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The critical state of compressible swirling flows in a finite-length straight circular pipe

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Abstract

Functional analysis techniques are used to rigorously determine the range of flow Mach number Ma_0 for the existence of the critical swirl ratio ω_1 for exchange of stability of a base columnar compressible swirling flow of a perfect gas in a finite-length straight circular pipe. For swirling flows with a monotonic circulation profile, it is first established that ω_1 definitely exists in the range $0 \leq Ma_0 \leq 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$, where $1 < \gamma \leq 5/3$ is the ratio of specific heat of the gas. Then, the existence of a limit Mach number $Ma_{0;lim}$ between $2\frac{\sqrt{\gamma-1}}{\gamma}$ and 1 is proven; i.e. ω_1 does not exist and the base flow is stable for all swirl levels when Ma_0 is above $Ma_{0;lim}$. In addition, the analytical solution of ω_1 as a function of Ma_0, γ and pipe length x_0 for a solid-body rotation flow with uniform axial velocity and temperature is also derived. For all $1 < \gamma \leq 5/3$, the $Ma_{0;lim}$ of this flow increases from $2\frac{\sqrt{\gamma-1}}{\gamma}$ to $\sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}} < 1$ as x_0 increases from 0 to infinity. This result matches with the numerical computations of Rusak & Lee (2002).

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1 Introduction

The stability of compressible swirling flows is an important problem for a variety of technological applications such as the aerodynamics of slender wings operating at high angles of incidence (Peckham & Atkinson 1957, Rusak *et al.* 1983, Delery 1994, Mitchell & Delery 2001), combustion chambers ((Umeh, Rusak & Gutmark 2010, 2012)), inlets and nozzles of jet engines ((Mclelland, MacManus & Sheaf 2015)), and other high-speed flow devices where swirl has a dominant influence. The study of this problem may also shed light on complicated stability and breakdown phenomena that appear in numerous problems of geophysical and meteorological significance. In all of these cases, the flow Mach number is not small and may reach values of 0.2 to 0.9, and the effect of compressibility is an essential part of the flow dynamics and influences the conditions for the appearance of instabilities and transition (breakdown) phenomena.

The phenomenon of vortex breakdown in incompressible flows was studied theoretically, numerically and experimentally by Wang & Rusak (1996, 1997), Rusak (1996), Malkiel *et al.* (1996), Rusak *et al.* (1998), Rusak & Lamb (1999), Rusak & Judd (2001) and Rusak & Meder (2004). Rusak & Lee (2002) extended these studies and investigated the effect of compressibility on the critical swirl level for breakdown of subsonic vortex flows in a straight circular pipe of finite length. The work extended the critical-state concept of Benjamin (1962) to include the influence of Mach number on the flow behavior. The analysis was based on a linearized version of the equations for the motion of a steady axisymmetric inviscid and compressible swirling flow of a prefect gas. The relationships between the velocity, density, temperature and pressure perturbations to a base columnar flow state were derived. An eigenvalue problem was formulated to determine the first critical level of swirl ω_1 as a function of incoming flow Mach number Ma_0 at which a special mode of a non-columnar small disturbance may appear on the base flow. It was found that when the characteristic Mach number of the base flow tends to zero the eigenvalue problem and ω_1 are the same as defined by Wang & Rusak (1996, 1997) (see also Rusak 2000) in their study of incompressible swirling flows in straight circular pipes. As the characteristic Mach number is increased, ω_1 increases and the flow perturbation expands in the radial direction. As the Mach number approaches a certain limit value $Ma_{0;lim}$ related to the core size of the vortex, ω_1 reaches very large values and becomes singular. These results indicated that the axisymmetric breakdown of high-Reynolds-number compressible vortex flows may be delayed to higher levels of the swirl ratio with the increase of the flow Mach number Ma_0. This is similar to results found in the numerical simulations of Melville (1996) and Herrada *et al.* (2003).

In a follow up study, Rusak & Lee (2004) studied the linear stability of a compressible inviscid axisymmetric and rotating columnar flow of a prefect gas in a finite-length straight circular pipe is investigated. A well-posed model of the unsteady motion of a swirling flow, with inlet and outlet conditions that may reflect the physical situation, is formulated. The linearized equations of motion for the evolution of infinitesimal axially symmetric disturbances are derived. A general mode of disturbance, that is not limited to the axial-Fourier mode, is introduced and an eigenvalue problem is developed. It is found that a neutral mode of disturbance exists at the critical swirl ratio for a compressible vortex flow, $\omega_1(Ma_0)$. The flow changes its stability characteristics as the swirl ratio increases across this critical level. When the swirl ratio is below the critical level (supercritical flow), the flow is asymptotically stable and, when it is above the critical level (subcritical flow), the flow unstable. When the characteristic Mach number of the base flow tends to zero, the results are the same as found for incompressible swirling flows in pipes. The growth rate ratio is positive but decreases as Mach number is increased. This ratio vanishes at the limit Mach number $Ma_{0;lim}$ at which the critical swirl tends to infinity.

In the present paper, functional analysis techniques are used to rigorously determine the range of flow Mach number Ma_0 for the existence of the critical swirl ratio ω_1. For swirling flows with a monotonic circulation profile. The mathematical formulation is described in section 2. It is first established that ω_1 definitely exists in the range $0 < Ma_0 < 2\sqrt{\gamma - 1}/\gamma < 1$, where $\gamma > 1$ is the ratio of specific heats of the gas (section 3). Then, the existence of a limit Mach number $Ma_{0;lim}$ between $2\sqrt{\gamma - 1}/\gamma$ and 1 is proven for a subclass of swirling flows; i.e. ω_1 does not exist and the flow is stable for all swirl level when Ma_0 is beyond $Ma_{0;lim}$ (section 3). For example, $0.9035 < Ma_{0;lim} < 1$ when $\gamma = 1.4$. In particular, the analytical solution of ω_1 as a function of Ma_0, γ and pipe length L for a solid-body rotation flow with a uniform axial velocity and temperature is also derived. The asymptotic behavior of this solution as Ma_0 tends to zero matches the results of Renac *et al.* (2007) (section 4). In addition, $Ma_{0;lim}$ of this flow is between 0.9035 and 0.9283 for $\gamma = 1.4$; it increases from 0.9035 to 0.9283 as the pipe non-dimensional length L increases from 1 and tends to infinity. This result matches with the numerical computations of Rusak & Lee (2002). Results are concluded in section 5. Specifically, the present results shed light on the stability of base columnar vortex flows and the evolution to vortex breakdown with the increase of the flow Mach number.

