Indefinite Theta Functions and Zeta Functions

by

Gene S. Kopp

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2017

Gene S. Kopp

gkopp@umich.edu

ORCID: 0000-0003-1285-4532

ACKNOWLEDGEMENTS

Thank you to the University of Michigan and to the National Science Foundation for funding. The work that went into this thesis was partially supported by NSF grant DMS-1401224, NSF grant DMS-1701576, and NSF RTG grant 1045119.

Thank you to Jeff Lagarias, my Ph.D. advisor. Learning from Jeff has helped be mature as a mathematician, and he has nurtured and deepened my existing passions for number theory and experimental mathematics while giving me a new passion for analysis. Jeff has been available to meet frequently and has been very accommodating, and he contributed major help to rewriting the introduction to this thesis and to making the thesis readable. I also thank Jeff for many mathematical discussions about the SIC-POVM problem.

Thank you to my second reader Kartik Prasanna for helpful comments and corrections and for interesting directions for future research. Thank you to the other members of my committee for taking the time to review my thesis and attend my defense.

Thank you to everyone I have done math with at the University of Michigan, especially Lara Du, Cameron Franc, Julian Rosen, and John Wiltshire-Gordon. I look forward to more discussions and collaborations with all of you.

Thank you to all those who shaped me into person capable of being a research mathematician. First were my parents, who encouraged and supported my interests from a young age. I am grateful to Linda Kelley, who provided me a space in school where I could think for myself and actualize those thoughts. I am indebted to the Ross Mathematics Program, and specifically to Dan Shapiro and Matt Satriano, for introducing me to proof-based mathematics and to number theory, and for teaching me "to think deeply of simple things." Finally, I thank Steven J. Miller, who gave me my first opportunity to do original research in a supportive setting, and John Wiltshire-Gordon, who had the gumption to collaborate with me on an unadvised undergraduate research project.

Thank you to my parents for helping out in many ways, large and small, emotionally and materially. Thank you to Natalie for being a wellspring of encouragement.

TABLE OF CONTENTS

ABSTRACT

Indefinite Theta Functions and Zeta Functions

by
Gene S. Kopp

Chair: Jeffrey C. Lagarias

We define an indefinite theta function in dimension g and index 1 whose modular parameter transforms by a symplectic group, generalizing a construction of Sander Zwegers used in the theory of mock modular forms. We introduce the indefinite zeta function, defined from the indefinite theta function using a Mellin transform, and prove its analytic continuation and functional equation. We express certain zeta functions attached to ray ideal classes of real quadratic fields as indefinite zeta functions (up to gamma factors). A Kronecker limit formula for the indefinite zeta function—and by corollary, for real quadratic fields—is obtained at $s = 1$. Finally, we discuss two applications related to Hilbert's 12th problem: numerical computation of Stark units in the rank 1 real quadratic case, and computation of fiducial vectors of Heisenberg SIC-POVMs.

CHAPTER I

Introduction

The goal of this thesis is to introduce new transcendental functions and prove new formulas for special values of L-functions of interest to Hilbert's 12th problem. This chapter begins with a discussion of the history of that problem. Afterwards, we give an overview of the most important definitions and the main theorems of the thesis. Finally, we discuss applications to the Stark conjectures and to the construction of SIC-POVMs in quantum information theory.

1.1 Hilbert's 12th problem

In the year 1900, David Hilbert published a list[1] of 23 open problems then inspired a great deal of mathematical development over many decades. Hilbert's 12th problem asks for an "Extension of Kronecker's Theorem on Abelian Fields to any Algebraic Realm of Rationality." "Kronecker's theorem"—more commonly known as the Kronecker-Weber theorem—states that the abelian extensions of the rational number field $\mathbb{Q}$ are obtained by adjoining the values of the complex exponential function $e(z) = e^{2\pi i z}$ when z is a rational number. It was also known to Hilbert that special values of elliptic functions generated abelian extensions of imaginary

[1]Hilbert's problems were translated into English by Mary Frances Winston Newson in 1902 [23].

quadratic fields.[2] Hilbert asks for "the extension of Kronecker's theory to the case that, *in place of the realm of rational number or of the imaginary quadratic field, any algebraic field whatever is laid down of as realm of rationality.*" He poses the challenge of "finding and discussing those functions which play the part for any algebraic number field corresponding to the exponential function in the real field and the elliptic modular function in the imaginary quadratic number field." Hilbert's 12th problem is sometimes referred to as "Kronecker's Jugendtraum," because Kronecker (in a letter to Dedekind) described the sought-after proof that the elliptic functions generated the abelian extensions of imaginary fields as "meinen liebsten Jugendtraum," or "my favorite youthful dream."

Class field theory over an arbitrary number field was mostly developed during the 1920s. Takagi defined the "ray class groups" and proved the existence of the corresponding "ray class fields" with his Takagi existance theorem. Artin's reciprocity law specified the isomorphism between a ray class groups and the Galois group of the ray class field as coming from a product of local Frobenius maps. Later developments included the introduction of the algebraic objects that appear in a modern treatment of the subject—Brauer groups by Brauer, class formations by Artin and Tate, idèles by Chevalley.

Abstract class field theory does not give a procedure to actually construct class fields. Explicit constructions of class fields beyond the imaginary quadratic case did not come until Shimura extended the theory of complex multiplication from elliptic curves to abelian varieties. Shimura constructed class fields of CM fields, that is, totally complex quadratic extensions of totally real fields. This explicit construction used special values of analytic functions—certain modular functions of

[2] Although Hilbert hints that elliptic function are enough to generate all such extensions, in fact, the j-invariant is also needed.

several variables—as Hilbert desired. The Shimura reciprocity law relates the Galois action on the special values to the action of the modular group on the functions themselves.

In a series of papers [42, 43, 44, 45], Harold Stark suggested a new approach to Hilbert's 12th problem using special values of derivatives of L-functions. Stark formulated a series of conjectures about the leading term of the Taylor series of an Artin L-function at $s = 1$ or $s = 0$. If $\rho : \mathrm{Gal}(L/K) \to \mathbf{GL}_n(\mathbb{C})$ is an irreducible Galois representation and $L(s, \rho)$ vanishes to order r at $s = 0$, the Stark conjectures predict the existence of a "Stark regulator" attached to ρ, a determinant of an $r \times r$ matrix of linear forms of logarithms of algebraic units (more generally, S-units) generalizing the regulator of a number field appearing in the class number formula. In the case when L/K is an abelian extension, any Artin L-function $L(s, \rho)$ is equal to the Hecke L-function of a finite-order Hecke character—specified by data internal to K—and the units are predicted to live in the corresponding class field. The abelian Stark conjectures could thus provide an answer to Hilbert's 12th Problem, constructing class fields explicitly from special values of derivatives of L-functions. Thus is especially true in the "rank 1" case ($r = 1$), when the Stark units may be recovered from the derivative L-value by exponentiation.

1.1.1 Kronecker's first limit formula and imaginary quadratic L-values

The abelian Stark conjectures are known over $\mathbb{Q}$ and over any imaginary quadratic field. In the imaginary quadratic case, the proof uses the first and second Kronecker limit formulas for real analytic Eisenstein series.

The real analytic Eisenstein series

$$E(\tau, s) := \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{\mathrm{Im}(\tau)^s}{|m\tau + n|^{2s}} \tag{1.1}$$

is closely related to the zeta function of an imaginary quadratic ideal class

$$\zeta(s,A) := \sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s}. \tag{1.2}$$

Specifically, if A is an ideal class of the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K, and we choose any $\mathfrak{b} \in A^{-1}$ such that $\mathfrak{b}\cap\mathbb{Q} = \mathbb{Z}$ and write $\mathfrak{b} = \mathbb{Z}+\tau\mathbb{Z}$ for $\mathrm{Im}(\tau) > 0$, then

$$\begin{aligned}
N(\mathfrak{b})^{-s}\zeta(s,A) &= \sum_{\mathfrak{a}\in A} N(\mathfrak{b}\mathfrak{a})^{-s} && (1.3)\\
&= \sum_{\alpha\in\mathfrak{b}/\mathcal{O}_K^\times} N(\alpha)^{-s} && (1.4)\\
&= \left|\mathcal{O}_K^\times\right| \sum_{\alpha\in\mathfrak{b}} N(\alpha)^{-s} && (1.5)\\
&= \left|\mathcal{O}_K^\times\right| \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} |m\tau+n|^{-s} && (1.6)\\
&= \frac{\left|\mathcal{O}_K^\times\right|}{\mathrm{Im}(\tau)^s} E(\tau,s). && (1.7)
\end{aligned}$$

Write $\tau = x+yi$ for real numbers x,y. The real analytic Eisenstein series has a Fourier series in x (see, e.g., [10], chapter 1, pages 67–69). We write it using the completed Eisenstein series $E^*(\tau,s) := \frac{1}{2}\pi^{-s}\Gamma(s)E(\tau,s)$ and the completed Riemann zeta function $\hat{\zeta}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

$$\begin{aligned}
E^*(\tau,s) &= \hat{\zeta}(2s)y^s + \hat{\zeta}(2s-1)y^{1-s} && (1.8)\\
&\quad + 2\sqrt{y}\sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{s-\frac{1}{2}}\sigma_{1-2s}(|m|)K_{s-\frac{1}{2}}(2\pi|m|y)e(mx). && (1.9)
\end{aligned}$$

By sending $s\to 1$ in this Fourier expansion, we obtain the *first Kronecker limit formula*.

$$\lim_{s\to1}\left(E^*(\tau,s) - \frac{1}{s-1}\right) = 2\log|\eta(\tau)|. \tag{1.10}$$

Here, $\eta(\tau)$ is the Dedekind eta function $\eta(\tau) = e(\tau/24)\prod_{d=1}^{\infty}(1 - e(d\tau))$, a modular form of weight $\frac{1}{2}$. A detailed proof of the first Kronecker limit formula may be found in [29], chapter 20, pages 273–275.

From this formula, we can obtain the constant term of the Taylor expansion of $\zeta(s, A)$ at $s = 1$, or, equivalently, the value of $\zeta'(s, A)$ at $s = 0$. Using results from the theory of elliptic curves with complex multiplication, one can show that integral linear combinations of the $\zeta'(0, A)$ whose coefficients sum to zero are logarithms of algebraic numbers (as they're logarithms of absolute values of modular functions evaluated at moduli of CM elliptic curves). Moreover, one may show that these algebraic numbers are algebraic units satisfying the conditions desired by Stark. Stark does so in the first paper of his series [42].

We will discuss and prove Kronecker's second limit formula later; it is Proposition IV.3. The two Kronecker limit formulas, together with the theory of complex multiplication and singular moduli, are the essential ingredients in the proof of the main theorems of [42].

1.1.2 "Kronecker limit formulas" for other fields

Several mathematicians have found analogues of the Kronecker limit formula in other settings. With an eye toward the Stark conjectures, we are particularly interested in analogues for other number fields beyond the imaginary quadratic case.

Hecke found a Kronecker limit formula for real quadratic fields in the case of a wide (modulus 1) ideal class. An exposition of Hecke's formula may be found in Siegel's Tata lectures [41] (p. 90–93) as well as in a paper of Zagier [51]. A Kronecker limit formula for narrow (modulus 1) ideal classes of real quadratic fields case was found by Herglotz [22] and rederived in a different form by Zagier [51].

The first problem one runs into in trying to find a Kronecker limit formula for number fields is that, for any number field K other than $\mathbb{Q}$ or an imaginary quadratic field, the group of units $\mathcal{O}_K^\times$ is infinite, so eq. (1.5) doesn't make sense. Shintani [40] resolved this issue for any totally real field, by choosing a finite set of integral cones that tile Minkowski space under the action of $\mathcal{O}_K^\times$, and writing $\zeta(s, A)$ as a sum of several Dirichlet series on cones. The Shintani decomposition of the unit group for any totally real number field and the Shintani zeta functions are exposed in [32], chapter VII, §9.

Shintani [38, 39] gives a Kronecker limit formula for ray class zeta functions of real quadratic fields in the rank 1 case (zero of multiplicity 1 at $s = 0$) and also proves results for more general totally real fields [40]. Shintani's main theorem for real quadratic fields (as stated in [39]) is

Theorem I.1 (Shintani's Kronecker limit formula)**.** *Let K be a real quadratic field, and let A be a narrow ray ideal class modulo $\mathfrak{f}$ in $\mathcal{O}_K$. Let R be the ray ideal class of all $a\mathcal{O}_k$ with a totally positive and $a \equiv -1 \pmod{\mathfrak{f}}$. Then,*

$$\zeta'(0, A) - \zeta'(0, RA) = \log X_{\mathfrak{f}}(A). \tag{1.11}$$

The quantity $X_{\mathfrak{f}}(A)$ is defined to be a certain finite product of special values of $F(z, \omega) = \frac{\Gamma_2(z,\omega)}{\Gamma_2(\omega_1+\omega_2-z,\omega)}$, where Γ_2 is the *double gamma function* introduced by Barnes [7]. The function $F(z, \omega)$ was later named the *double sine function* by Kurokawa and Koyama [27]. Shintani uses his formula to prove the (rank 1, real quadratic) Stark conjecture in the case when the ray class field is a degree 2 extension of a totally abelian field [39].

More recent work on Kronecker limit formulas by Yamamoto [49, 50], Vlasenko and Zagier [47], and Liu and Masri [30] builds on the earlier results of Shintani, Herglotz, and Zagier. This work has not yet led to proofs of new cases of the Stark conjectures.

Kronecker limit formulas for Eisenstein series $E_j^\Gamma(z,s)$ for noncongruence subgroups $\Gamma \leq \mathbf{SL}_2(\mathbb{Z})$ have been considered by Posingies [33]. These $E_j^\Gamma(z,s)$ are not known to specialize to Artin L-functions or related functions; nonetheless, Posingies's formulas may have unexplored applications to explicit class field theory. They express the constant term of $E_j^\Gamma(z,s)$ at $s=1$ in terms of the absolute value of a noncongruence modular function. When $\mathbb{Q}(z)$ is an imaginary quadratic field, the modular function will evaluate to an algebraic number in a (generally non-abelian) extension of $\mathbb{Q}(z)$.

1.1.3 From indefinite theta functions to a new Kronecker limit formula

We present a new approach to deriving a formula for $\zeta'(0,A) - \zeta'(0,RA)$ for real quadratic fields. The existing literature is based on Shintani decomposition—splitting up the zeta function into finitely many sums over cones or double cones. Shintani zeta functions are Dirichlet series that interpolate between arithmetically interesting zeta functions. Instead, we set out to interpolate in a way that preserves the functional equation, but were willing to give up the interpolating functions being Dirichlet series. The tool for the job is the indefinite theta functions introduced by Sander Zwegers.

Zwegers introduced the indefinite theta functions in his Ph.D. thesis [56]. He used them to construct harmonic weak Maass forms whose holomorphic parts are the mock theta functions of Ramanujan. Part of this work was contained in an earlier paper [55]. Zwegers's work triggered an explosion of interest in mock modular forms, with applications to partition identities [9], "quantum modular forms" and "false theta functions" [16], period integrals of the j-invariant [14], sporadic groups

[15], and quantum black holes [12]. A summary of Zwegers's thesis and some of the work by others that immediately followed is given by Zagier [52].

This thesis makes no direct use of mock modular forms. Rather, we are interested in certain Mellin transforms of indefinite theta functions, which we call indefinite zeta functions (even though they only sometimes have Dirichlet series). In dimension 2, the indefinite zeta functions interpolate between certain zeta functions associated to real quadratic fields, just as Eisenstein series do for imaginary quadratic fields. By computing certain Fourier series with respect to the action of a one-parameter unipotent subgroup $\{T^\xi\}$, a Kronecker limit formula for indefinite zeta functions—thus, for real quadratic fields—emerges.

We also generalize Zwegers's construction by introducing more general indefinite theta functions transforming by a symplectic group. We allow complex values of the parameters c_1 and c_2 that Zwegers treats as real vectors defining the boundary of a cone.

1.2 Terminology and definitions

This dissertation uses many special functions and a few nonstandard pieces of notation. We list some of the most commonly-used notation that may need clarification.

- $e(z) := \exp(2\pi i z)$ is the complex exponential, and this notation is used for $z \in \mathbb{C}$ not necessarily real.
- $\mathfrak{H} = \{\tau : \operatorname{Im}\tau > 0\}$ is the complex upper half-plane.
- Non-transposed vectors $v \in \mathbb{C}^g$ are always column vectors; the transpose $v^\top$ is a row vector.
- If M is a $g \times g$ matrix, then $M^\top$ is its transpose, and (when M is invertible)

$M^{-\top}$ is a shorthand for $(M^{-1})^{\top}$.

- $Q_M(v)$ denotes the quadratic form $Q_M(v) = \frac{1}{2}v^{\top}Mv$, where M is a $g \times g$ matrix, and v is a $g \times 1$ column vector.
- $f(c)|_{c=c_1}^{c_2} = f(c_2) - f(c_1)$, where f is any function taking values in an additive group.
- If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$ and f is a function of $\mathbb{C}^2$, we may write $f(v) = f\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ rather than $f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$.
- We will often need to express $\Omega = N + iM$ where N, M are real $g \times g$ symmetric matrices; N and M will always have real entries even when we do not say so explicitly.

We turn now to the definitions required to state the main results of this thesis.

1.2.1 Siegel intermediate half-space

The space on which the modular parameter of an indefinite theta function lives is $\mathfrak{H}_g^{(1)}$, where $\mathfrak{H}_g^{(k)}$ is defined as follows.

Definition II.14. For $0 \le k \le g$, we define the *Siegel intermediate half-space* of genus g and index k to be

$$\mathfrak{H}_g^{(k)} = \{\Omega \in \mathbf{M}_g(\mathbb{C}) : \Omega = \Omega^{\top} \text{ and } \operatorname{Im}(\Omega) \text{ has signature } (g-k,k)\}. \tag{1.12}$$

1.2.2 Incomplete Gaussian transform

Definition II.21. For any complex number α and any holomorphic test function f, define the *incomplete Gaussian transform*

$$\mathcal{E}_f(\alpha) = \int_0^{\alpha} f(u)e^{-\pi u^2}\, du, \tag{1.13}$$

along any contour from 0 to α. In particular, for $\mathbb{1}(z) = 1$, set

$$\mathcal{E}(\alpha) := \mathcal{E}_{\mathbb{1}}(\alpha) = \int_0^{\alpha} e^{-\pi u^2}\, du. \tag{1.14}$$

When α is real, define $\mathcal{E}_f(\alpha)$ for an arbitrary continuous test function f:

$$\mathcal{E}_f(\alpha) = \int_0^{\alpha} f(u) e^{-\pi u^2}\, du. \tag{1.15}$$

In terms of the similar function used by Zwegers [56], $\mathcal{E}(\alpha) = \frac{1}{2}E(\alpha)$.

1.2.3 Indefinite theta functions and indefinite theta nulls with characteristics

The incomplete Gaussian transform provides variable coefficients used to define an indefinite theta function.

Definition II.22. Let $\Omega = N + iM$ be a complex symmetric matrix whose imaginary part has signature $(g-1,1)$; that is, $\Omega \in \mathfrak{H}_g^{(1)}$. Define the *indefinite theta function*

$$\Theta^{c_1,c_2}[f](z,\Omega) = \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top\Omega n + n^\top z\right), \tag{1.16}$$

where $z \in \mathbb{C}^g$, $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top M c_1 < 0$, $\overline{c_2}^\top M c_2 < 0$, and $f(\xi)$ is a continuous function of one variable satisfying the growth condition $\log|f(\xi)| = o\left(|\xi|^2\right)$. If the c_j are not both real, also assume that f is holomorphic.

Set $\Theta^{c_1,c_2}(z,\Omega) := \Theta^{c_1,c_2}[\mathbb{1}](z,\Omega)$.

Zwegers's theta function is defined in arbitrary dimension g for real c_j when N is a scalar multiple of M. More precisely, if M is real symmetric matrix of signature $(g-1,1)$, $\tau \in \mathfrak{H}$, and $c_1, c_2 \in \mathbb{R}^g$, then $\Theta^{c_1,c_2}(Mz, \tau M)$ is equal up to an exponential factor to the function $\vartheta_M^{c_1,c_2}(z,\tau)$ introduced by Zwegers on page 27 of [56].

Definition II.27. Let $\Omega = N + iM \in \mathfrak{H}_g^{(1)}$. Define the *indefinite theta null with* *characteristics* $p, q \in \mathbb{R}^g$:

$$\Theta_{p,q}^{c_1,c_2}[f](\Omega) = e\left(\frac{1}{2}q^\top \Omega q + p^\top q\right) \Theta_f^{c_1,c_2}\left(p + \Omega q; \Omega\right); \tag{1.17}$$

$$\Theta_{p,q}^{c_1,c_2}(\Omega) = e\left(\frac{1}{2}q^\top \Omega q + p^\top q\right) \Theta^{c_1,c_2}\left(p + \Omega q; \Omega\right). \tag{1.18}$$

where $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top M c_1 < 0$, $\overline{c_2}^\top M c_2 < 0$, and $f(\xi)$ is a continuous function of one variable satisfying the growth condition $\log|f(\xi)| = o\left(|\xi|^2\right)$. If the c_j are not both real, also assume that f is holomorphic.

1.2.4 Definite and indefinite zeta functions

We define the definite zeta function using a Mellin transform of the Riemann zeta function $\Theta_{p,q}(\Omega)$ with real characteristics.

Definition III.1. Let $\Omega = N + iM \in \mathfrak{H}_g^{(0)}$ and $p, q \in \mathbb{R}^g$, and suppose $q \notin \mathbb{Z}^g$. Define the **definite zeta function**

$$\hat{\zeta}_{p,q}(\Omega, s) = \int_0^\infty \Theta_{p,q}(t\Omega) t^s \frac{dt}{t}. \tag{1.19}$$

We define the indefinite zeta function using a Mellin transform of the indefinite theta function with characteristics.

Definition III.2. Let $\Omega = N + iM \in \mathfrak{H}_g^{(1)}$. The *indefinite zeta function* is

$$\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = \int_0^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t}, \tag{1.20}$$

where $p, q \in \mathbb{R}^g$, and $c_1, c_2 \in \mathbb{C}^g$ are parameters satisfying $\overline{c_1}^\top M c_1 < 0$ and $\overline{c_2}^\top M c_2 < 0$.

1.2.5 Ray class zeta functions and differenced ray class field zeta functions

We now define two Dirichlet series, $\zeta_A(s)$ and $Z_A(s)$, attached to a ray ideal class A of the ring of integers of a number field.

Definition III.13 (Ray class zeta function). Let K be any number field, and let $\mathfrak{c}$ be an ideal of the maximal order $\mathcal{O}_K$. Let S be a subset of the real places of K (i.e., the embeddings $K \hookrightarrow \mathbb{R}$). Let A be a ray ideal class modulo $\mathfrak{c} \cup S$, that is, an element of the group

$$\mathrm{Cl}_{\mathfrak{c}\cup S}(\mathcal{O}_K) := \frac{\{\text{nonzero fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{c}\}}{\{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}}. \tag{1.21}$$

Define the *zeta function of* A to be

$$\zeta(s, A) = \sum_{\mathfrak{a} \in A} N(\mathfrak{a})^{-s}. \tag{1.22}$$

This function has a simple pole at $s = 1$ with residue independent of A. The pole may be eliminated by considering the function $Z_A(s)$, defined as follows.

Definition III.14 (Differenced ray class zeta function). Let R be the element of $C_{\mathfrak{c}\cup S}$ defined by

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}. \tag{1.23}$$

Define the *differenced zeta function of* A to be

$$Z_A(s) = \zeta(s, A) - \zeta(s, RA). \tag{1.24}$$

The function $Z_A(s)$ is holomorphic at $s = 1$.

1.3 Statement of results

In this section, we summarize the main results of this thesis, roughly in the order they appear. These results rely on the definitions in section 1.2.

1.3.1 Indefinite theta functions

We begin with the results on indefinite theta functions from chapter II. The first two results describe the elliptic and modular transformation properties, respectively, of the indefinite theta function.

Proposition II.24 (Elliptic transformation laws). The indefinite theta function satisfies the following transformation law with respect to the z variable, for $a + \Omega b \in \mathbb{Z}^g + \Omega\mathbb{Z}^g$:

$$\Theta_f^{c_1,c_2}(z + a + \Omega b; \Omega) = e\left(-\frac{1}{2}b^\top \Omega b - b^\top z\right)\Theta_f^{c_1,c_2}(z; \Omega). \tag{1.25}$$

Theorem II.26 (Modular transformation laws). The indefinite theta function satisfies the following transformation laws with respect to the Ω variable, where $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, $B = B^\top$:

(1) $\Theta_f^{c_1,c_2}(z; A^\top \Omega A) = \Theta_f^{Ac_1,Ac_2}(A^{-\top} z; \Omega)$.

(2) $\Theta_f^{c_1,c_2}(z; \Omega + 2B) = \Theta_f^{c_1,c_2}(z; \Omega)$.

(3) $\Theta^{c_1,c_2}(z; -\Omega^{-1}) = \frac{e\left(\frac{1}{2}z^\top \Omega z\right)}{\sqrt{\det(i\Omega^{-1})}}\Theta^{-\overline{\Omega}^{-1}c_1, -\overline{\Omega}^{-1}c_2}(\Omega z; \Omega)$.

Of special interest are indefinite theta functions satisfying a particular sort of symmetry, which we call *P-stability*.

Definition II.30. Let $P \in \mathbf{GL}_g(\mathbb{Z})$ be fixed. Let $z \in \mathbb{C}^g$, $\Omega \in \mathfrak{H}_g^{(1)}$, $c_1, c_2 \in \mathbb{R}^g$ satisfying $c_j^\top \operatorname{Im}(\Omega) c_j < 0$. The quadruple (c_1, c_2, z, Ω) is called *P-stable* if $P^\top \Omega P = \Omega$, $Pc_1 = c_2$, and $P^\top z \equiv z \pmod{\mathbb{Z}^2}$.

The condition of P-stability is also related to holomorphy in the τ-variable for Zwegers's indefinite theta functions. The indefinite theta function attached to a P-stable quadruple satisfies the following remarkable condition.

Theorem II.31 (P-stability theorem). Let $P \in \mathbf{GL}_g(\mathbb{Z})$. Let $z \in \mathbb{C}^g$, $\Omega \in \mathfrak{H}_g^{(1)}$, $c_1, c_2 \in \mathbb{R}^g$ satisfying $c_j^\top \operatorname{Im}(\Omega) c_j < 0$.If (c_1, c_2, z, Ω) is P-stable, then for any $r \in \mathbb{C}$ with $\operatorname{Re}(r) > 1$, we have $\Theta^{c_1,c_2}(z, \Omega) = \Theta^{c_1,c_2}[f_r](z, \Omega)$ for $f_r(u) = \frac{\pi^{\frac{r+1}{2}}}{\Gamma\left(\frac{r+1}{2}\right)}|u|^r$.

1.3.2 Indefinite zeta functions

Now we state the results on indefinite zeta functions from chapter III. The indefinite zeta function has an analytic continuation and functional equation.

Theorem III.3 (Analytic continuation and functional equation). The function $\hat{\zeta}_{a,b}^{c_1,c_2}(\Omega, s)$ may be analytically continued to an entire function on $\mathbb{C}$. It satisfies the functional equation

$$\hat{\zeta}_{a,b}^{c_1,c_2}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(a^\top b)}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-b,a}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(\Omega, s). \tag{1.26}$$

The indefinite zeta function may be specialization to differenced zeta functions attached to ray ideal classes of real quadratic fields.

Theorem III.15 (Specialization of indefinite zeta). For each $A \in C_{\mathfrak{c}\cup\{\infty_1,\infty_2\}}$ and integral ideal $\mathfrak{b} \in A^{-1}$, there exists a real symmetric matrix M of signature $(1,1)$, along with $c_1, c_2, q \in \mathbb{C}^2$, such that

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s)Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s). \tag{1.27}$$

The indefinite zeta function also has a general series expansion—given in Theorem III.11—which involves hypergeometric functions and is not a Dirichlet series.

1.3.3 Kronecker limit formulas

In Chapter IV, we derive a Kronecker limit formula for indefinite zeta functions in dimension $g = 2$. The classical "second" Kronecker limit formula for definite zeta functions, stated in our notation, is as follows.

Proposition IV.3 (Second Kronecker limit formula). Let $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ and $\Omega = iM = \frac{i}{\mathrm{Im}(\tau)} \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$ for $\tau \in \mathfrak{H}$. Then,

$$\hat{\zeta}_{p,0}(\Omega, 1) = -2\log\left| u^{p_1^2/2+1/12}\left(v^{1/2} - v^{-1/2}\right) \prod_{d=1}^{\infty}\left(1 - u^d v\right)\left(1 - u^d v^{-1}\right)\right| \tag{1.28}$$

where $u = e(\tau)$ and $v = e(p_2 - p_1\tau)$. This formula may be written more compactly as

$$\hat{\zeta}_{p,0}(\Omega, 1) = -2\log\left|\frac{\vartheta_{\frac{1}{2}+p_2,\frac{1}{2}-p_1}(\tau)}{\eta(\tau)}\right|. \tag{1.29}$$

This thesis generalizes Proposition IV.3 to arbitrary $\Omega \in \mathfrak{H}_2^{(0)}$.

Theorem IV.1 (Generalized second Kronecker limit formula). Let $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$ with $0 \leq p_1, p_2 < 1$, and let $\Omega = N + iM \in \mathfrak{H}_2^{(0)}$. Let $z = \tau_1$ and $z = \tau_2$ be the solutions of $Q_\Omega\begin{pmatrix} z \\ 1 \end{pmatrix} = 0$ in the upper and lower half-planes, respectively. Then,

$$\hat{\zeta}_{p,0}(\Omega, 1) = \frac{-1}{\sqrt{\det(-i\Omega)}}\left(\left(\log f_p\right)(\tau_1) + \left(\log f_p\right)(-\tau_2)\right), \tag{1.30}$$

where the function $f_p : \mathfrak{H} \to \mathbb{C}$ may be written either of the following ways,

$$f_p(\tau) = e\left(-\frac{p_2}{2}\right) u_\tau^{p_1^2/2+1/12}\left(v_\tau^{1/2} - v_\tau^{-1/2}\right)\prod_{d=1}^{\infty}\left(1 - u_\tau^d v_\tau\right)\left(1 - u_\tau^d v_\tau^{-1}\right) \tag{1.31}$$

$$= \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right)\vartheta_{\frac{1}{2}+p_2,\frac{1}{2}-p_1}(\tau)}{\eta(\tau)}, \tag{1.32}$$

where $u_\tau = e(\tau)$, $v_\tau = e(p_2 - p_1\tau)$, ϑ is the Jacobi theta function, and η is the Dedekind eta function. Here $\operatorname{Log} f_p$ is the branch satisfying

$$(\operatorname{Log} f_p)(\tau) \sim \pi i\left(p_1^2 - p_1 + \frac{1}{6}\right)\tau \text{ as } \tau \to i\infty. \tag{1.33}$$

Our main result in chapter IV is the following new Kronecker limit formula for indefinite zeta functions. It involves an integral of a rapidly convergent infinite product against a function $\kappa_\Omega^c\begin{pmatrix} \xi \\ 1 \end{pmatrix}$ built out polynomials and square roots.

Definition IV.5. Suppose $\Omega = N + iM \in \mathfrak{H}_2^{(1)}$, $c \in \mathbb{C}^2$, and $s \in \mathbb{C}$. Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)}Mcc^\top M$. Then, we define, for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$,

$$\kappa_\Omega^c(v) = \frac{c^\top M v}{4\pi i\sqrt{-Q_M(c)}Q_\Omega(v)\sqrt{-2iQ_{\Lambda_c}(v)}}. \tag{1.34}$$

The formula is as follows.

Theorem IV.6 (Indefinite Kronecker limit formula). Let $\Omega = N + iM \in \mathfrak{H}_2^{(1)}$, $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$, and $c_1, c_2 \in \mathbb{C}^2$ such that $\overline{c_j}^\top M c_j < 0$. For $c = c_1, c_2$, factor the quadratic form

$$Q_{\Lambda_c}\begin{pmatrix} \xi \\ 1 \end{pmatrix} = \alpha(c)(\xi - \tau^+(c))(\xi - \tau^-(c)), \tag{1.35}$$

where $\tau^+(c)$ is in the upper half-plane and $\tau^-(c)$ is in the lower half-plane. Then,

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = I^+(c_2) - I^-(c_2) - I^+(c_1) + I^-(c_1), \tag{1.36}$$

where

$$\begin{aligned} I^\pm(c) = {} & -\operatorname{Li}_2(e(\pm p_1))\kappa_\Omega^c\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + 2i \int_0^\infty \left(\operatorname{Log}\varphi_{p_1,\pm p_2}\right)(\pm\tau^\pm(c) + it)\kappa_\Omega^c\begin{pmatrix} \pm(\tau^\pm(c) + it) \\ 1 \end{pmatrix} dt. \end{aligned} \tag{1.37}$$

The function $\varphi_{p_1,p_2} : \mathfrak{H} \to \mathbb{C}$ is defined by the a product expansion,

$$\varphi_{p_1,p_2}(\xi) := (1 - e(p_1\xi_t + p_2)) \prod_{d=1}^\infty \frac{1 - e\left((d + p_1)\xi + p_2\right)}{1 - e\left((d - p_1)\xi - p_2\right)}, \tag{1.38}$$

and its logarithm $\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)$ is the unique continuous branch with the property

$$\lim_{\xi \to i\infty} \left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi) = \begin{cases} \log(1 - e(p_2)) & \text{if } p_1 = 0, \\ 0 & \text{if } p_1 \neq 0. \end{cases} \tag{1.39}$$

Here $\log(1 - e(p_2))$ is the standard principal branch.

