First Mover or Higher Quality? Optimal Product Strategy in Markets with Positive Feedbacks

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A Proofs of Analytic Results

Recall that we denote the expectation of any variable by using a bar, and dependence on the initial state and the transition weights using square brackets. For example,

$$\bar{X}^t[S;\rho]$$

denotes the expected outcome vector at time t with initial state S and transition weights ρ.

Theorem 1. *Strictly increasing the initial market presence of firm j strictly increases the expected total sales of firm j after T periods for any $T \geq 1$.*

Proof. By Lemma 1 from the main text, without loss of generality we can assume that there is only one other competing firm, which we will call firm k. We as-

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sume that the transition weights take on values with certainty, i.e. ρ_j and ρ_k are constant, and drop them from our notation since they are assumed to be the same throughout. The general proof follows similarly. It suffices to show that for any initial state S,

$$\bar{X}_j^T[S] \; < \bar{X}_j^T[S+\delta_j]$$

where $S+\delta_j = (S_j+\delta, S_k)$ denotes the vector S with an arbitrary positive value δ added to the j component. The proof is by induction. For $T=1$, the market share for firm j equals one if the consumer in period one buys from j, and zero otherwise. Therefore, the expected total sales for firm j equals the probability that the period one consumer buys from j, and so

$$\bar{X}_j^1[S] = R_j[S] = \frac{S_j}{S_j+S_k} < \frac{S_j+\delta}{S_j+S_k+\delta} = R_j[S+\delta_j] = \bar{X}_j^1[S+\delta_j].$$

We next assume that for any initial state S and any δ_j as above,

$$\bar{X}_j^t[S] \; < \bar{X}_j^t[S+\delta_j] \tag{1}$$

for $t=T$, and we show that the claim holds for $t=T+1$. Note that (1) implies

$$\bar{X}_k^t[S] \; < \bar{X}_k^t[S+\delta_j] \tag{2}$$

since $\bar{X}_k^t[S] = t - \bar{X}_j^t[S]$ for any t.

Let $S+\rho_i$ denote the vector S with ρ_i added to the i component, $i=j,k$. The expected sales after $T+1$ periods can be written in terms of the expected sales in the first period and the sum of the expected sales in the following T periods:

$$\bar{X}_j^{T+1}[S] = R_j[S] + R_j[S] \cdot \bar{X}_j^T[S+\rho_j] + R_k[S] \cdot \bar{X}_j^T[S+\rho_k]$$

and similarly,

$$\bar{X}_j^{T+1}[S+\delta_j] = R_j[S+\delta_j] + R_j[S+\delta_j] \cdot \bar{X}_j^T[S+\delta_j+\rho_j] + R_k[S+\delta_j] \cdot \bar{X}_j^T[S+\delta_j+\rho_k].$$

As shown in the initial case, $R_j[S] < R_j[S+\delta_j]$, so it is sufficient to show,

$$R_j[S] \cdot \bar{X}_j^T[S+\rho_j] + R_k[S] \cdot \bar{X}_j^T[S+\rho_k] <$$
$$R_j[S+\delta_j] \cdot \bar{X}_j^T[S+\delta_j+\rho_j] + R_k[S+\delta_j] \cdot \bar{X}_j^T[S+\delta_j+\rho_k].$$

By the inductive hypothesis (1),

$$R_j[S] \cdot \bar{X}_j^T[S+\rho_j] + R_k[S] \cdot \bar{X}_j^T[S+\rho_k] <$$
$$R_j[S] \cdot \bar{X}_j^T[S+\delta_j+\rho_j] + R_k[S] \cdot \bar{X}_j^T[S+\delta_j+\rho_k].$$

Therefore, it suffices to show,

$$R_j[S] \cdot \bar{X}_j^T[S+\delta_j+\rho_j] + R_k[S] \cdot \bar{X}_j^T[S+\delta_j+\rho_k] <$$
$$R_j[S+\delta_j] \cdot \bar{X}_j^T[S+\delta_j+\rho_j] + R_k[S+\delta_j] \cdot \bar{X}_j^T[S+\delta_j+\rho_k],$$

or equivalently,

$$0 < \bar{X}_j^T[S+\delta_j+\rho_j]\big(R_j[S+\delta_j]-R_j[S]\big)+\bar{X}_j^T[S+\delta_j+\rho_k]\big(R_k[S+\delta_j]-R_k[S]\big). \tag{3}$$

Now,

$$\big(R_k[S+\delta_j]-R_k[S]\big) = (1-R_j[S+\delta_j])-(1-R_j[S]) = -\big(R_j[S+\delta_j]-R_j[S]\big).$$

Thus, equation (3) can be rewritten as,

$$\begin{aligned}0 &< \bar{X}_j^T[S+\delta_j+\rho_j]\big(R_j[S+\delta_j]-R_j[S]\big)-\bar{X}_j^T[S+\delta_j+\rho_k]\big(R_j[S+\delta_j]-R_j[S]\big)\\ &= \big(\bar{X}_j^T[S+\delta_j+\rho_j]-\bar{X}_j^T[S+\delta_j+\rho_k]\big)\big(R_j[S+\delta_j]-R_j[S]\big)\end{aligned}$$

which follows from the inductive hypothesis and the fact that $R_j[S+\delta_j] > R_j[S]$.