2 The compressible flow critical-state problem

We consider a base, steady, inviscid, compressible, axisymmetric, parallel, swirling flow of a perfect gas in a straight, finite-length circular pipe of radius $\tilde{r}_0$ and length $\tilde{x}_0$. Distances are scaled with the pipe radius $\tilde{r}_0$ and pipe non-dimensional length is $x_0 = \tilde{x}_0/\tilde{r}_0$. The cylindrical coordinate system (x, r, θ) is used to describe the flow, where x measures non-dimensional distance from the pipe inlet along the pipe centerline ($0 \leq x \leq x_0$), r measures non-dimensional radial distance from the centerline ($0 \leq r \leq 1$), and θ is the azimuth angle ($0 \leq \theta < 2\pi$). Let $y = r^2/2$ where $0 \leq y \leq 1/2$. The corresponding base flow non-dimensional axial, radial and circumferential velocity components $w,\ u,\ v$ (scaled with the inlet centerline axial velocity $\tilde{U}_0$) are given by the profiles $w(x, y, \theta) = w_0(y)$, $v(x, y, \theta) = \omega v_0(y)$ and $u(x, y, \theta) = 0$ for $0 \leq x \leq x_0$ and $0 \leq y \leq 1/2$. Here $\omega > 0$ is the swirl ratio of the flow. The flow non-dimensional temperature field (scaled with the inlet centerline temperature $\tilde{T}_0$) is given by $T(x, y, \theta) = T_0(y)$. The flow non-dimensional pressure field (scaled with the inlet centerline pressure $\tilde{p}_0$) is given by $p(x, y, \theta) = p_0(y) = \exp\left(\frac{\gamma Ma_0^2\omega^2}{2}\int_0^y \frac{v_0^2(\overline{y})}{\overline{y}T_0(\overline{y})}d\overline{y}\right)$. We focus in this paper on the family of flows where the axial velocity and the temperature are uniform profiles, $w_0(y) = T_0(y) = 1$. In addition, the Mach number is $Ma_0 = \tilde{U}_0/\sqrt{\gamma R\tilde{T}_0}$ where $R = \mathcal{R}/M$ is the specific gas constant, $\mathcal{R} = 8.3145$ J/(mol K) is the universal gas constant and M is the gas molecular weight. Also, γ is the ratio of specific heat of the perfect gas, $1 < \gamma \leq 5/3$ (Thompson 1988, p. 80), i.e., $\gamma = 5/3$ for a monatomic gas (He, Ne, Ar), $\gamma = 7/5$ for diatomic gases ($H_2, N_2,\ O_2$, Air) at a temperature below 300 K, and decreases to 9/7 with the increase of temperature (above 1000 K). Moreover, γ approaches unity for gases with a large number of degrees of freedom and extreme molecular complexity (Butane, Octane, Fluorocarbons and Hydrofluorocarbons), see Cramer (1989), Cramer & Tarkenton (1992), and Schnerr & Leidner (1993).

We define the base flow circulation function as $K_0(y) = \sqrt{2y}v_0(y)$. We also assume that $K_0(y)$ is continuous with a continuous first derivative, i.e. $K_0 \in C^1[0, 1/2]$, and that $K_0(y)$ is a strictly monotonic function, i.e. $K_0(y) > 0$, $K_{0y} > 0$ with the properties $\left(\frac{K_0(y)}{y}\right)_y \leq 0$ for $0 < y < 1/2$ and $K_0(y) \sim cy$ when $y \to 0^+$, where $c \neq 0$ is a constant. Note that under these conditions the base flow is neutrally stable according to Rayleigh's stability criterion for swirling flows in a pipe with periodic inlet-outlet conditions. However, these flows are unstable according to Rusak & Lee (2004) and Rusak *et al.* (2007) when non-periodic inlet-outlet conditions are used and $\omega > \omega_1(Ma_0, x_0, \gamma)$.

Following Rusak & Lee (2002), the critical state of this base flow is determined by the first eigenvalue $\Omega = \Omega_1 = \omega_1^2$ of the following eigenvalue problem:

$$\Phi_{yy} + \Phi_y Q(y) + \Phi S(y) = 0 \ \text{ with boundary conditions } \ \Phi(0) = \Phi(1/2) = 0, \tag{1}$$

where

$$Q(y) = -\frac{\gamma Ma_0^2 \Omega K_0^2}{4y^2},$$
$$S(y) = (1 - Ma_0^2)\left(\frac{\Omega K_0 K_{0y}}{2y^2} - \frac{\pi^2}{8x_0^2 y}\right) + (\gamma - 1)Ma_0^2\left(\frac{\Omega K_0^2}{4y^2}\right)^2 - (\gamma - 1)Ma_0^2\left(\frac{\Omega K_0^2}{4y^2}\right)_y. \tag{2}$$

Here $\Phi(y)$ is the corresponding eigenfunction. The critical swirl ω_1 is a function of incoming flow Mach number $0 \leq Ma_0 < 1$, pipe non-dimensional length x_0, and the ratio of specific heat $1 < \gamma \leq 5/3$, i.e., $\omega_1(Ma_0, x_0, \gamma)$. Note that this eigenvalue problem is not self-adjoint. To convert the problem into a self-adjoint problem, we let $\Phi(y) = \exp\left(-\frac{1}{2}\int Q \mathrm{d}y\right)\psi(y)$. Substituting it into (1) we obtain

$$\psi_{yy} + P(y)\psi = 0 \ \text{ with boundary conditions } \ \psi(0) = \psi(1/2) = 0, \tag{3}$$

where

$$\begin{aligned} P(y) &= (1 - Ma_0^2)\left(\frac{\Omega K_0 K_{0y}}{2y^2} - \frac{\pi^2}{8x_0^2 y}\right) \\ &+ \left[-\frac{\gamma^2 Ma_0^4}{4} + (\gamma - 1)Ma_0^2\right]\left(\frac{\Omega K_0^2}{4y^2}\right)^2 + \left(1 - \frac{\gamma}{2}\right)Ma_0^2\left(\frac{\Omega K_0^2}{4y^2}\right)_y. \end{aligned} \tag{4}$$