The following specialization looks somewhat simpler and contains all of the cases of arithmetic zeta functions $Z_A(s)$ associated to real quadratic fields.

Theorem IV.7 (Indefinite Kronecker limit formula, pure imaginary case). Let M be a 2×2 real matrix of signature $(1,1)$, and let $\Omega = iM$. Let $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$, and $c_1, c_2 \in \mathbb{R}^2$ such that $c_j^\top M c_j < 0$.

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = 2i \operatorname{Im}\left(I(c_2) - I(c_1)\right), \tag{1.40}$$

where

$$I(c) = -\operatorname{Li}_2(e(p_1))\kappa_\Omega^c\begin{pmatrix}1\\0\end{pmatrix} \tag{1.41}$$

$$+ 2i\int_0^\infty \left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\tau(c)+it)\kappa_\Omega^c\begin{pmatrix}\tau(c)+it\\1\end{pmatrix} dt. \tag{1.42}$$

Here, $\operatorname{Log}\varphi_{p_1,p_2}$ and κ_Ω^c are defined as in the statement of Theorem IV.6, and $\xi = \tau(c)$ is the unique root of the quadratic polynomial $Q_{\Lambda_c}\begin{pmatrix}\xi\\1\end{pmatrix}$ in the upper half plane.

1.4 Applications to the Stark conjectures

The rank 1 abelian Stark conjecture is known when $K = \mathbb{Q}$ or K is an imaginary quadratic field. It is not known for any other particular base field (e.g., it is open for $K = \mathbb{Q}(\sqrt{3})$). We give a statment of the rank 1 abelian Stark conjecture for real quadratic fields in terms of the functions $Z_A(s)$. Precisely, the following is a restatement of Conjecture 1 from [44] in the real quadratic case, along with two addional requirements—Conjecture 2 of [44] and the assumption (included in the general conjecture in [45]) that the isomorphism between the ray class group and the Galois group is the Artin map.

Conjecture I.2 (Stark conjecture, rank 1 real quadratic case)**.** *Let $\mathfrak{c}$ be a nonzero ideal of the ring of integers of a real quadratic number field K with the property that, if $\varepsilon \in \mathcal{O}_K$ such that $\varepsilon \equiv 1 \pmod{\mathfrak{c}}$, then one of ε or $-\varepsilon$ is totally positive. Let A be a ray ideal class in $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}}$. Let H_j be the ray class field of K modulo $\mathfrak{c}\cup\{\infty_j\}$, and let ρ_j be an embedding of H_j that embeds K using the jth real place, so that $\rho_1(H_2) = \rho_2(H_1)$ is a real field and $\rho_1(H_1) = \rho_2(H_2)$ is complex. Then,*

(1) $Z_A'(0) = \log(\rho_1(\varepsilon_A))$ for a unit $\varepsilon_A \in H_2$.

(2) The units ε_A are compatible with the Artin map $\mathrm{Art} : \mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}} \to \mathrm{Gal}(H/K)$. Specifically, $\varepsilon_A = \varepsilon_I^{\mathrm{Art}(A)}$.

Our Kronecker limit formula for indefinite zeta functions specializes to an analytic formula for rank 1 "Stark units" over a real quadratic base field. It deals with the same cases as Shintani's Kronecker limit formula [38], although our formula is very different. It can be used for numerical computation of special values. So far, we have not been able to obtain any results on algebraicity by these methods.

1.4.1 Stark conjecture example

Now we consider an example. Let $K = \mathbb{Q}(\sqrt{3})$. The ring of integers is $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$, and $\mathcal{O}_K$ has class number 1. A rational prime $p \neq 3$ splits in K if and only if $p \equiv \pm 1 \pmod{12}$, by quadratic reciprocity. In particular, (5) is inert, so $\mathfrak{c} = 5\mathbf{O}_K$ is a prime ideal in K. Let ρ_1 be the real embedding sending $\sqrt{3} \mapsto \sqrt{3}$ (determining the infinite place ∞_1), and let ρ_2 be the real embedding sending $\sqrt{3} \mapsto -\sqrt{3}$ (determining the infinite place ∞_2).

The fundamental unit of $\mathcal{O}_K$ is $\varepsilon_K := 2 + \sqrt{3}$. Since $\mathcal{O}_K$ has class number 1, $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}}$ may be identified with $(\mathcal{O}_K/\mathfrak{c})^\times \times \mathbb{R}^\times/\mathbb{R}^\times_+$ modulo the action of the unit group $\{\pm(2+\sqrt{3})^n : n \in \mathbb{Z}\}$. We can use -1 to get into $\mathbb{R}^\times_+$, so we're left with $(\mathcal{O}_K/\mathfrak{c})^\times / \left\langle 2+\sqrt{3}\right\rangle$. But $(\mathcal{O}_K/\mathfrak{c})^\times$ is a cyclic group of order 24, and $(2+\sqrt{3})^3 = 26 + 15\sqrt{3} \equiv 1 \pmod 5$ so $2+\sqrt{3}$ has order 3 modulo 5; thus, $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}} \cong \mathbb{Z}/8\mathbb{Z}$.

Let H_2 be the ray class field of $\mathcal{O}_K$ for $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}}$. The field H_2 is unramified at ∞_1—that is, a real field with respect to any embedding extending ρ_1—but ramified at ∞_2—that is, complex with respect to some (indeed, all) embeddings extending ρ_2. We calculated (with the help of Magma) the intermediate fields between K and H_2, each a quadratic extension of the previous one.

- $K = \mathbb{Q}(\sqrt{3})$,
- $L = K(\sqrt{5})$,

- $M = L\left(\sqrt{2\left(5+\sqrt{5}\right)}\right)$,
- $H_2 = M\left(\sqrt{-5+10\sqrt{3}+\sqrt{5}+2\sqrt{15}+(3-\sqrt{3}+\sqrt{5})\sqrt{2(5+\sqrt{5})}}\right)$.

As expected, that L and M are totally real, whereas H_2 is real but not totally real.

In chapter III, we will check the Stark conjecture numerically in this case using a rapidly convergent formula for the analytic continuation of indefinite zeta functions. We will see that, if I is the identity element of $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}}$, then

$$\exp\left(Z_I'(0)\right) = 3.890861713943079255337 6... \tag{1.43}$$

is equal (to 100 digits) to an algebraic unit, specifically, a root of the polynomial

$$\begin{aligned} &x^8 - (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 + (225+130\sqrt{3})x^4 \\ &- (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 - (8+5\sqrt{3})x + 1. \end{aligned} \tag{1.44}$$

This unit generates the field H_2 over K.

In Chapter IV, we will numerically check our Kronecker limit formula in this case and observe at least 30 decimal places of agreement.

1.5 Applications to SIC-POVMs

The existence of symmetric informationally complete positive operator-valued measures (SIC-POVMs) in every dimension was conjectured by Zauner in 1999 [53] and remains open. Much of the progress on this problem has been in the form of numerical investigations—enumerating all or some of the SIC-POVMs in particular dimensions. The numerical evidence strongly supports a surprising connection between SIC-POVMs and Hilbert's 12th problem for real quadratic fields discovered numerically by Appleby, Flammia, McConnell and Yard [5, 6].

An SIC-POVM is a set of d^2 "equiangular complex lines" in d-dimensional Hilbert space. In other words, it is a set of one-dimensional subspaces $\mathbb{C}v_1, \mathbb{C}v_2, \ldots, \mathbb{C}v_{d^2} \subset$

$\mathbb{C}^d$ such that $\left|\frac{\langle v_i,v_j\rangle^2}{\langle v_i,v_i\rangle\langle v_j,v_j\rangle}\right|$ takes the same value for all $i \neq j$. It is known that at most d^2 complex lines can be equiangular in $\mathbb{C}^d$. Moreover, it is known that $\left|\frac{\langle v_i,v_j\rangle^2}{\langle v_i,v_i\rangle\langle v_j,v_j\rangle}\right| = \frac{1}{d+1}$ in this case.

It is conjectured that SIC-POVMs exist in every dimension, and that there are only finitely many in each dimension except for $d = 3$. Moreover, it is conjectured that, excluding exceptions in dimensions $d = 2, 4, 8$, all SIC-POVMs are unitary-equivalent to *Heisenberg SIC-POVMs*, which are the orbit of a *fiducial vector* under the action of a certain Heisenberg group.

SIC-POVMs were introduced by Zauner in 1999 in his Ph.D. thesis [53] (translated [54] into English from German in 2011). SIC-POVMs appear in quantum information processing (e.g., [46, 11]) and quantum foundations (specifically the theory of quantum Baysianism [18]), and they have been connected to Lie and Jordan algebras [3, 4]. Computer calculations by Scott and Grassl have found at least one SIC-POVM in every dimension up to $d = 121$ [37, 36]. The case $d = 4$ is described in detail is [8]. An overview of the SIC-POVM problem is provided by the preprint [17].

1.5.1 SIC-POVM example

The numerical example for the Stark conjecture discussed in section 1.4.1 corresponds to the ray class field associated to the $d = 5$ Heisenberg SIC-POVM according to conjectures of Appleby et. al. [6], which are verified in this case. We found numerically that the derivative differenced zeta values $Z'_A(0)$ for the narrow ray class group of $\mathbb{Z}[\sqrt{3}]$ modulo $(5)\infty_2$ can be related to the phase factors of a fiducial vector for a $d = 5$ Heisenberg SIC-POVM. This work will be described in chapter V of this thesis.

CHAPTER II

Indefinite Theta Functions

In this chapter, we give a theory of indefinite theta functions. For comparison, we first provide an overview of the classical theory of Riemann (definite) theta functions, which are attached to complex symmetric matrices whose imaginary part defines a quadratic form of signature $(g, 0)$. We then define analogous indefinite theta functions attached to complex symmetric matrices whose imaginary part defines a quadratic form of signature $(g-1, 1)$. Our definition is a generalization of the definition of indefinite theta functions provided in Zwegers's thesis [56].

This thesis treats theta functions as explicit functions of several complex variables and doesn't rely formally on any results from algebraic geometry. However, we will give an overview of the geometric role of these functions to provide context.

2.1 Riemann theta functions

The definite theta function—or Riemann theta function—of genus g is a function of an elliptic parameter z and a modular parameter Ω. Riemann's theory generalizes the "genus 1" case of Jacobi theta functions. The elliptic parameter z lives in $\mathbb{C}^g$, but may (almost) be treated as an element of a complex torus $\mathbb{C}^g/\Lambda$, which happens to be an abelian variety. The parameter Ω is written as a complex $g \times g$ matrix and lives in the Siegal upper half-space $\mathfrak{H}_g$, whose definition imposes a condition on

$M = \mathrm{Im}(\Omega)$.

2.1.1 Definitions and geometric context

An *abelian variety* over a field K is a connected projective algebraic group; it follows from this definition that the group law of is abelian. (See [31] as a reference for all results mentioned in this discussion.) A *principal polarization* on an abelian variety A is an isomorphism between A and the dual abelian variety $A^\vee$. Over $K = \mathbb{C}$, every principally polarized abelian variety of dimension g is a complex torus of the form $A(\mathbb{C}) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, where Ω is in the *Siegel upper half-space* (sometimes called the Siegel upper half-plane, although it is a complex manifold of dimension $\frac{g(g+1)}{2}$).

Definition II.1. The *Siegel upper half-space* of genus g is defined to be the following open subset of the space $\mathbf{M}_g(\mathbb{C})$ of symmetric $g \times g$ complex matrices.

$$\mathfrak{H}_g^{(0)} = \mathfrak{H}_g = \{\Omega \in \mathbf{M}_g(\mathbb{C}) : \Omega = \Omega^\top \text{ and } \mathrm{Im}(\Omega) \text{ is positive-definite}\}. \tag{2.1}$$

When $g = 1$, we recover the usual upper half-plane $\mathfrak{H}_1 = \mathfrak{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

Definition II.2. The *definite (Riemann) theta function* is, for $z \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$,

$$\Theta(z; \Omega) = \sum_{n \in \mathbb{Z}^g} e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right). \tag{2.2}$$

Definition II.3. When $g = 1$, the definite theta functions is called a *Jacobi theta function* and is denoted by $\vartheta(z, \tau) = \Theta([z], [\tau])$ for $z \in \mathbb{C}$ and $\tau \in \mathfrak{H}$.

It is a theorem that the complex structure on $A(\mathbb{C})$ determines the algebraic structure on $A_{\mathbb{C}}$. The functions $\Theta(z + t; \Omega)$ for representatives $t \in \mathbb{C}^g$ of 2-torsion points of $A(\mathbb{C})$ may be used to define an explicit holomorphic embedding of A as an algebraic locus in complex projective space. These shifts t are called *characteristics*. More details may be found in Chapter VI of [28], in particular pages 104–108.

The positive integer g is called the "genus" because the Jacobian $\mathrm{Jac}(C)$ of an algebraic curve of genus g is a principally polarized abelian variety of dimension g. Not all principally polarized abelian varieties are Jacobians of curves; the question of characterizing the locus of Jacobians of curves inside the moduli space of all principally polarized abelian varieties is known as the *Schottky problem*.

2.1.2 A canonical square root

On the Siegel upper half-space $\mathfrak{H}_g$, $\det(-i\Omega)$ has a canonical square root.

Lemma II.4. *Let* $\Omega \in \mathfrak{H}_g$. *Then,*

$$\left(\int_{x\in\mathbb{R}^g} e\left(\frac{1}{2}x^\top \Omega x\right)\, dx\right)^2 = \frac{1}{\det(-i\Omega)}. \tag{2.3}$$

Proof. Equation (2.3) holds for Ω diagonal and purely imaginary by reduction to the one-dimensional case $\int_{-\infty}^{\infty} e^{-\pi a x^2}\, dx = \frac{1}{\sqrt{a}}$. Consequently, eq. (2.3) holds for any purely imaginary Ω by a change of basis, using spectral decomposition.

Consider the two sides of eq. (2.3) as holomorphic functions in $\frac{g(g+1)}{2}$ complex variables (the entries of Ω); they agree whenever those $\frac{g(g+1)}{2}$ variables are real. Because they are holomorphic, it follows by analytic continuation that they agree everywhere. $\square$

Definition II.5. Lemma II.4 provides a canonical square root of $\det(-i\Omega)$:

$$\sqrt{\det(-i\Omega)} := \left(\int_{x\in\mathbb{R}^g} e\left(\frac{1}{2}x^\top \Omega x\right)\, dx\right)^{-1}. \tag{2.4}$$

Whenever we write "$\sqrt{\det(-i\Omega)}$" for $\Omega \in \mathfrak{H}_g$, we will be referring to this square root.

We will later need to use this square root to evaluate a shifted version of the integral that defines it.

Corollary II.6. *Let $\Omega \in \mathfrak{H}_g$ and $c \in \mathbb{C}^g$. Then,*

$$\int_{x\in\mathbb{R}^g} e\left(\frac{1}{2}(x+c)^\top\Omega(x+c)\right)\,dx = \frac{1}{\sqrt{\det(-i\Omega)}}. \tag{2.5}$$

Proof. Fix Ω. The left-hand side of eq. (2.5) is constant for $c \in \mathbb{R}^g$, by Lemma II.4. Because the left-hand side is holomorphic in c, it is in fact constant for all $c \in \mathbb{C}^g$. □

Note that, if $\Omega \in \mathfrak{H}_g$, then Ω is invertible and $-\Omega^{-1} \in \mathfrak{H}_g$. This is a special case of Proposition II.15, which says, in particular, that $\mathfrak{H}_g$ is closed under the fractional linear transformation action of the symplectic group,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1} \text{ for } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}_g(\mathbb{R}). \tag{2.6}$$

In particular, $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \cdot \Omega = -\Omega^{-1}$.

The behavior of our canonical square root under the modular transformation $\Omega \mapsto -\Omega^{-1}$ is given by the following proposition.

Proposition II.7. *If $\Omega \in \mathfrak{H}_g$, then $\sqrt{\det(-i\Omega)}\sqrt{\det(i\Omega^{-1})} = 1$.*

Proof. This follows from Definition II.5 by plugging in $\Omega = iI$, because the expression $\sqrt{\det(-i\Omega)}\sqrt{\det(i\Omega^{-1})}$ is a continuous function of Ω, and $\mathfrak{H}_g$ is connected. □

2.1.3 Transformation laws of definite theta functions

Proposition II.8. *The definite theta function for $z \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$ satisfies the following transformation law with respect to the z variable, for $a + \Omega b \in \mathbb{Z}^g + \Omega\mathbb{Z}^g$:*

$$\Theta(z + a + \Omega b, \Omega) = e\left(-\frac{1}{2}b^\top\Omega b - b^\top z\right)\Theta(z, \Omega). \tag{2.7}$$

Proof. The proof is a straightforward calculation. It may be found (using slightly different notation) as Theorem 4 on page 8–9 of [34]. □

Theorem II.9. *The definite theta function for $z \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$ satisfies the following transformation laws with respect to the Ω variable, where $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, $B = B^\top$:*

(1) $\Theta(z; A^\top \Omega A) = \Theta(A^{-\top} z; \Omega)$.

(2) $\Theta(z; \Omega + 2B) = \Theta(z; \Omega)$.

(3) $\Theta(z; -\Omega^{-1}) = \frac{e\left(\frac{1}{2} z^\top \Omega z\right)}{\sqrt{\det(i\Omega^{-1})}} \Theta(\Omega z; \Omega)$.

Proof. The proof of (1) and (2) is a straightforward calculation. A more powerful version of this theorem, combining (1)–(3) into a single transformation law, appears as Theorem A on pages 86–87 of [34].

To prove (3), we apply the Poisson summation formula directly to the theta series. The Fourier transforms of the terms are given as follows.

$$\int_{\mathbb{R}^g} e\left(Q_\Omega(n) + n^\top z\right) e\left(-n^\top \nu\right)\, dn$$

$$= \int_{\mathbb{R}^g} e\left(Q_\Omega(n) + n^\top (z - \nu)\right) \tag{2.8}$$

$$= e\left(-Q_{-\Omega^{-1}}(z-\nu)\right) \int_{\mathbb{R}^g} e\left(Q_\Omega\left(n + \Omega^{-1}(z-\nu)\right)\right) \tag{2.9}$$

$$= \frac{e\left(-Q_{-\Omega^{-1}}(z-\nu)\right)}{\sqrt{\det(-i\Omega)}}. \tag{2.10}$$

In the last line, we used Lemma II.4 and Definition II.5. Now, by the Poisson summation formula,

$$\Theta(z, \Omega) = \sum_{\nu \in \mathbb{Z}^g} \frac{e\left(-Q_{-\Omega^{-1}}(z-\nu)\right)}{\sqrt{\det(-i\Omega)}} \tag{2.11}$$

$$= \frac{e\left(Q_{-\Omega^{-1}}(z)\right)}{\sqrt{\det(-i\Omega)}} \sum_{\nu \in \mathbb{Z}^g} e\left(Q_{-\Omega^{-1}}(\nu) + \nu^\top \Omega^{-1} z\right) \tag{2.12}$$

$$= \frac{e\left(Q_{-\Omega^{-1}}(z)\right)}{\sqrt{\det(-i\Omega)}} \sum_{\nu \in \mathbb{Z}^g} e\left(Q_{-\Omega^{-1}}(\nu) - \nu^\top \Omega^{-1} z\right) \text{ (sending } \nu \mapsto -\nu) \tag{2.13}$$

$$= \frac{e\left(-\frac{1}{2} z^\top \Omega^{-1} z\right)}{\sqrt{\det(-i\Omega)}} \Theta\left(-\Omega^{-1} z, -\Omega^{-1}\right). \tag{2.14}$$

If Ω is replaced by $-\Omega^{-1}$, we obtain (3). $\square$

As was mentioned, it is possible to combine all of the modular transformations into a single theorem describing the transformation of Ω under the action of $\mathbf{Sp}_{2g}(\mathbb{Z})$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}. \tag{2.15}$$

This is already complicated in genus $g = 1$, where the tranformation law involves Dedekind sums. The general case is done in Chapter III of [34], with the main theorems stated on pages 86–90.

2.1.4 Definite theta functions with characteristics

There is another notation for theta functions, using "characteristics," and it will be necessary to state the transformation laws using this notation as well. We replace z with $z = p + \Omega q$ for real variables $p, q \in \mathbb{R}^g$. The reader is cautioned that the literature on theta functions contains conflicting conventions, and some authors may use notation identical to this one to mean something slightly different.

Definition II.10. Define the *definite theta null with real characteristics* $p, q \in \mathbb{R}^g$, for $\Omega \in \mathfrak{H}_g$:

$$\Theta_{p,q}(\Omega) = e\left(\frac{1}{2}q^\top \Omega q + p^\top q\right) \Theta\left(p + \Omega q, \Omega\right). \tag{2.16}$$

The transformation laws for $\Theta_{p,q}(\Omega)$ follow from those for $\Theta(z, \Omega)$.

Proposition II.11. *Let $\Omega \in \mathfrak{H}_g$ and $p, q \in \mathbb{R}^g$. The elliptic transformation law for the definite theta null with real characteristics is given by*

$$\Theta_{p+a,q+b}(\Omega) = e\left(a^\top (q + b)\right) \Theta_{p,q}(\Omega). \tag{2.17}$$

for $a, b \in \mathbb{Z}^g$.

Proposition II.12. *Let $\Omega \in \mathfrak{H}_g$ and $p, q \in \mathbb{R}^g$. The modular transformation laws for the definite theta null with real characteristics are given as follows, where $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, and $B = B^\top$.*

(1) $\Theta_{p,q}(A^\top \Omega A) = \Theta_{A^{-\top}p, Aq}(\Omega)$.

(2) $\Theta_{p,q}(\Omega + 2B) = e(-q^\top Bq)\Theta_{p+2Bq,q}(\Omega)$.

(3) $\Theta_{p,q}\left(-\Omega^{-1}\right) = \frac{e\left(p^\top q\right)}{\sqrt{\det(i\Omega^{-1})}}\Theta_{-q,p}(\Omega)$.

2.2 Indefinite theta functions

If we allow $\mathrm{Im}(\Omega)$ to be indefinite, the series expansion in eq. (2.2) no longer converges anywhere. We want to remedy this problem by inserting a variable coefficient into each term of the sum. In Chapter 2 of his Ph.D. thesis [56], Sander Zwegers found—in the case when Ω is purely imaginary—a choice of coefficients that preserves the transformation properties of the theta function.

The results of this section generalize Zwegers's work by replacing Zwegers's indefinite theta function $\vartheta_M^{c_1,c_2}(z,\tau)$ by the indefinite theta function $\Theta_\Omega^{c_1,c_2}[f](z,\Omega)$. The function has been generalized in the following ways.

- Replacing τM for $\tau \in \mathfrak{H}$ and $M \in M_g(\mathbb{R})$ real symmetric in of signature $(g-1, 1)$ by $\Omega \in \mathfrak{H}_g^{(1)}$. (Adds $\frac{g(g+1)}{2} - 1$ real dimensions.)
- Allowing c_1, c_2 to be complex. (Adds $2g - 2$ real dimensions.)
- Allowing a test function $f(u)$, which must be specialized to $f(u) = 1$ for all the modular transformation laws to hold.

One motivation for introducing a test function f is to find transformation laws for a more general class of test functions (e.g., polynomials). We may investigate the behavior of test functions under modular transformations in future work.

2.2.1 The Siegel intermediate half-space

Definition II.13. If $M \in \mathbf{GL}_g(\mathbb{R})$ and $M = M^\top$, the *signature* of M (or of the quadratic form Q_M) is a pair (j,k), where j is the number of positive eigenvalues of M, and k is the number of negative eigenvalues (so $j+k=g$).

Definition II.14. For $0 \leq k \leq g$, we define the *Siegel intermediate half-space* of genus g and index k to be

$$\mathfrak{H}_g^{(k)} = \{\Omega \in \mathbf{M}_g(\mathbb{C}) : \Omega = \Omega^\top \text{ and } \operatorname{Im}(\Omega) \text{ has signature } (g-k,k)\}. \tag{2.18}$$

We call a complex torus of the form $T_\Omega = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ for $\Omega \in \mathfrak{H}_g^{(k)}$, $k \neq 0, g$, an *intermediate torus.*

Intermediate tori are usually *not* algebraic varieties. An example of intermediate tori in the literature are the intermediate Jacobians of Griffiths [19, 20, 21]. Intermediate Jacobians generalize Jacobians of curves, which are abelian varieties, but those defined by Griffiths are usually not algebraic. (In contrast, the intermediate Jacobians defined by Weil [48] are algebraic.)

The symplectic group $\mathbf{Sp}_{2g}(\mathbb{R})$ acts on the set of $g \times g$ complex symmetric matrices by the fractional linear transformation action,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}. \tag{2.19}$$

Proposition II.15. *If* $\Omega \in \mathfrak{H}_g^{(k)}$ *and* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}_{2g}(\mathbb{R})$, *then* $(A\Omega + B)(C\Omega + D)^{-1} \in \mathfrak{H}_g^{(k)}$. *Moreover, the* $\mathfrak{H}_g^{(k)}$ *are the open orbits of the* $\mathbf{Sp}_{2g}(\mathbb{R})$*-action on the set of* $g \times g$ *complex symmetric matrices.*

Proof. Trivial for $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$. For $\begin{pmatrix} A^\top & 0 \\ 0 & A^{-1} \end{pmatrix}$, this is Sylvester's Law of Inertia. For $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, we have $\operatorname{Im}(-\Omega^{-1}) = \frac{1}{2i}(-\Omega^{-1} + \overline{\Omega}^{-1}) = \frac{1}{2i}\overline{\Omega}^{-1}(-\overline{\Omega} + \Omega)\Omega^{-1} =$

$\overline{\Omega}^{-1}\operatorname{Im}(\Omega)\Omega^{-1} = \left(\overline{\Omega}^{-1}\right)^{\top}\operatorname{Im}(\Omega)\Omega^{-1}$, so $\operatorname{Im}(-\Omega^{-1})$ and $\operatorname{Im}(\Omega)$ have the same signature. These three types of matrices generate $\mathbf{Sp}_{2g}(\mathbb{R})$.

Now suppose $\Omega_1, \Omega_2 \in \mathfrak{H}_g^{(k)}$. There exists a matrix $A \in \mathbf{GL}_g(\mathbb{R})$ such that $A^{\top}\operatorname{Im}(\Omega_1)A = \operatorname{Im}(\Omega_2)$. For an appropriate choice of real symmetric $B \in \mathbf{M}_g(\mathbb{R})$, we thus have $A^{\top}\Omega_1 A + B = \Omega_2$. That is,

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} A^{\top} & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot \Omega_1 = \Omega_2, \tag{2.20}$$

so Ω_1 and Ω_2 are in the same $\mathbf{Sp}_{2g}(\mathbb{R})$-orbit.

Thus, the $\mathfrak{H}_g^{(k)}$ are the open orbits of the $\mathbf{Sp}_{2g}(\mathbb{R})$-action on the set of $g \times g$ symmetric complex matrices. $\square$

2.2.2 More canonical square roots

From now on, we will focus on the case of index $k = 1$, which is signature $(g-1, 1)$. The construction of modular theta series for $k \geq 2$ utilizes higher-order error functions arising from physics [1]. More research is needed to develop the higher index theory.

Lemma II.16. *Let M be a real symmetric matrix of signature $(g-1,1)$. On the region $R_M = \{z \in \mathbb{C}^g : \overline{z}^{\top}Mz < 0\}$, there is a canonical choice of holomorphic function $g(z)$ such that $g(z)^2 = -z^{\top}Mz$.*

Proof. By Sylvester's law of inertia, there is some $P \in \mathbf{GL}_g^{+}(\mathbb{R})$ (i.e., with $\det(P) > 0$) such that $M = P^{\top}JP$, where

$$J = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{2.21}$$

The region $S = \{(z_2, \dots, z_g) \in \mathbb{C}^{g-1} : |z_2|^2 + \cdots |z_g|^2 < 1\}$ is simply connected (as it is a solid ball) and does not intersect $\{(z_2, \dots, z_g) \in \mathbb{C}^{g-1} : z_2^2 + \cdots + z_g^2 = 1\}$ (because, if it did, we'd have $1 = \left|z_2^2 + \cdots + z_g^2\right| \leq |z_2|^2 + \cdots |z_g|^2 < 1$, a contradiction). Thus, there exists a unique continuous function $\sqrt{1 - z_2^2 - \cdots - z_g^2}$ on S sending $(0, \dots, 0) \mapsto 1$; this function is also holomorphic. For $z \in R_J$, define

$$g_J(z) := z_1 \sqrt{1 - \left(\frac{z_2}{z_1}\right)^2 - \cdots - \left(\frac{z_g}{z_1}\right)^2}. \tag{2.22}$$

This g_J is holomorphic and satisfies $g_J(z)^2 = -z^\top J z$, $g_J(\alpha z) = \alpha g_J(z)$, and $g_J(e_1) = 1$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.23}$$

Conversely, if we have a continuous function $g(z)$ satisfying $g(z)^2 = -z^\top J z$ and $g(e_1) = 1$, it follows that $g(\alpha z) = \alpha g(z)$, and thus $g(z) = g_J(z)$.

Now, we'd like to define $g_M(z) := g_J(Pz)$, so that we have $g_M(z)^2 = -z^\top M z$. We need to check that this definition does not depend on the choice of P. Suppose $M = P_1^\top J P_1 = P_2^\top J P_2$ for $P_1, P_2 \in \mathbf{GL}_g^+(\mathbb{R})$. So $J = \left(P_2 P_1^{-1}\right)^\top J \left(P_2 P_1^{-1}\right)$, that is, $P_2 P_1^{-1} \in \mathbf{O}(g-1, 1)$. But $\det(P_2 P_1^{-1}) = \det(P_2)\det(P_1)^{-1} > 0$, so, in fact, $P_2 P_1^{-1} \in \mathbf{SO}(g-1, 1)$.

For any $Q \in \mathbf{SO}(g-1, 1)$, we have $g_J(Qe_1)^2 = 1$. The function $Q \mapsto g_J(Qe_1)$ must be either the constant 1 or the constant -1, because $\mathbf{SO}(g-1, 1)$ is connected. Since $g_J(e_1) = 1$ ($Q = I$), we have $g_J(Qe_1) = 1$ for all $Q \in SO(1, g-1)$. The function $z \mapsto g_J(Qz)$ is a continuous square root of $-z^\top J z$ sending e_1 to 1, so $g_J(Qz) = g_J(z)$. Taking $Q = P_2 P_1^{-1}$ and replacing z with $P_1 z$, we have $g_J(P_2 z) = g_J(P_1 z)$, as desired. $\square$

Definition II.17. If M is a real symmetric matrix of signature $(g-1,1)$, we will write $\sqrt{-z^\top Mz}$ for the function $g_M(z)$ in Lemma II.16. We may also use similar notation, such as $\sqrt{-\frac{1}{2}z^\top Mz} := \frac{1}{\sqrt{2}}\sqrt{-z^\top Mz}$.

