + \cdots + \mu_l) - (\tau_1 + \cdots + \tau_l).$$

Then $\delta_1, \dots, \delta_c$ are nonnegative integers such that

$$\begin{aligned} (\tau_1, \tau_2, \dots, \tau_c) &= (\mu_1, \mu_2, \dots, \mu_c) + \delta_1(-1, 1, 0, \dots, 0) + \delta_2(0, -1, 1, 0, \dots, 0) + \cdots + \\ &\quad + \delta_{c-1}(0, \dots, 0, -1, 1) + \delta_c(0, \dots, 0, -1). \end{aligned}$$

This in turn tells us that our inequality is the sum of some of the chamber inequalities along with the additive inequality:

$$\rho_1\alpha_1 + \cdots + \rho_a\alpha_a + \sigma_1\beta_1 + \cdots + \sigma_b\beta_b \geq \mu_1\gamma_1 + \cdots + \mu_c\gamma_c$$

We therefore have that every inequality $\bar{\mathcal{H}}_{abc}$ satisfies is implied by a collection of inequalities of our two types, so $\bar{\mathcal{H}}_{abc}$ is precisely the cone defined by these inequalities. \square

As a direct consequence of the decomposition of the inequality at the end of the proof, we get the following corollary.

Corollary II.25. *Every face of $\bar{\mathcal{H}}_{abc}$ is an intersection of a single face given by an additive inequality with a collection of faces given by chamber inequalities.*

2.4 Faces of $\bar{\mathcal{H}}_{abc}$

Now that we have a set of defining inequalities for $\bar{\mathcal{H}}_{abc}$, we can use them to study the faces of $\bar{\mathcal{H}}_{abc}$ along with the properties of the integer triples contained in a face. We will start by looking at those faces which are given by an additive inequality.

Definition II.26. An *additive face* of $\bar{\mathcal{H}}_{abc}$ is a face given by an additive inequality.

Suppose that

$$(2.13) \quad \rho_1\alpha_1 + \rho_2\alpha_2 + \cdots + \rho_a\alpha_a + \sigma_1\beta_1 + \sigma_2\beta_2 + \cdots + \sigma_b\beta_b \geq \mu_1\gamma_1 + \mu_2\gamma_2 + \cdots + \mu_c\gamma_c$$

is an additive inequality, and let F be the preimage in \mathcal{I}_{abc} of the face this defines.

Proposition II.27. *A point $(T, \gamma) \in \mathbb{R}_{\geq 0}^{ab+c}$ is in F if and only if it satisfies the following conditions:*

(c1) For any $m \in \{\mu_1, \dots, \mu_c\}$,

$$\sum_{k: \mu_k=m} \gamma_k = \sum_{(i,j): \rho_i+\sigma_j=m} T(i,j).$$

(c2) The coordinates have γ greater than or equal to $\text{co}(T)$ in the dominance order.

Proof. One direction is straight forward. If (T, γ) is a point satisfying these two conditions then (c2) guarantees that this point is in \mathcal{I}_{abc} , while (c1) implies that the image of (T, γ) in $\bar{\mathcal{H}}_{abc}$ has equality in (2.13).

For the other direction, first consider the case when $(T, \gamma) = (T_s, 1^n)$ is a pair defining a ray of \mathcal{I}_{abc} . Let s consist of the pairs $(i_1, j_1), \dots, (i_n, j_n)$, ordered so that $\rho_{i_1} + \sigma_{j_1} \leq \rho_{i_2} + \sigma_{j_2} \leq \dots \leq \rho_{i_n} + \sigma_{j_n}$. Consider plugging in the image of $(T_s, 1^n)$ into the additive inequality (2.13). Each pair (i_l, j_l) will contribute $\rho_{i_l} + \sigma_{j_l}$ to the sum on the left hand side. The right hand side will be equal to $\mu_1 + \mu_2 + \dots + \mu_n$, but by how the μ_i are defined these are the smallest n values occurring amongst the $\rho_i + \sigma_j$. In order for the left hand side to not be strictly larger than the right hand side, we need $\rho_{i_1} + \sigma_{j_1}, \dots, \rho_{i_n} + \sigma_{j_n}$ to also be the smallest n terms amongst the $\rho_i + \sigma_j$. By how we ordered the pairs, we must then have that $\rho_{i_l} + \sigma_{j_l} = \mu_l$ for $l = 1, \dots, n$. This is precisely what is required for $(T_s, 1^n)$ to satisfy (c1). The second condition (c2) is trivially satisfied since $\text{co}(T) = 1^n = \gamma$. Hence, any of the points $(T_s, 1^n)$ giving the rays of \mathcal{I}_{abc} which are in \mathcal{F} must satisfy the two conditions.

Both of these conditions are preserved under taking $\mathbb{R}_{\geq 0}$ -linear combinations of points satisfying the two conditions. We know that on each ray bounding F there is a point satisfying these two conditions, so by taking $\mathbb{R}_{\geq 0}$ combinations of these points on the rays we have that every point in F satisfy the conditions. \square

Remark II.28. As stated, Proposition II.27 only applies to preimages of additive faces. Corollary II.25 tells us a general face of $\bar{\mathcal{H}}_{abc}$ is the intersection of one of these additive faces with some of the faces given by the chamber inequalities. For these general faces only a slight addition needs to be made for Proposition II.27 to hold; one just needs add the extra condition

(c3) The entries of γ satisfy the equalities forced by the chamber inequalities for the face.

Condition (c2) in Proposition II.27 only depends on the point and does not depend on the additive face chosen. Condition (c1) only depends on triples (i, j, k) which have $\rho_i + \sigma_j = \mu_k$. This means that two additive inequalities correspond to the same face if and only if they have the same set of such triples. This leads us to the following natural definitions and corollaries.

Definition II.29. Let F be a face of $\bar{\mathcal{H}}_{abc}$ given by an additive inequality

$$\rho_1\alpha_1 + \rho_2\alpha_2 + \cdots + \rho_a\alpha_a + \sigma_1\beta_1 + \sigma_2\beta_2 + \cdots + \sigma_b\beta_b \geq \mu_1\gamma_1 + \mu_2\gamma_2 + \cdots + \mu_c\gamma_c$$

- Define $R(F) = R(\rho, \sigma)$ be the set of ordered quadruples (i, j, i', j') , $(i, j) <_L (i', j')$, such that $\rho_i + \sigma_j = \rho_{i'} + \sigma_{j'}$. We call $R(F)$ the *set of equalities for F* .
- Define $Q(F) = Q(\rho, \sigma)$ to be the set of ordered quadruples (i, j, i', j') such that $\rho_i + \sigma_j < \rho_{i'} + \sigma_{j'}$.

Corollary II.30. *An additive face F is uniquely determined by the pair of sets $(Q(F), R(F))$.*

Proof. Using $R(F)$, we can define an equivalence relation \sim on $\{1, \dots, a\} \times \{1, \dots, b\}$ by saying $(i, j) \sim (i', j')$ if (i, j, i', j') or (i', j', i, j) is in $R(F)$. Let E_1, \dots, E_k be the equivalence classes of \sim . We can use $Q(F)$ to produce a linear order on these equivalence classes by saying $E_r < E_t$ if $(i, j, i', j') \in Q(F)$ for any $(i, j) \in E_r$ and $(i', j') \in E_t$. We may assume that the equivalence classes are labeled so that $E_1 < E_2 < \cdots < E_k$. Set $e_1 := 0$, and for $r = 2, \dots, k$, let $e_r = |E_1| + |E_2| + \cdots + |E_{r-1}| =$ number of points in equivalence classes smaller than $|E_r|$. Condition (c1)

in Proposition II.27 is equivalent to the equalities:

$$\sum_{k=e_r+1}^{e_{r+1}} \gamma_k = \sum_{(i,j) \in E_r} T(i,j)$$

for $r = 1, \dots, k$. Hence the first condition only depends on $Q(F)$ and $R(F)$. Condition (c2) does not depend on the face chosen and so it can be checked even if we only know $Q(F)$ and $R(F)$. \square

Corollary II.31. *If F and F' are two additive faces, then F' is a subface of F if and only if $R(F') \subset R(F)$ and $Q(F) \subset Q(F')$.*

Proof. First, suppose that F' is a subface of F . By Proposition II.27, each of the sets $\{(i,j) : \rho'_i + \sigma'_j = m\}$, $m \in \{\mu_1, \dots, \mu_c\}$, for F' will be a subset of one of these sets for F . The entries of $R(F)$ and $R(F')$ are formed by taking pairs of (i,j) which are elements of the same one of these sets. Hence, every pair occurring in $R(F')$ must also be in $R(F)$, and so $R(F') \subset R(F)$. The sets $Q(F)$ and $Q(F')$ correspond to the pairs in the complements of $R(F)$ or $R(F')$ respectively. The only extra information is that the pairs are ordered by their values of $\rho_i + \sigma_j$. In order for the points in F' to satisfy the conditions of Proposition II.27 for F , this ordering has to be preserved between F and F' , so we also get that $Q(F) \subset Q(F')$ (the complement of the containment for the R).

Next, suppose that $R(F') \subset R(F)$ and $Q(F) \subset Q(F')$. Let \sim_F and $\sim_{F'}$ be the equivalence relations defined by $R(F)$ and $R(F')$ as in the proof of Corollary II.30. Let E_1, \dots, E_r and E'_1, \dots, E'_t be the equivalence classes for \sim_F and $\sim_{F'}$ and let $e_1, \dots, e_r, e'_1, \dots, e'_t$ be as in the proof of Corollary II.30. If $(i,j) \sim_{F'} (i^*, j^*)$ then $(i,j) \sim_F (i^*, j^*)$ since $R(F') \subset R(F)$. Hence, any E_i is a disjoint union of the E'_j . In addition, $Q(F) \subset Q(F')$ gives us that if $E_i < E_j$ then $E'_{i'} < E'_{j'}$ for any i', j' with $E'_{i'} \subset E_i$ and $E'_{j'} \subset E_j$. The first condition of Proposition II.27 for F is equivalent

to

$$\sum_{k=e_r+1}^{e_{r+1}} \gamma_k = \sum_{(i,j) \in E_r} T(i,j)$$

By what was just shown,

$$\begin{aligned} \sum_{k=e_r+1}^{e_{r+1}} \gamma_k &= \sum_{t: E'_t \subset E_r} \sum_{k=e'_t+1}^{e'_{t+1}} \gamma_k, \\ \sum_{(i,j) \in E_r} T(i,j) &= \sum_{E'_t \subset E_r} \sum_{(i,j) \in E'_t} T(i,j), \end{aligned}$$

hence

$$\sum_{t: E'_t \subset E_r} \sum_{k=e'_t+1}^{e'_{t+1}} \gamma_k = \sum_{t: E'_t \subset E_r} \sum_{(i,j) \in E'_t} T(i,j).$$

This is a sum of the equations we get from (c1) for F' , so if a point is in F' then it must also be in F . \square

Since $(Q(F), R(F))$ is enough to specify an additive face, a natural question then is if we are given an ordered pair (Q, R) , can we find an inequality for the additive face producing this pair. Let F be an additive face of $\bar{\mathcal{H}}_{abc}$. We want to find the collection of coefficients $\rho_1, \dots, \rho_a, \sigma_1, \dots, \sigma_b$ ($\rho_1 = \sigma_1 = 0$) which give rise to an additive inequality corresponding to this face. By Corollary II.30, the coefficients will give rise to this face if and only if they satisfy:

$$(2.14) \quad \rho_i + \sigma_j = \rho_{i'} + \sigma_{j'} \text{ if } (i, j, i', j') \in R(F).$$

$$(2.15) \quad \rho_i + \sigma_j < \rho_{i'} + \sigma_{j'} \text{ if } (i, j, i', j') \in Q(F).$$

Definition II.32. Let $P(F)$ be the cone defined by (2.14), $\rho_1 = \sigma_1 = 0$, and

$$(2.16) \quad \rho_i + \sigma_j \leq \rho_{i'} + \sigma_{j'} \text{ if } (i, j, i', j') \in Q(F).$$

Note: A point (ρ, σ) will produce the coefficients for an additive inequality corresponding to F if and only if it is in the interior of this cone.

This cone is able to tell us more, namely it can also be used to find additive inequalities for any face containing F .

Proposition II.33. *There is a one-to-one correspondence between the faces of $P(F)$ and the additive faces F' of $\bar{\mathcal{H}}_{abc}$ which contain F .*

Proof. Consider the boundary of this cone. If F' is face given by (ρ, σ) on the boundary of $P(F)$, then $(Q(F'), R(F'))$ will differ from $(Q(F), R(F))$ by having some of the inequalities for F become equalities for F' , so we will have that $R(F) \subset R(F')$ and $Q(F') \subset Q(F)$. By Corollary II.31, this means that F is a subface of F' . Similarly, if (ρ, σ) produce a face F' of $\bar{\mathcal{H}}_{abc}$ which contains F , then $R(F) \subset R(F')$ and $Q(F') \subset Q(F)$ guarantee that (ρ, σ) satisfies (2.14) and (2.16) respectively, so $(\rho, \sigma) \in P(F)$. Hence the cone $P(F)$ consists of all points (ρ, σ) such that (ρ, σ) produces an additive face which contains F .

If two points (ρ, σ) and (ρ', σ') are in the same face of $P(F)$ then they will produce faces with the same (Q, R) pair, so they will both produce the same face. We therefore have that each face of $P(F)$ corresponds to a single face of $\bar{\mathcal{H}}_{abc}$. If F' is a face of $\bar{\mathcal{H}}_{abc}$ corresponding to a face G of $P(F)$, then

$$R(F') = R(F) \cup \{(i, j, i', j') \in Q(F): \text{the points in } G \text{ have equality in (2.16)}\},$$

so $R(F')$ is enough to determine which face of $P(F)$ this G is. Hence, the correspondence between faces of $P(F)$ and faces of $\bar{\mathcal{H}}_{abc}$ which contain F is one-to-one. \square

Remark II.34. Suppose G_1 and G_2 are faces of $P(F)$ with $G_1 \subset G_2$. If F_1 and F_2 are the faces of $\bar{\mathcal{H}}_{abc}$ which G_1 and G_2 correspond to respectively, then $G_1 \subset G_2$ means that every $(i, j, i', j') \in Q(F)$ for which (2.16) is an equality for points in G_1 must also have equality for points in G_2 , so $R(F_2) \subset R(F_1)$. In addition, this forces every (2.16) which is strict for G_2 to be strict for G_1 also, so $Q(F_1) \subset Q(F_2)$. Hence,

$F_2 \subset F_1$. The correspondence therefore reverses the partial order on the faces given by inclusion.

This provides us with a template for finding the additive facets of $\bar{\mathcal{H}}_{abc}$. The origin in $P(F)$ corresponds to the face which is the entire cone $\bar{\mathcal{H}}_{abc}$, so the rays of $P(F)$ correspond to faces of $\bar{\mathcal{H}}_{abc}$ which are not strictly contained in any proper face of $\bar{\mathcal{H}}_{abc}$, i.e. the rays correspond to facets of $\bar{\mathcal{H}}_{abc}$. Each ray is half of a 1-dimensional intersection of planes of the form $\rho_i + \sigma_j - \rho_{i'} - \sigma_{j'} = 0$. Each 1-dimensional intersection of such planes gives rise to two facets; if the intersection is spanned by (ρ, σ) then both it and $-(\rho, \sigma)$ give rise to $R(F)$ consisting of a single ray, but the facets they correspond to have different $Q(F)$ (the inequalities are flipped between the two).

Theorem II.35. *Let $x_2, \dots, x_a, y_2, \dots, y_b$ be indeterminates where we also set $x_1 = y_1 = 0$. Let W be the vector space spanned by $x_2, \dots, x_a, y_2, \dots, y_b$. There is a 2-to-1 correspondence*

$$\{\text{facets of } \bar{\mathcal{H}}_{abc}\} \leftrightarrow \{\text{codimension 1 subspaces of } W \text{ spanned by elements of the form } x_i + y_j - x_{i'} - y_{j'} \text{ for } 1 \leq i, i' \leq a, 1 \leq j, j' \leq b\}.$$

Proof. Let F be a facet of $\bar{\mathcal{H}}_{abc}$. By Proposition II.33, we know that the intersection of the planes of the form $\rho_i + \sigma_j - \rho_{i'} - \sigma_{j'} = 0$ for $(i, j, i', j') \in R(F)$ has to be 1-dimensional, so the subspace of W spanned by $x_i + y_j - x_{i'} - y_{j'}$ for $(i, j, i', j') \in R(F)$ must have codimension 1. This means that the map

$$F \mapsto \text{Span}(x_i + y_j - x_{i'} - y_{j'} | (i, j, i', j') \in R(F))$$

is between the two claimed sets. For each codimension 1 subspace W' , we can solve for the facets F mapping to it by finding solutions (x, y) to the system of equations

$$w(x, y) = 0, w \in W'$$

and setting $\rho = x$ and $\sigma = y$. The space of solutions is 1-dimensional so there are precisely 2 choices for $Q(R)$ which can arise from these solutions, one for either ray of the line. \square

Remark II.36. This proof tells us that for facets we do not need to remember all of $Q(F)$. All we need is $R(F)$ and a single element of $Q(F)$ to tell us which of the two options it corresponds to.

2.5 When is $H(\alpha\beta\gamma)$ 0-Dimensional?

We are almost ready to show how we can produce all of the triples of partitions $\alpha\beta\gamma$ where $H(\alpha\beta\gamma)$ is 0-dimensional. Once we have this, we will have asymptotically solved the question of how to produce all h -stable triples. We will determine when $H(\alpha\beta\gamma)$ is 0-dimensional by using our results on the faces of $\bar{\mathcal{H}}_{abc}$ along with the following crucial proposition.

Proposition II.37. *If F is a face of $\bar{\mathcal{H}}_{abc}$, then $\dim(H(\alpha\beta\gamma))$ is constant on the interior of F .*

Proof. For any $(\alpha, \beta, \gamma) \in \bar{\mathcal{H}}_{abc}$, the polytope $H(\alpha\beta\gamma)$ is the preimage of this point in \mathcal{H}_{abc} . Let F be a face of $\bar{\mathcal{H}}_{abc}$ and let G be the preimage of F in \mathcal{H}_{abc} . For any point $(\alpha, \beta, \gamma) \in F$, $H(\alpha\beta\gamma)$ is the intersection of G with the affine planes

$$(2.17) \quad \sum_{j', k'} x(i, j', k') = \alpha_i, \quad \sum_{i', k'} x(i', j, k') = \beta_j, \quad \sum_{i', j'} x(i', j', k) = \gamma_k.$$

Let H_1, \dots, H_t be the hyperplanes corresponding to the inequalities of \mathcal{H}_{abc} which G has equality for. If (α, β, γ) is a point in the interior of F , then the dimension of $H(\alpha\beta\gamma)$ is equal to the dimension of

$$H_1 \cap \dots \cap H_t \cap \{\text{Affine planes of (2.17)}\}.$$

Since H_1, \dots, H_t are hyperplanes, each of these intersections is a translation of the intersection for $\alpha, \beta, \gamma = 0$ (the origin is not in the interior of F but we can still form this intersection). Hence, the dimension of this intersection does not depend on (α, β, γ) , and so the dimension of $H(\alpha\beta\gamma)$ is independent of (α, β, γ) in the interior of F . \square

This provides us with a blue print for producing all 0-dimensional $H(\alpha\beta\gamma)$. Every face of $\bar{\mathcal{H}}_{abc}$ is the intersection of an additive face with some chamber faces, so we will begin by working with an additive face and then we will show how to choose a collection of chamber faces whose intersection with the additive face will produce a face where $H(\alpha\beta\gamma)$ 0-dimensional.

In order for $H(\alpha\beta\gamma)$ to be 0-dimensional, we need the preimage of (α, β, γ) in \mathcal{I}_{abc} to be a single point (T, γ) , and then we need the preimage of this point in \mathcal{H}_{abc} to be zero dimensional. The preimage of the point (T, γ) is a polytope which up to permuting the coordinates is the Gelfand-Testlin polytope $GT(\gamma, \text{co}(T))$. We therefore have that $H(\alpha\beta\gamma)$ is 0-dimensional if and only if the following two conditions hold:

- (H1) There is a unique matrix T occurring in the preimage of (α, β, γ) in \mathcal{I}_{abc} .
- (H2) For this matrix T , the polytope $GT(\gamma, \text{co}(T))$ contains a single point.

For (H2), we need to know conditions for when the polytope $GT(\gamma, \text{co}(T))$ is zero dimensional. We will make use of the following result of Stembridge which is a particular case of a result of Berenstein and Zelevinsky [1].

Lemma II.38 (Corollary 5.4 of [14]). *Let α and β be partitions. The polytope $GT(\alpha, \beta)$ is 0-dimensional if and only if $\alpha \geq \beta$ and every primitive factor of (α, β) has a shape with at most two distinct part sizes, one of which only occurs once.*

Definition II.39. A *primitive factor* of (α, β) is a pair (α^*, β^*) such that $\alpha^* = (\alpha_i, \alpha_{i+1}, \dots, \alpha_j)$ and $\beta^* = (\beta_i, \beta_{i+1}, \dots, \beta_j)$ for some i, j , and

$$\alpha_1 + \alpha_2 + \dots + \alpha_{i-1} = \beta_1 + \dots + \beta_{i-1},$$

$$\alpha_i + \alpha_{i+1} + \dots + \alpha_k \neq \beta_i + \beta_{i+1} + \dots + \beta_k \text{ for } k = i, \dots, j-1,$$

$$\alpha_i + \alpha_{i+1} + \dots + \alpha_j = \beta_i + \beta_{i+1} + \dots + \beta_j.$$

If $\alpha \geq \beta$, then the primitive pairs are the maximal sequential subsequences of α and β where the inequalities

$$\alpha_1 + \dots + \alpha_k \geq \beta_1 + \dots + \beta_k$$

are strict (Note: these are the inequalities for checking if $\alpha \geq \beta$ in the dominance order). We call α^* the *shape* of the primitive factor.

Remark II.40. In order to check if $\alpha \geq \beta$ in the dominance order, we only need to check that $\alpha^* \geq \beta^*$ for each of the primitive factors.

Let F be an additive face of $\bar{\mathcal{H}}_{abc}$ and suppose that (T, γ) is a point in \mathcal{I}_{abc} whose image is in the interior of F . Let $\nu_1 < \nu_2 < \dots < \nu_k$ be an ordering of the elements in $\{\mu_1, \dots, \mu_c\}$ and set $n_i := \#\{j : \mu_j \leq \nu_i\}$ (where $n_0 = 0$ by convention).

Proposition II.41. *Let G be a face of $\bar{\mathcal{H}}_{abc}$. Points (T, γ) in the preimage of G will have $GT(\gamma, \text{co}(T))$ be 0-dimensional if and only if G is a subface of the intersection of an additive face F with the chamber faces*

$$\gamma_{n_{d-1}+1} = \gamma_{n_{d-1}+2}, \dots, \gamma_{n_d-2} = \gamma_{n_d-1}, \text{ or}$$

$$\gamma_{n_{d-1}+2} = \gamma_{n_{d-1}+3}, \dots, \gamma_{n_d-1} = \gamma_{n_d},$$

for $d = 1, \dots, k$.

Proof. By Proposition II.27 we have both that $\gamma \geq \text{co}(T)$ and for $d = 1, \dots, k$,

$$\sum_{i=n_{d-1}+1}^{n_d} \gamma_i = \sum_{(i,j): \rho_i + \sigma_j = \nu_d} T(i, j)$$

where the right hand side is a sum of $n_d - n_{d+1}$ terms in T . The inequality $\gamma \geq \text{co}(T)$ tells us that $\sum_{i=1}^{n_1} \gamma_i$ is greater than or equal to the sum of any n_1 terms of T , so in order for there to be equality for $d = 1$ we must have that the $T(i, j)$ with $\rho_i + \sigma_j = \nu_1$ are the largest n_1 terms of T . Hence $\sum_{i=1}^{n_1} \gamma_i = \sum_{i=1}^{n_1} \text{co}(T)_i$. Repeating this argument for ν_2 , then ν_3 , and so on we get that for $d = 1, \dots, k$

$$\sum_{i=n_{d-1}+1}^{n_d} \gamma_i = \sum_{i=n_{d-1}+1}^{n_d} \text{co}(T)_i.$$

The point (T, γ) is in the interior of the preimage of F so by Proposition II.27 it cannot have any other equalities amongst the partial sums of this form, so we have that the

$$((\gamma_{n_{d-1}+1}, \dots, \gamma_{n_d}), (\text{co}(T)_{n_{d-1}+1}, \dots, \text{co}(T)_{n_d}))$$

for $d = 1, \dots, k$, are the primitive factors of $(\gamma, \text{co}(T))$.

By Lemma II.38, $GT(\gamma, \text{co}(T))$ will be 0-dimensional if and only if we force there to be at most two lengths for γ amongst each of these primitive factors, where one of the lengths occurs at most once. The points in the interior of G will have this property on their primitive factors if and only if for $d = 1, \dots, k$ one of the two chains of chamber equalities from the statement of the proposition holds. \square

Remark II.42. A corollary of the proof of Proposition II.41 is that (T, γ) will be the preimage of a point in the interior of an additive face if and only if γ dominates $\text{co}(T)$ and the primitive factors of $(\gamma, \text{co}(T))$ are precisely the pairs

$$((\gamma_{n_{d-1}+1}, \dots, \gamma_{n_d}), (\text{co}(T)_{n_{d-1}+1}, \dots, \text{co}(T)_{n_d}))$$

for $d = 1, \dots, k$.

Next, we want to find a way of determining when there is a unique matrix in the preimage of a face so that (H1) is satisfied. Suppose (α, β, γ) is in the interior of an additive face F of $\bar{\mathcal{H}}_{abc}$, and let T be a generic matrix occurring in the preimage of (α, β, γ) .

Lemma II.43. *For small ϵ , $T + \epsilon S$ will also occur as a matrix in $H(\alpha\beta\gamma)$ if and only if S has 0 row and column sums and satisfies*

$$(2.18) \quad \sum_{k:\mu_k=m} \gamma_k = \sum_{(i,j):\rho_i+\sigma_j=m} S(i,j), \quad \forall m \in \{\mu_1, \dots, \mu_c\}.$$

Proof. From Remark II.42, we know that the image of (T, γ) will be a point in the interior of F if and only if γ dominates $\text{co}(T)$ and the primitive factors of $(\gamma, \text{co}(T))$ are precisely the pairs

$$\left((\gamma_k)_{\mu_k=m}, \text{co}(T(i,j))_{\rho_i+\sigma_j=m} \right)$$

for $m \in \{\mu_1, \dots, \mu_c\}$.

Since these are the primitive factors, the only equalities amongst the partial sums of γ and $\text{co}(T)$ used to compare γ and $\text{co}(T)$ in the dominance order are the ones that show that the size of both parts of these primitive factors are the same. Hence for ϵ small enough, γ will dominate $\text{co}(T + \epsilon S)$ as long as the primitive factors of $(\gamma, \text{co}(T))$ and $(\gamma, \text{co}(T + \epsilon S))$ are the same subsequences.

For any matrix S and ϵ small enough, $(\gamma, \text{co}(T + \epsilon S))$ will still have these as the primitive factors if and only if the (2.18) equalities hold. Hence, $T + \epsilon S$ will be a matrix in $H(\alpha\beta\gamma)$ if and only if the (2.18) equalities hold for S . \square

There will be a unique matrix in the preimage of (α, β, γ) if and only if there is no matrix S with 0 row and column sums satisfying (2.18). As before, we will enforce this by taking an additive face and showing which chamber faces we must

intersect it with in order for there to be no such S . Let F be an additive face given by the coefficients $\rho_1, \dots, \rho_a, \sigma_1, \dots, \sigma_b$ with equalities $R = R(\rho, \sigma)$. We will proceed by constructing a weighted graph which can be used to produce S with 0 row and column sums that satisfy (2.18), and then use this graph to determine which chamber equalities are necessary to eliminate these S .

Construct a graph $\bar{G}(R)$ by letting it have a vertex for each pair (i, j) and an edge between (i, j) and (i', j') if (i, j, i', j') or $(i', j', i, j) \in R$. Each edge in $\bar{G}(R)$ corresponds to an equality $\rho_i + \sigma_j = \rho_{i'} + \sigma_{j'}$, so by the transitive property every connected component of $\bar{G}(R)$ will be a complete graph. We do not need all of the equalities in order to determine R ; we only need a spanning tree of each connected component in order to determine R . Pick a maximal subforest, L , of $\bar{G}(R)$.

Definition II.44. Given a set of equalities R for a face and a maximal subforest L of $\bar{G}(R)$, let $G(R)$ be the graph with b vertices v_1, \dots, v_b such that there is an edge between v_j and $v_{j'}$ if there is an edge between (i, j) and (i', j') in L for some i, i' (we allow edges to be loops from a vertex to itself and pairs of vertices to have multiple edges between them).

This graph will just serve as a convenient way of keeping track of the equalities R , and so in practice the computations the graph is used for will not depend on which choice of L is used. For this reason, we refer to this graph as $G(R)$ and not $G(R, L)$.

This graph $G(R)$ is formed by taking L and merging all the vertices with the same second coordinate.

Example II.45. Consider the additive face given by $\rho = (10, 0, 7, 4)$ and $\sigma = (6, 10, 3, 0)$.

As we can see in Figure 2.1, this choice of ρ and σ produces

		σ			
		6	10	3	0
ρ	10	16	20	13	10
	0	6	10	3	0
	7	13	17	10	7
	4	10	14	7	4

Figure 2.1: An example of values for ρ and σ along with the μ they produce.

$\mu = (0, 3, 4, 6, 7, 7, 10, 10, 10, 10, 13, 13, 14, 16, 17, 20)$, with equalities

$R = ((1, 3, 3, 1), (1, 4, 2, 2), (1, 4, 3, 3), (1, 4, 4, 1), (2, 2, 3, 3), (2, 2, 4, 1), (3, 3, 4, 1),$

$(3, 4, 4, 3))$. By introducing an edge for each entry of R , we obtain the $\bar{G}(R)$ of Figure

2.2. This graph $\bar{G}(R)$ is a union of complete graphs (a K_4 , two K_1 's, and the rest

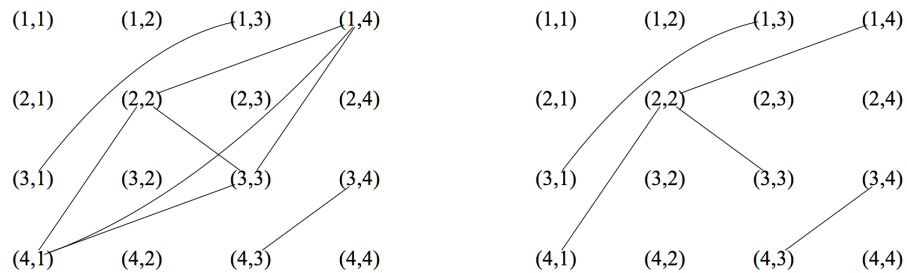


Figure 2.2: An example of the $\bar{G}(R)$ graph (left) and a choice of L (right).

are single vertices). To construct a maximal subforest L we need to remove edges from the K_4 to get a spanning tree. In Figure 2.2, the $(3, 3, 4, 1)$ and $(2, 4, 3, 3)$ edges were removed to produce L . Now, in order to construct $G(R)$ from L , we merge each vertical column of vertices (i.e. ones with the same second coordinate), producing the graph of Figure 2.3. The edges of Figure 2.3 have been assigned weights based on the first coordinates of the vertices incident to the corresponding edge in L (we will see how this is done in Notation II.49).

Our main focus will be the cycles of $G(R)$. We will assign weights to the edges

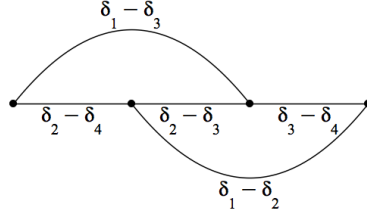


Figure 2.3: An example of the $G(R)$ graph.

of $G(R)$ and then use the weights of the cycles to find a necessary and sufficient condition for a face producing $G(R)$ to satisfy (H1).

Definition II.46. Recall from Definition II.29 that if $(i, j, i', j') \in R$, then $(i, j) <_L (i', j')$. Our choice of ordering on the pairs in the equality is equivalent to choosing how to orient the edge formed by the equality in $\bar{G}(R)$, and so it descends to an orientation on the edge in $G(R)$: if an edge of $G(R)$ is given by the equality $(i, j, i', j') \in R$ then the edge is oriented from v_j to $v_{j'}$. We will say that a directed path traverses an edge in the *positive direction* if it follows this orientation, and that it traverses the edge in the *negative direction* if it goes against the orientation.

In Figure 2.2, the positive direction for all of the edges in $\bar{G}(R)$ is the downwards direction. This descends to the positive direction of all the edges of $G(R)$ in Figure 2.3 being from right to left except for the edge between the two middle vertices is from left to right.

Notation II.47. To any cycle C of $G(R)$, assign a formal sum of edges

$$E(C) := \sum_{e \in C} \epsilon_e e$$

where $\epsilon_e = 1$ if the cycle traverses e in the positive direction and $\epsilon_e = -1$ if it traverses e in the negative direction.

Definition II.48. Call a collection of cycles C_1, \dots, C_n a *cycle basis* if $E(C_1), \dots, E(C_n)$ form a basis for the space $\text{Span}\{E(C) : C \text{ a cycle in } G(R)\}$.

Notation II.49. Assign weights to the edges of $G(R)$ as follows: for each edge $((i, j), (i', j'))$ in L , assign the corresponding edge in $G(R)$ the weight $w(e) = \delta_i - \delta_{i'}$, where $\delta_1, \dots, \delta_a$ are indeterminates. To any cycle C in $G(R)$, assign it the weight

$$w(C) := \sum_{e \in C} \epsilon_e w(e)$$

where ϵ_e is the same as for $E(C)$.

Theorem II.50. *Let F be an additive face, let $R = R(F)$ be the set of equalities for F , and let C_1, \dots, C_n be a cycle basis for $G(R)$. The points in F have a unique matrix in their preimage if and only if $w(C_1), \dots, w(C_n)$ are linearly independent.*

Theorem II.50 is our desired necessary and sufficient condition for (H1) to hold, and so our goal now is to prove this theorem. We will do this by showing how to construct a matrix S with 0 row and column sums which satisfies (2.18) if the weights of the cycles are not linearly independent.

Notation II.51. For any quadruple (i, j, i', j') , let $S(i, j, i', j')$ be the matrix whose entries are all 0, except at (i, j) it is 1 and (i', j') it is -1. Additionally, for any edge e in $G(R)$, set $S(e) := S(i, j, i', j')$ where (i, j, i', j') is the equality in R that e corresponds to.

Remark II.52. For every $(i, j, i', j') \in R$, since we chose L maximal, there is a unique path P between v_j and $v_{j'}$. For this path

$$\sum_{e \in P} \epsilon_e S(e) = S(i, j, i', j').$$

Notation II.53. For any oriented cycle C in $G(R)$, set

$$S(C) := \sum_{e \in C} \epsilon_e S(e)$$

where ϵ_e is as in Notation II.47.

Each $S(e)$ satisfies (2.18), so $S(C)$ does too. Every vertex in C has indegree = outdegree, so the column sums of $S(C)$ will also be 0. Therefore each $S(C)$ is a matrix satisfying (2.18) with 0 column sums.

Lemma II.54. *The space of all matrices which have column sums 0 and satisfy (2.18) is precisely the set $\text{Span}\{S(C) : C \text{ is a cycle in } G(R)\}$.*

Proof. Taking linear combinations of matrices with 0 column sums and satisfying (2.18) preserves both of these properties, so we already have that this span is contained in the set. We now just have to show the opposite containment.

Let S be a matrix with column sums 0 that also satisfies (2.18). Let (i_0, j_0) be an entry of S with smallest nonzero absolute value, say $\epsilon = S(i_0, j_0)$. By taking the negative of S if necessary, we may assume that $\epsilon = S(i_0, j_0) > 0$. Our matrix S satisfies (2.18), so there must be some (k_1, j_1) with $S(k_1, j_1) < 0$ and (i_0, j_0, k_1, j_1) or $(k_1, j_1, i_0, j_0) \in R$. Let $S' = S - \epsilon S(i_0, j_0, k_1, j_1)$ if $(i_0, j_0) <_L (k_1, j_1)$ and $S' = S + \epsilon S(i_0, j_0, k_1, j_1)$ if $(i_0, j_0) >_L (k_1, j_1)$. This matrix S' still satisfies (2.18) but no longer has 0 column sums; its j_0 and j_1 columns now have a negative and positive sum respectively. By our choice of (i_0, j_0) and (k_1, j_1) we also have that

$$\sum_{i,j} |S(i, j)| < \sum_{i,j} |S'(i, j)|.$$

Replace S by S' . The matrix S now has a positive j_1 column sum, so there is (i_1, j_1) with $S(i_1, j_1) > 0$. Then again by (2.18) we get (k_2, j_2) with $S(k_2, j_2) < 0$ and $(i_1, j_1, k_2, j_2) \in R$. Again replace S by $S \pm \epsilon S(i_1, j_1, k_2, j_2)$. Repeat this process until we reach an l with $j_l = j_0$. This must occur since $\sum |S(i, j)|$ is decreasing by a fixed amount each time so it will eventually reach 0. If the sum gets to 0 then we must have reached column j_0 again since it has negative column sum after the first step. For $t = 0, \dots, l-1$, let P_t be the path in $G(R)$ corresponding to

$(i_t, j_t, k_{t+1}, j_{t+1})$ or $(k_{t+1}, j_{t+1}, i_t, j_t) \in R$ (see Remark II.52), oriented to go from v_{j_t} to $v_{j_{t+1}}$. Paths P_t and P_{t+1} have an end vertex $v_{j_{t+1}}$ in common and P_{l-1}, P_0 have end vertices $v_{j_0} = v_{j_l}$ in common. Concatenating these paths together therefore forms a cycle in $G(R)$. Let S be the starting matrix and S_{final} be the matrix we obtain after this process. Our method of replacing S was chosen so that $S = S_{final} + \epsilon S(C)$ where $\sum |S_{final}(i, j)| < \sum |S(i, j)|$. Repeating this process of producing cycles by now using S_{final} as our initial matrix, we will eventually end up with the zero matrix. Once we have this, then S was just the sum of the $\epsilon S(C)$ which we've found. \square

Next, consider the linear map $W : Span\{S(C)\} \rightarrow Span\{\delta_1, \dots, \delta_a\}$ given by $S(C) \mapsto w(C)$. The $S(e)$ matrices have row sums all 0 except the sum for rows i and i' are 1 and -1 respectively. Comparing this to our definition of $w(e)$, we find that the i th row sum of $S(C)$ is equal to the coefficient of δ_i in $w(C)$. The map therefore only depends on the matrix in $Span(S(C))$ and not which linear combination of $S(C)$ it is equal to, and so is well defined.