The problem (3) is rearranged in the form

$$-\psi_{yy} + C(y;x_0;Ma_0)\psi + \Omega G_1(K_0, K_{0y}, y; Ma_0;\gamma)\psi + \Omega^2 G_2(K_0, y; Ma_0;\gamma)\psi = 0, \qquad (5)$$

$$\psi(0) = \psi(1/2) = 0,$$

where

$$\begin{aligned} C(y;x_0;Ma_0) &= (1-Ma_0^2)\frac{\pi^2}{8x_0^2 y}, \\ G_1(K_0, K_{0y}, y; Ma_0;\gamma) &= (Ma_0^2-1)\frac{K_0K_{0y}}{2y^2} + \left(\frac{\gamma}{2}-1\right)Ma_0^2\left(\frac{K_0^2}{4y^2}\right)_y, \\ G_2(K_0, y; Ma_0;\gamma) &= \left[\frac{\gamma^2 Ma_0^4}{4} - (\gamma-1)Ma_0^2\right]\left(\frac{K_0^2}{4y^2}\right)^2. \end{aligned} \qquad (6)$$

In the Appendix A, we define the functional spaces $L^2\left(0,\frac{1}{2}\right)$ and $H_0^1\left(0,\frac{1}{2}\right)$ to be used in the following analysis. Based on (6), let

$$\mathcal{L}(\cdot) \equiv -\frac{\mathrm{d}^2}{\mathrm{d}y^2}(\cdot),\ \mathcal{A} \equiv \mathcal{L}^{-1},\ \mathcal{G}_1 \equiv -G_1\mathcal{A},\ \mathcal{G}_2 \equiv -G_2\mathcal{A}.$$

be linear operators on L^2. Positive definiteness of $\mathcal{L}$ and Hardy's inequality (17) imply that $\mathcal{G}_1$ and $\mathcal{G}_2$ are well-defined. Notice that the operators $\mathcal{A}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are self-adjoint and compact; in particular, $\mathcal{A}$ is positive definite.

According to Eisenfeld (1968), let $\xi \equiv \mathcal{L}^{1/2}\psi$. We obtain an equivalent form of (5)

$$(\mathcal{I} + C\mathcal{A})\xi - \Omega\mathcal{G}_1\xi - \Omega^2\mathcal{G}_2\xi = 0. \qquad (7)$$

Here $\mathcal{I}$ is the identity operator on $\mathfrak{H}$ (defined in Appendix A). In the following sections, we prove the range of Ma_0 for the existence of Ω_1 based on the two forms of the eigenvalue problem, (5) or (7).

3 Existence of Ω_1 for subsonic swirling flows

From (6), we find that $G_2(K_0, y; Ma_0;\gamma) \le 0$ for all $0 < y \le 1/2$ if and only if $\frac{\gamma^2}{4}Ma_0^4 - (\gamma-1)Ma_0^2 \le 0$ or $0 \le Ma_0 \le 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$ when $1 < \gamma \le 5/3$. In addition, we show in Appendix B that if $x_0 \ge \frac{\sqrt{1-Ma_0^2}}{4}$ then the operator $\mathcal{I}+C\mathcal{A}$ is invertible and $(\mathcal{I}+C\mathcal{A})^{-1}$ is self-adjoint, bounded and positive definite for each $0 \le Ma_0 < 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$.

In Appendix C we prove the following theorem:

Theorem 1: The critical swirl Ω_1 exists for each Mach number Ma_0 and pipe non-dimensional length x_0 if $x_0 \geq \frac{1}{4}\sqrt{1 - Ma_0^2}$ and $0 \leq Ma_0 < 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$. Moreover, the critical swirl for a finite-length pipe $\Omega_1(Ma_0, x_0, \gamma)$ is greater than the critical swirl $\Omega_1(Ma_0, x_0 \to \infty, \gamma)$ for an infinitely-long pipe.

It should be noted, however, that given a monotonic circulation function $K_0(y)$, the critical swirl $\Omega_1(Ma_0, x_0, \gamma)$ may not always exist for all Ma_0 in the range $2\frac{\sqrt{\gamma-1}}{\gamma} \leq Ma_0 < 1$. For example, we rigorously show in section 4 that for a solid-body rotation flow where $K_0(y) = 2y$, the critical swirl Ω_1 does not exist when $\sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}} < Ma_0 < 1$ with $1 < \gamma \leq 5/3$.

For each circulation function $K_0(y)$ and pipe length x_0, we define the limit Mach number $Ma_{0,lim}$ as

$$Ma_{0,lim} = \sup\{Ma \geq 0 : \Omega_1 \text{ exists for each } 0 \leq Ma_0 < Ma\}. \tag{8}$$

Theorem 1 shows that for any monotonic circulation function, $Ma_{0,lim} \geq 2\frac{\sqrt{\gamma-1}}{\gamma}$.

In Appendix D we prove the following theorem:

Theorem 2: If there exists $0 < y^* < 1/2$ such that

$$y^* K_{0y}(y^*) > \frac{2(\gamma-1)}{\gamma} K_0(y^*) \tag{9}$$

and

$$y K_{0y}(y) \neq \frac{2(\gamma-1)}{\gamma} K_0(y) \tag{10}$$

almost everywhere over $0 < y < 1/2$, then Ω_1 exists at $Ma_0 = 2\frac{\sqrt{\gamma-1}}{\gamma}$.

In particular, Theorem 2 shows that for both the solid-body rotation flow, where $K_0(y) = 2y$, and the Lamb-Oseen vortex flow, where $K_0(y) = 1 - \exp(-2by)$ (where $b > 0$ is the vortex core parameter, $r_c = 1.12/\sqrt{b}$), the critical swirl Ω_1 exists at $Ma_0 = 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$, but takes different values for each flow.

In Appendix E we prove the following theorem:

Theorem 3: The critical swirl Ω_1 does not exist in a vicinity of $Ma_0 = 1^-$ for every monotonic circulation function $K_0(y)$ for which $\left(\frac{K_0(y)}{y}\right)_y < 0$ almost everywhere in the domain $0 < y < 1/2$.

