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Dynamic Pricing with Point Redemption

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Many sellers allow consumers to pay with reward points instead of cash or credit card. While the revenue implications of cash and credit card purchases are transparent, the implication of reward sales are not trivial, especially when a firm that issues points is not a seller. In this case, a seller receives a monetary compensation or reimbursement from the point issuer when a consumer purchases the good by redeeming points. In this paper, we examine how reward sales influence a seller's pricing and inventory decisions. In particular, we consider a consumer who can choose to pay with cash or points based on her attributes – reservation price, point balance, and the perceived value of a point. Then, we incorporate this consumer choice model into a dynamic pricing model where a seller earns revenues from both cash and reward sales. In contrast to an intuition that adding reward sales will increase sales and revenue, we show that the effect of reward sales on the seller's price is non-trivial as the seller could either add a premium or offer a discount depending on the inventory level, time, and the reimbursement rate. Furthermore, such price adjustments (premium or discount) can attenuate the optimal mark-up or mark-down level, and reduce the price fluctuation caused by inventory level and remaining time. We also investigate several settings where the seller has a different level of operational control over reward sales and show that the seller blocks the reward sales only when the reimbursement rate is very low. That is, allowing reward sales is still better even when the revenue from the reward sales is smaller than the cash sales. In addition, we find that a seller with an ability to control reward availability (i.e., allow a reward sale or not) can achieve a revenue similar to the revenue of a seller with ability to change point requirements and price at the time time.

Key words: Dynamic Pricing, Reward Point Redemption, Consumer Choice

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1. Introduction

In several industries including hospitality, financial service, transportation, and retailing, firms allow consumers to make purchases with their reward or loyalty points instead of cash or credit cards; point or miles (e.g. Hilton's Honors, and Member's Rewards by American Express) are awarded for consumers' past purchases, and the accumulated points can be later redeemed for additional product or services. In a sense, these reward points function as a currency that consumers can spend, and as a result, point redemption (or reward sales) can have significant impacts on both consumers and the seller. The main goal of this paper is to understand how the reward sales affect consumers' decisions as well as the seller's pricing and inventory policy.

Loyalty or reward program memberships are widespread in the U.S.: In 2014, there are more than 3 billion memberships, an average of 10 memberships per person (Colloquy 2015). With the growth in membership, points that are issued and redeemed in each year are enormous. For instance, passengers of Southwest Airline redeemed 8.3 million award flight tickets, which represented 13% of revenue passenger miles flown in 2016, up from 11% in 2014 (Southwest 2016). As an another example, Hilton Honors program awarded 5.4 million reward nights and more than 95,800 items through the Honors Global Shopping Mall in 2015 (HiltonWorldwide 2015a).

When consumers use points to purchase products, the financial implication to the seller varies depending on who issues points and who sells products. When the seller is indeed the point issuer, the seller entirely bears the cost of a reward sale without earning actual cash revenue[1]. In contrast, when the seller who actually provides goods or services is different from the point issuer (e.g., corporate-run reward programs with franchisees, joint/coalition reward programs), the seller receives compensatory revenue (or reimbursement) from the firm who runs the reward program and issues points. Such cases can be easily found in many franchised industries (such as hotels, fast food, services). For example, in the hotel industry, a large proportion of the hotel properties are franchised and operated by the property management companies: about 70% of Marriott

[1] Reward points represent a promise for future service, and their monetary value thus counts as a liability on the point issuing firm's balance sheets. For the accounting standards governing the reward liabilities calculations and its implications, see, e.g., Chapple et al. 2010, Chun et al. 2015.

6080 properties are franchised or licensed (Marriot 2016), and 85% of Hilton's 4300 properties are franchised (HiltonWorldwide 2015b). Thus, when consumers redeem brand loyalty points to stay in the franchised properties, the franchisee (the seller) receives compensatory revenue from the hotel-brand headquarter who issues reward points.

One interesting aspect to examine is the extent to which sellers can control these reward sales, which varies significantly in practice. In some cases, a seller should always allow point redemptions (under so-called "no black-out policy") while a point issuer determines a point requirement. In other cases, a seller can control the availability and/or point requirement of rewards, and the compensation amount can depend upon points redeemed. For instance, properties participating the Hyatt Gold Passport program are not allowed to have any restricted periods for standard-room free night awards[2], but properties participating IHG reward club may restrict award room inventory[3]. As an another example, the point requirement for one night stay (standard room) at Park Hyatt Beaver Creek, CO is fixed to 30,000 points (by the Hyatt headquarter) while the cash price for the same room is dynamically controlled by the seller and ranges from \$160 to \$550 (Milecards 2016). On the other hand, both the cash price and point requirement for one night stay at Hilton Ocean City in Maryland ranged from \$169 to \$529 and from 50,000 to 80,000 points respectively (BoardingArea 2015). In practice, reward terms and conditions vary significantly within and across industries. As a consequence, different sellers have different levels of discretion over reward sales. Given large volume of reward sales (e.g., more than 5.4 millions room nights for Hilton), the extent of operational control over reward sales certainly affect inventory availability and profits.

In this paper, we consider a seller who offers point redemption as a method of payment and receives reimbursement from a point issuer. We study the implications of such reward redemptions and reimbursements on the seller's pricing and inventory rationing policies compared to the seller who does not allow reward sales. We further investigate how the seller's operational control over reward sales (e.g., availability of reward sales or point requirements) affects the seller's policy and

[2] https://world.hyatt.com/content/gp/en/member-benefits.html

[3] https://www.ihg.com/content/us/en/customer-care/member-tc#rewardnights

revenue. Specifically, we examine the following: (*i*) how do reward sales affect the seller's posted (cash) price?, (*ii*) if the seller can dynamically ration inventory for reward sales, when should the seller block/allow reward sales?, and (*iii*) if the seller can also set point requirements, how should the seller change price and requirements and by how much does it benefit the seller?

In order to address our research questions, we model a consumer's choice (whether to buy or not, and whether to pay for the product with cash or point) based on the three attributes that are intrinsic to each consumer: willingness to pay, point balance, and the perceived value of a point (i.e., the dollar value that consumers subjectively associated with one point). Then, we characterize a consumer's decision as a function of price and point requirement. We then incorporate our model into a standard dynamic pricing (and rationing) model for the seller who maximizes the total expected revenue: the sum of the revenues from cash sales and reward sales. This allows us to closely examine how reward sales influence the seller's price.

In order to account for the different levels of seller's operational control over reward sales, we consider several settings. We first examine the setting where the seller can only change cash price and has no discretion over the reward sales. In this case, no rationing is allowed (no black-out), and both point requirement and reimbursement rate are fixed. We then extend our analysis to the case where the seller can dynamically block inventory for reward sales (*black out*) while adjusting the price. Lastly, we generalize our analysis to the case where the seller can change both (cash) price and point requirement, and the reimbursement rate depends on the point requirement.

1.1. Summary of Contributions

We study the dynamic pricing and inventory control problem of a seller whose products can be purchased with cash or reward points. We consider several important factors that influence a consumer's decision on whether to buy or not, and whether to pay in cash or with points in response to the (cash) price, point requirement for redemption, and availability of rewards and/or points. We model this decision process based on consumer heterogeneity with respect to reservation price, point balance, and the perceived value of a point. Incorporating consumer's model into the seller's problem, we examine the implications of reward sales where a seller receives a compensatory revenue from a point issuer, and we study how redemption and reimbursement rate influence the seller's dynamic pricing and inventory decisions.

We show that consumers view two different purchase options (cash vs. point) as imperfect substitutes in the following sense. As cash price (or point requirement) increases, some consumers switch from paying cash to redeeming points (or *vice versa*), but not all. We examine the overall implication on total demand. Since consumers in our model are heterogeneous in three aforementioned attributes, the resulting seller's problem is not tractable under IGFR assumption widely used in the classic dynamic pricing literature that captures a single attribute (often, willingness to pay): see Ziya et al. 2004 and Lariviere 2006 for more details. We show that, under reasonable assumptions, the revenue function is well-behaved (e.g., the revenue function is unimodal in price and the price-point pair is monotone in inventory and time). We believe that this is one of the first papers in dynamic pricing literature that study the case where consumers are heterogeneous in several attributes and have multiple payment options.

One may think that allowing reward sales increases the overall chance of sales in each period thus increases price (since a product can be sold in many different ways). We find that this is not the case. We show that the seller's price can be either higher or lower than the price of the seller who does not allow a reward sale (which we call *cash-only price*). For instance, when the reimbursement rate is very low relative to the *cash-only price*, the seller offers a discount in order to induce more cash sales (compared to the cash-only price). In the opposite case, the seller increases price (adds a premium) in order to induce more reward sales while reducing cash sales. Furthermore, we show that such price adjustments (premium or discount) attenuate the extent of mark-up or mark-down in the cash-only price, and thus reduces the price fluctuation.

If the seller has discretionary control over reward sales, we show that it is optimal to block a reward sale (i.e., disallow point redemptions) only when inventory level (relative to the remaining selling season) or a reimbursement rate is "very" low. In both cases, the *cash-only price* will be significantly higher than a reimbursement rate, and thus the seller completely block the reward sales and only sells products in cash. Interestingly, we also show that allowing reward sales are still better for the seller even when the reimbursement rate is quite lower than the *cash-only price*. This is because the seller can still benefit from the increased likelihood of total purchase/sales (both in cash and points) when the reward sales are allowed. In that case, the seller offers a price discount in order to induce more cash sales while allowing reward sales. We also analyze the case where the seller can dynamically change both price (for cash sales) and point requirement (for reward sales), and characterize the optimal policy. In this case, although points and cash are substitutable, we find that the optimal price is not necessarily increasing in point requirement. When inventory level changes, it is possible that the price goes up while the point requirement goes down (and vice versa). While the optimal policy could be quite complicated in general, we provide a condition under which the optimal policy (price and point requirement) is monotone. Through numerical study, we demonstrate how much additional benefit the seller can realize in flexibility over reward sales. In particular, we show that, while the seller significantly benefits from having an option to block reward sales, the option to control point requirements (further segmenting reward sales with multiple tiers of point requirements) can only marginally improves the revenue.

The remainder of this paper is organized as follows. Section 2 provides a survey of relevant literature. Section 3 presents the consumer and seller models and provide analytical results on the effect of reward sales. In Sections 4 and 5, we provide extensions of our base model to the cases in which the seller further controls availability of reward sales and/or point requirements along with price.

2. Related Literature

Our paper is closely related to the literature on the loyalty reward programs. A substantial body of marketing literature focuses on the impact of reward programs on customer retention and lifetime value of a consumer. For example, Lewis (2004), Liu (2007) and Lederman (2007) analyze the data (from retailers and airlines) and show their reward programs have a positive effect on annual purchases. Similarly, using a financial services firm's customer database, Verhoef (2003) demonstrates that its program affects both customer retention and share-of-wallet. Different from these studies, we focus on the impact of reward sales/point redemptions on a seller as such practice affects inventory availability and thus price.

The perceived value of a point and redemption behavior have also been studied. In empirical studies, Liston-Heyes (2002) and Basumallick et al. (2013) demonstrate that the perceived value of points are quite different across consumers: some consumers may opt to value a point to a cash value they would have paid and others may think of a purchasing power of the entire points they possess. As a result, the way that a consumer uses points is quite different for different consumers. Similarly, Kivetz and Simonson (2002a) and Kivetz and Simonson (2002b) suggest that the point value varies and can depend on an individual's efforts to obtain points or on the intrinsic guilt about hedonic consumption. Recently, Chun and Hamilton (2017) also confirm that consumers are heterogenous in perceived value of a point through a series of lab experiments. Kopalle et al. (2012) and Dorotic et al. (2014) demonstrate that reward sales can cannibalize cash sales and revenue. Building upon these empirical findings, our model incorporates the variability of point value and captures a choice behavior including a cannibalization effect.

Our paper also contributes to a growing literature on managing (or operating) reward programs. Kim et al. (2004) study reward programs in the context of capacity management as the reward sales could be used to reduce excess capacities during the period of low demand. Sun and Zhang (2015) study the problem of setting optimal duration for a reward and investigate the tradeoff between long and short expiration dates. Chun and Ovchinnikov (2015) consider a recent change where several firms switch their programs from the “quantity-based” toward ”spending-based” design, and study its implication. Lu and Su (2015) also study the same two designs for a firm setting capacity limits for rewards in a classical Littlewood model. Chun et al. (2015) study the problem of optimally setting the monetary value of points in view of inherent liabilities that points create. None of the above mentioned studies, however, explicitly study the impact of different terms of reward sales and reimbursement revenue in a multi-period setting.