Lemma II.55. *A matrix S satisfies (2.18) and has 0 row and column sums if and only if $S \in \ker(W)$.*

Proof. We know that the coefficient of δ_i in the image of a matrix under this map is equal to the i th row sum of the matrix. Hence, S will be in this kernel if and only if it has 0 row sums. Applying this to Lemma II.54 gives us that this kernel is precisely the set of S we are interested in. \square

Proof of Theorem II.50. Suppose C_1, \dots, C_n is a cycle basis. We will show that $S(C_1), \dots, S(C_n)$ forms a basis for $Span\{S(C)\}$. Given this fact, W will have trivial kernel if and only if $w(C_1), \dots, w(C_n)$ are linearly independent, and so the theorem will follow from Lemma II.55.

Consider the map $\text{Span}\{E(C)\} \rightarrow \text{Span}\{S(C)\}$ given by $e \mapsto S(e)$ for the edges e of $G(R)$. This map sends $E(C)$ to $S(C)$ for any cycle C in $G(R)$, so the map is surjective. Suppose that the map is not injective. That means that there is a nontrivial formal sum $\sum_e \tau_e e$ such that its image, $\sum_{e \text{ edge of } G} \tau_e S(e)$, is the zero matrix. Restricting this sum to the (i, j) entry of the matrices, we get that $\sum_e \tau_e = 0$ where this new sum is over edges e of $G(R)$ whose corresponding edge in L (the maximal forest used to construct $G(R)$) is incident to the vertex (i, j) . If (i, j) is a leaf in L , then the sum for (i, j) consists of a single edge, so we must have that $\tau_e = 0$ for any edge incident to a leaf. Consider the graph formed by removing all the edges of L for which τ_e is zero for the corresponding edge in $G(R)$. This subgraph of L must have no leaves, but L is a forest so its only subgraph with no leaves is the edgeless graph. Hence, $\tau_e = 0$ for every edge e of $G(R)$. We therefore have that no such nontrivial linear combination exists, so the map is injective.

We therefore have that the map $\text{Span}\{E(C)\} \rightarrow \text{Span}\{S(C)\}$ is a linear isomorphism. The formal sums $E(C_1), \dots, E(C_n)$ form a basis for $\text{Span}\{E(C)\}$, so their images $S(C_1), \dots, S(C_n)$ form a basis for $\text{Span}\{S(C)\}$. \square

Remark II.56. If $\tau_1 w(C_1) + \dots + \tau_n w(C_n) = 0$ is a nontrivial dependence relation, then $\tau_1 S(C_1) + \dots + \tau_n S(C_n)$ will be a nonzero matrix satisfying (2.18) with 0 row and column sums, so it is a witness to the additive face not satisfying (H1).

Now that we have a nice method for checking for matrices S with 0 row and column sums which satisfy (2.18), the question is which chamber facets do we need to intersect our additive face with so that these S can no longer occur. Suppose we have such an S for an additive face F . Let $(T, \gamma) \in \mathcal{I}_{abc}$ be in the preimage of a point in the interior of F . We need to intersect F with chamber facets so that $(T + \epsilon S, \gamma)$ is not in the preimage of the new face. Since S satisfies (2.18), $T + \epsilon S$ will

always satisfy (c1) of Proposition II.27 for the intersection of F with some chamber facets, so by the proposition in order for $(T + \epsilon S, \gamma)$ to not be in the preimage of the intersection it must fail to satisfy (c2). Hence, we need to introduce chamber equalities so that γ no longer dominates $\text{co}(T + \epsilon S)$.

From Remark II.42, we know that the primitive factors of $(\gamma, \text{co}(T))$ are the pairs

$$(2.19) \quad \left((\gamma_k)_{\mu_k=m}, \text{co}(T(i, j))_{\rho_i+\sigma_j=m} \right)$$

for $m \in \{\mu_1, \dots, \mu_c\}$. Since S satisfies (2.18), we have that for any m ,

$\sum_{(i,j):\rho_i+\sigma_j=m} T(i, j) = \sum_{(i,j):\rho_i+\sigma_j=m} (T + \epsilon S)(i, j)$, so the primitive factors for $(\gamma, \text{co}(T + \epsilon S))$ will be subsequences of those in (2.19). Provided ϵ is small enough, any of the inequalities used to check if γ dominates $\text{co}(T)$ which are strict inequalities will still be strict inequalities when checking if γ dominates $\text{co}(T + \epsilon S)$. Hence, the primitive factors of $(\gamma, \text{co}(T + \epsilon S))$ will be the same subsequences as those of (2.19) and γ will dominate $\text{co}(T + \epsilon S)$. As long as the pairs of (2.19) are the primitive factors for points in the interior of our face, γ will dominate $T + \epsilon S$ for small ϵ and there will not be a unique matrix in the preimage of the face. We therefore need to intersect F with chamber facets so that the pairs of (2.19) are no longer the primitive factors for points in the interior of the face.

The only way for the (2.19) pair indexed by m to not be a primitive factor is to intersect our face with chamber equalities so that $(\gamma_k)_{\mu_k=m}$ is the minimum partition of its size in the dominance order (i.e. all of its parts are the same length). For this new face, in order for $(\gamma_k)_{\mu_k=m}$ to dominate $\text{co}(T(i, j))_{\rho_i+\sigma_j=m}$ they must be the same partition which has a single part length. Now, $(\gamma_k, \text{co}(T)_k)$ is a primitive factor for any k with $\mu_k = m$ and so the primitive factor indexed by m has been broken up into smaller ones. Therefore, to prevent γ from dominating $T + \epsilon S$, for one of the

primitive factors where $(\text{co}(T)_{(i,j)})_{\rho_i+\sigma_j=m} \neq (\text{co}(T+\epsilon S)_{(i,j)})_{\rho_i+\sigma_j=m}$ we must force $(\gamma_k)_{\mu_k=m}$ to be the minimum partition of its size.

We therefore have that for some m where the T and $T+\epsilon S$ portions differ, we must intersect F with the chamber facets given by the chamber equalities $\gamma_i = \gamma_{i+1}$ where $\mu_i = \mu_{i+1} = m$. Suppose $\tau_1 w(C_1) + \dots + \tau_n w(C_n)$ is a nontrivial dependence relation amongst a cycle basis for $G(R)$ and e is an edge which has a nonzero coefficient in $\tau_1 E(C_1) + \dots + \tau_n E(C_n)$. If S is the matrix produced by this dependence relation (Remark II.56), then $S(i, j)$ will be nonzero for some entry in $\{(i, j) : \rho_i + \sigma_j = m\}$. By what was shown in the previous paragraph, if we intersect F with the chamber facets $\gamma_i = \gamma_{i+1}$ for $\mu_i = \mu_{i+1} = m$, then $T + \epsilon S$ will not be a matrix in the preimage of this intersection. In order for the intersection to have a unique matrix in its preimage, we must do this for every nontrivial independence relation.

There will be overlap amongst the dependence relations for which edges occur with nonzero coefficient, so it will be useful to reconstruct $G(R)$ after we have intersected the face with some chamber equalities to see which dependence relations remain. To do this, once we have intersected F with chamber facets so that $\gamma_i = \gamma_{i+1}$ for all i with $\mu_i = \mu_{i+1} = m$ for some fixed m , we must remove all of the edges from $G(R)$ which are produced by equalities of the form $\rho_i + \sigma_j = \rho_{i'} + \sigma_{j'} = m$ (or alternatively remove all the equalities from R and use the new R to form $G(R)$). Once we've done this, any S produced by a dependence relation in the new $G(R)$ will have all of its entries (i, j) with $\rho_i + \sigma_j = m$ be 0, so it will not have nonzero terms amongst the primitive factors where we have forced the γ part to be minimal. This means γ will dominate $\text{co}(T + \epsilon S)$ and the dependence relation will still produce a witness to the face not satisfying (H1).

Remark II.57. This provides us with a recipe for finding faces where $H(\alpha\beta\gamma)$ is 0-

dimensional. Begin with an additive face F given by coefficients ρ, σ with equalities $R = R(\rho, \sigma)$.

1. Form the graph $G(R)$ and find a cycle basis C_1, \dots, C_n for the graph.
2. Check if the weights of the cycles in the basis are linearly independent. If they are not, then there is a nontrivial dependence relation $\tau_1 w(C_1) + \dots + \tau_k w(C_n) = 0$. Pick an edge e with nonzero coefficient in the sum $\tau_1 E(C_1) + \dots + \tau_n E(C_n)$; it will correspond to some value $m \in \{\mu_1, \dots, \mu_c\}$. Intersect F with the chamber faces $\gamma_i = \gamma_{i+1}$ for all i with $\mu_i = \mu_{i+1} = m$. Remove all the equalities $(i, j, i', j') \in R$ with $\rho_i + \sigma_j = \rho_{i'} + \sigma_{j'} = m$. The edge, and hence the cycle, will no longer appear in $G(R)$.
3. Repeat Steps 1 and 2 until the weights are linearly independent.
4. Take all the elements of $\{\mu_1, \dots, \mu_c\}$ which have not been used yet and set all but the first or last of the corresponding γ_i to be equal by intersecting the face with the appropriate chamber facets.

Once you have completed these steps the constraints describe a face where the polytope $H(\alpha\beta\gamma)$ is 0-dimensional.

Finally, we are interested in the case when α, β and γ are partitions. So far though, everything we have done has been for α and β which not have to be nonincreasing. What we are truly interested in then is $\bar{\mathcal{H}}'_{abc} =$ the intersection of $\bar{\mathcal{H}}_{abc}$ with the additional chamber equalities for α and β :

$$\alpha_i \geq \alpha_{i+1}, \quad \beta_j \geq \beta_{j+1}.$$

The faces of $\bar{\mathcal{H}}'_{abc}$ which do not involve these new chamber inequalities correspond to a subset of the faces of $\bar{\mathcal{H}}_{abc}$. Using the same reasoning as in Theorem II.23

we can express every inequality as sum of these new chamber inequalities with an additive inequality where $\rho_1 \leq \rho_2 \leq \dots \leq \rho_a$ and $\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_b$ (we can again use the fact α and β are partitions of the same number to make $\rho_1 = \sigma_1 = 0$). The inequalities for $\bar{\mathcal{H}}'_{abc}$ are then these ordered additive inequalities along with our expanded chamber inequalities.

What this means for the equalities, $R(F)$, of the additive faces of $\bar{\mathcal{H}}'_{abc}$ is that we only need to be concerned with equalities of the form (i, j, i', j') with $i' \leq i, j \leq j'$. An equality not of this type can only arise if we have $\rho_i = \rho_{i'}$ and $\sigma_j = \sigma_{j'}$, in which case it is implied by equalities of this form. In addition, for Theorem II.35, the map is now injective since only one ray can have increasing ρ and σ .

2.6 Examples of 0-dimensional $H(\alpha\beta\gamma)$

We close this discussion on the $H(\alpha\beta\gamma)$ polytopes with three examples of using the procedure from Remark II.57 to find triples where $H(\alpha\beta\gamma)$ is 0-dimensional.

Example II.58. Consider the additive face F_1 of $\bar{\mathcal{H}}_{4,4,16}$ given by the choice of $\rho = (0, 1, 3, 4)$ and $\sigma = (0, 1, 2, 6)$ for coefficients in the additive inequality (2.13). This produces the inequality

$$\begin{aligned} \alpha_2 + 3\alpha_3 + 4\alpha_4 + \beta_2 + 2\beta_3 + 6\beta_4 \geq & \gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 + 3\gamma_6 + 3\gamma_7 + 4\gamma_8 + \\ & + 4\gamma_9 + 5\gamma_{10} + 5\gamma_{11} + 6\gamma_{12} + 6\gamma_{13} + 7\gamma_{14} + 9\gamma_{15} + 10\gamma_{16} \end{aligned}$$

For Step 1 of Remark II.57, $\bar{G}(R)$ is the top graph of Figure 2.5. This graph $\bar{G}(R)$ is already a forest so L is just $\bar{G}(R)$. Merging all of the vertices with the same second coordinate results in $G(R)$, the bottom graph of Figure 2.5. Since the entires of ρ and σ are weakly increasing, the positive direction (Definition II.46) for all of the edges of $\bar{G}(R)$ is down and to the left so the positive direction for each of the edges in $G(R)$ is right to left.

		σ			
		0	1	2	6
ρ	0	0	1	2	6
	1	1	2	3	7
	3	3	4	5	9
	4	4	5	6	10

Figure 2.4: The entries of ρ and σ of F_1 along with the entries of μ they produce.

There are two linearly dependent cycles in $G(R)$: going from v_1 (the leftmost vertex) to v_2 along the top edge and then back to v_1 using the bottom edge produces a cycle with weight $-\delta_1 + \delta_2 + \delta_4 - \delta_3$ and the analogous cycle between v_2 and v_3 produces a loop with the same weight. For Step 2, we therefore need to introduce chamber equalities so that we can remove one of the edges of these two cycles. We will introduce equalities to remove the top edge of the first cycle. This edge corresponds to the pair of entries equal to 1 in Figure 2.4, so we need to set the corresponding γ values to be equal to each other. Hence we need to intersect our face with the chamber equality

$$\gamma_2 = \gamma_3.$$

Doing this will remove the top edge between v_1 and v_2 from $G(R)$, and the resulting graph will have linearly independent cycle weights. This means that the points in the intersection of F_1 with $\gamma_2 = \gamma_3$ have a unique matrix in their preimage in \mathcal{I}_{abc} . Finally, we must introduce chamber equalities from Step 4 in order to make the Gelfand-Tsetlin polytope for the point in \mathcal{I}_{abc} be 0-dimensional. All of the primitive factors are of length 1 or 2, so we do not need to intersect F_1 with any more chamber faces.

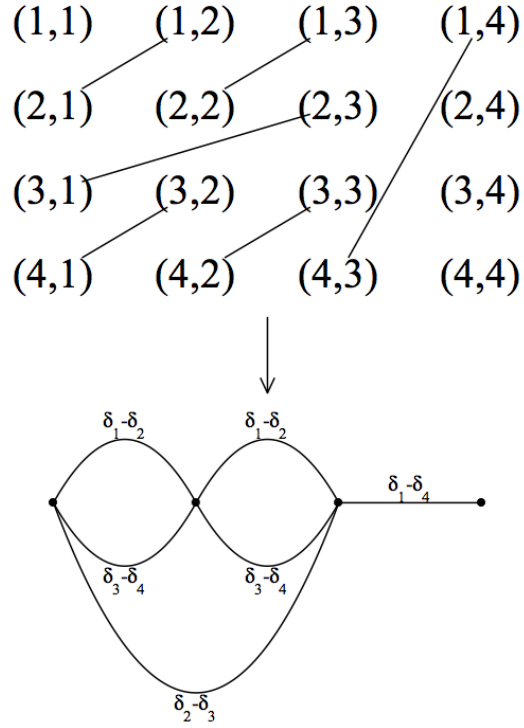


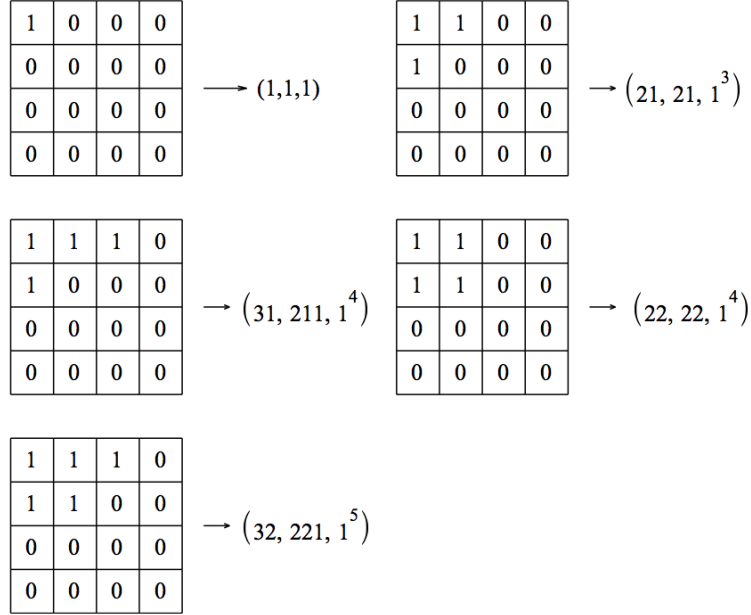
Figure 2.5: The graph $\bar{G}(R)$ obtained using $R(F_1)$ and the resulting graph $G(R)$ obtained by collapsing all the points with the same second coordinate to a single point.

We therefore have that the points in the face

$$F_1 \cap \{\gamma_2 = \gamma_3\}$$

will have 0-dimensional $H(\alpha\beta\gamma)$.

Figure 2.6 consists of the T matrices from Proposition II.20 which correspond to the extreme rays of the preimage of this face in \mathcal{I}_{abc} with $l(\gamma) \leq 5$. Next to each matrix is the extreme ray of our face of $\bar{\mathcal{H}}_{abc}$ that is the image of the ray of \mathcal{I}_{abc} . The first two entries of the $\bar{\mathcal{H}}_{abc}$ rays are the row and column sums of the matrix while the third entry is $\text{co}(T)$. These matrices are formed by filling in the matrix with 1's, starting with the entries with the smallest $\rho_i + \sigma_j$ (its value in Figure 2.4) and then filling in the entries with the next largest value and so on. Whenever there are two entries with the same $\rho_i + \sigma_j$ value, they can be filled in either order unless we have introduced chamber equalities for this $\rho_i + \sigma_j$ value. If we have introduced chamber

Figure 2.6: Extreme rays of $F_1 \cap \{\gamma_2 = \gamma_3\}$ with $l(\gamma) \leq 5$

equalities from Step 2, then all of the values of γ corresponding to that $\rho_i + \sigma_j$ must be equal, so we have to fill in all of the entries with this value of $\rho_i + \sigma_j$ at the same time. This is why there is no ray with $\gamma = 1^2$, the $\rho_i + \sigma_j = 1$ entries must have the same value since $\gamma_2 = \gamma_3$. If we have introduced chamber inequalities for Step 4, then one of the entries with the $\rho_i + \sigma_j$ value can be different from the others. In this case, depending on which chain of chamber equalities were chosen, either one entry is filled first then the rest must all be at filled it at the same time, or all but one are filled in at once and then the final is filled in (see Figure 2.10 for an example of this).

Any nonnegative linear combination of these five extreme rays $((1, 1, 1), (21, 21, 1^3), (31, 211, 1^4), (22, 22, 1^4), \text{ and } (32, 221, 1^5))$ will produce a triple $\alpha\beta\gamma$ with 0-dimensional $H(\alpha\beta\gamma)$.

Example II.59. Consider the additive face F_2 of $\bar{\mathcal{H}}_{4,4,16}$ given by the choice of $\rho = (0, 2, 3, 5)$ and $\sigma = (0, 1, 3, 3)$ for coefficients in the additive inequality (2.13).

This produces the inequality

$$2\alpha_2 + 3\alpha_3 + 5\alpha_4 + \beta_2 + 3\beta_3 + 3\beta_4 \geq \gamma_2 + 2\gamma_3 + 3\gamma_4 + 3\gamma_5 + 3\gamma_6 + 3\gamma_7 + 4\gamma_8 + \\ + 5\gamma_9 + 5\gamma_{10} + 5\gamma_{11} + 6\gamma_{12} + 6\gamma_{13} + 6\gamma_{14} + 8\gamma_{15} + 8\gamma_{16}$$

		σ			
		0	2	3	5
ρ	0	0	2	3	5
	1	1	3	4	6
	3	3	5	6	8
	3	3	5	6	8

Figure 2.7: The entries of ρ and σ for F_2 along with the entries of μ they produce.

For Step 1 of Remark II.57, $\bar{G}(R)$ is the top graph of Figure 2.8. Here, L is chosen to be the subgraph formed by the solid edges and the resulting $G(R)$ is shown at the bottom of Figure 2.8. Same as Example II.58, the positive direction for each of the edges of $G(R)$ is right to left.

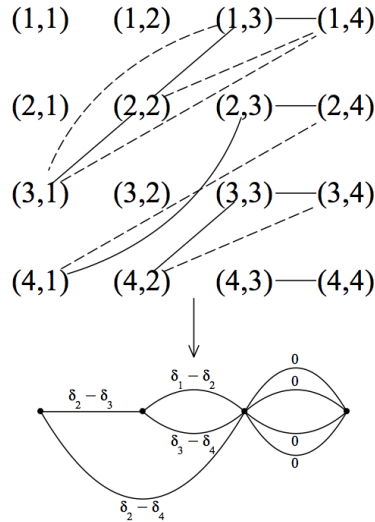


Figure 2.8: The $\bar{G}(R)$ and $G(R)$ graphs for F_2 .

There are four edges between the last two edges with 0 weight. As long as at least

two of these edges remain there will be a cycle of weight 0, so we need to introduce chamber equalities to remove three of these edges. These edges are in the complete subgraphs corresponding to $\rho_i + \sigma_j = 3, 5, 6,$ and 8 (the value of the (i, j) box in Figure 2.7). We will introduce chamber equalities to remove the $\rho_i + \sigma_j = 5, 6,$ and 8 edges. This corresponds to intersecting F_2 with the chamber equalities

$$5 : \quad \gamma_9 = \gamma_{10} = \gamma_{11}$$

$$6 : \quad \gamma_{12} = \gamma_{13} = \gamma_{14}$$

$$8 : \quad \gamma_{15} = \gamma_{16}.$$

We now must remove the edges from $\bar{G}(R)$ corresponding to the equalities of the form $\rho_i + \sigma_j = \rho_{i'} + \sigma_{j'} = 5, 6, 8$. This removes all of the edges from Figure 2.8 except for those in the K_4 subgraph (the complete subgraph corresponding to $\rho_i + \sigma_j = 3$). Using the remaining three solid edges for L , we obtain the $G(R)$ of Figure 2.9.

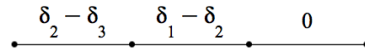


Figure 2.9: The $G(R)$ after edges have been removed for Step 2 of Remark II.57.

This new $G(R)$ has no cycles; performing Step 2 to remove the cycles for the weight 0 edges also removed the weight 0 cycle formed by going from v_1 (the leftmost vertex) to v_2 to v_3 using the bottom edge and then back to v_1 . We are just left with introducing the chamber equalities for Step 4. There is one $\rho_i + \sigma_j$ value remaining (i.e. not 5,6, or 8) which has more than two entries, namely 3. Hence we need to introduce one of the following two collections of chamber equalities:

$$\gamma_4 = \gamma_5 = \gamma_6 \text{ OR } \gamma_5 = \gamma_6 = \gamma_7.$$

We will introduce the first collection of chamber equalities, $\gamma_4 = \gamma_5 = \gamma_6$. We

therefore have that points in the face

$$F_2 \cap \{\gamma_4 = \gamma_5 = \gamma_6\} \cap \{\gamma_9 = \gamma_{10} = \gamma_{11}\} \cap \{\gamma_{12} = \gamma_{13} = \gamma_{14}\} \cap \{\gamma_{15} = \gamma_{16}\}$$

have 0-dimensional $H(\alpha\beta\gamma)$.

Figure 2.10 consists of the T matrices of Proposition II.20 which produce the extreme rays of the preimage of the face in \mathcal{I}_{abc} with $l(\gamma) \leq 8$, along with the point in $\bar{\mathcal{H}}_{abc}$ each of these rays is mapped to. In the rays for this face, since we chose the chamber equalities $\gamma_4 = \gamma_5 = \gamma_6$ for Step 5, when filling the entries with $\rho_i + \sigma_j = 3$ three of them must be done at the same time, then the final one can be filled in (if we had chosen $\gamma_5 = \gamma_6 = \gamma_7$ then one would have to be filled first followed by the rest being filled in at the same time).

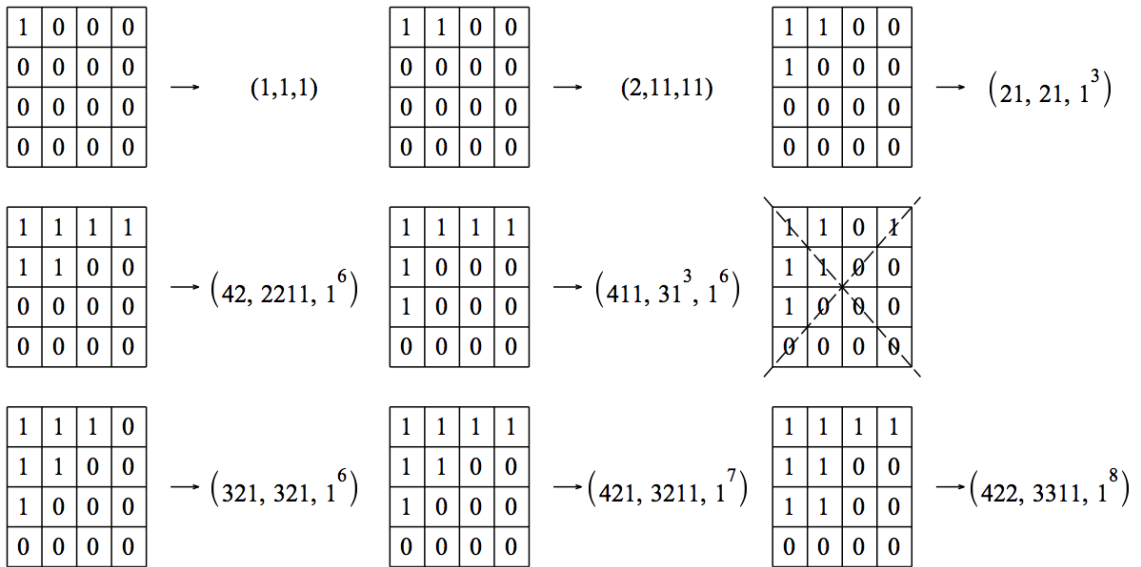


Figure 2.10: Extreme rays of our face with $l(\gamma) \leq 8$

One of the tables is crossed out since it produces a β which is not weakly decreasing, so we don't include it when taking nonnegative linear combinations to find triples of partitions. Taking nonnegative linear combinations of the eight remaining points in Figure 2.10 will produce triples of partitions $\alpha\beta\gamma$ where $H(\alpha\beta\gamma)$ is 0-dimensional.

Example II.60. For the final example, consider the additive face F_3 of $\bar{\mathcal{H}}_{4,4,8}$ given by the choice of $\rho = (0, 1, 3, 3)$ and $\sigma = (0, 2, 3, 5)$ for the coefficients in the additive equality (2.13). These are the coefficients of F_2 from Example II.59 where ρ and σ have been swapped. Since $\bar{\mathcal{H}}_{abc}$ is symmetric in α and β , F_3 intersected with the same chamber equalities as Example II.59 will produce a face where $H(\alpha\beta\gamma)$ is 0-dimensional, and so applying the steps of Remark II.57 should produce these chamber equalities.

The table of $\rho_i + \sigma_j$ values will be the transpose of the table of Figure 2.7. The μ_i values for F_3 are the same as those for F_2 . Computing $\bar{G}(R)$ for F_3 , we get the ‘transpose’ of the $\bar{G}(R)$ graph of Figure 2.8. Letting L again be the solid edges of Figure 2.8, we obtain the $G(R)$ of Figure 2.11.

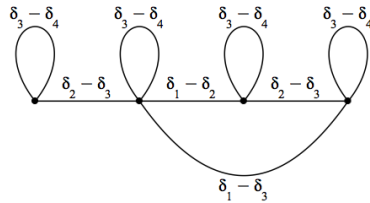


Figure 2.11: The $G(R)$ graph for F_3 .

There are four loops with the weight $\delta_3 - \delta_4$. As long as two of the loops remain, then by taking the difference of the weights of the two loops we obtain a nontrivial dependence relation. For Step 2, we must intersect F_3 with chamber equalities to remove three of these edges. Same as the previous example, we will remove the edges corresponding to $\rho_i + \sigma_j = 5, 6,$ and 8 . This corresponds to intersecting F_3 with the chamber equalities

$$5 : \quad \gamma_9 = \gamma_{10} = \gamma_{11}$$

$$6 : \quad \gamma_{12} = \gamma_{13} = \gamma_{14}$$

$$8 : \quad \gamma_{15} = \gamma_{16}.$$

Removing the edges from $\bar{G}(R)$ given by equalities of the form $\rho_i + \sigma_j = \rho_{i'} + \sigma_{j'} = 5, 6, 8$ again removes all of the edges but those in the K_4 connected component. Using the three remaining solid edges for L we obtain the $G(R)$ of Figure 2.12.

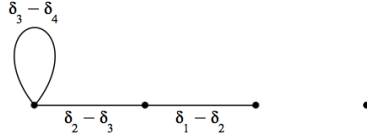


Figure 2.12: The $G(R)$ graph after edges have been removed for Step 2 of Remark II.57.

There are no cycles remaining in this $G(R)$; removing the loops also removed the cycle with 0 weight formed by going from v_4 (the rightmost vertex) to v_3 to v_2 and then back to v_4 using the bottom edge (this cycle produces the same S as the weight 0 (v_1, v_2, v_3, v_1) cycle from the previous example). All that remains is to perform Step 4. There is only one $\rho_i + \sigma_j$ value with more than two entries which hasn't already had chamber equalities introduced for it, namely 3. For Step 4, we introduce the chamber equalities $\gamma_4 = \gamma_5 = \gamma_6$. We therefore have that points in the face

$$F_3 \cap \{\gamma_4 = \gamma_5 = \gamma_6\} \cap \{\gamma_9 = \gamma_{10} = \gamma_{11}\} \cap \{\gamma_{12} = \gamma_{13} = \gamma_{14}\} \cap \{\gamma_{15} = \gamma_{16}\}$$

have 0-dimensional $H(\alpha\beta\gamma)$, which is the same set of chamber equalities as in Example II.59.

The extreme rays for this face will be the same as rays from Figure 2.10, except that the matrices are now transposed and the α and β entries of the rays are swapped.

CHAPTER III

Kronecker Stability and the $\bar{\mathcal{F}}_{a,l}$ Polytope

3.1 The $F(\alpha\beta\gamma)$ polytope

Our goal for this chapter is to study the f -coefficients of Definition II.3. Similar to Chapter II, we want to find which triples are stable, and it again will correspond to a related polytope being 0 dimensional. We will determine when this polytope is 0 dimensional in a similar manner to Chapter II. We will first find a polytope $F(\alpha\beta\gamma)$ such that $f(\alpha\beta\gamma)$ is equal to the number of integer points in the polytope. We will then use $F(\alpha\beta\gamma)$ to produce the polytope of all triples $\alpha\beta\gamma$ of specified lengths for which $f(\alpha\beta\gamma)$ is nonzero, and by studying this cone we will be able to say for which triples the $F(\alpha\beta\gamma)$ polytope is 0 dimensional.

Lets begin by finding $F(\alpha\beta\gamma)$. To do this we will use the polytope of Stembridge (Lemma 7.2 of [14]) and will now replicate his argument here. Recall from Proposition I.43 that

$$(3.1) \quad h_\alpha * s_\beta = \sum \prod_{i=1}^a s_{\mu^i/\mu^{i-1}},$$

where $a = l(\alpha)$ and the sum is over all sequences $(\mu^0, \mu^1, \dots, \mu^a)$ of partitions such that $\emptyset = \mu^0 \subset \mu^1 \subset \dots \subset \mu^a = \beta$ and $|\mu^i/\mu^{i-1}| = \alpha_i$ for all i . If $\alpha\beta\gamma$ is a triple of partitions of size n , then

$$f(\alpha\beta\gamma) = \langle h_\alpha * s_\beta, s_\gamma \rangle = \text{multiplicity of } I_\gamma \text{ in } A_\alpha \otimes I_\beta = \dim(A_\beta \otimes I_\beta \otimes I_\gamma)^{S_n}$$

where I_λ denotes the irreducible representation of S_n indexed by the partition λ and A_α is the permutation representation induced by the Young subgroup of type α . Notice that the right hand side is symmetric in β and γ , so we have that

$$f(\alpha\beta\gamma) = \langle h_\alpha * s_\gamma, s_\beta \rangle = f(\alpha\gamma\beta)$$

Applying (3.1) to this equation we find that if $l(\alpha) = a$ then

$$(3.2) \quad f(\alpha\beta\gamma) = \left\langle \sum s_{\mu^1} s_{\mu^2/\mu^1} \cdots s_{\gamma/\mu^{a-1}}, s_\beta \right\rangle$$

where the sum is over the all sequences of partitions $\emptyset = \mu^0 \subset \mu^1 \subset \mu^2 \subset \cdots \subset \mu^a = \gamma$ and $|\mu^i/\mu^{i-1}| = \alpha_i$ for $i = 1, \dots, a$.

Let $<_L$ be the lexicographical order on \mathbb{Z}^2 (Notation II.7). Similar to when we defined $H(\alpha\beta\gamma)$ (Definition II.8), we will work with tableaux with entries in $(\mathbb{Z}^{>0})^2$. As before, the *bicontent* of a tableau S is the pair of compositions (λ, μ) where λ_i and μ_i are the numbers of times that i occurs in the first and second coordinates respectively of the entries of S . Use the ordering from $<_L$ to define when these tableaux are semistandard. Given a tableau S , for any i , the boxes of S assigned entries of the form $(i, *)$ will form a skew shape. We can think of S as a gluing together of semistandard Young tableaux (one for each value of i) where we fill the boxes in this case with the second coordinate of the normal \mathbb{Z}^2 entry. We will use this model for the S interchangeably with the model where the tableaux are filled with entries in \mathbb{Z}^2 . Using this new model, let the *reading word* of a tableau S be the word formed by starting with the reverse row reading word (Definition I.45) for the skew shape for $i = 1$, followed by the reading word for the skew shape for $i = 2$, and so on. We say that the tableau satisfies the Yamanouchi condition if its reading word is a lattice word (Definition I.46) with respect to the second coordinate. Applying

the Littlewood-Richardson Rule (Theorem I.47) to (3.2) we find that

$$(3.3) \quad f(\alpha\beta\gamma) = \{\text{semistandard Young tableau of shape } \gamma \text{ with bicontent } (\alpha, \beta) \\ \text{whose reading word satisfies the Yamanouchi condition}\}$$

Let S be a semistandard Young tableau of shape γ and bicontent (α, β) whose reading word satisfies the Yamanouchi condition. Let $x(i, j, k)$ be the number times (i, j) occurs as an entry in row k of S .

Definition III.1. Fix positive integers a, b, c . For any triple of partitions $\alpha\beta\gamma$ of lengths at most a, b, c respectively, let $F(\alpha\beta\gamma)$ be the polytope defined by the following set of inequalities:

$$(3.4) \quad x(i, j, k) \geq 0 \quad (1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c),$$

$$(3.5) \quad \sum_{j^*, k^*} x(i, j^*, k^*) = \alpha_i, \quad \sum_{i^*, k^*} x(i^*, j, k^*) = \beta_j, \quad \sum_{i^*, j^*} x(i^*, j^*, k) = \gamma_k,$$

$$(3.6) \quad \sum_{(i^*, j^*) <_L (i, j)} x(i^*, j^*, k) \geq \sum_{(i^*, j^*) \leq_L (i, j)} x(i^*, j^*, k+1),$$

$$(3.7) \quad \sum_{(i^*, k^*) <_L (i, k)} x(i^*, j, k^*) \geq \sum_{(i^*, k^*) \leq_L (i, k)} x(i^*, j+1, k^*).$$

Note: In these equations, i^*, j^*, k^* are summation variables whereas i, j, k parameterize the inequalities.

Proposition III.2 (Lemma 7.2 of [14]). *The number of integer points in the polytope $F(\alpha\beta\gamma)$ is $f(\alpha\beta\gamma)$.*

Proof. As before, S is uniquely determined by the $x(i, j, k)$. From before, we have that S being of shape γ with bicontent (α, β) and S being semistandard are equivalent to the $x(i, j, k)$ satisfying (3.5) and (3.6) respectively. All that is left is to find a condition on the $x(i, j, k)$ for the reading word for S to satisfy the Yamanouchi

condition. For any j , consider the subword formed by entries of the form $(*, j)$ and $(*, j + 1)$. It will be a word consisting of $x(1, j + 1, 1)$ $j + 1$'s, followed by $x(1, j, 1)$ j 's, followed by $x(2, j + 1, 1)$ $j + 1$'s, followed by $x(2, j, 1)$ j 's, and so on. Namely, the j and $j + 1$ for $x(i, j, k)$ and $x(i, j + 1, k)$ will occur before those for $x(i^*, j, k^*)$ and $x(i^*, j + 1, k^*)$ if and only if $(i, k) <_L (i^*, k^*)$. In order for the word to be Yamanouchi we need the number of j 's to be more than the number of $j + 1$'s at any position in the word. In terms of the $x(i, j, k)$ and $x(i, j + 1, k)$, this says that we must have

$$\sum_{(i^*, k^*) <_L (i, k)} x(i^*, j, k^*) \geq \sum_{(i^*, k^*) \leq_L (i, k)} x(i^*, j + 1, k^*),$$

which is (3.7). □

Remark III.3. The inequalities (3.6) and (3.7) which are due to the semistandardness and Yamanouchi conditions force β and γ to be non increasing in order for $F(\alpha\beta\gamma)$ to be nonempty. There is, however, no such restriction on the components of α .

We do not require α, β, γ to be partitions for $F(\alpha\beta\gamma)$ to be well defined; they can instead be any points in $\mathbb{R}^a, \mathbb{R}^b, \mathbb{R}^c$ respectively. We therefore relax our definition of $F(\alpha\beta\gamma)$ to cover this more general case. Similar to our study of the h -coefficients, we will again focus on the cone of all triples for which $F(\alpha\beta\gamma)$ is nonempty. It will turn out that the inequalities will be simplest in the case when $b = c = l$ for some l and there is no loss of generality regarding Kronecker coefficients, so we will assume this is the case. In our study of this cone we will need to make use of the hives and honeycombs of Knutson and Tao to study the faces of this cone. In order to do this, we will need to introduce a change of coordinates so that the integer points of in the $F(\alpha\beta\gamma)$ will correspond to hives.

Definition III.4. For any i and $x \in F(\alpha\beta\gamma)$, let $\rho_\beta^i = \rho_\beta^i(x)$ be the l -tuple

$$\rho_\beta^i = \left(\sum_{(i',k') \leq_L (i,l)} x(i', 1, k'), \sum_{(i',k') \leq_L (i,l)} x(i', 2, k'), \dots, \sum_{(i',k') \leq_L (i,l)} x(i', l, k') \right).$$

Let ρ_γ^i be the analogously defined l -tuple where the role of the second and third coordinates of the $x(i, j, k)$ are swapped.