Theorem 3 shows that for every circulation function $K_0(y)$ which fulfills the theorem assumption, its $Ma_{0,lim} < 1$. The limit Mach number $Ma_{0,lim}$ may be close to unity or quite below unity, depending on the values of γ and x_0. Notice that the solid-body rotation flow $K_0(y) = 2y$ does not fulfill the assumption in Theorem 3. However, as we show in the next section, that the above conclusion still holds for this flow.

4 The critical swirl for a solid-body rotation flow

The base circulation function of a solid-body rotation flow in a rotating pipe is $K_0(y) = 2y$. Substituting this into (5) and multiplying both sides by y we obtain

$$y\Phi_{yy} - \gamma Ma_0^2 \Omega y \Phi_y + \left[(\gamma-1)Ma_0^2\Omega^2 y + (1-Ma_0^2)\left(2\Omega - \frac{\pi^2}{8x_0^2}\right)\right]\Phi = 0, \tag{11}$$
$$\Phi(0) = \Phi(1/2) = 0.$$

Introducing the substitution $\Phi = \frac{\eta}{\Omega} w(\eta)$ where $\eta = \Omega y$ gives an equivalent equation for $w(\eta)$,

$$\eta w_{\eta\eta} + (2 - \gamma Ma_0^2 \eta) w_\eta + \left[(\gamma-1)Ma_0^2\eta + (1-Ma_0^2)\left(2 - \frac{\pi^2}{8x_0^2\Omega}\right) - \gamma Ma_0^2\right] w = 0.$$

This is a second-order ordinary differential equation and its general solution consists of a linear combination of two linearly independent solutions of the form (Slater (1960)

$$w(\eta) = A\exp\left(\frac{\gamma Ma_0^2}{2}\eta\right)\exp\left(-\frac{\sqrt{\Delta}}{2}\eta\right){}_1F_1\left(1 - \frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}}\ ;2\ ;\ \eta\sqrt{\Delta}\right)$$
$$+B\exp\left(\frac{\gamma Ma_0^2}{2}\eta\right)\exp\left(\frac{\sqrt{\Delta}}{2}\eta\right)\Psi\left(1 + \frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}}\ ;\ 2\ ;-\eta\sqrt{\Delta}\right). \tag{12}$$

Here $\Delta = 4G_2$ and ${}_1F_1$ and Ψ are Kummer's and Tricomi's functions, respectively. The coefficients A and B are constant. Assuming that w is bounded, then the boundary condition $\Phi(0) = 0$ is automatically satisfied. However, because we are interested in bounded real-valued eigenmodes $\Phi(y)$ of (11), we need to be specific on the choice of the coefficients A and B. In Appendix F we prove that we must have $B = 0$. Then, the eigenmode $\Phi(y)$ that corresponds to the first eigenvalue Ω_1 of (11) is given by

$$\Phi(y) = y\exp\left(\frac{\gamma Ma_0^2\Omega_1}{2}y\right)\exp\left(-\frac{\sqrt{\Delta}}{2}\Omega_1 y\right){}_1F_1\left(1 - \frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega_1}\right)}{\sqrt{\Delta}};2;\Omega_1\sqrt{\Delta}y\right). \tag{13}$$

Using the boundary condition $\Phi(1/2) = 0$, the critical swirl Ω_1 is the least positive root of the following characteristic equation in terms of Ω

$$ {}_1F_1\left(1 - \frac{(1-Ma_0^2)\left(2 - \frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}}\ ; 2\ ;\ \frac{\Omega}{2}\sqrt{\Delta}\right) = 0. \tag{14} $$

We consider three cases according to the sign of Δ:

Case (I): When $\Delta < 0$, then $0 < Ma_0 < \frac{2\sqrt{\gamma-1}}{\gamma}$. From Theorem 1 in section 3 we find that Ω_1 exists in this range of Ma_0.

Case (II): When $\Delta = 0$, then either $Ma_0 = 0$ or $Ma_0 = \frac{2\sqrt{\gamma-1}}{\gamma}$. For this case, (14) is reduced to either $J_1\left(\sqrt{2\left(2\Omega_1 - \frac{\pi^2}{8x_0^2}\right)}\right) = 0$ when $Ma_0 = 0$ or $J_1\left(\sqrt{2(1-4\frac{\gamma-1}{\gamma^2})\left(2\Omega_1 - \frac{\pi^2}{8x_0^2}\right)}\right) = 0$ when $Ma_0 = \frac{2\sqrt{\gamma-1}}{\gamma}$. We obtain $\omega_1 = \sqrt{\frac{j_1^2}{4} + \frac{\pi^2}{16x_0^2}}$ when $Ma_0 = 0$, where j_1 is the first positive zero of the Bessel function J_1. This is the critical swirl of an incompressible solid-body rotation flow in a finite-length straight circular pipe (first defined in Wang & Rusak 1996). Also, $\omega_1 = \sqrt{\frac{\gamma^2 j_1^2}{4(2-\gamma)^2} + \frac{\pi^2}{16x_0^2}}$ when $Ma_0 = \frac{2\sqrt{\gamma-1}}{\gamma}$.

Case (III): When $\Delta > 0$, then $\frac{2\sqrt{\gamma-1}}{\gamma} < Ma_0 < 1$ for $1 < \gamma \leq 5/3$ and $\sqrt{\Delta}$ is real. First, when $x_0 \to \infty$ equation (14) becomes

$$ {}_1F_1\left(1 - \frac{2(1-Ma_0^2)}{\sqrt{\Delta}}\ ; 2\ ;\ \frac{\Omega_1}{2}\sqrt{\Delta}\right) = 0. \tag{15} $$

When the first argument of ${}_1F_1$ is negative, $1 - \frac{2(1-Ma_0^2)}{\sqrt{\Delta}} < 0$, we have $Ma_0 < \sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}}$. Then, we use the distribution of zeros of Kummer's function (Bateman *et al.* 1955) to deduce the existence of a solution $\Omega_1 > 0$ of (15). On the other hand, when $1 - \frac{2(1-Ma_0^2)}{\sqrt{\Delta}} \geq 0$ we have $\sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}} \leq Ma_0 < 1$ for $1 < \gamma \leq 5/3$. Then from the series representation (29) we know that the left-hand side of (15) is greater than 1 for all $\Omega > 0$, and thus Ω_1 does not exist.