Lemma II.18. *Suppose M is a real symmetric matrix of signature $(g-1,1)$, and $c \in \mathbb{C}^g$ such that $\overline{c}^\top Mc < 0$. Then, $M + M\,\mathrm{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1} cc^\top\right) M$ is well-defined (that is, $c^\top Mc \neq 0$) and positive definite.*

Proof. Because M has signature $(g-1,1)$ and $\overline{c}^\top Mc < 0$,

$$\left(\overline{c}^\top Mc\right)^2 - \left|c^\top Mc\right|^2 = \det\begin{pmatrix} \overline{c}^\top Mc & c^\top Mc \\ \overline{c}^\top M\overline{c} & c^\top M\overline{c} \end{pmatrix} < 0. \tag{2.24}$$

Thus, $\left|c^\top Mc\right| > \left(\overline{c}^\top Mc\right)^2 > 0$, so $c^\top Mc \neq 0$ and $M + M\,\mathrm{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1} cc^\top\right) M$ is well defined. Let

$$A = M + M\,\mathrm{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1} cc^\top\right) M \tag{2.25}$$

$$= M - M\left(c^\top Mc\right)^{-1} cc^\top M - M\left(\overline{c}^\top M\overline{c}\right)^{-1} \overline{cc}^\top M. \tag{2.26}$$

On the $(g-1)$-dimensional subspace $W = \{w \in \mathbb{C}^g : \overline{c}^\top Mw = 0\}$, the sesquilinear form $w \mapsto \overline{w}^\top Mw$ is positive definite; this follows from the fact that $\overline{c}^\top Mc < 0$, because M has signature $(g-1,1)$. For nonzero $w \in W$,

$$\overline{w}^\top Aw = \overline{w}^\top Mw - (c^\top Mc)^{-1}(\overline{w}^\top Mc)(c^\top Mw) - (\overline{c}^\top M\overline{c})^{-1}(\overline{w}^\top M\overline{c})(\overline{c}^\top Mw) \tag{2.27}$$

$$= \overline{w}^\top Mw - (c^\top Mc)^{-1}(0)(c^\top Mw) - (\overline{c}^\top M\overline{c})^{-1}(\overline{w}^\top M\overline{c})(0) \tag{2.28}$$

$$= \overline{w}^\top Mw > 0. \tag{2.29}$$

Moreover,

$$c^\top Aw = c^\top Mw - (c^\top Mc)^{-1}(c^\top Mc)(c^\top Mw) - (\overline{c}^\top M\overline{c})^{-1}(c^\top M\overline{c})(\overline{c}^\top Mw) \tag{2.30}$$

$$= c^\top Mw - c^\top Mw - (\overline{c}^\top M\overline{c})^{-1}(c^\top M\overline{c})(0) \tag{2.31}$$

$$= 0, \tag{2.32}$$

and

$$c^\top A\overline{c} = c^\top M\overline{c} - (c^\top Mc)^{-1}(c^\top Mc)(c^\top M\overline{c}) - (\overline{c}^\top M\overline{c})^{-1}(c^\top M\overline{c})(\overline{c}^\top M\overline{c}) \tag{2.33}$$

$$= c^\top M\overline{c} - c^\top M\overline{c} - c^\top M\overline{c} \tag{2.34}$$

$$= -c^\top M\overline{c} \tag{2.35}$$

$$= -\overline{c}^\top Mc > 0. \tag{2.36}$$

We have now shown that A is positive definite, as it is positive definite on subspaces W and $\mathbb{C}\overline{c}$, and these subspaces span $\mathbb{C}^g$ and are perpendicular with respect to A. □

Lemma II.19. *Let $\Omega = N + iM$ be an invertible complex symmetric $g \times g$ matrix. Consider $c \in \mathbb{C}^g$ such that $\overline{c}^\top Mc < 0$. The following identities hold:*

(1) $M\Omega^{-1} = \overline{\Omega}\,\mathrm{Im}\,(-\Omega^{-1})$.

(2) $M - 2iM\Omega^{-1}M = \overline{\Omega}\,\mathrm{Im}\,(-\Omega^{-1})\,\overline{\Omega}$.

(3) $\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right) = \det(-i\Omega)\frac{c^\top\overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}c}{c^\top Mc}$.

Proof. Proof of (1):

$$M\Omega^{-1} = \frac{1}{2i}(\Omega - \overline{\Omega})\Omega^{-1} \tag{2.37}$$

$$= \frac{1}{2i}(I - \overline{\Omega}\Omega^{-1}) \tag{2.38}$$

$$= \overline{\Omega}\frac{1}{2i}(\overline{\Omega}^{-1} - \Omega^{-1}) \tag{2.39}$$

$$= \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right). \tag{2.40}$$

Proof of (2):

$$M - 2iM\Omega^{-1}M = M\Omega^{-1}\left(\Omega - 2iM\right) \tag{2.41}$$

$$= \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\left(\Omega - (\Omega - \overline{\Omega})\right) \text{ using (1)} \tag{2.42}$$

$$= \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}. \tag{2.43}$$

Proof of (3): Note that $\det(I+A) = 1+\mathrm{Tr}(A)$ for any rank 1 matrix A. Thus,

$$\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right) \tag{2.44}$$

$$= \det(-i\Omega)\det\left(I + \frac{2i}{c^\top Mc}(\Omega Mc)(Mc)^\top\right) \tag{2.45}$$

$$= \det(-i\Omega)\left(1 + \mathrm{Tr}\left(\frac{2i}{c^\top Mc}(\Omega Mc)(Mc)^\top\right)\right) \tag{2.46}$$

$$= \det(-i\Omega)\left(1 + \left(\frac{2i}{c^\top Mc}c^\top M\Omega^{-1}Mc\right)\right) \tag{2.47}$$

$$= \det(-i\Omega)\frac{-c^\top\left(M - 2iM\Omega^{-1}M\right)c}{-c^\top Mc} \tag{2.48}$$

$$= \det(-i\Omega)\frac{-(\overline{\Omega}c)^\top\,\mathrm{Im}\left(-\Omega^{-1}\right)(\overline{\Omega}c)}{-c^\top Mc}, \tag{2.49}$$

using (2) in the last step. $\square$

Definition II.20 (Canonical square root). If $\Omega \in \mathfrak{H}_g^{(1)}$, then we define $\sqrt{\det(-i\Omega)}$ as follows. Write $\Omega = N + iM$ for $N, M \in \mathbf{M}_g(\mathbb{R})$, and choose any c such that $\overline{c}^\top Mc < 0$. By Lemma II.18, $M + M\,\mathrm{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1}cc^\top\right)M$ is positive definite. Write $M + M\,\mathrm{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1}cc^\top\right)M = \mathrm{Im}\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)$. By part (3) of Lemma II.19,

$$\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right) = \det(-i\Omega)\frac{-(\overline{\Omega}c)^\top\,\mathrm{Im}\left(-\Omega^{-1}\right)(\overline{\Omega}c)}{-c^\top Mc}. \tag{2.50}$$

We can thus define $\sqrt{\det(-i\Omega)}$ as follows:

$$\sqrt{\det(-i\Omega)} := \frac{\sqrt{-c^\top Mc}\sqrt{\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right)}}{\sqrt{-(\overline{\Omega}c)^\top\,\mathrm{Im}\left(-\Omega^{-1}\right)(\overline{\Omega}c)}}, \tag{2.51}$$

where the square roots on the RHS are as defined in Definition II.5 and Definition II.17. This definition does not depend on the choice of c because the set $\{c \in \mathbb{C}^g : \overline{c}^\top Mc < 0\}$ is connected.

2.2.3 Definition of indefinite theta functions

Definition II.21. For any complex number α and any entire test function f, define the *incomplete Gaussian transform*

$$\mathcal{E}_f(\alpha) = \int_0^\alpha f(u) e^{-\pi u^2}\, du, \tag{2.52}$$

where the integral may be taken along any contour from 0 to α. In particular, for the constant functions $\mathbb{1}(u) = 1$, set

$$\mathcal{E}(\alpha) := \mathcal{E}_{\mathbb{1}}(\alpha) = \int_0^\alpha e^{-\pi u^2}\, du = \frac{\alpha}{2|\alpha|} \int_0^{|\alpha|^2} t^{-1/2} e^{-\pi(\alpha/|\alpha|)^2 t}\, dt. \tag{2.53}$$

When α is real, define $\mathcal{E}_g(\alpha)$ for an arbitrary continuous test function f:

$$\mathcal{E}_f(\alpha) = \int_0^\alpha f(u) e^{-\pi u^2}\, du. \tag{2.54}$$

Definition II.22. Define the *indefinite theta function attached to the test function* f to be

$$\Theta^{c_1,c_2}[f](z;\Omega) = \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2} c^\top \operatorname{Im}(\Omega) c}} \right) \Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2} n^\top \Omega n + n^\top z \right), \tag{2.55}$$

where $\Omega \in \mathfrak{H}_g^{(1)}$, $z \in \mathbb{C}^g$, $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top M c_1 < 0$, $\overline{c_2}^\top M c_2 < 0$, and $f(\xi)$ is a continuous function of one variable satisfying the growth condition $\log|f(\xi)| = o\left(|\xi|^2\right)$. If the c_j are not both real, also assume that f is entire.

Also define the *indefinite theta function* $\Theta^{c_1,c_2}(z;\Omega) := \Theta^{c_1,c_2}[\mathbb{1}](z;\Omega)$.

The function $\Theta^{c_1,c_2}(z;\Omega) = \Theta^{c_1,c_2}[\mathbb{1}](z;\Omega)$ is the function we are most interested in, because it will turn out to satisfy a symmetry in $\Omega \mapsto -\Omega^{-1}$. We will also show that the functions $\Theta^{c_1,c_2}[u \mapsto |u|^r](z;\Omega)$ are equal (up to a constant) for certain special values of the parameters.

Before we can prove the transformation laws of our theta functions, we must show that the series defining them converges.

Proposition II.23. *The indefinite theta series attached to f (eq. (2.55)) converges absolutely and uniformly for $z \in \mathbb{R}^g + iK$, where K is a compact subset of $\mathbb{R}^g$ (and for fixed Ω, c_1, c_2, and f).*

Proof. Let $M = \operatorname{Im}\Omega$. We may multiply c_1 and c_2 by any complex scalar without changing the terms of the series eq. (2.55), so we may assume without loss of generality that $\operatorname{Re}(\overline{c_1}^\top M c_2) < 0$.

For $\lambda \in [0,1]$, define the vector $c(\lambda) = (1-\lambda)c_1 + \lambda c_2$ and the real symmetric matrix $A(\lambda) := M + M \operatorname{Re}\left(\left(-\frac{1}{2}c(\lambda)^\top M c(\lambda)\right)^{-1} c(\lambda)c(\lambda)^\top\right) M$. Note that $\overline{c(\lambda)}^\top M c(\lambda) = (1-\lambda)^2 \overline{c_1}^\top M c_1 + 2\lambda(1-\lambda)\operatorname{Re}(\overline{c_1}^\top M c_2) + \lambda^2 \overline{c_2}^\top M c_2 < 0$ because each term is negative (except when $\lambda = 0$ or 1, in which case one term is negative and the others are zero). By Lemma II.18, $A(\lambda)$ is well-defined and positive definite for each $\lambda \in [0,1]$.

Consider $(x,\lambda) \mapsto x^\top A(\lambda) x$ as a positive real-valued continuous function on the compact set that is the product of the unit ball $\{x^\top x = 1\}$ and the interval $[0,1]$. It has a global minimum $\varepsilon > 0$.

The parametrization $\gamma : [0,1] \to \mathbb{C}$, $\gamma(\lambda) := \frac{c(\lambda)^\top (Mn+y)}{\sqrt{-\frac{1}{2}c(\lambda)^\top M c(\lambda)}}$, defines a countour from $\frac{c_1^\top (Mn+y)}{\sqrt{-\frac{1}{2}c_1^\top M c_1}}$ to $\frac{c_2^\top (Mn+y)}{\sqrt{-\frac{1}{2}c_2^\top M c_2}}$, so that

$$\mathcal{E}_f\left(\frac{c^\top (Mn+y)}{-\frac{1}{2}c^\top M c}\right)\Bigg|_{c=c_1}^{c_2} = \int_\gamma f(u) e^{-\pi u^2}\, du. \tag{2.56}$$

We give an upper bound for

$$\max_{\lambda\in[0,1]} \left| e^{-\pi\gamma(\lambda)^2} e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right)\right| \tag{2.57}$$

$$= e^{\pi y^\top M^{-1} y} \max_{\lambda\in[0,1]} e^{\frac{-\pi}{-\frac{1}{2}c(\lambda)^\top M c(\lambda)}\left(c(\lambda)^\top M\left(n+M^{-1}y\right)\right)^2} e^{-\pi\left(n+M^{-1}y\right)^\top M\left(n+M^{-1}y\right)} \tag{2.58}$$

$$= e^{\pi y^\top M^{-1} y} \max_{\lambda\in[0,1]} e^{-\pi\left(n+M^{-1}y\right)^\top A(\lambda)\left(n+M^{-1}y\right)} \tag{2.59}$$

$$\le e^{\pi y^\top M^{-1} y} e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}. \tag{2.60}$$

Thus,

$$\left|\mathcal{E}_f\left(\frac{c^\top(Mn+y)}{-\frac{1}{2}c^\top Mc}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top\Omega n+n^\top z\right)\right|$$

$$\leq \int_{\gamma_n} |f(u)|\, e^{\pi y^\top M^{-1}y} e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}\, du \tag{2.61}$$

$$\leq p(n)e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}, \tag{2.62}$$

where $\log p(n) = o\left(\|n\|^2\right)$. Thus, the terms of the series are $o\left(e^{-\frac{\pi\varepsilon}{2}\left(\|n\|^2+\|M^{-1}y\|\right)}\right)$, and so the series converges absolutely and uniformly for $x\in\mathbb{R}^g$ and $y\in K$. $\square$

2.2.4 Transformation laws of indefinite theta functions

We will now prove the elliptic and modular transformation laws for indefinite theta functions. In all of these results, we assume that $z\in\mathbb{C}^g$, $\Omega\in\mathfrak{H}_g^{(1)}$, $c_j\in\mathbb{C}^g$ satisfying $\overline{c_j}^\top\operatorname{Im}(\Omega)c_j$, and f is a function of one variable satisfying the conditions specified in Definition II.22.

Proposition II.24. *The indefinite theta function attached to f satisfies the following transformation law with respect to the z variable, for $a+\Omega b\in\mathbb{Z}^g+\Omega\mathbb{Z}^g$:*

$$\Theta^{c_1,c_2}[f](z+a+\Omega b;\Omega) = e\left(-\frac{1}{2}b^\top\Omega b - b^\top z\right)\Theta^{c_1,c_2}[f](z;\Omega). \tag{2.63}$$

Proof. By definition,

$$\Theta^{c_1,c_2}[f](z+a+\Omega b;\Omega)$$

$$= \sum_{n\in\mathbb{Z}^g}\mathcal{E}_f\left(\frac{c^\top\operatorname{Im}(\Omega n+(z+a+\Omega b))}{-\frac{1}{2}c^\top\operatorname{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n)+n^\top(z+a+\Omega b)\right). \tag{2.64}$$

Because $a \in \mathbb{Z}^g$, $\mathrm{Im}(a)$ is zero and $e(n^\top a) = 1$, so

$$\begin{aligned}
&\Theta^{c_1,c_2}[f](z + a + \Omega b; \Omega) \\
&= \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \mathrm{Im}(\Omega(n+b)+z)}{-\frac{1}{2}c^\top \mathrm{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n) + n^\top(z+\Omega b)\right) \qquad (2.65)\\
&= e\left(-\frac{1}{2}b^\top\Omega b\right)\sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \mathrm{Im}(\Omega(n+b)+z)}{-\frac{1}{2}c^\top \mathrm{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n+b) + n^\top z\right) \qquad (2.66)\\
&= e\left(-\frac{1}{2}b^\top\Omega b\right)\sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \mathrm{Im}(\Omega n+z)}{-\frac{1}{2}c^\top \mathrm{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n) + (n-b)^\top z\right) \qquad (2.67)\\
&= e\left(-\frac{1}{2}b^\top\Omega b - b^\top z\right)\Theta[f]^{c_1,c_2}(z;\Omega). \qquad (2.68)
\end{aligned}$$

The identity is proved. ∎

Proposition II.25. *The indefinite theta function satisfies the following condition with respect to the c variable:*

$$\Theta^{c_1,c_3}[f](z;\Omega) = \Theta^{c_1,c_2}[f](z;\Omega) + \Theta^{c_2,c_3}[f](z;\Omega) \qquad (2.69)$$

Proof. Add the series termwise. ∎

Theorem II.26. *The indefinite theta function satisfies the following transformation laws with respect to the Ω variable, where $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, $B = B^\top$:*

(1) $\Theta^{c_1,c_2}[f](z; A^\top \Omega A) = \Theta^{Ac_1,Ac_2}[f](A^{-\top}z;\Omega)$.

(2) $\Theta^{c_1,c_2}[f](z;\Omega + 2B) = \Theta^{c_1,c_2}[f](z;\Omega)$.

(3) In the case where $f(u) = \mathbb{1}(u) = 1$, we have

$$\Theta^{c_1,c_2}(z;-\Omega^{-1}) = \frac{e^{\pi i z^\top \Omega z}}{\sqrt{\det(i\Omega^{-1})}}\Theta^{-\overline{\Omega}^{-1}c_1,-\overline{\Omega}^{-1}c_2}(\Omega z;\Omega). \qquad (2.70)$$

Proof. The proof of (1) is a direct calculation.

$$\Theta^{c_1,c_2}[f](z;A^\top\Omega A)$$

$$= \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(A^\top\Omega A n+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top A^\top \Omega A n + n^\top z\right) \tag{2.71}$$

$$= \sum_{m\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(A^\top\Omega m+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}m^\top \Omega m + \left(A^{-1}m\right)^\top z\right) \tag{2.72}$$

by the change of basis $m = An$, so

$$\Theta^{c_1,c_2}[f](z;A^\top\Omega A)$$

$$= \sum_{m\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{(Ac)^\top \operatorname{Im}(\Omega m+A^{-\top}z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}m^\top \Omega m + m^\top A^{-\top} z\right) \tag{2.73}$$

$$= \Theta^{Ac_1,Ac_2}[f](A^{-\top}z;\Omega). \tag{2.74}$$

The proof of (2) is also a direct calculation.

$$\Theta^{c_1,c_2}[f](z;\Omega+2B)$$

$$= \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}((\Omega+2B)n+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top (\Omega+2B)n + n^\top z\right) \tag{2.75}$$

$$= \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top (\operatorname{Im}((\Omega)n+z)) + 2\operatorname{Im}(B)n}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n) + n^\top B n + n^\top z\right) \tag{2.76}$$

$$= \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}((\Omega)n+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n) + n^\top z\right) \tag{2.77}$$

$$= \Theta^{c_1,c_2}[f](z;\Omega); \tag{2.78}$$

where $e\left(n^\top B n\right) = 1$ because the $n^\top B n$ are integers, and $\operatorname{Im}(B) = 0$ because B is a real matrix.

The proof of (3) is more complicated, and, like the proof of the analogous property for definite (Jacobi and Riemann) theta functions, uses Poisson summation. The argument that follows is a modification of the argument that appears in the proof of Lemma 2.8 of Zwegers's thesis [56].

We will find a formula for the Fourier transform of the terms of our theta series. Most of the work is done in the calculation of the integral that follows. In this calculation, $M = \operatorname{Im}\Omega$, and $z = x + iy$ for $x, y \in \mathbb{C}^g$. The differential operator ∇_x is a row vector with entries $\frac{\partial}{\partial x_j}$, and similarly for ∇_n.

$$\nabla_x \left(\int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top M n + c^\top y}{\sqrt{-\frac{1}{2}c^\top M c}} \right) \Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n + \Omega^{-1}z\right)\right)\, dn \right)$$

$$= \int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top M n + c^\top y}{\sqrt{-\frac{1}{2}c^\top M c}} \right) \Bigg|_{c=c_1}^{c_2} \nabla_x \left(e\left(Q_\Omega\left(n + \Omega^{-1}z\right)\right)\right) dn \tag{2.79}$$

$$= \left(\int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top M n + c^\top y}{\sqrt{-\frac{1}{2}c^\top M c}} \right) \Bigg|_{c=c_1}^{c_2} \nabla_n \left(e\left(Q_\Omega\left(n + \Omega^{-1}z\right)\right)\right) dn \right) \Omega^{-1} \tag{2.80}$$

$$= \left(-\int_{n\in\mathbb{R}^g} \nabla_n \left(\mathcal{E}\left(\frac{c^\top M n + c^\top y}{\sqrt{-\frac{1}{2}c^\top M c}} \right) \right) \Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n + \Omega^{-1}z\right)\right) dn \right) \Omega^{-1} \tag{2.81}$$

$$= \left(k \int_{n\in\mathbb{R}^g} e\left(\frac{i}{-c^\top M c} \left(c^\top \operatorname{Im}(\Omega) n\right)^2 \right) e\left(Q_\Omega(n + a_z)\right) dn \right) c^\top M \Omega^{-1} \Bigg|_{c=c_1}^{c_2}, \tag{2.82}$$

where $k = \frac{-2}{\sqrt{-\frac{1}{2}c^\top M c}} \in \mathbb{C}$, $a_z = \Omega^{-1}z - M^{-1}y \in \mathbb{C}^g$, and integration by parts was used in eq. (2.81). Continuing the calculation,

$$\nabla_x \left(\int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top M n + c^\top y}{\sqrt{-\frac{1}{2}c^\top M c}} \right) \Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n + \Omega^{-1}z\right)\right) dn \right)$$

$$= k \left(\int_{n\in\mathbb{R}^g} e\left(Q_{\Omega - \frac{2i}{c^\top M c} M c c^\top M}(n) + a^\top \Omega n + \frac{1}{2} a^\top \Omega a \right) dn \right) c^\top M \Omega^{-1} \Bigg|_{c=c_1}^{c_2} \tag{2.83}$$

$$= k e\left(-\frac{1}{2} a^\top \Omega \left(\Omega - \frac{2i}{c^\top M c} M c c^\top M \right)^{-1} \Omega a + \frac{1}{2} a^\top \Omega a \right) I^{(c)} c^\top M \Omega^{-1} \Bigg|_{c=c_1}^{c_2}, \tag{2.84}$$

where

$$I^{(c)} = \int_{n\in\mathbb{R}^g} e\left(Q_{\Omega-\frac{2i}{c^\top Mc}Mcc^\top M}\left(n+\left(\Omega-\frac{2i}{c^\top Mc}Mcc^\top M\right)^{-1}\Omega a\right)\right)\,dn \tag{2.85}$$

$$= \frac{1}{\det\sqrt{-i\left(\Omega-\frac{2i}{c^\top Mc}Mcc^\top M\right)}} \tag{2.86}$$

by Lemma II.4.

We can check (by multiplication) that

$$\left(\Omega-\frac{2i}{c^\top Mc}Mcc^\top M\right)^{-1} = \Omega^{-1} - \frac{2i}{c^\top Mc - 2ic^\top M\Omega^{-1}Mc}\Omega^{-1}Mcc^\top M\Omega^{-1}. \tag{2.87}$$

Thus,

$$\Omega - \Omega\left(\Omega-\frac{2i}{c^\top Mc}Mcc^\top M\right)^{-1}\Omega = \frac{2i}{c^\top Mc - 2ic^\top M\Omega^{-1}Mc}Mcc^\top M. \tag{2.88}$$

Now compute, using Lemma II.19, $Ma = M\Omega^{-1}z - y = \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)z - y = \overline{\Omega}\left(\mathrm{Im}\left(-\Omega^{-1}\right)z - \overline{\Omega}^{-1}y\right) = \frac{1}{2i}\overline{\Omega}\left(\left(-\Omega^{-1}+\overline{\Omega}^{-1}\right)z - \overline{\Omega}^{-1}(z-\overline{z})\right) = \frac{1}{2i}\overline{\Omega}\left(-\Omega^{-1}z+\overline{\Omega}^{-1}\overline{z}\right) = \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}z\right).$ Also by Lemma II.19, $M - 2iM\Omega^{-1}M = \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}$, and

$$\sqrt{\det\left(-i\left(\Omega-\frac{2i}{c^\top Mc}Mcc^\top M\right)\right)} = \sqrt{\det(-i\Omega)}\frac{\sqrt{-c^\top\overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}c}}{\sqrt{-c^\top Mc}}. \tag{2.89}$$

We have now shown that

$$\nabla_x\left(\int_{n\in\mathbb{R}^g}\mathcal{E}\left(\frac{c^\top\,\mathrm{Im}\left(\Omega n+z\right)}{\sqrt{-\frac{1}{2}c^\top Mc}}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\,dn\right)]nn \tag{2.90}$$

$$= \frac{-2e\left(\frac{i}{(\overline{\Omega}c)\,\mathrm{Im}(-\Omega^{-1})(\overline{\Omega}c)}(c^\top Ma)^2\right)}{\sqrt{\det(-i\Omega)}\sqrt{-\frac{1}{2}(\overline{\Omega}c)\,\mathrm{Im}(-\Omega^{-1})(\overline{\Omega}c)}}c^\top M\Omega^{-1}\Bigg|_{c=c_1}^{c_2} \tag{2.91}$$

$$= \frac{-2e\left(\frac{i}{(\overline{\Omega}c)\,\mathrm{Im}(-\Omega^{-1})(\overline{\Omega}c)}(c^\top Ma)^2\right)}{\sqrt{\det(-i\Omega)}\sqrt{-\frac{1}{2}(\overline{\Omega}c)\,\mathrm{Im}(-\Omega^{-1})(\overline{\Omega}c)}}(\Omega c)^\top\,\mathrm{Im}(\Omega^{-1})\Bigg|_{c=c_1}^{c_2} \tag{2.92}$$

$$= \frac{1}{\sqrt{\det(-i\Omega)}}\nabla_x\mathcal{E}\left(\frac{(\overline{\Omega}c)^\top\left(\mathrm{Im}(-\Omega^{-1})n+\mathrm{Im}(-\Omega^{-1}z)\right)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\,\mathrm{Im}(-\Omega^{-1})(\overline{\Omega}c)}}\right)\Bigg|_{c=c_1}^{c_2}. \tag{2.93}$$

Define the following function on $\mathbb{C}^g$,

$$C(z) = C_\Omega^{(c)}(z) := \int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2}c^\top \Omega c}}\right) e\left(Q_\Omega\left(n + \Omega^{-1}z\right)\right)\, dn \tag{2.94}$$

$$-\frac{1}{\sqrt{\det(i\Omega)}}\mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}}\right), \tag{2.95}$$

suppressing the dependence of $C(z)$ on Ω and c. We have just showed that $\Delta_x C(z) = 0$, so $C(z+a) = C(z)$ for any $a \in \mathbb{R}^g$. By inspection, $C(z+\Omega^{-1}b) = C(z)$ for any $b \in \mathbb{R}^g$. It follow from both of these properties that $C(z)$ is constant. Moreover, by inspection, $C(-z) = -C(z)$; therefore, $C(z) = 0$. In other words,

$$\int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2}c^\top \Omega c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n + \Omega^{-1}z\right)\right)\, dn$$

$$= \frac{1}{\sqrt{\det(-i\Omega)}}\,\mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}}\right)\Bigg|_{c=c_1}^{c_2}. \tag{2.96}$$

Now set $g(z) := \Theta^{c_1,c_2}(z;\Omega)$, which has Fourier coefficients

$$c_n(g)(z) = \mathcal{E}\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2}c^\top \Omega c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top \Omega n + n^\top z\right). \tag{2.97}$$

By plugging in $z-\nu$ for z in eq. (2.96) and multiplying both sides by $e\left(-\frac{1}{2}(z-\nu)^\top\Omega^{-1}(z-\nu)\right)$, we obtain the following expression for the Fourier coefficients of $\hat{g}$:

$$c_\nu(\hat{g})(z) = \int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2}c^\top \Omega c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top \Omega n + n^\top z\right) e(-n^\top \nu)\, dn \tag{2.98}$$

$$= \frac{e\left(-\frac{1}{2}(z-\nu)^\top\Omega^{-1}(z-\nu)\right)}{\sqrt{\det(-i\Omega)}}\,\mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}\nu - \Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}}\right)\Bigg|_{c=c_1}^{c_2} \tag{2.99}$$

$$= \frac{e\left(-\frac{1}{2}z^\top\Omega^{-1}z\right)}{\sqrt{\det(-i\Omega)}}\,\mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}(-\nu) - \Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}}\right)\Bigg|_{c=c_1}^{c_2} \tag{2.100}$$

$$\cdot\, e\left(\frac{1}{2}\nu^\top(-\Omega^{-1})\nu + (-\nu)^\top(-\Omega^{-1}z)\right). \tag{2.101}$$

It follows by Poisson summation that

$$\Theta^{c_1,c_2}(z;\Omega) = \sum_{\nu\in\mathbb{Z}^g} c_\nu\left(\hat{g}\right)(z) \tag{2.102}$$

$$= \frac{e\left(-\frac{1}{2}z^\top\Omega^{-1}z\right)}{\sqrt{\det(-i\Omega)}}\Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}z;-\Omega^{-1}\right). \tag{2.103}$$

We obtain (3) by replacing Ω with $-\Omega^{-1}$. □

2.2.5 Indefinite theta functions with characteristics

Now we restate the transformation laws using "characteristics" notation, which will be used when we define indefinite zeta functions in chapter III.

Definition II.27. Define the *indefinite theta null with characteristics* $p, q \in \mathbb{R}^g$:

$$\Theta^{c_1,c_2}_{p,q}[f](\Omega) = e^{2\pi i\left(\frac{1}{2}q^\top\Omega q+p^\top q\right)}\Theta^{c_1,c_2}[f]\left(p+\Omega q;\Omega\right); \tag{2.104}$$

$$\Theta^{c_1,c_2}_{p,q}(\Omega) = e^{2\pi i\left(\frac{1}{2}q^\top\Omega q+p^\top q\right)}\Theta^{c_1,c_2}\left(p+\Omega q;\Omega\right). \tag{2.105}$$

The transformation laws for $\Theta^{c_1,c_2}_{p,q}[f](\Omega)$ follow from the transformation laws for $\Theta^{c_1,c_2}[f](z;\Omega)$.

Proposition II.28. *The elliptic transformation law for the indefinite theta null with characteristics is:*

$$\Theta^{c_1,c_2}_{p+a,q+b}[f](\Omega) = e(a^\top(q+b))\Theta^{c_1,c_2}_{p,q}[f](\Omega). \tag{2.106}$$

Proposition II.29. *The modular transformation laws for the indefinite theta null with characteristics are as follows.*

(1) $\Theta^{c_1,c_2}_{p,q}[f](A^\top\Omega A) = \Theta^{Ac_1,Ac_2}_{A^{-\top}p,Aq}[f](\Omega)$.

(2) $\Theta^{c_1,c_2}_{p,q}[f](\Omega+2B) = e(-q^\top Bq)\Theta^{c_1,c_2}_{p+2Bq,q}[f](\Omega)$.

(3) $\Theta^{c_1,c_2}_{p,q}(-\Omega^{-1}) = \frac{e(p^\top q)}{\sqrt{\det(i\Omega^{-1})}}\Theta^{-\overline{\Omega}^{-1}c_1,-\overline{\Omega}^{-1}c_2}_{-q,p}(\Omega)$.

2.2.6 P-stable indefinite theta functions

We now introduce a special property of the parameters (c_1, c_2, z, Ω), which we call *P-stability.* In this section, c_1, c_2 will always be real vectors.

Definition II.30. Let $P \in \mathbf{GL}_g(\mathbb{Z})$ be fixed. Let $z \in \mathbb{C}^g$, $\Omega \in \mathfrak{H}_g^{(1)}$, $c_1, c_2 \in \mathbb{R}^g$ satisfying $c_j^\top \operatorname{Im}(\Omega) c_j < 0$. The quadruple (c_1, c_2, z, Ω) is called *P-stable* if $P^\top \Omega P = \Omega$, $Pc_1 = c_2$, and $P^\top z \equiv z \pmod{\mathbb{Z}^2}$.

Remarkably, P-stable indefinite theta functions attached to $f(u) = |u|^r$ turn out to be independent of r (up to a constant factor).