Finally, our paper is related to the literature on dynamic pricing (for a general review, see, e.g., Bitran and Caldentey 2003 and Elmaghraby and Keskinocak 2003). Many studies on the topic have focused on how the seller should adjust the price based on remaining time and inventory throughout the selling horizon for a single product (e.g., Gallego and Van Ryzin 1994 and Bitran and Mondschein 1997), for multiple products (e.g., Zhang and Cooper 2005 and Maglaras and Meissner 2006), in the presence of strategic consumers (e.g., Aviv and Pazgal 2008, Ahn et al. 2007), and with negotiation (e.g., Kuo et al. 2011). By design, we use a standard dynamic pricing model (by Gallego and Van Ryzin 1994) and study how reward sales affect prices over a course of sales season. Up to our best knowledge, this is the first paper that examines the interaction between price and reward purchase (point redemption) in a multi-period dynamic pricing setting with a detailed consumer choice model.

3. The Model

We consider a seller with limited inventory of a product over a predetermined selling horizon, which is divided into T periods, indexed by $t \in \{1, 2, ..., T\}$. Each period is short enough so that at most one consumer arrives in a given period with probability $\lambda \in (0, 1)$. In each period, the seller can sell its product either in cash (*cash sale*) or with points (*reward sale*). That is, a consumer can purchase a product either by paying p in cash or redeeming q reward points (where q is a *point requirement*, the number of points required for a reward). Given both price and point requirement, a consumer decides whether to buy or not and whether to pay with cash or point. For each cash sale, the seller gets the revenue p, but for each reward sale, the seller receives the reimbursement from the point issuer. The amount of reimbursement depends on point requirement, q, and it is reasonable to assume that the reimbursement rate (amount), $R(\cdot)$, is a non-decreasing function of q. The seller's goal is to maximize its total expected revenue (the revenue from cash sales plus the revenue from reward sales) throughout the selling season.

Our basic setup utilizes the classical dynamic pricing model considered in Gallego and Van Ryzin (1994) and Bitran and Mondschein (1997) (and many others). This enables us to highlight how reward sales affect the seller's pricing (and rationing) policy compared to the seller who does not offer point redemption options. In particular, we consider several settings by varying how much the seller can control reward sales. This will help us to understand the value of operational flexibility in operating reward sales. We first consider a seller who only controls the cash price of a product and always allows reward redemption while the point requirements (and thus reimbursement rate) are fixed and exogenously determined. In Section 4 and Section 5, we extend our analysis and consider the cases where the seller can also control the availability and/or point requirement of rewards along with cash price, and the reimbursement rate varies depending on the number of point redeemed.

3.1. Consumers' Problem

An arriving consumer observes cash price (p) and point requirement (q), and decides whether to buy or not, and how to pay for a product (either paying p in cash or redeeming q points). We assume that a consumer maximizes her utility, which depends on the following three random attributes: (i) reservation price, which is the maximum price that a customer is willing to pay for the product (ii) point balance, which determines whether a consumer is able to buy with points or not, and (iii) perceived value of a point, which is the dollar value a consumer subjectively associated with one reward point.

Specifically, we model the consumer's reservation price by a random variable, V, which follows distribution $F(\cdot)$ and density $f(\cdot)$. If cash is the only available form of payment, consumers with reservation price higher than the current price will purchase the product. Thus, $\bar{F}(p) := 1 - F(p)$ represents the probability that a consumer would prefer to purchase with cash (instead of not making any purchases).

The variability in point balance reflects the fact that not all consumers belong to a reward program (i.e., they have zero point balance) and, even among members, only some have enough points to buy with points (i.e., have more than q points). We model the point balance as a non-negative random variable, W, with distribution $\beta(\cdot)$. Then, $\bar{\beta}(q) := 1 - \beta(q)$ represents the proportion of consumers who have sufficient points for redemptions: We call them *reward consumers* and the remaining consumers (either a non-member or a member without enough points) *cash-only consumers*.

Several empirical observations introduced earlier (Liston-Heyes 2002, Kivetz and Simonson 2002a, Kivetz and Simonson 2002b and Basumallick et al. 2013), suggest that the perceived value of a point is highly subjective and can depend on many factors. For this reason, not all consumers with the same reservation price and point balance will behave the same. Many factors such as how they acquire points (e.g., by purchasing a specific product, by using credit card, by directly paying for points, or as a promotional gift) and/or how she prefers to use points (e.g., for utilitarian or hedonic rewards) will influence an individual consumer's assessment. For example, $80,000$ Chase Rewards points can be redeemed for a business-class ticket from the US to Europe (cash-value of $\$6,000$) or a $800 Amazon gift card (PointsGuy 2015). Although it is obvious that using points for a flight ticket in this case saves more money, not all consumers will choose this option. For this reason, we model the perceived value using a non-negative random variable Θ with distribution $G(\cdot)$ and density $g(\cdot)$. That is, a point-worth of $\Theta = \theta$ (1 point is equal to $\$\theta$) means that q points has the equivalent monetary value of $\$\theta q$ to that consumer, making her indifferent between paying $\$\theta q$ in cash and redeeming q points.

We assume that all three attributes – V, W, and Θ are independent as there is no clear empirical evidence or underlying cognitive mechanism to relate one attribute to another. For instance, even when two consumers have the same reservation price for a product, they are likely to have different point balances and have different values for points. Similarly, there is no clear relationship between point balance and reservation price (or between point balance and perceived point value). Hence, not all the consumers with the same point balance will have the same reservation price or same perceived value. In fact, several empirical papers generally support the lack of structured relationships and demonstrate that point balance will not not have a significant effect on a purchase and redemption decision as long as consumers have enough points to make a redemption (e.g., Drèze and Nunes 2004, Chun and Hamilton 2017).

Under these assumptions, for given p and q, consumer's utility (surplus) of each purchase option depends upon three attributes v, w, and θ (realizations of V, W, and Θ) as follows: If she obtains a product by paying p in cash, the corresponding utility is $v-p$. If she makes a reward purchase (q point redemption), her utility is $v-\theta q$. If she does not purchase, she earns the reservation utility of the outside option, which is set to be zero. In case of a *cash-only consumer* who does not have enough points (i.e., $w<q$), her decision is reduced to either buying with cash or not buying, i.e., $\max\{v-p,\, 0\}$. On the other hand, a *reward consumer* (who has enough points) compares three options (cash purchase, point purchase and no purchase) and makes a choice to maximize her utility, i.e., $\max\{v-p,\, v-\theta q,\, 0\}$.

Given this consumer model, the following Lemma and Figure 1 characterize the likelihood of each choice behavior.

LEMMA 1. *Given price p and point requirement q:*

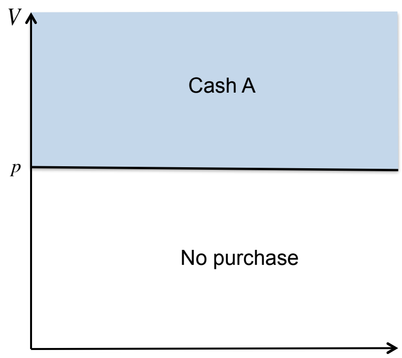
(a) *Consumers' decision falls into three cases:*

- *With probability $P(cash\ purchase) = \beta(q)\bar{F}(p) + \bar{\beta}(q)\bar{F}(p)\bar{G}(p/q)$, a consumer buys with cash.*
- *With probability $P(point\ purchase) = \bar{\beta}(q)\int_0^{p/q}\bar{F}(qx)dG(x)$, a consumer buys with points.*
- *With probability $1-P(cash\ purchase)-P(point\ purchase)$, a consumer does not purchase.*

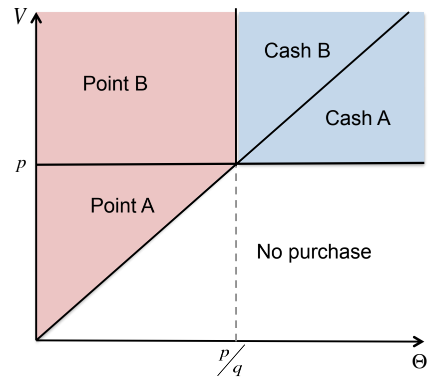
(b) *Compared to a cash-only consumer, a reward consumer buys with cash less likely, but overall (both using cash and points), buys a product more likely.*

$$P\left(cash\ purchase\middle|loyalty\right) \leq P\left(cash\ purchase\middle|cash\text{-}only\right) \leq P\left(cash\ \&\ point\ purchase\middle|loyalty\right).$$

With probability $\beta(q)$, a consumer is a *cash-only consumer*, thus, she purchases if and only if $v \geq p$ (as shown in Figure 1-(a)). With probability $\bar{\beta}(q)$, an arriving consumer has enough points, and her choice depends on both v and θ: a cash purchase is optimal if and only if $v \geq p$ and $\theta q \geq p$, a point redemption is optimal if and only if $v \geq \theta q$ and $p > \theta q$. When her reservation price is low and point worth is high, no purchase is optimal (as shown in Figure 1-(b)). Note again that all three attributes – reservation price, point balance, and point worth – affect a consumer's choice, and these choice probabilities are described in Part (a) of Lemma 1.



(a) *cash-only consumer*



(b) *reward consumer*

Figure 1 **Consumer behavior depending on their types (cash-only consumer or reward consumer), reservation price (V), and point-worth (Θ) given price p and point requirement q.**

Part (b) shows that a *reward consumer* is more likely to purchase (combining cash and point purchases) than a *cash-only consumer*. This implies that allowing point redemption increases the overall probability of a purchase, but decreases the probability of a cash purchase; allowing point redemption creates a new stream of demand from reward consumers who find the price p too high but redeeming q points not too costly (the region *Point A* in Figure 1-(b)). On the other hand, some consumers (those with high point balance and low perceived value of a point) who would have purchased with cash if there were no options to buy with points, can switch from a cash purchase to a point purchase (the region *Point B* in Figure 1-(b)).

The next result further characterizes how consumers respond to a change in price or point requirement. We note that the results hold regardless of distributions.

Lemma 2. (*a*) *For given point requirement q,*

$$\frac{dP(\textit{cash purchase})}{dp} \leq 0, \quad \frac{dP(\textit{point purchase})}{dp} \geq 0, \quad \frac{dP(\textit{total purchase})}{dp} \leq 0.$$

(*b*) *For given price p,*

$$\frac{dP(\textit{cash purchase})}{dq} \geq 0, \quad \frac{dP(\textit{point purchase})}{dq} \leq 0, \quad \frac{dP(\textit{total purchase})}{dq} \leq 0.$$

Lemma 2 shows that cash and point purchases are indeed imperfect substitutes as the increase of price (or point requirement) results in some consumers switching from a cash to point purchase (or vice versa) and others not buying.

Part (a) indicates that when price increases, a cash purchase becomes less attractive while a point purchase become relatively more attractive. However, decrease in cash sales is always bigger than increase in reward sales and, as a result, the probability of total purchase (in cash and points) decreases. This result is derived from different behaviors of consumers: 1) *cash-only consumers* simply get priced-out as p increases, 2) some *reward consumers* whose perceived value of a reward point is high enough will not consider point purchase (even though they have enough points) and drop out as p increases (the region *Cash A* in Figure 1-(b)).

Similarly, part (b) shows that when the point requirement increases, some consumers who buy with points would switch to either pay with cash or do not purchase at all. This result, however, is based on two different effects of the increased point requirement; as q increases, point-purchase becomes less attractive, and at the same time, fewer people become eligible to use points (i.e., $\bar{\beta}(q)$ decreases in q).

Remark 1. In some cases, consumers receive a slightly inferior product when purchasing with points, which makes the product they get with points less valuable. For instance, passengers who purchase award flight tickets are placed to a different priority for a seat upgrade. Our model can accommodate such scenarios with a slight modification; suppose that a consumer accrues $\bar{q}$ points only if she purchases with cash. With the perceived value of a point θ, disutility of a point purchase in this case can be represented as $\theta\bar{q}$ and thus, the utility of a point purchase becomes $v-\theta(q+\bar{q})$. We note that the modification does not alter the results and general insights.

3.2. The Seller's Problem

We now consider the seller's pricing problem in each period. We first start with a case where the seller only controls the price and always allows point redemption throughout a selling horizon with fixed point requirement and reimbursement rate. Later, we extend our analysis in two ways: in Section 4, we consider a seller who can block the reward sales (*black-out model*), and in Section 5, we consider a seller who chooses not only price but also point requirement in each period (*dynamic adjustment model*).

We incorporate the results in Lemma 1 and embed consumer choice for given (p,q) in a seller's dynamic pricing problem. Since q and R are fixed, for ease of notation, we use β for $\beta(q)$, $\bar{\beta}$ for $\bar{\beta}(q)$, and R for $R(q)$ throughout this section.