Combining cases (I), (II) and (III), we find that when $x_0 \to \infty$ the critical swirl ω_1 exists in the subsonic flow regime if and only if $0 \leq Ma_0 < \sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}}$. This implies that

$$ Ma_{0,lim}(x_0 \to \infty, \gamma) = \sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}}. \tag{16} $$

Note that $Ma_{0,lim}(x_0 \to \infty, \gamma = 1.4) = 0.9283$; this value is same as the one obtained numerically in Rusak & Lee (2002) and (2004). Figure 1 presents the results of $\omega_1 = \sqrt{\Omega_1}$

when $x_0 \to \infty$ as a function of $0 \le Ma_0 \le 1$ for various values of $\gamma = 1.4$, 1.1, and 1.02. The limit values $Ma_{0,lim}$ for each value of γ are also shown for reference. It can be seen that ω_1 increases with Ma_0 and tends to ∞ as Ma_0 approaches $Ma_{0,lim}(x_0 \to \infty, \gamma)$. Also, $Ma_{0,lim}(x_0 \to \infty, \gamma)$ decreases with the decrease of γ toward unity to a limit value of 0.8165.

Similarly, for a finite-length pipe with length x_0, the same argument implies the existence of $Ma_{0,lim}(x_0)$. Besides, since $(1-Ma_0^2)\left(2 - \frac{\pi^2}{8x_0^2\Omega_1}\right) < 2(1-Ma_0^2)$, we have $Ma_{0,lim}(x_0,\gamma) < Ma_{0,lim}(x_0 \to \infty, \gamma)$. Moreover, since the term $(1-Ma_0^2)\left(2 - \frac{\pi^2}{8x_0^2\Omega}\right)$ monotonically increases with x_0 for each $0 \le Ma_0 < 1$ and $\Omega > 0$, it follows that $Ma_{0,lim}(x_0,\gamma)$ is an increasing function of x_0 toward the limit value $Ma_{0,lim}(x_0 \to \infty, \gamma)$.

In summary, we find that there exists a limit Mach number $Ma_{0,lim}(x_0,\gamma) < 1$ such that Ω_1 exists if and only if $0 \le Ma_0 < Ma_{0,lim}$. Moreover, for each $0 < x_0 < \infty$ we have for the solid-body rotation flow that

$$\frac{2\sqrt{\gamma-1}}{\gamma} < Ma_{0,lim}(x_0,\gamma) < Ma_{0,lim}(x_0 \to \infty, \gamma) = \sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}}.$$

This is also demonstrated in figure 1 for various values of γ.

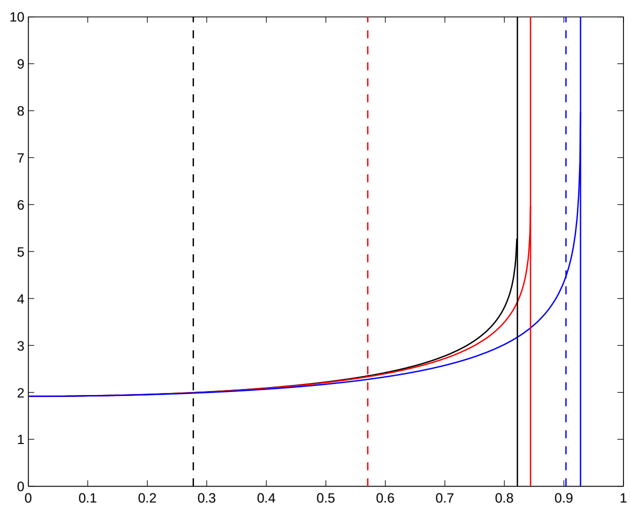

Figure 1: ω_1 versus Ma_0 plots for $\gamma = 1.02$ (black), $\gamma = 1.1$ (red) and $\gamma = 1.4$ (blue) when $x_0 \to \infty$. The solid vertical lines represent the respective values of $Ma_{0,lim}$ according to (16) and the dash lines represent the respective values of $Ma_0 = 2\frac{\sqrt{\gamma-1}}{\gamma}$.

5 Conclusions

Functional analysis techniques are used to rigorously determine the range of flow Mach number Ma_0 for the existence of the critical swirl ratio ω_1 for exchange of stability of a base columnar compressible swirling flow of a perfect gas in a finite-length straight circular pipe. For swirling flows with a monotonic circulation profile, it is first established that ω_1 definitely exists in the range $0 \leq Ma_0 \leq 2\sqrt{\gamma - 1}/\gamma$ (Theorems 1 and 2). Then, the existence of a limit Mach number $Ma_{0,lim}$ between $2\sqrt{\gamma - 1}/\gamma$ and 1 is proven, i.e. ω_1 does not exist when Ma_0 is above $Ma_{0,lim} < 1$. For example, $0.9035 < Ma_{0,lim} < 1$ when $\gamma = 1.4$. In addition, the analytical solution of ω_1 as a function of Ma_0, γ and pipe non-dimensional length x_0 for a solid-body rotation flow with a uniform axial velocity and temperature is also derived. The limit Mach number $Ma_{0,lim}$ of this flow increases from $2\frac{\sqrt{\gamma-1}}{\gamma}$ to $\sqrt{\frac{6-2\gamma-2\sqrt{2\gamma^2-6\gamma+5}}{4-\gamma^2}}$ as x_0 increases from 0 to infinity. This result matches with the numerical computations of Rusak & Lee (2002). The upper limit decreases with decreasing γ to unity and approaches a limit value of 0.8165.

The present results demonstrate that the axisymmetric breakdown of high-Reynolds-number compressible vortex flows may be delayed to higher swirl ratios with the increase of the incoming flow Mach number Ma_0 toward $Ma_{0,lim}$. It also indicates that there is no critical swirl and instability of the base columnar vortex flow at Mach numbers above $Ma_{0,lim}$ that is less than unity. Therefore, it is predicted that no vortex breakdown process can evolve under such conditions. Furthermore, the increase of molecular complexity of the operating gas decreases the value of the limit Mach number.