Remark III.5. The ρ_β and ρ_γ have a nice description in terms of the tableau S corresponding to x . Think of the S as being a gluing together of semistandard Young tableaux, and let $S_{\leq i}$ be the portion of S formed by boxes in the first i tableaux. Then ρ_β^i is $\text{co}(S_{\leq i})$, and ρ_γ^i is the partition whose j th part is the length of the j th row of $S_{\leq i}$. Looking at the $i = a$ case, we find that $\rho_\beta^a = \beta$ and $\rho_\gamma^a = \gamma$.

These partitions will show up when working with our hives and honeycombs, their main use being their relation to α : for $i = 1, \dots, a$,

$$\sum_{i' \leq i} \alpha_{i'} = |\rho_\beta^i| = |\rho_\gamma^i|.$$

Remark III.6. In order for the tableau formed by the $(1, *)$ entries to be semistandard and satisfy the Yamanouchi condition, it must be the tableau where each row j only consists of the entry j . Hence $x(1, i, j) = 0$ if $i \neq j$, so we will drop these entries from x . In addition, this also implies that $\rho_\beta^1 = \rho_\gamma^1 = (x(1, 1, 1), x(1, 2, 2), \dots, x(1, l, l))$. Notice that we get these are partitions only using $x(1, l, l) \geq 0$ and inequalities of type (3.6) or (3.7). The inequalities $x(1, t, t) \geq 0$ for $t < l$ are implied by these inequalities and are therefore not essential.

Consider the following change of coordinates: for $1 \leq i \leq a$, $0 \leq j, k \leq l$,

$$y(i, j, k) = \sum_{i' < i} (i - 1 - i')\alpha_{i'} + \sum_{j' \leq j, k' \leq k} x(i, j', k') + \sum_{j' \leq j} (\rho_\beta^{i-1})_{j'} + \sum_{k' \leq k} (\rho_\gamma^{i-1})_{k'}$$

where for $i = 1$ we set $\rho_\beta^0 = \rho_\gamma^0 = 0$. From the previous remark, we know that most of the entries $x(1, *, *)$ are 0. We only need $y(1, 1, l), y(1, 2, l), \dots, y(1, l, l)$ and

$$\begin{array}{ccccc}
\left(\rho_\gamma^{i-1}\right)_4 & x(i,1,4) & x(i,2,4) & x(i,3,4) & x(i,4,4) \\
\left(\rho_\gamma^{i-1}\right)_3 & x(i,1,3) & x(i,2,3) & x(i,3,3) & x(i,4,3) \\
\left(\rho_\gamma^{i-1}\right)_2 & x(i,1,2) & x(i,2,2) & x(i,3,2) & x(i,4,2) \\
\left(\rho_\gamma^{i-1}\right)_1 & x(i,1,1) & x(i,2,1) & x(i,3,1) & x(i,4,1) \\
\Sigma & \left(\rho_\beta^{i-1}\right)_1 & \left(\rho_\beta^{i-1}\right)_2 & \left(\rho_\beta^{i-1}\right)_3 & \left(\rho_\beta^{i-1}\right)_4
\end{array}$$

Figure 3.1: How to arrange the x coordinates when computing the y coordinates. The value of $y(i, 2, 3)$ is the sum of the terms in the rectangle.

$y(1, l, 1), y(1, l, 2), \dots, y(1, l, l-1)$ to reconstruct all of the $x(1, *, *)$ terms. For this reason, we will discard the rest of the $y(1, *, *)$ terms.

Pictorially, to find the $y(i, *, *)$ from the $x(i, *, *)$, arrange the terms for $x(i, *, *)$ in a square with the entries of ρ_β^{i-1} and ρ_γ^{i-1} along the bottom and left respectively along with $\Sigma := \sum_{i' < i-1} (i-2-i')\alpha_{i'}$ in the bottom left, like in Figure 3.1. Then $y(i, j, k)$ is the sum of all the terms in this arrangement which are below and to the left of $x(i, j, k)$, including $x(i, j, k)$ ($y(i, j, 0)$ and $y(i, 0, k)$ are the sums of the terms below and to the left of $(\rho_\beta^{i-1})_j$ and $(\rho_\gamma^{i-1})_k$ respectively).

Notation III.7. For $i = 1, \dots, a$, we will use π_β^i and π_γ^i to denote the l -tuples $(y(i+1, 0, 0), y(i+1, 1, 0), \dots, y(i+1, l, 0))$ and $(y(i+1, 0, 0), y(i+1, 0, 1), \dots, y(i, 0, l))$ respectively. These tuples can be expressed in terms of α and the ρ values as:

$$(\pi_\beta^i)_j = \sum_{i'=1}^i (i-i')\alpha_{i'} + \sum_{j'=1}^j (\rho_\beta^i)_{j'}, \quad (\pi_\gamma^i)_k = \sum_{i'=1}^i (i-i')\alpha_{i'} + \sum_{k'=1}^k (\rho_\gamma^i)_{k'}$$

By how the y coordinates are defined, π_β^i and π_γ^i are also equal to the sequences $(y(i, 0, l), y(i, 1, l), \dots, y(i, l, l))$ and $(y(i, l, 0), y(i, l, 1), \dots, y(i, l, l))$ respectively; i.e. if we arrange the $y(i, j, k)$ values like the $x(i, j, k)$ values in Figure 3.1, then the top and right sides of the i th square are equal to the bottom and left sides of the $i+1$ st square respectively.

Let $F_y(\alpha\beta\gamma)$ be the image of $F(\alpha\beta\gamma)$ under this linear transformation (where we

are enforcing $b = c = l$). Dropping the non-essential inequalities from Remark III.6, $F_y(\alpha\beta\gamma)$ is given by the points $y(i, j, k)$, $2 \leq i \leq a$, $0 \leq j, k \leq l$, satisfying

$$(3.8) \quad y(i, 0, 0) = (i - 2) \cdot \alpha_1 + (i - 3) \cdot \alpha_2 + \cdots + 1 \cdot \alpha_{i-2},$$

$$(3.9) \quad y(i, j, l) = y(i + 1, j, 0) \left(= (\pi_\beta^i)_j \right), \quad j = 0, \dots, l,$$

$$(3.10) \quad y(i, l, k) = y(i + 1, 0, k) \left(= (\pi_\gamma^i)_k \right), \quad k = 0, \dots, l,$$

$$(3.11) \quad y(a, j, l) = \sum_{i' < a} (a - i') \alpha_{i'} + \sum_{j' \leq j} \beta_{j'}, \quad y(a, l, k) = \sum_{i' < a} (a - i') \alpha_{i'} + \sum_{k' \leq k} \gamma_{k'},$$

$$(3.12) \quad y(2, 0, t) = y(2, t, 0), \quad t = 1, \dots, l,$$

$$(3.13) \quad y(2, 0, l) - y(2, 0, l - 1) \geq 0,$$

$$(3.14) \quad y(i, j, k) + y(i, j - 1, k - 1) - y(i, j - 1, k) - y(i, j, k - 1) \geq 0,$$

$$(3.15) \quad y(i, j, k) + y(i, j - 1, k - 2) - y(i, j, k - 1) - y(i, j - 1, k - 1) \geq 0,$$

$$(3.16) \quad y(i, j, k) + y(i, j - 2, k - 1) - y(i, j - 1, k) - y(i, j - 1, k - 1) \geq 0.$$

Where (3.14),(3.15),(3.16) are only for when all the entries are defined.

Remark III.8. The equalities (3.9) and (3.10) are due to some redundancy in how we defined the y coordinates. The (3.11) equalities correspond to (3.5) while the (3.12) equalities are due to the fact that $\rho_\beta^1 = \rho_\gamma^1$. The inequalities (3.14), (3.15), (3.16) are the results of applying the transformation to (3.4),(3.6),(3.7) respectively. The only inequality which doesn't get transformed in this manner is $x(1, l, l) \geq 0$, which is what (3.13) corresponds to. This inequality will be very important in what follows since it is the only inequality which is not originally included in the hive and honeycomb models.

Definition III.9. • Let $F_{a,l}$ be the cone given by the inequalities for $F_y(\alpha\beta\gamma)$, except for the (3.8) and (3.11) equalities.

- Let $\mathcal{F}_{a,l}$ be the image of $F_{a,l}$ under the linear map

$$y \mapsto (\rho^1, \rho_\beta^2, \dots, \rho_\beta^l, \rho_\gamma^2, \dots, \rho_\gamma^l).$$

where $\rho^1 = \rho_\beta^1 = \rho_\gamma^1$.

- Let $\bar{\mathcal{F}}_{a,l}$ be the cone consisting of all triples (α, β, γ) , $\alpha \in \mathbb{R}^a$ and $\beta, \gamma \in \mathbb{R}^l$, such that $F(\alpha\beta\gamma)$ is nonempty.

The cone $F_{a,l}$ corresponds to removing the condition that our tableaux have bi-content (α, β) and shape γ , so it is formed by taking the union of all the $F_y(\alpha\beta\gamma)$. Each $F_y(\alpha\beta\gamma)$ is the intersection of this cone with some hyperplanes. The cone $\bar{\mathcal{F}}_{a,l}$ is the image of $F_{a,l}$ and $\mathcal{F}_{a,l}$ under the maps

$$y \mapsto (\alpha, \beta, \gamma) \text{ and}$$

$$(\rho^1, \rho_\beta^2, \dots, \rho_\beta^l, \rho_\gamma^2, \dots, \rho_\gamma^l) \mapsto (|\rho^1|, |\rho_\beta^2| - |\rho^1|, \dots, |\rho_\beta^l| - |\rho_\beta^{l-1}|), \rho_\beta^l, \rho_\gamma^l)$$

respectively.

Remark III.10. Our definition of $f(\alpha\beta\gamma)$ is symmetric in the entries of α since h_α is. This is maintained by the polytope $\bar{\mathcal{F}}_{a,l}$; if (α, β, γ) is a point in $\bar{\mathcal{F}}_{a,l}$, then so is (α', β, γ) for any permutation α' of the entries of α . We are only interested in the case when α is weakly increasing, but when studying $\bar{\mathcal{F}}_{a,l}$ we do not require this and so will not assume it. When we ultimately look for stable triples, we will reinstate this condition.

3.2 Hives and Honeycombs

As with the h -stable triples, we will need to find inequalities corresponding to faces of $\bar{\mathcal{F}}_{a,l}$. In order to do this, we will make use of the hive and honeycomb models of Knutson and Tao. Knutson and Tao first introduced hives and honeycombs in

order to study the polytope of all triples $\alpha\beta\gamma$ such that the Littlewood-Richardson coefficient $c_{\beta\gamma}^\alpha$ is nonempty ([5], [6], [7]). Unsurprisingly, since our f coefficients can be expressed in terms of Littlewood-Richardson coefficients, the honeycomb and hive models are also applicable to our coefficients.

Our change of coordinates $x(i, j, k) \mapsto y(i, j, k)$ will allow us to produce a bijection between the integer points of our polytopes and a slight modification of the hives. Once we have our hives we will be able to reduce to working with honeycombs. This in turn enables us to use the techniques of [6] which Knutson, Tao, and Woodward employed in order to study the Littlewood-Richardson coefficients.

Let W_l be the set of points

$$W_l := \{(i, j, k) \in \mathbb{Z}^3 : i + j + k = 2l, l \geq i \geq 0, l \geq j \geq 0\}.$$

Define a graph G_l by letting the points in W_l be the vertices, and adding in an edge between each pair of vertices a (Euclidean) distance of 1 apart. The graph G_l is a union of unit equilateral triangles. Each interior edge of G_l gives rise to a rhombus by joining the two equilateral triangles sharing that edge. For each interior edge e , let $v_1(e)$ and $v_2(e)$ be the two vertices of e and let $w_1(e)$ and $w_2(e)$ be the other two vertices of the rhombus given by e ; i.e. the edge between $v_1(e)$ and $v_2(e)$ is the short diagonal of the rhombus and the edge between $w_1(e)$ and $w_2(e)$ is the long diagonal of the rhombus.

Definition III.11. A *hive* (or l -hive) is a pair (G_l, f) where f is an integer valued function on the vertices of G_l satisfying

$$(3.17) \quad f(v_1(e)) + f(v_2(e)) - f(w_1(e)) - f(w_2(e)) \geq 0, \quad e \text{ an internal edge.}$$

We will call the inequalities of (3.17) the *rhombus inequalities*, and for any internal

edge e we will set $f(e)$ to be the value on the left hand side of the corresponding rhombus inequality.

Remark III.12. These are very similar to the hives of Knutson and Tao [7], the difference being that instead of triangles our hives are rhombuses formed by joining two of theirs together.

For $i = 1, \dots, k-1$ and $y \in F_{a,l}$, we can map the points $y(i+1, *, *)$ to a function f_i on the vertices of G_l by setting $f_i(s, t, 2l - s - t) := y(i+1, s, t)$. When we depict G_l , we will draw it so that $(0, 0, 2l)$ is the bottom vertex with the $(1, 1, -2)$ direction pointing upwards and the $(1, 0, -1)$ and $(0, 1, -1)$ directions pointing 60° to the clockwise and counterclockwise of vertical respectively (see Figure 3.2). Pictorially, this map from $y(i+1, *, *)$ to f_i corresponds to laying out the $y(i, *, *)$ in a similar fashion to the $x(i, *, *)$ in Figure 3.1, rotating the square counterclockwise 45° and then matching the points up with the triangular grid of G_l in Figure 3.2. Once this is done, the entries on the southeast, southwest, northeast, and northwest sides of the i th hive will be the entries of $\pi_\beta^i, \pi_\gamma^i, \pi_\gamma^{i+1}$, and π_β^{i+1} respectively (see Figure 3.2).

Proposition III.13. *For $i = 1, \dots, a-1$, let f_i be the integer valued function on the vertices of G_l defined by $f_i(s, t, 2l - s - t) = y(i+1, s, t)$. Then the map*

$$y \mapsto (G_l, f_1), \dots, (G_l, f_{a-1})$$

is a bijection between the integer points of $F_{a,l}$ and collections of hives (H_1, \dots, H_{a-1}) satisfying

- *The entries of H_i on the northeast (resp. northwest) side of the hive are the same as the entries of H_{i+1} on the southwest (resp. southeast) side; i.e. $f_i(s, l, l-s) = f_{i+1}(s, 0, 2l-s)$ and $f_i(l, t, l-t) = f_{i+1}(0, t, 2l-t)$.*

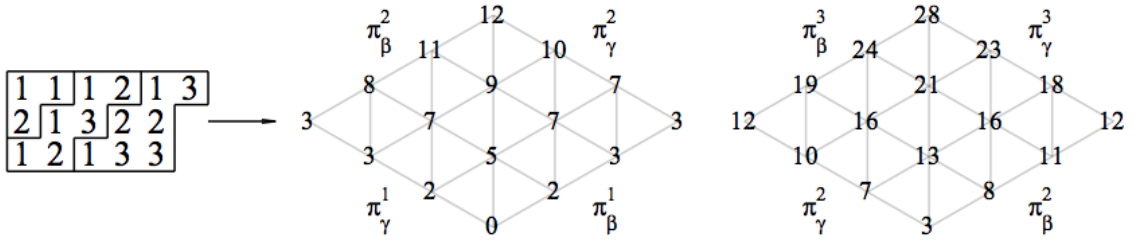


Figure 3.2: Example of the pair of hives obtained from a tableau.

- In H_1 , the entries on the southeast side of the hive are equal to the entries on the southwest side, i.e. $f_1(r, 0, 2l - r) = f_1(0, r, 2l - r)$.
- $f_1(l, 0, l) - f_1(l - 1, 0, l + 1) \geq 0$ (or equivalently $f_1(0, l, l) - f_1(0, l - 1, l + 1) \geq 0$).
- $f_1(0, 0, 2l) = 0$.

Proof. The inequalities (3.14), (3.15), and (3.16) are precisely the rhombus inequalities, so the f_i are hives. The first condition corresponds to the (3.9) and (3.10) equalities, the second and third conditions correspond to (3.12) and (3.13) respectively, while the fourth corresponds to (3.8) for $i = 1$. \square

To each of these hives we will associate a honeycomb, and it will be properties of these honeycombs which we will use to study the faces of $\bar{\mathcal{F}}_{a,l}$. In order to define honeycombs, we will need to introduce tinkertoys. Tinkertoys will require a nonstandard type of directed graph. Here we will think of a directed graph Γ as a quadruple $(V_\Gamma, E_\Gamma, head, tail)$ where $head$ and $tail$ are maps from the edges E_Γ to the vertices V_Γ and may only be partially defined, i.e. some the edges can have just a head, just a tail, or neither. When an edge is missing a head or a tail we will regard it as an infinite edge. When taking a subgraph of Γ , we restrict the $tail$ and $head$ maps to the remaining edges whose head or tail are in the subgraph (Note: some finite edges

can become infinite edges in a subgraph).

Definition III.14. A tinkertoy τ is a triple (B, Γ, d) consisting of a vector space B , a directed graph Γ (possibly with some one-ended edges), and a map $d : \text{Edges}(\Gamma) \rightarrow B$ assigning every edge of Γ a ‘direction’.

All of our tinkertoys will be in the space $B = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$, so we will assume B is fixed from now on. Each of the directions $d(e)$ will be one of $(0, -1, 1), (1, 0, -1), (-1, 1, 0)$.

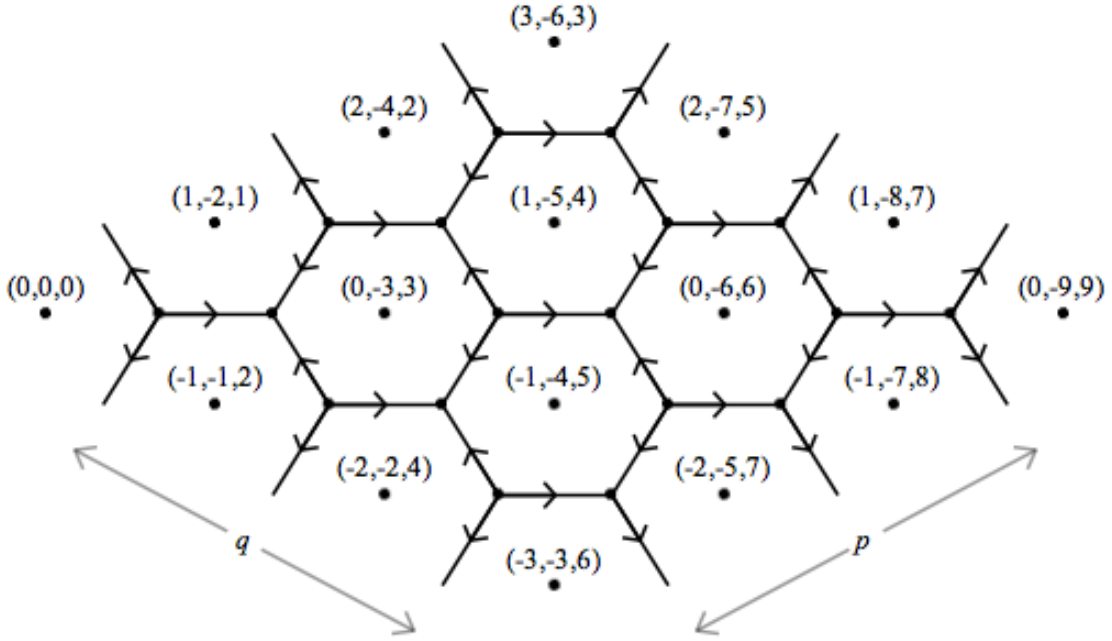
A subtinkertoy of (B, Γ, d) is a tinkertoy (B, Δ, d') where Δ is a subgraph of Γ and the direction map d' is the restriction of the direction map d to the subgraph.

Definition III.15. A *configuration* of a tinkertoy τ is a function $h : V_\Gamma \rightarrow B$ such that for any finite edge e , $h(\text{head}(e)) - h(\text{tail}(e)) \in \mathbb{R} \cdot d(e)$. A configuration can be thought of as an embedding of the graph Γ of the tinkertoy into the plane B such that each edge e is sent to a (possibly infinite) line segment in the direction $d(e)$ (here infinite edges are sent to rays starting at $h(\text{head}(e))/h(\text{tail}(e))$ and going in the $+/-d(e)$ direction). When traversing an edge in a configuration, one of the coordinates will remain constant; we will call this the *constant coordinate* of the edge in the configuration.

We will be interested in configurations of a specific type of tinkertoys (the GL_n -tinkertoys of [5]). Let V be the set of points

$$V := \{(i, j, k) \in \mathbb{Z}^3 : i + j + k = 0 \text{ and } 3 \text{ doesn't divide } 2i + j\}.$$

Construct a directed graph Γ_0 by letting the points in V be its vertices, and for each vertex $(i, j, k) \in V$ with $2i + j \equiv 2 \pmod{3}$, add three outwardly directed edges from (i, j, k) to the points $(i - 1, j + 1, k), (i, j - 1, k + 1), (i + 1, j, k - 1)$. For each

Figure 3.3: The $\tau_{3,3}$ tinkertoy

edge $e \in \Gamma_0$ set its direction $d(e)$ to be $(\text{head}(e) - \text{tail}(e))$. The *infinite honeycomb tinkertoy* is the triple (B, Γ_0, d) .

Definition III.16. The $\tau_{p,q}$ tinkertoy is the subtinkertoy of the infinite honeycomb tinkertoy whose underlying graph $\Gamma_{p,q}$ consists of the vertices of Γ_0 contained in the rhombus $i \leq k \leq i + 3q, j \leq i \leq j + 3p$.

Of particular interest will be the tinkertoy $\tau_l := \tau_{l,l}$ (see Figure 3.3). This tinkertoy has $4l$ infinite edges; each side of the rhombus will have l edges passing through it. We will call these edges the *boundary edges* of τ_l .

Definition III.17. A *honeycomb* (or τ_l -*honeycomb*) is a configuration of the τ_l tinkertoy.

Remark III.18. We can associate a honeycomb to any hive (G_l, f) as follows: Embed the graph G_l into the plane $x + y + z = 0$ by mapping each vertex $(i, j, k) \in G_l$ to the point $(i + j - l, i - 2j - l, -2i + j + 2l)$, and superimpose it on the trivial

embedding of Γ_l (the subgraph of Γ_0 for the tinkertoy τ_l) in this plane. The G_l graph and this trivial embedding of Γ_l are dual graphs (see the left hand diagram of Figure 3.6). The vertices of G_l are precisely the integer points inside the rhombus bounding Γ_l which are not vertices of Γ_l ; i.e. around each interior vertex of G_l there is a hexagon in Γ_l . Each edge of Γ_l intersects an edge of G_l at a right angle. We build up the honeycomb by starting with choosing $v_0 = h(0, -1, 1)$, then specifying the lengths of all of the finite edges in the honeycomb. Once we know the edge lengths, we know the direction, $d(e)$, of each edge so given the coordinates for one of the vertices an edge is incident to we can find the coordinates for the other vertex the edge is incident to, and so we can progressively build up the honeycomb from this starting vertex v_0 .

Send the vertex $(0, -1, 1)$ of Γ_l to the point

$$v_0 = (f(0, l-1, l+1) - f(1, l, l-1), f(1, l, l-1) - f(0, l, l), f(0, l, l) - f(0, l-1, l+1)).$$

Building up from this vertex, for each finite edge $e \in \Gamma_l$, let e' be the associated interior edge of G_l . Send e to an edge of length $f(e')\sqrt{2}$ (see Definition III.11) in the direction $d(e)$ (so the points for the two vertices will have two of their entries differ by $f(e')$ while the constant coordinate for the edges is the same for both vertices). See Figures 3.5 and 3.6 for an example of this construction ($v_0 = h(0, -1, 1) = (-12, 9, 3)$ is the leftmost vertex in the honeycomb).

In order for this to be well defined, we need that traversing any of the fundamental hexagons of the Γ_l results in us returning to the original point.

It is easy to check that the edge lengths will close up and produce a hexagon if and only if the following three equalities hold in Figure 3.4:

$$(3.18) \quad a + b = d + e, \quad b + c = e + f, \quad \text{and} \quad c + d = f + a.$$

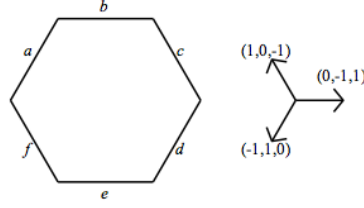


Figure 3.4: A generic hexagon in a honeycomb. The labels are the lengths of the edges.

Each of the edge lengths in one of the hexagons is given by the left hand side of a rhombus inequality. By plugging in the rhombus inequalities into these three conditions it is straight forward to see that they are satisfied for our edge lengths. We therefore have that this does in fact produce a honeycomb.

If we do this construction for the H_i have obtained from a point $x(i, j, k)$, then our initial point v_0 $(-12, 9, 3)$ for the example in Figure 3.6) was chosen so that the constant coordinates of the two infinite edges incident to the vertex were $(\rho_\beta^i)_l$ and $(\rho_\gamma^{i+1})_1$. Building up from this vertex and keeping track of the coordinates of the vertices in terms of the $y(i, j, k)$, one will find that the constant coordinates of the infinite edges is a difference of two $y(i, j, k)$ which is equal to an entry of ρ_β^i , ρ_β^{i+1} , ρ_γ^i , or ρ_γ^{i+1} . The entries of ρ_β^i form the constant coordinates for the infinite edges in the direction $(1, 0, -1)$ (the southeast infinite edges in Figure 3.6), the entries of ρ_β^{i+1} form the constant coordinates for the infinite edges in the direction $(1, 0, -1)$ (the northwest infinite edges), the entries of ρ_γ^i form the constant coordinates for the infinite edges in the direction $(-1, 1, 0)$ (the southwest infinite edges), and the entries of ρ_γ^{i+1} form the constant coordinates of the infinite edges in the direction $(1, -1, 0)$ (the northeast infinite edges).

Remark III.19. When depicting honeycombs, we will always draw them as in Figure 3.6: the $(0, -1, 1)$ direction will be to the right, the $(1, 0, -1)$ direction will be 120° counterclockwise and the $(-1, 1, 0)$ direction will be 120° clockwise. The honeycomb

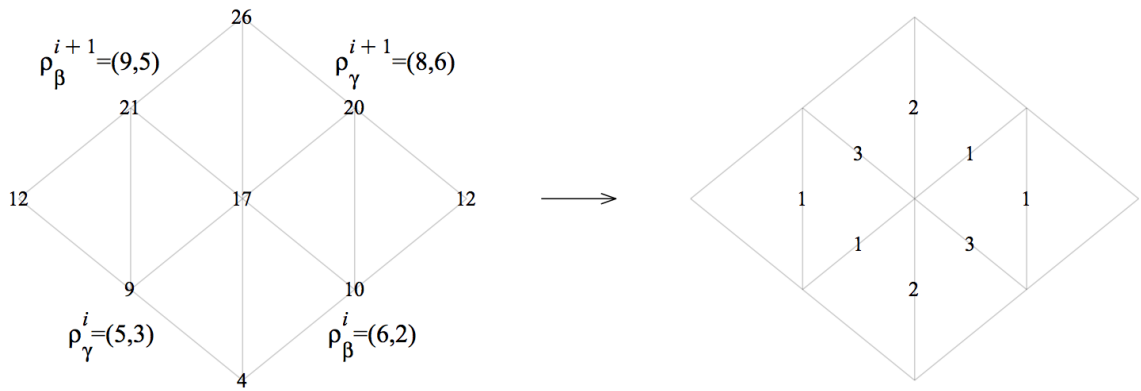


Figure 3.5: A hive and the $f(e)$ values (the left hand side of the rhombus inequality associated to e) for each interior edge of the hive.

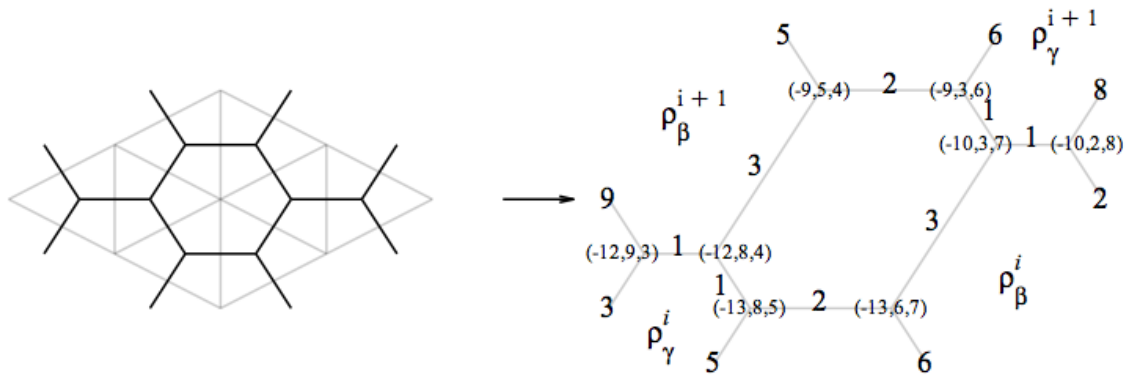


Figure 3.6: The bijection between the edges of the honeycomb and the edges of the hive, along with the honeycomb obtained from the hive in 3.5. The interior edges are labeled with the $f(e)$ value of the corresponding edge in the hive and the boundary values are the constant coordinates of the boundary edges.

associated to the hive H_i for a point x will have the constant coordinates of its southwest, southeast, northwest, and northeast boundary edges be the entries of ρ_γ^i , ρ_β^i , ρ_β^{i+1} , and ρ_γ^{i+1} respectively (see Figure 3.6).

Proposition III.20. *The cone $\mathcal{F}_{a,l}$ is precisely the set of points*

($\rho_\beta^1, \rho_\beta^2, \dots, \rho_\beta^l, \rho_\gamma^1, \rho_\gamma^2, \dots, \rho_\gamma^l$) which occur as the constant coordinates of the boundary edges of a collection of honeycombs (h_1, \dots, h_{a-1}) satisfying the conditions:

- *The southern boundary edges in h_1 will have nonnegative constant coordinates (equivalently the northern most amongst the southeastern boundary edges will have a nonnegative constant coordinate).*
- *In h_1 , the set of constant coordinates for the southeastern boundary edges will be the same as the set of constant coordinates for the southwestern boundary edges.*
- *The northwestern (resp. northeastern) boundary edges of h_{i-1} will have the same set of constant coordinates as the southeastern (resp. southwestern) boundary edges of h_i .*

Moreover, $\bar{\mathcal{F}}_{a,l}$ is the set of points $((|\rho_\beta^1|, |\rho_\beta^2| - |\rho_\beta^1|, \dots, |\rho_\beta^l| - |\rho_\beta^{l-1}|), \rho_\beta^l, \rho_\gamma^l)$ given by these collections of honeycombs.

Proof. This construction is reversible: given a honeycomb we can reconstruct the hive by first assigning the boundary points of the hive the appropriate sum of constant coordinates of the boundary edges of the honeycomb. We can then fill in the values of the hive using the fact that the edge lengths of the honeycomb are equal to the values of the left hand side of rhombus inequalities. We therefore have a bijection between our hives and certain honeycombs.

Converting the conditions of Proposition III.13 to the honeycombs produces precisely these three conditions. \square

3.3 Working with honeycombs

Honeycombs will be our main tool for studying the points in $\bar{\mathcal{F}}_{a,l}$. They provide a natural way to determine the degrees of freedom of the polytopes $F(\alpha\beta\gamma)$ using only the shape of the graph which does not depend on the actual configuration. As we shall see, we can also easily determine the underlying graph (i.e tinkertoy) for the honeycomb just by knowing which face of $\bar{\mathcal{F}}_{a,l}$ the point the honeycomb corresponds to is in. To do this, we will make use of some useful constructions on honeycombs. This section will be devoted to a brief introduction to the constructions of [5] and [6] which we will need for our analysis.

Boundary points of $\bar{\mathcal{F}}_{a,l}$ will have equality in some of the rhombus inequalities. Their corresponding honeycombs will therefore have some edges of length 0. Hence, when studying the faces of $\bar{\mathcal{F}}_{a,l}$ we will be interested in honeycombs where some of the edges are length 0. It will therefore be useful to get a better understanding of these types of honeycombs.

Definition III.21. We say an edge e of Γ_l is *degenerate* in a honeycomb h if $h(\text{head}(e)) = h(\text{tail}(e))$ (i.e. the edge is mapped to an edge of length 0). We say this degeneracy is *simple* if the edge does not share any vertices with another degenerate edge.

Suppose h is a τ_l -honeycomb which has a simple degenerate edge e . Consider the embedding in the plane for h . The two endpoints of e are mapped to the same point which is incident to four edges, two incoming edges and two outgoing edges. By our choice of directions for our tinkertoys, each of the incoming edges has an

outgoing edge in the same direction. We can therefore remove the vertex and merge the edges to form two intersecting edges. We can also do this to the corresponding edges and vertices in our tinkertoy τ to produce a new tinkertoy $\bar{\tau}$. The directions of the two new edges are well defined in $\bar{\tau}$ since the two edges it is replacing have the same direction. This new honeycomb is a configuration for $\bar{\tau}$ and it produces the same constant coordinates in its boundary edges. Every honeycomb in which e is degenerate is equivalent to a configuration of this new tinkertoy $\bar{\tau}$. This modification of the tinkertoy is called *eliding* the edge e , and the resulting tinkertoy $\bar{\tau}$ is called the *post-elision tinkertoy*.

Eliding simple degeneracies is useful since working with the post elision tinkertoy enables us to better find degrees of freedom in the honeycomb construction. Consider a τ_l honeycomb and let $\bar{\tau}$ be a post-elision tinkertoy for this honeycomb. Let γ be a non self-intersecting undirected loop in the underlying graph of $\bar{\tau}$ containing only non degenerate vertices of $\bar{\tau}$. Each of the vertices γ passes through must be trivalent since they are non degenerate. Choose an orientation for traversing γ . At every vertex, γ either turns left or right. For any $\epsilon > 0$, replace every vertex v in the loop by the vertex $v \pm \epsilon d(e_v)$ where e is the edge incident to v which is not part of the loop, and the \pm is chosen so that we are decreasing the length of e_v if the loop turns left and increasing its length if the loop turns right. For ϵ small enough, this will result in another valid configuration of τ_l where the constant coordinates of the infinite edges of the honeycomb are left unchanged and so it will correspond to the same point in $\bar{\mathcal{F}}_{a,l}$.

Definition III.22. This construction is called *breathing* the loop γ (see Figure 3.7).

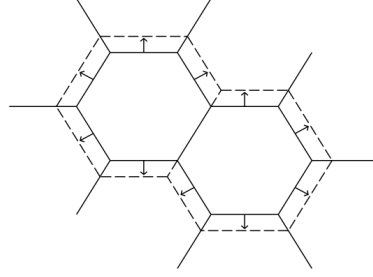


Figure 3.7: Example of breathing a loop.

Definition III.23. We can perform a similar construction on any non self-intersecting path between two boundary edges of the honeycomb and still produce a valid honeycomb. This is called the *trading construction*.

The trading construction is so named since when we do it we are trading ϵ between the constant coordinates of two of the boundary edges while keeping the rest unchanged. Doing this will result in a honeycomb for a different point in $\bar{\mathcal{F}}_{a,l}$, but crucially it will still have the same degeneracies as the original honeycomb.

Next consider a honeycomb with a non simple degeneracy. More than two vertices of Γ_l will be mapped to the same point in the configuration, so we may now have multiple edges of Γ_l mapped to the same edge. The underlying graph of the honeycomb will now consist of vertices with a valence of up to 6 as in Figure 3.8. The multiplicities of the edges incident to any vertex have to satisfy the relations: $a + b = d + e$, $b + c = e + f$, $c + d = f + a$. This honeycomb can be considered as a configuration of the tinkertoy formed using this underlying graph. This provides us with a more general notion of tinkertoy where we now allow edges to have multiplicities as long as they satisfy these relations. This new tinkertoy is called the *degeneracy tinkertoy* of the honeycomb.

Definition III.24. A *gentle path* in a honeycomb or tinkertoy is a path which only goes straight or turns 60° at any vertex, and must go straight at any cross vertex

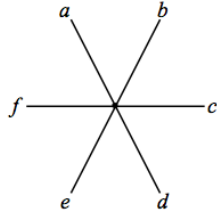


Figure 3.8: Generic vertex in a honeycomb where the edges are labeled by their multiplicities.



Figure 3.9: Degree 4 vertex which can be elided.

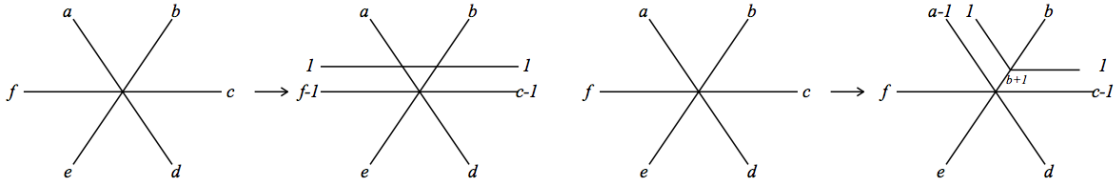


Figure 3.10: The two shifts needed to perform the trade constructions on a path of multiplicity > 1

(see Figure 3.9). Every path in a degeneracy free honeycomb or tinkertoy is a gentle path.

Remark III.25. In [5], Knutson and Tao only used these constructions when there are only simple degeneracies, but they can be performed in the more general case. We can still elide degeneracies at any vertex which is a degree 4 crossing as in Figure 3.9. We can still breathe any loop as long as the loop is a gentle loop and we can still perform the trade construction on any gentle path using the shifts of Figure 3.10.

Given a tinkertoy τ with multiplicity > 1 edges, it will be useful to produce a ‘minimal’ tinkertoy with only simple degeneracies which does not contain any degeneracies not in τ . To do this we will make use of *molting* in honeycombs.

Definition III.26. Given a maximal gentle path of edges with the same multiplicity $m > 1$, then we can *molt* the path to reduce the multiplicity of the path while producing a new gentle cycle/path of multiplicity 1 surrounding the path. This is done using the constructions of Figure 3.11.