Given y units in inventory in period t, the sellers problem of setting the (posted) price is given by the following optimality equations:

$$V_t(y) = \max_{p\geq 0} \; J_t(p,y) \quad \text{ for } y>0,\; t=1,2,\dots,T, \text{ and} \tag{1}$$
$$V_0(y)=0 \text{ for } y\geq 0, \quad \text{and} \quad V_t(0)=0 \text{ for } t=1,2,\dots,T,$$

where

$$\begin{aligned} J_t(p,y) = \lambda\bar{\beta}\Big[\bar{F}(p)\bar{G}(p/q)(p+V_{t-1}(y-1)) + \int_0^{p/q} \bar{F}(qx)dG(x)(R+V_{t-1}(y-1)) \\ + \left(1-\bar{F}(p)\bar{G}(p/q) - \int_0^{p/q}\bar{F}(qx)dG(x)\right)V_{t-1}(y)\Big] \\ + \lambda\beta\Big[\bar{F}(p)(p+V_{t-1}(y-1)) + F(p)V_{t-1}(y)\Big] + (1-\lambda)V_{t-1}(y). \end{aligned}$$

We define $\Delta_t(y) = V_t(y) - V_t(y-1)$, which represents the marginal value of inventory, and rewrite $J_t(p,y)$ as

$$\begin{aligned} J_t(p,y) = \lambda\bar{\beta}\Big[\bar{F}(p)\bar{G}(p/q)(p-\Delta_{t-1}(y)) + \int_0^{p/q}\bar{F}(qx)dG(x)\,(R-\Delta_{t-1}(y))\Big] \\ + \lambda\beta\Big[\bar{F}(p)\,(p-\Delta_{t-1}(y))\Big] + V_{t-1}(y). \end{aligned} \tag{2}$$

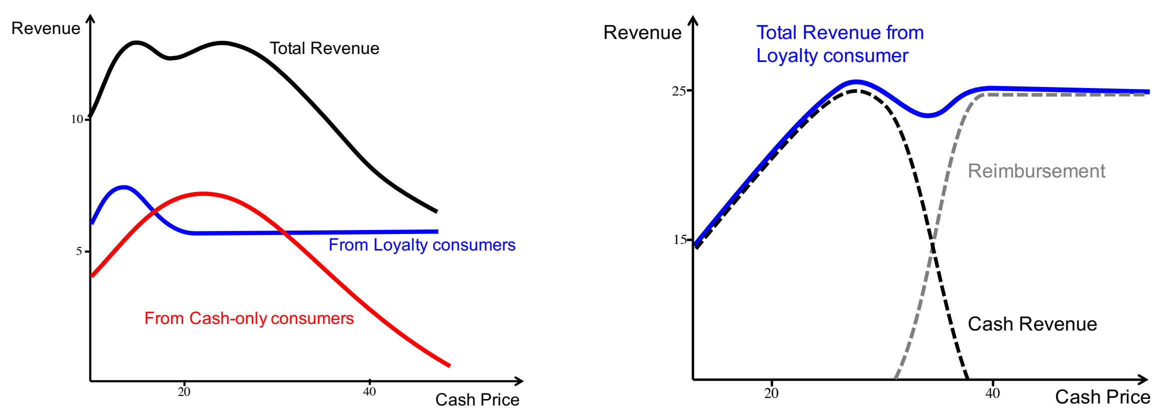
From (2), we observe that $J_t(p,y)$ has three sources of revenue.

i) cash-sales revenue from a *cash-only consumer* : $\lambda\beta\bar{F}(p)(p-\Delta_{t-1}(y))$

ii) cash-sales revenue from a *reward consumer* : $\lambda\bar{\beta}\bar{F}(p)\bar{G}(p/q)(p-\Delta_{t-1}(y))$

iii) reward-sales revenue from a *reward consumer* : $\lambda\bar{\beta}\int_0^{p/q}\bar{F}(qx)dG(x)\,(R-\Delta_{t-1}(y))$

Note that the first term, $\lambda\beta\bar{F}(p)(p-\Delta_{t-1}(y))$, has been extensively studied in the literature, as this is the revenue term in classical dynamic pricing problems. It has been shown that this is unimodal in p if $F(\cdot)$ has an increasing generalized failure rate (IGFR), see Ziya et al. (2004) and Lariviere (2006) for more details. Similarly, we can show that cash-sales revenue from a *reward* *consumer* (the second term) is also unimodal in p if both $F(\cdot)$ and $G(\cdot)$ are IGFR. Lastly, from Lemma 2(a), we know that more consumers would choose to use points as price increases. Thus, the reward-sales revenue (the third term) increases in p, and thus unimodel as well. However, $J_t(p,y)$, which is the sum of there three terms, is not necessarily unimodal in p although each term is unimodel under a simple condition (IGFR). This imposes a critical challenge in characterizing optimal policy.



(a) *cash-only consumers* + *reward consumers* (b) Cash revenue + Reimbursement

Figure 2 **Examples of not unimodal expected revenue-to-go functions in price when both** $F(\cdot)$ **and** $G(\cdot)$ **are truncated normal (i.e., IGFR):** $F \sim truncated\, N(30,10)$**,** $q=1$**,** $t=1$**, and** $y=1$**. In figure (a),** $\bar{\beta}=0.6$**,** $G \sim truncated\, N(15,2)$ **and** $R=10$**. In figure (b),** $\bar{\beta}=1$**,** $G \sim truncated\, N(28,1)$ **and** $R=30$**.**

Figure 2 illustrates two examples showing how the seller's revenue changes in price. One may think that the lack of unimodality comes from the fact that there are two types of consumers, *cash-only consumers* and *reward consumers*, as illustrated in Figure 2-(a). This intuition, however, is not correct. The expected revenue function is not necessarily unimodal even when all consumers are *reward consumers* as shown in Figure 2-(b). Note again that *reward consumers* have two payment options (cash and point), and as price increases, some of those consumers whose valuations are close to p either substitute to buy with points or decide not to purchase at all. Thus, the change of seller's revenue in price not only depends on $F(\cdot)$ and $G(\cdot)$, but also on the interaction between two. As a result, the expected revenue-to-go function could be non-unimodal even when both $F(\cdot)$ and $G(\cdot)$ are IGFR.

Nonetheless, we identify sufficient conditions under which the revenue function is unimodal in price and characterize the optimal price and its properties.

THEOREM 1. *If the following three conditions are satisfied for a given q:*

A1. $F(\cdot)$ *and* $G(\cdot)$ *are with an increasing failure rate (IFR),*

A2. $\dfrac{f(p)}{\bar{F}(p)} \geq \dfrac{g(p/q)}{\bar{G}(p/q)}$ *for any* p,

A3. $\dfrac{f(p)}{\bar{F}(p)} \Big/ \dfrac{\bar{\beta} g(p/q)}{1-\bar{\beta} G(p/q)}$ *is non-decreasing in* p,

then there exists a unique optimal price, $p_t^*(y)$, *which satisfies the first-order condition for* $J_t(p,y)$.

As discussed, the condition *A1* in Theorem 1 is widely assumed in the pricing literature (see Ziya et al. 2004 and Lariviere 2006) and is satisfied by a large range of probability distributions including uniform, normal, and exponential and their truncated versions. This condition guarantees that cash-sales revenues from *cash-only* and *reward consumers* are unimodal, separately. However, because of the substitution between and cash and reward purchases discussed above, this does not necessarily ensure the unimodality of the revenue function. Conditions *A2* and *A3* address this issue.

Recall from Lemma 2-(a) that an increase in price induces some consumers who pay with cash either not to purchase at all (*price-out effect*) or to switch to a point-purchase (*substitution effect*). Condition *A2* implies that, as price increases, the number of consumers who are priced out increases faster than the number of consumers who switch to a point purchase. In other words, the *price-out effect* dominates the *substitution effect*. Condition *A3* further implies that the *price-out effect* grows faster than the *substitution effect* as price increases. Both conditions suggest that an increase in price always has a bigger impact on the price-out effect than the substitution effect.

It can be also shown that conditions *A2* and *A3* are satisfied if the condition in the following corollary is satisfied, and in that case, the revenue function is unimodal even when all consumers are *reward consumers* ($\beta = 1$).

COROLLARY 1. *The sufficient conditions for A2 and A3 are as follows: for a given q, the hazard ratio of the reservation price distribution to the q point-worth distribution,* $\dfrac{f(p)}{\bar{F}(p)} \Big/ \dfrac{g(p/q)}{\bar{G}(p/q)}$, *is greater* *than* 1 *and increasing in price.*

In fact, these conditions presented have appealing practical implications. In practice, only about a quarter of newly issued frequent-flyer miles (or equivalently 6.3% of the accumulated miles) is redeemed in 2004, while the rest are accumulated or expired (Economist 2005). Another study shows that at least one third of issued points ($15 billion worth) in the United States each year are never redeemed and go expired (Colloquy 2011). This might be because of high point requirements (difficult to redeem) or complex redemption rules. However, even when points are easy to redeem and there is no incentive for the point accumulation, consumers tend to accumulate and stockpile points rather than using (Stourm et al. 2015). While there could be several plausible explanations including cognitive cost and psychological motivation (e.g., a high value on the possession of points itself), all of these empirical evidence suggest that *A2* and *A3* are consistent with these examples since both conditions are easily satisfied when consumers are hesitant to use points (i.e., stockpile points).

Based on Theorem 1, we now examine the properties associated with the optimal price. In order to facilitate the analysis on this matter, we first define an auxiliary optimization problem,

$$V_t^c(y) = \max_p \left\{ \lambda \bar{F}(p)\,(p + V_{t-1}(y-1)) + \left(1 - \lambda \bar{F}(p)\right) V_{t-1}(y) \right\},\ y > 0,\ t = 1, \ldots, T, \tag{3}$$

which represents a pricing problem of the seller who does not offer a reward redemption in period t, but follows the optimal policy with reward sales from period $t-1$ and onward. That is, the revenue-to-go from period $t-1$ onward is still $V_{t-1}(\cdot)$ as defined in (1).

Let $p_t^c(y)$ be the solution to (3), which we call *cash-only price*, the best price the seller can charge if only a cash purchase is available in period t. We also denote the optimal price (the solution to (2)) that the seller can charge considering both cash and reward sales by $p_t^*(y)$. When a reward sale is available, the marginal value of inventory may increase (as the seller has an additional channel to sell products), and if this intuition holds, the optimal price should be always higher than the *cash-only price*: i.e., $p_t^*(y) > p_t^c(y)$ for any t and y. This logic is further supported by the fact that, for a given price, the likelihood of total sales is always higher when the reward sales are available (Lemma 1-b). However, the next result shows that this is not the case.

Proposition 1. *Given y units of inventory and t periods to go until the end of the season:*
(a) If the reimbursement rate is higher than the cash-only price price ($R > p_t^c(y)$), the reward sales induce the seller to add a premium on the cash-only price: $p_t^c(y) < p_t^*(y) < R$.
(b) If the reimbursement rate is lower than the cash-only price ($R < p_t^c(y)$), the reward sales induce the seller to offer a discount on the cash-only price: $R < p_t^*(y) < p_t^c(y)$.

This proposition indicates that the optimal price $p_t^*(y)$ can be higher or lower than the *cash-only price* $p_t^c(y)$ depending on whether $p_t^c(y)$ is higher or lower than R (Note that $p_t^c(y)$ is not fixed and monotonically changes in y and t). To elaborate this further, consider the case when $R > p_t^c(y)$. Suppose that the seller charges the *cash-only price*, $p_t^c(y)$, but allows a reward purchase. Since $R > p_t^c(y)$, the seller gains more from a reward sale than a cash sale, and we know from Lemma 2-(a) that the seller can increase reward sales by increasing price; in other words, the seller adds a premium: $p_t^c(y) < p_t^*(y)$. Interestingly, the amount of a premium is chosen so that $p_t^*(y)$ does not exceed R. To understand why, recall that the primary reason for a premium is to induce some consumers with a relatively lower willingness to pay in cash to buy with points while still earning a reasonable amount of revenue from cash-sales. However, if this premium becomes too high ($p > R$), very few consumers will buy in cash in the first place, thus the seller makes very poor cash-sales to the point that the total revenue decreases (decrease in cash-sales revenue is larger than increase in reward-sales revenue). Hence, $p_t^*(y)$ should always lie between $p_t^c(y)$ and R.

Similarly, when $R < p_t^c(y)$, the seller prefers cash sales to reward sales. If the seller charges $p_t^c(y)$ while allowing reward sales, the seller will get the lower revenue from any consumer who buy with points. In anticipation of this, the seller offers a discount to induce more consumers to buy with cash. Once again, $p_t^*(y)$ should always lie between R and $p_t^c(y)$ because, if price falls below R, the seller does not earn much from both cash and reward sales.

The following result further shows that the region that the seller offers a discount or premium is monotone in inventory level and remaining periods.

Corollary 2. *Given y units of inventory and t periods to go until the end of the season:*

(a) If it is optimal to offer a discount on the cash-only price, then it is also optimal to offer a discount with $y-1$ units of inventory or/and $t+1$ periods to go.

(b) If it is optimal to add a premium on the cash-only price, then it is also optimal to add a premium with $y+1$ units of inventory or/and $t-1$ periods to go.