Appendix A

The space of square-integrable functions on $(0, 1/2)$ is defined by

$$L^2\left(0, \frac{1}{2}\right) = \left\{ f : \left(0, \frac{1}{2}\right) \to \mathbb{R} \;\middle|\; f \text{ is measurable, } \int_0^{1/2} |f|^2 \mathrm{d}y < \infty \right\}$$

with the associated function norm

$$\|f\|_{L^2\left(0,\frac{1}{2}\right)} = \left(\int_0^{1/2} |f|^2 \mathrm{d}y \right)^{1/2}.$$

Let

$$H_0^1\left(0,\frac{1}{2}\right)=\left\{f\in L^2\left(0,\frac{1}{2}\right)\;\middle|\; f(0)=0=f\left(\frac{1}{2}\right),\;\int_0^{1/2}\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2\mathrm{d}y<\infty\right\}$$

be the usual Sobolev space induced by $L^2(0,1/2)$ with the norm

$$\|f\|_{H_0^1(0,1/2)}=\left(\int_0^{1/2}\left(|f|^2+|f'|^2\right)\mathrm{d}y\right)^{1/2}.$$

For brevity, we denote $H_0^1\equiv H_0^1(0,1/2)$ and $L^2\equiv L^2(0,1/2)$ hereafter. The following version of Hardy's inequality is adopted in this study (see Davies 1995): Let $0<\beta<\infty$, $-\infty<\alpha<1$, and let $f\in H_0^1(0,\beta)$. Then

$$\int_0^{\beta} y^{\alpha-2}|f(y)|^2\mathrm{d}y\le\frac{4}{(1-\alpha)^2}\int_0^{\beta} y^{\alpha}|f'(y)|^2\mathrm{d}y. \tag{17}$$

In this study, we refer the term Hardy's inequality to the above inequality with parameters $\beta=1/2$ and $\alpha=0$.

In addition, for the problem (7), we denote $\mathfrak{H}\equiv\mathcal{L}^{1/2}(H_0^1)$. Let $A,B\in\{H_0^1,L^2\}$. We endow each bounded linear operator $\mathcal{T}$ which maps from $(A,\|\cdot\|_A)$ into $(B,\|\cdot\|_B)$ with the standard operator norm

$$\|\mathcal{T}\|_{op}=\inf\{c\ge 0:\|\mathcal{T}x\|_B\le c\|x\|_A \text{ for all } x\in A\}.$$

Appendix B

For each $0\le Ma_0<2\frac{\sqrt{\gamma-1}}{\gamma}<1$, if $x_0\ge\frac{\sqrt{1-Ma_0^2}}{4}$, then the operator $\mathcal{I}+C\mathcal{A}$ is invertible and $(\mathcal{I}+C\mathcal{A})^{-1}$ is self-adjoint, bounded and positive definite.

Proof: The first (minimal positive) eigenvalue λ_1 of $\mathcal{L}$ is determined by the Dirichlet problem $\frac{\mathrm{d}^2\psi}{\mathrm{d}y^2}+\lambda_1\psi=0$ subject to $\psi(0)=\psi(1/2)=0$. Solving this Dirichlet problem implies that $\lambda_1=4\pi^2$. Thus, $\|\mathcal{A}\|_{op}=\|\mathcal{L}^{-1}\|_{op}\le\frac{1}{4\pi^2}$. Besides, using the definition of C in (6), we find that for each ψ that belongs to H_0^1 with $\|\psi\|_{H_0^1}\le 1$,

$$\|C\psi\|_{L^2}=\frac{(1-Ma_0^2)\pi^2}{8x_0^2}\left(\int_0^{1/2}\frac{\psi^2}{y^2}\mathrm{d}y\right)^{1/2}<\frac{(1-Ma_0^2)\pi^2}{4x_0^2}\|\psi\|_{H_0^1}\le\frac{(1-Ma_0^2)\pi^2}{4x_0^2}.$$

Here, the first inequality is derived from Hardy's inequality. Therefore, $\|C\mathcal{A}\|_{op}<\frac{1-Ma_0^2}{16x_0^2}$.

In accordance with the Neumann series theorem (see Tosio 1966), $\mathcal{I}+C\mathcal{A}$ is invertible if $\|C\mathcal{A}\|_{op} < 1$, which requires $\frac{1-Ma_0^2}{16x_0^2} \leq 1$ or equivalently $x_0 \geq \frac{\sqrt{1-Ma_0^2}}{4}$.

Under this condition, the self-adjointness of $(\mathcal{I}+C\mathcal{A})^{-1}$ follows from the self-adjointness of $(\mathcal{I}+C\mathcal{A})$, the boundedness of $(\mathcal{I}+C\mathcal{A})^{-1}$ follows from the bounded inverse theorem (see Rudin 1991), and the positive definiteness of $(\mathcal{I}+C\mathcal{A})^{-1}$ follows from the positive definiteness of $(\mathcal{I}+C\mathcal{A})$.

For each $0 \leq Ma_0 < 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$, the improper integrals $\int_0^{1/2} C\psi^2 \mathrm{d}y$, $\int_0^{1/2} G_1\psi^2 \mathrm{d}y$, and $\int_0^{1/2} G_2\psi^2 \mathrm{d}y$ are convergent (i.e. they have finite values) for every $\psi \in H_0^1$, where C, G_1, and G_2 are given by (6). Besides, we have the following estimations

$$\int_0^{1/2} |G_1|\psi^2 \mathrm{d}y \leq g_1 \int_0^{1/2} \frac{\psi^2}{y^2}\mathrm{d}y, \qquad \int_0^{1/2} |G_2|\psi^2 \mathrm{d}y \leq g_2 \int_0^{1/2} \frac{\psi^2}{y^2}\mathrm{d}y, \tag{18}$$

where

$$\begin{aligned} g_1 &= \frac{1}{2}\left[\left(1-\frac{\gamma}{2}Ma_0^2\right) \sup_{0<y\leq 1/2}(K_0K_{0y}) + \left(1-\frac{\gamma}{2}\right) Ma_0^2 \sup_{0<y\leq 1/2}\left(\frac{K_0^2}{y}\right)\right], \\ g_2 &= \frac{1}{16}\left[-\frac{\gamma^2}{4}Ma_0^4 + (\gamma-1)Ma_0^2\right]\left(\sup_{0<y\leq 1/2}\left(\frac{K_0^2}{y}\right)\right)^2. \end{aligned} \tag{19}$$