□

Theorem 2. *Strictly increasing firm j's transition weight ρ_j strictly increases the expected total sales of firm j after T periods for $T > 1$.*

Proof. As in the previous proof we assume that the transition weights take on values with certainty, i.e. each ρ_i is a constant. The general proof follows similarly. Let ρ' equal ρ with ρ_j replaced by $\rho'_j = \rho_j + \delta$ for some $\delta > 0$. It suffices to show that

$$\bar{X}_j^T[S;\rho] < \bar{X}_j^T[S;\rho'].$$

The proof is by induction. We begin with the case $T = 2$. Let $\Sigma = \sum_{i=1}^{M} S_i$.

The expected total sales for firm j after two periods given transition weights ρ is

$$\bar{X}_j^2[S;\rho] = \left(\frac{S_j}{\Sigma}\right)\left(\frac{S_j+\rho_j}{\Sigma+\rho_j}\right) + \frac{S_j}{\Sigma} + \frac{S_j}{\Sigma}\sum_{i\neq j}\frac{S_i}{\Sigma+\rho_i}. \tag{4}$$

With transition weights ρ', the expected total sales for firm j after two periods equals

$$\bar{X}_j^2[S;\rho'] = \left(\frac{S_j}{\Sigma}\right)\left(\frac{S_j+\rho'_j}{\Sigma+\rho'_j}\right) + \frac{S_j}{\Sigma} + \frac{S_j}{\Sigma}\sum_{i\neq j}\frac{S_i}{\Sigma+\rho_i}. \tag{5}$$

Expression (5) is larger than expression (4) if and only if

$$\frac{S_j+\rho_j}{\Sigma+\rho_j} \;<\; \frac{S_j+\rho'_j}{\Sigma+\rho'_j},$$

which follows from $\rho'_j > \rho_j$ and $\Sigma > S_j$.

We next assume that

$$\bar{X}_j^t[S;\rho] \;< \bar{X}_j^t[S;\rho'],$$

for $t = T$ and show that the claim holds for $t = T+1$. As in the previous proof, the expected sales after $T+1$ periods can be written in terms of the expected sales in the first period and the sum of the expected sales in the following T periods:

$$\bar{X}_j^{T+1}[S;\rho] = R_j[S;\rho] + \sum_{i=1}^{M} R_i[S;\rho]\cdot \bar{X}_j^T[S+\rho_i;\rho] \tag{6}$$

and

$$\bar{X}_j^{T+1}[S;\rho'] = R_j[S;\rho'] + \sum_{i=1}^{M} R_i[S;\rho']\cdot \bar{X}_j^T[S+\rho'_i;\rho'].$$

The transition weights do not affect the initial state, so $R_i[S;\rho] = R_i[S;\rho']$, and therefore

$$\bar{X}_j^{T+1}[S;\rho'] = R_j[S;\rho] + R_j[S;\rho] \cdot \bar{X}_j^T[S+\rho'_j;\rho'] + \sum_{i \neq j} R_i[S;\rho] \cdot \bar{X}_j^T[S+\rho_i;\rho']. \tag{7}$$

Subtracting $R_j[S;\rho]$ from (6) and (7), it remains to show that

$$\sum_{i=1}^{M} R_i[S;\rho] \cdot \bar{X}_j^T[S+\rho_i;\rho] < R_j[S;\rho] \cdot \bar{X}_j^T[S+\rho'_j;\rho'] + \sum_{i \neq j} R_i[S;\rho] \cdot \bar{X}_j^T[S+\rho_i;\rho'].$$

By the inductive hypothesis,

$$\bar{X}_j^T[S+\rho_i;\rho] < \bar{X}_j^T[S+\rho_i;\rho'].$$

So, it suffices to show that

$$\bar{X}_j^T[S+\rho_j;\rho] < \bar{X}_j^T[S+\rho'_j;\rho'].$$

By Theorem 1,

$$\bar{X}_j^T[S+\rho_j;\rho] < \bar{X}_j^T[S+\rho'_j;\rho],$$

and by the inductive hypothesis

$$\bar{X}_j^T[S+\rho'_j;\rho] < \bar{X}_j^T[S+\rho'_j;\rho'],$$

so the claim follows by induction. $\square$