Theorem II.31 (P-Stability Theorem)**.** *Set* $\Theta_r^{c_1,c_2}(z;\Omega) := \frac{\pi^{\frac{r+1}{2}}}{\Gamma\left(\frac{r+1}{2}\right)} \Theta^{c_1,c_2}[f](z;\Omega)$ *when* $f(u) = |u|^r$ *for* $\operatorname{Re}(r) > -1$*. If* (c_1, c_2, z, Ω) *is P-stable, then* $\Theta_r^{c_1,c_2}(z,\Omega)$ *is independent of* r*.*

Proof. Let $M = \operatorname{Im}(\Omega)$ and $y = \operatorname{Im}(z)$. If $\alpha \in \mathbb{R}$ and $\operatorname{Re}(r) > 1$, then

$$\mathcal{E}_r(\alpha) = \int_0^\alpha |u|^r e^{-\pi u^2}\, du \tag{2.107}$$

$$= \operatorname{sgn}(\alpha) \int_0^{|\alpha|} u^r e^{-\pi u^2}\, du \tag{2.108}$$

$$= -\frac{\operatorname{sgn}(\alpha)}{2\pi} \int_0^{|\alpha|} u^{r-1}\, d\left(e^{-\pi u^2}\right) \tag{2.109}$$

$$= -\frac{\operatorname{sgn}(\alpha)}{2\pi} \left(u^{r-1} e^{-\pi u^2} \Big|_{u=0}^{|\alpha|} - \int_0^{|\alpha|} e^{-\pi u^2}\, d\left(u^{r-1}\right) \right) \tag{2.110}$$

$$= -\frac{\operatorname{sgn}(\alpha)}{2\pi} \left(|\alpha|^{r-1} e^{-\pi\alpha^2} - (r-1) \int_0^{|\alpha|} u^{r-2} e^{-\pi u^2}\, du \right) \tag{2.111}$$

$$= \frac{1}{2\pi} \left(-\operatorname{sgn}(\alpha)\, |\alpha|^{r-1} e^{-\pi\alpha^2} + (r-1)\mathcal{E}_{r-2}(\alpha) \right). \tag{2.112}$$

Let $\alpha_n^c = \frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-Q_M(c)}}$. Set $A^c := M + M \operatorname{Re}\left((-Q_M(c))^{-1} cc^\top\right) M$, so that A^{c_1} and A^{c_2} are positive definite, as in the proof of Proposition II.23. Thus,

$$\Theta_r^{c_1,c_2}(z,\Omega) = -\frac{\pi^{r/2}}{\Gamma\left(\frac{r+1}{2}\right)} S + \Theta_{r-2}^{c_1,c_2}(z,\Omega), \tag{2.113}$$

where

$$S=\sum_{n\in\mathbb{Z}^g}\operatorname{sgn}\left(\alpha_n^c\right)\left|\alpha_n^c\right|^{r-1}\exp\left(-\pi\left(\alpha_n^c\right)^2\right)\Big|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top\Omega n+n^\top z\right). \tag{2.114}$$

The c_1 and c_2 terms in this sum decay exponentially, because

$$\left|\exp\left(-\pi\left(\alpha_n^c\right)^2\right) e\left(\frac{1}{2}n^\top\Omega n+n^\top z\right)\right|=\exp\left(-2\pi Q_{A^c}\left(n+M^{-1}y\right)\right). \tag{2.115}$$

Thus, the series may be split as a sum of two series:

$$S=\sum_{n\in\mathbb{Z}^g}\operatorname{sgn}\left(\alpha_n^{c_2}\right)\left|\alpha_n^{c_2}\right|^{r-1}\exp\left(-\pi\left(\alpha_n^{c_2}\right)^2\right) e\left(\frac{1}{2}n^\top\Omega n+n^\top z\right) \tag{2.116}$$

$$-\sum_{n\in\mathbb{Z}^g}\operatorname{sgn}\left(\alpha_n^{c_1}\right)\left|\alpha_n^{c_1}\right|^{r-1}\exp\left(-\pi\left(\alpha_n^{c_1}\right)^2\right) e\left(\frac{1}{2}n^\top\Omega n+n^\top z\right). \tag{2.117}$$

Now we use the P-symmetry to show that these two series are, in fact, equal. Note that $\operatorname{Im}(P^\top z)=\operatorname{Im}(z)$ because $P^\top z\equiv z \pmod{\mathbb{Z}^2}$, so

$$\alpha_{Pn}(c_2)=\frac{(Pc_1)^\top\operatorname{Im}(\Omega Pn+z)}{\sqrt{-Q_M(Pc_1)}} \tag{2.118}$$

$$=\frac{c_1^\top\operatorname{Im}(P^\top\Omega Pn+P^\top z)}{\sqrt{-Q_{P^\top MP}(c_1)}} \tag{2.119}$$

$$=\frac{c_1^\top\operatorname{Im}(\Omega n+z)}{\sqrt{-Q_M(c_1)}} \tag{2.120}$$

$$=\alpha_n(c_1). \tag{2.121}$$

Moreover,

$$\frac{1}{2}(Pn)^\top\Omega(Pn)+(Pn)^\top z=\frac{1}{2}n^\top(P^\top\Omega P)n+n^\top(P^\top z) \tag{2.122}$$

$$\equiv\frac{1}{2}n^\top\Omega n+n^\top z \pmod{\mathbb{Z}^2}. \tag{2.123}$$

Thus, we may substitute Pn for n in the first series (involving c_2) to obtain the second (involving c_1).

We've now shown the periodicity relation

$$\Theta_r^{c_1,c_2}(z,\Omega)=\Theta_{r-2}^{c_1,c_2}(z,\Omega). \tag{2.124}$$

Note that this identity provides an analytic continuation of $\Theta_r^{c_1,c_2}(z,\Omega)$ to the entire r-plane. To show that it is constant in r, we will show that it is bounded on vertical strips in the r-plane. As in the proof of Proposition II.23, bound $(x,\lambda)\mapsto x^\top A(\lambda)x$, considered as a positive real-valued continuous function on the product of the unit ball $\{x^\top x = 1\}$ and the interval $[0,1]$, from below by its global minimum $\varepsilon > 0$. Thus,

$$\left| \mathcal{E}_r\left(\frac{c^\top(Mn+y)}{\sqrt{-\frac{1}{2}c^\top Mc}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top\Omega n + n^\top z\right)\right|$$

$$\leq \left|\int_{\frac{c_1^\top \operatorname{Im}(\Omega n+z)}{\sqrt{-\frac{1}{2}c_1^\top \operatorname{Im}(\Omega)c_1}}}^{\frac{c_2^\top \operatorname{Im}(\Omega n+z)}{\sqrt{-\frac{1}{2}c_2^\top \operatorname{Im}(\Omega)c_2}}} |u|^{\operatorname{Re}(r)}\, du\right| e^{\pi y^\top M^{-1}y} e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2} \tag{2.125}$$

$$\leq p_{\operatorname{Re}(r)}(n)e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}, \tag{2.126}$$

where $p_{\operatorname{Re}(r)}(n)$ is a polynomial independent of $\operatorname{Im}(r)$. Hence, $\Theta_r^{c_1,c_2}(z,\Omega)$ is bounded on the line $\operatorname{Re}(r)=\sigma$ by $\sum_{n\in\mathbb{Z}^g} p_\sigma(n)e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}$. It follows that it is bounded on any vertical strip. Along with periodicity, this implies that $\Theta_r^{c_1,c_2}(z,\Omega)$ as a function of r is bounded and entire, thus constant. $\square$

CHAPTER III

Indefinite Zeta Functions and Real Quadratic Fields

In this chapter, we consider the Mellin transforms of definite and indefinite theta functions. In the definite case in dimension 2, they are generalize analytic Eisenstein series, and they specialize to zeta functions of imaginary quadratic ideal classes. In the indefinite case in dimension 2, we recover certain L-series attached to (ideals of orders of) real quadratic fields. The class of L-series we recover spans the same vector space as those Hecke L-functions attached to Hecke characters of finite order ramified at exactly one infinite place.

3.1 Definite zeta functions and real analytic Eisenstein series

We define the definite zeta function as a Mellin transform of the indefinite theta null with real characteristics.

Definition III.1. Let $\Omega \in \mathfrak{H}_g^{(0)}$ and $p, q \in \mathbb{R}^g$. The *definite zeta function* is

$$\hat{\zeta}_{p,q}(\Omega, s) = \begin{cases} \int_0^\infty \Theta_{p,q}(t\Omega) t^s \frac{dt}{t} & \text{if } q \notin \mathbb{Z}^g, \\ \int_0^\infty \left(\Theta_{p,q}(t\Omega) - 1\right) t^s \frac{dt}{t} & \text{if } q \in \mathbb{Z}^g. \end{cases} \tag{3.1}$$

By direct calculation, $\hat{\zeta}_{p,q}(\Omega, s)$ has a Dirichlet series expansion.

$$\hat{\zeta}_{p,q}(\Omega, s) = (2\pi)^{-s}\Gamma(s) \sum_{\substack{n \in \mathbb{Z}^g \\ n \neq -q}} e(p^\top(n+q)) Q_{-i\Omega}(n+q)^{-s}, \tag{3.2}$$

where $Q_{-i\Omega}(n+q)^{-s}$ is defined using the standard branch of the logarithm (with a branch cut on the negative real axis).

Now, suppose $g=2$, $\Omega = iM$ for some real symmetric, positive definite matrix M, $p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $q \notin \mathbb{Z}^2$. Then the definite zeta function may be written as follows.

$$\hat{\zeta}_{0,q}(\Omega, s) = (2\pi)^{-s}\Gamma(s) \sum_{n\in\mathbb{Z}^2} Q_M(n+q)^{-s} \tag{3.3}$$

$$= (2\pi)^{-s}\Gamma(s) \sum_{n\in\mathbb{Z}^2+q} Q_M(n)^{-s}. \tag{3.4}$$

Up to scaling, M is of the form $M = \frac{1}{\operatorname{Im}(\tau)}\begin{pmatrix} 1 & \operatorname{Re}(\tau) \\ \operatorname{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$ for some $\tau \in \mathfrak{H}$; scaling M by $\lambda \in \mathbb{R}$ simply scales $\hat{\zeta}_{p,q}(\Omega, s)$ by λ^{-s}, so we assume M is of this form. Write

$$Q_M\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \frac{1}{2\operatorname{Im}(\tau)}\left(n_1^2 + 2\operatorname{Re}\tau n_1 n_2 + \tau\overline{\tau} n_2^2\right) \tag{3.5}$$

$$= \frac{1}{2\operatorname{Im}(\tau)}\left|n_1 + n_2\tau\right|^2 \tag{3.6}$$

Thus,

$$\hat{\zeta}_{0,q}(\Omega, s) = \pi^{-s}\Gamma(s)\operatorname{Im}(\tau)^s \sum_{\left(\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right)\in\mathbb{Z}^2+q} \left|n_1\tau + n_2\right|^{-2s}. \tag{3.7}$$

If $q \in \mathbb{Q}^2$ and the gcd of the denominators of the entries of q is N, this is essentially an Eisenstein series of associated to $\Gamma_1(N)$. Choose $k, \ell \in \mathbb{Z}$ such that $q \equiv \begin{pmatrix} k/N \\ \ell/N \end{pmatrix} \pmod 1$ and $\gcd(k, \ell) = 1$. Then, we have

$$\hat{\zeta}_{0,q}(\Omega, s) = (\pi N)^{-s}\Gamma(s)\operatorname{Im}(\tau)^s \sum_{\substack{c\equiv k \pmod N \\ d\equiv \ell \pmod N}} \left|c\tau + d\right|^{-2s}. \tag{3.8}$$

The Eisenstein series associated to the cusp ∞ of $\Gamma_1(N)$ is

$$E^{\infty}_{\Gamma_1(N)}(\tau, s) = \sum_{\gamma\in\Gamma_1^{\infty}(N)\backslash\Gamma_1(N)} \operatorname{Im}(\gamma\cdot\tau)^s \tag{3.9}$$

$$= \operatorname{Im}(\tau)^s \sum_{\substack{c\equiv 0 \pmod N \\ d\equiv 1 \pmod N}} \left|c\tau + d\right|^{-2s}. \tag{3.10}$$

Here, $\Gamma_1^\infty(N)$ is the stabilizer of ∞ under the fractional linear transformation action; that is, $\Gamma_1^\infty(N) = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$.

Choose $u, v \in \mathbb{Z}$ such that $\det \begin{pmatrix} u & v \\ k & \ell \end{pmatrix} = 1$. We have

$$E^\infty_{\Gamma_1(N)}\left(\frac{u\tau+v}{k\tau+\ell}, s\right) = \operatorname{Im}\left(\frac{u\tau+v}{k\tau+\ell}\right)^s \sum_{\substack{c\equiv 0 \pmod N \\ d \equiv 1 \pmod N}} \left| c\left(\frac{u\tau+v}{k\tau+\ell}\right) + d\right|^{-2s} \tag{3.11}$$

$$= \operatorname{Im}(\tau)^s \sum_{\substack{c\equiv 0 \pmod N \\ d \equiv 1 \pmod N}} \left|(cu+dk)\tau + (cv+d\ell)\right|^{-2s} \tag{3.12}$$

$$= \operatorname{Im}(\tau)^s \sum_{\substack{c'\equiv k \pmod N \\ d' \equiv \ell \pmod N}} \left|c'\tau + d'\right|^{-2s}. \tag{3.13}$$

Combining eq. (3.8) and eq. (3.13), we see that

$$\hat{\zeta}_{0,q}(\Omega, s) = (\pi N)^{-s}\Gamma(s) E^\infty_{\Gamma_1(N)}\left(\frac{u\tau+v}{k\tau+\ell}, s\right). \tag{3.14}$$

3.2 Indefinite zeta functions: definition, analytic continuation, and functional equation

As usual, let $\Omega \in \mathfrak{H}_g^{(1)}$, $p, q \in \mathbb{R}^g$, $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top M c_1 < 0$, $\overline{c_2}^\top M c_2 < 0$.

We define the indefinite zeta function using a Mellin transform of the indefinite theta function with characteristics.

Definition III.2. The indefinite zeta function is

$$\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s) = \int_0^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega) t^s \frac{dt}{t}. \tag{3.15}$$

The terminology "zeta function" here should not be taken to mean that $\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s)$ has a Dirichlet series—it (usually) doesn't (although it does have an analogous series expansion involving hypergeometric functions, as we'll see in section 3.3). Rather, we think of it as a zeta function by analogy with the definite case, and (as we'll see) because is sometimes specializes to certain classical zeta functions.

By defining the zeta function as a Mellin transform, we've set things up so that a proof of the functional equation is a natural first step. Analytic continuation and a functional equation will follow from Theorem II.26 by standard techniques. Our analytic continuation will also converge quickly everywhere, unlike eq. (3.15) or the series expansion in section 3.3.

Theorem III.3. *The function* $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s)$ *may be analytically continued to an entire function on* $\mathbb{C}$*. It satisfies the functional equation*

$$\hat{\zeta}_{p,q}^{c_1,c_2}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}, s\right). \tag{3.16}$$

Proof. Fix $r > 0$, and split up the Mellin transform integral into two pieces,

$$\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = \int_0^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t} \tag{3.17}$$

$$= \int_r^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t} + \int_0^r \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t}. \tag{3.18}$$

Replacing t by t^{-1}, and then using part (3) of Theorem II.26, the second integral is

$$\int_0^r \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t} = \int_{r^{-1}}^\infty \Theta_{p,q}^{c_1,c_2}(t^{-1}\Omega) t^{-s} \frac{dt}{t} \tag{3.19}$$

$$= \int_{r^{-1}}^\infty \frac{e(p^\top q)}{\sqrt{\det(-it\Omega)}} \Theta_{-q,p}^{t\overline{\Omega}c_1,t\overline{\Omega}c_2}(-(t^{-1}\Omega)^{-1}) t^{-s} \frac{dt}{t} \tag{3.20}$$

$$= \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^\infty \Theta_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(t(-\Omega^{-1})) t^{\frac{g}{2}-s} \frac{dt}{t}. \tag{3.21}$$

(Recall that scaling the c_j does not affect the value of $\Theta_{p,q}^{c_1,c_2}(\Omega)$.) Putting it all together, we have

$$\begin{aligned} \hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = &\int_r^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t} \\ &+ \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^\infty \Theta_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(t(-\Omega^{-1})) t^{\frac{g}{2}-s} \frac{dt}{t}. \end{aligned} \tag{3.22}$$

As we showed in the proof of Proposition II.23, the Θ-functions in both integrals decay exponentially as $t \to \infty$, so the right-hand side converges for all $s \in \mathbb{C}$. The right-hand side is obviously analytic for all $s \in \mathbb{C}$, so we've analytically continued $\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s)$ to an entire function of s. Finally, we must prove the functional equation. If we plug $\frac{g}{2} - s$ for s in eq. (3.22), factor out the coefficient of the second term, and switch the order of the two terms, we obtain

$$\begin{aligned}\hat{\zeta}^{c_1,c_2}_{p,q}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \Bigg(&\int_{r^{-1}}^{\infty} \Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}(t(-\Omega^{-1}))t^s \frac{dt}{t} \\ &- \frac{e(-p^\top q)}{\sqrt{\det(i\Omega^{-1})}} \int_r^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega) t^{\frac{g}{2}-s} \frac{dt}{t} \Bigg).\end{aligned} \tag{3.23}$$

Reusing eq. (3.22) on $\hat{\zeta}^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}(-\Omega^{-1}, s)$, and appealing to the fact that $\Theta^{c_1,c_2}_{p,q}(\Omega) = -\Theta^{c_1,c_2}_{-p,-q}(\Omega)$, we have

$$\begin{aligned}\hat{\zeta}^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}\left(-\Omega^{-1}, s\right) = &\int_{r^{-1}}^{\infty} \Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}(t(-\Omega^{-1}))t^s \frac{dt}{t} \\ &- \frac{e(-p^\top q)}{\sqrt{\det(i\Omega^{-1})}} \int_r^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega) t^{\frac{g}{2}-s} \frac{dt}{t}.\end{aligned} \tag{3.24}$$

The functional equation now follows from eq. (3.23) and eq. (3.24). □

The formula for the analytic continuation is useful in itself. In particular, we have used this formula for computer calculations, as it may be used to compute the indefinite zeta function to arbitrary precision in polynomial time.

Corollary III.4. *The following expression is valid on the entire s-plane.*

$$\begin{aligned}\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s) = &\int_r^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega) t^s \frac{dt}{t} \\ &+ \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^{\infty} \Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}(t(-\Omega^{-1}))t^{\frac{g}{2}-s} \frac{dt}{t}.\end{aligned} \tag{3.25}$$

Proof. The is eq. (3.22). □

3.3 Series expansion of indefinite zeta function

In this section, we give a series expansion for indefinite zeta functions, under the assumption that c_1 and c_2 are real. Specifically, we write $\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s)$ as a sum of three series, the first of which is a Dirichlet series and the others of which involve hypergeometric functions. This expansion is related to the decomposition of a weak harmonic Maass form into its holomorphic "mock" piece and a nonholomorphic piece obtained from a "shadow" form in another weight. However, we don't describe the relationship here.

To proceed, we will need to introduce some special functions and review some of their properties.

3.3.1 Hypergeometric functions and modified beta functions

Let a, b, c be complex numbers, c not a negative integer or zero. If $z \in \mathbb{C}$ with $|z| < 1$, the power series

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} \tag{3.26} $$

converges. Here we are using the Pochhammer symbol $(w)_n := w(w+1)\cdots(w+n-1)$.

Proposition III.5. *There is an identity*

$$ {}_2F_1(a, b; c; z) = (1-z)^{-b} {}_2F_1\left(b, c-a; c; \frac{z}{z-1}\right), \tag{3.27} $$

valid about $z = 0$ and using the principal branch for $(1-z)^{-b}$.

Proof. This is part of Theorem 2.2.5 of [2]. □

Using this identity, we extend the domain of definition of ${}_2F_1(a, b; c; x)$ from the unit disc $\{|z| < 1\}$ to the union of the unit disc and a half-plane $\{|z| < 1\} \cup \{\mathrm{Re}(z) < \frac{1}{2}\}$. At the boundary point $z = 1$, the hypergeometric series converges when $\mathrm{Re}(c) > \mathrm{Re}(a+b)$, and its evaluation is a classical theorem of Gauss.

Proposition III.6. *If* $\mathrm{Re}(c) > \mathrm{Re}(a+b)$*, then*

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.28} $$

Proof. This is Theorem 2.2.2 of [2]. □

Of particular interest to us will be a special hypergeometric function which is a modified version of the beta function.

Definition III.7. Let $x \in \mathbb{R}$ and $a, b \in \mathbb{C}$. The *beta function* is

$$B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \tag{3.29}$$

and the *modified beta function* is

$$\tilde{B}(x; a, b) = \int_0^x t^{a-1}(1+t)^{b-1}\, dt. \tag{3.30}$$

The following proposition enumerates some properties of the modified beta function.

Proposition III.8. *Let $x \in \mathbb{R}$, and let a, b be complex numbers with $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ and $\mathrm{Re}(a+b) < 1$. Then,*

(1) $\tilde{B}(x; a, b) = B\left(\dfrac{x}{x+1}; a, 1-a-b\right)$,

(2) $\tilde{B}(x; a, b) = \dfrac{1}{a} x^a {}_2F_1(a, 1-b; a+1; -x)$,

(3) $\tilde{B}\left(\dfrac{1}{x}; a, b\right) = \dfrac{\Gamma(a)\Gamma(1-a-b)}{\Gamma(1-b)} - \tilde{B}(x; 1-a-b, b)$, *and*

(4) $\tilde{B}(+\infty; a, b) = B(1; a, 1-a-b) = \dfrac{\Gamma(a)\Gamma(1-a-b)}{\Gamma(1-b)}$.

Proof. To prove (1), we use the substitution $t = \frac{u}{1-u}$.

$$\tilde{B}(x; a, b) = \int_0^x t^{a-1}(1+t)^{b-1}\, dt \tag{3.31}$$

$$= \int_0^{\frac{x}{x+1}} \left(\frac{u}{1-u}\right)^{a-1} \left(1 + \frac{u}{1-u}\right)^{b-1} \frac{du}{(1-u)^2} \tag{3.32}$$

$$= \int_0^{\frac{x}{x+1}} u^{a-1}(1-u)^{-a-b}\, du \tag{3.33}$$

$$= B\left(\frac{x}{x+1}; a, 1-a-b\right). \tag{3.34}$$

To prove (2), expand $G(x;a,b)$ as a power series in x (up to a non-integral power).

$$\tilde{B}(x;a,b) = \int_0^x t^{a-1}(1+t)^{b-1}\,dt \tag{3.35}$$

$$= \int_0^x \sum_{n=0}^{\infty} \binom{b-1}{n} t^{n+a-1}\,dt \tag{3.36}$$

$$= \sum_{n=0}^{\infty} \binom{b-1}{n} \frac{1}{n+a} x^{n+a} \tag{3.37}$$

$$= \sum_{n=0}^{\infty} \frac{(b-n)\cdot(b-n+1)\cdots(b-1)}{n!} \cdot \frac{1}{n+a} x^{n+a} \tag{3.38}$$

$$= x^a \sum_{n=0}^{\infty} \frac{(-1)^n(1-b)\cdot(2-b)\cdots(n-b)}{n+a} \cdot \frac{x^n}{n!} \tag{3.39}$$

$$= x^a \sum_{n=0}^{\infty} \frac{(a)_n(1-b)_n}{a(a+1)_n} \cdot \frac{(-x)^n}{n!} \tag{3.40}$$

$$= \frac{1}{a} x^a {}_2F_1(a, 1-b; a+1; -x). \tag{3.41}$$

To prove (3), use the substitution $t = \frac{1}{u}$.

$$\tilde{B}\left(\frac{1}{x}; a, b\right) = \int_0^{1/x} t^{a-1}(1+t)^{b-1}\,dt \tag{3.42}$$

$$= \int_\infty^x u^{-a+1}\left(1+\frac{1}{u}\right)^{b-1}\left(-\frac{du}{u^2}\right) \tag{3.43}$$

$$= \int_x^\infty u^{-a-b}(1+u)^{b-1}\,du \tag{3.44}$$

$$= G(+\infty, 1-a-b, b) - G(x, 1-a-b, b) \tag{3.45}$$

To complete the proof of (3), we need to prove (4). Note that it follows from (4) that $\tilde{B}(+\infty, 1-a-b, b) = \frac{\Gamma(a)\Gamma(1-a-b)}{\Gamma(1-b)}$. The first equality of (4) follows from (1) with $x \to +\infty$; we will now derive the second. By (2),

$$\tilde{B}(x;a,b) = \frac{1}{a}x^a {}_2F_1(a, 1-b; a+1; -x) \tag{3.46}$$

$$= \frac{1}{a}x^a {}_2F_1(1-b, a; a+1; -x) \tag{3.47}$$

$$= \frac{1}{a}x^a \cdot (1-(-x))^{-a} {}_2F_1\left(a, (a+1)-(1-b); a+1; \frac{-x}{(-x)-1}\right) \tag{3.48}$$

$$= \frac{1}{a}\left(\frac{x}{1+x}\right)^a {}_2F_1\left(a, a+b; a+1; \frac{x}{x+1}\right) \tag{3.49}$$

Proposition III.5 was used in eq. (3.48). Sending $x \to +\infty$ and applying Proposition III.6 yields the second equality of (4). $\square$

Lemma III.9. *Let* $\lambda, \mu > 0$, *and* $\mathrm{Re}(s) > 0$. *Then*

$$\int_0^\infty \mathcal{E}(\sqrt{\lambda t}) \exp(-\mu t) t^s \frac{dt}{t} = \frac{1}{2}\pi^{-1/2}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\pi\lambda}{\mu}; \frac{1}{2}, \frac{1}{2}-s\right). \tag{3.50}$$

Proof. First of all, note that the left-hand side of Equation (3.50) converges: The integrand is $\exp(-O(t))$ as $t \to \infty$ and $O\left(t^{\mathrm{Re}\, s - \frac{1}{2}}\right)$ as $t \to 0$. Write $\mathcal{E}(\sqrt{\lambda t}) = \frac{1}{2}\int_0^{\lambda t} u^{-1/2}e^{-\pi u}du$. The left-hand side of Equation (3.50) may be rewritten, using the substitution $u = \frac{\mu t v}{\pi}$ in the inner integral, as

$$\int_0^\infty \mathcal{E}(\sqrt{\lambda t}) \exp(-\mu t) t^s \frac{dt}{t} = \frac{1}{2}\int_0^\infty \int_0^{\lambda t} u^{-1/2}e^{-(\pi u + \mu t)}t^s \, du \frac{dt}{t} \tag{3.51}$$

$$= \frac{1}{2}\int_0^\infty \int_0^{\frac{\pi\lambda}{\mu}} \left(\frac{\mu t v}{\pi}\right)^{-1/2} e^{-(\mu t v + \mu t)}t^s \, \frac{\mu t}{\pi} dv \frac{dt}{t}. \tag{3.52}$$

The double integral is absolutely convergent (indeed, the integrand is nonnegative, and we already showed convergence), so we may swap the integrals. We compute

(3.53)
$$\int_0^\infty \mathcal{E}(\sqrt{\lambda t})\exp(-\mu t)t^s\frac{dt}{t} = \frac{1}{2}\left(\frac{\mu}{\pi}\right)^{1/2}\int_0^{\frac{\pi\lambda}{\mu}} v^{-1/2}\left(\int_0^\infty e^{-\mu t(v+1)}t^{s+\frac{1}{2}}\,\frac{dt}{t}\right)dv$$

$$= \frac{1}{2}\left(\frac{\mu}{\pi}\right)^{1/2}\int_0^{\frac{\pi\lambda}{\mu}} v^{-1/2}\left(\Gamma\left(s+\frac{1}{2}\right)(\mu(v+1))^{-(s+\frac{1}{2})}\right)dv \tag{3.54}$$

$$= \frac{1}{2}\pi^{-1/2}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\int_0^{\frac{\pi\lambda}{\mu}} v^{-1/2}(v+1)^{-(s+\frac{1}{2})}\,dv \tag{3.55}$$

$$= \frac{1}{2}\pi^{-1/2}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\pi\lambda}{\mu};\frac{1}{2},\frac{1}{2}-s\right). \tag{3.56}$$

This proves Equation (3.50). □

Lemma III.10. *Let* $\nu_1, \nu_2 \in \mathbb{R}$ *and* $\mu \in \mathbb{C}$ *satisfying* $\mathrm{Re}(\mu) > -\pi\max\{\nu_1^2, \nu_2^2\}$ *if* $\mathrm{sgn}(\nu_1) = \mathrm{sgn}(\nu_2)$ *and* $\mathrm{Re}(\mu) > 0$ *otherwise. Then,*

$$\begin{aligned}&\int_0^\infty \mathcal{E}\left(\nu t^{1/2}\right)\Big|_{\nu=\nu_1}^{\nu_2}\exp(-\mu t)t^s\frac{dt}{t}\\ &= \frac{1}{2}\left(\mathrm{sgn}(\nu_2)-\mathrm{sgn}(\nu_1)\right)\Gamma(s)\mu^{-s}\\ &\quad - \frac{\mathrm{sgn}(\nu_2)}{2s}\pi^{-(s+\frac{1}{2})}\Gamma\left(s+\frac{1}{2}\right)|\nu_2|^{-2s}\,{}_2F_1\left(s, s+\frac{1}{2}, s+1; -\frac{\mu}{\pi\nu_2^2}\right)\\ &\quad + \frac{\mathrm{sgn}(\nu_1)}{2s}\pi^{-(s+\frac{1}{2})}\Gamma\left(s+\frac{1}{2}\right)|\nu_1|^{-2s}\,{}_2F_1\left(s, s+\frac{1}{2}, s+1; -\frac{\mu}{\pi\nu_1^2}\right).\end{aligned} \tag{3.57}$$

Proof. Initially, consider $\lambda, \mu > 0$, as in Lemma III.9. We have

$$\int_0^\infty \mathcal{E}\left(\sqrt{\lambda t}\right)\exp(-\mu t)t^s\frac{dt}{t}$$

$$= \frac{1}{2}\pi^{-\frac{1}{2}}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\pi\lambda}{\mu};\frac{1}{2},\frac{1}{2}-s\right) \tag{3.58}$$

$$= \frac{1}{2}\pi^{-\frac{1}{2}}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\left(\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})} - \tilde{B}\left(\frac{\mu}{\pi\lambda};s,1-s\right)\right) \tag{3.59}$$

$$= \frac{1}{2}\Gamma(s)\mu^{-s} - \frac{1}{2}\pi^{-\frac{1}{2}}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\mu}{\pi\lambda};s,1-s\right) \tag{3.60}$$

$$= \frac{1}{2}\Gamma(s)\mu^{-s} - \frac{1}{2s}\pi^{-(s+\frac{1}{2})}\Gamma\left(s+\frac{1}{2}\right)\lambda^{-s}{}_2F_1\left(s, s+\frac{1}{2}, s+1; -\frac{\mu}{\pi\lambda}\right), \tag{3.61}$$

using parts (2) and (3) of Proposition III.8. Equation (3.57) follows for positive real μ. But the integral on the left-hand side of eq. (3.57) converges for $\mathrm{Re}(\mu) > -\pi \max\{\nu_1^2, \nu_2^2\}$ if $\mathrm{sgn}(\nu_1) = \mathrm{sgn}(\nu_2)$ and $\mathrm{Re}(\mu) > 0$ otherwise, and both sides are analytic functions in μ on this domain. Thus, eq. (3.57) holds in general by analytic continuation. □

3.3.2 The series expansion

Theorem III.11. *If* $c_1, c_2 \in \mathbb{R}$, *and* $\mathrm{Re}(s) > 1$, *then the indefinite zeta function may be written as*

$$(3.62)\quad \hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s) = \pi^{-s}\Gamma(s)\zeta^{c_1,c_2}_{p,q}(\Omega, s) - \pi^{-(s+\frac{1}{2})}\Gamma\left(s + \frac{1}{2}\right)\left(\xi^{c_2}_{p,q}(\Omega, s) - \xi^{c_1}_{p,q}(\Omega, s)\right),$$

where $M = \mathrm{Im}(\Omega)$ *and*

$$(3.63)\qquad \zeta^{c_1,c_2}_{p,q}(\Omega, s) = \frac{1}{2}\sum_{n\in\mathbb{Z}^2+q}\left(\mathrm{sgn}(c_1^\top Mn) - \mathrm{sgn}(c_2^\top Mn)\right) e\left(p^\top n\right) Q_{-i\Omega}(n)^{-s},$$

$$\xi^{c}_{p,q}(\Omega, s) = \frac{1}{2}\sum_{\nu\in\mathbb{Z}^2+q}\mathrm{sgn}(c^\top Mn) e\left(p^\top n\right)\left(\frac{(c^\top Mn)^2}{Q_M(c)}\right)^{-s}$$

$$(3.64)\qquad \cdot\, {}_2F_1\left(s, s+\frac{1}{2}, s+1; \frac{2Q_M(c)Q_{-i\Omega}(n)}{(c^\top Mn)^2}\right).$$

Proof. Take the Mellin transform of the theta series term-by-term, and apply Lemma III.10. Note that the series for $\xi^c_{p,q}(\Omega, s)$ converges absolutely, so the series may be split up like this. □

The function $\zeta^{c_1,c_2}_{p,q}(\Omega, s)$ here is a sum over a double cone, with the boundaries of the cone weighted by $\frac{1}{2}$. Even ignoring the weighting, this is not generally a difference of two Shintani zeta functions. To be a Shintani zeta function (up to minor changes), the boundary of the double cone would need to contain a basis for $\mathbb{Z}^g$.

Theorem III.12. *Suppose* $(c_1, c_2, p+\Omega q, \Omega)$ *is* P*-stable. Then,* $\xi^{c_1}_{p,q}(\Omega, s) = \xi^{c_2}_{p,q}(\Omega, s)$ *and* $\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s) = \pi^{-s}\Gamma(s)\zeta^{c_1,c_2}_{p,q}(\Omega, s)$.

Proof. The equality of the $\xi_{p,q}^{c_j}(\Omega, s)$ follows by the substitution $n \mapsto Pn$ and the definition of P-stability. The equation

$$\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = \pi^{-s}\Gamma(s)\zeta_{p,q}^{c_1,c_2}(\Omega, s) \tag{3.65}$$

then follows from Theorem III.11. □

3.4 Zeta functions of ray ideal classes in real quadratic fields

In this section, we will specialize indefinite zeta functions to obtain certain zeta functions to obtain certain zeta functions attached to real quadratic fields. We define two Dirichlet series, $\zeta_A(s)$ and $Z_A(s)$, attached to a ray ideal class A of the ring of integers of a number field.