Proof: Based on (6), for each $\psi \in H_0^1$ we have the following estimations

$$\begin{aligned} |G_1|\psi^2 &\leq \left|\frac{1}{2}(Ma_0^2-1)K_0K_{0y}\frac{\psi^2}{y^2}\right| + \left|\frac{1}{2}\left(\frac{\gamma}{2}-1\right) Ma_0^2\left(K_0K_{0y} - \frac{K_0^2}{y}\right)\frac{\psi^2}{y^2}\right| \\ &\leq \frac{1}{2}\left(1-\frac{\gamma}{2}Ma_0^2\right) K_0K_{0y}\frac{\psi^2}{y^2} + \frac{1}{2}\left(1-\frac{\gamma}{2}\right) Ma_0^2\frac{K_0^2}{y}\frac{\psi^2}{y^2} \leq g_1\frac{\psi^2}{y^2}, \end{aligned}$$

and

$$|G_2|\psi^2 = \frac{1}{16}\left[-\frac{\gamma^2}{4}Ma_0^4 + (\gamma-1)Ma_0^2\right]\left(\frac{K_0^2}{y}\right)^2\frac{\psi^2}{y^2} \leq g_2\frac{\psi^2}{y^2}.$$

Besides, since$\int_0^{1/2} \frac{\psi^2}{y^2}\mathrm{d}y$ is convergent, which is by virtue of Hardy's inequality (17). We conclude that $\int_0^{1/2} |G_1|\psi^2 \mathrm{d}y$ and $\int_0^{1/2} |G_2|\psi^2 \mathrm{d}y$ are convergent, and so do $\int_0^{1/2} G_1\psi^2 \mathrm{d}y$ and $\int_0^{1/2} G_2\psi^2 \mathrm{d}y$ (absolute convergence implies convergence for improper integrals). Moreover, the convergence of $\int_0^{1/2} C\psi^2 \mathrm{d}y$ follows from the convergence of $\int_0^{1/2} \frac{\psi^2}{y}\mathrm{d}y$, which is in turn by virtue of the comparison $\frac{\psi^2}{y} < \frac{\psi^2}{y^2}$ for every $0 < y \leq 1/2$.

Appendix C

Proof of Theorem 1: By virtue of (7) and Appendix B, the boundary value problem (BVP) (7) is equivalent to

$$\xi - \Omega(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_1\xi - \Omega^2(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_2\xi = 0.$$

Appendix B also implies that the operators $(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_2$ and $(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_1$ are compact (the product of a bounded and a compact operator is compact, see for example Tosio (1966). Besides, the operator $(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_2$ is positive definite if $0 < Ma_0 < 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$.

Case (I): When $Ma_0 = 0$, the incompressible flow case, the BVP (7) becomes

$$\xi - \Omega(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_1\xi = 0.$$

Since $G_1(K_0, K_{0y}, y; 0) = -\frac{K_0K_{0y}}{2y^2} < 0$ for every $0 < y < 1/2$, the operator $\mathcal{G}_1$ is positive definite. Then, BVP (7) is put into an equivalent form

$$(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_1\xi = \frac{1}{\Omega}\xi, \quad \xi \in \mathfrak{H}. \tag{20}$$

In accordance with the spectral theorem by Rudin (1991), (20) has a countable set of non-negative eigenvalues with $\Omega = 0^+$ being its only possible accumulation point. Moreover, we have the following definition of Ω_1 derived from the Rayleigh quotient of the BVP (5) (Hadeler 1967)

$$\Omega_1 = \inf_{\psi \in H_0^1 - \{0\}} \frac{\int_0^{1/2} \psi_y^2 \mathrm{d}y + \int_0^{1/2} \frac{\pi^2}{8x_0^2 y}\psi^2 \mathrm{d}y}{\int_0^{1/2} \frac{K_0K_{0y}}{2y^2}\psi^2 \mathrm{d}y}.$$

We next show that the Ω_1 defined above is positive, and hence it is an eigenvalue of the BVP (7). It is sufficient to show that this is the case for $x_0 \to \infty$. Let $m_1 \equiv \sup\limits_{0<y\leq 1/2} (K_0K_{0y}) > 0$. For each $\psi \in H_0^1$ we have

$$\int_0^{1/2} \psi_y^2 \mathrm{d}y \geq \frac{1}{4}\int_0^{1/2} \frac{\psi^2}{y^2}\mathrm{d}y = \frac{1}{2m_1}\int_0^{1/2} \frac{m_1\psi^2}{2y^2}\mathrm{d}y \geq \frac{1}{2m_1}\int_0^{1/2} \frac{K_0K_{0y}}{2y^2}\psi^2 \mathrm{d}y,$$

in which the first inequality is the Hardy's inequality (17). Therefore, $\Omega_1 \geq \frac{1}{2m_1} > 0$ and thus Ω_1 is the least positive eigenvalue of the BVP (7).

Case(II): When $0 < Ma < 2\frac{\sqrt{\gamma-1}}{\gamma} < 1$, the subsonic compressible case, then the operator $(\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_2$ is positive definite, compact, and self-adjoint. The BVP (7) is equivalent to the following system

$$\begin{bmatrix} (\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_1 & ((\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_2)^{\frac{1}{2}} \\ ((\mathcal{I} + C\mathcal{A})^{-1}\mathcal{G}_2)^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \nu \end{bmatrix} = \frac{1}{\Omega}\begin{bmatrix} \xi \\ \nu \end{bmatrix}, \quad (\xi, \nu) \in \mathfrak{H} \times \mathfrak{H}, \tag{21}$$

where

$$\nu \equiv \Omega \left((\mathcal{I} + C\mathcal{A})^{-1} \mathcal{G}_2 \right)^{\frac{1}{2}} \xi.$$