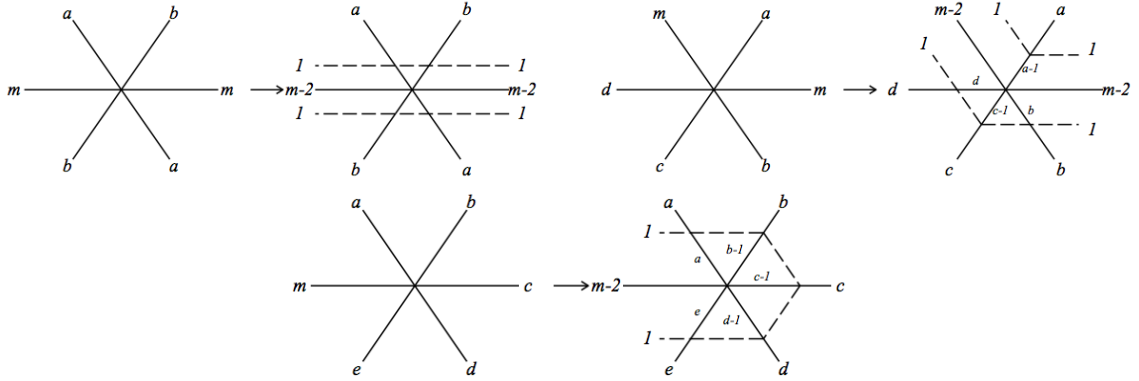


Figure 3.11: Constructions needed to molt a path

Remark III.27. In the construction for producing a honeycomb from a hive (Remark III.18), we associated to each edge e of a honeycomb an edge e' of the hive (namely the edge it intersected in the overlay of Γ_l and G_l , see Figure 3.6). The length of the edge e is equal to $f(e')\sqrt{2}$, the left hand side of the rhombus inequality associated to e' . That edge in the honeycomb is degenerate if and only if the hive has equality for the corresponding rhombus inequality. Hence, any edge of the honeycomb is degenerate if and only if the corresponding point in $F_{a,l}$ is in a specific face of the polytope. Faces of $F_{a,l}$, and by extension $\bar{\mathcal{F}}_{a,l}$, therefore correspond to honeycombs with a predetermined set of degenerate edges. For any face F of $F_{a,l}$, consider the degeneracy tinkertoy formed by introducing degeneracies for the rhombus inequalities where F has equality and then eliding any remaining simple degeneracies. This produces a sequence of tinkertoys $\tau_F^1, \dots, \tau_F^{a-1}$ such that every point in F corresponds to configurations of these tinkertoys (possibly with the additional requirement that the configuration has $(\rho_\beta^1)_l = (\rho_\gamma^1)_l = 0$), and a generic point in this face corresponds to a configuration with no degenerate edges. In addition, any sequence of honeycombs resulting as a configuration of this sequence of tinkertoys must have a corresponding hive with equality in all the rhombus inequalities defining F , so it must be a point in F . Hence, the points of F are precisely those points which can be produced by

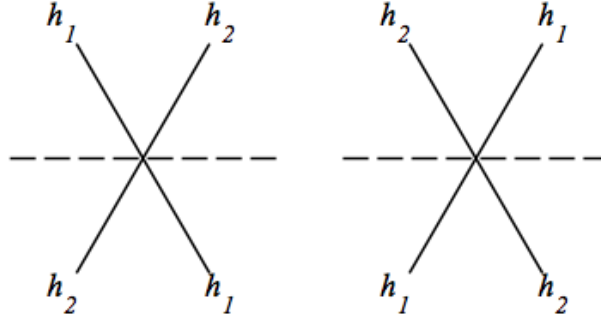


Figure 3.12: The possible intersections of a transverse overlay of two honeycombs. On the left h_1 turns clockwise to h_2 while on the right h_2 turns clockwise to h_1 .

configurations of $\tau_F^1, \dots, \tau_F^{a-1}$.

There is one final construction enabled by eliding which we will need. An *overlay* of two honeycombs or tinkertoys is the honeycomb or tinkertoy formed by placing one on top of the other. Suppose we have an overlay of two non degenerate honeycombs h_1 and h_2 . Suppose the honeycombs do not intersect at any of their vertices and that any intersection of their edges is transverse. Let p be a point where they intersect. Up to rotation, there two possibilities for the intersection (see Figure 3.12).

Definition III.28. We say that h_1 *turns clockwise to* h_2 if at the intersection the edges of h_2 are 60° clockwise from the edges of h_1 .

Remark III.29. Suppose h_1 turns clockwise to h_2 , and let P_1 and P_2 be non self-intersecting paths from p to a boundary edge of h_1 and h_2 respectively. We can perform the trade construction on the path formed by traversing P_1 in reverse and then following P_2 in order to change the constant coordinates of these two boundary edges while leaving the rest unchanged (Figure 3.13).

Note: With the original trade construction we can traverse the path backwards to shift the opposite direction, but that is not possible with this construction.

Proposition III.30. *Suppose that $\bar{\tau}$ is obtained from τ_l by eliding simple degenera-*

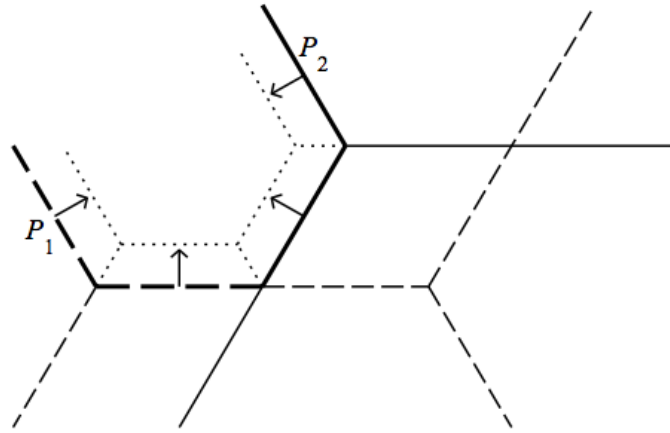


Figure 3.13: An example of the trade construction on the intersection of two honeycombs. The solid and dashed honeycombs intersect transversely with P_2 turning clockwise to P_1 .

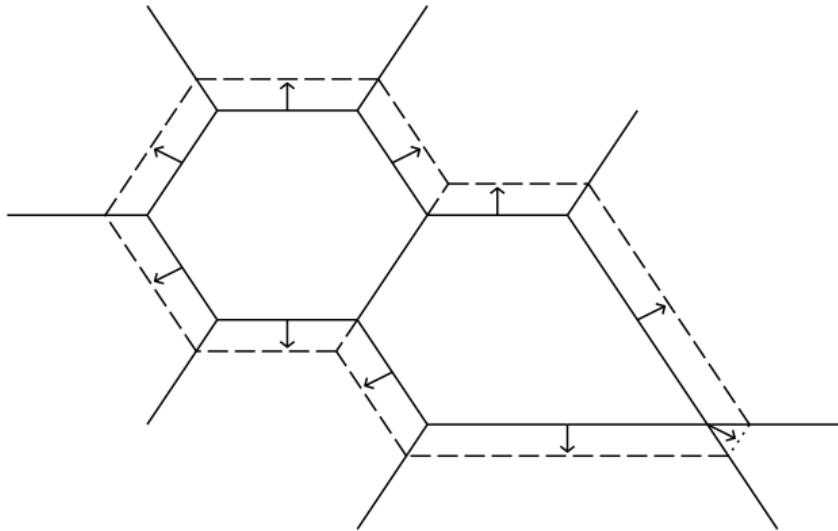


Figure 3.14: Performing the trade construction on a self intersecting path to remove a degeneracy (the southeast dotted edge).

cies (so $\bar{\tau}$ has no multiplicity > 1 edges). Then there is a tinkertoy $\bar{\tau}'$ which is an overlay of subtinkertoys isomorphic to some $\tau_{p,q}$ such that the constant coordinates obtainable by $\bar{\tau}$ -honeycombs are the same as those obtainable by $\bar{\tau}'$.

Proof. Consider the connected components of $\bar{\tau}$. Each connected component is a subtinkertoy. These subtinkertoys cannot intersect at vertices, and their edges must intersect transversely.

Let τ' be one of these connected subtinkertoys. In τ_l , each vertex v of the tinkertoy is incident to three edges, say e_1, e_2, e_3 , and the sum of the directions $d(e_1) + d(e_2) + d(e_3)$ is $(0, 0, 0)$. The sum of the directions of the edges incident to a vertex being 0 is maintained both when we elide degeneracies and also is maintained in the degeneracy tinkertoy. Hence, if we chose a configuration of τ' such that all its interior edges are degenerate, we will obtain a honeycomb which is a degree 4 cross as in Figure 3.9. This means that the number of infinite edges of τ' through any of the sides of the rhombus cutting out τ_l has to equal the number of infinite edges passing through the parallel side. This tinkertoy τ' is therefore a $\tau_{p,q}$, possibly with some elided degeneracies.

Suppose τ' has an elided degeneracy. Take any gentle path between the two edges containing the elided vertex. By performing the trade construction on this transverse intersection we can remove the degeneracy while keeping the constant coordinates of the infinite edges unchanged (Figure 3.14). We can therefore replace τ' by the honeycomb without the degeneracy and this does not change the possible values for the constant coordinates of the infinite edges of $\bar{\tau}$. Un-eliding all the simple degeneracies in τ' will result in a tinkertoy isomorphic to $\tau_{p,q}$ for some $p, q \leq l$. \square

Notation III.31. We call these subtinkertoys isomorphic to $\tau_{p,q}$ of Proposition III.30 the *irreducible tinkertoys* of $\bar{\tau}$.

Proposition III.32. *If h is a configuration of $\tau_{p,q}$, then*

$$(3.19) \quad \sum_{e \text{ boundary edge of } \tau_{p,q}} \epsilon_e C_e = 0$$

where C_e is the value of the constant coordinate of e and ϵ_e is 1 if $d(e) = (1, 0, -1)$ or $(-1, 1, 0)$ and -1 otherwise. Thus, $\epsilon_e = 1$ if the edge is an eastern boundary edge (in h_i these are the boundary edges whose constant coordinates are the entries of ρ_β^{i+1} and ρ_γ^i) and $\epsilon_e = -1$ if the edge is a western boundary edge (in h_i these are the boundary edges whose constant coordinates are the entries of ρ_γ^{i+1} and ρ_β^i).

Proof. This clearly holds for a honeycomb in which every finite edge is degenerate (it will be a degree 4 vertex as in Figure 3.9). Up to a shift in the plane, we can produce h from this maximally degenerate honeycomb using molting, un-eliding simple degeneracies, and using the trade construction on intersections to undo degeneracies. All of these procedures maintain this property, so h must also satisfy the condition. \square

3.4 Regular faces of $\bar{\mathcal{F}}_{2,l}$ are given by puzzles

Our goal for the next two sections will be to produce an analogue of the puzzles of [6] for our cones. Each puzzle will correspond to an inequality which $\bar{\mathcal{F}}_{a,l}$ must satisfy. For simplicity, we will focus on the case $a = 2$ for now and will generalize to larger a later. For $a = 2$, we have $\rho_\beta^1 = \rho_\gamma^1$, $\rho_\beta^2 = \beta$, and $\rho_\gamma^2 = \gamma$, so $\rho_\beta^1 = \rho_\gamma^1$ is the only additional partition occurring as the constant coordinates of boundary edges.

Notation III.33. For the rest of this section, we will use ρ to denote the partition

$$\rho := \rho_\beta^1 = \rho_\gamma^1.$$

Definition III.34. A *puzzle* is an edge-labeled subgraph of G_l (the hive graph) obtained by removing some interior edges of G_l such that each of the bounded faces in the graph is one of the following ‘pieces’:

- A unit equilateral triangle with all edges labeled with the same value.
- A unit rhombus with outer edges labeled j if clockwise from an obtuse angle and labeled i if clockwise from an acute angle for some $i < j$.

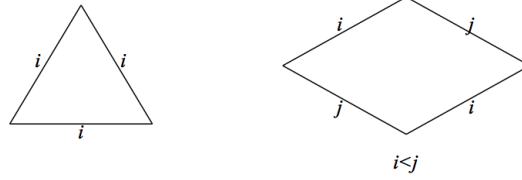


Figure 3.15: The two pieces allowed in a puzzle.

We will say that the puzzle is $\{0, 1\}$ if 0 and 1 are the only labels used in the puzzle.

Remark III.35. When constructing the hive for a point $x \in F_{2,l}$, the values for the points on the southwest, southeast, northwest, and northeast sides of G_l are the entries of π_γ^1 , π_β^1 , π_β^2 , and π_γ^2 respectively (see Figure 3.2). The difference between two consecutive terms in these π_β^i/π_γ^i is an entry in the corresponding $\rho_\beta^i/\rho_\gamma^i$, e.g. $(\pi_\beta^2)_{k+1} - (\pi_\beta^2)_k = (\rho_\beta^2)_k$. This produces a correspondence between the boundary edges of G_l and the entries of ρ_β^1 , ρ_γ^1 , $\beta = \rho_\beta^2$ and $\gamma = \rho_\gamma^2$: each edge corresponds to the $\rho_\beta^i/\rho_\gamma^i$ entry which is equal to the difference of the two π_β^i/π_γ^i entries for its vertices. When working with these puzzles, the label assigned to these boundary edges will be the coefficient of the corresponding $\rho_\beta^i/\rho_\gamma^i$ entry in a inequality (Proposition III.39).

Notation III.36. Given a puzzle, we will denote its boundary edges as follows:

- Let $e(\rho_\beta^1, 1), e(\rho_\beta^1, 2), \dots, e(\rho_\beta^1, l)$ denote the boundary edges on the southeastern side of the puzzle from left to right.
- Let $e(\beta, 1), e(\beta, 2), \dots, e(\beta, l)$ denote the boundary edges on the northwestern side of the puzzle from the left to right.

- Let $e(\rho_\gamma^1, 1), e(\rho_\gamma^1, 2), \dots, e(\rho_\gamma^1, l)$ denote the boundary edges on the southwestern side of the puzzle from right to the left.
- Let $e(\gamma, 1), e(\gamma, 2), \dots, e(\gamma, l)$ denote the boundary edges on the northeastern side of the puzzle from right to left.

Note: The southeastern and northwestern boundary edges are ordered from left to right while the southwestern and northeastern edges are ordered from right to left. This is done so that each boundary edge is indexed by the ρ entry the edge corresponds to in the previous remark, e.g. $e(\rho_\beta^1, j)$ corresponds to $(\rho_\beta^1)_j$.

Notation III.37. For any edge e in a puzzle P , we will use $\text{Lab}(e)$ to denote the label assigned to e in the puzzle and $\text{Lab}(P) = \{\text{Lab}(e) : e \text{ edge in } P\}$ to denote the set of all labels used in the puzzle.

Definition III.38. A puzzle is a $\bar{\mathcal{F}}_{2,l}$ -puzzle of degree d and type n if:

- For $j \leq n$, $\text{Lab}(e(\rho_\beta^1, j)) = \text{Lab}(e(\rho_\gamma^1, j)) - d$.
- For $j > n$, $\text{Lab}(e(\rho_\beta^1, j)) < \text{Lab}(e(\rho_\gamma^1, j)) - d$.

We will say that the puzzle is *proper* if $n = l$. Figure 3.16 is an example of a $\bar{\mathcal{F}}_{2,l}$ -puzzle.

Proposition III.39. Let P be a $\bar{\mathcal{F}}_{2,l}$ -puzzle of degree d . Then $\bar{\mathcal{F}}_{2,l}$ must satisfy the following inequality:

$$(3.20) \quad d\alpha_1 + \sum_j \text{Lab}(e(\beta, j)) \beta_j - \sum_j \text{Lab}(e(\gamma, j)) \gamma_j \leq 0$$

Proof. We say that the edges of a triangular piece are positively oriented if the triangle is oriented like the (+) triangles in Figure 3.17, and they are negatively oriented if the triangle is oriented like the (−) triangles. We say that an edge of a

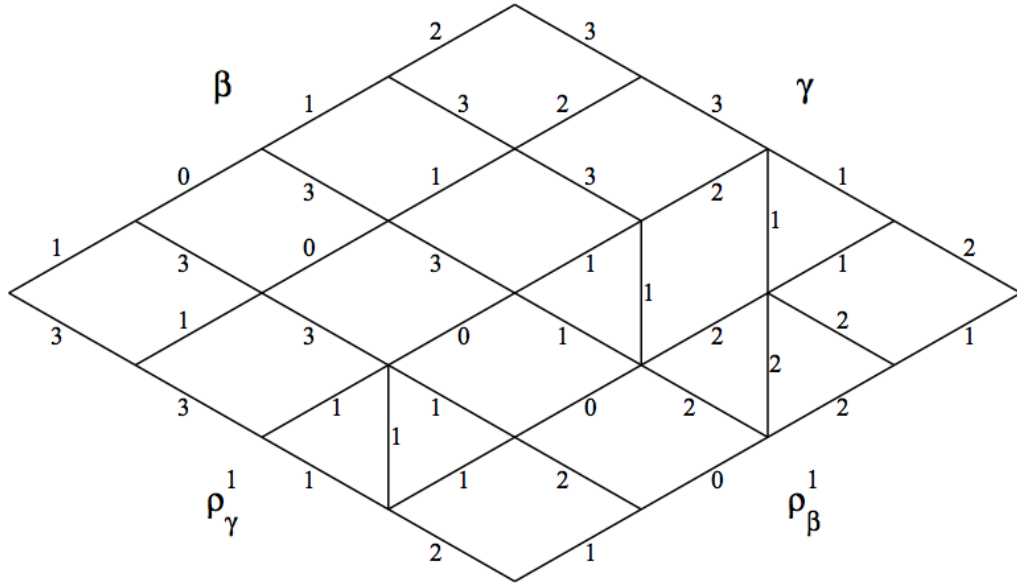


Figure 3.16: Example of a $\bar{\mathcal{F}}_{2,l}$ -puzzle of degree 1 and type 3 which produces the inequality $\alpha_1 + \beta_1 + \beta_3 + 2\beta_4 - 2\gamma_1 - \gamma_2 - 3\gamma_3 - 3\gamma_4 \leq 0$ in Proposition III.39.

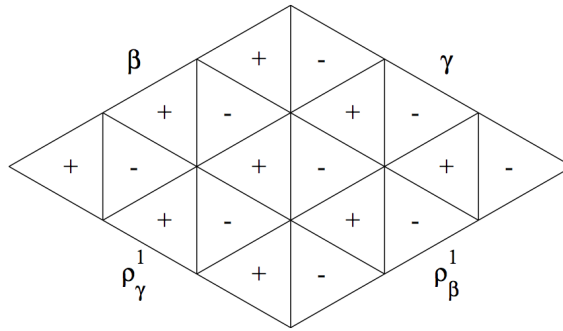


Figure 3.17: The orientation of the triangles in a puzzle.

rhombus piece is positively (resp. negatively) oriented if its outward normal is in the same direction as one of the outward normals of a positively (resp. negatively) oriented triangular piece. Let $\text{sgn}(e)$ equal $+1$ if the edge is positively oriented and -1 if it is negatively oriented.

Let h be a honeycomb for a point in $\bar{\mathcal{F}}_{a,l}$. Recall that every edge of G_l corresponds to a (possible degenerate) edge of h . For any edge e of G_l , let $c(e)$ be the constant coordinate of the edge in h corresponding to e . Consider the sum

$$S(h) = \sum_{p \text{ piece in } P} \sum_{e \text{ edge of } p} \text{sgn}(e) \text{Lab}(e)c(e)$$

where in the outer sum is over the triangular and rhombic pieces forming the interior faces of the puzzle.

Every interior edge of P contributes twice to this sum, once positively oriented and once negatively oriented, so its net contribution is 0. Removing the interior edges from the sum we find that $S(h)$ is equal to

$$S(h) = \sum_j \text{Lab}(e(\beta, j)) \beta_j - \sum_j \text{Lab}(e(\gamma, j)) \gamma_j - \\ - \sum_j \text{Lab}(e(\rho_\beta^1, j)) (\rho_\beta^1)_j + \sum_j \text{Lab}(e(\rho_\gamma^1, j)) (\rho_\gamma^1)_j.$$

Alternatively, each triangular piece of P corresponds to a vertex of h . The sum of the constant coordinates of the edges incident to a vertex is 0, so we have that

$\sum_{e \text{ edge of } p} \text{sgn}(e) \text{Lab}(e)c(e) = 0$ if p is a triangle. It is easy to check that if $u < v$ are the labels assigned to a rhombus piece p , then $\sum_{e \text{ edge of } p} \text{sgn}(e) \text{Lab}(e)c(e) = (u - v) \cdot f(e_p)$ where e_p is the edge of G_l which is the short diagonal of the rhombus and $f(e_p)$ is the left hand side of the rhombus inequality (3.17) associated to e_p (recall that the length of the edge in h corresponding to e_p is $f(e_p)\sqrt{2}$). We therefore have that $S(h)$ is a sum of non positive entries, i.e. $S(h) \leq 0$.

If P is a proper puzzle, then using the fact that $\rho_\gamma^1 = \rho_\beta^1 = \rho$, $\alpha_1 = |\rho|$, and $\text{Lab}(e(\rho_\beta^1, j)) = \text{Lab}(e(\rho_\gamma^1, j)) - d$ for $j = 1, \dots, l$, we obtain that $S(h)$ is equal to the right hand side of the inequality. If P is not proper, the same reasoning gives us that

$$(3.21) \quad 0 \geq S(h) = q_l \rho_l + q_{l-1} \rho_{l-1} + \dots + q_{n+1} \rho_{n+1} + \\ + \sum_j \text{Lab}(e(\beta, j)) \beta_j - \sum_j \text{Lab}(e(\gamma, j)) \gamma_j$$

for some $q_{n+1}, \dots, q_l \geq 0$. Adding the inequalities $0 \leq \rho_l \leq \dots \leq \rho_{n+1}$ as necessary, we obtain the desired inequality. \square

Our goal now is to show that the inequalities we obtain from these puzzles along with the chamber inequalities (defined below) form a defining set of inequalities for $\bar{\mathcal{F}}_{2,l}$. In doing so, we will show that these puzzles actually tell us what the tinkertoys are for the points in the face (Remark III.27), which will prove useful later when we try to determine when $F(\alpha\beta\gamma)$ is 0-dimensional. To do this, we will prove something analogous to Theorem 3 of [6] for faces of $\bar{\mathcal{F}}_{a,l}$, and so we will draw heavily on the techniques and ideas used in the proof of this theorem.

We already know the defining inequalities for some of the faces of $\bar{\mathcal{F}}_{2,l}$: β and γ have to be weakly decreasing. We will again call these inequalities the *chamber inequalities*:

$$\beta_j \geq \beta_{j+1}, \quad \gamma_k \geq \gamma_{k+1}$$

Definition III.40. A face of $\bar{\mathcal{F}}_{a,l}$ is *regular* if it is not contained in a chamber facet.

Let F be a regular face of $\bar{\mathcal{F}}_{2,l}$. From Remark III.27, we have that there is a tinkertoy τ_F such that every point in the interior of F corresponds to a configuration of τ_F with no additional degenerate edges and whose southeastern and southwestern infinite edges have the same sets of constant coordinates (recall that the constant coordinates of the southwestern and southeastern infinite edges will be the entries of ρ_γ^1 and ρ_β^1 respectively). The β and γ chamber inequalities will be strict for the points in F , but $\rho = \rho_\beta^1 = \rho_\gamma^1$ is not necessarily a strictly decreasing sequence for the points in F . It will prove useful to work with honeycombs for which ρ is strictly decreasing, so let G be the minimum regular face of $\bar{\mathcal{F}}_{2,l}$ containing the preimage of F , and let τ_G be the tinkertoy for G . In terms of tinkertoys, τ_F is obtained from τ_G by forcing some of the southern infinite edges to degenerate to the same edge. This τ_F does not encode whether $\rho_l = 0$ (i.e. whether F has equality for the non rhombus inequality (3.13)), so we will have to keep track of this separately.

Notation III.41. When working with honeycombs, the only coordinates which are always uniquely determined by which point in F it corresponds to are the constant coordinates for the northern infinite boundary edges since these must be the entries of β and γ . For this reason, for the rest of this section we will use *boundary edges* to reference just the northern infinite edges and not the southern infinite edges (whose constant coordinates are the entries of $\rho_\beta^1 = \rho_\gamma^1$ and contribute to the point (α, β, γ) for the honeycomb via $|\rho_\beta^1| = |\rho_\gamma^1| = \alpha_1$).

The set of constant coordinates for the southeastern infinite edges must be the same as the set of constant coordinates for the southwestern infinite edges (both must be the entries of ρ). Hence, for each southwestern infinite edge there must be a southeastern infinite edge with the same constant coordinate. We will say that these edges are *corresponding edges*.

Lemma III.42. *The tinkertoy τ_G has no edges of multiplicity greater than 1.*

Proof. A boundary edge of τ_G will have multiplicity greater than 1 if and only if G is contained in one of the β or γ chamber faces, which does not occur by assumption. Hence all the boundary edges must have multiplicity 1.

Consider the corresponding southern infinite edges as being connected. Suppose one of the interior edges has multiplicity larger than 1. Let h be any honeycomb of a point in the interior of G . Molt a maximal path containing the multiplicity greater than 1 edge (since we consider the corresponding southern edges as being connected, if the path has one it must have the corresponding one too). This will result in a honeycomb with none of the boundary edges moved and leaves the value of $\alpha_1 = |\rho|$ unchanged, so it will also be honeycomb for a point in G . Since $\rho_l > 0$ for the interior of G by how it was defined, as long as the molting is small enough we will maintain this property, hence the resulting honeycomb will still be a valid one

for $\mathcal{F}_{2,l}$. We therefore have a honeycomb for a point in G with fewer degeneracies than τ_G , contradicting how the tinkertoy τ_G is defined. \square

Corollary III.43. *There is an n such that for any point x in the interior of F*

$$0 = \rho_l = \rho_{l-1} = \cdots = \rho_{n+1} < \rho_n < \rho_{n-1} < \cdots < \rho_1.$$

Proof. We can use the same molting procedure on τ_F as in the proof of the Lemma. It will produce a valid honeycomb for a point in $\bar{\mathcal{F}}_{2,l}$ unless we molt a southern infinite edge whose constant coordinate is forced to be 0, since this would mean that $\rho_l < 0$. We therefore can molt away any instance of $\rho_j = \rho_{j+1}$, unless they are forced to be 0. \square

Definition III.44. We call the southern infinite edges whose constant coordinates are equal to ρ_1, \cdots, ρ_n the *proper* edges of F and $\rho_{n+1}, \cdots, \rho_l$ the *improper* edges of F .

By the proposition, the honeycombs for the points in the interior of G only have simple degeneracies and τ_G is obtained by eliding the degeneracies. Proposition III.30 tells us that τ_G is a transverse overlay of irreducible tinkertoys isomorphic to $\tau_{p,q}$ for some $p, q < l$. The τ_F tinkertoy is then an overlay (not necessarily transverse) of irreducible tinkertoys nearly isomorphic $\tau_{p,q}$: it will contain the degeneracies necessary to force some initial chain of the southeastern and southwestern infinite edges to all be equal. The τ_G tinkertoy contains similar information to τ_F (it is obtained by molting the improper edges of τ_f), its only advantage being that by using it we can distinguish which of the degenerate edges corresponds to each of $\rho_l, \cdots, \rho_{n+1}$. For this reason, they will be used fairly interchangeably, except when dealing with the improper edges.

Lemma III.45. *Every irreducible tinkertoy of τ_G must contain a proper infinite edge.*

Proof. If one of the irreducible tinkertoys contains only improper southern infinite edges, then it must be completely degenerate in τ_F (i.e. it will consist of just two edges which intersect at an elided vertex). The boundary edges of this irreducible tinkertoy will all have 0 constant coordinate, but this would mean $\beta = \gamma = 0$, so F is not regular. \square

Remark III.46. From Proposition III.32, we know that each of these irreducible tinkertoys produces an equality that the constant coordinates of its boundary edges must satisfy. It turns out that the inequalities obtained from puzzles are just sums of the inequalities for these irreducible tinkertoys. The rest of this section is essentially devoted showing how this is true (though this is not done explicitly) and what the consequences of this are for the puzzles.

Consider an inequality producing F . Using the equalities $\alpha_1 + \alpha_2 = \beta_1 + \dots + \beta_l = \gamma_1 + \dots + \gamma_l$ we can assume that the inequality is of the form

$$(3.22) \quad r_1\alpha_1 + s_1\beta_1 + \dots + s_l\beta_l - t_1\gamma_1 - \dots - t_l\gamma_l \leq 0$$

for $s_1, \dots, s_l, t_1, \dots, t_l \geq 0$. In addition, since $\bar{\mathcal{F}}_{2,l}$ is symmetric in α_1 and α_2 , by swapping the coefficients of α_1 and α_2 as necessary and then applying the equalities, we can reduce to the case when $r_1 \geq 0$.

For the next lemma, we will associate the boundary edges of τ_G with the value of the s/t coefficient in (3.22) of the entry of β/γ that equals the constant coordinate of the boundary edge.

Lemma III.47. *For any irreducible tinkertoy μ of τ_G , there is a k such that the set of boundary edges of μ is the set of boundary edges whose s/t coefficient equals k .*

Proof. Let e_1 and e_2 be two boundary edges both contained in μ . Suppose their s/t coefficients are not equal. There must be a path between e_1 and e_2 in this irreducible tinkertoy. Given a honeycomb for a generic point in G , we can perform the trade construction on this path to obtain another honeycomb of τ_G . This trade construction will either increase some β_i (resp. γ_i) while decreasing another β_j (resp. γ_j) by the same amount and leaving the other boundary edges and ρ unchanged, or it will increase (or decrease) some β_i and γ_j by the same amount while leaving the rest unchanged. The β_i/γ_j changed are the constant coordinates of e_1 and e_2 , so their coefficients in (3.22) are distinct and this will move the honeycomb out of the hyperplane given by equality in (3.22). The path can be shifted either direction using the trade construction so we can shift our honeycomb to either side of the hyperplane. This would mean that G contains a point which does not satisfy (3.22), so we therefore have that the s/t coefficients of the boundary edges of μ must be equal.

Next, suppose there is another irreducible tinkertoy whose boundary edges have the same value for their s/t coefficients as μ . If it intersects μ , then we can perform the trade construction on a point of intersection between them to trade values between a boundary edge of each. The s/t coefficients for these boundary edges will be equal, so this will result in a honeycomb whose left hand side of (3.22) will be unchanged, and hence it will still be in G . Performing the trading construction will introduce a new edge to the honeycomb which used to be degenerate. This would mean that we have an extra degeneracy in τ_G which does not occur for all the points in G , which contradicts how τ_G was defined. This means that the s/t coefficients of the boundary edges of irreducible tinkertoys must be distinct if they intersect. Two of the irreducible tinkertoys will fail to intersect only when they are parallel lines. In

this case, consider the set of parallel lines with the same s/t coefficients as forming a single tinkertoy. We therefore have that boundary edges for μ must be precisely the set of edges with the same s/t coefficient in (3.22). \square

Definition III.48. We call the value of k from Lemma III.47 the *st-value* of the irreducible tinkertoy.

Lemma III.49. *Suppose μ and ν are two of the irreducible tinkertoys of τ_G , and that at one of their intersections μ turns clockwise to ν . Then the st-value of μ must be smaller than the st-value of ν .*

Proof. Let k_μ and k_ν be the st-values of μ and ν respectively. Pick a boundary edge of each tinkertoy and consider a honeycomb for a generic point in F . Using the trade construction on a path between these boundary edges and their intersection, we can produce a new honeycomb whose constant coordinates for the boundary edges of μ and ν are changed by some $\epsilon > 0$. The values are traded so that the left hand side of (3.22) for this new honeycomb will equal $\epsilon(k_\mu - k_\nu)$. This must be negative since the new honeycomb will still be in the polytope, so we must have that $k_\mu < k_\nu$. \square

We obtain the following two useful results about the overlays of these irreducible tinkertoys as direct corollaries of Lemma III.49.

Corollary III.50. *Suppose μ and ν are two irreducible tinkertoys of τ_G such that at one of their intersections μ turns clockwise to ν . Then at every point in their intersection μ will turn clockwise to ν .*

Corollary III.51. *Suppose μ, ν , and σ are 3 irreducible tinkertoys of τ_G . If μ turns clockwise to ν and ν turns clockwise to σ , then μ must turn clockwise to σ .*

Definition III.52. Corollary III.51 tells us how to construct a partial order on the irreducible tinkertoys of τ_G :

$$\mu <_c \nu \text{ if } \mu \text{ turns clockwise to } \nu.$$

We will call this order the *clockwise ordering* on the irreducible tinkertoys.

By Lemma III.49, ordering the irreducible tinkertoys by their st -value is a linear extension of the clockwise ordering.

Remark III.53. We can use our decomposition of τ_G into a clockwise overlay of irreducible tinkertoys to produce a puzzle. Recall that τ_G only has simple degeneracies, so we can retrieve the tinkertoy τ_l from τ_G by de-eliding and removing the degeneracy at all the points of intersections of the irreducible tinkertoys. By overlaying τ_l on G_l (the hive graph) we get a correspondence between every edge of τ_l and every edge of G_l (see Figure 3.18). Assign each irreducible tinkertoy a value such that the ordering of the values is compatible with the clockwise ordering. Construct a puzzle P by removing every edge in G which corresponds to a degenerate edge in τ_G , and assigning the remaining edges the value of the irreducible tinkertoy containing the corresponding τ_G edge (see Figure 3.19). The triangular faces of P correspond to the vertices of τ_G and all its edges are assigned the same value as a result. The rhombus faces of P correspond to intersections between the irreducible tinkertoys. Lemma III.49 and Corollary III.50 tell us that the edges labels of the rhombus satisfy the condition of Definition III.34. If we can choose our values so that they satisfy the conditions to be a $\bar{\mathcal{F}}_{2,l}$ -puzzle (Definition III.38) where the type of the puzzle is equal to the n -value from Corollary III.43, then the inequality for P from Proposition 3.20 will be obtained as a positive linear combination of the rhombus inequalities corresponding to the degenerate edges of τ_G along with the ρ chamber

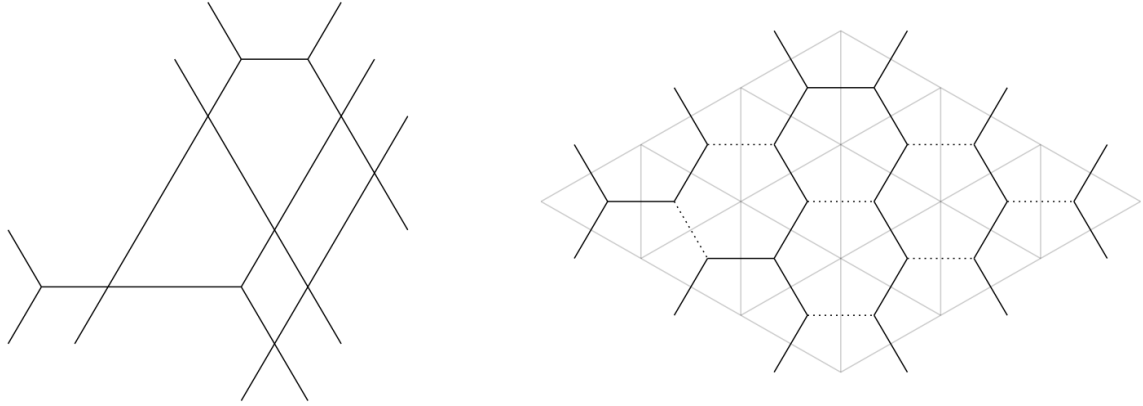


Figure 3.18: An example of a transverse overlay and the bijection between the edges of the honeycomb and the graph for the hives. The dotted edges are the degenerate edges of the honeycomb.

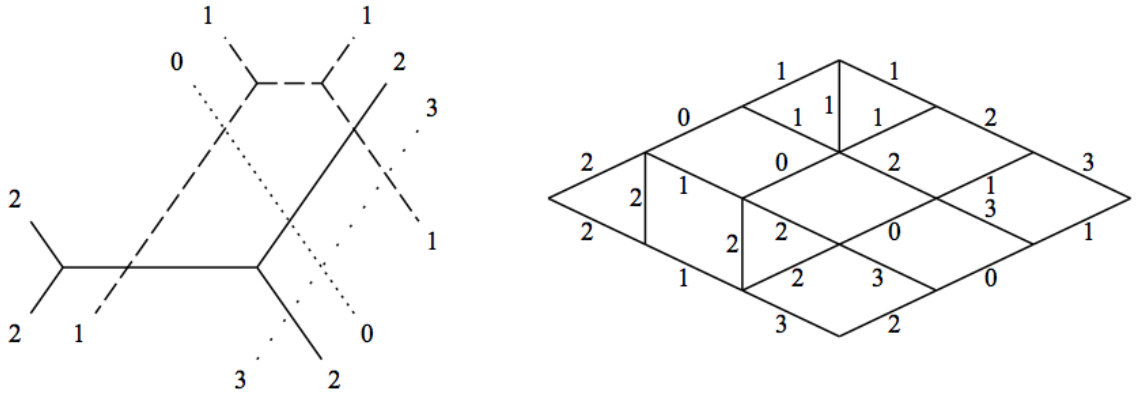


Figure 3.19: An assignment of values to the irreducible tinkertoys of Figure 3.18 which is compatible with the clockwise order and the puzzle this produces.

inequalities $0 \leq \rho_l \leq \rho_{l-1} \leq \dots \leq \rho_{n+1}$. The points of F are precisely the points whose honeycombs have equality for these inequalities, so F is therefore the face of $\bar{\mathcal{F}}_{2,l}$ given by the puzzle inequality for P .

Building off of the above remark, our goal now is to show that we can appropriately assign values to the irreducible tinkertoys in a way that is compatible with the clockwise order. To do this we need to show that we can choose the values so that the puzzle is a $\bar{\mathcal{F}}_{2,l}$ -puzzle, which means we need an understanding of how the tinkertoys containing corresponding southern infinite edges are related.

Lemma III.54. *Suppose μ and ν are two distinct irreducible tinkertoys of τ_G such*

that μ contains a proper southeastern infinite edge and ν contains the corresponding southwestern infinite edge. Then $\mu <_c \nu$.

Proof. The tinkertoys ν and μ must intersect since they cannot both be of the form $\tau_{0,q}$ or $\tau_{p,0}$, so suppose $\nu <_c \mu$. We can perform the trade construction on the intersection of μ and ν to decrease the constant coordinates of the southeastern infinite edge of μ and its corresponding southwestern infinite edge in ν while leaving the other infinite edges unchanged. This will result in a honeycomb with the same β and γ but a smaller α_1 . If $r_1 > 0$ in (3.22) then this will result in a point on the wrong side of the inequality, while if $r_1 = 0$ then this will result in a honeycomb for a point still in F but with fewer degeneracies. In either case, we have a contradiction. \square

Lemma III.55. *If μ is an irreducible tinkertoy of τ_G , then there is a unique irreducible tinkertoy ν containing the southwestern (resp. southeastern) infinite edges corresponding to the proper southeastern (resp. southwestern) infinite edges of μ .*

Proof. Suppose there are two distinct irreducible tinkertoys ν and σ containing southwestern edges corresponding to proper southeastern edges of μ . Denote the southwestern edges of ν and σ by e_ν^γ and e_σ^γ respectively and denote their corresponding southeastern edges in μ by e_ν^β and e_σ^β respectively. Let P_1 be a path in τ_F from a boundary edge of ν to e_ν^γ , let P_2 be a path from e_ν^β to e_σ^β , and let P_3 be a path from e_σ^γ to a boundary edge of σ (see Figure 3.20). By performing the trading construction simultaneously on these three paths, we can produce honeycombs where we have kept the value of α_1 unchanged while changing the constant coordinates of the boundary edges. We can do the trading constructions in either direction, so we can produce a honeycomb on the wrong side of the inequality of (3.22).