This result identifies when the reward sales will induce a mark-up or mark-down. In the standard dynamic pricing literature (e.g., Gallego and Van Ryzin 1994 and Bitran and Mondschein 1997), it is shown that a mark-up occurs, for instance, when the inventory level changes from y to $y-1$ given t periods to go. However, Proposition 1 and Corollary 2 indicate that in regions where the *cash-only price* is sufficiently high (compared to R), the seller must reduce such a mark-up (and keep the price not too high) and discourage some consumers from buying with points. Similarly, in regions where the *cash-only price* is very low (lower than R), the seller needs to reduce a mark-down (and keep the price not too low) and induce some consumers to purchase with points. Thus, the reward sales have non-trivial effects on the seller's price, and they also reduce the dynamic price fluctuation by attenuating the extent of a mark-up or mark-down. Figure 3 illustrates this point (cash-only price fluctuates more).

Our next result explores how the optimal price, $p_t^*(y)$, changes in the fraction of *reward consumers* and the reimbursement rate.

Proposition 2. *Given y units of inventory and t periods to go until the end of the season:*

(a) The amount of price adjustment (either premium or discount) due to reward sales becomes greater if more consumers are eligible to purchase with points in period t (the higher $\bar{\beta}_t$).

(b) The optimal price $p_t^*(y)$ and the resultant expected revenue increase if R increases, but consumers' utility do not increase.

Part (a) of this proposition indicates that the seller makes a more aggressive price adjustment if there are more *reward consumers*. This is simply because consumers' payment substitution (from cash to points, or *vice versa*) has a greater impact when there are more *reward consumers*. Part (b)

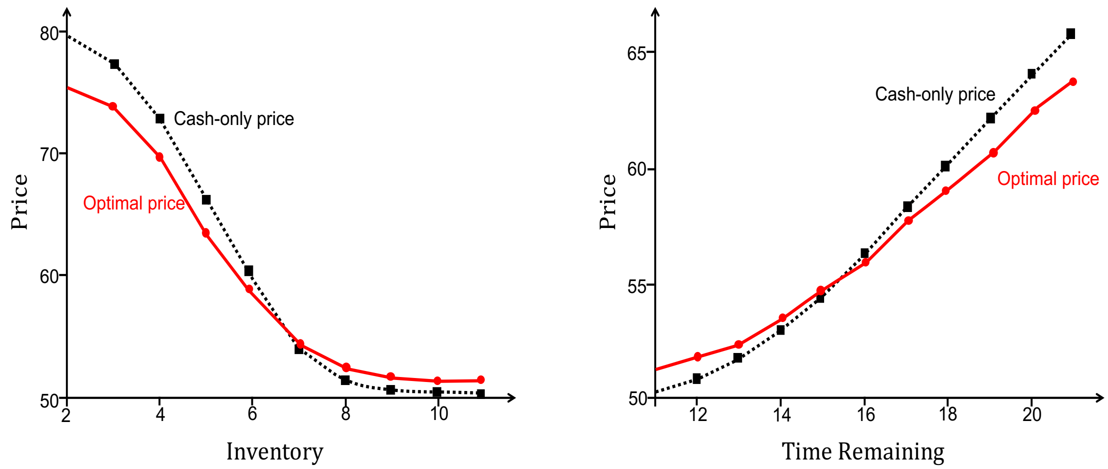


Figure 3 **The optimal price, $p_t^*(y)$, and the cash-only price, $p_t^c(y)$, depending on the inventory level (left figure; for given $t = 10$) and the remaining time (right figure; for given $y = 8$) when $\lambda = 0.8$, $\bar{\beta} = 0.7$, F ∼Uniform$[0, 100]$, G ∼Uniform$[0, 10]$, $q = 10$, and $R = 55$.**

states that as the reimbursement rate goes up, the seller increases the price so that more consumers buy with points, and the resulting revenue increases. This is rather intuitive, but not so obvious part is that no consumers are better off in spite of the increased reimbursement rate. This is because the only way the seller can induce more reward sales is to increase price as the point requirement q is fixed. While consumers' utility from a point purchase is independent of price, a high price negatively affects the utility of consumers who previously purchased with cash.

Remark 2. In Proposition 2, we assume that the change of a parameter (β or R) is limited only to period t. If the change is global (the parameter changes for all periods), it affects the marginal value of inventory in every period and, as a result, the value function. This makes analytically comparing two different policies difficult as one needs to know the exact difference of the value functions for two separate dynamic programming problems. Therefore, we have conducted an extensive numerical study and confirmed that Proposition 2 continues to hold globally. In our numerical experiments, we consider three different combinations of reservation price and point-worth distributions ($F(\cdot)$ and $G(\cdot)$ follow uniform over $[0, 100]$, exponential over $[0, 100]$ with mean 60, and truncated normal over $[0, 100]$ with mean 60 and standard deviation 20; where the point requirement is set at $q = 1$), three different values for the arrival probability ($\lambda \in \{0.3, 0.6, 0.9\}$), ten different values of *reward consumers*' fraction ($\bar{\beta} \in \{0.1, 0.2, \cdots, 1.0\}$), and ten different values of reimbursement rate ($R \in \{10, 20, \cdots, 100\}$), with inventory levels ranging from 1 to 20 with $T = 20$ periods to go until the end of the season. In all instances of global changes in β and R, the result of Proposition 2 continues to hold.

Finally, the following proposition presents the monotonicity result and its implication on the customers' redemption behavior. This is also depicted in Figure 3; the optimal price is monotone in the inventory level and remaining periods.

PROPOSITION 3. *The optimal price, $p_t^*(y)$, is decreasing in the inventory level y and increasing in the remaining periods t. Consequently, consumers are more likely to use points when the inventory level is low and/or more periods remain.*

The result is similar to findings in the typical dynamic pricing literature (without reward sales), the proposition above shows that the seller is willing to move inventory faster by lowering price as the inventory level increases or the remaining time decreases. In other words, the price adjustment (premium or discount) shown in Proposition 1 does not change the monotonicity of price in inventory and time while it reduces the seller's mark-up and mark-down. This is a direct consequence of Lemma 2-(a), which states that the probability of total sales (cash and reward sales) is decreasing in cash price for a given point requirement.

However, this result has an interesting implication on how the reward channel is utilized under the considered setting. As suggested in Kim et al. (2004), the seller could use reward sales channel to reduce excess capacity. That is, the seller would prefer reward sales when the marginal value of inventor is very low (e.g., high inventory level and short remaining period). With no control over the cash/reward sale availability and point requirement, however, the seller can only incur more reward sales by increasing price, and by doing so, it decreases the likelihood of total sales. Therefore, as stated in Proposition 3, the seller instead decreases the price (induce more cash sales) when the marginal value is low and increases the price (induce more reward sales) when the marginal value is high. In other words, consumers use points more likely when it is less desirable (e.g., when the seller could earn high revenue from cash sales) and use points less likely when it is more desirable (e.g., when the seller could earn high revenue from reward sales). This underscores the limitation of the seller's price-only control on reward sales. This is in contrast to the cases where the seller has control over the reward availability and/or point requirement, which are considered next.

4. The Black-Out Model

So far, we have considered the seller who does not have any discretion on the reward sales. In this section, we consider a seller who has the ability to block the reward sales and thus decides the price along with whether to allow consumers to use points (*open*) or not (*close*) in each period.

For this seller (*black-out seller*), we modify the dynamic programming problem of (1) as follows. Let $V_t^B(y)$ be the optimal value function of the black-out seller with y units of inventory in period t, which is given by the following optimality equation:

$$V_t^B(y) = \max \left\{ \tilde{V}_t^c(y),\, \tilde{V}_t^o(y) \right\} \quad \text{for } y>0,\ t=1,2,\ldots,T, \text{ and} \tag{4}$$
$$V_0^B(y)=0 \text{ for } y\geq 0, \quad \text{and} \quad V_t^B(0)=0 \text{ for } t=1,2,\ldots,T,$$

where

$$\tilde{V}_t^c(y) = \max_{p\geq 0} \left\{ \lambda\bar{F}(p)\Big(p+V_{t-1}^B(y-1)\Big) + \Big(1-\lambda\bar{F}(p)\Big)V_{t-1}^B(y) \right\} \text{ and} \tag{5}$$

$$\begin{aligned}\tilde{V}_t^o(y) = \max_{p\geq 0} \Bigg\{ &\lambda\bar{\beta}\Big[\,\bar{F}(p)\bar{G}(p/q)\Big(p+V_{t-1}^B(y-1)\Big) + \int_0^{p/q} \bar{F}(qx)dG(x)\Big(R+V_{t-1}^B(y-1)\Big) \\ &\qquad + \left(1-\bar{F}(p)\bar{G}(p/q) - \int_0^{p/q}\bar{F}(qx)dG(x)\right)V_{t-1}^B(y)\Big] \\ &+ \lambda\beta\Big[\bar{F}(p)\Big(p+V_{t-1}^B(y-1)\Big) + F(p)V_{t-1}^B(y)\Big] + (1-\lambda)V_{t-1}^B(y) \Bigg\}. \end{aligned} \tag{6}$$

$\tilde{V}_t^c(y)$ and $\tilde{V}_t^o(y)$ represent the value functions when the seller closes and opens the reward-sale channel in period t, respectively.

Analogous to $p_t^c(y)$ introduced in Section 3, we denote the solution of (5) by $\tilde{p}_t^c(y)$ to represent the *cash-only price* that maximizes the revenue if a reward sale was not available (*closed*) in period

t.[4] Also, let $\tilde{p}_t^o(y)$ be the solution of (6), the best price when a reward sale is allowed (open) in period t. We denote the optimal price by $p_t^B(y)$, which is either $\tilde{p}_t^c(y)$ or $\tilde{p}_t^o(y)$ depending on the open/close decision.

The first question to ask is when the seller should allow reward sales and when to block. The following proposition and Figure 4 answer to this question and further specify the effect of reward sales on price depending on the inventory level y and remaining periods t.

PROPOSITION 4. *Given t remaining periods, there exist two (inventory) thresholds $h_0(t)$ and $h_1(t)$, $h_0(t) \le h_1(t)$, such that*

(a) if $y < h_0(t)$, it is optimal to close reward sales and set $p_t^B(y) = \tilde{p}_t^c(y) > \tilde{p}_t^o(y)$.

(b) if $h_0(t) \le y < h_1(t)$, it is optimal to open and offer a discount, i.e., $p_t^B(y) = \tilde{p}_t^o(y) < \tilde{p}^c(t)$.

(c) if $h_1(t) \le y$, it is optimal to open reward sales and add a premium, i.e., $p_t^B(y) = \tilde{p}_t^o(y) > \tilde{p}^c(t)$.

Furthermore, the two thresholds, $h_0(t)$ and $h_1(t)$, are increasing in t.

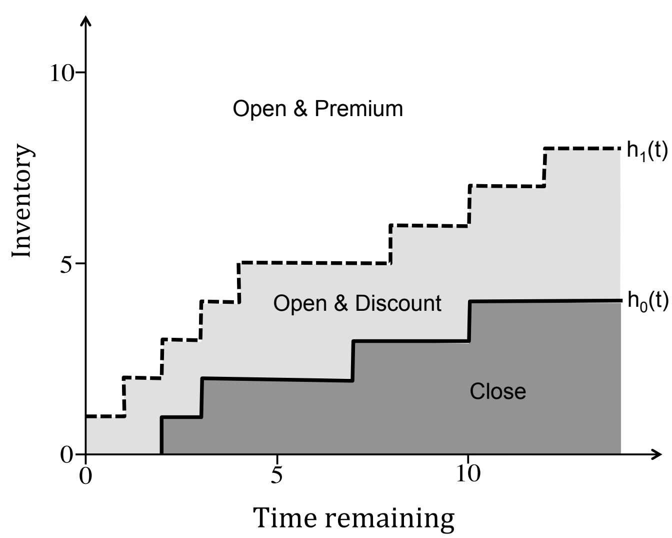


Figure 4 **The optimal strategy when the seller has y units of inventory with t periods to go until the end of the season:** $\lambda = 0.8$, $\bar{\beta} = 0.5$, $q = 10$, $R = 55$, $F(\cdot) \sim$ **Uniform**$[0, 100]$, $G \sim$ **Uniform**$[0, 10]$.

[4] We note that the solution of (3) is different from the solution of (5). In (3), the seller always allows reward sales from period $t-1$ onward, but in (5), the seller optimally controls the reward sale availability from period $t-1$ onward. From now on, we use the *cash-only price* to denote the solution of (5), $\tilde{p}_t^c(y)$.

Proposition 4 implies that the seller blocks reward sales only when the inventory level is "sufficiently" low compared to the remaining time, $y < h_0(t)$. From earlier discussion, the *cash only price* likely exceeds R in this case, making a cash buyer more valuable to the seller. To induce more cash sales, the seller has to offer a discount (Proposition 1-(b)) to make sure not too many consumers buy with points. If a discount becomes too deep, the seller instead closes reward sales and instead focuses only on cash sales. It should be noted, however, that it can still be optimal to open even when R is smaller than $p_t^o(y)$ (when $h_0(t) \leq y < h_1(t)$). Although the seller gets a lower revenue from a reward sale than a cash sale, opening reward sales increases the overall sales (Lemma 1-(b)). As long as the discount is not too big, the benefit of boosting sales still outweighs the discounted margin from cash sales. Hence, the seller still opens reward sales, but offer a price discount to induce more cash sales. When the inventory level is high relative to the number of periods left, opening reward sales is clearly better and the seller adds a premium to induce even more reward sales.