Definition III.13 (Ray class zeta function)**.** Let K be any number field and $\mathfrak{c}$ an ideal of the maximal order $\mathcal{O}_K$. Let S be a subset of the real places of K (i.e., the embeddings $K \hookrightarrow \mathbb{R}$). Let A be a ray ideal class modulo $\mathfrak{c} \cup S$, that is, an element of the group

$$\mathrm{Cl}_{\mathfrak{c}\cup S}(\mathcal{O}_K) := \frac{\{\text{nonzero fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{c}\}}{\{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}}. \tag{3.66}$$

Define the *zeta function of* A to be

$$\zeta(s, A) = \sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s}. \tag{3.67}$$

This function has a simple pole at $s = 1$ with residue independent of A. The pole may be eliminated by considering the function $Z_A(s)$, defined as follows.

Definition III.14 (Differenced ray class zeta function)**.** Let R be the element of $C_{\mathfrak{c}\cup S}$ defined by

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}. \tag{3.68}$$

Define the *differenced zeta function of A* to be

$$Z_A(s) = \zeta(s, A) - \zeta(s, RA). \tag{3.69}$$

The function $Z_A(s)$ is holomorphic at $s = 1$.

Now, specialize to the case where $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant D. Let $\mathcal{O}_K$ be the maximal order of K, and let $\mathfrak{c}$ be an ideal of $\mathcal{O}_K$. Let A be a narrow ray ideal class modulo $\mathfrak{c}$, that is, an element of the group

$$C_{\mathfrak{c}\cup\{\infty_1,\infty_2\}}(\mathcal{O}) := \frac{\{\text{ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{c}\}}{\{\text{principal ideals } a\mathcal{O}_K \text{ with } a \equiv 1 \pmod{\mathfrak{c}}\}}. \tag{3.70}$$

We show that the indefinite zeta function specializes to the L-series $Z_A(s)$ attached to a ray class of an order in a real quadratic field.

Theorem III.15. *For each $A \in C_{\mathfrak{c}\cup\{\infty_1,\infty_2\}}$ and integral ideal $\mathfrak{b} \in A^{-1}$, there exists a real symmetric 2×2 matrix M, vectors $c_1, c_2 \in \mathbb{R}^2$,and $q \in \mathbb{Q}^2$ such that*

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s)Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s). \tag{3.71}$$

Proof. The differenced zeta function $Z_A(s)$ is

$$Z_A(s) = \sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s} - \sum_{\mathfrak{a}\in RA} N(\mathfrak{a})^{-s}. \tag{3.72}$$

We have

$$N(\mathfrak{b})^{-s}Z_A(s) = \sum_{\mathfrak{a}\in A} N(\mathfrak{b}\mathfrak{a})^{-s} - \sum_{\mathfrak{a}\in RA} N(\mathfrak{b}\mathfrak{a})^{-s} \tag{3.73}$$

$$= \sum_{\substack{b\in\mathfrak{b} \\ (b)\in I \\ \text{up to units}}} N(b)^{-s} - \sum_{\substack{b\in\mathfrak{b} \\ (b)\in R \\ \text{up to units}}} N(b)^{-s}. \tag{3.74}$$

Write $\mathfrak{b}\mathfrak{c} = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$. The norm form $N(n_1\gamma_1 + n_2\gamma_2) = Q_M\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ for some real symmetric matrix M with integer coefficients. The signature of M is $(1, 1)$, just like the norm form for K. Since $\mathfrak{b}$ and $\mathfrak{c}$ are relatively prime (meaning $\mathfrak{b}+\mathfrak{c}=\mathcal{O}_K$), there exists by the Chinese remainder theorem some $b_0 \in \mathcal{O}_K$ such that $b \equiv b_0 \pmod{\mathfrak{bc}}$ if and only if $b \equiv 0 \pmod{\mathfrak{b}}$ and $b \equiv 1 \pmod{\mathfrak{c}}$. Express $b_0 = p_1\gamma_1 + p_2\gamma_2$ for rational numbers p_1, p_2, and set $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$.

Let ε_0 be the fundamental unit of $\mathcal{O}_K$, and let ε ($=\varepsilon_0^k$ for some k) be the smallest totally positive unit of $\mathcal{O}_K$ greater than 1 such that $\varepsilon \equiv 1 \pmod{\mathfrak{c}}$.

Choose any $c_1 \in \mathbb{R}^2$ such that $Q_M(c_1) < 0$. Let P be the matrix describing the linear action of ε on $\mathfrak{b}$ by multiplication, i.e., $\varepsilon(\beta^\top n) = \beta^\top(Pn)$. Set $c_2 = Pc_1$.

Thus, we have

$$N(\beta)^{-s} Z_A(s) = \frac{1}{2} \sum_{n \in \mathbb{Z}^2 + q} \left(\operatorname{sgn}(c_2^\top M n) - \operatorname{sgn}(c_1^\top M n)\right) Q_M(n). \tag{3.75}$$

Moreover, (c_1, c_2, p, Ω) is P-stable. So, by Theorem III.12, eq. (3.75) may be rewritten as

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s) Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s), \tag{3.76}$$

completing the proof. □

3.5 Example

Let $K = \mathbb{Q}(\sqrt{3})$, so $\mathbf{O}_K = \mathbb{Z}[\sqrt{3}]$, and let $\mathfrak{c} = 5\mathbf{O}_K$. This example was introduced in Chapter I, and the ray class group $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}} \cong \mathbb{Z}/8\mathbb{Z}$. The fundamental unit $\varepsilon = 2+\sqrt{3}$ is totally positive: $\varepsilon\varepsilon' = 1$. It has order 3 modulo 5: $\varepsilon^3 = 26 + 15\sqrt{3} \equiv 1 \pmod 5$. In this section, we use the analytic continuation eq. (3.25) for indefinite zeta functions to compute $Z_I'(0)$, where I is the principal ray class of $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}}$.

By definition, $Z_I = \zeta(s, I) - \zeta(s, R)$ where

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at } \infty_2\} \tag{3.77}$$

$$= \{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ is negative at } \infty_2\}. \tag{3.78}$$

Write $I = I_+ \sqcup I_-$ and $R = R_+ \sqcup R_-$, where $I_\pm$ and $R_\pm$ are the following ray ideal classes in $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_1,\infty_2\}}$:

$$I_\pm = \{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ has sign } \pm \text{ at } \infty_1 \text{ and } + \text{ at } \infty_2\}, \tag{3.79}$$

$$R_\pm = \{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ has sign } \pm \text{ at } \infty_1 \text{ and } - \text{ at } \infty_2\}. \tag{3.80}$$

Thus, $Z_I(s) = \zeta(s, I_+) + \zeta(s, I_-) - \zeta(s, R_+) - \zeta(s, R_-)$. The Galois automorphism $(a_1 + a_2\sqrt{3})^\sigma = (a_1 - a_2\sqrt{3})$ defines a norm-preserving bijection between I_- and R_+, so the middle terms cancel and

$$Z_I(s) = \zeta(s, I_+) - \zeta(s, R_-) = Z_{I_+}(s). \tag{3.81}$$

To the principal ray class I_+ of $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_1,\infty_2\}}$, we associate $\Omega = iM$ where $M = \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}$ and $q = \begin{pmatrix} 1/5 \\ 0 \end{pmatrix}$. We may choose $c_1 \in \mathbb{R}^2$ arbitrarily so long as $c_1^\top M c_1 < 0$; take $c_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The left action of ε on $\mathbb{Z} + \sqrt{3}\mathbb{Z}$ is given by the matrix $P = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. By Theorem III.15,

$$(2\pi)^{-s}\Gamma(s) Z_{I_+}(s) = \hat{\zeta}_{0,q}^{c_1, P^3 c_1}(iM, s). \tag{3.82}$$

Taking a limit as $s \to 0$, and using eq. (3.81), eq. (3.82) becomes

$$Z_I'(0) = Z_{I_+}'(s) = \hat{\zeta}_{0,q}^{c_1, P^3 c_1}(iM, 0). \tag{3.83}$$

For the purpose of making the numerical computation more efficient, we split up the right-hand side as

$$Z_I'(0) = \hat{\zeta}_{0,q}^{c_1, Pc_1}(iM, 0) + \hat{\zeta}_{0,q}^{Pc_1, P^2c_1}(iM, 0) + \hat{\zeta}_{0,q}^{P^2c_1, P^3c_1}(iM, 0) \tag{3.84}$$

$$= \hat{\zeta}_{0,q_0}^{c_1, Pc_1}(iM, 0) + \hat{\zeta}_{0,q_1}^{c_1, Pc_1}(iM, 0) + \hat{\zeta}_{0,q_2}^{c_1, Pc_1}(iM, 0), \tag{3.85}$$

where $q_0 = q = \frac{1}{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q_1 = q = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $q_2 = q = \frac{1}{5}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are obtained from the residues of $\varepsilon^0, \varepsilon^1, \varepsilon^2$ modulo 5.

Using eq. (3.25), we computed $Z'_I(0)$ to 100 decimal digits. The decimal begins

$$Z'_I(0) = 1.358630653392208162595113082300\ldots. \tag{3.86}$$

The conjectural Stark unit is $\exp(Z'_I(0)) = 3.890861713943079255337643959 62\ldots$. We used the `RootApproximant[]` function in Mathematica, which is uses lattice basis reduction internally, to find a degree 16 integer polynomial having this number as a root, and we factored that polynomial over $\mathbb{Q}(\sqrt{3})$. To 100 digits, $\exp(Z'_I(0))$ is equivalent to be the root of the polynomial

$$\begin{aligned} &x^8 - (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 + (225+130\sqrt{3})x^4 \\ &- (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 - (8+5\sqrt{3})x + 1. \end{aligned} \tag{3.87}$$

We have verified that this root generates the expected class field H_2, as discussed in chapter I.

We have also computed $Z'_I(0)$ a different way in PARI/GP, using its internal algorithms for computing Hecke L-values. We obtained the same numerical answer this way.

CHAPTER IV

Kronecker Limit Formulas

The goal of this chapter is to prove a Kronecker limit formula for the indefinite zeta function is dimension $g = 2$, that is, a formula for $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, 1)$ or $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, 0)$. Specifically, we will give such a formula at $s = 1$ when $q = 0$. (By the functional equation, one can then obtain a formula at $s = 0$ when $p = 0$.) We retain the term "Kronecker limit formula" for historical value, even though there is no pole to be removed and so no limit is being taken.

First, we work through a proof of the classical "second" Kronecker limit formula for real analytic Eisenstein series. In the process of doing so, we generalize the classical second limit formula to a limit formula for definite zeta functions. The method of proof is to compute the Fourier series in a single real variable ξ for a definite theta function with respect to an action by a one-parameter unipotent subgroup $\{T^\xi\}$ of $\mathbf{SL}_2(\mathbb{R})$. We then take the Mellin transform term-by-term and send $s \to 1$.

We will use complex logarithms throughout this chapter. If $f(\tau)$ is any nonvanishing holomorphic function on the upper half plane $\mathfrak{H}$, there is some holomorphic function $(\operatorname{Log} f)(\tau)$ such that $\exp((\operatorname{Log} f)(\tau)) = f(\tau)$, because $\mathfrak{H}$ is simply connected. Specifying a single value (or the limit as τ approaches some element of $\mathbb{R} \cup \{\infty\}$) specifies $\operatorname{Log} f$ uniquely. It won't necessarily be true that $(\operatorname{Log} f)(\tau) = \log(f(\tau))$.

4.1 Statement of results

Theorem IV.1 (Generalized second Kronecker limit formula at $s = 1$)**.** *Let* $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ *with* $0 \le p_1, p_2 < 1$, *and let* $\Omega = N + iM \in \mathfrak{H}_2^{(0)}$. *Let* $z = \tau_1$ *and* $z = \tau_2$ *be the solutions of* $Q_\Omega \begin{pmatrix} z \\ 1 \end{pmatrix} = 0$ *in the upper and lower half-planes, respectively. Then,*

$$\hat{\zeta}_{p,0}(\Omega, 1) = \frac{-1}{\sqrt{\det(-i\Omega)}} \left(\left(\operatorname{Log} f_p\right)(\tau_1) + \left(\operatorname{Log} f_p\right)(-\tau_2) \right), \tag{4.1}$$

where the function $f_p : \mathfrak{H} \to \mathbb{C}$ *may be written either of the following ways,*

$$f_p(\tau) = e\left(-\frac{p_2}{2}\right) u_\tau^{p_1^2/2+1/12} \left(v_\tau^{1/2} - v_\tau^{-1/2}\right) \prod_{d=1}^{\infty} \left(1 - u_\tau^d v_\tau\right)\left(1 - u_\tau^d v_\tau^{-1}\right) \tag{4.2}$$

$$= \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right) \vartheta_{\frac{1}{2}+p_2, \frac{1}{2}-p_1}(\tau)}{\eta(\tau)}, \tag{4.3}$$

where $u_\tau = e(\tau)$, $v_\tau = e(p_2 - p_1\tau)$, ϑ *is the Jacobi theta function, and* η *is the Dedekind eta function. Here* $\operatorname{Log} f_p$ *is the branch satisfying*

$$(\operatorname{Log} f_p)(\tau) \sim \pi i \left(p_1^2 - p_1 + \frac{1}{6}\right) \tau \text{ as } \tau \to i\infty. \tag{4.4}$$

Theorem IV.2 (Generalized second Kronecker limit formula at $s = 0$)**.** *Let* $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ *with* $0 \le q_1, q_2 < 1$, *and let* $\Omega = N + iM \in \mathfrak{H}_2^{(0)}$. *Let* $z = \tau_1$ *and* $z = \tau_2$ *be the solutions of* $Q_\Omega \begin{pmatrix} z \\ 1 \end{pmatrix} = 0$ *in the upper and lower half-planes, respectively. Then,*

$$\hat{\zeta}_{p,0}(\Omega, 1) = -\left(\left(\operatorname{Log} g_q\right)(\tau_1) + \left(\operatorname{Log} g_q\right)(-\tau_2) \right), \tag{4.5}$$

where the function $g_q : \mathfrak{H} \to \mathbb{C}$ *is given by*

$$g_q(\tau) = \frac{\vartheta_{\frac{1}{2}-q_1, \frac{3}{2}-q_2}(\tau)}{\eta(\tau)}. \tag{4.6}$$

Our formulas in the definite case specialize to the classical Kronecker second limit formula, which we state here at $s = 1$ and $s = 0$.

Proposition IV.3. *Let* $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2 \backslash \mathbb{Z}^2$ *and* $\Omega = iM = \frac{i}{\mathrm{Im}(\tau)} \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$ *for* $\tau \in \mathfrak{H}$. *Then,*

$$\hat{\zeta}_{p,0}(\Omega, 1) = -2 \log \left| \frac{\vartheta_{\frac{1}{2}+p_1, \frac{1}{2}-p_2}(\tau)}{\eta(\tau)} \right|. \tag{4.7}$$

Proposition IV.4. *Let* $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{R}^2$ *and* $\Omega = iM = \frac{i}{\mathrm{Im}(\tau)} \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$ *for* $\tau \in \mathfrak{H}$. *Then,*

$$\hat{\zeta}_{0,q}(\Omega, 0) = -2 \log \left| \frac{\vartheta_{\frac{1}{2}-q_1, \frac{3}{2}-q_2}(\tau)}{\eta(\tau)} \right|. \tag{4.8}$$

The method of proof in the indefinite case is the same—compute the Fourier series in ξ for an indefinite theta function with respect to an action by a one-parameter unipotent subgroup $\{T^\xi\}$ of $\mathbf{SL}_2(\mathbb{R})$. However, we take Mellin transforms and specialize some variables earlier in the calculation than in the definite case. Unlike in the definite case, the Fourier coefficients of the indefinite theta are not elementary functions. The final formula involves an integral.

The Kronecker limit formula at $s = 1$ for indefinite zeta functions requires the following definition of the function $\kappa_\Omega^c(v)$, which is the square root of a rational function and will appear as a factor in the integrand.

Definition IV.5. Suppose $\Omega = M + iN \in \mathfrak{H}_2^{(1)}$, $c \in \mathbb{C}^2$ satisfying $Q_M(c) < 0$, $v \in \mathbb{C}^2$, and $s \in \mathbb{C}$. Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)} M c c^\top M$. Then, we define

$$\kappa_\Omega^c(v) = \frac{c^\top M v}{4\pi i \sqrt{-Q_M(c)} Q_\Omega(v) \sqrt{-2iQ_{\Lambda_c}(v)}}. \tag{4.9}$$

We now state the formula.

Theorem IV.6 (Indefinite Kronecker limit formula)**.** *Let* $\Omega = N + iM =\in \mathfrak{H}_2^{(1)}$, $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, *and* $c_1, c_2 \in \mathbb{C}^2$ *such that* $\overline{c_j}^\top \operatorname{Im}\Omega c_j < 0$. *For* $c = c_1, c_2$, *factor the quadratic form*

$$Q_{\Lambda_c}\begin{pmatrix} \xi \\ 1 \end{pmatrix} = \alpha(c)(\xi - \tau_1(c))(\xi - \tau_2(c)), \tag{4.10}$$

where $\tau^+(c)$ *is in the upper half-plane and* $\tau^-(c)$ *is in the lower half-plane. Then,*

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = I^+(c_2) - I^-(c_2) - I^+(c_1) + I^-(c_1), \tag{4.11}$$

where

$$I^\pm(c) = -\operatorname{Li}_2(e(\pm p_1))\kappa_\Omega^c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.12}$$

$$+ 2i \int_0^\infty \left(\operatorname{Log}\varphi_{p_1,\pm p_2}\right)(\pm\tau^\pm(c) + it)\kappa_\Omega^c \begin{pmatrix} \pm\left(\tau^\pm(c) + it\right) \\ 1 \end{pmatrix} dt. \tag{4.13}$$

The function $\varphi_{p_1,p_2} : \mathfrak{H} \to \mathbb{C}$ *is defined by the a product expansion,*

$$\varphi_{p_1,p_2}(\xi) := (1 - e(p_1\xi_t + p_2)) \prod_{d=1}^\infty \frac{1 - e\left((d + p_1)\xi + p_2\right)}{1 - e\left((d - p_1)\xi - p_2\right)}, \tag{4.14}$$

and its logarithm $\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)$ *is the unique continuous branch with the property*

$$\lim_{\xi \to i\infty} \left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi) = \begin{cases} \log(1 - e(p_2)) & \text{if } p_1 = 0, \\ 0 & \text{if } p_1 \neq 0. \end{cases} \tag{4.15}$$

Here $\log(1 - e(p_2))$ *is the standard principal branch.*

The following specialization looks somewhat simpler and contains all of the cases of arithmetic zeta functions $Z_A(s)$ associated to real quadratic fields.

Theorem IV.7 (Indefinite Kronecker limit formula, pure imaginary case)**.** *Let* M *be a* 2×2 *real matrix of signature* $(1, 1)$, *and let* $\Omega = iM$. *Let* $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$, *and* $c_1, c_2 \in \mathbb{R}^2$ *such that* $c_j^\top M c_j < 0$. *Then,*

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = 2i \operatorname{Im}\left(I(c_2) - I(c_1)\right), \tag{4.16}$$

where

$$I(c) = -\operatorname{Li}_2(e(p_1))\kappa_\Omega^c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.17}$$

$$+\, 2i \int_0^\infty (\operatorname{Log} \varphi_{p_1,p_2})\,(\tau(c)+it)\kappa_\Omega^c \begin{pmatrix} \tau(c)+it \\ 1 \end{pmatrix} dt. \tag{4.18}$$

Here, $\operatorname{Log} \varphi_{p_1,p_2}$ *and* κ_Ω^c *are defined as in the statement of Theorem IV.6, and* $\xi = \tau(c)$ *is the unique root of the quadratic polynomial* $Q_{\Lambda_c}\begin{pmatrix} \xi \\ 1 \end{pmatrix}$ *in the upper half plane.*

4.2 Kronecker limit formulas for definite zeta functions

In this section, we'll find a formula for $\hat{\zeta}_{p,0}(\Omega, 1)$ in terms of logarithms of modular forms. This formula will specialize to the classical second Kronecker limit formula.

4.2.1 Fourier series of a unipotent transform of a definite theta function

Consider the definite (Riemann) theta function in dimension $g = 2$, for $z \in \mathbb{C}^2$ and $\Omega \in \mathfrak{H}_2^{(0)}$;

$$\Theta(z, \Omega) = \sum_{n \in \mathbb{Z}^2} e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right). \tag{4.19}$$

Let $T^\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ for $\xi \in \mathbb{R}$, and fix z and Ω. In this section, we will calculate the Fourier expansion of the function

$$g(\xi) = \Theta(\left(T^\xi\right)^\top z; \left(T^\xi\right)^\top \Omega T^\xi) = \sum_{n \in \mathbb{Z}^2} e\left(\frac{1}{2}\left(T^\xi n\right)^\top \Omega \left(T^\xi n\right) + \left(T^\xi n\right)^\top z\right). \tag{4.20}$$

We have $g(\xi + 1) = g(\xi)$ by an integral change of basis on $\mathbb{Z}^2$. We will compute the Fourier coefficients

$$a_k = \int_0^1 g(\xi) e(-k\xi)\, d\xi. \tag{4.21}$$

Proposition IV.8. *The Fourier coefficeints of $g(\xi)$ are given by the following formulas. If $k \neq 0$,*

(4.22)
$$a_k = \frac{e\left(\frac{2k\omega_{12}-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}} \sum_{\substack{d\in\mathbb{Z} \\ d|k}} e\left(\frac{1}{2\omega_{11}}\left((\det\Omega)d^2 + 2(\omega_{11}z_2 - \omega_{12}z_1)d + 2z_1\frac{k}{d} - \frac{k^2}{d^2}\right)\right).$$

For $k = 0$, and using $\vartheta(z,\omega)$ to denote the Jacobi theta function, we have

$$a_0 = \vartheta(z_1,\omega_{11}) + \frac{e\left(\frac{-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}}\left(\vartheta\left(\frac{\omega_{11}z_2 - \omega_{12}z_1}{\omega_{11}}, \frac{\det\Omega}{\omega_{11}}\right) - 1\right). \tag{4.23}$$

Proof. Express $\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix}$, $n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Write $g(\xi) = \sum_{n_2=-\infty}^{\infty} g_{n_2}(\xi)$, where $g_j(\xi)$ is the sum over the terms with $n_2 = j$. First, calculate $g_0(\xi)$:

$$g_0(\xi) = \sum_{n_1=-\infty}^{\infty} e\left(\frac{1}{2}\omega_{11}n_1^2 + n_1z_1\right) \tag{4.24}$$

$$= \vartheta(z_1,\omega_{11}), \tag{4.25}$$

where ϑ is the Jacobi theta function. We may write $g_{n_2}(\xi)e(-k\xi)$ as

$$\begin{aligned}
&g_{n_2}(\xi)e(-k\xi) \\
&= \sum_{n_1=-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}(n_1+n_2\xi)^2 + \omega_{12}n_2(n_1+n_2\xi)\right. \\
&\qquad\qquad \left. + \frac{\omega_{22}}{2}n_2^2 + (n_1+n_2\xi)z_1 + n_2z_2 - k\xi\right) && (4.26) \\
&= \sum_{n_1=-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}(n_1+n_2\xi)^2 + \left(\omega_{12}n_2 + z_1 - \frac{k}{n_2}\right)(n_1+n_2\xi)\right. \\
&\qquad\qquad \left. + \left(\frac{\omega_{22}}{2}n_2^2 + n_2z_2 + \frac{kn_1}{n_2}\right)\right) && (4.27) \\
&= \sum_{n_1=-\infty}^{\infty} b_{n_1,n_2} e\left(\frac{\omega_{11}}{2}\left((n_1+n_2\xi) + \frac{\omega_{12}n_2 + z_1 - k/n_2}{\omega_{11}}\right)^2\right), && (4.28)
\end{aligned}$$

where $b_{n_1,n_2} = e\left(\left(\frac{\omega_{22}}{2}n_2^2 + n_2z_2 + \frac{kn_1}{n_2}\right) - \frac{(\omega_{12}n_2+z_1-k/n_2)^2}{2\omega_{11}}\right)$. Note that $b_{n_1+n_2,n_2} =$

b_{n_1,n_2}, and, moreover,

(4.29)
$$g_{n_2}(\xi)e(-k\xi) = \sum_{n_1=0}^{n_2-1} b_{n_1,n_2} \sum_{j=-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}\left((n_1+n_2(\xi+j)) + \frac{\omega_{12}n_2+z_1-k/n_2}{\omega_{11}}\right)^2\right).$$

Thus,

$$\int_0^1 g_{n_2}(\xi)e(-k\xi)\,d\xi$$
$$= \sum_{n_1=0}^{n_2-1} b_{n_1,n_2}\int_{-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}\left((n_1+n_2\xi)+\frac{\omega_{12}n_2+z_1-k/n_2}{\omega_{11}}\right)^2\right)d\xi \tag{4.30}$$
$$= \sum_{n_1=0}^{n_2-1}\frac{b_{n_1,n_2}}{\sqrt{-i\omega_{11}n_2^2}}, \tag{4.31}$$

by Corollary II.6.

(4.32)
$$\int_0^1 g_{n_2}(\xi)e(-k\xi)\,d\xi = \frac{1}{\sqrt{-i\omega_{11}}\,|n_2|}\sum_{n_1=0}^{n_2-1} b_{n_1,n_2}$$
$$= \frac{e\left(\left(\frac{\omega_{22}}{2}n_2^2+n_2z_2\right)-\frac{(\omega_{12}n_2+z_1-k/n_2)^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}\,|n_2|}\sum_{n_1=0}^{n_2-1} e\left(\frac{kn_1}{n_2}\right) \tag{4.33}$$
$$= \begin{cases} \frac{1}{\sqrt{-i\omega_{11}}}e\left(\left(\frac{\omega_{22}}{2}n_2^2+n_2z_2\right)-\frac{(\omega_{12}n_2+z_1-k/n_2)^2}{2\omega_{11}}\right), & \text{if } n_2|k;\\ 0, & \text{else.}\end{cases} \tag{4.34}$$

Thus, for $k \neq 0$, we have $\int_0^1 \vartheta(z_1,\omega_{11})e(-k\xi)\,d\xi = 0$ and

$$a_k = \sum_{\substack{d\in\mathbb{Z}\\ d|k}} \frac{1}{\sqrt{-i\omega_{11}}} e\left(\left(\frac{\omega_{22}}{2}d^2+dz_2\right)-\frac{(\omega_{12}d+z_1-k/d)^2}{2\omega_{11}}\right) \tag{4.35}$$
$$= \frac{e\left(\frac{2k\omega_{12}-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}}\sum_{\substack{d\in\mathbb{Z}\\ d|k}} e\left(\frac{1}{2\omega_{11}}\left((\det\Omega)d^2+2(\omega_{11}z_2-\omega_{12}z_1)d\right.\right.$$
$$\left.\left.+2z_1\frac{k}{d}-\frac{k^2}{d^2}\right)\right). \tag{4.36}$$

For $k=0$, we have $\int_0^1 \vartheta(z_1,\omega_{11})e(-k\xi)\,d\xi = \vartheta(z_1,\omega_{11})$ and

$$a_0 = \vartheta(z_1,\omega_{11}) + \sum_{d\in\mathbb{Z}\setminus\{0\}} \frac{1}{\sqrt{-i\omega_{11}}} e\left(\left(\frac{\omega_{22}}{2}d^2 + dz_2\right) - \frac{(\omega_{12}d+z_1)^2}{2\omega_{11}}\right) \tag{4.37}$$

$$= \vartheta(z_1,\omega_{11}) + \frac{e\left(\frac{-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}} \sum_{d\in\mathbb{Z}\setminus\{0\}} e\left(\frac{\det\Omega}{\omega_{11}}d^2 + \frac{\omega_{11}z_2-\omega_{12}z_1}{\omega_{11}}d\right) \tag{4.38}$$

$$= \vartheta(z_1,\omega_{11}) + \frac{e\left(\frac{-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}} \left(\vartheta\left(\frac{\omega_{11}z_2-\omega_{12}z_1}{\omega_{11}}, \frac{\det\Omega}{\omega_{11}}\right) - 1\right). \tag{4.39}$$

This completes the proof of the proposition. □

From now on, we will use the notation $\sum_{d|k}$ in place of $\sum_{\substack{d\in\mathbb{Z}\\ d|k}}$. This is nonstandard—we are summing over all integral divisors of k, not just positive divisors. A sum over the divisors of 0 is a sum over all integers.

Use the definite theta with characteristics to define a function of $\xi, t \in \mathbb{R}$,

$$h(\xi,t) := \Theta_{(T^\xi)^\top p, T^{-\xi}q}(t(T^\xi)^\top \Omega T^\xi) \tag{4.40}$$

$$= e\left(\frac{t}{2}q^\top \Omega q + p^\top q\right)\Theta\left((T^\xi)^\top(p+t\Omega q), t(T^\xi)^\top\Omega T^\xi\right). \tag{4.41}$$

Write this function as a Fourier series,

$$h(\xi,t) = \sum_{k=-\infty}^{\infty} b_k(t)e(k\xi). \tag{4.42}$$

The Fourier coefficients of $h(\xi,t)$ are given by the following corollary.

Corollary IV.9. *If $k\neq 0$, then*

$$\begin{aligned} b_k(t) = \frac{t^{-1/2}}{\sqrt{-i\omega_{11}}}\sum_{d|k} e\Bigg(&\frac{(\det\Omega)(q_2+d)^2}{2\omega_{11}}t \\ &+ \frac{(\omega_{11}p_2 - \omega_{12}(p_1-k/d))(q_2+d)+\omega_{11}q_1k/d}{\omega_{11}} - \frac{(p_1-k/d)^2}{2\omega_{11}}t^{-1}\Bigg). \end{aligned} \tag{4.43}$$

For $k=0$, we have

(4.44)
$$b_0(t) = \sum_{n=-\infty}^{\infty} e\left(\left(\frac{\omega_{11}}{2}(q_1+n)^2 + \omega_{12}(q_1+n)q_2 + \frac{\omega_{22}}{2}q_2^2\right)t + (p_1q_1+p_2q_2+p_1n)\right)$$
$$+ \frac{t^{-1/2}}{\sqrt{-i\omega_{11}}} \sum_{d\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\det\Omega)(q_2+d)^2}{2\omega_{11}}t + \frac{(\omega_{11}p_2-\omega_{12}p_1)(q_2+d)}{\omega_{11}} - \frac{p_1^2}{2\omega_{11}}t^{-1}\right).$$

Proof. Follows from Proposition IV.8. □

4.2.2 Taking the Mellin transform term-by-term

Next, we will shift our focus from theta functions to zeta functions. We will need to take a Mellin transform term-by-term in an infinite sum, and, to do this, we will need an absolute convergence result. First, we need the following technical inequality.

Lemma IV.10. *Let* $\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix} \in \mathfrak{H}_2^{(0)}$. *Then*

$$\operatorname{Im}\left(\frac{-1}{\omega_{11}}\right)\operatorname{Im}\left(\frac{\det\Omega}{\omega_{11}}\right) > \left(\operatorname{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)^2. \tag{4.45}$$

Proof. Express Ω in terms of its real and imaginary parts,

$$\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix} + i\begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}. \tag{4.46}$$

Note that $m_{11} \neq 0$ because $m_{11}m_{22} - m_{12}^2 = \det M > 0$, and thus $\omega_{11} \neq 0$. By an algebraic calculation,

$$\operatorname{Im}\left(\frac{-1}{\omega_{11}}\right)\operatorname{Im}\left(\frac{\det\Omega}{\omega_{11}}\right) - \left(\operatorname{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)^2 = \frac{m_{11}m_{22}-m_{12}^2}{n_{11}^2+m_{11}^2}. \tag{4.47}$$

Now, $m_{11}m_{22} - m_{12}^2 = \det M$ is positive, and so is $n_{11}^2 + m_{11}^2$. Thus, the inequality eq. (4.45) holds. □

Here is another inequality that we will need later.

Lemma IV.11. *Let* $\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix} \in \mathfrak{H}_2^{(0)}$. *The two roots of* $Q_\Omega \begin{pmatrix} z \\ 1 \end{pmatrix} = 0$ *are* $\tau_1 = \frac{-\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}$ *and* $\tau_2 = \frac{-\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}$. *Then,* $\mathrm{Im}(\tau_1) > 0 > \mathrm{Im}(\tau_2)$.

Proof. We have $Q_\Omega \begin{pmatrix} z \\ 1 \end{pmatrix} = \omega_{11}z^2 + 2\omega_{12}z + \omega_{22}$, and the expressions for the roots come from the quadratic formula.

For any complex numbers $\alpha = a_1+ia_2$ and $\beta = b_1+ib_2$, $(\mathrm{Im}(\alpha\beta))^2 - \mathrm{Im}(\alpha^2)\,\mathrm{Im}(\beta^2) = (a_1b_2 - a_2b_1)^2 \geq 0$. Thus, $(\mathrm{Im}(\alpha\beta))^2 \geq \mathrm{Im}(\alpha^2)\,\mathrm{Im}(\beta^2)$.