The same construction can also be used when the southeastern and southwestern

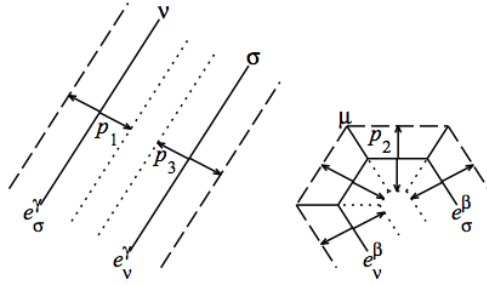


Figure 3.20: Trade construction used in the proof of Lemma III.55.

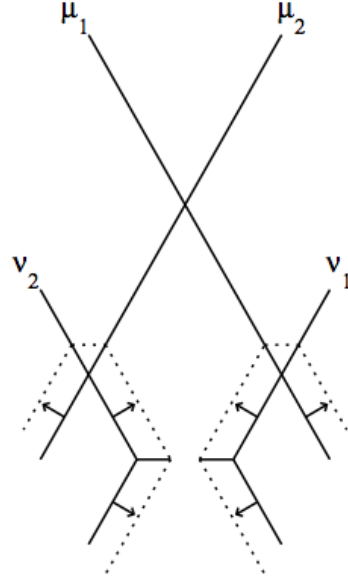


Figure 3.21: Trade construction used in the proof of Lemma III.62.

infinite edges are swapped. □

Lemma III.56. *An irreducible tinkertoy of τ_G cannot contain both improper southwestern and southeastern infinite edges.*

Proof. Suppose μ is an irreducible tinkertoy containing both an improper southwestern infinite edge and an improper southeastern infinite edge. Taking any path between these two edges, we can shift the path to increase the constant coordinate of both of these edges. This will increase α_1 while leaving β and γ unchanged. If $r_1 > 0$, then this will result in a point on the incorrect side of (3.22). If $r_1 = 0$, then this will result in a point which still has equality in (3.22), so it is still in F but it has fewer degeneracies than τ_F . In either case, we reach a contradiction. □

Lemma III.57. *If μ and ν are two irreducible tinkertoys of τ_G containing improper southeastern infinite edges and southwestern infinite edges respectively, then $\mu <_c \nu$.*

Proof. We can perform the same trade construction as in the proof of Lemma III.54.

□

Lemma III.58. *Either every irreducible tinkertoy of τ_G contains the southwestern infinite edges corresponding to its proper southeastern infinite edges, or none of the irreducible tinkertoys have this property.*

Proof. Suppose μ is an irreducible tinkertoy which has this property. Let P be path between one of its proper southeastern infinite edges and the corresponding southwestern infinite edge. We can shift this path to increase α_1 while leaving β and γ unchanged. If $r_1 > 0$, then this would result in a honeycomb on the incorrect side of (3.22). We therefore have that $r_1 = 0$ if any of the irreducible tinkertoys have this property.

Next, suppose $r_1 = 0$, but that there is a tinkertoy that does not have this property; i.e. there are distinct irreducible tinkertoys ν and σ such that ν contains the southwestern infinite edges corresponding to the proper southeastern infinite edges of σ . Let P be any path from a proper southeastern infinite edge of σ to a point of intersection of ν and σ and then back to the corresponding southwestern infinite edge. We can use the trade construction on this intersection in any honeycomb for a point in F to produce a honeycomb with the same β and γ values but a different α . This new honeycomb will still correspond to a point in F since $r_1 = 0$, but it will not have one of the degeneracies of τ_F , contradicting the definition of τ_F . This means that if $r_1 = 0$ then every tinkertoy must have this property.

We therefore must have that either every tinkertoy has this property (in which case $r_1 = 0$) or none of them have this property (in which case $r_1 > 0$). □

Definition III.59. We say that F is *r-free* if every irreducible tinkertoy contains the southwestern edges corresponding to its proper southeastern edges.

We are now ready to prove that F is given by a puzzle inequality. There are two possibilities for F with slightly different puzzles depending on whether or not F is r -free.

Theorem III.60. *If F is r -free then there is a $\bar{\mathcal{F}}_{2,l}$ -puzzle P of degree 0, type n (=number of pairs of corresponding proper southern edges), and with $\text{Lab}(P) = \{0, \dots, m\}$ for some m , such that F is the face of $\bar{\mathcal{F}}_{2,l}$ given by the inequality of P . Moreover, if F is a facet then P is $\{0, 1\}$ (i.e. $m = 1$).*

Proof. Let $\mu_0 < \mu_1 < \dots < \mu_k$ be any linear extension of the clockwise ordering on the irreducible tinkertoys of τ_G . Let P be the puzzle from Remark III.53 obtained by assigning μ_i a value of i . This assignment satisfies the conditions to be a $\bar{\mathcal{F}}_{2,l}$ -puzzle of degree 0 since for any j , $e(\rho_\beta^1, j)$ and $e(\rho_\gamma^1, j)$ will have the same label if the southern infinite edges with constant coordinate equal to ρ_j are proper, and Lemma III.57 guarantees that $e(\rho_\gamma^1, j)$ will have a large label than $e(\rho_\beta^1, j)$ if the southern improper edges with constant coordinate equal to ρ_j are improper. The number of proper edges is equal to the value of n from Corollary III.43, so F will exactly be the face given by the inequality for P .

Now, suppose that F is a facet. Let μ be one of the irreducible tinkertoys. We know that μ is isomorphic to $\tau_{p,q}$ for some p, q , so by Proposition III.32 the constant coordinates of the infinite edges of any configuration of μ satisfy an equality. Using the fact that $\rho_\beta^1 = \rho_\gamma^1$ and that the constant coordinate of the improper southern edges must be 0, we get that any τ_F honeycomb must satisfy the equality

$$\beta_{i_1} + \dots + \beta_{i_p} = \gamma_{j_1} + \dots + \gamma_{j_q}$$

where $\beta_{i_1}, \dots, \beta_{i_p}, \gamma_{j_1}, \dots, \gamma_{j_q}$ are the constant coordinates of the boundary edges of μ .

The only equalities which all of the $\bar{\mathcal{F}}_{2,l}$ satisfy are

$$\sum_i \alpha_i = \sum_i \beta_i = \sum_i \gamma_i.$$

If F has more than two irreducible tinkertoys, then we obtain at least two equalities which the honeycombs of τ_F must satisfy which are linearly independent from the equalities for all of $\bar{\mathcal{F}}_{2,l}$. This would mean that F has codimension at least 2 and hence not a facet. There being one irreducible tinkertoy corresponds to the trivial face which is the entire polytope, so there must be exactly two irreducible tinkertoys. Hence, the P obtained will be a $\{0, 1\}$ puzzle. \square

We are now left with the case when F is not r -free. As in the r -free case, the labeling will be determined by the connected components of τ_G , but the connected components are not as simple in this case. We will begin by getting a grasp on these connected components.

Consider τ_G where we have connected the corresponding proper southern infinite edges. Given an oriented loop on the underlying graph of τ_G , define a winding number of the oriented loop by assigning it $+1$ every time it goes from a southeastern edge to the corresponding southwestern edge and giving it -1 each time it does the reverse. Lemma III.54 guarantees that there are no nonzero winding number loops.

The connected components of τ_G are unions of the irreducible tinkertoys. Since there are no nonzero winding number loops, each connected component contains precisely one tinkertoy with no proper southeastern (alternatively southwestern) edges. This means that it cannot contain two tinkertoys consisting of parallel lines in the same direction, so all of the irreducible tinkertoys inside one of the connected components must intersect. Hence, the clockwise order is a linear order on the irreducible tinkertoys of a connected component.

Notation III.61. For any connected component C of τ_G , let $\mu_C^1 <_c \mu_C^2 <_c \dots$ be the ordering of the irreducible tinkertoys in C

Lemma III.62. *Suppose μ_1 is an irreducible tinkertoy containing a (not necessarily proper) southeastern infinite edge and μ_2 is an irreducible tinkertoy containing the corresponding southwestern infinite edge. If ν_1 and ν_2 are a similar pair where the southern infinite edges are proper, then at most one of $\mu_1 >_c \nu_1$ and $\nu_2 >_c \mu_2$ can be true.*

Proof. By performing a series of trade constructions as in Figure 3.21, we can produce a honeycomb corresponding to the same point but with fewer degeneracies than τ_F . □

Remark III.63. Let C be one of the connected component of τ_G and let μ_1 and μ_2 be two of the irreducible tinkertoys in C such that μ_2 contains the southwestern infinite edges corresponding to the proper southeastern infinite edges of μ_1 . By Lemma III.62 there cannot be consecutive irreducible tinkertoys $\nu_1 <_c \nu_2$ in a connected component other than C such that $\mu_1 >_c \nu_1$ and $\nu_2 >_c \mu_2$. This means that for any two connected components C and D there is an i such that

$$\mu_C^1 <_c \mu_C^2 <_c \dots <_c \mu_C^i <_c \mu_D^1 <_c \mu_C^{i+1} <_c \mu_D^2 <_c \dots$$

where the irreducible tinkertoys keep alternating in the clockwise order until one of the connected components runs out.

Lemma III.64. *For any connected component C of τ_G ,*

$$r_1 = (\text{st-value of } \mu_C^{i+1}) - (\text{st-value of } \mu_C^i).$$

Proof. Shift the union of any two paths from boundary edges of μ_C^{i+1} and μ_C^i to a pair of corresponding southwestern and southeastern infinite edges respectively. This will

result in a valid τ_F honeycomb, so the point will still be in F . This shifts the left hand side of (3.22) by $r_1 - (\text{st-value of } \mu_C^{i+1}) + (\text{st-value of } \mu_C^i)$, which must equal 0 since the point is still in F . \square

Lemma III.65. *If μ contains an improper southeastern infinite edge and ν contains the corresponding southwestern infinite edge then*

$$r_1 < (\text{st-value of } \mu) - (\text{st-value of } \nu)$$

Proof. Perform the same shift as in the previous lemma, making sure to increase the constant coordinate of the improper edges. This will result in a valid honeycomb for $\bar{\mathcal{F}}_{2,l}$ but not F since it removed an improper edge. This again shifts the left hand side of (3.22) by $r_1 - (\text{st-value of } \mu) + (\text{st-value of } \nu)$, so this value must be negative since the point must satisfy (3.22) with strict inequality. \square

Let s_0 be the minimal st -value of the irreducible tinkertoys. Lemma III.64 tells us that the value of

$$(3.23) \quad (\text{st-value of } \mu_C^i) - s_0 \pmod{r_1}$$

is constant along any of the connected components. Order the connected components based on their smallest nonnegative residue in (3.23) (i.e. the actual remainder when $(\text{st-value of } \mu_C^i) - s_0$ is divided by r_1), breaking any ties based using their st -values. We are now ready to assign values to the irreducible tinkertoys as in Remark III.53 to produce a puzzle for F .

Theorem III.66. *If F is not r -free, then there is a $\bar{\mathcal{F}}_{2,l}$ puzzle P of nonzero degree such that F is the face of $\bar{\mathcal{F}}_{2,l}$ given by the inequality of P . Moreover, if F is a facet, then P is proper and can be chosen to both have degree 1 and $\text{Lab}(P) = \{0, \dots, m\}$ for some m .*

Proof. Let $C_1 < C_2 < \dots < C_d$ be the ordering of the connected components. For any i , assign $\mu_{C_i}^1$ a value of $k \cdot d + i - 1$ where k is the nonnegative integer such that $k \cdot r_1 \leq (\text{st-value of } \mu_{C_i}^1) < (k + 1) \cdot r_1$. Assign $\mu_{C_i}^2$ a value d larger than this, $\mu_{C_i}^3$ a value d larger than $\mu_{C_i}^2$, and so on until all of the irreducible tinkertoys in C_i are labeled. This assignment is compatible with the clockwise order since it is compatible with the order given by the st -values. Using Remark III.53, we can create a puzzle using this assignment. This puzzle will be a $\bar{\mathcal{F}}_{2,l}$ puzzle by Lemmas III.64 and III.65, and it has degree equal to the number of connected components of τ_G .

Next, suppose that F is a facet. F must be the intersection of $\bar{\mathcal{F}}_{2,l}$ with the equality given by P . This equality will have $r_1 > 0$ since P has nonzero degree. Suppose there is a connected component of τ_G which contains both improper southwestern and improper southeastern infinite edges. Then by taking a path/loop between these two edges in τ_F , we can shift the path to increase the constant coordinates of the improper edges while producing a honeycomb for $\bar{\mathcal{F}}_{2,l}$. This loop will have nonzero winding number by Lemma III.57, so doing this construction will increase α_1 while leaving β and γ unchanged. This will result in a honeycomb which does not satisfy (3.22), so we therefore have that an improper southwestern infinite edge cannot be in the same connected component as an improper southeastern infinite edge.

Using the same reasoning as in Theorem III.60, we get that every connected component will produce an equality

$$\beta_{i_1} + \dots + \beta_{i_p} = \gamma_{j_1} + \dots + \gamma_{j_q}$$

that the points in F will satisfy, where $\beta_{i_1}, \dots, \beta_{i_p}, \gamma_{j_1}, \dots, \gamma_{j_q}$ are the constant coordinates of the boundary edges of the connected component. The inequality for P will have $r_1 > 0$, so the equality obtained from P will be distinct from this equality. If there are two connected components then P along with one of these equalities will

be a pair of equalities which are linearly independent from the equalities for all of $\bar{\mathcal{F}}_{2,l}$. This would mean that F has codimension at least two and is not a facet. We therefore have that τ_G has a single connected component. By what was just shown, this is only possible if τ_G has no improper edges, i.e. P is proper. Since τ_G has a single connected component, say C , when we do this assignment, each of the irreducible tinkertoys μ_C^i is assigned value of $i - 1$, which produces a puzzle where the set of labels used is precisely $\{0, \dots, m\}$ where $m = (\text{the number of irreducible tinkertoys}) - 1$. \square

Remark III.67. The proof of Theorem III.66 actually proves something stronger about the puzzle. It shows that the degree of the puzzle is the number of connected components of the tinkertoy for the face, provided the face is not r -free. We can determine the number of connected components from just the labels on the southern sides of the puzzle. Construct a graph by letting it have a vertex for every southern infinite edge of τ_G . Add an edge between the vertices for each corresponding pair of infinite edges. Then introduce an edge between any two vertices whose infinite edges are in the same irreducible tinkertoy. Two vertices in this graph are in the same connected component if and only if their infinite edges are in the same connected component of τ_G .

3.5 Puzzle chains and regular faces of $\bar{\mathcal{F}}_{a,l}$.

Now that we have an understanding of the $a = 2$ case, our goal is to generalize the previous section to $\bar{\mathcal{F}}_{a,l}$ for any $a \geq 2$. The points in $\bar{\mathcal{F}}_{a,l}$ correspond to sequences of hives/honeycombs which are related by some conditions on the boundary edges/constant coordinates of the infinite edges (Proposition III.13 and Proposition III.20). We can again use the clockwise overlays to produce puzzles (Remark III.53) for each of the honeycombs, but these puzzles will need to be related by certain

boundary edges being assigned the same labels.

Notation III.68. Let P_1, \dots, P_{a-1} be a sequence of puzzles, where P_i will be the puzzle associated with the i th hive for points in $\bar{\mathcal{F}}_{a,l}$. For any i :

- Let $\text{Lab}(P_i, \rho_\beta^+, j)$ be the label assigned to the j th boundary edge on the north-west side of P_i , where the edges on the side are ordered from left to right.
- Let $\text{Lab}(P_i, \rho_\beta^-, j)$ be the label assigned to the j th boundary edge on the south-east side of P_i , where the edges on the side are ordered from left to right.
- Let $\text{Lab}(P_i, \rho_\gamma^+, j)$ be the label assigned to the j th boundary edge on the north-east side of P_i , where the edges on the side are ordered from right to left.
- Let $\text{Lab}(P_i, \rho_\gamma^-, j)$ be the label assigned to the j th boundary edge on the south-west side of P_i , where the edges on the side are ordered from right to left.

In the i th hive, the difference between the values assigned to the two vertices incident to the boundary edge labeled with $\text{Lab}(P_i, \rho_\beta^+, j)$, $\text{Lab}(P_i, \rho_\beta^-, j)$, $\text{Lab}(P_i, \rho_\gamma^+, j)$, or $\text{Lab}(P_i, \rho_\gamma^-, j)$ is equal to $(\rho_\beta^{i+1})_j$, $(\rho_\beta^i)_j$, $(\rho_\gamma^{i+1})_j$, or $(\rho_\gamma^i)_j$ respectively (this is the analogue of Remark III.35 for the general case).

Definition III.69. Let δ be a partition of length $a - 1$ and n a nonnegative integer.

We say that the sequence P_1, \dots, P_{a-1} is a *chain of degree δ and type n* if it satisfies the following conditions:

- For $i = 2, \dots, a - 1$ and $j = 1, \dots, l$,

$$\text{Lab}(P_i, \rho_\beta^-, j) = \text{Lab}(P_{i-1}, \rho_\beta^+, j),$$

$$\text{Lab}(P_i, \rho_\gamma^-, j) = \text{Lab}(P_{i-1}, \rho_\gamma^+, j) + (\delta_i - \delta_{i+1}).$$

- For $j = 1, \dots, n$,

$$\text{Lab}(P_1, \rho_\gamma^-, j) = \text{Lab}(P_1, \rho_\beta^-, j) + (\delta_1 - \delta_2).$$

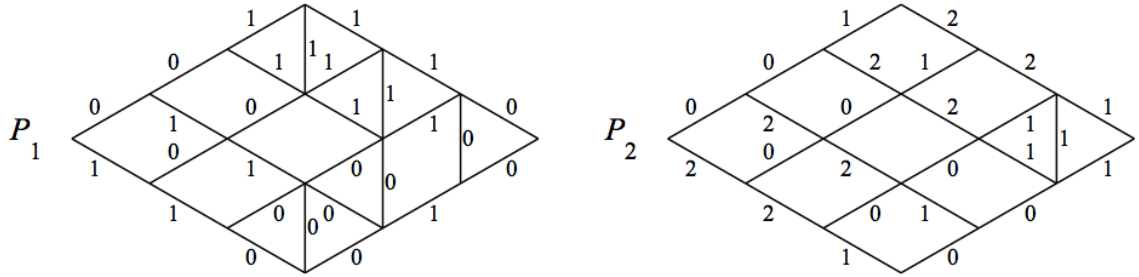


Figure 3.22: A $\bar{\mathcal{F}}_{3,3}$ puzzle chain with $n = 2$ for the inequality $\alpha_1 + \alpha_2 + \beta_3 - \gamma_1 - 2\gamma_2 - 2\gamma_3 \leq 0$.

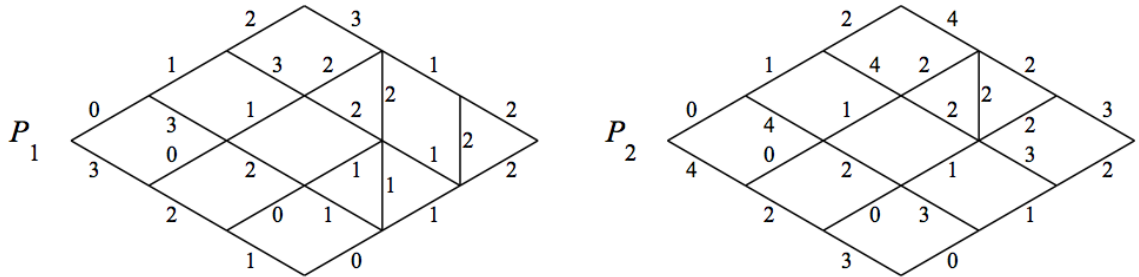


Figure 3.23: A proper (i.e. $n = 3$) $\bar{\mathcal{F}}_{3,3}$ puzzle chain for the inequality $2\alpha_1 + \alpha_2 + \beta_2 + 2\beta_3 - 3\gamma_1 - 2\gamma_2 - 4\gamma_3 \leq 0$

- For $j = n + 1, \dots, l$,

$$\text{Lab}(P_1, \rho_{\gamma}^-, j) > \text{Lab}(P_1, \rho_{\beta}^-, j) + (\delta_1 - \delta_2).$$

We again say that the puzzle chain is $\{0, 1\}$ if this is the set of labels used, and that it is *proper* if $n = l$. Figures 3.22 and 3.23 are examples of puzzle chains for $\bar{\mathcal{F}}_{3,3}$.

Proposition III.70. *Let P be a puzzle chain of degree δ . Then $\bar{\mathcal{F}}_{a,l}$ will satisfy the inequality:*

$$\sum_{i=1}^{a-1} (\delta_i - \delta_{i+1}) \alpha_i + \sum_{j=1}^l \text{Lab}(P_{a-1}, \rho_{\beta}^+, j) \cdot \beta_j - \sum_{j=1}^l \text{Lab}(P_{a-1}, \gamma, j) \cdot \gamma_j \leq 0$$

Proof. This follows by the same reasoning as the $a = 2$ case (Proposition III.39). \square

Similar to the previous section, we want show that the regular faces of $\bar{\mathcal{F}}_{a,l}$ have inequalities which are given by these puzzle chains. Our approach this time will be

less constructive. In the previous section, we showed how we could use the clockwise order on the irreducible tinkertoys to produce the labels for the puzzle. That approach can be done for the general case, but ends up being fairly tedious and not very enlightening. Instead, for this section we will show that any inequality for a regular face can be produced from a puzzle chain as long as the coefficient of α_a is 0 in the inequality (using $\sum_i \alpha_i = \sum_i \beta_i$ any inequality can be expressed in this form). This result is useful since in the next section we will see that in order for the puzzles to correspond to a regular face it must be uniquely determined by its coefficients. This means that as long as we know an inequality for the face, we can construct a puzzle chain for the face, which in turn can be used to construct the sequence of tinkertoys for the face.

Let F be a regular face of $\bar{\mathcal{F}}_{a,l}$. Using the equalities $\sum_i \alpha_i = \sum_i \beta_i = \sum_i \gamma_i$, we can reduce any defining inequality for F to be of the form

$$(3.24) \quad r_1 \alpha_1 + \cdots + r_{a-1} \alpha_{a-1} + s_1 \beta_1 + \cdots + s_l \beta_l - t_1 \gamma_1 - \cdots - t_l \gamma_l \leq 0$$

where $r_1, \dots, r_{a-1} \geq 0$ (no restrictions on the s_j and t_k). Using the fact that $\bar{\mathcal{F}}_{a,l}$ is symmetric in the parts of α , we will reduce to looking at the case when $r_1 \geq r_2 \geq \cdots \geq r_{a-1} \geq 0$.

Remark III.71. The results for this section will still hold for the inequality as long as the coefficient of α_a is 0. This order for the coefficients of the α_i is chosen since it produce faces with fewer improper edges than any other ordering.

Let $\tau_F^1, \dots, \tau_F^{a-1}$ be the sequence of tinkertoys associated to F . We want to show that we can assign the coefficients in the inequality to irreducible tinkertoys to produce puzzles forming a chain using the construction of Remark III.53. In order to do this we need tinkertoys where we have removed all the non simple degeneracies of the

τ_F^i . As before, let G be a face of $\mathcal{F}_{a,l}$ which is minimal amongst the faces containing the preimage of F and is not contained in any chamber faces. Let $\tau_G^1, \dots, \tau_G^{a-1}$ be the sequence of tinkertoys for G . Molting away the multiplicity > 1 edges like in Lemma III.42 and Corollary III.43, we again find that the τ_G^i have only simple degeneracies and that the τ_F^i are obtained from the τ_G^i by introducing the degeneracies forced by $0 = \rho_l^1 = \dots = \rho_{n+1}^1$ for some n . Hence we again have a notion of proper and improper edges of the τ_F^i/τ_G^i . Since the τ_G^i have only simple degeneracies, by Proposition III.30 they are transverse overlays of irreducible tinkertoys isomorphic to various $\tau_{p,q}$. We can then apply the same reasoning as Lemma III.49 to show that we have a clockwise order on the irreducible tinkertoys of each τ_G^i .

Recall that $\alpha_1 + \alpha_2 + \dots + \alpha_i = |\rho_\gamma^i| = |\rho_\beta^i|$. The constant coordinates of the northern infinite edges of τ_G^{a-1} are the only infinite edges whose entries are uniquely determined by the point $(\alpha, \beta, \gamma) \in \bar{\mathcal{F}}_{a,l}$ the honeycomb is for (they are equal to the entries of β and γ). For the rest of the infinite edges, they are only related to the point (α, β, γ) by the fact that the sum of all of the constant coordinates of the infinite edges in one of the honeycombs which point in the same direction will be equal to $|\rho_\gamma^i|$ or $|\rho_\beta^i|$ for some i .

Notation III.72. For every infinite edge in a sequence of honeycombs for $\tau_G^1, \dots, \tau_G^{a-1}$, except the northern ones in the honeycomb for τ_G^{a-1} , there is another infinite edge whose constant coordinate is equal to the same entry of $\rho_\beta^i/\rho_\gamma^i$. We will say that two such edges are *corresponding edges*.

Notation III.73. For an irreducible tinkertoy μ , let $B_\mu^+, B_\mu^-, \Gamma_\mu^+$, and Γ_μ^- denote the set of northwestern, southeastern, northeastern, and southwestern infinite edges of μ respectively.

Lemma III.74. *If μ is an irreducible tinkertoy of τ_G^i containing a northwestern (resp. northeastern) infinite edge, then there is an irreducible tinkertoy ν of τ_G^{i+1} such that $B_\mu^+ = B_\nu^-$ (resp. $\Gamma_\mu^+ = \Gamma_\nu^-$).*

Proof. Suppose that there are two tinkertoys ν and σ containing southeastern infinite edges, say e_ν^β and e_σ^β , whose corresponding infinite edges are in B_μ^+ (the Γ_μ^+ case will follow by the exact same argument).

Case 1: μ is not a set of parallel lines and ν, σ intersect.

Without loss of generality, $\nu <_c \sigma$. Take a proper path in μ between the corresponding infinite edges for e_ν^β and e_σ^β and proper paths in ν and σ from the intersection of ν and σ to e_ν^β and e_σ^β respectively. Use the trade construction on the paths in ν and σ to increase the constant coordinate of e_ν^β while decreasing the constant coordinate of e_σ^β , and shift the path in μ to perform the same trade on its edges corresponding to e_μ^β and e_ν^β . This will maintain (α, β, γ) , so it will still be a valid honeycomb for F and hence G . This will remove the degeneracy at the intersection of ν and σ , contradicting how the τ_G^i are defined.

Case 2: μ is not a set of parallel lines and ν, σ do not intersect.

Since they do not intersect, ν and σ are both sets of parallel lines. Consider the pairs of corresponding proper edges of $\tau_G^1, \dots, \tau_G^{a-1}$ as being connected. Let τ_G^i be the first irreducible tinkertoy where either ν or σ is connected to an irreducible tinkertoy which isn't a set of parallel lines with constant second coordinate. We can perform the analogue of the construction from case 1 to again show this is not possible. We therefore have that ν and σ are only connected to straight lines with constant second coordinate. Let β_i and β_j be entries of β corresponding to the boundary edges ν and σ are connected to respectively. By taking a path from the boundary edge for β_i to μ and from μ back to the boundary edge for β_j , we can shift this path

to increase/decrease β_i and decrease/increase β_j while keeping the other coordinates of (α, β, γ) unchanged. This will create a point on the incorrect side of (3.24) unless $s_i = s_j$, in which case we can consider ν and σ as being a single irreducible tinkertoy.

Case 3: μ is a set of parallel lines.

We can perform the same constructions as in case 2, we just have to go to earlier τ_G^i to find an intersection to perform the trade construction on. By the previous lemma, the connected components of ν and σ must extend to τ_G^1 where after passing from southeastern edges to southwestern edges of τ_G^1 , they must intersect. Do the analogue of the trade construction from the previous case by taking paths from boundary edges connected to σ and ν to this intersection. This case therefore follows by the same reasoning as the previous case.

We therefore have that there is a tinkertoy ν of τ_G^{i+1} such that $B_\mu^+ \subset B_\nu^-$. The same reasoning also gives us that $B_\nu^- \subset B_\mu^+$. \square

Notation III.75. If μ and ν are irreducible tinkertoys of τ_G^i and τ_G^{i+1} respectively with $B_\mu^+ = B_\nu^-$, then we say that ν *precedes* μ *in the β direction*. If $\Gamma_\mu^+ = \Gamma_\nu^-$ then we say that ν *precedes* μ *in the γ direction*.

Our goal now is to show that we can assign values to the irreducible tinkertoys in a way that is consistent with the clockwise ordering on the irreducible tinkertoys and can then be used to produce a chain of puzzles using Remark III.53. We can apply the same reasoning as Lemma III.47 to get that each of the irreducible tinkertoys of τ_G^{a-1} corresponds to a unique st -value of (3.24). Since we are trying to produce the inequality (3.24), a natural choice is to assign the irreducible tinkertoys of τ_G^{a-1} their st -value. Continue this by assigning every irreducible tinkertoy a value equal to the tinkertoy preceding it in the β direction. If an irreducible tinkertoy in τ_G^i does not have a tinkertoy preceding it in the β direction, then assign it a value equal to the

value of the tinkertoy preceding it in the γ direction minus $(r_{i+1} - r_{i+2})$. It is just left to check that this labeling produces a chain of puzzles of degree $\delta = (r_1, \dots, r_{a-1})$.

Lemma III.76. *If μ is an irreducible tinkertoy in τ_G^i , then it is assigned a value $(r_{i+1} - r_{i+2})$ smaller than the tinkertoy preceding it in the γ direction.*

Proof. We will show this using induction, starting with τ_G^{a-1} where it is vacuously true and working back to τ_G^1 . Consider corresponding pairs of infinite edges as being connected. If μ does not have an irreducible tinkertoy preceding it in the β direction then this holds trivially, so we may assume that this is not the case. Let ν and σ be the irreducible tinkertoys preceding μ in the β and γ directions respectively. There then must be path from μ to a northwestern edge of τ_G^{a-1} which only uses corresponding pairs of northwestern and southeastern infinite edges and no northeastern or southwestern edges between tinkertoys, and an analogous path using only corresponding pairs of northeastern and southwestern infinite edges. We know that μ and ν are assigned a value of s_p where β_p is this constant coordinate for the boundary edge the first path is connected to, and σ is assigned a value of $t_q - r_{i+2}$ where γ_q is constant coordinate of the boundary edge for the second path. By shifting these paths (see Figure 3.24) along with a path through μ in between them, we can increase/decrease the values of β_p, γ_q , and α_{i+1} . The resulting honeycombs are still configurations of the τ_G^i so there must be equality in (3.24). This shifts the left hand side of (3.24) by $r_{i+1} + s_p - t_q$, so this must be 0, which implies the desired result. \square

Lemma III.77. *Let μ be an irreducible tinkertoy in τ_G^1 .*

- *There is a unique irreducible tinkertoy ν containing the southwestern infinite edges corresponding to the proper southeastern edges of μ .*
- *This ν is assigned a value that is $r_1 - r_2$ larger than the value assigned to μ .*

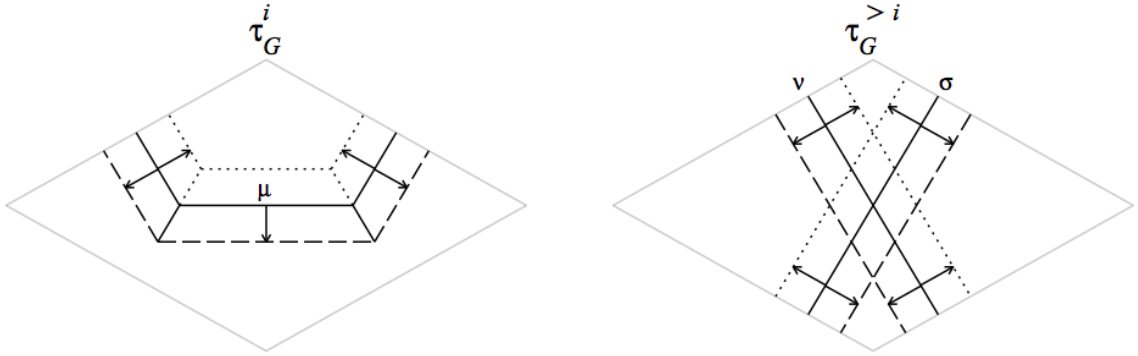


Figure 3.24: Trade construction used in the proof of Lemma III.76.

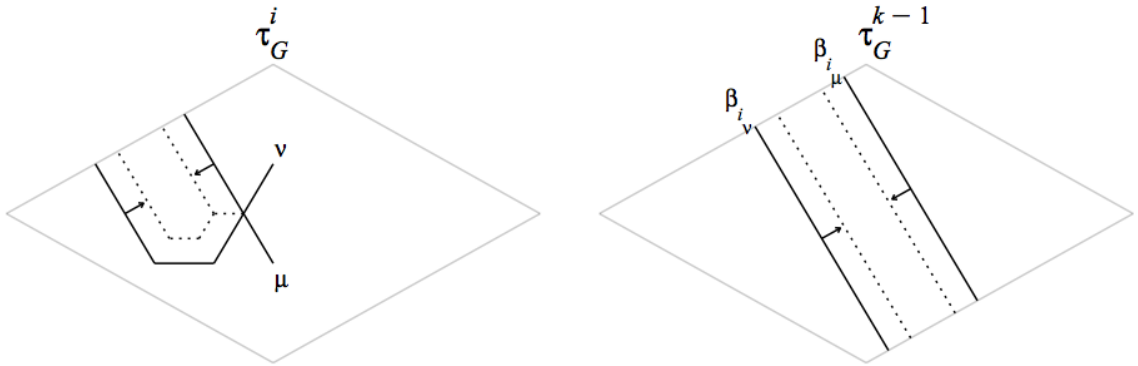


Figure 3.25: Trade construction used in the proof of Lemma III.78.

- If σ is an irreducible tinkertoy containing the southwestern infinite edge corresponding to an improper southeastern infinite edge of μ , then it is assigned a value more than $r_1 - r_2$ larger than μ .

Proof. These all follow by extending the constructions from the proofs of Lemmas III.54, III.55, III.64, and III.65 to a sequence of honeycombs in a similar manner to the proof of Lemma III.76. \square

Lemma III.78. *The chosen assignment is compatible with the clockwise ordering.*

Proof. Suppose $\mu <_c \nu$. Take any two paths from their intersection to northern boundary edges of τ_G^{a-1} where each path uses only corresponding pairs of northwestern and southeastern infinite edges or only corresponding pairs of northeastern and southwestern infinite edges between the tinkertoys. Perform the trade construction on their intersection and then propagate the shift through the paths to the boundary. If the paths lead to two northwestern infinite edges of τ_G^{a-1} , say ones whose constant coordinates are β_{i_μ} and β_{i_ν} , then this shift decreases β_{i_μ} and increases β_{i_ν} while leaving all of the α unchanged (see Figure 3.25). This will be a valid honeycomb for $\bar{\mathcal{F}}_{a,l}$, so we must have that $s_{i_\mu} \leq s_{i_\nu}$ since it satisfies (3.24). If there is equality then the resulting chain of honeycombs is still in F , but with one fewer degeneracies which contradicts the definition of the τ_F^i . We therefore have that $s_{i_\mu} < s_{i_\nu}$, which are the values assigned to these two tinkertoys, so the assignment is compatible in this case. The other possibilities for the boundary edges follow by the same reasoning. \square

Theorem III.79. *There is a puzzle chain whose inequality is (3.24).*

Proof. Use the chosen assignment to produce a sequence of puzzles using the method from Remark III.53. This sequence of puzzles satisfies the relations in Definition III.69 by Lemmas III.76 and III.77, hence it is a puzzle chain. This puzzle chain

has degree $\delta = (r_1, \dots, r_{a-1})$ by these two lemmas, so the inequality obtained from Proposition III.70 will be exactly (3.24). \square

Similar to the $a = 2$ case, there is more we can say more about the puzzle chains corresponding to facets. Again there are different possibilities depending on whether or not $\delta = 0$.

Proposition III.80. *If F is a facet with degree 0 (i.e. the r_i are all 0), then F is given by a $\{0, 1\}$ puzzle chain.*

Proof. Consider corresponding pairs of infinite edges as being connected. For each connected component, by summing the equalities we obtain from Proposition III.32 we obtain that the points in F must be in the hyperplane

$$\beta_{i_1} + \dots + \beta_{i_p} = \gamma_{j_1} + \dots + \gamma_{j_q}$$

where $\beta_{i_1}, \dots, \beta_{i_p}, \gamma_{j_1}, \dots, \gamma_{j_q}$ are the constant coordinates of the boundary edges of the connected component. Since $r_1 = \dots = r_{a-1} = 0$ in (3.24), every irreducible tinkertoy is assigned the st -value in (3.24) corresponding to the boundary edges of the connected component it is in. The equality for (3.24) is therefore a nonnegative linear combination of the equalities obtained from the connected components. The only equalities which all of $\bar{\mathcal{F}}_{a,l}$ satisfies are those generated by $\sum_i \alpha_i = \sum_i \beta_i = \sum_i \gamma_i$. The sum of the equalities for the connected components is the equality $\sum_i \beta_i = \sum_i \gamma_i$, but they are linearly independent from the other 2 equalities for $\bar{\mathcal{F}}_{a,l}$. Hence, in order for F to have codimension 1 it must have precisely 2 connected components, in which case the s_i and t_j can only take two values. We can use the equalities for $\bar{\mathcal{F}}_{a,l}$ to reduce (3.24) so that 1 is the only nonzero coefficient. The puzzle chain obtained is then $\{0, 1\}$. \square

Lemma III.81. *If F is a facet with $\delta \neq 0$ then the chain of tinkertoys τ_F^i has a single connected component.*

Proof. Consider corresponding pairs of infinite edges of the τ_F^i as being connected as in the previous proof. We again find that each connected component produces an equality which the points in F satisfy; namely

$$\beta_{i_1} + \cdots + \beta_{i_p} + \gamma_{j_1} + \cdots + \gamma_{j_q} = 0$$

where $\beta_{i_1}, \dots, \beta_{i_p}, \gamma_{j_1}, \dots, \gamma_{j_q}$ are the constant coordinates of the boundary edges of the connected component. Since $r_1 > 0$, the equality for (3.24) is linearly independent from the equalities for the connected components and the equalities $\sum_i \alpha_i = \sum_i \beta_i = \sum_i \gamma_i$ for all of $\bar{\mathcal{F}}_{a,l}$. Therefore, in order for F to be codimension 1 it must have 1 connected component. \square

In the $a = 2$ case, there being a single connected component in the graph of Remark III.67 for a puzzle is enough to guarantee that if the puzzle corresponds to a regular face, then that face is a facet. This does not hold for the general case; a condition stronger than connectivity is required. This condition can be stated in terms of a related labeled graph associated with the puzzle and only depends on the coefficients of the puzzle inequality.