Our next result explores how the optimal strategy changes in the reimbursement rate and the fraction of *reward consumers.*

PROPOSITION 5. *For a given period t, the thresholds, $h_0(t)$ and $h_1(t)$, decrease :*

(a) if the reimbursement in period t (R_t) increases.

(b) if the fraction of reward consumers in period t (β_t) increases for any reimbursement rate $R > 0$.

First note that when the two thresholds decrease (in Figure 4), a region of open decision expands (the seller tends to open more likely). With this observation, the first result makes sense since reward sales become more desirable as the reimbursement rate increases. On the other hand, the second result is not so obvious as it is conceivable that the seller may want to avoid the reward sales if the reimbursement rate is low and there are more *reward consumers.* However, the result indicates that even for the low reimbursement rate, the seller opens the reward sales more likely if there are more *reward consumers.* Recall that the seller's revenue is a weighted sum of the revenue from a *cash-only consumer* and a *reward consumer.* When the seller opens reward sales ($V_t^o(y) > \tilde{V}_t^c(y)$)

with price adjustment (premium or discount), the seller sacrifices a gain from a *cash-only consumer* to get more from a *reward consumer*. Therefore, if there are more *reward consumers* (the higher $\bar{\beta}$), then opening reward sales become more attractive even when the reimbursement rate is low.

We next proceed with the monotonicity result.

PROPOSITION 6. *(Analogous to Proposition 3) The optimal price, $p_t^B(y)$, is decreasing in the inventory level y and increasing in the number of remaining periods t.*

Observe from Figure 4 that the optimal policy changes from *close* to *open* as inventory level increases (or remaining time decreases). Hence, the optimal price switches from $\tilde{p}_t^c(y)$ to $\tilde{p}_t^o(y)$. It can be shown that $\tilde{p}_t^c(y)$ and $\tilde{p}_t^o(y)$ are decreasing in y (increasing in t), which immediately implies that the optimal price is indeed decreasing in y within each of the two regimes – *open* and *close*. However, it is not clear whether the monotonicity is preserved for all inventory level because $p_t^o(y)$ can be higher or lower than $p_t^c(y)$ depending on the inventory level (Proposition 4). This proposition addresses this question and states that the price is monotone in the inventory level and time. Notice that closing is beneficial only when R is sufficiently small so that the seller has to offer a deep discount in order to discourage reward sales: $\tilde{p}_t^c(y) > \tilde{p}_t^o(y)$ for $y < h_0(t)$. Therefore, we can see that the optimal price (either of $\tilde{p}_t^c(y)$ or $\tilde{p}_t^o(y)$) decrease in y. A similar result holds for the monotonicity of price in the number of periods left.

One interesting question to ask is when the seller gains the most from the ability to block reward sales. To this end, we conduct a numerical study and compare the following three different settings – (i) no reward sales in any period, (ii) open reward sales in every period (denoted with O), and (iii) discretionally open/close reward sales in each period (denoted with B). For the sellers who allow reward sales, we assume that point requirement is given at 10. We consider a 20-period selling season in which at most one consumer arrives in each period with a probability $\lambda = 0.9$. For each setting, we vary the initial inventory levels (ranging from 1 to 20) and consider three different reservation and point-worth distributions $(F(\cdot), G(\cdot))$ – uniform, exponential, normal. (Specifically, we consider (i) $F(\cdot) \sim U[0,100], G(\cdot) \sim U[0,10]$, (ii) $F(\cdot) \sim Exp(\text{mean} = 60), G(\cdot) \sim Exp(\text{mean} = 6)$,

Table 1 Summary statistics for percentage revenue change from allowing reward sales in every periods (O) and allowing only when it is optimal (B) compared to the bench-mark in which there is no reward sale. The number in the bracket represents the proportion of open decisions for the black-out seller.

		uniform		exponential		normal	
$\bar{\beta}$	R	O	B (open)	O	B (open)	O	B (open)
0.2	10	-13.20	0.00(0.00)	-12.73	0.00(0.00)	-11.66	0.00(0.00)
0.2	20	-9.92	0.00(0.00)	-7.42	0.00(0.00)	-9.48	0.00(0.00)
0.2	30	-6.58	0.00(0.00)	-2.00	2.13(0.65)	-7.19	0.00(0.00)
0.2	40	-3.17	0.96(0.55)	3.50	5.96(0.75)	-4.78	0.00(0.00)
0.2	50	0.30	2.93(0.70)	9.10	10.39(0.85)	-2.25	0.60(0.55)
0.2	60	3.85	5.33(0.80)	14.79	15.33(0.90)	0.41	1.76(0.75)
0.5	10	-27.76	0.00(0.00)	-26.94	0.00(0.00)	-22.21	0.00(0.00)
0.5	20	-20.87	0.00(0.00)	-15.40	0.00(0.00)	-18.41	0.00(0.00)
0.5	30	-13.68	0.00(0.00)	-3.49	5.18(0.60)	-14.17	0.00(0.00)
0.5	40	-6.20	2.35(0.55)	8.97	14.27(0.75)	-9.46	0.00(0.00)
0.5	50	1.55	7.08(0.65)	21.80	24.81(0.80)	-4.23	1.47(0.50)
0.5	60	9.59	12.81(0.75)	34.96	36.45(0.85)	1.51	4.31(0.70)
0.8	10	-38.59	0.00(0.00)	-37.40	0.00(0.00)	-28.91	0.00(0.00)
0.8	20	-29.11	0.00(0.00)	-21.44	0.00(0.00)	-24.27	0.00(0.00)
0.8	30	-18.97	0.00(0.00)	-4.10	8.15(0.60)	-18.89	0.00(0.00)
0.8	40	-8.22	3.72(0.50)	14.17	21.88(0.70)	-12.65	0.00(0.00)
0.8	50	3.11	10.95(0.65)	33.27	37.85(0.80)	-5.42	2.32(0.50)
0.8	60	15.01	19.71(0.70)	53.00	55.47(0.85)	2.82	6.75(0.70)

$F(\cdot) \sim N(60,20), G(\cdot) \sim N(6,2)$. For normal and exponential, we consider the truncated versions.) We also vary fraction of *reward consumers* ($\bar{\beta} \in \{0.2, 0.5, 0.8\}$), and reimbursement rate ($R \in \{10, 20, 30, 40, 50, 60\}$). For each scenario, we measure the percentage revenue increase/decrease compared to the seller with no reward sales. The results are summarized in Table 1.

If R is high, allowing reward sales is indeed beneficial as it increases the total sales and total revenue. However, reward sales do not always increase the seller's revenue. As expected, when R is low, always allowing reward sales can be quite costly to the seller. This reiterates the limitation of the seller with no control over reward sales (as discussed at the end of Section 3) and demonstrates that reward sales with a low reimbursement rate could cannibalize the cash-sales revenue. Therefore, having an ability to block reward sales helps the seller only when R is very low. If R is relatively high, reward sales are hardly blocked, and hence the additional benefit of discretion becomes relatively small.

5. Dynamic Adjustment Model

We now extend our model to the case where the seller can control both price and the point requirement in each period. Specifically, we consider a seller who chooses a point requirement from a set, $Q = \{q_1, q_2, ..., q_N\}$ where $q_i < q_j$ for any $i < j$, and we also allow that the seller receives different reimbursement revenue depending on the point requirement (amount of points redeemed) with non-decreasing reimbursement rate, $R(q_i) \leq R(q_j)$ for all $i < j$. This general framework captures various cases possible. For instance, if there is a fixed reimbursement rate for each point redeemed, $R(q)$ is linear in q. When $q_N = \infty$ (or sufficiently large), it is equivalent to blocking reward sales entirely. If $N = 1$ and $q_1 < \infty$, it represents the seller with a fixed point requirement. Thus, the dynamic adjustment model subsumes the previous two models as special cases.

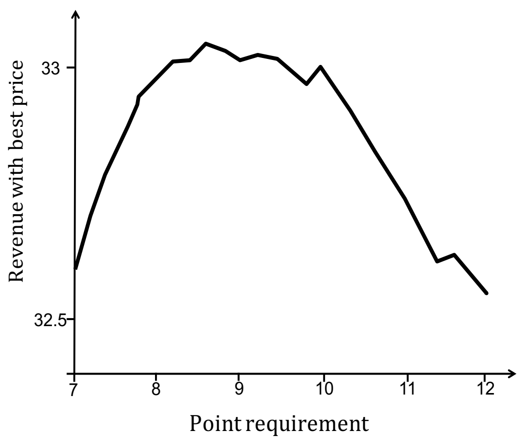
The seller's problem of choosing the optimal price and point requirement, $(p_t^Q(y), q_t^Q(y))$, with remaining period t and inventory level y can be written as follows:

$$V_t^Q(y) = \max_{q \in Q,\, p \geq 0} J_t^Q(p, q, y) \quad \text{for } y > 0,\ t > 0, \text{ and } V_t^Q(y) = 0 \text{ if } t = 0 \text{ or } y = 0,$$

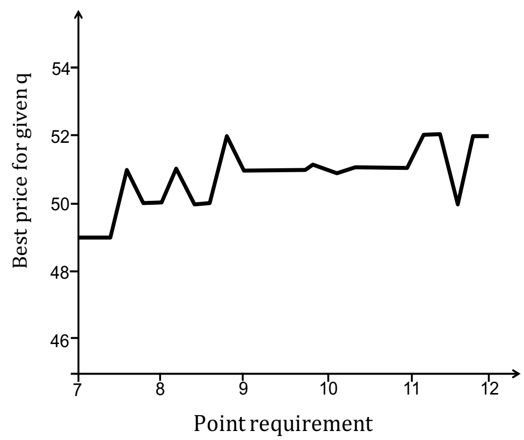
where

$$\begin{aligned} J_t^Q(p,q,y) = \lambda\bar{\beta}(q)\Bigg[\, & \bar{F}(p)\bar{G}(p/q)\Big(p + V_{t-1}^Q(y-1)\Big) + \int_0^{p/q} \bar{F}(qx)dG(x)\Big(R(q) - V_{t-1}^Q(y-1)\Big) \\ & + \left(1 - \bar{F}(p)\bar{G}(p/q) - \int_0^{p/q} \bar{F}(qx)dG(x)\right) V_{t-1}^Q(y)\Bigg] \\ & + \lambda\beta(q)\Big[\bar{F}(p)\Big(p + V_{t-1}^Q(y-1)\Big) + F(p)V_{t-1}^Q(y)\Big] + (1-\lambda)V_{t-1}^Q(y). \end{aligned} \tag{7}$$

From Theorem 1, $J_t^Q(p, q, y)$ is unimodal in p for a given q if the conditions *A1-A3* hold. Thus, for each $q \in \{q_1, q_2, ..., q_N\}$, there exists a unique price (denoted by $\tilde{p}_t(y, q)$) that maximizes the revenue. However, since q is also a decision variable, conditions *A1-A3* are no longer sufficient to guarantee nice analytical properties. This is because the change in q triggers non-trivial changes in both consumers' and seller's decisions. For instance, suppose that a seller increases a point requirement (q increases). Then, a smaller fraction of consumers will buy with points for two different reasons: (i) the proportion of eligible consumers decreases (as $\bar{\beta}(q)$ decreases in q) and (ii) buying with points becomes less attractive (as $G(p/q)$ decreases in q). On the other hand, the seller receives a higher revenue from a reward sale as $R(q)$ increases in q, and this also changes the optimal price to charge. That is, dynamic adjustment is further complicated by three heterogeneous attributes – reservation price $F(\cdot)$, point balance $\beta(\cdot)$, and point worth $G(\cdot)$). For this reason, the resulting revenue function is not necessarily well-behaved. Let $\tilde{p}_t(q,y)$ be the best price that maximizes the revenue function, $J_t^Q(p,q,y)$ for given q. Figure 5 demonstrates that the revenue function could be multi-modal (subplot (a)), and the best price $\tilde{p}_t(q,y)$ for each q changes non-monotonically (subplot (b)), i.e., $J_t^Q(p,q,y)$ is not jointly unimodal in (p,q).



(a) Maximum revenue for given q



(b) Best price for given q

Figure 5 **Examples of best revenue ($\max_p J_t^Q(p,q,y)$) and best price ($\arg\max_p J_t^Q(p,q,y)$) for each given point requirement when $F \sim N(60,30)$, $G \sim N(6,3)$, $q \in \{7.0, 7.2, 7.4, \cdots, 12.0\}$, $\bar{\beta}(q) = \frac{1}{\log q}$, $R(q) = 8q$, $t = 1$, and $y = 1$.**

However, since $|Q|$ is finite, one can utilize the fact that the revenue function is unimodal for given q and find the optimal price and point requirement combination, $(p_t^Q(y), q_t^Q(y))$, by comparing the revenue at $p = \tilde{p}_t(q_i, y)$ for each q_i. Further, we next present conditions under which the seller's optimal policy can be analytically characterized.