In particular, taking $\alpha = \frac{1}{\sqrt{-\omega_{11}}}$ and $\beta = \frac{\sqrt{\det(-i\Omega)}}{\sqrt{-\omega_{11}}}$ (for any choice of $\sqrt{-\omega_{11}}$), we obtain the inequality

$$\left(\mathrm{Im}\left(\frac{\sqrt{\det(-i\Omega)}}{\omega_{11}}\right)\right)^2 \geq \mathrm{Im}\left(\frac{-1}{\omega_{11}}\right)\mathrm{Im}\left(\frac{\det(-i\Omega)}{-\omega_{11}}\right) \tag{4.48}$$

$$= \mathrm{Im}\left(\frac{-1}{\omega_{11}}\right)\mathrm{Im}\left(\frac{\det(\Omega)}{\omega_{11}}\right). \tag{4.49}$$

Appealing to Lemma IV.10, we see by transitivity that

$$\left(\mathrm{Im}\left(\frac{\sqrt{\det(-i\Omega)}}{\omega_{11}}\right)\right)^2 > \left(\mathrm{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)^2. \tag{4.50}$$

By subtracting the left-hand side and factoring, this inequality may be rewritten as $0 > \mathrm{Im}(\tau_1)\,\mathrm{Im}(\tau_2)$. So $\mathrm{Im}(\tau_1)$ and $\mathrm{Im}(\tau_2)$ are always nonzero real numbers with opposite signs. In the special case $\Omega = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $\tau_1 = i$ and $\tau_2 = -i$. Since $\mathfrak{H}_2^{(0)}$ is connected, we always have $\mathrm{Im}(\tau_1) > 0 > \mathrm{Im}(\tau_2)$. ☐

We will encounter Bessel functions in both our absolute convergence argument and our calculation of the Fourier coefficients in the next section. Let K_s denote the K-Bessel function. That is, for $\mathrm{Re}(\alpha) > 0$,

$$K_s(\alpha) := \frac{1}{2}\int_0^\infty \exp\left(-\frac{\alpha}{2}\left(t + t^{-1}\right)\right) t^s \,\frac{dt}{t}. \tag{4.51}$$

This function satisfies the identities $K_s(\alpha) = K_{-s}(\alpha)$ and $K_{\frac{1}{2}}(\alpha) = \sqrt{\frac{\pi}{2\alpha}}e^{-\alpha}$. It also has exponential decay in α; specifically, $|K_s(\alpha)| = o(\exp(-\alpha))$ as $\mathrm{Re}(\alpha) \to \infty$. (See p. 66 of [10] and chapter 4 of [2].)

We can use the Bessel function to write certain integrals in a more compact form.

Lemma IV.12. *Suppose $a, b \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$. Taking the standard branch of all power functions with a branch cut along the negative real axis,*

$$\int_0^\infty \exp\left(-\left(at + bt^{-1}\right)\right) t^s \,\frac{dt}{t} = 2(b/a)^{s/2} K_s(2\sqrt{ab}). \tag{4.52}$$

Proof. Substitute $t = \sqrt{\frac{b}{a}}u$.

$$\begin{aligned}
&\int_0^\infty \exp\left(-\left(at + bt^{-1}\right)\right) t^s \,\frac{dt}{t} \\
&= (b/a)^{s/2} \lim_{N\to\infty} \int_0^{\sqrt{\frac{b}{a}}N} \exp\left(-\sqrt{ab}\left(u + u^{-1}\right)\right) u^s \,\frac{du}{u} && (4.53) \\
&= (b/a)^{s/2} \lim_{N\to\infty} \left(\int_0^{\sqrt{\left|\frac{b}{a}\right|}N} + \int_{\sqrt{\left|\frac{b}{a}\right|}N}^{\sqrt{\frac{b}{a}}N}\right) \exp\left(-\sqrt{ab}\left(u + u^{-1}\right)\right) u^s \,\frac{du}{u} && (4.54) \\
&= (b/a)^{s/2} \lim_{N\to\infty} \int_0^{\sqrt{\left|\frac{b}{a}\right|}N} \exp\left(-\sqrt{ab}\left(u + u^{-1}\right)\right) u^s \,\frac{du}{u} && (4.55) \\
&= 2(b/a)^{s/2} K_s(2\sqrt{ab}). && (4.56)
\end{aligned}$$

In eq. (4.54), we used the bound

$$\left|\int_{\sqrt{\left|\frac{b}{a}\right|}N}^{\sqrt{\frac{b}{a}}N} \exp\left(-\sqrt{ab}\left(u + u^{-1}\right)\right) u^s \,\frac{du}{u}\right| \leq \exp(-bN)\mathrm{poly}(N) \to 0 \tag{4.57}$$

as $N \to \infty$, where $\mathrm{poly}(N)$ denotes some polynomial. □

Now, we'll prove an absolute convergence result that will allow us to reverse the order of summation/integration.

Proposition IV.13. *For any $\sigma \in \mathbb{R}$,*

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} < \infty. \tag{4.58}$$

Proof. We bound $b_k(t)$ by

$$|b_k(t)| \le |\omega_{11}|^{-\frac{1}{2}} t^{-\frac{1}{2}} \sum_{d|k} \exp\left(-\pi \left(\mathrm{Im}\left(\frac{\det \Omega}{\omega_{11}}\right)(d+q_2)^2 t\right.\right. \tag{4.59}$$

$$\left.\left. +2\,\mathrm{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)(k/d-p_1)(d+q_2) + \mathrm{Im}\left(\frac{-1}{\omega_{11}}\right)(k/d-p_1)^2 t^{-1}\right)\right). \tag{4.60}$$

Thus, we have (for some polynomial function $p(k)$)

$$\int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} \le p(k) \exp\left(2\pi\,\mathrm{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)(k/d-p_1)(d+p_2)\right)$$

$$\cdot K_\sigma\left(2\pi \sqrt{\mathrm{Im}\left(\frac{-1}{\omega_{11}}\right)\mathrm{Im}\left(\frac{\det\Omega}{\omega_{11}}\right)}\,|(k/d-p_1)(d+p_2)|\right) \tag{4.61}$$

$$\le p(k) \exp\left(-2\pi\left(\sqrt{\mathrm{Im}\left(\frac{-1}{\omega_{11}}\right)\mathrm{Im}\left(\frac{\det\Omega}{\omega_{11}}\right)} \pm \mathrm{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)\right.$$

$$\left. |(k/d-p_1)(d+p_2)|\right). \tag{4.62}$$

In the second line, we used the fact that, as $\alpha \to \infty$, $K_\sigma(\alpha) = o(\exp(-\alpha))$. Now, by Lemma IV.10, there is a constant $\varepsilon > 0$ so that

$$\int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} \le \exp\left(\varepsilon\, |(k/d-p_1)(d+p_2)|\right). \tag{4.63}$$

Thus,

$$\sum_{k=-\infty}^{\infty} \int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} \le \sum_{d_1 \ne 0} \sum_{d_2 \ne 0} \exp\left(\varepsilon\, |(d_1-p_1)(d_2+p_2)|\right) < \infty. \tag{4.64}$$

This completes the proof of the proposition. $\square$

Now we may compute the Fourier series in ξ for $\zeta_{(T^\xi)^\top p, T^{-\xi} q}\left((T^\xi)^\top \Omega T^\xi, s\right)$.

Proposition IV.14. *The Fourier coefficients $\beta_k(s)$ of $\zeta_{(T^\xi)^\top p, T^{-\xi} q}\left((T^\xi)^\top \Omega T^\xi, s\right)$ are* *given by the following formulas. If $k \neq 0$, then*

$$\beta_k(s) = \frac{2}{\sqrt{-i\omega_{11}}} \sum_{d|k} e\left(\frac{(\omega_{11}p_2 - \omega_{12}(p_1 - k/d))(q_2 + d) + \omega_{11}q_1 k/d}{\omega_{11}}\right)$$

$$\cdot (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \left|\frac{p_1 - k/d}{q_2 + d}\right|^{s-\frac{1}{2}}$$

$$\cdot K_{s-\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1 - k/d)(q_2 + d)|\right). \tag{4.65}$$

For $k = 0$,

$$\int_0^\infty b_0(t) t^s \frac{dt}{t}$$

$$= (2\pi)^{-s}\Gamma(s) \sum_{n=-\infty}^{\infty} e(p_1 q_1 + p_2 q_2 + p_1 n) Q_\Omega \begin{pmatrix} q_1 + n \\ q_2 \end{pmatrix}^{-s} \tag{4.66}$$

$$+ \frac{2}{\sqrt{-i\omega_{11}}} \sum_{d \in \mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2 - \omega_{12}p_1)(q_2 + d)}{\omega_{11}}\right) (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \tag{4.67}$$

$$\cdot \left|\frac{p_1}{q_2 + d}\right|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|p_1(q_2 + d)|\right). \tag{4.68}$$

Proof. It follows from Proposition IV.13 that we may take the Mellin transform term-by-term.

$$\zeta_{(T^\xi)^\top p, T^{-\xi} q}\left((T^\xi)^\top \Omega T^\xi, s\right) = \int_0^\infty h(\xi, t) t^s \frac{dt}{t} = \sum_{k=-\infty}^{\infty} \left(\int_0^\infty b_k(t) t^s \frac{dt}{t}\right) e(k\xi). \tag{4.69}$$

The formulas follow by Lemma IV.12. □

4.2.3 Proof of the Kronecker limit formulas

We will need a standard result on the values of the polylogarithm $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} k^{-s} z^k$ at positive integers $s = n$.

Proposition IV.15. *Suppose $n \in \mathbb{Z}$, $n \geq 1$, $x \in \mathbb{R}$, and $0 \leq x \leq 1$. Then,*

$$\mathrm{Li}_n(e(x)) + (-1)^n \mathrm{Li}_n(e(-x)) = -\frac{(2\pi i)^n}{n!} B_n(x), \tag{4.70}$$

where $B_{2n}(x)$ is the $(2n)$th Bernoulli polynomial.

Proof. A proof may be found in [2]. □

We will only need this result at $s = 2$.

Corollary IV.16. *If $x \in \mathbb{R}$, and $\{x\}$ denotes the fractional part of x, then*

$$\mathrm{Li}_2(e(x)) + \mathrm{Li}_2(e(-x)) = 2\pi^2 \left(\{x\}^2 - \{x\} + \frac{1}{6} \right). \tag{4.71}$$

Proof. Plug $n = 1$ into Proposition IV.15. □

Using the change of variables $(d_1, d_2) = (\frac{n}{d}, d)$, we have

$$\begin{aligned}
&\hat{\zeta}_{p,q}(\Omega, s) \\
&= (2\pi)^{-s}\Gamma(s) \sum_{n=-\infty}^{\infty} e(p_1q_1 + p_2q_2 + p_1n) Q_{-i\Omega} \begin{pmatrix} q_1 + n \\ q_2 \end{pmatrix}^{-s} \\
&\quad + \frac{2}{\sqrt{-i\omega_{11}}} \sum_{k\in\mathbb{Z}} \sum_{\substack{d|k \\ d\neq 0}} e\left(\frac{(\omega_{11}p_2 - \omega_{12}(p_1 - k/d))(q_2 + d) + \omega_{11}q_1k/d}{\omega_{11}} \right) \\
&\qquad\qquad \cdot (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \left| \frac{p_1 - k/d}{q_2 + d} \right|^{s-\frac{1}{2}} \\
&\qquad\qquad \cdot K_{s-\frac{1}{2}} \left(\frac{2\pi i}{\omega_{11}} \sqrt{\det(-i\Omega)}\, |(p_1 - k/d)(q_2 + d)| \right) \qquad (4.72) \\
&= (2\pi)^{-s}\Gamma(s) \sum_{n=-\infty}^{\infty} e(p_1q_1 + p_2q_2 + p_1n) Q_{-i\Omega} \begin{pmatrix} q_1 + n \\ q_2 \end{pmatrix}^{-s} \\
&\quad + \frac{2}{\sqrt{-i\omega_{11}}} \sum_{d_1\in\mathbb{Z}} \sum_{d_2\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2 - \omega_{12}(p_1 - d_1))(q_2 + d_2) + \omega_{11}q_1d_1}{\omega_{11}} \right) \\
&\qquad\qquad \cdot (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \left| \frac{p_1 - d_1}{q_2 + d_2} \right|^{s-\frac{1}{2}} \\
&\qquad\qquad \cdot K_{s-\frac{1}{2}} \left(\frac{2\pi i}{\omega_{11}} \sqrt{\det(-i\Omega)}\, |(p_1 - d_1)(q_2 + d_2)| \right). \qquad (4.73)
\end{aligned}$$

Specialize to the case when $s=1$, $q=\begin{pmatrix}0\\0\end{pmatrix}$, and $p_1p_2\neq 0$. Now,

$$\begin{aligned}
\hat{\zeta}_{p,0}(\Omega,1) &= (2\pi)^{-1}\Gamma(1)\sum_{n\in\mathbb{Z}\setminus\{0\}} e(p_1n)\left(\frac{-i\omega_{11}}{2}n^2\right)^{-1}\\
&\quad+\frac{2}{\sqrt{-i\omega_{11}}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right)(\det(-i\Omega))^{-\frac{1}{4}}\\
&\qquad\cdot\left|\frac{p_1-d_1}{d_2}\right|^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right) \qquad (4.74)\\
&= \frac{1}{-\pi i\omega_{11}}\left(\mathrm{Li}_2(e(p_1))-\mathrm{Li}_2(e(-p_1))\right)\\
&\quad+\frac{2}{\sqrt{-i\omega_{11}}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right)(\det(-i\Omega))^{-\frac{1}{4}}\\
&\qquad\cdot\left|\frac{p_1-d_1}{d_2}\right|^{\frac{1}{2}}\left(\frac{\sqrt{-i\omega_{11}}}{2}\det(-i\Omega)^{-1/4}\,|(p_1-d_1)d_2|^{-1/2}\right)\\
&\qquad\cdot e\left(\frac{-1}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right) \qquad (4.75)\\
&= \frac{1}{-\pi i\omega_{11}}2\pi^2\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right)\\
&\quad+\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}}\frac{1}{|d_2|}e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right.\\
&\qquad\left.-\frac{1}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right) \qquad (4.76)\\
&= \frac{2\pi}{-i\omega_{11}}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right)\\
&\quad+\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}}\frac{1}{|d_2|}e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right.\\
&\qquad\left.-\frac{1}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right). \qquad (4.77)
\end{aligned}$$

Split the series up into four pieces.

$$\begin{aligned}
\hat{\zeta}_{p,0}(\Omega,1) &= \frac{2\pi}{-i\omega_{11}}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right)\\
&\quad+\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\sum_{d_2>0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2}\\
&\quad+\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\sum_{d_2>0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2}\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\sum_{d_2<0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2}\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\sum_{d_2<0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2} \qquad (4.78)\\
&= \frac{2\pi}{-i\omega_{11}}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\log\left(1-e\left(p_2+\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\log\left(1-e\left(p_2+\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\log\left(1-e\left(-p_2-\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\log\left(1-e\left(-p_2-\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right) \qquad (4.79)
\end{aligned}$$

Let $\tau_1 = \frac{-\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}$ and $\tau_2 = \frac{-\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}$, so that $Q_\Omega\begin{pmatrix} z \\ 1\end{pmatrix} = \frac{\omega_{11}}{2}(z-\tau_1)(z-\tau_2)$. Then $\frac{\tau_1-\tau_2}{2i} = \frac{\sqrt{\det(-i\Omega)}}{-i\omega_{11}}$, and

(4.80)

$$\begin{aligned}
-\sqrt{\det(-i\Omega)}\hat{\zeta}_{p,0}(\Omega,1) &= -2\pi\frac{\tau_1-\tau_2}{2i}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right)\\
&+\sum_{d_1>p_1}\log\left(1-e\left(p_2-\tau_2(d_1-p_1)\right)\right)+\sum_{d_1<p_1}\log\left(1-e\left(p_2-\tau_1(d_1-p_1)\right)\right)\\
&+\sum_{d_1>p_1}\log\left(1-e\left(-p_2+\tau_1(d_1-p_1)\right)\right)+\sum_{d_1<p_1}\log\left(1-e\left(-p_2+\tau_2(d_1-p_1)\right)\right).
\end{aligned}$$

Assume that $0 \le p_1 < 1$. The first term may be rewritten as

$$-2\pi\frac{\tau_1-\tau_2}{2i}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right) = \log\left(e(-p_2/2)e(\tau_1)^{p_1^2/2+1/12}e(p_2-p_1\tau_1)^{1/2}\right)$$
$$+\log\left(e(-p_2/2)e(-\tau_2)^{p_1^2/2+1/12}e(p_2+p_1\tau_2)^{1/2}\right) \tag{4.81}$$

So, the whole thing can be written

$$-\sqrt{\det(-i\Omega)}\hat{\zeta}_{p,0}(\Omega,1) = \left(\mathrm{Log}\, f_{p_1,p_2}\right)(\tau_1) + \left(\mathrm{Log}\, f_{p_1,p_2}\right)(-\tau_2), \tag{4.82}$$

where

$$f_{p_1,p_2}(\tau) = e(-p_2/2)e(\tau)^{p_1^2/2+1/12}\left(e(p_2-p_1\tau)^{1/2}-e(p_2-p_1\tau)^{-1/2}\right) \tag{4.83}$$
$$\cdot\prod_{d=1}^{\infty}\left(1-e(\tau)^d e\left(p_2-p_1\tau\right)\right)\left(1-e(\tau)^d e\left(p_2-p_1\tau\right)^{-1}\right), \tag{4.84}$$

and $(\mathrm{Log}\, f_{p_1,p_2})(\tau)$ is its unique holomorphic function on $\mathfrak{H}$ with the properties that $\exp\left((\mathrm{Log}\, f_{p_1,p_2})(\tau)\right) = f_{p_1,p_2}(\tau)$ and

$$(\mathrm{Log}\, f_p)(\tau) \sim \pi i\left(p_1^2-p_1+\frac{1}{6}\right)\tau \text{ as } \tau\to i\infty. \tag{4.85}$$

This proves the first part of Theorem IV.1.

Now rewrite $f_{p_1,p_2}(\tau)$ as a ϑ-function. The Jacobi triple product identity says,

Theorem IV.17. *For $z, w \in C$, $|z|<1$, $w \neq 0$, the following identity holds:*

$$\prod_{d=1}^{\infty}\left(1-z^{2d}\right)\left(1-wz^{2d-1}\right)\left(1-w^{-1}z^{2d-1}\right) = \sum_{n=-\infty}^{\infty}(-1)^n w^n z^{n^2}. \tag{4.86}$$

Proof. See Theorem 10.4.1 of [2]. □

Proposition IV.18. *If $0 \le p_1, p_2 < 1$ and $\tau \in \mathfrak{H}$, then*

$$f_{p_1,p_2}(\tau) = \frac{e\left(\left(p_1-\frac{1}{2}\right)\left(p_2+\frac{1}{2}\right)\right)\vartheta_{\frac{1}{2}+p_2,\frac{1}{2}-p_1}(\tau)}{\eta(\tau)}. \tag{4.87}$$

Proof. Let $u = e(\tau)$ and $v = e(p_2 - p_1\tau)$, and rewrite this formula as

$$f_{p_1,p_2}(\tau) = e\left(-\frac{p_2}{2}\right) u^{p_1^2/2+1/12} \left(v^{1/2} - v^{-1/2}\right) \prod_{d=1}^{\infty} \left(1 - u^d v\right)\left(1 - u^d v^{-1}\right). \tag{4.88}$$

Now, use the Jacobi triple product identity to rewrite the product as a sum.

$$\left(v^{1/2} - v^{-1/2}\right) \prod_{d=1}^{\infty} \left(1 - u^d v\right)\left(1 - u^d v^{-1}\right)$$

$$= \frac{v^{\frac{1}{2}}}{\prod_{d=1}^{\infty}(1-u^d)} \prod_{d=1}^{\infty} (1 - (u^{\frac{1}{2}})^{2d})(1 - (u^{\frac{1}{2}})^{2d-1}(u^{\frac{1}{2}}v))(1 - (u^{\frac{1}{2}})^{2d-1}(u^{\frac{1}{2}}v)^{-1}) \tag{4.89}$$

$$= \frac{v^{1/2}u^{1/24}}{\eta(\tau)} \sum_{n=-\infty}^{\infty} (-1)^n u^{n^2/2+n/2} v^n, \tag{4.90}$$

using Theorem IV.17 in the last line. Thus,

$$f_{p_1,p_2}(\tau) = e\left(-\frac{p_2}{2}\right) \frac{v^{1/2}u^{p_1^2/2+1/8}}{\eta(\tau)} \sum_{n=-\infty}^{\infty} u^{n^2/2+n/2} v^n \tag{4.91}$$

$$= \frac{e\left(-\frac{p_2}{2}\right)}{\eta(\tau)} \sum_{n=-\infty}^{\infty} u^{n^2/2+n/2+p_1^2/2+1/8} v^{n+1/2}. \tag{4.92}$$

We have

$$\begin{aligned}
&(-1)^n u^{n^2/2+n/2+p_1^2/2+1/8} v^{n+1/2} \\
&= e\left(\left(\frac{n^2}{2} + \frac{n}{2} + \frac{p_1^2}{2} + \frac{1}{8}\right)\tau + \left(n + \frac{1}{2}\right)(p_2 - p_1\tau) + \frac{n}{2}\right) \\
&= e\left(\left(n^2 - 2\left(p_1 - \frac{1}{2}\right)n + p_1^2 - p_1 + \frac{1}{4}\right)\frac{\tau}{2} + \left(n + \frac{1}{2}\right)p_2 + \frac{n}{2}\right) \\
&= e\left(\left(n - \left(p_1 - \frac{1}{2}\right)\right)^2 \frac{\tau}{2} + \left(n + \frac{1}{2}\right)p_2 + \frac{n}{2}\right)
\end{aligned}$$

$$= e\left(p_1p_2 + \frac{p_1}{2} - \frac{1}{4}\right) e\left(\left(n - p_1 + \frac{1}{2}\right)^2 \frac{\tau}{2} + \left(n - p_1 + \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right). \tag{4.93}$$

Thus,

$$\begin{aligned}
f_{p_1,p_2}(\tau) = \frac{e\left(p_1p_2 + \frac{p_1}{2} - \frac{p_2}{2} - \frac{1}{4}\right)}{\eta(\tau)} \sum_{n=-\infty}^{\infty} e\Bigg(&\left(n - p_1 + \frac{1}{2}\right)^2 \frac{\tau}{2} \\
&+ \left(n - p_1 + \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\Bigg)
\end{aligned} \tag{4.94}$$

$$= \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right) \vartheta_{\frac{1}{2}+p_2, \frac{1}{2}-p_1}(\tau)}{\eta(\tau)}, \tag{4.95}$$

completing the proof of the proposition. $\square$

This completes the proof of Theorem IV.1.

Now we will prove Theorem IV.2, the Kronecker limit formula at $s=0$. If we set $\begin{pmatrix} \tilde{\omega}_{11} & \tilde{\omega}_{12} \\ \tilde{\omega}_{12} & \tilde{\omega}_{22} \end{pmatrix} := -\Omega^{-1} = \frac{1}{\det(-i\Omega)} \begin{pmatrix} \omega_{22} & -\omega_{12} \\ -\omega_{12} & \omega_{11} \end{pmatrix}$, $\tilde{\tau}_1 = \frac{-\tilde{\omega}_{12} - \sqrt{\det(i\Omega^{-1})}}{\tilde{\omega}_{11}}$, and $\tilde{\tau}_2 = \frac{-\tilde{\omega}_{12} + \sqrt{\det(i\Omega^{-1})}}{\tilde{\omega}_{11}}$, it is easy to show that $\tilde{\tau}_1 = -1/\tau_1$ and $\tilde{\tau}_2 = -1/\tau_2$. Moreover,

(4.96)
$$f_{p_1,p_2}(-1/\tau) = \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right) \vartheta_{\frac{1}{2}+p_2, \frac{1}{2}-p_1}(-1/\tau)}{\eta(-1/\tau)}$$

$$= \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right) e\left(\left(\frac{1}{2} + p_2\right)\left(\frac{1}{2} - p_1\right)\right) \sqrt{-i\tau}\vartheta_{p_1 - \frac{1}{2}, p_2 + \frac{1}{2}}(\tau)}{\sqrt{-i\tau}\eta(\tau)} \tag{4.97}$$

$$= \frac{\vartheta_{p_1 - \frac{1}{2}, p_2 + \frac{1}{2}}(\tau)}{\eta(\tau)}. \tag{4.98}$$

Thus, using the functional equation for the definite zeta function,

$$\hat{\zeta}_{0,q}(\Omega, 0) = \frac{1}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-q,0}(-\Omega^{-1}, 1) \tag{4.99}$$

$$= \frac{-1}{\sqrt{\det(-i\Omega)}\sqrt{\det(i\Omega^{-1})}} \left(\left(\operatorname{Log} f_{1-q_1,1-q_2}\right)(-1/\tau_1)\right.$$

$$\left. + \left(\operatorname{Log} f_{1-q_1,1-q_2}\right)(1/\tau_2)\right) \tag{4.100}$$

$$= -\left(\left(\operatorname{Log} g_{q_1,q_2}\right)(\tau_1) + \left(\operatorname{Log} g_{q_1,q_2}\right)(-\tau_2)\right), \tag{4.101}$$

where $g_{q_1,q_2}(\tau) = \frac{\vartheta_{\frac{1}{2}-q_1, \frac{3}{2}-q_2}(\tau)}{\eta(\tau)}$. This completes the proof of Theorem IV.2.

Proposition IV.3 and Proposition IV.4 follows by specialization of the variables.

4.3 Kronecker limit formulas for indefinite zeta functions

In this section, we'll find a formula for $\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1)$ in terms of the dilogarithm function and a rapidly convergent integral.

Let $c_1, c_2 \in \mathbb{C}^2$ satisfying $\overline{c_j}^\top M c_j < 0$, and consider the indefinite theta $\hat{\Theta}_{p,q}^{c_1,c_2}$ with characteristics $p, q \in \mathbb{R}^2$, introduced in Chapter II. Let $t > 0$, $\Omega \in \mathfrak{H}_2^{(1)}$, and

$M = \mathrm{Im}(\Omega)$. Write the indefinite theta of $t\Omega$ as

$$\hat{\Theta}_{p,q}^{c_1,c_2}(t\Omega) = \sum_{n\in\mathbb{Z}^2} \rho_{\mathrm{Im}(t\Omega)}^{c_1,c_2}\,(n+q)\,e\left(Q_\Omega(n+q)t + p^\top(n+q)\right) \tag{4.102}$$

$$= \sum_{n\in\mathbb{Z}^2} \rho_M^{c_1,c_2}\left((n+q)t^{1/2}\right)e\left(Q_\Omega(n+q)t + p^\top(n+q)\right), \tag{4.103}$$

where

$$\rho_M^{c_1,c_2}(v) = \mathcal{E}\left(\frac{c_2^\top Mv}{\sqrt{-\frac{1}{2}c_2^\top Mc_2}}\right) - \mathcal{E}\left(\frac{c_1^\top Mv}{\sqrt{-\frac{1}{2}c_1^\top Mc_1}}\right), \tag{4.104}$$

and

$$\mathcal{E}(z) = \int_0^z e^{-\pi u^2}\,du. \tag{4.105}$$

4.3.1 Some integrals involving $\mathcal{E}(u)$

We will now prove a few integral formulas that we will need.

Lemma IV.19. *Suppose that $\alpha, \beta \in \mathbb{C}$ satisfy $\mathrm{Re}\,(\alpha^2 - 2i\beta) > 0$. Then, using the standard branch of the square root function,*

$$\int_0^\infty \mathcal{E}(\alpha t^{1/2})e(\beta t)\,dt = \frac{-\alpha}{4\pi i\beta\sqrt{\alpha^2 - 2i\beta}}. \tag{4.106}$$

Proof. By integration by parts,

$$\int_0^\infty \mathcal{E}(\alpha t^{1/2})e(\beta t)\,dt = \frac{1}{2\pi i\beta}\int_0^\infty \mathcal{E}(\alpha t^{1/2})\,\frac{d\left(e(\beta t)\right)}{dt}\,dt \tag{4.107}$$

$$= \frac{1}{2\pi i\beta}\left(\mathcal{E}(\alpha t^{1/2})e(\beta t)\big|_{t=0}^\infty - \int_0^\infty e^{-\pi\alpha^2 t}\frac{\alpha}{2}t^{-1/2}e(\beta t)\right) \tag{4.108}$$

$$= \frac{-\alpha}{4\pi i\beta}\int_0^\infty \exp\left(-\left(\pi\alpha - 2\pi i\beta\right)t\right)t^{1/2}\,\frac{dt}{t} \tag{4.109}$$

$$= \frac{-\alpha}{4\pi i\beta}\int_C \exp(-u)\left(\frac{u}{\pi\alpha^2 - 2\pi i\beta}\right)^{1/2}\frac{du}{u} \tag{4.110}$$

$$= \frac{-\alpha}{4\pi^{3/2}i\beta\sqrt{\alpha^2 - 2i\beta}}\int_C e^{-u}u^{1/2}\,\frac{du}{u}, \tag{4.111}$$

where the contour C is a ray from the origin through the point $\alpha^2 - 2i\beta$. If $z \in \mathbb{C}$ with $x = \mathrm{Re}(z) > 0$, $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 0$, and $[z_1, z_2]$ denotes the oriented line segment from z_1 to z_2, then

$$\lim_{N\to\infty} \int_{[0,Nz]} e^{-u} u^s \frac{du}{u} = \lim_{N\to\infty} \left(\int_{[0,Nx]} e^{-u} u^s \frac{du}{u} + \int_{[Nx,Nz]} e^{-u} u^s \frac{du}{u} \right) \tag{4.112}$$

$$= \Gamma(s) + \lim_{N\to\infty} \int_{[Nx,Nz]} e^{-u} u^s \frac{du}{u} \tag{4.113}$$

$$= \Gamma(s) + \lim_{N\to\infty} O\left(e^{-Nx} N^{\sigma}\right) \tag{4.114}$$

$$= \Gamma(s). \tag{4.115}$$

Thus, in particular, $\int_C e^{-u} u^{1/2} \frac{du}{u} = \Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$. Plugging this into eq. (4.111) gives eq. (4.106). □

As usual, let $M = \mathrm{Im}(\Omega)$. Define the following auxiliary function, which will appear as a factor in the integral in the indefinite Kronecker limit formula.

Definition IV.20. For $v \in \mathbb{C}^2$ and $s \in \mathbb{C}$, set

$$\kappa_\Omega^c(v, s) := -\int_0^\infty \rho_M^c\left(vt^{1/2}\right) e\left(Q_\Omega(v)t\right) t^s \frac{dt}{t}. \tag{4.116}$$

Also, set

$$\kappa_\Omega^{c_1,c_2}(v, s) := \kappa_\Omega^{c_2}(v, s) - \kappa_\Omega^{c_1}(v, s) \tag{4.117}$$

$$= \int_0^\infty \rho_M^{c_1,c_2}\left(vt^{1/2}\right) e\left(Q_\Omega(v)t\right) t^s \frac{dt}{t}. \tag{4.118}$$

In the case $s = 1$, we will leave out s and set $\kappa_\Omega^c(v) := \kappa_\Omega^c(v, 1)$, $\kappa_\Omega^{c_1,c_2}(v) := \kappa_\Omega^{c_1,c_2}(v, 1)$.

In particular,

Corollary IV.21. *Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)} M c c^\top M$. Note that $\Lambda_c \in \mathfrak{H}_2^{(0)}$ by Lemma II.18. Then,*

$$\kappa_\Omega^c(v) = \frac{c^\top M v}{4\pi i \sqrt{-Q_M(c)} Q_\Omega(v) \sqrt{-2i Q_{\Lambda_c}(v)}}. \tag{4.119}$$

Proof. Follows from Lemma IV.19. □

The following lemma will be needed to evaluate certain integrals.