Definition III.82. Let F be a face of $\bar{\mathcal{F}}_{a,l}$ given by an inequality of the form (3.24) with $r_1 > 0$. Construct a weighted graph $G(F)$ as follows. The vertices of $G(F)$ are the sets of s_i and t_j which are equal to the same value; i.e. each vertex is a maximal set of the form $\{s_{i_1}, s_{i_2}, \dots, t_{j_1}, t_{j_2}, \dots\}$ such that $s_{i_1} = s_{i_2} = \dots = t_{j_1} = t_{j_2} = \dots$. Let $\Delta_2, \dots, \Delta_a$ be indeterminates which we will use for the edge weights of the graph. For $i > 2$, there is an edge between a vertex v and a vertex w of weight Δ_i whenever

$k_v - k_w = r_1 - r_i$, where k_v and k_w are the values of the s/t coefficients associated with v and w respectively.

Remark III.83. For any face, knowing the labels on the infinite edges of any τ_G^i is enough to determine how many infinite edges each irreducible tinkertoy has, which in turn determines which $\tau_{p,q}$ the irreducible tinkertoys are isomorphic to. This is precisely the information encoded by $G(F)$. The labels for the southwestern edges in τ_F^i are $s_j + r_1 - r_{i+1}$, so for each edge labeled with a Δ_i between v and w there is an irreducible tinkertoy in τ_F^{i-1} isomorphic to $\tau_{\{\#s \text{ in } v\}, \{\#t \text{ in } w\}}$. Every other irreducible tinkertoy not represented by an edge in $G(F)$ consists of parallel lines. We can also determine the labels of the improper edges since if there is a mismatch in the number of edges for corresponding irreducible tinkertoys in τ_G^1 , then by Lemma III.56 the number of improper edges in the irreducible tinkertoy with more edges is the difference.

Let C be any cycle in $G(F)$. Assign the cycle a weight of $w(C) = \sum_{e \text{ edge in } C} \epsilon_e w(e)$ where $\epsilon_e = 1$ if the cycle traverses e from the vertex with the smaller s/t value to the vertex with the larger s/t value and -1 otherwise. If we set $\Delta_i = r_1 - r_{i+1}$, then any cycle in $G(F)$ produces an equality $w(C) = 0$ which these Δ_i must satisfy.

Let $C(F) \in \mathbb{R}^{a+l-2}$ be the cone of points $(\Delta_2, \dots, \Delta_a, \sigma_2, \dots, \sigma_l)$ defined by the constraints

$$\begin{aligned} \sigma_i &= 0 & \text{if } s_i &= s_1, \\ \sigma_i &= \sigma_j & \text{if } s_i &= s_j, \\ \sigma_i + \Delta_j &= \sigma_k & \text{if } s_i + r_1 - r_j &= s_k, \\ \sigma_i + \Delta_j &> \sigma_k & \text{if } s_i + r_1 - r_j &> s_k, \\ \sigma_i + \Delta_j &< \sigma_k & \text{if } s_i + r_1 - r_j &< s_k. \end{aligned}$$

Lemma III.84. *A face F is a facet if and only if $C(F)$ is 1-dimensional.*

Proof. A sequence s_1, \dots, s_k will produce a valid labeling for the τ_F^i with degree $\delta = (r_1 - r_2, r_2 - r_3, \dots, r_{a-1})$ if and only if the point $\sigma_i = s_1 - s_i$, $\Delta_i = r_1 - r_i$ is contained in $C(F)$. Each point in $C(F)$ along with a choice for s_1 and r_1 produces a hyperplane that the points in F must satisfy. Our face F is a facet if and only if all of these hyperplanes produce the same hyperplane when intersected with the hyperplanes $\sum_i \alpha_i = \sum_i \beta_i = \sum_i \gamma_i$. Two of these hyperplanes will produce the same intersection if and only if their $C(F)$ points are multiples of each other. We therefore have that F is a facet if and only if $C(F)$ is 1-dimensional. \square

Proposition III.85. *A face F is a facet if and only if $\text{Span}\{w(C) : C \text{ cycle in } G(F)\}$ has codimension 1.*

Proof. Let (Δ, σ) be a point in $C(F)$. By choosing values for r_1 and s_1 , we can construct a labeling for the puzzle chain for F by letting $r_i = r_1 - \Delta_i$ and $s_i = s_1 - \sigma_i$. We know that these Δ must satisfy $w(C) = 0$ for any cycle in $G(F)$, hence it must be in the kernel of the map $\Delta \mapsto (w(C))_C$. We therefore have that the dimension of $C(F)$ is less than or equal to the dimension of the kernel of this map.

Next, let $(\Delta_2, \dots, \Delta_k)$ be any point satisfying $w(C) = 0$ for every cycle C in $G(F)$. Assign values to the σ_i by setting $\sigma_i = \Delta_j + \sigma_k$ whenever there is an edge in $G(F)$ of weight Δ_j between the vertices containing s_i and s_k . By Lemma III.81, $G(F)$ is a connected graph so every σ_i will be assigned a value. Any path between the vertices for s_1 and s_i will produce the same value since $w(C) = 0$ for any cycle in $G(F)$. This produces a point (Δ, σ) satisfying the equalities for $C(F)$. We therefore have that the dimension of the kernel of $\Delta \mapsto (w(C))_C$ is less than or equal to the dimension of $C(F)$.

Putting these together, we find that the dimension of $C(F)$ is the dimension of the kernel of $\Delta \mapsto (w(C))_C$. By the previous lemma, this means that F is a facet if and only if this kernel has dimension 1. This kernel has dimension 1 if and only if $\text{Span}\{w(C) : C \text{ cycle in } G(F)\}$ has codimension 1. \square

3.6 Rigid puzzles

Now that we know that all of the faces of $\bar{\mathcal{F}}_{a,l}$ which are not contained in a chamber face are given by a chain of puzzles, the natural follow up question is for which chains of partitions is there a corresponding regular face of $\bar{\mathcal{F}}_{a,l}$. For the puzzles in [4] and [6], the obstructions to puzzles corresponding to a regular face are caused by gentle loops in the puzzles. We will see that with one extra condition, this is again the case. The arguments of [4] for rigid puzzles will all apply to our puzzles, the only additional information which is needed is how to handle gentle loops passing from one puzzle to another and the improper edges of P_1 .

Definition III.86. Two puzzle pieces in a puzzle of the same type (including the same labels) are said to be *in the same region* if they share an edge. Decompose the puzzles into *regions* which are the transitive closure of this. Each region is either a union of triangles with the same label i , which we will call an *i -region*, or a union of rhombi with the same labels i, j , which we will call an *(i, j) -region*.

The (i, j) -regions are parallelograms, while the generic i -region is a hexagon. Each i -region can only border (i, j) -regions, and not other regions of triangles. Consider the graph formed by taking the graph for a puzzle and deleting the edges in the interior of each of the regions. Orient the remaining edges (the boundary edges of the regions) as follows:

- If the edge is between a i -triangle and a (i, j) -rhombus, $i < j$, then orient the

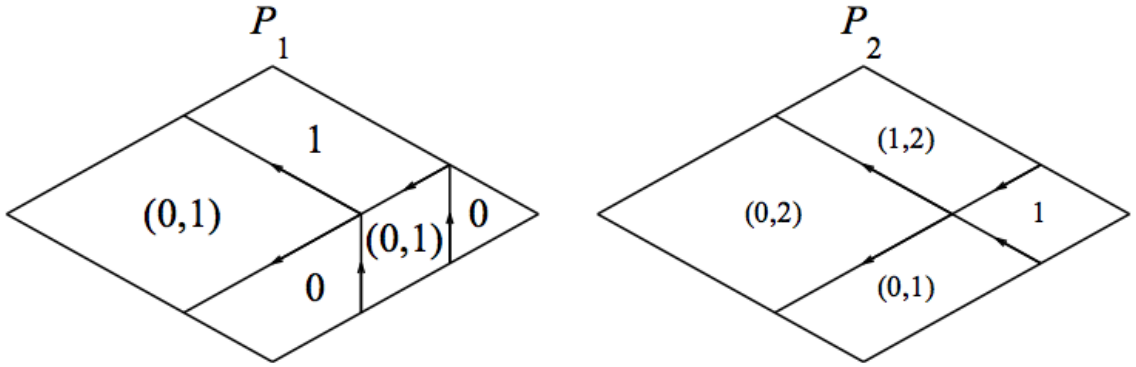


Figure 3.26: Puzzle regions for the puzzles in the chain from Figure 3.22.

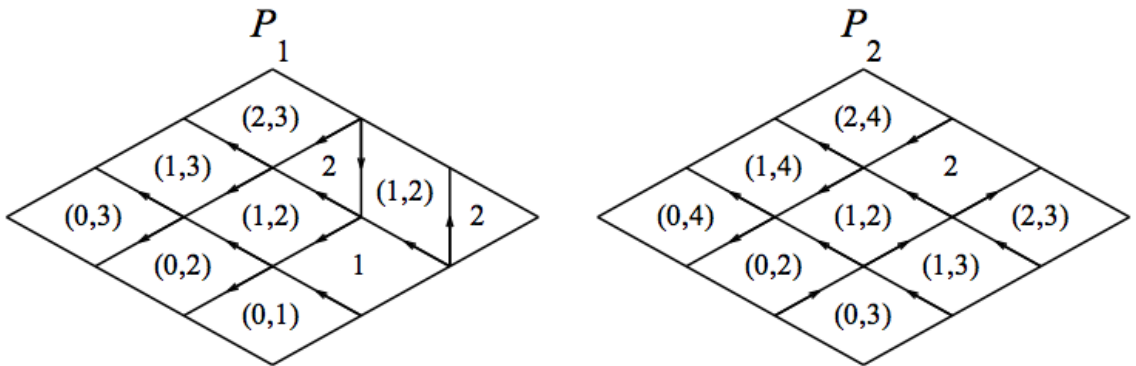


Figure 3.27: Puzzle regions for the puzzles in the chain from Figure 3.23.

edge so that the rhombus is to its right.

- If the edge is between a i -triangle and a (i, j) -rhombus, $i > j$, then orient the edge so that the triangle is to its right.
- If the edge is between a (i, j) -rhombus and a (i, k) -rhombus, $i > j > k$, then orient the edge so that the (i, k) rhombus is to its right.

Recall that a gentle loop (Definition III.24) is a path which either goes straight or turns 60° at any vertex, with the additional requirement that at the intersection of two straight lines it must go straight. We will be interested in the gentle loops on this oriented region graph. By looking at the various possibilities for the configuration of honeycombs which produce a vertex in this graph, it is possible to show that as a

gentle loop is traversed in this graph, the corresponding edges in the honeycomb are shrinking in length.

Lemma III.87 (Proposition 5 of [4]). *Let P be a puzzle occurring in a puzzle chain for a regular face of $\bar{\mathcal{F}}_{a,l}$, let γ be a gentle loop in the region graph for P , and let $\tilde{\gamma}$ be the corresponding sequence of edges for a generic honeycomb of P . Then the lengths of the edges $\tilde{\gamma}_i$ are weakly decreasing.*

We can extend this to a puzzle chain by introducing edges between the puzzles as follows (see Figure 3.28):

- Add an edge between corresponding vertices along the northwestern and southeastern sides of the region graphs for P_i and P_{i+1} . Orient these edges so that of the two boundary edges of the puzzles incident to the vertex, the larger label is to the right.
- Add an edge between corresponding vertices along the southwestern and southeastern sides of the region graph for P_1 , where the two boundary edges of P_1 adjacent to the vertices are both proper. Again orient these edges so that the boundary edge with the larger label is to the right.

Corollary III.88. *Let P_1, \dots, P_{a-1} be a puzzle chain corresponding to a regular face of $\bar{\mathcal{F}}_{a,l}$, let γ be a gentle loop in this modified region graph for the chain, and let $\tilde{\gamma}$ be the corresponding sequence of edges for a generic sequence of honeycombs for the chain. Then the lengths of the edges $\tilde{\gamma}_i$ are weakly decreasing.*

Proof. We prove this by showing how to concatenate two puzzles to form a new puzzle where we can apply the previous lemma. We will start by looking into how to concatenate honeycombs corresponding to the puzzles.

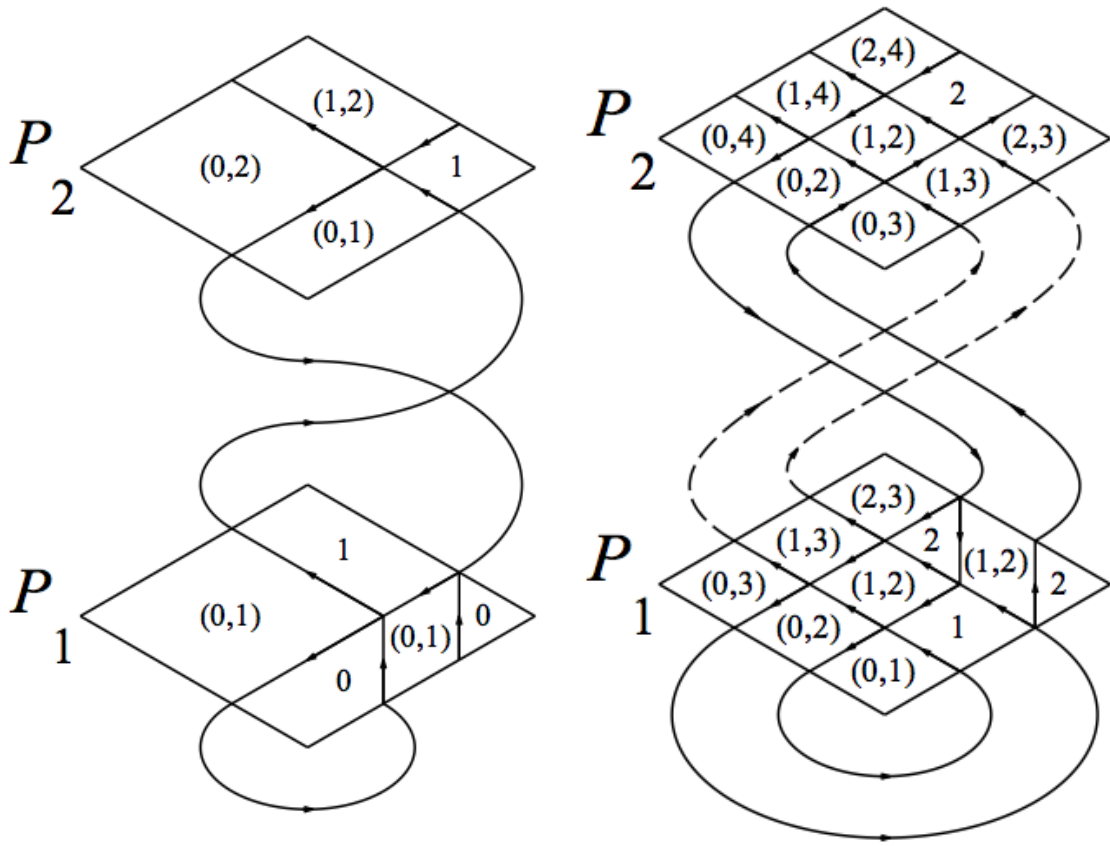


Figure 3.28: Region graphs for the two puzzle chains from Figures 3.26 and 3.27.

We know that for a sequence of honeycombs given by a puzzle chain, the constant coordinates of some of the infinite edges of consecutive honeycombs in the sequence have to be the same. This means that by shifting the honeycombs, we can concatenate them along either the corresponding northwestern and southeastern or the corresponding northeastern and southwestern infinite edges between the two honeycombs (see Figure 3.29). Introduce a straight edge traversing this concatenation; this edge allows us to keep track of how far apart the infinite edges of the puzzles are. In terms of puzzles, this concatenation using northwestern and southeastern edges corresponds to taking two puzzles P_i and P_{i+1} in the chain, lining them up along their northwestern and southeastern sides and introducing a line of rhombi joining them where the new edges of these rhombi are labeled using a value larger than any label occurring in the two puzzles (see Figure 3.30). Concatenating along the northeastern and southwestern edges corresponds to lining up P_i and P_{i+1} along their northeastern and southwestern sides, adding $(r_{i+2} - r_{i+1})$ to the labels of P_i so that the corresponding edges along the concatenated sides of both puzzles have the same labels, and then introducing a line of rhombi joining them where the new edges of these rhombi are labeled using a value smaller than any label occurring in either puzzle. Taking the region graph of these concatenations produces the first set of new edges.

For the second set of new edges, concatenate a honeycomb for P_1 with itself by rotating one copy of itself so that the corresponding southeastern and southwestern infinite edges line up, then concatenate only using the proper infinite edges. Translating this to the region graph on the puzzles results in the desired new edges. \square

Corollary III.89. *A puzzle chain corresponding to a face of $\bar{\mathcal{F}}_{a,l}$ has no gentle loops.*

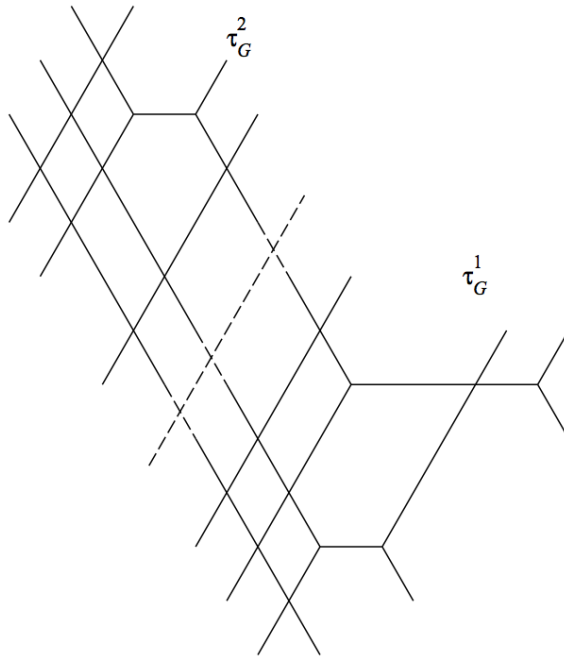


Figure 3.29: The concatenation of honeycombs along the northwestern and southeastern edges for the face given by the chain from Figure 3.23.

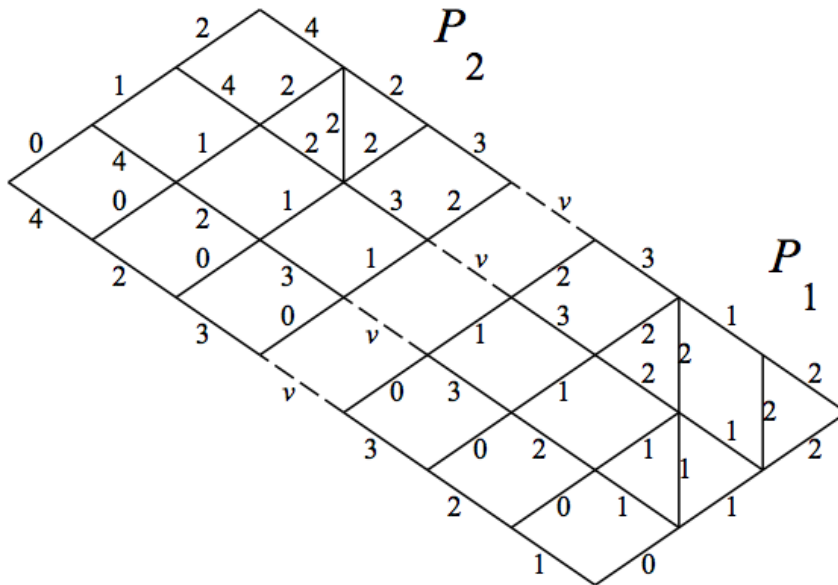


Figure 3.30: The concatenation of puzzles along the northwestern and southeastern sides for the puzzle chain from Figure 3.23. The constant v can be any value larger than 4.

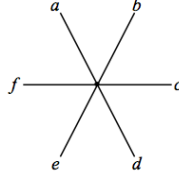


Figure 3.32: Generic vertex in the region graph of a puzzle chain.

as in Figure 3.32 satisfies the equalities

$$a + b = d + e, \quad b + c = e + f, \quad \text{and} \quad c + d = a + f.$$

Construct a honeycomb by taking the tinkertoy whose dual graph is the region graph restricted to one of the puzzles, and assigning each edge to have a length equal to the value assigned to the corresponding edge in the region graph. For each vertex in the region graph, we obtain a degeneration of a hexagon. The above relations on the number of gentle paths is precisely the condition necessary for the hexagon to close up (see (3.18)), so this will produce a valid honeycomb.

Note: We still need to specify the coordinates of one of the vertices in the honeycomb if we want to actually produce a honeycomb. Any choice of coordinates for any of the vertices will produce a honeycomb with the desired degeneracy tinkertoy. In Figure 3.33, the coordinates are chosen for the points p_1 and p_2 . The point p_1 is chosen so that the two infinite edges incident to it have the same constant coordinate, while p_2 is chosen so that the first entries of ρ_β^2 and ρ_γ^2 match between the two honeycombs.

This means that we can produce honeycombs for each puzzle in the tinkertoy, but in order for the constant coordinates to satisfy the conditions to be a honeycomb for a point in $\bar{\mathcal{F}}_{a,l}$ (see Prop III.20) the initial points for the honeycomb must be properly chosen.

Corollary III.89 tells us that these gentle loops are obstructions to the puzzle chain

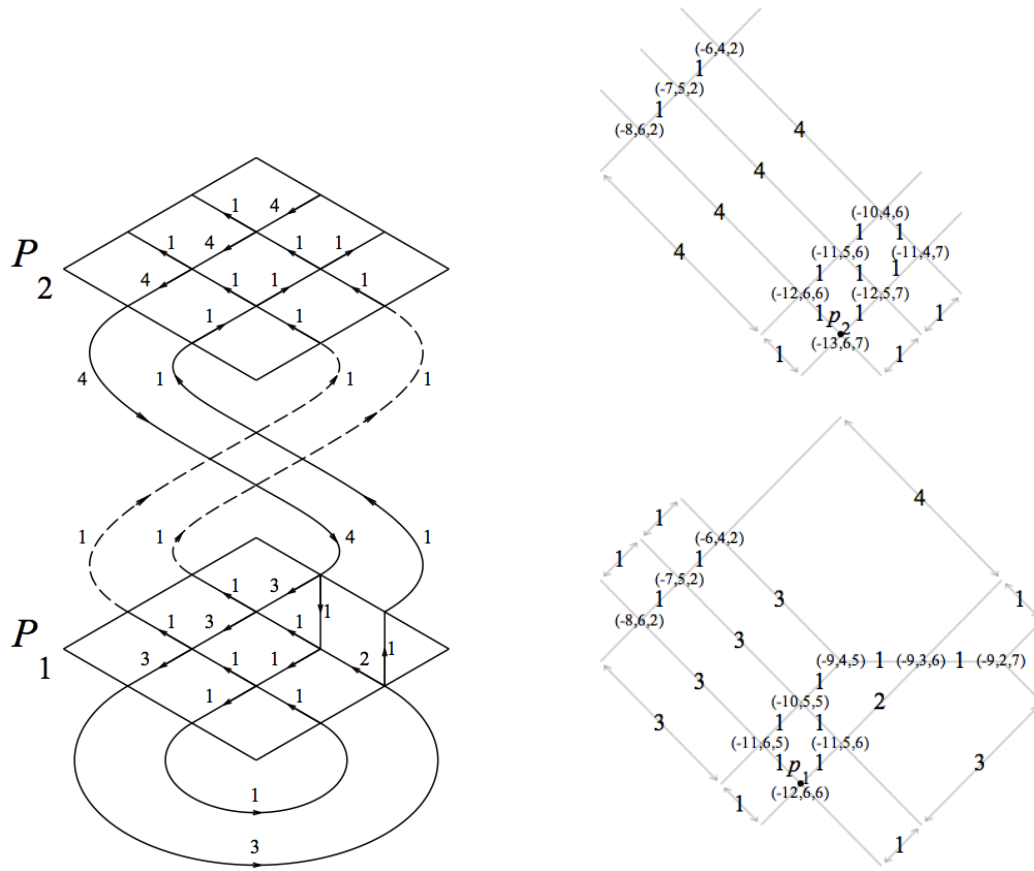


Figure 3.33: The region graph on the right of Figure 3.28 with its edges labeled by the number of gentle loops along with a honeycomb chain produced from it.

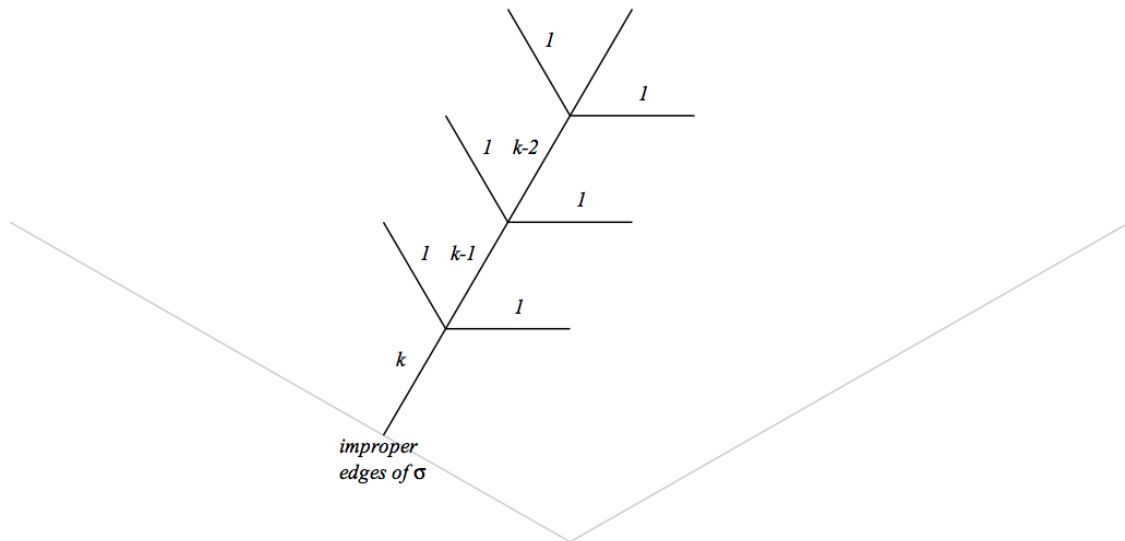


Figure 3.34: The improper portion of a tinkertoy in τ_G^1 . The edges are labeled by their multiplicities.

producing a regular face. There is one final obstruction to a chain of puzzles giving a regular face. It is possible that in the honeycombs for the τ_G^i from before, forcing the improper edges of τ_G^1 to have 0 constant coordinate will force one of the boundary edges corresponding to a β_i or γ_j to have a 0 constant coordinate. Suppose we have a chain of tinkertoys $\tau_G^1, \dots, \tau_G^{a-1}$ which are the moltings of the tinkertoys for a regular face. Let σ be an irreducible tinkertoy in τ_G^1 and let $\text{im}(\sigma, \beta)$ and $\text{im}(\sigma, \gamma)$ denote the number of improper southeastern and southwestern edges of σ respectively. After introducing into σ the degeneracies forced by the improper edges degenerating to the same edge, σ will be of the form of Figure 3.34. If the number of southwestern edges of σ is less than $\text{im}(\sigma, \beta)$, then there will be a northwestern infinite edge which is forced to have constant coordinate 0. Continuing this, suppose the tinkertoys in τ_G^i succeeding σ in the β and γ directions are isomorphic to $\tau_{p_i^\beta, q_i^\beta}$ and $\tau_{p_i^\gamma, q_i^\gamma}$ respectively. There will be a β_i or γ_j forced to be 0 precisely when one of the following inequality holds:

$$\begin{aligned} \text{im}(\sigma, \beta) &> p_1^\beta + p_2^\beta + \dots + p_{a-1}^\beta \\ \text{im}(\sigma, \gamma) &> q_1^\gamma + q_2^\gamma + \dots + q_{a-1}^\gamma. \end{aligned}$$

Consider what this corresponds to in the puzzle chain. For any label n of an improper edge in P_1 , let $\text{Im}(n, \beta)$ and $\text{Im}(n, \gamma)$ be the number of improper southeastern and southwestern edges respectively with this label. Let $P(n, i)$ be the number of southeastern edges with label n in P_i and let $Q(n, i)$ be the number of southwestern edges with label n in P_i . One of the β_i or γ_j will be forced to be 0 if and only if one

of the following inequality holds for some n :

$$\text{Im}(n, \beta) > P(n, 1) + P(n, 2) + \cdots + P(n, k - 1)$$

$$\begin{aligned} \text{Im}(n, \gamma) > Q(n, 1) + Q(n + (r_2 - r_3), 2) + Q(n + (r_2 - r_4), 3) + \cdots + \\ + Q(n + r_2, a - 1) \end{aligned}$$

Definition III.92. We say that a puzzle chain *has proper boundary edges* if neither of the above inequalities holds for any n .

Proposition III.93. *A puzzle chain produces the inequality for a regular face of $\bar{\mathcal{F}}_{a,l}$ if and only if it has no gentle loops and has proper boundary edges.*

Proof. We just need to show that we can produce a honeycomb for a point in $\bar{\mathcal{F}}_{a,l}$ which is a configuration of the tinkertoys obtained from the puzzle chain. Consider the honeycombs obtained from the region graph as in Remark III.91. First, suppose that our puzzle chain has no improper edges. Choose an initial point for the first honeycomb so that the constant coordinates of the infinite edges for $(\rho_\beta^1)_l$ and $(\rho_\gamma^1)_l$ are equal. When concatenating the puzzles, we introduced a new edge transverse to the infinite edges which were being concatenated. This edge is dual to the edges between the boundaries of the puzzles, and so its length in the resulting honeycomb is equal to the number of gentle paths through its dual edge. This means that the distance between two infinite edges in the honeycomb is given by the number of gentle paths through an edge between puzzles in the region graph. For the southeastern and southwestern infinite edges, the distance between corresponding adjacent pairs both are equal to the number of gentle paths through the same edge, hence the constant coordinates for the infinite edges for $(\rho_\beta^1)_i$ and $(\rho_\gamma^1)_i$ will be the same for all i since it holds for $i = l$.

For the later puzzles, choose the honeycomb so that it has the appropriate $(\rho_\beta^i)_1$

and $(\rho_\gamma^i)_1$ constant coordinates. By the same reasoning, the rest of the constant coordinates will match those of the previous honeycomb. This will produce a sequence of honeycombs satisfying the last two conditions of Proposition III.20. If it does not satisfy the first condition (i.e. $(\rho_\beta^1)_l = (\rho_\gamma^1)_l$ must be positive), then simply scale all of the edge lengths down or choose a starting point with larger constant coordinates until the constant coordinates on the first honeycomb are all positive. This will produce a sequence of honeycombs corresponding to a point in $\bar{\mathcal{F}}_{a,l}$. By molting this sequence of honeycombs to produce honeycombs with only multiplicity one edges and then performing trade constructions to de-elide edges as necessary, we can obtain a honeycomb for a point in $\bar{\mathcal{F}}_{a,l}$ which has precisely the degeneracies given by the puzzle chain. Hence, the puzzle chain does produce a regular face of $\bar{\mathcal{F}}_{a,l}$.

Next, suppose that our puzzle chain has improper edges. Produce a sequence of honeycombs as in the proper case (see Figure 3.35 for the honeycomb obtained from the left hand region graph of Figure 3.28). This sequence of honeycombs will satisfy the first and third conditions of Proposition III.20, but it will not satisfy the second condition for the improper edges in the first honeycomb. Shift the improper edges so that they have 0 constant coordinate and do a ‘one sided molt’ as in Figure 3.36 on any edges which contain both improper and proper edges (i.e. use the shifts of Figure 3.10). We can do these shifts while maintaining the underlying shape of the graph since all of the proper finite edges are to one side of the improper edges and these shifts are only on the opposite side. Molting away the multiplicity > 1 edges which are proper edges, we can again obtain a honeycomb with the desired degeneracies. \square

Definition III.94. A puzzle chain is called *rigid* if it is uniquely determined by the

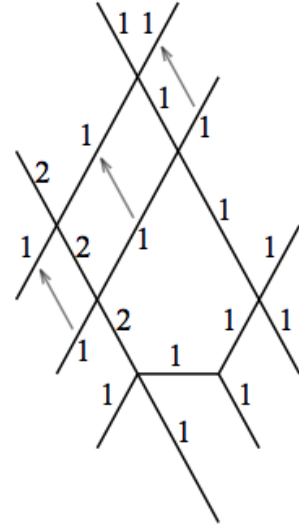
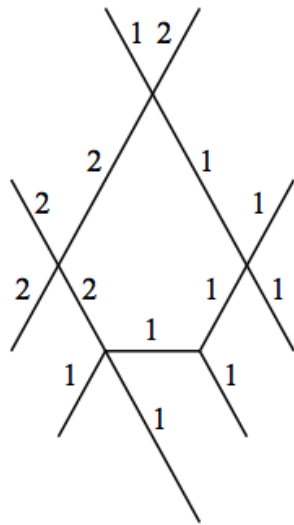
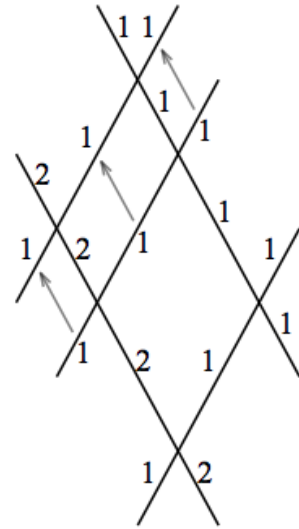
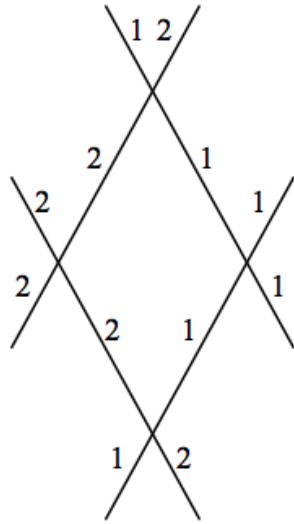


Figure 3.35: The honeycomb obtained from the left hand region graph in Figure 3.28.

Figure 3.36: The 'one sided molting' done on this chain of honeycombs.

values of the coefficients of its inequality.

Every regular face must be given by a rigid puzzle chain, since the puzzle chain is uniquely determined by which of the inequalities of $\mathcal{F}_{a,l}$ the points in the face have equality for. Our final use for the gentle loops will be to show that the converse of this also holds.

Theorem III.95. *A puzzle chain produces the inequality for a regular face of $\bar{\mathcal{F}}_{a,l}$ if and only if it is rigid and has proper boundary edges.*

Proof. In Lemma 6 of [6], the authors show how to swap puzzle pieces along a gentle loop to form another puzzle. These same swaps work for our puzzle chains, so we get that any puzzle with a gentle loops will not be rigid. If there are no gentle loops, then by Proposition III.93, the puzzle chain will produce an inequality for a regular face and so it will be rigid. We therefore have that a puzzle chain is rigid if and only if it has no gentle loops. The theorem then follows by Proposition III.93. \square

Remark III.96. A puzzle chain with a gentle loop will still produce an inequality for a face of $\bar{\mathcal{F}}_{a,l}$, but that face will not be regular.

Remark III.97. Theorem III.95 tells us that if we somehow already know the inequality for a regular face, then by starting with the boundary we are able to fill in a puzzle chain in order to produce the one for the face. In the next section, we will see how to use the tinkertoy chain corresponding to this puzzle chain to determine the dimension of the $F(\alpha\beta\gamma)$ polytope for points in the face. This therefore provides us with a way of finding the dimension of $F(\alpha\beta\gamma)$ for points in a regular face from just knowing the inequality for the face.

In addition, this also provides us with a means to test if an inequality corresponds to a regular face by checking if filling in the puzzles starting at the northern sides of

P_{a-1} produces multiple possible puzzle chains.

3.7 0-Dimensional $F(\alpha\beta\gamma)$

We are finally equipped to answer our original question: how do you produce triples where $F(\alpha\beta\gamma)$ 0-dimensional? Our procedure will largely follow the same blueprint as the $H(\alpha\beta\gamma)$ case. We start by noting that the argument for Proposition II.37 is directly applicable to $\bar{\mathcal{F}}_{a,l}$, so we have the analogous result.

Proposition III.98. *If F is a face of $\bar{\mathcal{F}}_{a,l}$, then $\dim(F(\alpha\beta\gamma))$ is constant on the interior of F .*

The first step is to find a checkable condition on a face that forces $F(\alpha\beta\gamma)$ to be 0-dimensional. We know that every face has an associated sequence of tinkertoys, τ_F^i , so we will search for such a condition in terms of the tinkertoys.

Consider the sequence of tinkertoys τ_F^i for a face F of $\bar{\mathcal{F}}_{a,l}$. Identify the ends of corresponding infinite edges of the tinkertoys whose constant coordinates are forced to be equal. Given a gentle loop in this chain of tinkertoys, we say that the loop traverses an edge between tinkertoys in the positive direction if it traverses the edge going from τ_F^{i-1} to τ_F^i , and say it traverses in the negative direction for the reverse. For a corresponding pair of southern infinite edges of τ_F^1 , the positive direction is from the southwestern infinite edge to the southeastern infinite edge.

Definition III.99. Given a gentle loop, the *1st winding number* is the number of times the loop traverses an edge between corresponding southern infinite edges of τ_F^1 in the positive direction minus the number of times the loop traverses such an edge in the negative direction. Similarly, for $i > 1$ the *i th winding number* of the loop is the number of times the loop traverses an edge between a northeastern infinite edge of τ_G^i and a southwestern infinite edge of τ_G^{i+1} in the positive direction

minus the number of times the loop traverses the edge in the negative direction. The *winding-tuple* is the $a - 1$ -tuple consisting of all the winding numbers.

Choose an orientation of the edges in a sequence of tinkertoys for a face. To any gentle loop, associate the formal sum $\sum_{e \text{ edge in the loop}} \epsilon_e e$ where $\epsilon_e = 1$ if the loop traverses the edge in the direction it is oriented, and is -1 otherwise.

Definition III.100. We say that a gentle loop is *proper* if it does not traverse any edges whose constant coordinate is uniquely determined by the β_i, γ_j , or the improper edges whose constant coordinate is one of the $(\rho_\beta^1)_i = (\rho_\gamma^1)_i = 0$. A *nontrivial sum of loops* is a formal sum of proper gentle loops such that replacing each loop by its formal sum of edges produces a nonzero sum. The winding-tuple of a proper sum of loops is the weighted sum of the winding-tuples for the individual loops.