Theorem 2. *$\tilde{p}_t(q,y)$ increases in q if the conditions A1–A3, and the following two conditions hold:*

A4. For a given p, $\frac{1}{q} \cdot \frac{\bar{\beta}(q) g\left(p/q\right)}{1-\bar{\beta}(q) G(p/q)}$ *decreases in* $q \in Q$*. That is, the substitution effect (from cash to points) decreases in q in the hazard rate order.*

A5. The reimbursement rate is smaller than the cash-only price, i.e., $\max_{q \in Q} R(q) \leq \tilde{p}_t^c(y)$*. This condition is automatically satisfied when* $R(q_N)$ *is smaller than the cash-only price for a single-period problem, i.e.,* $\max_{q \in Q} R(q) \leq \arg\max_p p\bar{F}(p)$*.*

The condition *A4* reflects the fact that, as q increases, a point purchase become less and less attractive (compared to paying p in cash). In particular, *A4* implies that the proportion of marginal consumers who are indifferent between using q points and paying p in cash decreases in q. In other words, the proportion of consumers who prefer paying with points to cash decreases sharply as the point requirement increases.

Condition *A5* states that, as long as the reimbursement rate is low enough (i.e., the margin from a reward sale is considerably smaller than the margin from a cash sale), the best price is increasing in q. In practice, it is known that the reimbursement rates could be a lot lower than the (cash) price. For instance, Marriott Rewards reimburses \$201 for one award night at Ritz-Carlton South Beach (a premier Marriott property) while the cash price of the same room is \$599 (Ollila 2012 and Ollila 2013). As we discussed, if the reimbursement rate is low, the seller tries to discourage reward sales by offering a discount on the *cash-only price* ($p < p^c$). However, this can be done differently when the seller can also change the point requirement. When q increases, 1) the seller gets higher reimbursement, as $R(q)$ increases in q, and 2) the proportion of consumers who are eligible to buy with points decreases. Therefore, the seller can effectively discourage the reward sales even with a higher price (offering a smaller discount). That is, these two effects contribute to an increased price in q.

Now consider the case when condition *A5* does not hold. In this case, the seller needs to add a premium (on the *cash-only price*) to induce more reward sales. Again, this can be done by decreasing q, which has two effects: 1) decreases $R(q)$, and 2) increases the proportion of consumers who are eligible to buy with points, and the importance of revenues from reward sales. If the first effect is dominant, the the seller increases a premium amount and charge a even higher price. However, if this second effect is sufficiently strong, then the seller's price will decrease toward a *cash-only price*. Thus, the seller's price may increase or decreases in q.

PROPOSITION 7. *If conditions A1 $\sim$ A5 hold, the optimal price and point requirement, $p_t^Q(y)$ and $q_t^Q(y)$, are both decreasing in the inventory level, y, and increasing in the remaining periods, t.*

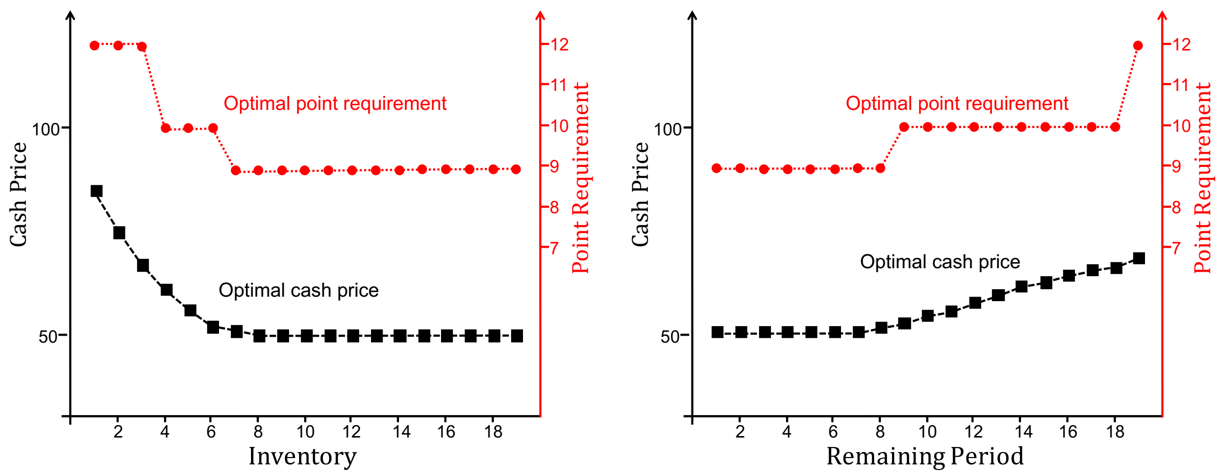


Figure 6 **The optimal price and point requirement depending on the remaining inventory level and period when** $\lambda = 0.9$, $Q = \{7, 8, 9, 10, 12\}$, $\bar{\beta}(q) = 0.6 - 0.05q$, $R(q) = 5q$, $F \sim$ **Uniform** $[0, 100]$, $G \sim$ **Uniform** $[0, 10]$, $t = 9$ **(left), and** $y = 6$ **(right).**

Proposition 7 (which generalizes Proposition 4) shows that both price and point requirement move in the same monotonic direction (also shown in Figure 6). For instance, as inventory level y increases (or the number of remaining period t decreases), the seller wants to move inventory faster, and thus choose the lower price and lower point requirement. This result might explain why several firms in practice are changing both cash price and point requirement in a monotone schedule (BoardingArea 2015).

Regarding the seller's discretion on the point requirement, one question to ask is whether a full capability of dynamic adjustment is necessary, and if so, how much the seller could benefit from that. In order to investigate this issue, we further conduct a numerical study as follows: we compare 4 different cases, ranging from no discretion to full discretion (see below for details). For each case, we consider the same settings considered in Section 4 (20 periods of selling season, $\lambda = 0.9$, starting inventory = $\{1, ..., 20\}$, (F, G) = uniform, exponential, and normal), but now specify three different point-balance distributions ($\bar{\beta}(q) = 0.6 - q/20$, $1.35 - q/8$, and $2.1 - q/5$) and four different reimbursement-rate function ($R(q) = 40 + q$, $30 + 2q$, $20 + 3q$, and $10 + 4q$). For each setting, we consider the following different cases to study the effect of dynamic adjustment (multiple-levels) on the revenue:

- Q_0: No black-out and the static point requirement $\bar{q} = \arg\min_{q \in \{6,7,8,9,10\}} V_t(y; q)$,
- Q_1: No black-out and the static point requirement $\bar{q} = \arg\max_{q \in \{6,7,8,9,10\}} V_t(y; q)$,
- Q_2: Black-out and the static point requirement $\bar{q} = \arg\max_{q \in \{6,7,8,9,10\}} V_t^B(y; q)$, and
- Q_3: Black-out and the dynamic point control among 5 point levels, $\{6, 7, 8, 9, 10, \infty\}$.

The first two sellers (Q_0 and Q_1) have no discretion at all, and a static point requirement is used throughout a planning horizon (no black-out policy). But, the difference between Q_0 and Q_1 is on which point-requirement is being used: The point requirement of seller Q_0 (which is a benchmark seller) is set to be the worst among $\{6, 7, 8, 9, 10\}$ (i.e., minimizes its revenue), while that of seller Q_1 is the best static point requirement that maximizes its revenue. By comparing these two sellers, we investigate the importance of choosing the best static point requirement in terms of the seller's revenue. On the other hand, the seller Q_2 has an option to block reward sales in each period, and the seller Q_3 can dynamically control point requirement and price in each period. For each scenario, we solve dynamic programming problems associated with different seller cases, measure the average (over different starting inventories) percentage revenue improvement compared to the benchmark seller (Q_0), and summarize the results in Table 2.

A number of observations are in order. First, we observe that by optimizing a static point requirement (the seller Q_1), the seller can improve the revenue by more than $5 \sim 6\%$ on average compared to the seller with the worst point requirement (bench-mark Q_0). Second, as we observe from Section 4, the option to block reward sales could be quite beneficial even compared to the

Table 2 **Summary statistics for percentage revenue improvement from the seller's discretion at multiple levels.**

$\bar{\beta}(q)$	$R(q)$	Uniform Q_1	Uniform Q_2	Uniform Q_3	Exponential Q_1	Exponential Q_2	Exponential Q_3	Normal Q_1	Normal Q_2	Normal Q_3
$0.6-q/20$, $\bar{\beta}(6)=0.3$, $\bar{\beta}(10)=0.1$	$40+q$	3.20	8.98	9.02	5.05	8.19	8.24	3.65	9.96	9.97
	$30+2q$	5.05	8.19	8.24	3.65	9.96	9.97	3.02	10.08	10.21
	$20+3q$	3.65	9.96	9.97	3.02	10.08	10.21	4.44	8.52	8.73
	$10+4q$	3.02	10.08	10.21	4.44	8.52	8.73	3.94	11.75	11.81
$1.35-q/8$, $\bar{\beta}(6)=0.6$, $\bar{\beta}(10)=0.1$	$40+q$	5.42	16.55	16.63	7.69	13.86	13.96	6.10	17.58	17.60
	$30+2q$	7.69	13.86	13.96	6.10	17.58	17.60	5.34	19.10	19.34
	$20+3q$	6.10	17.58	17.60	5.34	19.10	19.34	6.91	14.89	15.33
	$10+4q$	5.34	19.10	19.34	6.91	14.89	15.33	6.86	21.17	21.29
$2.1-q/5$, $\bar{\beta}(6)=0.9$, $\bar{\beta}(10)=0.1$	$40+q$	7.23	23.27	23.37	8.86	17.74	17.98	8.27	24.09	24.11
	$30+2q$	8.86	17.74	17.98	8.27	24.09	24.11	7.17	27.05	27.40
	$20+3q$	8.27	24.09	24.11	7.17	27.05	27.40	8.16	19.71	20.42
	$10+4q$	7.17	27.05	27.40	8.16	19.71	20.42	9.58	29.34	29.52
Avg. improvement		5.37	14.99	15.11	5.72	16.44	16.60	6.15	18.29	18.48

seller Q_1, who can choose the optimal fixed point requirement. Third, surprisingly, we find that the additional benefit of dynamic adjustment is minimal: the black-out seller Q_2 achieves almost the same revenues of the dynamic adjustment seller Q_3 who can further control point requirements. That is, the ability to further segment reward sales with multiple tiers of point requirements has a marginal impact on the revenue (compared to the black-out seller case).

Remark 3. Note that the dynamic adjustment model allows the reimbursement rate to change in point requirement. If the reimbursement rate is constant, i.e., $R(q)=R$ for any q, changing point requirement decision only affects the consumers' choice, not the seller's compensation scheme. We note that all of our results presented in this section hold without conditions *A4* and *A5* for this simpler case.

6. Conclusion

In this paper we study the impact of reward sales on pricing and inventory decisions. Since the seller earns different amounts of revenue from cash sales and reward sales, the option of buying with points and consumers' purchase payment choice (pay with cash or points) directly affect the seller's revenue. We show that, only when the reimbursement rate is substantially low (compared to the cash price), reward sales can indeed hurt the seller's revenue (i.e., point redemption cannibalize cash sales). On the other hand, reward sales can benefit the seller since consumers who can not afford a cash price can purchase with points (i.e. total sales increases).

In order to examine the trade-off in detail, we first model the consumer choice behavior capturing three heterogenous attributes: reservation price, point balance, and perceived value of a reward point. Then, we incorporate a resultant consumer choice into the seller's problem. In particular, we study three different cases by varying the level of sellers' discretion over the reward sales: (1) the seller who has no discretion, (2) the seller who can block reward sales, and (3) the seller who can change both the price and point requirement in each period, and analyze the impact of seller's discretion on the optimal policy and revenue.

We find that the seller adds a premium or offers a discount depending on the gap between the reimbursement rate and the *cash-only price*. For instance, when the reimbursement rate is low relative to the *cash-only price*, the seller offers a discount in order to convert some of reward sales to cash sales. In the opposite case, the seller offers a premium to induce more reward sales. When a seller can further control the availability of reward sales, we show that it is optimal to block reward sales only when the inventory level (relative to the remaining selling season) or the reimbursement rate is very low. Notice that, in both cases, the seller can avoid a deep price discount by simply closing the reward sales. However, unless the gap between the *cash-only price* and the reimbursement rate is significantly large, allowing reward sales is generally better for the seller as doing so increases the total sales and revenue. Thus, even when the reimbursement rate is lower than the cash-only price, allowing reward sales with a discount can be still optimal. In the case where the seller can dynamically change both price (for cash sales) and point requirement (for reward sales), we show that price and point requirement do not necessarily move in the same direction. We also show that most of the gains obtained from having the full discretion can be captured with a simple black-out policy, which may explain why many sellers use static point requirement coupled with availability constraint.