Lemma IV.22. *For any real number $\alpha \in \mathbb{R}$,*

$$\int_0^\infty \rho_M^{c_1,c_2}\left(v\alpha t^{1/2}\right) e\left(Q_\Omega(v)\alpha^2 t\right) t^s \frac{dt}{t} = -\frac{\operatorname{sgn}(\alpha)}{|\alpha|^{2s}} \kappa_\Omega^{c_1,c_2}(v,s). \tag{4.120}$$

Proof. Follows from the definition of $\kappa_\Omega^{c_1,c_2}(v,s)$. □

4.3.2 Fourier series of a unipotent transform of an indefinite theta function

Consider the function of $\xi \in \mathbb{R}$ (although ξ will be allowed to be complex later on) and $t \in \mathbb{R}_{\geq 0}$,

$$h(\xi,t) := \hat{\Theta}^{T^{-\xi}c_1, T^{-\xi}c_2}_{\left(T^\xi\right)^\top p, T^{-\xi}q}\left(t\left(T^\xi\right)^\top \Omega T^\xi\right) \tag{4.121}$$

$$= \sum_{n\in\mathbb{Z}^2} \rho_\Omega^{c_1,c_2}\left(\left(T^\xi n + q\right)t^{1/2}\right) e\left(Q_\Omega(T^\xi n + q)t + p^\top(T^\xi n + q)\right). \tag{4.122}$$

Write this function as a Fourier series,

$$h(\xi,t) = \sum_{k=-\infty}^{\infty} b_k(t) e(k\xi). \tag{4.123}$$

We are ultimately interested in the Mellin transform of this function,

$$\hat{\zeta}^{T^{-\xi}c_1, T^{-\xi}c_2}_{\left(T^\xi\right)^\top p, T^{-\xi}q}\left(\left(T^\xi\right)^\top \Omega T^\xi, s\right) = \int_0^\infty h(\xi,t) t^s \frac{dt}{t} \tag{4.124}$$

$$= \sum_{k=-\infty}^{\infty} \beta_k(s) e(k\xi), \tag{4.125}$$

where, as we will show,

$$\beta_k(s) := \int_0^\infty b_k(t) t^s \frac{dt}{t}. \tag{4.126}$$

Express $\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix}$, $n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$, $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. Write $h(\xi,t) = \sum_{n_2=-\infty}^{\infty} h_{n_2}(\xi,t) = h_0(\xi,t) + \tilde{h}(\xi,t)$, where $h_j(\xi,t)$ is the sum over the terms with $n_2 = j$, and $\tilde{h}(\xi,t)$ is the sum over all the terms where $n_2 \neq 0$.

Also, assume that $q_1 = q_2 = 0$.

First, calculate $h_0(\xi, t)$:

$$h_0(\xi,t) = \sum_{n_1=-\infty}^{\infty} \rho_\Omega^{c_1,c_2}\begin{pmatrix} n_1 t^{1/2} \\ 0 \end{pmatrix} e\left(\frac{1}{2}\omega_{11} n_1^2 t + p_1 n_1\right). \tag{4.127}$$

The $n_1 = 0$ term of this sum vanishes.

We write, for $n_2 \neq 0$,

$$\begin{aligned}
&\int_0^1 h_{n_2}(\xi,t) e(-k\xi)\, d\xi \\
&= \int_0^1 \sum_{n_1=-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} n_1 + n_2\xi \\ n_2 \end{pmatrix} t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} n_1 + n_2\xi \\ n_2 \end{pmatrix} t + p^\top \begin{pmatrix} n_1 + n_2\xi \\ n_2 \end{pmatrix}\right) e(-k\xi)\, d\xi && (4.128)\\
&= \sum_{n_1=0}^{n_2-1} \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} n_1 + n_2\xi \\ n_2 \end{pmatrix} t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} n_1 + n_2\xi \\ n_2 \end{pmatrix} t + p^\top \begin{pmatrix} n_1 + n_2\xi \\ n_2 \end{pmatrix}\right) e(-k\xi)\, d\xi && (4.129)\\
&= \sum_{n_1=0}^{n_2-1} \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} n_2\xi \\ n_2 \end{pmatrix} t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} n_2\xi \\ n_2 \end{pmatrix} t + p^\top \begin{pmatrix} n_2\xi \\ n_2 \end{pmatrix}\right) e\left(-k\left(\xi - \frac{n_1}{n_2}\right)\right) d\xi && (4.130)\\
&= \left(\sum_{n_1=0}^{n_2-1} e\left(\frac{kn_1}{n_2}\right)\right) \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2\right) e\left(-k\xi\right)\, d\xi. && (4.131)
\end{aligned}$$

The exponential sum $\displaystyle\sum_{n_1=0}^{n_2-1} e\left(\frac{kn_1}{n_2}\right)$ evaluates to $|n_2|$ if $n_2 | k$, and to 0 otherwise.

Thus, for all $k \in \mathbb{Z}$ (including $k = 0$),

(4.132)
$$\int_0^1 \tilde{h}(\xi,t)e(-k\xi)\,d\xi$$
$$= \sum_{n_2|k} |n_2| \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix}\xi\\1\end{pmatrix} n_2 t^{1/2}\right) e\left(Q_\Omega\begin{pmatrix}\xi\\1\end{pmatrix} n_2^2 t + p^\top \begin{pmatrix}\xi\\1\end{pmatrix} n_2\right) e\left(-k\xi\right)\,d\xi.$$

Recall that, by our convention, a sum over $n_2|k$ ranges over both positive and negative n_2 (and over all integers when $k = 0$).

4.3.3 Shifting the contour vertically

Fix a positive real number λ to be specified later. Let C^+ (C^-) be the contour consisting of the horizontal line $\mathrm{Im}(z) = \lambda$ ($\mathrm{Im}(z) = -\lambda$), oriented towards the right half-plane. For each $d_1, d_2 \in \mathbb{Z}$, $d_2 \neq 0$, let $C(d_1,d_2)$ be C^+ if $d_1 d_2 > 0$ or $d_1 = 0$ and $d_2 > 0$; let $C(d_1,d_2)$ be C^- if $d_1 d_2 < 0$ or $d_1 = 0$ and $d_2 < 0$. The integrands in eq. (4.132) approach zero as $\mathrm{Re}(\xi) \to \pm\infty$, so we may rewrite this formula using contour integrals

(4.133)
$$\int_0^1 \tilde{h}(\xi,t)e(-k\xi)\,d\xi$$
$$= \sum_{n_2|k} |n_2| \int_{C\left(\frac{k}{n_2},n_2\right)} \rho_M^{c_1,c_2}\left(\begin{pmatrix}\xi\\1\end{pmatrix} n_2 t^{\frac{1}{2}}\right) e\left(Q_\Omega\begin{pmatrix}\xi\\1\end{pmatrix} n_2^2 t + p^\top \begin{pmatrix}\xi\\1\end{pmatrix} n_2\right) e\left(-k\xi\right)\,d\xi.$$

4.3.4 Taking Mellin transforms term-by-term

To calculate the Mellin transform of $h_0(\xi,t)$, we need to check absolute convergence to justify reversing the order of summation/integration.

Proposition IV.23. *If* $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, *then*

$$\int_0^\infty \sum_{n_1=-\infty}^{\infty} \left| \rho_\Omega^{c_1,c_2}\begin{pmatrix} n_1 t^{1/2} \\ 0 \end{pmatrix} e\left(\frac{1}{2}\omega_{11} n_1^2 t + p_1 n_1\right)\right| t^\sigma \,\frac{dt}{t} < \infty. \tag{4.134}$$

Proof. We bound the integral as follows.

$$\int_0^\infty \sum_{n_1=-\infty}^{\infty} \left| \rho_\Omega^{c_1,c_2} \begin{pmatrix} n_1 t^{1/2} \\ 0 \end{pmatrix} e\left(\frac{1}{2}\omega_{11} n_1^2 t + p_1 n_1\right) \right| t^\sigma \frac{dt}{t} \tag{4.135}$$

$$= \int_0^\infty \sum_{n_1=-\infty}^{\infty} \left| \rho_\Omega^{c_1,c_2} \begin{pmatrix} t^{1/2} \\ 0 \end{pmatrix} e\left(\frac{1}{2}\omega_{11} t\right) \right| \left(\frac{t}{n_1^2}\right)^\sigma \frac{dt}{t} \tag{4.136}$$

$$= \left(\sum_{n_1=-\infty}^{\infty} |n_1|^{-2\sigma}\right) \left(\int_0^\infty \left| \rho_\Omega^{c_1,c_2} \begin{pmatrix} t^{1/2} \\ 0 \end{pmatrix} e\left(\frac{1}{2}\omega_{11} t\right) \right| t^\sigma \frac{dt}{t}\right) \tag{4.137}$$

$$< \infty. \tag{4.138}$$

The sum converges for $\sigma > \frac{1}{2}$, and the integral converges for $\sigma > 0$ (as the integrand approaches a constant at $t \to 0$ and decays exponentially as $t \to \infty$). $\square$

Therefore, we can switch the sum and the integral, and by Lemma IV.19 and dropping the subscript on n_1,

$$\int_0^\infty h_0(\xi, t) t^s \frac{dt}{t} = - \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\operatorname{sgn}(n) e(p_1 n)}{|n|^{2s}} \kappa_\Omega^{c_1,c_2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, s \right) \tag{4.139}$$

$$= - \left(\operatorname{Li}_{2s}(e(p_1)) - \operatorname{Li}_{2s}(e(-p_1))\right) \kappa_\Omega^{c_1,c_2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, s \right). \tag{4.140}$$

Next, we're going to calculate the Mellin transform of $\tilde{h}(\xi, t)$. We need an absolute convergence result to justifty our calculation here, too.

Proposition IV.24. *Suppose $\sigma = \operatorname{Re}(s) > \frac{1}{2}$. Then,*

$$\sum_{k \in \mathbb{Z}} \sum_{\substack{n_2 | k \\ n_2 \neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2} \left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2} \right) \right.$$

$$\left. \cdot e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 \right) e\left(-k\xi\right) t^s \right| \frac{dt}{t} d\xi < \infty. \tag{4.141}$$

Proof. Let

$$K^\pm = \int_0^\infty \int_{C^\pm} \left| \rho_M^{c_1,c_2} \left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} t^{1/2} \right) e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} t \right) \right| t^\sigma \, d\xi \frac{dt}{t} \tag{4.142}$$

$$< \infty. \tag{4.143}$$

Set $K = \max\{K^+, K^-\}$. We have

$$\sum_{k\in\mathbb{Z}} \sum_{\substack{n_2|k \\ n_2\neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2} \left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2} \right) \cdot e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 \right) e\left(-k\xi\right) t^s \right| \frac{dt}{t} d\xi$$

$$= \sum_{k\in\mathbb{Z}} \sum_{\substack{n_2|k \\ n_2\neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2} \left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2} \right) e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t \right) \right| \cdot e^{-2\pi\lambda k} t^\sigma \frac{dt}{t} d\xi \tag{4.144}$$

$$= \sum_{k\in\mathbb{Z}} \sum_{\substack{n_2|k \\ n_2\neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2} \left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} t^{1/2} \right) e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} t \right) \right| \cdot e^{-2\pi\lambda k} \left(\frac{t}{n_2^2} \right)^\sigma \frac{dt}{t} d\xi \tag{4.145}$$

$$\leq K \sum_{k\in\mathbb{Z}} \sum_{\substack{n_2|k \\ n_2\neq 0}} e^{-2\pi\lambda k} n_2^{-2\sigma} \tag{4.146}$$

$$= K \sum_{d_1\in\mathbb{Z}} \sum_{d_2\in\mathbb{Z}\setminus\{0\}} e^{-2\pi\lambda|d_1 d_2|} d_2^{-2\sigma} \tag{4.147}$$

$$< \infty. \tag{4.148}$$

The proposition is proved. $\square$

Now we may justify taking the Mellin transform of the Fourier series term-by-term. It follows from Proposition IV.24 that

$$\hat{\zeta}^{T^{-\xi}c_1, T^{-\xi}c_2}_{\left(T^\xi\right)^\top p, 0} \left(\left(T^\xi\right)^\top \Omega T^\xi, s \right) = \int_0^\infty h(\xi,t) t^s \frac{dt}{t} \tag{4.149}$$

$$= \sum_{k=-\infty}^{\infty} \beta_k(s) e(k\xi), \tag{4.150}$$

where $\beta_k(s) := \int_0^\infty b_k(t) t^s \frac{dt}{t}$. Define $\tilde{\beta}_k(s) := \int_0^\infty \tilde{b}_k(t) t^s \frac{dt}{t}$; then,

(4.151)
$$\beta_k(s) = \begin{cases} -\left(\mathrm{Li}_{2s}(e(p_1)) - \mathrm{Li}_{2s}(e(-p_1))\right) \kappa_\Omega^{c_1,c_2}\left(\begin{pmatrix}1\\0\end{pmatrix}, s\right) + \tilde{\beta}_0(s) & \text{if } k = 0, \\ \tilde{\beta}_k(s) & \text{if } k \neq 0. \end{cases}$$

Proposition IV.24 also implies that we can switch the order of integration to compute

(4.152)
$$\tilde{\beta}_k(s) = \int_0^\infty \int_0^1 \tilde{h}(\xi, t) e(-k\xi)\, d\xi\, t^s \frac{dt}{t}$$

$$= \sum_{n_2 | k} |n_2| \int_{C\left(\frac{k}{n_2}, n_2\right)} e\left(n_2 p^\top \begin{pmatrix}\xi\\1\end{pmatrix} - k\xi\right) \left(-\operatorname{sgn}(n_2)\, |n_2|^{-2s}\, \kappa_\Omega^{c_1,c_2}(\xi, s)\right) d\xi \tag{4.153}$$

$$= -\sum_{n_2 | k} \frac{\operatorname{sgn}(n_2)}{|n_2|^{2s-1}} \int_{C\left(\frac{k}{n_2}, n_2\right)} e\left(n_2(p_1\xi + p_2) - k\xi\right) \kappa_\Omega^{c_1,c_2}(\xi, s)\, d\xi. \tag{4.154}$$

4.3.5 Series manipulations

In this subsection, we set $\xi = 0$ in eq. (4.150). We will manipulate the right-hand side of this equation to prove Theorem IV.6. First of all, we have

(4.155)
$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, s) = \sum_{k=-\infty}^{\infty} \beta_k(s)$$

$$= -\left(\mathrm{Li}_{2s}(e(p_1)) - \mathrm{Li}_{2s}(e(-p_1))\right) \kappa_\Omega^{c_1,c_2}\left(\begin{pmatrix}1\\0\end{pmatrix}, s\right) + \sum_{k=-\infty}^{\infty} \tilde{\beta}_k(s). \tag{4.156}$$

We will rewrite the sum of the $\tilde{\beta}_k(s)$ using the substitution $(d_1, d_2) = (\frac{k}{n_2}, n_2)$. The following manipulation is legal by Proposition IV.24.

(4.157)
$$\sum_{k=-\infty}^{\infty} \tilde{\beta}_k(s) = -\sum_{k\in\mathbb{Z}} \sum_{\substack{n_2 | k \\ n_2 \neq 0}} \frac{\operatorname{sgn}(n_2)}{|n_2|^{2s-1}} \int_{C\left(\frac{k}{n_2}, n_2\right)} e\left(n_2(p_1\xi + p_2) - k\xi\right) \kappa_\Omega^{c_1,c_2}(\xi, s)\, d\xi$$

$$= -\sum_{d_1\in\mathbb{Z}} \sum_{d_2\in\mathbb{Z}\setminus\{0\}} \frac{\operatorname{sgn}(d_2)}{|d_2|^{2s-1}} \int_{C(d_1,d_2)} e\left(d_2(p_1\xi + p_2) - d_1 d_2\xi\right) \kappa_\Omega^{c_1,c_2}(\xi, s)\, d\xi. \tag{4.158}$$

Split up the series into four pieces.

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}\tilde{\beta}_k(s) = &-\sum_{d_1>0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^-} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1>0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&-\sum_{d_1\le 0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1\le 0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^-} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \qquad (4.159)\\
= &-\sum_{d_1>0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi \\
&+\sum_{d_1>0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&-\sum_{d_1\le 0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1\le 0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi \qquad (4.160)\\
= &-\sum_{d_1>0}\sum_{d_2>0}\frac{e(d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi \\
&+\sum_{d_1>0}\sum_{d_2>0}\frac{e(-d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&-\sum_{d_1\ge 0}\sum_{d_2>0}\frac{e(d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1+p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1\ge 0}\sum_{d_2>0}\frac{e(-d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1+p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi. \qquad (4.161)
\end{aligned}$$

Now, move the contour integral outside the sums, and rewrite the series as

(4.162)

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}\tilde{\beta}_k(s) = \int_{C^+}\Bigg(&\sum_{d_2\ge 0}\frac{e(-p_2+p_1\xi)^{d_2}}{d_2^{2s-1}}\kappa_\Omega^{c_1,c_2}(-\xi,s) - \sum_{d_2\ge 0}\frac{e(p_2+p_1\xi)^{d_2}}{d_2^{2s-1}}\kappa_\Omega^{c_1,c_2}(\xi,s) \\
&+\sum_{d_1>0}\sum_{d_2>0}\frac{1}{d_2^{2s-1}}\bigg(\Big(-e\left((d_1-p_1)\xi+p_2\right)^{d_2}+e\left((d_1+p_1)\xi-p_2\right)^{d_2}\Big)\kappa_\Omega^{c_1,c_2}(-\xi,s) \\
&+\Big(e\left((d_1-p_1)\xi-p_2\right)^{d_2}-e\left((d_1+p_1)\xi+p_2\right)^{d_2}\Big)\kappa_\Omega^{c_1,c_2}(\xi,s)\bigg)\Bigg)\,d\xi.
\end{aligned}$$

Setting $s=1$, we obtain

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}\tilde{\beta}_k(1) \\
&= \int_{C+}\Bigg(-\log(1-e(-p_2+p_1\xi))\kappa_\Omega^{c_1,c_2}\begin{pmatrix}-\xi\\1\end{pmatrix}+\log(1-e(p_2+p_1\xi))\kappa_\Omega^{c_1,c_2}\begin{pmatrix}\xi\\1\end{pmatrix} \\
&\quad+\sum_{d_1=1}^{\infty}\Bigg(\left(\log\left(1-e\left((d_1-p_1)\xi+p_2\right)\right)-\log\left(1-e\left((d_1+p_1)\xi-p_2\right)\right)\right)\kappa_\Omega^{c_1,c_2}\begin{pmatrix}-\xi\\1\end{pmatrix} \\
&\quad\left(-\log\left(1-e\left((d_1-p_1)\xi-p_2\right)\right)+\log\left(1-e\left((d_1+p_1)\xi+p_2\right)\right)\right)\kappa_\Omega^{c_1,c_2}\begin{pmatrix}\xi\\1\end{pmatrix}\Bigg)\Bigg)\,d\xi.
\end{aligned}
\tag{4.163}
$$

We want to write this sum of logarithms as a logarithm of a product, but there is the issue of the choice of branch. In order to make a clear choice, let

$$
\varphi_{p_1,p_2}(\xi) := \left(1-e\left(p_1\xi+p_2\right)\right)\prod_{d=1}^{\infty}\frac{1-e\left((d+p_1)\xi+p_2\right)}{1-e\left((d-p_1)\xi-p_2\right)}
\tag{4.164}
$$

for $\xi\in\mathfrak{H}$. This is a function on the upper half-plane which is never zero, and the upper half-plane is simply connected, so it has a choice of continuous logarithm. Let $\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)$ be the branch such that

$$
\lim_{\xi\to i\infty}\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)=\begin{cases}\log(1-e(p_2)) & \text{if } p_1=0,\\ 0 & \text{if } p_1\neq 0.\end{cases}
\tag{4.165}
$$

Here $\log(1-e(p_2))$ is the standard principal branch. Thus,

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty}\tilde{\beta}_k(1) = \int_{C+}\Bigg(&-\left(\operatorname{Log}\varphi_{p_1,-p_2}\right)(\xi)\cdot\kappa_\Omega^{c_1,c_2}\begin{pmatrix}-\xi\\1\end{pmatrix} \\
&+\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)\cdot\kappa_\Omega^{c_1,c_2}\begin{pmatrix}\xi\\1\end{pmatrix}\Bigg)\,d\xi.
\end{aligned}
\tag{4.166}
$$

Adding back the other piece of $\beta_0(1)$ into $\hat{\zeta}^{c_1,c_2}_{p,0}(\Omega,1) = \sum_{k=-\infty}^{\infty} \beta_k(1)$, we obtain

$$\hat{\zeta}^{c_1,c_2}_{p,0}(\Omega,1) = -\left(\mathrm{Li}_2(e(p_1)) - \mathrm{Li}_2(e(-p_1))\right)\kappa^{c_1,c_2}_{\Omega}\begin{pmatrix}1\\0\end{pmatrix} \tag{4.167}$$

$$+\int_{C+}\left(-\left(\operatorname{Log}\varphi_{p_1,-p_2}\right)(\xi)\cdot\kappa^{c_1,c_2}_{\Omega}\begin{pmatrix}-\xi\\1\end{pmatrix}\right. \tag{4.168}$$

$$\left.+\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)\cdot\kappa^{c_1,c_2}_{\Omega}\begin{pmatrix}\xi\\1\end{pmatrix}\right)d\xi. \tag{4.169}$$

4.3.6 Collapsing the contour onto the branch cuts

We could declare ourselves done at this point. Equation (4.167) is a formula for $\hat{\zeta}^{c_1,c_2}_{p,0}(\Omega,1)$, as we desired, and it appears very difficult to evaluate or simplify the contour integral in any way. However, eq. (4.167) is not a useful formula for computation because the integral converges slowly. The integrand decays polynomially as $\xi \to \pm\infty$ along the horocycle C^+.

We will obtain a Kronecker limit formula with rapid convergence by shifting the contour so that the integrand decays exponentially. In doing so, we will also split up the formula as a difference of a c_1-piece and a c_2-piece. The movement of the contour is shown in Section 4.3.6.

Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)} M c c^\top M$ for $c = c_1, c_2$, as we did in Corollary IV.21. Factor the quadratic polynomial $\mathbb{Q}_{\Lambda_c}\begin{pmatrix}\xi\\1\end{pmatrix}$ in ξ,

$$\mathbb{Q}_{\Lambda_c}\begin{pmatrix}\xi\\1\end{pmatrix} = \alpha(c)(\xi-\tau_1(c))(\xi-\tau_2(c)). \tag{4.170}$$

Since $\Lambda_c \in \mathfrak{H}_2^{(0)}$ by Lemma II.18, we know by Lemma IV.11 that we may choose $\tau_1(c)$ to be in the upper half-plane and $\tau_2(c)$ in the lower half-plane.

The complex function $\xi \mapsto \kappa^c_\Omega\begin{pmatrix}\xi\\1\end{pmatrix}$ has branch cuts along the vertical ray from $\tau_1(c)$ to $i\infty$ and the vertical ray from $\tau_2(c)$ to $-i\infty$. We check that this function is holomorphic away from these branch cuts. Since $\kappa_\Omega^c\begin{pmatrix}\xi\\1\end{pmatrix}$ has simple poles at the roots $\xi = r_1, r_2$ of $Q_\Omega\begin{pmatrix}\xi\\1\end{pmatrix} = 0$, we must check that the residues at the poles cancel when taking the difference $\kappa_\Omega^{c_1,c_2}\begin{pmatrix}\xi\\1\end{pmatrix} = \kappa_\Omega^{c_2}\begin{pmatrix}\xi\\1\end{pmatrix} - \kappa_\Omega^{c_1}\begin{pmatrix}\xi\\1\end{pmatrix}$. We have

$$\operatorname{res}_{\xi\to r_1} \kappa_\Omega^c\begin{pmatrix}\xi\\1\end{pmatrix}$$

$$= \lim_{\xi\to r_1} (\xi - r_1)\frac{c^\top M\begin{pmatrix}\xi\\1\end{pmatrix}}{2\pi i Q_\Omega\begin{pmatrix}\xi\\1\end{pmatrix}\sqrt{\left(c^\top M\begin{pmatrix}\xi\\1\end{pmatrix}\right)^2 - 2iQ_M(c)Q_\Omega\begin{pmatrix}\xi\\1\end{pmatrix}}} \tag{4.171}$$

$$= \lim_{\xi\to r_1} \frac{c^\top M\begin{pmatrix}\xi\\1\end{pmatrix}}{\pi i\omega_{11}(\xi - r_2)\sqrt{\left(c^\top M\begin{pmatrix}\xi\\1\end{pmatrix}\right)^2 - 2iQ_M(c)Q_\Omega\begin{pmatrix}\xi\\1\end{pmatrix}}} \tag{4.172}$$

$$= \frac{1}{\pi i\omega_{11}(r_1 - r_2)}, \tag{4.173}$$

and similarly, $\operatorname{res}_{\xi\to r_2} \kappa_\Omega^c\begin{pmatrix}\xi\\1\end{pmatrix} = \frac{1}{\pi i\omega_{11}(r_2-r_1)}$. These residues do not depend on c, so they cancel, and $\kappa_\Omega^{c_1,c_2}\begin{pmatrix}\xi\\1\end{pmatrix}$ is holomorphic at r_1 and r_2.

Move the countours of integration above the zeros of $Q_\Omega\begin{pmatrix}\pm\xi\\1\end{pmatrix}$. Now we may safely split up the integral into a term for c_1 and a term for c_2.

Now we retract the integral onto the branch cut. As $\xi = \pm\tau^\pm + \varepsilon$ and $\varepsilon \to 0$, the denominator of the integrand blows up like $\varepsilon^{1/2}$, so the integral converges. The integrand changes sign when we cross the branch cut. Thus, eq. (4.167) becomes

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = I^+(c_2) - I^-(c_2) - I^+(c_1) + I^-(c_1), \tag{4.174}$$

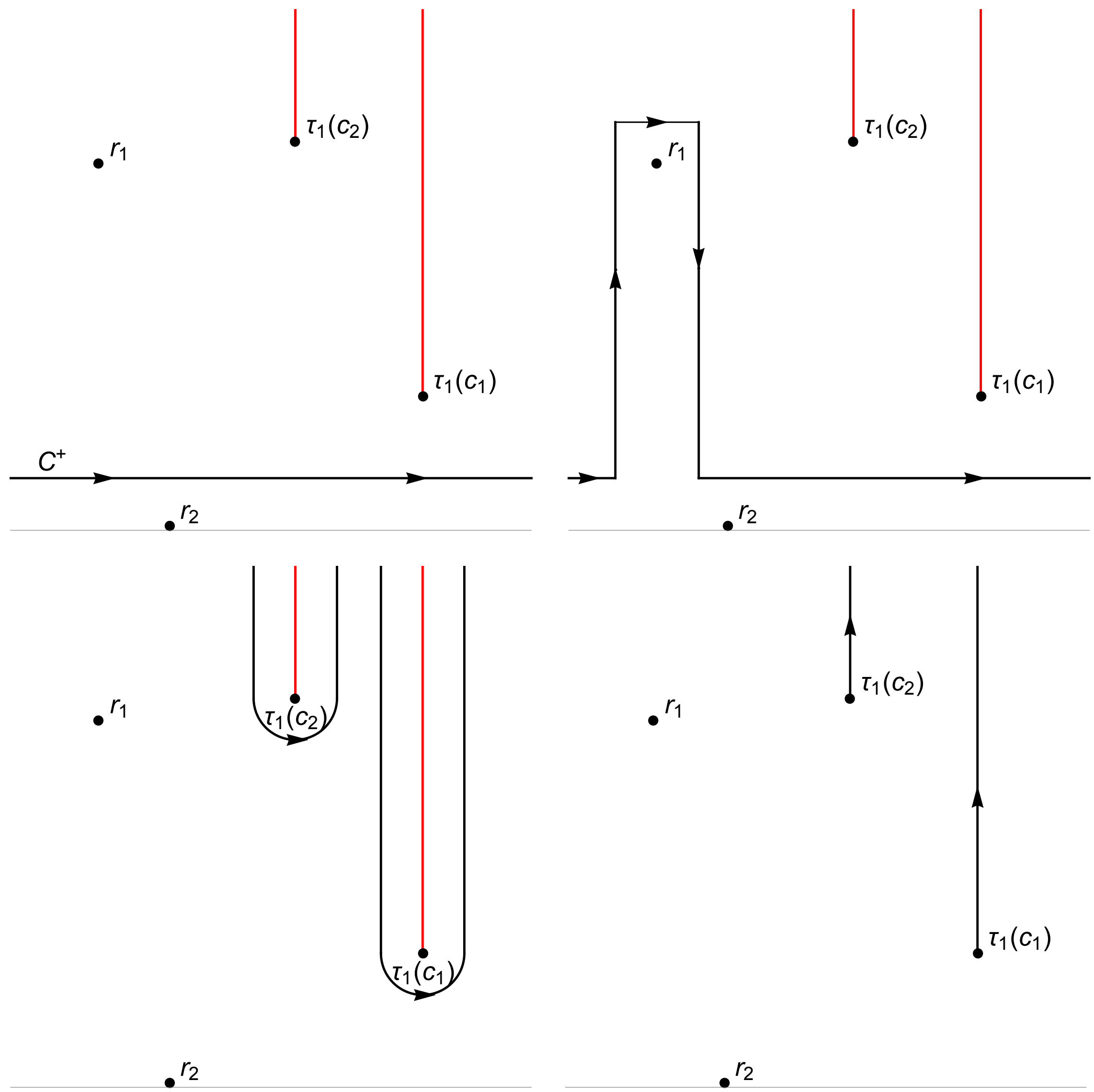

Figure 4.1: The contour C^+ is moved above the poles of $\kappa^c_\Omega\begin{pmatrix}\xi\\1\end{pmatrix}$, then collapsed onto branch cuts.

where

$$I^{\pm}(c) = -\operatorname{Li}_2(e(\pm p_1))\kappa_\Omega^c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ 2i \int_0^\infty (\operatorname{Log}\varphi_{p_1,\pm p_2})\,(\pm\tau^{\pm}(c) + it)\kappa_\Omega^c \begin{pmatrix} \pm(\tau^{\pm}(c) + it) \\ 1 \end{pmatrix} dt. \tag{4.175}$$

We have now proven Theorem IV.6. Theorem IV.7 follows by specialization of the variables, setting $\Omega = iM$ and restricting to $c_1, c_2 \in \mathbb{R}^g$.

4.4 Example

We will continue our running example with $K = \mathbb{Q}(\sqrt{3})$ and $\mathfrak{c} = 5\mathcal{O}_K$, which appeared previously in Chapter I and Chapter III. In this section, we use the Kronecker limit formula for indefinite zeta functions to compute $Z_I'(0)$, where I is the principal ray class of $\mathrm{Cl}_{\mathfrak{c}\cup\{\infty_2\}}$.

By the discussion in Chapter III, we have

$$Z_I'(0) = \hat{\zeta}_{0,q}^{c_1,P^3c_1}(iM, 0). \tag{4.176}$$

Use the functional equation for indefinite zeta functions to write $Z_I'(0)$ in terms of an indefinite zeta value at $s = 1$:

$$Z_I'(0) = \frac{1}{\sqrt{-12}}\hat{\zeta}_{-q,0}^{-iMc_1,-iMP^3c_1}(iM^{-1}, 1). \tag{4.177}$$

We have $Mc_1 = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = -6c_1$. Let $\tilde{P} = MPM^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. We may rescale the c_j without changing the value of the indefinite zeta function. Thus,

$$Z_I'(0) = \frac{-i}{2\sqrt{3}}\hat{\zeta}_{-q,0}^{c_1,\tilde{P}^3c_1}(iM^{-1}, 1). \tag{4.178}$$

Now we want to use Theorem IV.7 to compute the right-hand side of eq. (4.178). If we try to do so directly, we obtain $\tilde{P}^3c_1 = \begin{pmatrix} -15 \\ 26 \end{pmatrix}$ and $\kappa_\Omega^{\tilde{P}^3c_1}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix}, 1\right) =$

$\frac{6\sqrt{2}(45\xi+26)}{\pi(3\xi^2-1)\sqrt{4053\xi^2+4680\xi+1351}}$. The branch point of $\kappa_{\Omega}^{\tilde{P}^3 c_1}\left(\begin{pmatrix}\xi\\1\end{pmatrix},1\right)$ in the upper half-plane is $\xi = \frac{-2340+i\sqrt{3}}{4053}$, which is very close to the real axis. That means we'd need to use about $\frac{\log(10)N}{\pi\sqrt{3}/4053} \approx 1700N$ terms in the product expansion of $\varphi_{p_1,p_2}(\xi)$ to compute $Z_I'(0)$ to N decimal places of accuracy. We technically have exponential decay, but it's not very useful.

It is much more practical to break up the zeta function into pieces. We can also improve the rate of convergence by choosing c_1 optimally; here, we will use $c = \begin{pmatrix}1\\3\end{pmatrix}$, $\tilde{P}c = \begin{pmatrix}-1\\3\end{pmatrix}$.

$$\hat{\zeta}_{-q,0}^{c_1,\tilde{P}^3 c_1}(-\Omega^{-1},1) = \hat{\zeta}_{-q,0}^{c,\tilde{P}^3 c}(-\Omega^{-1},1) \tag{4.179}$$

$$= \hat{\zeta}_{-q,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q,0}^{\tilde{P}c,\tilde{P}^2 c}(-\Omega^{-1},1) + \hat{\zeta}_{-q,0}^{\tilde{P}^2 c,\tilde{P}^3 c}(-\Omega^{-1},1) \tag{4.180}$$

$$= \hat{\zeta}_{-q_0,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q_1,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q_2,0}^{c,\tilde{P}c}(-\Omega^{-1},1), \tag{4.181}$$

where $q_0 = q = \frac{1}{5}\begin{pmatrix}1\\0\end{pmatrix}$, $q_1 = q = \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix}$, and $q_2 = q = \frac{1}{5}\begin{pmatrix}2\\4\end{pmatrix}$ are obtained from the residues of $\varepsilon^0, \varepsilon^1, \varepsilon^2$ modulo 5.