Theorem III.101. *The polytope $F(\alpha\beta\gamma)$ is 0-dimensional for points in the interior of a face F if and only if the τ_F^i do not have a nontrivial sum of loops with zero winding-tuple.*

Proof. First, notice that if the sequence does contain a nontrivial sum of loops with zero winding-tuple, then breathing all of the loops produces a new honeycomb corresponding to the same point. We therefore have that if $\dim(F(\alpha\beta\gamma))$ is 0-dimensional then τ_F^i does not contain such a sum.

Next, suppose that there is no nontrivial sum of loops with zero winding-tuple. Assign weights to the edges of the τ_F^i as follows. First, assign the infinite edges whose constant coordinates are the β_i and γ_i a weight of 0, and the improper southern infinite edges of τ_F^1 a weight of 0. Progressively go through and assign weights to the edges using the following rules:

- If three edges with distinct constant coordinates meet at a vertex and two of

them are weighted, then assign the unweighted edge a weight equal to minus the sum of the other two edges.

- If an edge shares a vertex with a weighted edge which has the same constant coordinate, then assign it the weight of this weighted edge.

This labeling was done so that every edge which has been assigned a weight of 0 has its constant coordinate uniquely determined by β , γ , and the improper edges. Next, pick any unweighted infinite edge and assign it a value of p_1 . Continue to assign weights using the same rule until every edge that can be labeled has been. Again pick any unweighted infinite edge, assign it a weight of p_2 , and then assign all possible weights. Continue this until all of the infinite edges have been assigned a weight.

Suppose that there is an edge which has not been assigned a weight. If it shares a vertex with an edge with the same constant coordinate, then that edge must also be unweighted. Otherwise, it must share a vertex with edges which have the other two coordinate constants. These two edges cannot both be weighted. This means that we can find a gentle loop only using unweighted edges since we can always leave a vertex a path has entered. This gentle loop cannot use any infinite edges so it will have 0 winding-tuple, contradicting our assumption. We therefore have that every edge has been assigned a weight.

Any honeycomb for the τ_F^i tinkertoys is uniquely determined by the values of β, γ and the constant coordinates of the infinite edges which were the first to be assigned each of the p_i . To see this, notice that if we set each p_i to be equal to a number and make the constant coordinate of the infinite edge first assigned p_i to be this number, then the value of the constant coordinate of any edge will have to equal to its weight plus a linear function in the entries of β and γ . If we can show that each of the

p_i is uniquely determined by the α_i , then we will have that there can be only one honeycomb corresponding to the triple $\alpha\beta\gamma$ (i.e. $F(\alpha\beta\gamma)$ will be 0-dimensional for any point in F).

To do this, construct gentle loops in the tinkertoys as follows. Pick any edge and choose one of the p_i occurring with nonzero coefficient in its weight. Construct a path by orienting this edge and at the endpoint of the edge either going straight or turning 60° to an edge whose p_i coefficient has the opposite sign of the current edge. Repeat this at each vertex until the path repeats an edge, producing a loop C . Modify the edge weights by subtracting $\pm p_i$ for each of the edges in the loop, where the sign is determined by the sign of p_i on the edge. Continue to do this until every edge has weight 0. Consider the vector

$$(3.25) \quad \sum_C p_C \cdot \text{winding-tuple}(C)$$

where $p_C = p_i$ if p_i is the value used to produce C . This vector can be expressed as a matrix, say M , times the vector (p_1, p_2, \dots) . This matrix M must have trivial kernel by our assumption, and hence is invertible. Choose values for the p_i and for each i set the constant coordinate of the first edge assigned p_i to be equal to this value for p_i . The i th entry in the sum (3.25) is equal to the total contribution to α_i of the southwestern edges of τ_G^i which are in the loops. Hence α_i is equal to the i th entry of this sum plus a constant depending on the entries of β and γ . This produces a linear system that can be used to solve for the values of p_1, p_2, \dots in order for the sequence of honeycombs to be for the point (α, β, γ) . Since M is invertible, we have that the values of the p_i are uniquely determined by the point (α, β, γ) , so the entire chain of honeycombs is uniquely determined by (α, β, γ) . \square

Remark III.102. The proof of Theorem III.101 also tells us how to determine the

dimension of $F(\alpha\beta\gamma)$ for the points in the interior of a face. It is the number of ‘independent’ loops in the interior of each puzzle plus the dimension of the kernel of the matrix M obtained from (3.25). Both of these depend only on the structure of the irreducible tinkertoys and how they are connected, which is entirely encoded in the graph $G(F)$ (Definition III.82), which in turn can be constructed only using the inequality for the face. Hence, it is possible to determine the dimension of a regular face just from the inequality for the face (the formula is fairly messy due to the impact of the improper edges), though $G(F)$ does not directly translate over to non regular faces.

We can produce the sequence of tinkertoys for a regular face using the chain of puzzles for the face. Every face of $F(\alpha\beta\gamma)$ is the intersection of a regular face with some chamber faces. We can therefore produce the sequence tinkertoys for any face by taking the sequence for the regular face and then introducing the degeneracies forced by the chamber inequalities for which it has equality.

Remark III.103. This provides us with a recipe for producing faces of $\bar{\mathcal{F}}_{a,l}$ for which $F(\alpha\beta\gamma)$ is 0-dimensional:

- Find a puzzle chain for a regular face.
- Produce the sequence of tinkertoys using the puzzle chain.
- Introduce chamber equalities until there is no nontrivial sum of loops with 0 winding-tuple.

We therefore know how to produce all of the triples for which $F(\alpha\beta\gamma)$ is 0-dimensional.

We will finish this chapter by doing an example of this procedure.

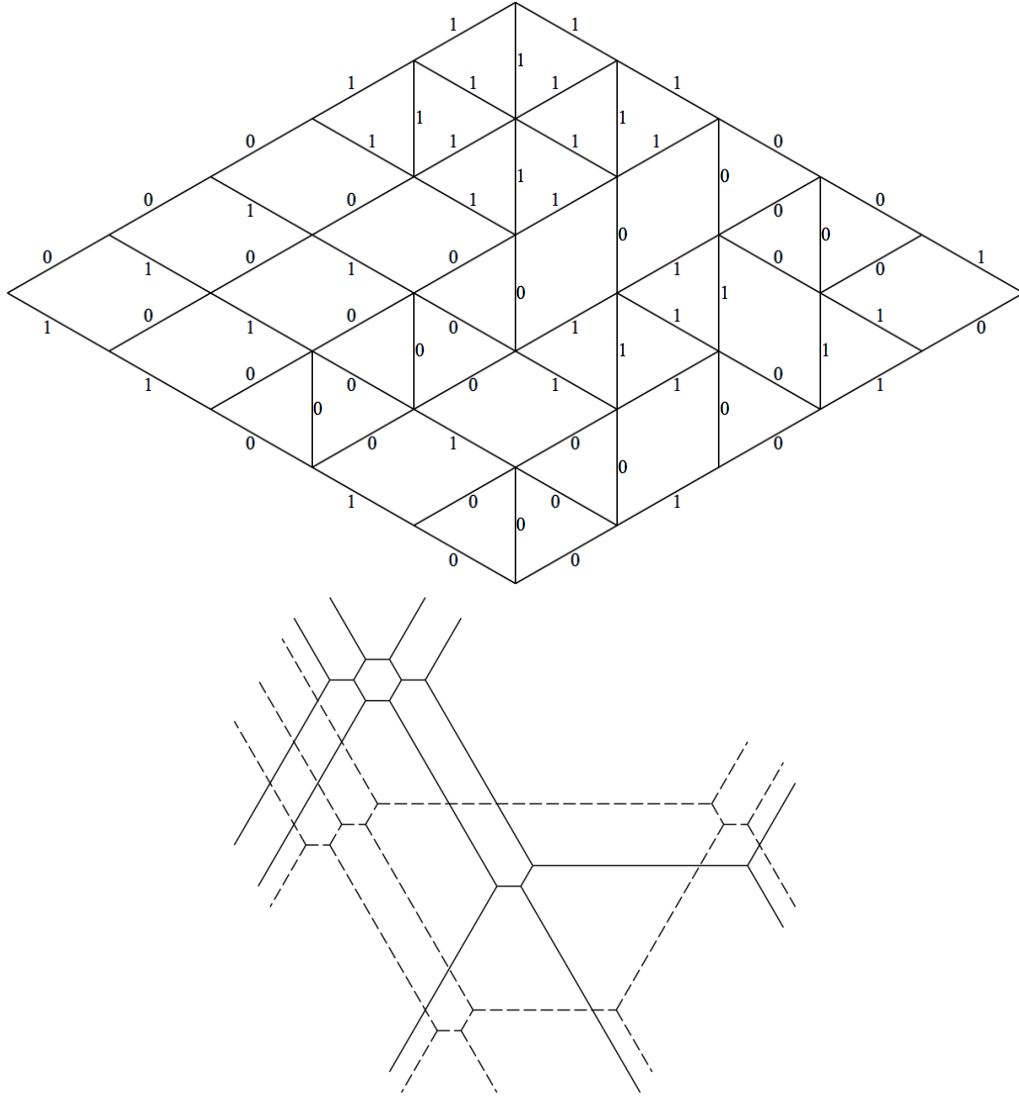


Figure 3.37: Puzzle and tinkertoy for a face of $\bar{\mathcal{F}}_{2,5}$.

Example III.104. Consider the puzzle of Figure 3.37 for the face of $\bar{\mathcal{F}}_{2,5}$ given by the inequality $\beta_1 + \beta_2 - \gamma_1 - \gamma_4 - \gamma_5 \leq 0$.

The tinkertoy for this face (the bottom diagram in Figure 3.37) has two irreducible tinkertoys, one isomorphic to $\tau_{3,2}$ (the dotted irreducible tinkertoy which corresponds to the 0 labels of the puzzle) and the other isomorphic to $\tau_{2,3}$ (the solid irreducible tinkertoy which corresponds to the 1 labels of the puzzle). The tinkertoy has one pair of improper infinite edge, namely the edges whose constant coordinates are

$(\rho_\beta^1)_5 = (\rho_\gamma^1)_5 = 0$. There are 8 ‘independent’ proper loops in the tinkertoy, 2 in each irreducible tinkertoy only using interior edges and 2 in each irreducible tinkertoy with winding number 1, hence the dimension of $F(\alpha\beta\gamma)$ for the points in the interior of this face is 7 (there are 4 interior loops and the kernel of the matrix M obtained from (3.25) has dimension 3 since all the winding numbers are the same).

Consider the subspace given by enforcing the chamber equalities $\beta_1 = \beta_2 = \beta_3$. It is easiest to see how this effects the tinkertoy by working with the dual graph. Start with the puzzle (the dual graph to the tinkertoy for the face) and merge the edges corresponding to β_1 and β_2 into one. Propagate this through the graph using the following rules:

- Once two edges have been merged, remove any remaining edges incident to the vertex where the two merged edges meet.
- Whenever a vertex has a 180° angle between consecutive incident edges, merge the two edges forming this angle.
- Remove any vertex and its incident edges if the vertex has 3 or more consecutive incident edges removed.

While doing this, two edges being merged means that the corresponding edges in the tinkertoy have merged to the same edge. Whenever this happens, the other edges of the hexagon corresponding to the point must become degenerate and length 0. Any removed edges will correspond to edges in the honeycomb which are degenerate.

The result of doing this for our example are the dual graph and tinkertoy of Figure 3.38.

There are now 2 proper interior gentle loops and 3 proper gentle loops with winding number 0 (there are no longer 4 since $(\rho_\beta^1)_1 = (\rho_\gamma^1)_1$ is now forced to be equal to

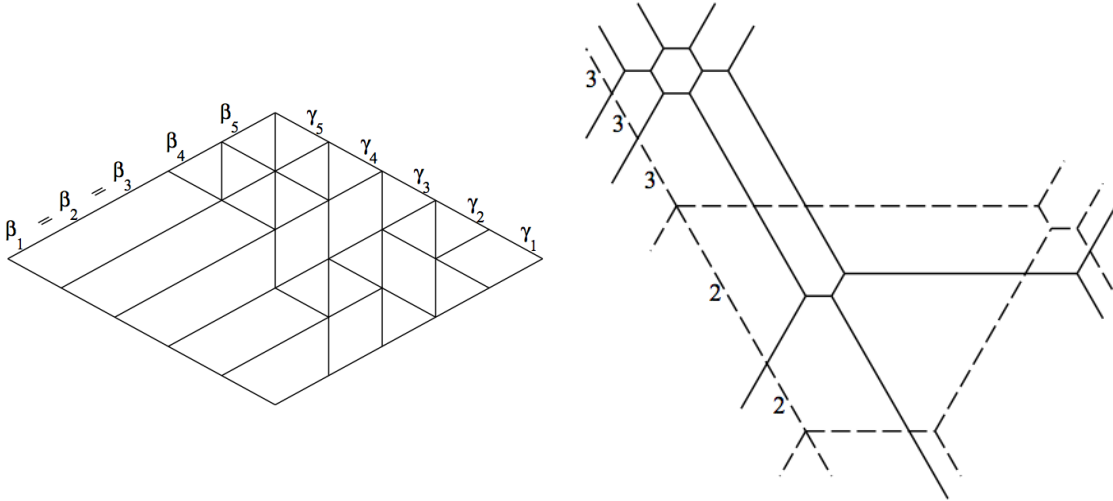


Figure 3.38: Dual graph and tinkertoy for the puzzle from Figure 3.37 after the chamber equalities $\beta_1 = \beta_2 = \beta_3$ have been enforced.

β_1). Hence the dimension of $F(\alpha\beta\gamma)$ for this subspace is 4. We can remove the top interior gentle loop by forcing $\gamma_5 = 0$. In terms of the dual graph to the tinkertoy, enforcing this chamber equality behaves differently than the other. Take the north-west most path between the boundary edge with constant coordinate $(\rho_\gamma^1)_5$ and the boundary edge with constant coordinate γ_5 . For each edge along this path which is not in the same direction as the two end edges, remove the corresponding edge in the dual graph (see Figure 3.39). This corresponds to forcing all of the edges to have 0 length, so the resulting honeycomb will have a straight line between the two edges (see Figure 3.40).

Finally, enforce $\gamma_2 = \gamma_3$ and $\gamma_4 = \gamma_5$ (see Figure 3.41). The bold loop in the figure is the only proper gentle loop, and it has winding number 1. Hence, the points produced as configurations of this tinkertoy will have $F(\alpha\beta\gamma)$ which are 0-dimensional. We therefore have that points in the face

$$\bar{\mathcal{F}}_{2,5} \cap \{\beta_1 + \beta_2 - \gamma_1 - \gamma_4 - \gamma_5 = 0\} \cap \{\beta_1 = \beta_2 = \beta_3\} \cap \{\gamma_2 = \gamma_3\} \cap \{\gamma_4 = \gamma_5 = 0\}$$

will all have 0-dimensional $F(\alpha\beta\gamma)$.

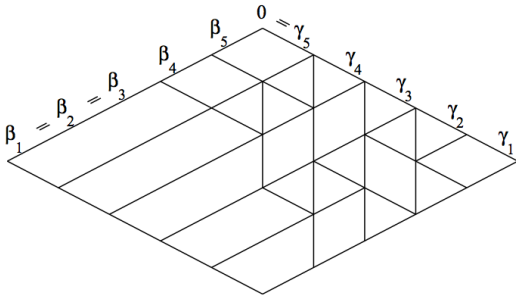


Figure 3.39: Dual graph for the puzzle from Figure 3.37 after the chamber equalities $\beta_1 = \beta_2 = \beta_3$ and $\gamma_5 = 0$ have been enforced.

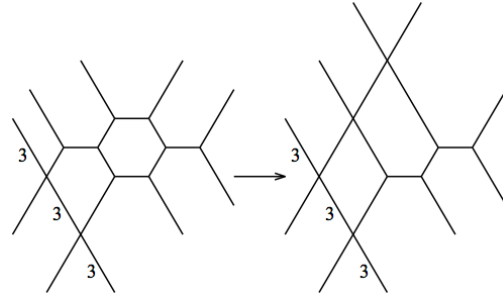


Figure 3.40: The effect forcing $\gamma_5 = 0$ has on the northwest portion of the tinkertoy.

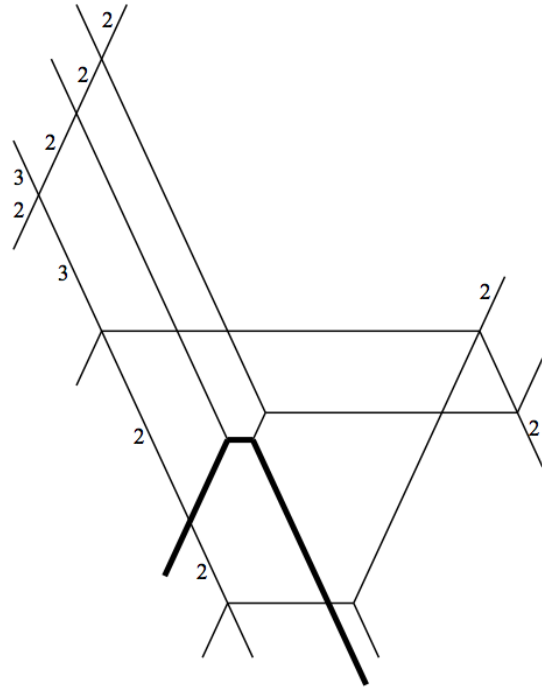
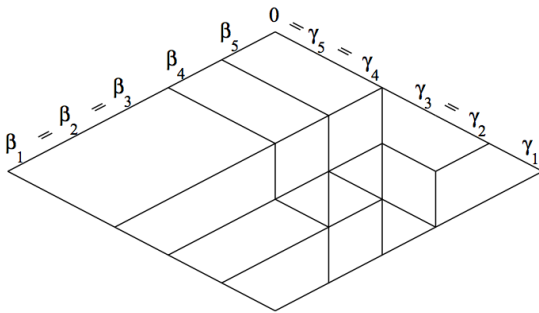


Figure 3.41: Dual graph and tinkertoy for the puzzle from Figure 3.37 after the chamber equalities $\beta_1 = \beta_2 = \beta_3$, $\gamma_2 = \gamma_3$, and $\gamma_4 = \gamma_5 = 0$ have been enforced.

CHAPTER IV

Weights of Plethysms

For the final chapter, we will be stepping away from Kronecker coefficients and will instead look into plethysms. Our primary goal will be to find an algorithm which can be used to find weights which are maximal in the dominance order. We will do this by studying the weight polytope for these representations. As we shall see, in general the weight polytope is not as well behaved as it is for other representations; there can be integer points in the polytope which do not correspond to weights of the representation. A side effect of this will be that the algorithm produced may not find all of the maximal weights, though in the computations done so far this issue has not arisen.

One particularly nice feature of the algorithm is that once it has been done for one plethysm, that knowledge can be easily be translated to other plethysms. In this way we will see that there is a generic shape for the weight polytope of plethysms, and many of the weights which are maximal in the dominance order are determined by this shape.

4.1 The weight polytope $W_{\lambda,\mu}$

Consider a plethysm of schur functions (Definition I.34), say $s_\lambda[s_\mu]$. We want to construct a polytope by taking the convex hull of the weights of $s_\lambda[s_\mu]$. To take this

convex hull, we first need to restrict which weights are allowed so that we can embed the weights into a finite dimensional space. To do this, fix a positive integer d and restrict $s_\lambda[s_\mu]$ to just the weights whose lengths are at most d (i.e. we regard these as weight vectors for representations of GL_d). Throughout this chapter, d will be a fixed positive integer and whenever we talk about weights we will mean only those of length at most d . Embed the weights in \mathbb{R}^d by appending 0's to the end of them as necessary.

Definition IV.1. Let $W_{\lambda,\mu}$ be the polytope formed by taking the convex hull of the weights of the GL_d representation whose character is the plethysm $s_\lambda[s_\mu]$. We will call this polytope the *(d -dimensional) weight polytope of $s_\lambda[s_\mu]$* .

Consider the faces of $W_{\lambda,\mu}$. Every face is an intersection of the form

$$W_{\lambda,\mu} \cap \{x : f(x) = \max f(W_{\lambda,\mu})\}$$

for some linear functional $f : \mathbb{R}^d \rightarrow \mathbb{R}$.

Notation IV.2. For any f , we use H_f to denote the hyperplane

$$H_f := \{x : f(x) = \max f(W_{\lambda,\mu})\},$$

and use F_f to denote the face $F_f := W_{\lambda,\mu} \cap H_f$. We will call F_f the face corresponding to f .

To study the weights which are maximal in the dominance order, we can reduce to looking at the intersection of $W_{\lambda,\mu}$ with the dominant chamber $(x_1 \geq x_2 \geq \cdots \geq x_d)$. For any face F_f intersecting the dominant part of the polytope, the linear functional $f(x) = a_1x_1 + \cdots + a_dx_d$ corresponding to the face can be chosen so that

$$(4.1) \quad a_1 \geq a_2 \geq \cdots \geq a_d.$$

Moreover, for faces which are entirely contained in the dominant chamber, f can be chosen so that

$$(4.2) \quad a_1 > a_2 > \cdots > a_d,$$

and it is these faces which we will be most interested in.

Definition IV.3. We say that a linear functional

$$f(x) = a_1x_1 + \cdots + a_dx_d$$

is *dominant* if its coefficients satisfy (4.2), and we say that the face F_f is a *dominant face*.

Lemma IV.4. *Every weight in a dominant face is maximal in the dominance order.*

Proof. Suppose that f satisfies (4.2). Any weight w which is larger in the dominance order than a point in F_f will have $f(w) > f(F_f) = \max f(W_{\lambda,\mu})$ by (4.2). This is not possible, so the weights inside the face must be maximal in the dominance order. \square

In order for a dominant weight to be maximal in the dominance order, adding any of the vectors

$$(1, -1, 0, \dots, 0), (0, 1, -1, 0, \dots, 0), \dots, (0, \dots, 0, 1, -1)$$

to the weight must result in a point which is not a weight for the plethysm. The dominant faces are precisely the faces for which all of these vectors point outside of the polytope. This means that in order for a weight to be maximal in the dominance order, it should be close to a dominant face to limit the possibility that there is a weight in one of these directions. It is therefore natural to look for maximal points ‘near’ to the dominant faces. If we know that moving in any of these directions from a weight produces a point on the other side of a dominant face, then that weight

will be maximal. This property can be easily expressed in terms of dominant linear functionals.

Definition IV.5. Let $f(x) = a_1x_1 + \cdots + a_dx_d$ be a dominant linear functional with $m = \max(f(x) : x \in W_{\lambda,\mu})$. Define the *index of f* to be

$$\text{Ind}(f) = \min(a_i - a_{i+1}).$$

We say that an integer point x is *near to F_f* if it satisfies $f(x) > m - \text{Ind}(f)$.

Lemma IV.6. *If x is a weight of $s_\lambda[s_\mu]$ that is near to a dominant face, then w is maximal.*

Proof. Any integer point larger than x in the dominance order will have f -value larger than $\max(f(W_{\lambda,\mu}))$. □

Remark IV.7. The notion of being near to a face depends on the linear functional which was chosen to represent the face. A direct corollary of Theorem IV.9 will be that if a point is near to a dominant face, then it is near to a dominant facet. Facets are given by a unique linear functional (up to scaling), so in practice this inconsistency will not be an issue.

Both Lemmas IV.4 and IV.6 are for maximal points whose maximality can be detected just by knowing the weight polytope, and not knowing which integer points in the polytope are actually weights. In general, the weight polytope will have holes (integer points which are not weights) which can potentially result in maximal weights which are not covered by these two cases. Empirically, however, $W_{\lambda,\mu}$ seems to have no holes when λ is not too long (regardless of choice of d), and even when there are holes it can still be the case that these are all the maximal weights (this is covered in more detail in Section 4.5).

Definition IV.8. The polytope $W_{\lambda,\mu}$ is *saturated* if all of the integer points in $W_{\lambda,\mu}$ are weights of $s_\lambda[s_\mu]$.

When $W_{\lambda,\mu}$ is saturated, finding the maximal weights is equivalent to finding the integer points in $W_{\lambda,\mu}$ which are maximal in the dominance order. The following result tells us that the maximal integer points for $W_{\lambda,\mu}$ are precisely the points in or near a dominant face, so when our polytope is saturated we now have conditions we can use to determine all of the maximal weights.

Theorem IV.9. *If an integer point is maximal in the dominance order among the integer points of $W_{\lambda,\mu}$, then it is near to a dominant facet.*

The proof of this theorem will rely on the use of weight schemes and as such will be postponed to the end of section 4.2.

4.2 Weight Schemes

We now have necessary (and sometimes sufficient) conditions for a weight of $s_\lambda[s_\mu]$ to be maximal. Our next goal is then to find a way to construct the weights of $s_\lambda[s_\mu]$ which are near to a dominant face. To do this, we will modify a typical combinatorial model for producing weights of a plethysm. This modified scheme will more naturally work with the dominant linear functionals.

Consider the tableau of tableaux model for the weights of $s_\lambda[s_\mu]$ (Remark I.35). Let $T_1 < \dots < T_k$ be a linear ordering of the semistandard Young tableau of shape μ using entries in $\{1, \dots, d\}$, and let $\omega_i \in \mathbb{Z}^d$ be the content of T_i . For any semistandard Young tableau S of shape λ whose entries are the T_i , let $n(S, i)$ denote the number of times T_i occurs as an entry in S .

Definition IV.10. For $\sigma \in \mathbb{R}^l$, the *permutohedron* in \mathbb{R}^l given by the point σ is the

convex hull of all of the points formed by permuting the entries of σ . We will use P_σ to denote this permutohedron.

Lemma IV.11. *The weight polytope $W_{\lambda,\mu}$ is the image of P_λ under the linear map*

$$n \rightarrow \sum_i n_i \omega_i$$

(here λ is considered as a point in \mathbb{R}^k , where k is the number of semistandard Young tableau of shape μ). Moreover, this map sends integer points in P_λ to integer points in $W_{\lambda,\mu}$.

Proof. Summing up the contribution of each tableau of shape μ to the total weight, we find that every weight of $W_{\lambda,\mu}$ occurs as a sum $\sum_i n(S, i) \omega_i$ for some tableau S of shape λ . A point $n \in \mathbb{Z}^k$ will occur as the content of a semistandard Young tableau S of shape λ if and only if it is contained in P_λ . This means that as long as an integer point $n \in \mathbb{R}^k$ is inside of P_λ , there will be a semistandard Young tableau of shape λ with content $n(S, i) = n_i$. We therefore have that any weight of $s_\lambda[s_\mu]$ can be found as a sum $\sum_i n_i \omega_i$ for some $n_i \in P_\lambda$. Hence the map sends the integer points of P_λ to the weights of $W_{\lambda,\mu}$. Extending this to the entire polytopes, we obtain the desired result. \square

When finding the image of a point in P_λ , we do not need to know which semistandard Young tableau of μ each coordinate corresponds to, we only need to know the content of the tableau. Each integer point in P_μ corresponds to an achievable content for a semistandard Young tableau of shape μ using entries $1, \dots, d$. At each integer point $\alpha \in P_\mu$ choose an ordering for the $K_{\mu,\alpha}$ semistandard Young tableau of μ of content α . We can represent a point $x \in P_\lambda$ by assigning to each integer point $\alpha \in P_\mu$ a $K_{\mu,\alpha}$ -tuple consisting of the coordinates of x corresponding to the tableaux of μ with content α .

Given such an assignment to the integer points of P_μ , the weight of $s_\lambda[s_\mu]$ of the corresponding point in P_λ is just the sum

$$(4.3) \quad \sum_{\alpha \in P_\mu} |n_\alpha| \alpha,$$

where n_α is the tuple assigned to α and $|n_\alpha|$ is the sum of the entries of n_α . The exact ordering of the tableaux at each integer point of P_μ has no effect on the sum in (4.3), so we will not need to keep track of these orderings. This leads us to the following construction for producing the weights of $s_\lambda[s_\mu]$.

Definition IV.12. Assign to each integer point $\alpha \in P_\mu$ a $K_{\mu,\alpha}$ -tuple of nonnegative integers, subject to the condition that the point $n = (n_\alpha)$ formed by taking the concatenation of the tuples and then ordering the entries is dominated by λ . We call such a construction a $(s_\lambda[s_\mu])$ *weight scheme* and will denote it by (n_α) .

A $(s_\lambda[s_\mu])$ *permutahedron scheme* is a weight scheme where we relax the assignments so that the $K_{\mu,\alpha}$ -tuples can consist of nonnegative real numbers.

The *content* of one of these schemes, (n_α) , is the sum (4.3).

Proposition IV.13. *The weights of $s_\lambda[s_\mu]$ are the contents of the $(s_\lambda[s_\mu])$ weight schemes, and $W_{\lambda,\mu}$ is the set of contents of $(s_\lambda[s_\mu])$ permutahedron schemes.*

Proof. This follows from Lemma IV.11 since the weight schemes correspond to points in P_λ , and the content of a weight scheme is precisely the image of the point under the linear map. □

Remark IV.14. These schemes allow us to reduce our study of $W_{\lambda,\mu}$ to working with permutohedrons, which are a well studied class of polytopes.

The rest of this section will be devoted to looking at weight schemes for various interesting types of points in $W_{\lambda,\mu}$, followed by the proof of Theorem IV.9. The

‘simplest’ weight schemes are those for the vertices of $W_{\lambda,\mu}$, so we will begin with them. The vertices of $W_{\lambda,\mu}$ are all weights by the definition of the polytope, so any vertex will be given by a weight (and not permutohedron) scheme. The vertices are the faces of the polytope which are given by a generic linear functional f ; i.e. the set of vertices is the set of intersections $W_{\lambda,\mu} \cap H_f$ for a sufficiently generic f . We will choose our f so that it takes a distinct value on each of the integer points of P_μ . This restriction enables us to use f to produce a linear order $\alpha_1 >_f \cdots >_f \alpha_k$ on the integer points of P_μ .

Lemma IV.15. *Let f be a generic linear functional with associated linear order $\alpha_1 >_f \cdots >_f \alpha_k$ on the points of P_μ . Set $r_i := \sum_{j=1}^{i-1} K_{\mu,\alpha_j}$ to be the number of semistandard Young tableaux of shape μ with content preceding α_i in the linear order given by f , and let (n_α) be the weight scheme produced by assigning the tuple $(\lambda_{r_i+1}, \dots, \lambda_{r_{i+1}})$ to α_i . Then the content of (n_α) is the vertex of $W_{\lambda,\mu}$ corresponding to f .*

Proof. The vertex of $W_{\lambda,\mu}$ corresponding to f is the point of $W_{\lambda,\mu}$ which has maximum f -value, and this (n_α) is the permutahedron scheme with maximum possible f value of its content. \square

Remark IV.16. Lemma IV.15 provides us with a simple recipe for producing vertices of $W_{\lambda,\mu}$. Obtain a linear ordering of the integer points of P_μ associated with a generic linear functional. Assign the entries of λ , starting with λ_1 , to the integer points of P_μ by first filling up the tuple for the maximum point of P_μ , then filling up the tuple of the second largest point and so on until all the tuples have been filled with entries of λ .

Note: Every vertex of $W_{\lambda,\mu}$ is a permutation of the coordinates of a vertex that

is maximal in the dominance order, so we only need to find the weight schemes for vertices given by dominant generic linear functionals in order to find all the vertices of $W_{\lambda,\mu}$.

These vertex weight schemes are simple enough that we can use them to produce a nice combinatorial rule for the multiplicity of the weight of the scheme. Let $M(a_1, a_2, \dots, a_l)$ be the multinomial coefficient which is equal to the number of distinct ordered l -tuples that can be formed using each of a_1, a_2, \dots, a_l once.

Proposition IV.17. *Let f be a generic linear functional, and let α_i and r_i be as in Lemma IV.15. The multiplicity of the vertex corresponding to f is*

$$(4.4) \quad \prod_{i=1}^k M(\lambda_{r_i+1}, \dots, \lambda_{r_i+1}).$$

Proof. The multiplicity of the weight is the number of tableaux of tableaux which produce that weight. By Lemma IV.11, we know that the tableaux of tableaux correspond to integer points in P_λ . There is only one weight scheme whose content is the vertex weight, so we just need to count the number of points in P_λ which produce this weight scheme.

Given an ordering S_1, S_2, \dots, S_k of the tableaux of shape μ , we obtain a point in P_λ from a weight scheme by associating each S_i with an entry in the tuple for the point $\text{content}(S_i)$ in P_μ and then setting the i th coordinate of the point of P_λ to be equal to this entry. Any choice of associating the S_i with an entry in the $\text{content}(S_i)$ tuple produces a point. Given such an assignment, we can produce another assignment corresponding to the same point in P_λ by taking two tableaux S_i and S_j with the same content which are also associated with tuple entries with the same value, and swapping which entry they correspond to. Any two assignments will produce the same point in P_λ if and only if they are linked by a chain of such swaps.

Hence, for the point α_i in P_μ which is assigned the tuple $(\lambda_{i+1}, \dots, \lambda_{i+1})$, there are $M(\lambda_{i+1}, \dots, \lambda_{i+1})$ ways to associate the entries of this tuple to tableaux of μ and obtain a different point in P_λ . The number of points of P_μ which can be produced using these assignments is then the product of these multinomial coefficients. \square

Next, we can use similar reasoning to find a description for the weight schemes producing points on the boundary of $W_{\lambda,\mu}$. Let f be any linear functional. We can no longer choose f so that it gives a linear order on the points of P_μ . Instead, we produce a partial order on the points of P_μ by saying two points are incomparable if they have the same f -value and ordering them by f -value otherwise. The maximal antichains of this partial order are the hyperplane sections $f^{-1}(m) \cap W_{\lambda,\mu}$, where m can be any integer, and therefore can be viewed as level sets of f . For any two of these level sets, all the points in one of the level sets will be larger than all of the points the other level set, so f gives a linear ordering on these level sets.

Lemma IV.18. *Let f be any linear functional, and let $I_1 >_f I_2 >_f \dots >_f I_l$ be the total ordering of the level sets. For $i = 1, \dots, l$, set $s_i := \sum_{j < i} \sum_{\alpha \in I_j} K_{\mu,\alpha}$ to be the number of tableaux whose content occur in level sets larger than I_i . A permutahedron scheme (n_α) will correspond to a point in F_f if and only if*

$$(4.5) \quad \sum_{\alpha \in I_i} |n_\alpha| = \sum_{j=s_i+1}^{s_{i+1}} \lambda_j.$$

Proof. A point will be in F_f if and only if it has the maximum f value amongst points in $W_{\lambda,\mu}$. The points in I_1 have the largest associated f -value, so for a point to be in F_f it must have the largest possible sum of the tuple for points in I_1 . Subject to maximizing I_1 , we next want to maximize the sum of the values in the tuples for $I_1 \cup I_2$, and so on. These sums are maximized precisely when (4.5) holds for $i = 1, 2, \dots$. \square

Definition IV.19. By equation (4.5), in order to check if the entire assignment is dominated by λ (so that it is a permutahedron scheme), it is sufficient to check if on each level set I_i the collection of order tuples is dominated by $(\lambda_{s_i+1}, \dots, \lambda_{s_i+1})$ since these are the primitive factors (see Definition II.39) of the pair $((n_\alpha), \lambda)$. For this reason, we will call $\lambda_{s_i+1}, \dots, \lambda_{s_i+1}$ the *entries of λ corresponding to I_i* .

Remark IV.20. For $i = 1, 2, \dots$, let F_i be the polytope which is the set of all possible contents for I_i :

$$F_i = \left\{ \sum_{\alpha \in I_i} |n_\alpha| \alpha : n = (n_\alpha) \text{ is dominated by } (\lambda_{s_i+1}, \dots, \lambda_{s_i+1}) \right\}.$$

A direct consequence of only having to check the dominance order using each level set and the entries of λ corresponding to it is that the face F_f is the Minkowski sum of these polytopes: i.e. $F_f = F_1 + F_2 + \dots$.

When working with these permutahedron schemes for boundary points of $W_{\lambda, \mu}$, there are two types of level sets.

Definition IV.21. A level set is *inactive* if the associated entries of λ are all equal ($\lambda_{s_i+1} = \dots = \lambda_{s_i+1}$). A level set is *active* if the associated entries of λ are not all equal.

The inactive level sets are so called since there is only one choice for how to fill their tuples in a permutahedron scheme: every entry of the tuples must be the value that the entries of λ are all equal to (hence F_i is a single point). The shape and dimension of a face F_f is therefore entirely determined by the active level sets. If one of the active level sets has codimension 1, then F_f will be a facet of $W_{\lambda, \mu}$.

Note: If I_i consists of a single point then F_i will be a single point, so active level sets have to have at least 2 points in order to contribute to the shape of F_f .

Next, we want to take a look at weight schemes for our final type of maximal weights: those near to a dominant face. From Lemma IV.6, we have that a weight is maximal if there is a dominant linear functional f such that the f value of the weight is within $\text{Ind}(f)$ of $\max(f(W_{\lambda,\mu}))$. Fixing f , we are interested in the weight schemes whose f values are within $\text{Ind}(f)$ of the weight schemes of the face F_f .

Definition IV.22. The *depth* of a weight scheme (n_α) with respect to f is how far below the maximum the f value of the content of the scheme is: i.e.

$$\text{Dep}_f((n_\alpha)) := \max(f(W_{\lambda,\mu})) - f(\text{content}(n_\alpha)).$$

Definition IV.23. Let $I_1 >_f I_2 >_f \dots$ be the ordering of the level sets. A *trade* on a weight scheme is when an entry in a tuple for a point in I_i is reduced by 1 while a tuple for a point in I_{i+1} is increased by 1. Performing a trade will increase the depth of the weight scheme by $f(I_i) - f(I_{i+1})$.

Lemma IV.24. *A weight scheme will be near to F_f if and only if it can be obtained from a weight scheme for F_f using a sequence of trades whose total depth is at most $\text{Ind}(f)$,*

Proof. A weight will be near to F_f if and only if its depth is less than $\text{Ind}(f)$, so a weight scheme will be near to F_f if it can be obtained from such a sequence of trades.

Next, suppose that a weight scheme corresponds to a point near to F_f . We will show that it can be produced in this manner using induction on the depth of the scheme. If the depth is 0, then the weight this corresponds to is in F_f then we are done. Assume that it the weight is not in F_f . There must be a level set I_i such that the total amount assigned to the tuples for points in this set is less than the sum of its associated λ terms. Choose i so that I_i is the first level set with this property.

We can perform the reverse of the trade construction to add 1 to the smallest entry occurring in a tuple for a point in I_i while removing 1 from any tuple of a point in I_{i+1} . The resulting assignment will still be dominated by λ and so it will still be a weight scheme. Doing this reduces the depth, so by induction the resulting weight scheme can be obtained using a sequence of trades, and hence so can the original weight scheme. \square

We will conclude our study of weight schemes by looking at the weight schemes for $\lambda = (1^n)$. We will do this since the weight schemes for these λ are in a sense the most basic weight schemes and, as we shall see, maximal weight schemes in this case can be used as a foundation to find maximal weight schemes for any general λ . If $\lambda = 1^n$, then in any weight scheme all of the entries in the tuples are either 0 or 1, so we can reconstruct the tuple just using the number of nonzero entries it has. This enables us to not have to keep track of the tuples at all; at each point of P_μ we only need to keep track of the number of 1's assigned to it.