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Appendix. 1. Proofs

Proof of Lemma 1

(a) and (b) Straightforward from Figure 1.

$$P(\text{cash purchase}|\text{cash-only}) - P(\text{cash purchase}|\text{loyalty}) = \bar{F}(p)G(p/q) > 0 \text{ and}$$
$$P(\text{total purchase}|\text{loyalty}) - P(\text{cash purchase}|\text{cash-only}) = \int_0^{p/q} \bar{F}(qx)dG(x) > 0.$$

Proof of Lemma 2

(a) It can be shown by the derivatives of probabilities from Lemma 1 with respect to p:

$$\partial P(\text{cash purchase}|p) \Big/ \partial p = -f(p)\bar{G}(p/q) - \bar{F}(p)g(p/q) < 0$$
$$\partial P(\text{point purchase}|p) \Big/ \partial p = \bar{F}(p)g(p/q) > 0$$
$$\partial P(\text{total purchase}|p) \Big/ \partial p = -f(p)\bar{G}(p/q) < 0.$$

(b) Suppose that the point requirement is increased from q to $q+\epsilon$ for any $\epsilon > 0$. Under the higher point requirement, the proportion of reward consumer is smaller, i.e., $\bar{\beta}(q) \geq \bar{\beta}(q+\epsilon)$. $P(\text{cash purchase})$ increases and $P(\text{point purchase})$ decreases, as there are more cash-only consumers (who buy with cash more likely than *reward consumers*) and less *reward consumers* (who can buy with points). Also, for *reward consumers*, increasing q makes $P(\text{point purchase}|\text{loyalty})$ smaller.

Proof of Theorem 1

First, we only need to consider any p greater than or equal to $\Delta_{t-1}(y)$. If $\Delta_{t-1}(y) > p$, it also implies that $\Delta_{t-1}(y) > R$. Together, it means that the seller makes the negative contributions to the expected revenue by making any sales (both the cash sales and the reward sales). That means, $V_t(y) < V_{t-1}(y)$, which contradicts to the property of the value function. Note that the first order condition is given by

$$\frac{\partial J_t(p,y)}{\partial p} = \lambda\Big(1-\bar{\beta}G(p/q)\Big)\bar{F}(p)\left[1 - \frac{f(p)}{\bar{F}(p)}\Big(p - \Delta_{t-1}(y)\Big) - \frac{\bar{\beta}g(p/q)/q}{1-\bar{\beta}G(p/q)}\Big(p-R\Big)\right] = 0 \tag{8}$$

and the second order condition (the second-order derivative is negative for any stationary point) is given by

$$\frac{\partial^2 J_t(p,y)}{\partial p^2}\Big|_{p=p^*} = \lambda\Big(1-\bar{\beta}G(p/q)\Big)\bar{F}(p)\Bigg[-\left(\frac{f'(p)}{\bar{F}(p)} + \frac{f(p)^2}{\bar{F}(p)^2}\right)\Big(p-\Delta_{t-1}(y)\Big) - \frac{f(p)}{\bar{F}(p)}$$
$$- \left(\frac{\bar{\beta}g'(p/q)/q^2}{1-\bar{\beta}G(p/q)} + \frac{\bar{\beta}^2 g(p/q)^2/q^2}{(1-\bar{\beta}G(p/q))^2}\right)\Big(p-R\Big) - \frac{\bar{\beta}g(p/q)/q}{1-\bar{\beta}G(p/q)}\Bigg]$$

$$= \lambda\Big(1-\bar{\beta}G(p/q)\Big)\bar{F}(p)\Bigg[-\left(\frac{f'(p)}{\bar{F}(p)}+\frac{f(p)^2}{\bar{F}(p)^2}\right)\Big(p-\Delta_{t-1}(y)\Big)-\frac{f(p)}{\bar{F}(p)}$$
$$-\left(\frac{g'(p/q)/q}{g(p/q)}+\frac{\bar{\beta}g(p/q)/q}{1-\bar{\beta}G(p/q)}\right)\left(1-\frac{f(p)}{\bar{F}(p)}\Big(p-\Delta_{t-1}(y)\Big)\right)-\frac{\bar{\beta}g(p/q)/q}{1-\bar{\beta}G(p/q)}\Bigg]<0, \quad (9)$$

which is equivalent to

Note that the assumption *A3* implies that $\left(\frac{g'(p/q)/q}{g(p/q)}+\frac{\bar{\beta}g(p/q)/q}{1-\bar{\beta}G(p/q)}\right)-\left(\frac{f'(p)}{f(p)}+\frac{f(p)}{\bar{F}(p)}\right)$ is non-positive, which further implies that the LHS is non-positive because $p>\Delta_{t-1}(y)$. On the other hand, the RHS can be shown to be positive under the assumption *A1* and *A2*. Particularly, the assumption *A2* implies that the hazard rate of $F(\cdot)$ is no less than that of $G(\cdot)$. Then, after replacing $\frac{f(p)}{\bar{F}(p)}$ by $\frac{g(p/q)/q}{\bar{G}(p/q)}$, the RHS is shown to be positive by the IFR condition of $G(\cdot)$.

Proof of Corollary 1

The proof is immediate from Theorem 1 and the monotonicity of the marginal value of inventory in time and inventory level. The monotonicity of the marginal value follows the proof of Lemma 2 in Bitran et al. (1993).

We first define the following inequalities:

$$I1(y,t): V_{t+1}(y+1)-V_{t+1}(y)\geq V_t(y+1)-V_t(y) \qquad \forall t,y \quad (10)$$

$$I2(y,t): V_{t+1}(y)-V_t(y)\geq V_{t+2}(y)-V_{t+1}(y) \qquad \forall t,y \quad (11)$$

$$I3(y,t): V_t(y+1)-V_t(y)\geq V_t(y+2)-V_t(y+1) \qquad \forall t,y \quad (12)$$

The inequalities $I1(y,t)$, $I2(y,t)$, and $I3(y,t)$ hold trivially for $t=0$ and $y=0$. We assume that the three inequalities are satisfied for any $t+y<k$ and we prove that they hold for $t+y=k$.

i) We first prove that $I1(y,t)$ holds for $t+y=k$. We know that $I1(0,t)$ holds for any t. Suppose that $y>0$ and let $\bar{p}$ be the optimal price for $J_{t+1}(p,y)$, that is,

$$V_{t+1}(y)=\lambda\bar{\beta}\Bigg[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)(\bar{p}+V_t(y-1))+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)(R+V_t(y-1))$$
$$+\left(1-\bar{F}(\bar{p})\bar{G}(\bar{p}/q)-\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)\right)V_t(y)\Bigg]$$
$$+\lambda\beta\Big[\bar{F}(\bar{p})(\bar{p}+V_t(y-1))+F(\bar{p})V_t(y)\Big]+(1-\lambda)V_t(y).$$

Subtracting $V_t(y)$ from both sides, we have

$$V_{t+1}(y)-V_t(y)=\lambda\bar{\beta}\Bigg[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)(\bar{p}+V_t(y-1)-V_t(y))+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)(R+V_t(y-1)-V_t(y))\Bigg]$$

$$+\lambda\beta\Big[\bar{F}(\bar{p})(\bar{p}+V_t(y-1)-V_t(y))\Big]. \tag{13}$$

Because $\bar{p}$ is feasible for $V_{t+1}(y+1)$, we know that $V_{t+1}(y+1)\geq J_{t+1}(\bar{p},y+1)$ and

$$V_{t+1}(y+1)-V_t(y+1)\geq\lambda\bar{\beta}\Big[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)(\bar{p}+V_t(y)-V_t(y+1))+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)(R+V_t(y)-V_t(y+1))\Big]$$
$$+\lambda\beta\Big[\bar{F}(\bar{p})(\bar{p}+V_t(y)-V_t(y+1))\Big].$$

From $I3(y-1,t)$, we know that $V_t(y)-V_t(y+1)\geq V_t(y-1)-V_t(y)$. Replacing it in the inequality above, we have:

$$V_{t+1}(y+1)-V_t(y+1)\geq\lambda\bar{\beta}\Big[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)(\bar{p}+V_t(y-1)-V_t(y))+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)(R+V_t(y-1)-V_t(y))\Big]$$
$$+\lambda\beta\Big[\bar{F}(\bar{p})(\bar{p}+V_t(y-1)-V_t(y))\Big]. \tag{14}$$

Combining (13) and (14), we prove:

$$I1(y,t): V_{t+1}(y+1)-V_{t+1}(y)\geq V_t(y+1)-V_t(y).$$

ii) We next prove that $I2(y,t)$ holds for $t+y=k$.

Let some $\bar{p}$ satisfies $V_{t+2}(y)=J_{t+2}(\bar{p},y)$. Then, we have

$$V_{t+2}(y)-V_{t+1}(y)=\lambda\bar{\beta}\Big[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)(\bar{p}+V_{t+1}(y-1)-V_{t+1}(y))+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)(R+V_{t+1}(y-1)-V_{t+1}(y))\Big]$$
$$+\lambda\beta\Big[\bar{F}(\bar{p})(\bar{p}+V_{t+1}(y-1)-V_{t+1}(y))\Big]. \tag{15}$$

Because $\bar{p}$ is feasible for $V_{t+1}(y)$, we know that $V_{t+1}(y)\geq J_{t+1}(\bar{p},y)$ and

$$V_{t+1}(y)-V_t(y)\geq\lambda\bar{\beta}\Big[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)(\bar{p}+V_t(y-1)-V_t(y))+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)(R+V_t(y-1)-V_t(y))\Big]$$
$$+\lambda\beta\Big[\bar{F}(\bar{p})(\bar{p}+V_t(y-1)-V_t(y))\Big].$$

From $I1(y-1,t)$, we know that $V_t(y-1)-V_t(y)\geq V_{t+1}(y-1)-V_{t-1}(y)$. Replacing it in the inequality above:

$$V_{t+1}(y)-V_t(y)\geq\lambda\bar{\beta}\Big[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)(\bar{p}+V_{t+1}(y-1)-V_{t+1}(y))+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)(R+V_{t+1}(y-1)-V_{t+1}(y))\Big]$$
$$+\lambda\beta\Big[\bar{F}(\bar{p})(\bar{p}+V_{t+1}(y-1)-V_{t+1}(y))\Big]. \tag{16}$$

Combining (15) and (16), we prove:

$$I2(y,t): V_{t+1}(y)-V_t(y)\geq V_{t+2}(y)-V_{t+1}(y)$$

iii) We finally prove that $I3(y,t)$ holds for $t+y=k$.

Let some $\bar{p}$ satisfies $V_t(y+2)=J_t(\bar{p},y+2)$. Then, we have

$$V_t(y+2)-V_{t-1}(y+1)=\lambda\bar{\beta}\Big[\bar{F}(\bar{p})\bar{G}(\bar{p}/q)\bar{p}+\int_0^{\bar{p}/q}\bar{F}(qx)dG(x)R$$

$$+ \left(1 - \bar{F}(\bar{p})\bar{G}(\bar{p}/q) - \int_0^{\bar{p}/q} \bar{F}(qx)dG(x)\right)(V_{t-1}(y+2) - V_{t-1}(y+1))\Bigg] + \lambda\beta\Big[\bar{F}(\bar{p})\bar{p} + F(\bar{p})(V_{t-1}(y+2) - V_{t-1}(y+1))\Big] + (1-\lambda)(V_{t-1}(y+2) - V_{t-1}(y+1)) \tag{17}$$

and

$$\begin{aligned} V_{t+1}(y+1) - V_t(y) \geq \lambda\bar{\beta}\Bigg[& \bar{F}(\bar{p})\bar{G}(\bar{p}/q)\bar{p} + \int_0^{\bar{p}/q} \bar{F}(qx)dG(x)R \\ & + \left(1 - \bar{F}(\bar{p})\bar{G}(\bar{p}/q) - \int_0^{\bar{p}/q} \bar{F}(qx)dG(x)\right)(V_t(y+1) - V_t(y))\Bigg] \\ & + \lambda\beta\Big[\bar{F}(\bar{p})\bar{p} + F(\bar{p})(V_t(y+1) - V_t(y))\Big] + (1-\lambda)(V_t(y+1) - V_t(y)). \end{aligned} \tag{18}$$

From $I1(y, t-1)$ and $I3(y, t-1)$, we have

$$V_t(y+1) - V_t(y) \geq V_{t-1}(y+1) - V_{t-1}(y) \geq V_{t-1}(y+2) - V_{t-1}(y+1).$$

Replacing it in (18),

$$\begin{aligned} V_{t+1}(y+1) - V_t(y) \geq \lambda\bar{\beta}\Bigg[& \bar{F}(\bar{p})\bar{G}(\bar{p}/q)\bar{p} + \int_0^{\bar{p}/q} \bar{F}(qx)dG(x)R \\ & + \left(1 - \bar{F}(\bar{p})\bar{G}(\bar{p}/q) - \int_0^{\bar{p}/q} \bar{F}(qx)dG(x)\right)(V_{t-1}(y+2) - V_{t-1}(y+1))\Bigg] \\ & + \lambda\beta\Big[\bar{F}(\bar{p})\bar{p} + F(\bar{p})(V_{t-1}(y+2) - V_{t-1}(y+1))\Big] + (1-\lambda)(V_{t-1}(y+2) - V_{t-1}(y+1)). \end{aligned} \tag{19}$$

Combining (17) and (19), we have

$$V_{t+1}(y+1) - V_t(y) \geq V_t(y+2) - V_{t-1}(y+1). \tag{20}$$

Additionally, by $I2(y+1, t-1)$, we have

$$V_t(y+1) - V_{t+1}(y+1) \geq V_{t-1}(y+1) - V_t(y+1). \tag{21}$$

Combining (20) and (21), we prove

$$I3(y,t) : V_t(y+1) - V_t(y) \geq V_t(y+2) - V_t(y+1).$$

Proof of Proposition 1

The optimal price, p^*, satisfies the following first-order condition:

$$1 - \frac{f(p)}{\bar{F}(p)}(p - \Delta) - \frac{\bar{\beta} \cdot g(p/q)/q}{1 - \bar{\beta}G(p/q)}(p - R) = 0, \tag{22}$$

and the *cash-only price*, p^c, satisfies the following first-order condition:

$$1 - \frac{f(p^c)}{\bar{F}(p^c)}(p^c - \Delta) = 0. \tag{23}$$

Suppose that $p^c < R$ and the seller chooses the *cash-only price* p^c, then, the optimal first-order condition (22) will be positive. On the other hand, any $p > R$ makes (22) to be negative. From the unimodal property, this implies that $p^* \in (p_c, R)$.