Now, we have $\kappa_{\Omega}^{c}\begin{pmatrix}\xi\\1\end{pmatrix} = \frac{-3\sqrt{6}(x-1)}{\pi(3x^2-1)\sqrt{3x^2-3x+1}}$ and $\kappa_{\Omega}^{\tilde{P}c}\begin{pmatrix}\xi\\1\end{pmatrix} = \frac{3\sqrt{6}(x+1)}{\pi(3x^2-1)\sqrt{3x^2+3x+1}}$, which is much more manageable. We computed the following values in Mathematica using 40 terms of the product expansion of φ_{p_1,p_2}.

$$\begin{aligned} I_0(\tilde{P}c) - I_0(c) \approx & -0.059238439175444883293545079 87\\ & + 3.6568783902031178613289385023 9i \end{aligned} \tag{4.182}$$

$$\begin{aligned} I_1(\tilde{P}c) - I_1(c) \approx & -1.3373302108594346921068501489 9\\ & + 0.5247781252942466338755689916 7i \end{aligned} \tag{4.183}$$

$$\begin{aligned} I_2(\tilde{P}c) - I_2(c) \approx & \ 2.6405758727192221245648419060 7\\ & + 0.5247781252942466338755689916 7i \end{aligned} \tag{4.184}$$

We now obtain

$$Z_I'(0) = \frac{-i}{2\sqrt{3}} \left(\hat{\zeta}^{c,\tilde{P}c}_{-q_0,0}(-\Omega^{-1},1) + \hat{\zeta}^{c,\tilde{P}c}_{-q_1,0}(-\Omega^{-1},1) + \hat{\zeta}^{c,\tilde{P}c}_{-q_2,0}(-\Omega^{-1},1)\right) \tag{4.185}$$

$$= \frac{1}{2\sqrt{3}} \operatorname{Im}\left((I_0(\tilde{P}c) - I_0(c)) + (I_1(\tilde{P}c) - I_1(c)) + (I_2(\tilde{P}c) - I_2(c))\right) \tag{4.186}$$

$$\approx 1.358630653392208162595113082 30. \tag{4.187}$$

This agrees (to 30 decimal digits) with the computations described in Chapter III.

CHAPTER V

Connections to the SIC-POVM Problem

In this chapter, we discuss a geometric problem having an interpretation in quantum information theory. A SIC-POVM (symmetric informationally complete positive operator-valued measure) is a set of d^2 equiangular lines in d-dimensional Hilbert space. Such configurations are conjectured to exist in all dimensions, and have been proven to exist (by explicit construction) in dimensions $d \leq 151$. In 2016, Appleby, Flammia, McConnell, and Yard [5] numerically described a connection between SIC-POVMs and Hilbert's 12th problem for real quadratic fields. We connect our running numerical example for $\mathbb{Q}(\sqrt{3})$ to a known SIC-POVM in dimension $d = 5$. By working out this example in detail, we discover a suggestive relationship between the conjectural Stark unit $\exp(Z_I'(0)) \approx 3.89086171394307925533376$ and a 5-dimensional SIC-POVM.

5.1 Equiangular complex lines

In d-dimensional complex Hilbert space $\mathbb{C}^d$, there is a well-defined notion of a configuration of equiangular lines, which yields a mathematically rich theory. A complex line in $\mathbb{C}^d$ is a one-dimensional complex subspace. For two complex lines $\mathbb{C}v$ and $\mathbb{C}w$, the angle between them is defined to be $\angle(v, w) = \arccos\left(\left|\frac{\langle v,w\rangle}{\sqrt{\langle v,v\rangle}\sqrt{\langle w,w\rangle}}\right|\right)$, as for real lines—but using the Hermitian inner product $\langle v, w\rangle = \overline{v}^\top w$.

Definition V.1. A *set of equiangular complex lines* is a set of one-dimensional subspaces $\mathbb{C}v_1, \ldots, \mathbb{C}v_2$ such that $\angle(v_i, v_j)$ takes the same value whenever $i \neq j$.

We also define the *overlap* of v and w to be $\frac{\langle v,w\rangle}{\sqrt{\langle v,v\rangle}\sqrt{\langle w,w\rangle}}$, so the absolute value of the overlap is $\cos(\angle(v,w))$. We may therefore write

$$\frac{\langle v, w\rangle}{\sqrt{\langle v, v\rangle}\sqrt{\langle w, w\rangle}} = \cos(\angle(v, w))\, e^{i\theta}, \tag{5.1}$$

where $e^{i\theta}$ is called the *overlap phase*.

The maximal number of equiangular complex lines possible in $\mathbb{C}^d$ is d^2; this was originally proved in 1975 by Delsarte, Goethals, and Seidel using orthogonal polynomials [13].

Proposition V.2 (Delsarte, and Goethals, and Seidel [13])**.** *Let $\alpha > 0$. Consider a set V of unit vectors in $\mathbb{C}^d$ spanning equiangular lines; that is, $|\langle v, w\rangle|^2 = \alpha$ whenever $v, w \in V$ and $v \neq w$. Then, $|V| \leq d^2$.*

Proof. The following proof is due to Koornwinder [26]. Let $\mathbb{C}^d$ have coordinates $(z_1, \ldots, z_d)$, and let $M(1,1)$ be the complex vector space of bihomogeneous polynomials of degree $(1,1)$ in $(z_1, \ldots, z_d; \overline{z}_1, \ldots, \overline{z}_d)$. A basis for $M(1,1)$ is given by $\{z_i\overline{z}_j\}$, so $\dim M(1,1) = d^2$.

Consider the polynomials in $M(1,1)$ given by $F_v(z) = \frac{1}{1-\alpha}\left(|\langle z, v\rangle|^2 - \alpha\langle z, z\rangle\right)$ for $v \in V$. If $w \in V$, then $F_v(w) = 0$ if $v \neq w$ and $F_v(w) = 1$ if $v = w$. Thus, $\{F_v(z) : v \in V\}$ is a set of linearly independent elements of $M(1,1)$. Therefore, $|V| \leq d^2$. $\square$

It was also shown by Delsarte, Goethals, and Seidel [13] that for any set of d^2 equiangular complex lines,

$$\alpha = \frac{1}{d+1}, \tag{5.2}$$

and thus the common angle is $\arccos\left(\frac{1}{\sqrt{d+1}}\right)$.

5.2 Definition of SIC-POVMs

The presence of SIC-POVMs in quantum information theory is due to Zauner's 1999 thesis [53] (see English translation [54]). The term *SIC-POVM* was attached to the concept in 2004 by Renes, Blume-Kohout, Scott, and Caves [35].

Definition V.3 (SIC-POVM)**.** A *symmetric informationally complete positive operator-valued measure (SIC-POVM)* is a set of d^2 equiangular complex lines in d-dimensional Hilbert space. In other words, by the discussion in the previous section, it is a set of one-dimensional subspaces $\mathbb{C}v_1, \mathbb{C}v_2, \ldots, \mathbb{C}v_{d^2}$ in $\mathbb{C}^d$ such that $\left|\frac{\langle v_i, v_j\rangle^2}{\langle v_i, v_i\rangle\langle v_j, v_j\rangle}\right| = \frac{1}{d+1}$ for all $i \neq j$.

There are two types of operators on $\mathbb{C}^d$ preserving the SIC-POVM property. A SIC-POVM $\{\mathbb{C}v_1, \ldots, \mathbb{C}v_{d^2}\}$ may be "rotated" by any unitary matrix $U \in \mathbf{U}(d) = \{U \in \mathbf{GL}(\mathbb{C}^d) : U\overline{U}^\top = 1\}$ to obtain another SIC-POVM $\{\mathbb{C}Uv_1, \ldots, \mathbb{C}Uv_{d^2}\}$. Moreover, if C_d is the complex conjugation operator on $\mathbb{C}^d$, so that $C_d v := \overline{v}$, then C_d also preserves the SIC-POVM property (and the same holds for any "antiunitary" operator of the form $C_d U$). These may be collected together by considering the extended unitary group.

Definition V.4. Define the *extended unitary group* $\mathbf{EU}(d) := \mathbf{U}(d) \sqcup C_d\mathbf{U}(d)$

Lemma V.5. *The action of* $\mathbf{EU}(d)$ *takes SIC-POVMs to SIC-POVMs.*

Proof. The action of $\mathbf{U}(d)$ preserves the Hermitian inner product; $C_d\mathbf{U}(d)$ conjugates the Hermitian inner product. Thus, both preserve its absolute value and thus preserve the SIC-POVM property. $\square$

5.3 Definition of Heisenberg SIC-POVMs

Heisenberg SIC-POVMs are a special class of SIC-POVMs. Let $\zeta_d = e\left(\frac{1}{d}\right) = \exp\left(\frac{2\pi i}{d}\right)$ be a dth root of unity.

Definition V.6 (Heisenberg group)**.** Let $d' = d$ if d is odd, $d' = 2d$ if d is even. Let I be the $d \times d$ identity matrix. The Heisenberg group $\mathrm{H}(d)$ is the finite group of order $d'd^2$ generated by the $d \times d$ scalar matrix $\zeta_{d'} I$ and the $d \times d$ matrices

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_d & 0 & \cdots & 0 \\ 0 & 0 & \zeta_d^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_d^{d-1} \end{pmatrix}. \tag{5.3}$$

The Heisenberg group spans the vector space $\mathbf{M}_d(\mathbb{C})$ of $d \times d$ complex matrices, and a canonical basis is given as follows.

Definition V.7 (Heisenberg basis)**.** The set of d^2 matrices $\Delta_{mn} = \zeta_d^{\frac{d+1}{2}mn} X^m Z^n$ for $0 \leq m, n \leq d-1$ forms a basis of $\mathbf{M}_d(\mathbb{C})$ over $\mathbb{C}$.

Empirically, many known SIC-POVMs are orbits of the Heisenberg group action, and this observation motivates the following definition.

Definition V.8 (Heisenberg SIC-POVM)**.** A *Heisenberg SIC-POVM* is a SIC-POVM of the form $\{\mathbb{C}\Delta_{mn} v : 0 \leq m, n \leq d-1\}$ for some vector $v \in \mathbb{C}^d$. This v is called a *fiducial vector.*

The elements of $\mathbf{EU}(d)$ that preserve the property of being a *Heisenberg* SIC-POVM are restricted to a finite group, the *extended Clifford group* $\mathrm{EC}(d)$, defined to be the normalizer of $\mathrm{H}(d)$ inside $\mathbf{EU}(d)$.

Lemma V.9. *If v is a fiducial vector for a Heisenberg SIC-POVM, and $\gamma \in \mathrm{EC}(d)$, then γv is also a fiducial vector for a Heisenberg SIC-POVM. Conversely, if v and w are* $\mathbf{EU}(d)$*-equivalent fiducial vectors, they are in fact* $\mathrm{EC}(d)$*-equivalent.*

Proof. See Scott and Grassl [37]. □

5.4 Main conjectures about SIC-POVMs

The following two conjectures concerning SIC-POVMs are due to Zauner [53].

Conjecture V.10 (Existence of SIC-POVMs)**.** *SIC-POVMs exist in every dimension $d \geq 1$*

Conjecture V.11 (Existence of Heisenberg SIC-POVMs)**.** *Heisenberg SIC-POVMs exist in every dimension $d \geq 1$. Moreover, a Heisenberg SIC-POVM exists with fiducial vector v an eigenvector with eigenvalue 1 of a particular $d \times d$ unitary matrix (the "Zauner matrix" [37]) having order 3, specified in [53].*

Scott and Grassl's extensive computations [37, 36] enumerating $\mathrm{EC}(d)$-orbits of Heisenberg SIC-POVMs find a "putatively complete" list with high probability when $d \leq 90$. Their computations strongly support the following conjecture, which is remarked on my Fuchs, Huang, and Stacey [17].

Conjecture V.12 (Number of Heisenberg SIC-POVMs)**.** *There are finitely many* $\mathrm{EC}(d)$*-orbits of Heisenberg SIC-POVMs for every d, with the exception of $d = 3$, when there is a continuous one-parameter family of orbits.*

In a fixed dimension d, enumerating Heisenberg SIC-POVMs is much more computationally feasible than enumerating all SIC-POVMs. Conjecture V.11 has focused the search for SIC-POVMs on the Heisenberg covariant case. According to Fuchs, Hoang, and Stacey [17], the complete set of SIC-POVMs is only known unconditionally in dimensions 2 and 3, and the following observation "could be an artifact" of the methods used to find SIC-POVMs. They indicate that all currently known SIC-POVMs are $\mathbf{EU}(d)$-equivalent to Heisenberg SIC-POVMs except in dimension $d = 8$, where the Hoggar lines [24, 25] give rise to a single sporadic $\mathbf{EU}(d)$-orbit.

A fiducial vector v is only determined up to multiplication λv by a complex scalar $\lambda \in \mathbb{C}^\times$. The ratios $\frac{v_i}{v_j} = \frac{\lambda v_i}{\lambda v_j}$ are independent of his choice. Additionally, the overlap phases $e^{i\theta_{m,n}}$ on the unit circle defined by

$$\frac{\langle v, \Delta_{m,n} v\rangle}{\langle v, v\rangle} = \frac{e^{i\theta_{m,n}}}{\sqrt{d+1}} \tag{5.4}$$

are independent of the choice of scalar λ.

5.5 SIC-POVMs and number theory

Much of the progress on the SIC-POVM problem has been in the form of numerical investigations. The work of Scott and Grassl [37], Scott [36], and Fuchs, Hoang, and Stacey [17] has produced Heisenberg SIC-POVMs in every dimension up to $d = 151$ and some higher dimensions. In many cases, exact as well as numerical solutions have been given, making exploration of the algebraic and Galois-theoretic properties of SIC-POVMs feasible.

In 2016, Appleby, Flammia, McConnell, and Yard [5, 6] numerically discovered a surprising connection between SIC-POVMs and Hilbert's 12th problem for real quadratic fields. For all Heisenberg SIC-POVMs they were able to check, they found that the ratios of the entries of the fiducial vector lie in an abelian extension of the real quadratic field $\mathbb{Q}(\sqrt{(d+1)(d-3)})$. This field contains the unit $\varepsilon = \frac{(d-1)+\sqrt{(d+1)(d-3)}}{2}$, which need not be a fundamental unit.

Let $K = \mathbb{Q}(\sqrt{(d+1)(d-3)})$, and let E be the field generated by the ratios of the entries of the fiducial vector along with the d'th roots of unity, where $d' = d$ if d is odd, and $d' = 2d$ if d is even. If v is a Heienberg fiducial vector and $\sigma \in \mathrm{Gal}(E/K)$, then v^σ is also a Heisenberg fiducial vector; v^σ may or may not lie in the same $\mathrm{EC}(d)$ orbit as v. This Galois action respects orbits because $(\gamma v)^\sigma = \gamma^\sigma v^\sigma$ and $\mathrm{EC}(d)$ is Galois-closed.

Definition V.13. The set of all those $\mathrm{EC}(d)$-orbits of fiducial vectors, which are Galois equivalent to a given $\mathrm{EC}(d)$-orbit, is called a *multiplet*.

Appleby et. al. [6] made the following conjecture about the "minimal multiplet" (i.e., the multiplet of smallest cardinality).

Conjecture V.14 (Field of the minimal multiplet, Appleby et. al. [6])**.** *For $d > 3$, there is a unique "minimal multiplet" in dimension d, of cardinality the class number of h_K of the real quadratic field $K = \mathbb{Q}(\sqrt{(d+1)(d-3)})$. Suppose that v is a fiducial vector belonging to a class in the minimal multiplet. Then, the field E generated by the ratios of the entries of the fiducial vector along with the d'th roots of unity is the ray class field of K modulo $d'\infty_1\infty_2$, where ∞_1, ∞_2 are the two real places of K. (Here, $d' = d$ if d is odd, and $d' = 2d$ if d is even.)*

5.6 The case $d = 5$

In this section, we show a striking relationship between the $d = 5$ Heisenberg SIC-POVM (there is only one up to $\mathrm{EC}(d)$ equivalence) and the (conjecturally) algebraic unit $\exp(Z_I'(0)) \approx 3.89086171394307925533 76$ appearing in the running example throughout this thesis.

The Heisenberg SIC-POVM of dimension 5 appears in Zauner's thesis [53]. Scott and Grassl [37] show that there is only one $\mathrm{EC}(d)$ orbit as part of their calculations of exact fiducial vectors.

5.6.1 Fiducial vector

The case $d = 5$ is small enough that the fiducial vectors v of Heisenberg SIC-POVMs may be enumerated by brute force. The conditions defining a Heisenberg SIC-POVMs become algebraic if we "complexify" by regarding v_j and $\overline{v}_j$ as independent variables. Using Mathematica's symbolic computation capabilities, we solved these algebraic equations and (as expected) recovered the single EC(d) orbit described by Scott and Grassl [37].

By trial and error, we found the following fiducial vector, whose overlap phases are particularly nice. Our fiducial vector—normalized so that $v_1 = 1$— is given as

$$v = \begin{pmatrix} 1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} \approx \begin{pmatrix} 1.000000000 \\ -0.5929852324 + 0.5068571680i \\ 1.1020919741 - 0.5049907538i \\ -0.728472837 + 1.654872222i \\ 0.3374000837 + 0.2797530365i \end{pmatrix}, \tag{5.5}$$

where v_2, v_3, v_4, v_5 are roots of the polynomial

$$\begin{aligned} &25x^{32} + 200x^{31} + 800x^{30} + 2300x^{29} + 5500x^{28} + 11000x^{27} + 19600x^{26} \\ &\quad + 27150x^{25} + 30020x^{24} + 23820x^{23} + 20930x^{22} + 26860x^{21} + 57325x^{20} \\ &\quad + 66970x^{19} + 114230x^{18} + 87840x^{17} + 133821x^{16} + 42124x^{15} \\ &\quad + 100500x^{14} + 1700x^{13} + 47440x^{12} - 12878x^{11} + 22678x^{10} - 11240x^{9} \\ &\quad + 9505x^{8} - 4520x^{7} + 2653x^{6} - 1078x^{5} + 450x^{4} - 130x^{3} + 35x^{2} \\ &\quad - 6x + 1. \end{aligned} \tag{5.6}$$

Observe that the v_i are 5-units; this seems to be related to the fact that we will be allowing ramification at the prime (5).

5.6.2 Overlap phases

Let $K = \mathbb{Q}(\sqrt{3})$, $\varepsilon = \varepsilon_3 = 2 + \sqrt{3}$, and consider the ray class group $\mathrm{Cl}_{(5)\cup\{\infty_2\}}$ of $\mathcal{O}_K$ modulo $(5) \cup \{\infty_2\}$. Here, ∞_1 is the infinite place associated to the real embedding $\rho_1 : \sqrt{3} \to \sqrt{3}$, and ∞_2 is associated to the real embedding $\rho_2 : \sqrt{3} \to -\sqrt{3}$. As discussed in section 1.4.1, $\mathrm{Cl}_{(5)\cup\{\infty_2\}} \cong \mathbb{Z}/8\mathbb{Z}$. Its elements may be enumerated as follows, for $0 \leq m, n \leq 4$ not both zero, with each class appearing three times because of the action of $\langle \varepsilon \rangle$:

$$A_{m+\varepsilon n} = \{\alpha\mathcal{O}_K : \alpha \equiv m + \varepsilon n \pmod{(5)} \text{ and } \rho_2(\alpha) > 0\}. \tag{5.7}$$

Let δ be the root of the polynomial

$$\begin{aligned} &x^8 - (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 + (225+130\sqrt{3})x^4 \\ &- (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 - (8+5\sqrt{3})x + 1. \end{aligned} \tag{5.8}$$

that is given approximately by $\delta \approx 3.89086171394307925533 76$. As discussed in section 1.4.1, the field $H_2 = K(\delta)$ is the class field associated to $\mathrm{Cl}_{(5)\cup\{\infty_2\}}$ by class field theory. As noted in section 3.5, the Stark conjectures predict that, if I is the identity element of $\mathrm{Cl}_{(5)\cup\{\infty_2\}}$, then

$$\delta = \exp(Z_I'(0)). \tag{5.9}$$

This is true to 100 digits, but unproved.

Let $\mathrm{Art} : \mathrm{Cl}_{(5)\cup\{\infty_2\}} \to \mathrm{Gal}(H/K)$ be the Artin map of class field theory. The eight Galois conjugates of δ over K are of the form $\delta^{\mathrm{Art}(A_{m+n\varepsilon})}$ and are real numbers. The other eight Galois conjugates of δ over $\mathbb{Q}$ are complex numbers on the unit circle.

Our results are summarized in the next two observations.

Observation V.15. *The squares of the overlap phases of the fiducial vector $v \in \mathbb{C}^5$ in eq.* (5.5) *are Galois conjugates of δ over $\mathbb{Q}$. Specifically, there is a Galois* *automorphism* $\sigma \in \mathrm{Gal}(H_2/\mathbb{Q})$ *such that, for* $0 \le m, n \le 4$ *not both zero,*

$$\left(\frac{\langle v, \Delta_{mn} v\rangle}{\langle v, v\rangle}\right)^2 = \frac{\delta^{\mathrm{Art}(A_{m+n\varepsilon})\sigma}}{d+1}. \tag{5.10}$$

We have discovered an apparent relationship between L-values and overlap phases, as follows.

Observation V.16. *Let* $\delta \approx 3.8908617139$ *be the algebraic unit given as a root of eq.* (5.8)*, let* $v \in \mathbb{C}^5$ *be the fiducial vector given by eq.* (5.5)*, and let* $A_{m+n\varepsilon}$ *(for* $0 \le m, n \le 4$ *not both zero) be the ray classes defined in eq.* (5.7)*. Then the eight real Galois conjugates of* δ *appear to be* $\exp(Z'_{A_{m+n\varepsilon}}(0))$ *(at least to 100 digits), each appearing three times. The eight remaining Galois conjugates of* δ *over* $\mathbb{Q}$ *on the unit circle are the squares of the overlap phases of the Heisenberg SIC-POVM generated by* v*. (Each overlap phase occurs 3 times in the list of* $5^2 - 1 = 24$ *overlap phases.)*

5.7 SIC-POVMs and orders

Appleby et. al. [6] recognize that SIC-POVMs generate class fields in all known cases, but they only predict the precise class field attached to the minimal multiplet. In this section, we give a new conjecture about the class field of an arbitrary multiplet.

Conjecture V.17. *Fix an integer* $d > 3$*. Consider the real quadratic field* $K = \mathbb{Q}\left(\sqrt{(d+1)(d-3)}\right)$ *and the associated unit* $\varepsilon_d = \frac{(d-1)+\sqrt{(d+1)(d-3)}}{2} \in \mathcal{O}_K$*.*

(1) The multiplets of SIC-POVMs in dimension d *are in one-to-one correspondence with the orders* $\mathcal{O}$ *of* K *satisfying* $\mathbb{Z}[\varepsilon] \subseteq \mathcal{O} \subseteq \mathcal{O}_K$*. (In particular, the "minimal multiplet" of Appleby et. al. [6] is the one corresponding to* $\mathcal{O}_K$*.)*

(2) The size of the multiplet corresponding to $\mathcal{O}$ *is the class number of* $\mathcal{O}$*.*

(3) The field $E_{\mathcal{O}}$ *generated by the multiplet corresponding to* $\mathcal{O}$ *is the compositum of the ray class field of* K *modulo* $d' \cup \{\infty_1, \infty_2\}$ *and the ring class field of* $\mathcal{O}$*.*

(Here, $d' = d$ if d is odd, and $d' = 2d$ if d is even.)

We have accumulated the following evidence in favor of this conjecture. Points (1) and (2) have been checked numerically for all SIC-POVMs in Scott and Grassl's first list [37]. Point (3) has been checked for a few small d.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Sergei Alexandrov, Sibasish Banerjee, Jan Manschot, and Boris Pioline. Indefinite theta series and generalized error functions. *arXiv preprint arXiv:1102.4353*, 2011.

[2] George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[3] D. M. Appleby, Steven T. Flammia, and Christopher A. Fuchs. The Lie algebraic significance of symmetric informationally complete measurements. *Journal of Mathematical Physics*, 52(2):022202, 2011.

[4] D. M. Appleby, Christopher A. Fuchs, and H. Zhu. Group theoretic, Lie algebraic and Jordan algebraic formulations of the sic existence problem. *Quantum Inf. Comput.*, 15:6194, 2015.

[5] Marcus Appleby, Steven Flammia, Gary McConnell, and Jon Yard. Generating ray class fields of real quadratic fields via complex equiangular lines. *arXiv preprint arXiv:1604.06098*, 2016.

[6] Marcus Appleby, Steven Flammia, Gary McConnell, and Jon Yard. SICs and algebraic number theory. *Foundations of Physics*, pages 1–18, 2017.

[7] Ernest W. Barnes. The theory of the double gamma function. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 196:265–387, 1901.

[8] Ingemar Bengtsson. The number behind the simplest SIC–POVM. *Foundations of Physics*, 47(8):1031–1041, 2017.

[9] Kathrin Bringmann and Ken Ono. The $f(q)$ mock theta function conjecture and partition ranks. *Inventiones mathematicae*, 165(2):243–266, 2006.

[10] Daniel Bump. *Automorphic forms and representations*, volume 55. Cambridge University Press, 1998.

[11] Bin Chen, Tao Li, and Shao-Ming Fei. General SIC measurement-based entanglement detection. *Quantum Information Processing*, 14(6):2281–2290, 2015.

[12] Atish Dabholkar, Sameer Murthy, and Don Zagier. Quantum black holes, wall crossing, and mock modular forms. *arXiv preprint arXiv:1208.4074*, 2012.

[13] Philippe Delsarte, J. M. Goethals, and J. J. Seidel. Bounds for systems of lines, and Jacobi polynomials. *Philips Research Reports*, 30:91, 1975.

[14] William Duke, Özlem Imamolu, and Árpad Tóth. Cycle integrals of the j-function and mock modular forms. *Annals of Mathematics*, pages 947–981, 2011.

[15] John F. R. Duncan, Michael J. Griffin, and Ken Ono. Proof of the umbral moonshine conjecture. *Research in the Mathematical Sciences*, 2(1):26, 2015.

[16] Amanda Folsom, Ken Ono, and Robert C. Rhoades. Mock theta functions and quantum modular forms. In *Forum of mathematics, Pi*, volume 1. Cambridge University Press, 2013.

[17] Christopher A. Fuchs, Michael C. Hoang, and Blake C. Stacey. The SIC question: history and state of play. *arXiv preprint arXiv:1703.07901*, 2017.

[18] Christopher A. Fuchs and Rüdiger Schack. Quantum-bayesian coherence. *Reviews of modern physics*, 85(4):1693–1715, 2013.

[19] Phillip A. Griffiths. Periods of integrals on algebraic manifolds, I (construction and properties of the modular varieties). *American Journal of Mathematics*, 90(2):568–626, 1968.

[20] Phillip A. Griffiths. Periods of integrals on algebraic manifolds, II (local study of the period mapping). *American Journal of Mathematics*, 90(3):805–865, 1968.

[21] Phillip A. Griffiths. Periods of integrals on algebraic manifolds, III (some global differential-geometric properties of the period mapping). *Publications Mathématiques de l'IHÉS*, 38:125–180, 1970.

[22] Gustav Herglotz. Über die Kroneckersche grenzformel für reelle, quadratische Körper. I, II. *Berichte uber die Verhandl. d. S achsischen Akad. der Wiss. zu*, 75:3–14,31–37, 1923.

[23] David Hilbert. Mathematical problems. *Bull. Amer. Math. Soc.*, 8(10):437–479, 1902.

[24] Stuart G. Hoggar. Two quaternionic 4-polytopes. In *The Geometric Vein*, pages 219–230. Springer, 1981.

[25] Stuart G. Hoggar. 64 lines from a quaternionic polytope. *Geometriae Dedicata*, 69(3):287–289, 1998.

[26] Tom H Koornwinder. A note on the absolute bound for systems of lines. In *Indagationes Mathematicae (Proceedings)*, volume 79, pages 152–153. Elsevier, 1976.

[27] Nobushige Kurokawa and Shin-ya Koyama. Multiple sine functions. In *Forum mathematicum*, volume 15, pages 839–876. De Gruyter, 2003.

[28] Serge Lang. *Introduction to Algebraic and Abelian Functions*, volume 89 of *Graduate Texts in Mathematics*. Springer, 2nd edition, 1982.

[29] Serge Lang. *Elliptic Functions*, volume 112 of *Graduate Texts in Mathematics*. Springer, 2nd edition, 1987.

[30] Sheng-Chi Liu and Riad Masri. A Kronecker limit formula for totally real fields and arithmetic applications. *Research in Number Theory*, 1(8):1–20, 2015.

[31] James S. Milne. Abelian varieties. `http://www.jmilne.org/math/CourseNotes/AV.pdf`, 2008.

[32] Jürgen Neukirch. *Algebraic Number Theory*, volume 322. Springer Science & Business Media, 2013.

[33] Anna Posingies. Kronecker limit formulas and scattering constants for Fermat curves. *arXiv preprint arXiv:1111.1132*, 2011.

[34] Harry E. Rauch and Hershel M. Farkas. *Theta Functions with Applications to Riemann Surfaces*. The Williams & Wilkins Company, 1974.

[35] Joseph M. Renes, Robin Blume-Kohout, Andrew J. Scott, and Carlton M. Caves. Symmetric informationally complete quantum measurements. *Journal of Mathematical Physics*, 45(6):2171–2180, 2004.

[36] Andrew J. Scott. SICs: Extending the list of solutions. *arXiv preprint arXiv:1703:03993*, 2017.

[37] Andrew J. Scott and Markus Grassl. Symmetric informationally complete positive-operator-valued measures: A new computer study. *Journal of Mathematical Physics*, 51(4):042203, 2010.

[38] Takuro Shintani. On Kronecker limit formula for real quadratic fields. *Proceedings of the Japan Academy*, 52(7):355–358, 1976.

[39] Takuro Shintani. On certain ray class invariants of real quadratic fields. *Journal of the Mathematical Society of Japan*, 30(1):139–167, 1978.

[40] Takuro Shintani. On special values of zeta functions of totally real algebraic number fields. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 591–597. Acad. Sci. Fennica, Helsinki, 1980.

[41] Carl L. Siegel. *Advanced Analytic Number Theory.* Tata Intstitute of Fundamental Research, 1980.

[42] Harold M. Stark. Values of L-functions at $s = 1$. I. L-functions for quadratic forms. *Advances in Mathematics*, 7(3):301–343, 1971.

[43] Harold M. Stark. L-functions at $s = 1$. II. Artin L-functions with rational characters. *Advances in Mathematics*, 17(1):60–92, 1975.

[44] Harold M. Stark. L-functions at $s = 1$. III. Totally real fields and Hilbert's twelfth problem. *Advances in Mathematics*, 22(1):64–84, 1976.

[45] Harold M. Stark. L-functions at $s = 1$. IV. First derivatives at $s = 0$. *Advances in Mathematics*, 35(3):197–235, 1980.

[46] Anna Szymusiak and Wojciech Słomczyński. Informational power of the Hoggar symmetric informationally complete positive operator-valued measure. *Physical Review A*, 94(1):012122, 2016.

[47] Maria Vlasenko and Don Zagier. Higher Kronecker "limit" formulas for real quadratic fields. *J. Reine Angew. Math.*, 679:23–64, 2013.

[48] Andre Weil. On Picard varieties. *American Journal of Mathematics*, 74(4):865–894, 1952.

[49] Shuji Yamamoto. On Kronecker limit formulas for real quadratic fields. *Journal of Number Theory*, 128(2):426–450, 2008.

[50] Shuji Yamamoto. On Shintani's ray class invariant for totally real number fields. *Mathematische Annalen*, 346(2):449–476, 2010.

[51] Don Zagier. A Kronecker limit formula for real quadratic fields. *Math. Ann.*, 213:153–184, 1975.

[52] Don Zagier. Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann). *Astérisque*, (326):Exp. No. 986, vii–viii, 143–164 (2010), 2009. Séminaire Bourbaki. Vol. 2007/2008.

[53] Gerhard Zauner. *Quantendesigns: Grundzüge einer nichtkommutativen Designtheorie.* PhD thesis, University of Vienna, Vienna, Austria, 1999.

[54] Gerhard Zauner. Quantum designs: Foundations of a noncommutative design theory. *International Journal of Quantum Information*, 9(1):445–507, 2011.

[55] Sander Zwegers. Mock ϑ-functions and real analytic modular forms. In *q-Series with Applications to Combinatorics, Number Theory, and Physics*, volume 291, pages 269–277, 2001.

[56] Sander Zwegers. *Mock Theta Functions.* PhD thesis, Universiteit Utrecht, Utrecht, The Netherlands, October 2002.