Definition IV.25. A *reduced weight scheme* (for $e_n[s_\mu]$) is an assignment (m_α) , where each point $\alpha \in P_\mu$ is assigned a nonnegative integer m_α less than or equal to $K_{\mu,\alpha}$. Similarly, a *reduced permutahedron scheme* (for $e_n[s_\mu]$) is an assignment (m_α) where each point $\alpha \in P_\mu$ is assigned a nonnegative number less than or equal to $K_{\mu,\alpha}$. For reduced schemes, the content of the scheme is

$$\sum_{\alpha} m_{\alpha} \alpha.$$

Lemma IV.26. *The weights of $e_n[s_\mu]$ are the contents of the $e_n[s_\mu]$ weight schemes, and $W_{1^n,\mu}$ is the set of contents of $e_n[s_\mu]$ permutahedron schemes.*

Remark IV.27. Several of our results about weight schemes have simpler descriptions in the $\lambda = 1^n$ case.

- Reduced weight schemes for vertices consist of every point being assigned either $K_{\mu,\alpha}$ or 0, except possibly one point. Using the notation of Lemma IV.15, if $r_i < n < r_{i+1}$ then α_i will be assigned both $\lambda_n = 1$ and $\lambda_{n+1} = 0$, so it will be assigned a value less than $K_{\mu,\alpha}$, while every point after α_i will be assigned 0 and every point before α_i will be assigned the Kostka number.
- In equation (4.4), the multinomial coefficient for every point but α_i will be equal to 1 while the multinomial coefficient for α_i is the binomial coefficient $\binom{K_{\mu,\alpha_i}}{m_{\alpha_i}}$. We therefore have that the multiplicity of a vertex is $\binom{K_{\mu,\alpha_i}}{m_{\alpha_i}}$, where i is such that $r_i \leq n < r_{i+1}$.
- There is at most one active level set for weight schemes for points in a face of $W_{1^n,\mu}$, namely the one containing α_i may be active. This means that F_f is a translation of F_i , and so the dimension of F_f is the dimension of I_i .

Note: If we replace (1^n) by any rectangle (m^n) , then these results all still hold true except that now integers assigned in the reduced weight schemes have to be less than or equal to $m \cdot K_{\mu,\alpha}$.

We will conclude this section by using weight schemes to prove Theorem IV.9

Proof of Theorem IV.9. Consider the image of $W_{\lambda,\mu}$ under the linear map $\mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ defined by

$$x \mapsto (x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots, x_1 + \dots + x_{d-1}).$$

This maps the integer points of $W_{\lambda,\mu}$ injectively to \mathbb{Z}^{d-1} . An integer point of $W_{\lambda,\mu}$ is maximal in the dominance order if and only if its image is maximal under componentwise comparison (i.e. $x \leq y$ if $x_i \leq y_i$ for $i = 1, \dots, d-1$). If y is in the image of $W_{\lambda,\mu}$ under the map, then y is maximal under component wise comparison if and only if $y + e_1, \dots, y + e_{d-1}$ are not in the image of $W_{\lambda,\mu}$ where e_1, \dots, e_{d-1} is the

standard basis. In the image, a linear functional $f(y) = a_1y_1 + \cdots + a_{d-1}y_{d-1}$ is now dominant when the a_i are all positive. For now on in this proof we will work in the image of this map, so whenever we talk about $W_{\lambda,\mu}$ or an integer point we will mean the image of it under this map.

Let x be an integer point of $W_{\lambda,\mu}$ which is maximal in the dominance order. Then the points $x + e_1, \cdots, x + e_{d-1}$ must all be outside of $W_{\lambda,\mu}$. Consider the simplex with vertices $x, x + e_1, x + e_2, \cdots, x + e_{d-1}$. Let F_1, F_2, \cdots, F_l be the facets of $W_{\lambda,\mu}$ which intersect this simplex and let f_1, \cdots, f_l be their respective linear functionals. These facets must have common face which intersects Δ . For each of $x + e_1, \cdots, x + e_{d-1}$, there must be an F_i such that the hyperplane containing F_i separates the point from $W_{\lambda,\mu}$. The theorem is a corollary of the following lemma about the behavior of these facets:

Lemma IV.28. *If there are $i \neq j$ and $a \neq b$ such that $f_a(e_i - e_j) > 0$ and $f_b(e_i - e_j) < 0$ then x is in the boundary of $W_{\lambda,\mu}$.*

Before proving the lemma, let us show why it is sufficient. Suppose the lemma holds. Let F_a be the facet whose hyperplane separates the largest number of $x + e_1, \cdots, x + e_{d-1}$ from x . If the hyperplane separates all of these points from x then x is near to this facet. So suppose that there is an i such that $x + e_i$ is on the same side of the hyperplane as x . There must be a facet F_b which separates $x + e_i$ from x . F_b cannot separate more points from x than F_a , so there must be a j such that F_a separates $x + e_j$ from x while F_b does not. Going in the direction $(x + e_i) - (x + e_j) = e_i - e_j$ from F_a will cause you to leave $W_{\lambda,\mu}$, so $f_a(e_i - e_j) > 0$. Similarly, we have that $f_b(e_i - e_j) < 0$, but then by the lemma this is only possible if x is in the boundary of $W_{\lambda,\mu}$. We therefore have that every dominant point is either

in the boundary of $W_{\lambda,\mu}$ or is near to a facet, and only the dominant facets can have dominant weights near to them and not be in the facet. \square

All that is left to do now is to prove the lemma.

Proof of Lemma IV.28. Without loss of generality, $i = 1$, $j = 2$, $a = 1$, and $b = 2$. Let F be a face in the intersection $F_1 \cap F_2$ and consider the weight schemes for F . The level sets from F_1 and F_2 are unions of the level sets of F , and the λ entries assigned to the level sets for F must be compatible with both of the orderings given by the f_i on the level sets. This face F cannot be a vertex since it is inside the simplex which has no interior points, so F must have an active level set. Let I be an active level set for F .

Suppose there is a level set J such that $(e_1 - e_2) \in I - J$ (i.e. there is a point in J which differs from a point in I by $(e_1 - e_2)$). Then $f_1(J) > f_1(I)$, so J must be assigned entries of λ which are at least as large as the entries for I , but $f_2(J) < f_2(I)$ so it must be assigned entries of λ which are no larger than the entries for I . Since I is active, it is assigned at least 2 distinct values in λ , so this is not possible. Hence there can be no points in the direction $(e_1 - e_2)$ from I . The same reasoning also gives us that there can be no point in the $(e_2 - e_1)$ direction for I . We therefore have that I can have no points in P_μ that are in either of the $\pm(e_1 - e_2)$ directions, which is only possible if I is in one of the two faces of P_μ which is the intersection of facets given by $g_1(y) = y_1$ and $g_2(y) = y_2$ ($= x_1 + x_2$ before the transformation done at the beginning).

The same reasoning also gives us that if I is any inactive level set containing a nonzero entry, then any inactive level set in either of the $\pm(e_1 - e_2)$ directions must have its tuples assigned the same value as those of I . This means that the assignment

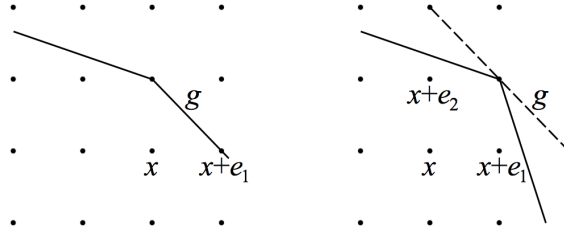


Figure 4.1: The possibilities for the image of $W_{\lambda, \mu}$ under the map.

of λ values is consistent with the order given by the linear functional $g(y) = y_1 + y_2$. We therefore have that $F_1 \cap F_2$ must be contained in the face of $W_{\lambda, \mu}$ given by g . This g does not depend on y_3, y_4, \dots , so we can project $W_{\lambda, \mu}$ onto its first two coordinates and $F_1 \cap F_2$ will map to a face in the image. Since the face given by g intersects the simplex, $x + e_1$ and $x + e_2$ are in the polytope if and only if they are in its image under this projection.

The image of $W_{\lambda, \mu}$ is 2-dimensional so the image of g is either 1-dimensional or a vertex. If it is one dimensional (the left diagram of Figure 4.1), then either x is in the face or one of $x + e_1$ or $x + e_2$ must be in the face. If the face is a vertex (the right diagram in Figure 4.1), then the cone at this vertex has to be larger than the cone made by the linear functionals $g_1(y) = y_1$ and $g_2(y) = y_2$ since these are faces of $W_{\lambda, \mu}$. In which case, either x is the vertex or both $x + e_1$ and $x + e_2$ must be in $W_{\lambda, \mu}$. In either case, we must have that x is in the face since $x + e_1$ and $x + e_2$ cannot be in the polytope. \square

4.3 Finding dominant vertices of $W_{\lambda, \mu}$

Our goal for this section will be to produce an algorithm that can be used to find the vertices of $W_{\lambda, \mu}$. In doing so we will also see how to find the inequalities for faces containing any given vertex, so we can also use the algorithm to find the defining inequalities for $W_{\lambda, \mu}$. We will do this by showing how to systematically build up

the weight schemes for the vertices. We will first do this for the case when λ is a rectangle (m^n) to take advantage of the reduced weight schemes, and we will then proceed to show how to modify the algorithm to find the vertices for any λ .

Suppose $\lambda = (m^n)$ is a rectangle. Fix a dominant vertex v of $W_{m^n, \mu}$, and let f be a linear functional producing v . Consider the weight scheme producing v given by this f . In order to specify the weight scheme we only need to keep track of which integer points of P_μ have been assigned nonzero values along with which point λ_n is assigned to.

Notation IV.29. Let $X(v)$ be the set of integer points of P_μ which are assigned nonzero integers in the weight scheme for v .

The integer point with λ_n will necessarily be the smallest point in $X(v)$ under the ordering given by f . This set $X(v)$ is separated from the rest of the points in P_μ by a hyperplane, namely a hyperplane of the form $f(x) = c$. Since v is a dominant vertex, $X(v)$ must also be an order filter for the dominance order on P_μ (i.e. X is closed under increasing in the dominance order). In fact, these conditions are sufficient for a set X to arise from a vertex.

Lemma IV.30. *If X is an order filter of the integer points of P_μ (using the dominance order) such that X and its complement can be separated by a hyperplane, then there exists an n such that $X = X(v)$ for a vertex v of $W_{m^n, \mu}$.*

Proof. Suppose X is such a order filter. Let f be any generic linear functional such that $\{f(x) = c\}$ is a hyperplane separating X from X^c for some c . Let $\tilde{\alpha}$ be the minimum of X under the linear ordering $<_f$. Then as long as n satisfies

$$\sum_{\alpha \in X - \{\tilde{\alpha}\}} K_{\mu, \alpha} < n \leq \sum_{\alpha \in X} K_{\mu, \alpha}$$

we will have $X = X(v)$ where v is the vertex of $W_{m^n, \mu}$ corresponding to f . □

Given an order filter, we can reduce the question of whether there is a hyperplane separating it from its complement to a question of whether or not a polyhedral cone is full dimensional.

Notation IV.31. To any order filter X of the points in P_μ , we associate a cone $C(X) \in \mathbb{R}^{d+1}$ given by the inequalities:

$$\begin{aligned} x_1 a_1 + \cdots + x_d a_d - c &\geq 0, x \in V, \\ x_1 a_1 + \cdots + x_d a_d - c &\leq 0, x \in V^c. \end{aligned}$$

Note: For these inequalities a_1, \dots, a_d and c are the indeterminates.

Lemma IV.32. *Given an order filter X , there is a hyperplane separating X and its complement if and only if $C(X)$ is full dimensional.*

Proof. Any hyperplane $a_1 x_1 + \cdots + a_d x_d = c$ will separate X and its complement with X on its ‘positive’ side if and only if $a_1 x_1 + \cdots + a_d x_d > c$ for all points $(x_1, \dots, x_d) \in X$ and $a_1 x_1 + \cdots + a_d x_d < c$ for all points $(x_1, \dots, x_d) \in X^c$. Any (a_1, \dots, a_d, c) will satisfy these inequalities if and only if it is in the interior of $C(X)$. \square

Remark IV.33. We can reduce the number of inequalities by using the fact that X is an order filter and v is dominant. Since v is dominant, we can choose the linear functional producing v to be dominant, i.e. $a_i > a_{i+1}$. We can obtain any of the inequalities for $x \in V$ by adding positive multiples of $a_i \geq a_{i+1}$ to the inequality for a point in $\min(V) = \{\text{minimal points of } V \text{ under the dominance order}\}$. Similarly, we can obtain any of the inequalities for $x \in V^c$ from the inequalities for points in

$\max(V^c)$. We can therefore reduce our set of inequalities to

$$x_1 a_1 + \cdots + x_d a_d - c \geq 0, x \in \min(X)$$

$$x_1 a_1 + \cdots + x_1 a_d - c \leq 0, x \in \max(X^c)$$

$$a_i - a_{i+1} \geq 0, i = 1..d$$

Remark IV.34. Observe that these order filters can be built up one vertex at a time. Suppose $X = X(v)$ is the order filter for a vertex given by the linear functional f . Let $x_1 >_f x_2 >_f \cdots >_f x_n$ be the ordering of the points of X , and for $i = 1 \cdots n$, let $X_i = \{x_1, \cdots, x_i\}$. Then X_i is the order filter for the vertex given by f for λ of length $\sum_{j=1}^{i-1} K_{\mu, x_j} < l(\lambda) \leq \sum_{j=1}^i K_{\mu, x_j}$. This means that we have a sequence of order filters $X_1 \subset X_2 \subset \cdots \subset X_n = X$ such that each of the order filters is the filter for a vertex of $W_{\lambda, \mu}$ for some λ . We therefore can build up these order filters recursively and be certain that we have reached all of them.

Putting these observations together provides us with a natural algorithm for finding these vertex order filters for P_μ . We will do this by building up all the order filters from the unique filter containing a single point, keeping track of the sets $\min(X)$ and $\max(X^c)$ as we do this. The algorithm is then:

- Start out with $X = \{\mu\}$, the order filter consisting of the maximal point of P_μ .
- Given a vertex filter X , produce all the vertex filters of size one larger which contain X :
 - Look at the filter $X' = X \cup \{x\}$ for each point $x \in \max(X^c)$.
 - Produce $\min(X')$ by adding x to $\min(X)$ and then removing any points which are larger than x .

- Produce $\max(X^\epsilon)$ by removing x from $\max(X^\epsilon)$ and then adding the points of P_μ covered by x which are not below any of the points in $\max(X^\epsilon)$.
- Use $\min(X')$ and $\max(X^\epsilon)$ to check if $C(X')$ is full dimensional. If it is then X' is an order filter containing one more point than X , otherwise X' is not an order filter and can be discarded.

Remark IV.35. By being a little bit more careful, we can reduce the number of points in $\max(X^\epsilon)$ for which we need to check $C(X')$. Suppose x_1, \dots, x_k is an arithmetic sequence in $\max(X^\epsilon)$ with difference Δx . If $x_1 - \Delta x \in X$ then we must have $f(\Delta x) < 0$ for any linear functional separating X and X^ϵ . This in turn means that $f(x_1) > f(x_2) > \dots > f(x_k)$, so x_1 will be added to the filter before any of x_2, \dots, x_k . Similarly, if $x_1 - \Delta x$ is in $X^\epsilon - \max(X^\epsilon)$ then we must have $f(\Delta x) > 0$, so x_k will be added to the filter before any of x_1, \dots, x_{k-1} . Even if we have neither of these conditions on $x_1 - \Delta x$, we still know that $f(\Delta x)$ has to be nonzero, and depending on its sign that means that either x_1 or x_k has to be added next to filter and not x_2, \dots, x_{k-1} . Therefore, if we keep track of the lines in $\max(X^\epsilon)$, we can eliminate the need to check $C(X')$ for any point in the interior of a line or at an endpoint which is guaranteed to have smaller f value.

Note: In order to use these order filters to find the vertices for λ a rectangle, we also need to keep track of what the smallest point in the filter was with respect to the linear functional. This smallest point is simply the last point that was added to the filter, so the algorithm can be used to exactly produce the vertices of $W_{m^n, \mu}$.

Next, we'll take a closer look at these cones $C(X)$ we used to determine if a filter corresponds to a vertex. These cones contain more information about the vertex with that filter; they can be used to determine the inequalities for the faces of $W_{m^n, \mu}$ which contain the vertex. Suppose v is a vertex of $W_{m^n, \mu}$. There are two possibilities

depending on the level sets of the reduced weight scheme for v . From Remark IV.27, we know that there is at most one point of $X(v)$ which is not assigned the maximum possible value in the reduced weight scheme for v . Let $\alpha(v)$ denote this point provided it exists. Recall that each ray $\mathbb{R} \cdot (a_1, \dots, a_d, c) \in C(X)$ corresponds to a hyperplane $a_1x_1 + \dots + a_dx + d = x$. Let H_1, H_2, \dots be the hyperplanes corresponding to the extremal rays of the cone $C(X(v))$.

Lemma IV.36. *Suppose v is a dominant vertex of $W_{m^n, \mu}$. If $\alpha(v)$ exists, then the facets of $W_{m^n, \mu}$ containing v are precisely the intersections $H_i \cap W_{m^n, \mu}$ for H_i containing $\alpha(v)$. If $\alpha(v)$ does not exist, then the facets of $W_{m^n, \mu}$ containing v are precisely the intersections $H_i \cap W_{m^n, \mu}$ for H_i containing at least one point from each of $\min(X)$ and $\max(X^c)$.*

Proof. First, consider the case when $\alpha(v)$ exists. Suppose that H_i contains $\alpha(v)$. Let f be a linear functional producing H_i . Then $H_i \cap P_\mu$ will be a level set for f . Since this level set contains $\alpha(v)$, it is the unique active level set from Remark IV.27. This level set has codimension 1, so the face corresponding to f (i.e. $H_i \cap W_{m^n, \mu}$) is a facet.

Now let f be a linear functional corresponding to a facet of $W_{m^n, \mu}$ containing v . From Remark IV.27, there is a single active level set for the reduced weight scheme producing v , and it must contain $\alpha(v)$ and have codimension 1. This means that the active level set is the intersection of $W_{m^n, \mu}$ with a hyperplane of the form $f(x) = c$. Let H be this hyperplane. This hyperplane must weakly separate X and X^c (i.e. it separates $X \setminus H$ and $X^c \setminus H$) so it will correspond to a ray in $C(X)$. The extremal rays of $C(X)$ are precisely the hyperplanes which have codimension 1, so the ray corresponding to H will be an extremal ray.

We therefore have that if $\alpha(v)$ exists, then $H \cap W_{\lambda, \mu}$ will be a facet of $W_{m^n, \mu}$

containing v if and only $H = H_i$ for some i .

Finally, when there is no $\alpha(v)$, the only difference is that now, the active independent will be the set containing points in both X and X^c . Applying the same reasoning provides the desired result. \square

We will finish off this section by showing how to generalize this algorithm to any λ , not just rectangles. We can still work with these order filters for any λ . Now the order filters are for all the points in P_μ that are assigned a nonzero entry in their tuples. We still get the same conditions (Lemma IV.32) for an order filter to occur as $X(v)$ for some vertex v of $W_{\lambda,\mu}$. Hence, the set of vertex order filters for any λ of length n is the same as the set of vertex order filters for the rectangle case. The difference is that we need to know more information to reconstruct the vertex this is the order filter for. We need the additional information of the full ordering on the points of X in order to reproduce the vertex from X . It is possible modify our algorithm so that it keeps track of the full ordering while it is building up the order filter. To do this, we now need to keep track of the order in which the points of X were added to X . Now, when we are checking for which points in $\max(X^c)$ we can add to X , we also need to make sure that the point is compatible with the ordering we have on X . This means that before adding $x' \in \max(X^c)$ we need to check that there is a linear functional producing our ordering on the points of $X \cup \{x'\}$. As before, checking for this can be reduced to checking if a cone has a nonempty interior.

Notation IV.37. Suppose $x_1 \succ x_2 \succ \cdots \succ x_l$ is a linear extension of the dominance order on the points of an order filter X . Define a cone $C(X, \succ)$ by letting it be the

cone defined by the inequalities:

$$(x_{i,1} - x_{i+1,1})a_1 + \cdots + (x_{i,d} - x_{i+1,d})a_d \geq 0, \quad i = 1 \cdots, l-1$$

$$y_1 a_1 + \cdots + y_d a_d - c \geq 0, \quad y \in \min(X)$$

$$y_1 a_1 + \cdots + y_d a_d - c \leq 0, \quad y \in \max(X^c)$$

$$a_i - a_{i+1} \geq 0, \quad i = 1..d$$

Lemma IV.38. *Let $x_1 \succ x_2 \succ \cdots \succ x_l$ be a linear extension of the dominance order on the points of an order filter X . There is a linear functional f corresponding to a vertex v of $W_{\lambda,\mu}$ with $X(v) = X$ and $>_f$ coinciding with \succ on X if and only if $C(X, \succ)$ is full dimensional.*

Proof. A linear functional $f(y) = a_1 y_1 + \cdots + a_d y_d$ will be compatible with this ordering if and only if $f(x_i - x_{i+1}) > 0$ for $i = 1, \dots, l-1$. Putting these inequalities together with the inequalities for $C(X)$, we obtain $C(X, \succ)$. \square

Remark IV.39. The new algorithm is nearly the same as the original. The only differences being that we need to keep track of the order the points were added in, and then use this information to test if $C(X', \succ)$ is full dimensional instead of $C(X)$.

Now that we have modified the algorithm to work for any λ , our next question is if we can find the facets containing a given vertex as in the rectangle case. We will again be able to obtain these facets as rays of a cone, but the cone will be a modified version of $C(X, \succ)$.

Notation IV.40. Let $x_1 \succ x_2 \succ \cdots \succ x_l$ be a linear extension of the dominance order on the points of an order filter X . We let $C(X, \succ, \lambda)$ denote the cone given by

the set of inequalities:

$$(x_{i,1} - x_{i+1,1})a_1 + \cdots + (x_{i,d} - x_{i+1,d})a_d \geq 0, \text{ whenever } \lambda_i > \lambda_{i+1}$$

$$y_1 a_1 + \cdots + y_d a_d - c \geq 0, y \in \min(X)$$

$$y_1 a_1 + \cdots + y_d a_d - c \leq 0; y \in \max(X^c)$$

$$a_i - a_{i+1} \geq 0, i = 1..d$$

We again have that the rays of $C(X, \succ, \lambda)$ correspond to hyperplanes. Let H_1, H_2, \dots be the hyperplanes corresponding to the extremal rays of the cone $C(X, \succ, \lambda)$.

Lemma IV.41. *Let v be a dominant vertex of $W_{\lambda, \mu}$, and let \succ be the linear order on the points of $X(v)$ obtained from a linear functional f corresponding to v (i.e. let \succ be $>_f$). The facets of $W_{\lambda, \mu}$ containing v are precisely the intersections $H_i \cap W_{\lambda, \mu}$.*

Proof. Notice that a linear functional will correspond to v if the full order on the points of X is a linear extension of the order on the points obtained by ordering them by the λ value assigned to each point. The lemma then follows using the same reasoning as the proof of Lemma IV.36. \square

4.4 Relating Different $W_{\lambda, \mu}$

One particularly pleasing property of the algorithm from section 4.3 is that λ played only a minor role; the algorithm mostly depended on P_μ . The exact shape of the $W_{\lambda, \mu}$ is mostly determined by the hyperplane sections of P_μ with λ only serving to ‘scale’ the various faces of the polytope. This section will be devoted to the implications of this property: we will start by looking at how the $W_{\lambda, \mu}$ are related for different λ when μ is fixed and finish by looking at how they are related between different μ .

To start out with, fix a partition μ . From Lemma IV.41, the only information we need from λ in order to produce the hyperplanes for the facets of $W_{\lambda,\mu}$ are the positions with equal parts.

Definition IV.42. The *descent set* for a partition λ is the set $\text{des}(\lambda) := \{i : \lambda_i > \lambda_{i+1}\}$.

Proposition IV.43. Let f_1, \dots, f_k be the complete list of linear functionals producing facets of $W_{\lambda,\mu}$. If λ' is a partition with $\text{des}(\lambda') \subset \text{des}(\lambda)$ then the facets of $W_{\lambda',\mu}$ will be given by a sublist of the f_1, \dots, f_k .

Proof. Let f be a linear functional producing a facet of $W_{\lambda',\mu}$. The permutohedron schemes for the faces of $W_{\lambda,\mu}$ and $W_{\lambda',\mu}$ have the same level sets. Every active level set for $W_{\lambda',\mu}$ will be an active set for $W_{\lambda,\mu}$. By Remark IV.20, since the active level sets for $W_{\lambda',\mu}$ produce a face of codimension 1, the active level sets for $W_{\lambda,\mu}$ must also produce a face of codimension 1. \square

Recall that the l -staircase partition is the partition $\delta_l = (l-1, l-2, \dots, 1)$ (Definition I.7).

Corollary IV.44. Let f_1, \dots, f_k be the complete list of linear functionals producing the facets of $W_{\delta_l,\mu}$. For any λ of length less than l the complete set of facets of $W_{\lambda,\mu}$ will be given by a subset of f_1, \dots, f_k .

Corollary IV.45. Let f be a linear functional giving a facet of $W_{e_l,\mu}$. If λ is a partition of length l then f will also produce a facet of $W_{\lambda,\mu}$.

We can apply the same reasoning to the vertices of $W_{\lambda,\mu}$. From Lemma IV.38, we know that in order to produce a vertex of $W_{\lambda,\mu}$ we need both an order filter X and a linear extension \succ of the dominance order on the points in X . Both X and \succ

only depend on μ and not λ , so we can essentially find the vertices of $W_{\lambda,\mu}$ without knowing λ .

Proposition IV.46. *Let f_1, \dots, f_k be linear functionals producing all of the vertices of $W_{\lambda,\mu}$. If λ' is a partition of the same length as λ with $\text{des}(\lambda') \subset \text{des}(\lambda)$ then each vertex of $W_{\lambda',\mu}$ will be given by one of the f_i .*

Proof. From Lemma IV.38 we get a vertex from each valid choice of order filter X and linear extension \succ . Two distinct pairs (X, \succ) and (X', \succ') will produce distinct vertices of $W_{\lambda,\mu}$ unless $X = X'$ and the points $x_1, x_2 \in X$ with $x_1 \succ x_2$ and $x_2 \succ' x_1$ are assigned the same value of λ . Since $\text{des}(\lambda') \subset \text{des}(\lambda)$, any two pairs (X, \succ) and (X', \succ') which produce the same vertex of $W_{\lambda',\mu}$ will also produce the same vertex in $W_{\lambda,\mu}$. \square

Corollary IV.47. *Let f_1, \dots, f_k be linear functionals producing all the vertices of $W_{\delta_l,\mu}$. If λ is a partition of length at most $l - 1$, then each vertex of $W_{\lambda,\mu}$ will be given by one of the f_i .*

Corollary IV.48. *Let f be a linear functional giving a vertex of $W_{1^l,\mu}$. If λ is a partition of length l , then f will also produce a vertex of $W_{\lambda,\mu}$. Moreover, if g is another linear functional producing a different vertex of $W_{1^l,\mu}$ then f and g will produce distinct vertices of $W_{\lambda,\mu}$.*

Remark IV.49. These propositions and corollaries tell us that for any μ there is a fixed set of possible hyperplanes producing facets of the weight polytope $W_{\lambda,\mu}$. The only effect that λ has is that it will translate these hyperplanes forming the facets. This means that for fixed μ , the various weight polytopes $W_{\lambda,\mu}$ are all related to each other by ‘shifting’ their facets. In this sense, $W_{\delta_l,\mu}$ has the ‘full’ polytope while $W_{1^l,\mu}$ has the most ‘degenerate’ shape.

Next, suppose that we no longer fix μ . Whether or not a linear functional corresponds to a facet is entirely determined by the level sets in its permutahedron schemes. As long as we know that there are active level sets producing $d - 2$ degrees of freedom, then we can guarantee that the linear functional produces a facet, regardless of the exact value μ . Given a linear functional f , as long as P_μ is ‘large enough’ the active level sets will be codimension 1 and will produce a facet. This means that as long as we know μ' is larger than μ in some sense, then we can know that all of the linear functionals corresponding to facets of $W_{\delta_i, \mu}$ will also correspond to facets of $W_{\delta_i, \mu'}$.

Proposition IV.50. *If f is a linear functional producing a facet in $W_{\delta_i, \mu}$, then for any μ' with $|\mu| = |\mu'|$ and μ' larger than μ in the dominance order there exists l' such that f produces of facet of $W_{\delta_k, \mu'}$ for any $k \geq l'$.*

Proof. Since f corresponds to a facet, the Minkowski sum of the level sets of P_μ which are assigned nonzero tuples must have codimension 1. We know that $P_\mu \subset P_{\mu'}$ since μ' dominates μ . This means that the level sets of P_μ will be subsets of the level sets of μ' , so their Minkowski sum will also have codimension 1. The only obstacle to f producing a facet of $W_{\lambda, \mu'}$ is that we need λ long enough so that the points in $P_{\mu'}$ which are assigned nonzero tuples contains the set of points which are assigned nonzero tuples for P_μ . \square

Proposition IV.51. *If f is a linear functional producing a facet in $W_{\delta_i, \mu}$, then for any μ' with $\mu_i - \mu_{i+1} \leq \mu'_i - \mu'_{i+1}$ for $i = 1, \dots, d - 1$ there exists a l' such that f produces a facet of $W_{\delta_k, \mu'}$ for any $l' \leq k \leq$ number of integer points in P_μ .*

Proof. In this case we can translate P_μ by $\mu' - \mu$ to have their dominant vertices match up and $P_\mu + (\mu' - \mu) \subset P_{\mu'}$. The result now follows by the same reasoning as

Proposition IV.50. □

Remark IV.52. We can continue this reasoning even further. The polytope containing P_μ does not have to itself be a permutohedron in order for this reasoning to apply. Given a linear functional f , as long as we can guarantee that the intersection of one of its hyperplane slices of P_μ has codimension 1, then f will produce a facet of P_μ for λ of the proper length.

For any choice of μ , consider the following map f_μ . Let $p : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ be the map

$$x \mapsto (x_1, x_1 + x_2, \dots, x_1 + \dots + x_{d-1})$$

sending a point to its partial sums, let $d_\mu : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ be the translation $x \mapsto x - p(\mu)$, and let f_μ be the composition $f_\mu = d_\mu \circ p$. The restriction of this map f_μ to P_μ is injective, maps $\mathbb{Z}^d \cap P_\mu$ to \mathbb{Z}^{d-1} , maps μ to the origin, and maps P_μ to the negative orthant \mathbb{R}_-^{d-1} .

Under this map, a linear functional $g(y) = b_1 y_1 + \dots + b_{d-1} y_{d-1}$ will produce a dominant linear functional $f_\mu \circ g$ if and only if $b_i > 0$ for $i = 1, \dots, d-1$. If g is such a dominant linear functional where $\{y : g(y) = c\} \cap \mathbb{R}_-^{d-1}$ is $(d-2)$ -dimensional, then as long as $c/b_i \leq \mu_i - \mu_{i+1}$ for $i = 1, \dots, d-1$ this intersection $\{y : g(y) = c\} \cap \mathbb{R}_-^{d-1}$ will be contained in $f_\mu(P_\mu)$. This guarantees that $f_\mu \circ g$ will produce a facet of $W_{\lambda, \mu}$ if the last nonzero entry of λ is assigned to a point in this intersection $\{y : g(y) = c\} \cap \mathbb{R}_-^{d-1}$. We can therefore produce facets of various $W_{\lambda, \mu}$ by using knowledge of hyperplane slices of the negative orthant \mathbb{R}^{d-1} .

4.5 When is $W_{\lambda, \mu}$ saturated?

The algorithm from Section 4.3 produces all of the dominant vertices and facets of $W_{\lambda, \mu}$. Using the weight schemes for faces, once we know the facets of $W_{\lambda, \mu}$ we

can produce all of the weights that are in or near to a dominant facet. If $W_{\lambda,\mu}$ is saturated, then Theorem IV.9 tells us that we now have all of the weights of $s_\lambda[s_\mu]$ that are maximal in the dominance order. The natural follow up question then is when is $W_{\lambda,\mu}$ saturated? As a general guideline, holes (integer points that are not weights) start occurring when the level sets in the boundary weight schemes stop looking like generalized permutahedra and start forming less ‘nice’ shapes.

Example IV.53. One of the smallest examples of a non saturated weight polytope that was found is $W_{1^{25},5^1}$ for $d = 4$. Let f be the linear functional

$$f(x_1, x_2, x_3, x_4) = 15x_1 + 6x_2 + 3x_3.$$

Consider the reduced weight schemes for the face F_f of $W_{1^{25},5^1}$ corresponding to f . Since $\mu = (5)$ is a single row, the Kostka number $K_{\mu,\alpha}$ is 1 for each point $\alpha \in P_\mu$. Hence in the reduced weight schemes for $W_{1^{25},5^1}$, every point in $P_{(5,0,0,0)}$ will be assigned a value of either 0 or 1. There are 23 points in $P_{(5,0,0,0)}$ with an f -value larger than 30, so these points will all be assigned a value of 1 in the weight schemes for F_f . This contributes $(57, 27, 18, 13)$ to the contents of the weight schemes. The active level set for this face is

$$I = P_{(5,0,0,0)} \cap \{x : f(x) = 30\} = \{(0, 5, 0, 0), (1, 1, 3, 0), (1, 2, 1, 1), (2, 0, 0, 3)\}.$$

To produce a weight scheme for F_f , we assign a value of 1 to the 23 points with f -values larger than 30 and a value of 0 to the points with f -values smaller than 30, and then assign two points in I a value of 1 and the other two points a value of 0. The three weight schemes which have a value of 0 assigned to $(1, 2, 1, 1)$ have contents $(60, 28, 21, 16)$, $(59, 32, 18, 16)$, and $(58, 33, 21, 13)$, so these three vectors are weights of the plethysm $e_{25}[h_5]$. The point $(59, 31, 20, 15)$ is in the convex hull of these three

points (it is obtained by averaging the three points), so it is in the face F_f . There are not two points in I which sum to $(59, 31, 20, 15) - (57, 27, 18, 13) = (2, 4, 2, 2)$, so there is no weight scheme of F_f with content equal to $(59, 31, 20, 15)$. We therefore have an integer point, $(59, 31, 20, 15)$, which is in the polytope $W_{1^{25}, 5^1}$ but is not a weight of $e_{25}[h_5]$, so the weight polytope is not saturated in this case.

Remark IV.54. Even though $W_{1^{25}, (5)}$ is not saturated, all the maximal weights still end up being near boundary dominant weights and are found by the algorithm. This has been the case in all the non saturated $W_{\lambda, \mu}$ found so far.

When there are fewer dimensions or λ has a small length, then there aren't as many options for the hyperplane sections making up the level sets and $W_{\lambda, \mu}$ ends up being saturated. Doing a case by case analysis of the possibilities for the level sets, one can show that $W_{\lambda, \mu}$ is always saturated if $d = 3$ or $l(\lambda) \leq 4$. The holes that we have found for $d = 4$ all require that the points in the level sets have the same multiplicity, so we conjecture that $W_{\lambda, \mu}$ is still saturated for $d = 4$ if λ is a strict partition. The shortest λ where we found a hole is $W_{1^{23}, 5^1}$ for $d \geq 5$, so it seems likely that the actual upper bound on the length of λ for the weight polytope to always be saturated is larger than 4.

From Section 4.4, we know that the shape of $W_{\lambda, \mu}$ is essentially determined by μ . One can hope then that if we fix μ , then $W_{\lambda, \mu}$ being saturated for certain λ can tell us about whether the polytope is saturated for more general λ . The 'simplest' weight polytopes are those with $\lambda = 1^l$, and as we saw this is the easiest case to describe the polytope. It would therefore be convenient if we could check saturation just for the rectangle case and deduce it for more general λ .

Proposition IV.55. *For fixed μ and m , there exists a k such that if $W_{1^l, \mu}$ is saturated for all $l < k$ then $W_{\lambda, \mu}$ is saturated for all λ with $l(\lambda) \leq m$.*

Proof. Suppose f is a linear functional corresponding to a facet of $W_{\lambda,\mu}$. Let $h(f)$ be the largest number of nonzero assignments which occur in a permutahedron scheme for f . This $h(f)$ is equal to

$$h(f) = \sum_{i=1}^j \sum_{\alpha \in I_i} K_{\mu,\alpha}$$

where I_1, I_2, \dots is the ordered list of level sets for f and j is such that

$$\sum_{i=1}^{j-1} \sum_{\alpha \in I_i} K_{\mu,\alpha} < l(\lambda) \leq \sum_{i=1}^j \sum_{\alpha \in I_i} K_{\mu,\alpha}.$$

We will show that the k in the statement of the proposition is $k = h(f) + \text{Ind}(f)$.

Let λ be any partition with $l(\lambda) \leq m$. To check that $W_{\lambda,\mu}$ is saturated, it suffices to check that all its maximal near boundary integer points are weights. Let x be a maximal near boundary integer point of $W_{\lambda,\mu}$. Let (n_α) be a permutahedron scheme corresponding to x and let f be the linear functional corresponding to the facet x is near to. Let (n'_α) be the permutahedron scheme formed by replacing every value in the tuples by its fractional part if it is non integer and by 1 if it is a positive integer. Let $l' = |(n'_\alpha)|$ be the sum of all the entries of the assignment, and let $y \in \mathbb{R}^d$ be the content of (n'_α) . Then l' is an integer smaller than $h(f) + \text{Ind}(f)$ by the definition of $h(f)$, and y is an integer point since it differs from x by subtracting away integer values. Thus (n'_α) is a permutahedron scheme for $W_{1^{l'},\mu}$, so y is an integer point in $W_{1^{l'},\mu}$. Therefore y is a weight by our assumption, so it has a corresponding weight scheme (p'_α) . If we let (p_α) be the weight scheme formed by adding back everything that was removed from n_α to obtain n'_α , i.e. $p_\alpha = p'_\alpha + n_\alpha - n'_\alpha$, then (p_α) is a weight scheme for $W_{\lambda,\mu}$ corresponding to a near boundary point x . We therefore have that x is a weight for every near boundary point x of $W_{\lambda,\mu}$, hence $W_{\lambda,\mu}$ is saturated. We therefore have that the proposition holds for $k = h(f) + \text{Ind}(f)$. \square

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