The opposite case when $p^c > R$ can be proved by the same logic.

Proof of Corollary 2

The proof is immediate from Proposition 1 with the fact that the marginal value of inventory, $\Delta_t(y)$, is increasing in t and decreasing in y.

Proof of Proposition 2

(a) From Proposition 1, we have two cases to consider, in which the seller either adds a premium or offers a discount considering the reward sales. In the case of adding a premium ($p^c < p^* < R$), note that $1 - \frac{f(p^*)}{\bar{F}(p^*)}(p^* - \Delta) < 0$ and $-\frac{\bar{\beta}g(p^*/q)/q}{1-\bar{\beta}G(p^*/q)}(p^* - R) > 0$. If $\bar{\beta}$ increases, the first order of the revenue when $p = p^*$ becomes positive, which means that the optimal price should be also higher as $\bar{\beta}$ increases.

In the opposite case when the seller offers a discount ($R < p^* < p^c$), we have $1 - \frac{f(p^*)}{\bar{F}(p^*)}(p^* - \Delta) > 0$ and $-\frac{\bar{\beta}g(p^*/q)/q}{1-\bar{\beta}G(p^*/q)}(p^* - R) < 0$. Then, increasing $\bar{\beta}$ makes the first order to be negative. Therefore, the amount of discount increases in $\bar{\beta}_t$.

(b) Suppose that the optimal price is $\bar{p}$ when the reimbursement rate and the marginal value of inventory are R_t and Δ, respectively, and consider that R_t increases by $\epsilon > 0$. Then, the first order of the expected revenue at $p = \bar{p}$ becomes positive: $1 - \frac{f(\bar{p})}{\bar{F}(\bar{p})}(\bar{p} - \Delta) - \frac{\bar{\beta}g(\bar{p}/q)/q}{1-\bar{\beta}G(\bar{p}/q)}(\bar{p} - R_t - \epsilon) = \frac{\bar{\beta}g(\bar{p}/q)/q}{1-\bar{\beta}G(\bar{p}/q)}\epsilon > 0$. From Lemma 1, this implies that the optimal price should be higher as R_t increases. It is also trivial to see that the expected revenue increases in R_t.

From a consumer's perspective, their utilities are the maximum of three options. Given that the price increases, the utility from a cash purchase decreases while the utilities from a point purchase or no purchase remain the same. This implies the the consumer's utility remains the same or gets worse as a consequence to the increase of reimbursement rate. In particular, the utility of a consumer who previously purchase with cash decreases.

Proof of Proposition 3

We need to show that p^* increases in marginal value of inventory since the marginal value of inventory ($\Delta_t(y)$) increases in t and decreases in y. Suppose that p^* satisfies the first-order condition, given by (22), when the marginal value of inventory is Δ. If Δ increases by $\epsilon > 0$, then, the first order at p^* becomes positive : $1 - \frac{f(p^*)}{\bar{F}(p^*)}(p^* - \Delta - \epsilon) - \frac{\bar{\beta}g(p^*/q)/q}{1-\bar{\beta}G(p^*/q)}(p^* - R) = \frac{f(p^*)}{\bar{F}(p^*)}\epsilon > 0$. Thus, the optimal price should also increase if the marginal value of inventory increases.

Proof of Proposition 4

Given that the marginal value of inventory increases in t and decreases in y, we only need to show that there exist three regions separated by the two threshold values of Δ. Suppose that Δ is low enough such that the *cash-only price* is lower than the reimbursement rate (i.e., $p^c(\Delta) < R$), then, it is trivially true that allowing reward sales gives a higher expected revenue by $J^o(p^*) \geq J^o(p^c) > J^c(p^c)$. On the other hand, if Δ is high enough so that $\Delta > R$, the seller is always better off to block reward sales because any point-sale transaction will be a loss. In the middle of these two cases where $\Delta < R < p^* < p^c$,

$$\frac{dJ^o(p^*(\Delta), \Delta)}{d\Delta} = \frac{\partial J}{\partial p^*}\frac{\partial p^*}{\partial \Delta} + \frac{\partial J}{\partial \Delta} = 0 - \left(\bar{F}(p^*) + \bar{\beta}\int_0^{p/q} \bar{F}(xq)dG(x)\right) \tag{24}$$

while

$$\frac{dJ^c(p^c(\Delta), \Delta)}{d\Delta} = \frac{\partial J}{\partial p^c}\frac{\partial p^c}{\partial \Delta} + \frac{\partial J}{\partial \Delta} = 0 - \bar{F}(p^c) \tag{25}$$

Since $p^c > p^*$, it is obvious that the value-to-go function decreases faster in the marginal value of inventory when it is open compared to when closed, i.e., $J^o_\Delta < J^c_\Delta$. This implies that if the value of inventory increases (as the longer periods or the less inventories left), the value function with open decision becomes less favorable than the value function with black-out decision.

Proof of Proposition 5

(a) From Proposition 2 (b), we know that the expected revenue with reward sales increases in the reimbursement rate while the cash-only revenue remains the same. Thus, it is obvious that for some value of Δ, the optimal decision changes from close to open.

(b) Suppose that it is optimal to open when $\bar{\beta}$ is given, that is, $J^o(p^*) > J^c(p^c)$. Note that the expected revenue with allowing reward sales is a weighted sum of revenue from *reward consumers*, $\bar{F}(p^*)\bar{G}(p^*/q)(p^* - \Delta) + \int_0^{p^*/q} \bar{F}(qx)dG(x)(R-\Delta)$, and cash-only consumers, $\bar{F}(p^*)(p^* - \Delta)$. Let us write the expected revenue with black-out by $\bar{\beta}\bar{F}(p^c)(p^c - \Delta) + \beta\bar{F}(p^c)(p^c - \Delta)$ and compare part by part. Since p^c maximizes the revenue from cash-only consumers,

$$\beta\bar{F}(p^c)(p^c - \Delta) > \beta\bar{F}(p^*)(p^* - \Delta).$$

With the assumption that the open decision is optimal ($J^o(p^*) > J^c(p^c)$), this implies that the expected revenue from *reward consumers* is greater when allowing reward sales:

$$\bar{\beta}\bar{F}(p^c)(p^c - \Delta) < \bar{\beta}\Big[\bar{F}(p^*)\bar{G}(p^*)(p^* - \Delta) + \int_0^{p^*/q} \bar{F}(xq)dG(x)(R-\Delta)\Big].$$

Suppose that $\bar{\beta}$ increases. It is obvious that the expected revenue with allowing reward sales increases as $\bar{\beta}$ even under the same price p^* while the expected revenue with black-out decision remains the same. The further adjustment of the optimal price for changing $\bar{\beta}$ will make even higher revenue from allowing reward sales, and this completes the proof.

Proof of Proposition 6

We know that both $p_t^c(y)$ and $p_t^*(y)$ are non-decreasing in t and non-increasing in y (Gallego and Van Ryzin (1994) for p^c and Proposition 3 for p^*). From Proposition 4, we also know that the optimal price changes from $p_t^B(y) = p_t^c(y)$ to $p_t^B(y) = p_t^*(y)$ as t decreases and y increases. Suppose that the transition is made between t and $t-1$, that is, $p_t^B(y) = \tilde{p}_t^c(y)$ and $\tilde{p}_t^o(y+1) = p_t^B(y+1)$. From Proposition 4 (a), the *cash-only price* is higher than the point-selling price when it is optimal to close and we know that $p_t^o(y)$ decreases as t decreases, thus, we have $\tilde{p}_t^c(y) > \tilde{p}_t^o(y) > \tilde{p}_{t-1}^o(y)$. We can show the same result when the transition is made between y and $y-1$.

Proof of Theorem 2

Note that $\tilde{p}_t(q, y)$ satisfies the following first-order condition:

$$1 - \frac{f(p)}{\bar{F}(p)}(p - \Delta) - \frac{1}{q} \cdot \frac{\bar{\beta}(q) \cdot g(p/q)}{1 - \bar{\beta}(q) \cdot G(p/q)}(p - R(q) = 0. \tag{26}$$

By *A4* and *A5*, we know that the third term, $-\frac{1}{q}\frac{\bar{\beta}(q)\cdot g(p/q)}{1-\bar{\beta}(q)G(p/q)}(p-R(q)$, increases in q for a given p. This implies that the best price $\tilde{p}_t(q,y)$ should also increase to satisfy the first-order condition as q increases.

Proof of Proposition 7

First note that any (t,y) can be summarized by the corresponding marginal value of inventory. For simplicity of notation, we will use Δ for replacing t and y, i.e., we use $J(p,q,\Delta)$ for $J_t^Q(p,q,y)$ and $\tilde{p}_t(q,y)$ for $\tilde{p}(q,\Delta)$. We also assume that Δ is a continuous variable for this proof despite the fact that it takes the discrete value because t and y are integral. Now we want to show that, for any pair of point requirements (q_l and q_h with $q_l < q_h$), there exists a unique $\bar{\Delta}_{l,h}$ such that

$$J(\tilde{p}(q_l,\Delta),q_l,\Delta) > J(\tilde{p}(q_h,\Delta),q_h,\Delta) \text{ if } \Delta < \bar{\Delta}_{l,h} \text{ and}$$

$$J(\tilde{p}(q_l,\Delta),q_l,\Delta) < J(\tilde{p}(q_h,\Delta),q_h,\Delta) \text{ if } \Delta > \bar{\Delta}_{l,h}.$$

There are three different cases to consider depending on the value of Δ.

(i) Suppose that Δ is high enough such that $R(q_l) < R(q_h) < \Delta$. In this case, any reward sale (regardless of whether $q = q_l$ or q_h) negatively affects the seller's revenue. Since the probability of a reward sale decreases in q, the higher point requirement is better, i.e., $J(\tilde{p}(q_h,y),q_h,\Delta) < J(\tilde{p}(q_h,y),q_h,\Delta)$.

(ii) Suppose that Δ has some intermediate value such that $R(q_l) < \Delta < R(q_h)$. In this case, only a reward sale at $q = q_l$ negatively affects the seller's revenue. Thus, the value function with a higher point requirement is better, i.e., $J(\tilde{p}(q_h,y),q_h,\Delta) < J(\tilde{p}(q_h,y),q_h,\Delta)$.

(iii) Suppose that Δ is low enough such that $\Delta < R(q_1) < R(q_2)$. The derivative of value-to-go function with respect to Δ is given by the negative of the probability of a purchase:

$$\frac{dJ^Q(\tilde{p}(q,\Delta),q,\Delta)}{d\Delta} = -\left\{\bar{F}(\tilde{p}(q,\Delta)) - \bar{\beta}(q)\int_0^{\tilde{p}(q,\Delta)} G(x/q)dF(x)\right\} < 0.$$

From $\bar{\beta}(q_l) > \bar{\beta}(q_h)$ and $\tilde{p}(q_l,\Delta) < \tilde{p}(q_h,\Delta)$, we know that the probability of total sales is higher when the point requirement is low. That is, the value-to-go function decreases faster in Δ with a lower point requirement. With the continuity of the value-to-go function in Δ, this implies that there exists at most one threshold, $\bar{\Delta}_{q_l,q_h}$, such that q_h is better than q_l if $\Delta > \bar{\Delta}_{q_l,q_h}$, and vice versa. Along with the fact that the marginal value of inventory is increasing in t and decreasing in y, this completes the proof.