Since the coefficient matrix of (21) is compact and self-adjoint. In accordance with the spectral theorem, the BVP (7) has a countable set of non-negative eigenvalues with $\Omega = 0^+$ being its only possible accumulation point. Moreover, we have the following definition of Ω_1 derived from the positive quadratic Rayleigh functional of the BVP (5) (Hadeler 1967

$$\Omega_1 = \inf_{\psi \in H_0^1 - \{0\}} \frac{\int_0^{1/2} G_1 \psi^2 \mathrm{d}y + \sqrt{D}}{-2 \int_0^{1/2} G_2 \psi^2 \mathrm{d}y}, \tag{22}$$

where

$$\begin{aligned} D &= D(K_0, K_{0y}, x_0, y; Ma_0) \\ &\equiv \left(\int_0^{1/2} G_1 \psi^2 \mathrm{d}y \right)^2 - 4 \int_0^{1/2} G_2 \psi^2 \mathrm{d}y \int_0^{1/2} \left[\psi_y^2 + (1 - Ma_0^2) \frac{\pi^2}{8 x_0^2 y} \psi^2 \right] \mathrm{d}y. \end{aligned} \tag{23}$$

Similar to case (I), it is sufficient to show that the above variational infimum is bounded below by a positive constant. Again, it is sufficient to show that this is the case for $x_0 \to \infty$. Let g_1 and g_2 be the positive constants defined by (19), and let m_2 be the positive root of the quadratic equation

$$g_2 m_2^2 + g_1 m_2 - 1/4 = 0. \tag{24}$$

We claim that

$$\sqrt{D} \geq - \int_0^{1/2} G_1 \psi^2 \mathrm{d}y - 2 m_2 \int_0^{1/2} G_2 \psi^2 \mathrm{d}y \tag{25}$$

for each $\psi \in H_0^1$. We have

$$
\begin{aligned}
(25) &\Longleftarrow \sqrt{D} \geq \left| \int_0^{1/2} G_1 \psi^2 \mathrm{d}y + 2m_2 \int_0^{1/2} G_2 \psi^2 \mathrm{d}y \right| \\
&\Longleftrightarrow D \geq \left(\int_0^{1/2} G_1 \psi^2 \mathrm{d}y \right)^2 + 4m_2^2 \left(\int_0^{1/2} G_2 \psi^2 \mathrm{d}y \right)^2 + 4m_2 \left(\int_0^{1/2} G_1 \psi^2 \mathrm{d}y \right) \left(\int_0^{1/2} G_2 \psi^2 \mathrm{d}y \right) \\
&\Longleftrightarrow -4 \int_0^{1/2} G_2 \psi^2 \mathrm{d}y \int_0^{1/2} \psi_y^2 \mathrm{d}y \geq 4m_2^2 \left(\int_0^{1/2} G_2 \psi^2 \mathrm{d}y \right)^2 + 4m_2 \left(\int_0^{1/2} G_1 \psi^2 \mathrm{d}y \right) \left(\int_0^{1/2} G_2 \psi^2 \mathrm{d}y \right) \\
&\Longleftrightarrow \int_0^{1/2} \psi_y^2 \mathrm{d}y \geq -m_2^2 \int_0^{1/2} G_2 \psi^2 \mathrm{d}y - m_2 \int_0^{1/2} G_1 \psi^2 \mathrm{d}y \\
&\Longleftarrow \int_0^{1/2} \psi_y^2 \mathrm{d}y \geq m_2^2 \int_0^{1/2} |G_2| \psi^2 \mathrm{d}y + m_2 \int_0^{1/2} |G_1| \psi^2 \mathrm{d}y \\
&\Longleftarrow \int_0^{1/2} \psi_y^2 \mathrm{d}y \geq \int_0^{1/2} (g_2 m_2^2 + g_1 m_2) \frac{\psi^2}{y^2} \mathrm{d}y \quad \Longleftrightarrow \quad \int_0^{1/2} \psi_y^2 \mathrm{d}y \geq \frac{1}{4} \int_0^{1/2} \frac{\psi^2}{y^2} \mathrm{d}y,
\end{aligned}
$$

in which the last two lines are due to (18) and (24), respectively. Hardy's inequality (17) implies that the last line is valid. Therefore, $\Omega_1 \geq m_2$ and thus Ω_1 is the least positive eigenvalue of the BVP (7).

Appendix D

Notice that condition (9) does not contradict to the assumption $\left(\frac{K_0}{y}\right)_y \leq 0$ for every $1 < \gamma < 5/3$.

Proof of Theorem 2: Let $Ma_0 = \frac{2\sqrt{\gamma-1}}{\gamma}$. For every $0 < y < 1/2$ we have $G_2 = 0$ and

$$
G_1 = \frac{2-\gamma}{\gamma} \frac{K_0}{2y^2} \left[\frac{2(\gamma-1)}{\gamma} \frac{K_0}{y} - K_{0y} \right], \tag{26}
$$

where G_1, G_2 are given by (6) and are viewed as functions of y only. Observe that conditions (9) and (10) are equivalent to $G_1(y^*) < 0$ and $G_1 \neq 0$ a.e. over $0 < y < 1/2$, respectively.

First, if $G_1(y) < 0$ for every $y \in (0, 1/2)$. Let

$$
a = \frac{2-\gamma}{2\gamma} \sup_{0<y\leq 1/2} K_0 \sup_{0<y\leq 1/2} \left(\frac{2(\gamma-1)}{\gamma} \frac{K_0}{y} + K_{0y} \right),
$$

it is clear that for each $\psi \in H_0^1 - \{0\}$ we have $|G_1| \leq a\frac{\psi^2}{y^2}$. By carrying out the same argument as Case (I) in the proof of Theorem 1, we conclude that the BVP (7) has a countable set of non-negative eigenvalues. Besides, we have

$$\int_0^{1/2} \psi_y^2 \mathrm{d}y \geq \frac{1}{4}\int_0^{1/2} \frac{\psi^2}{y^2}\mathrm{d}y = \frac{1}{4a}\int_0^{1/2} \frac{a\psi^2}{y^2}\mathrm{d}y \geq \frac{1}{4a}\int_0^{1/2} (-G_1)\psi^2 \mathrm{d}y.$$

Thus, we obtain

$$\Omega_1 = \inf_{\psi \in H_0^1 - \{0\}} \frac{\int_0^{1/2} \psi_y^2 \mathrm{d}y + \int_0^{1/2} \frac{\pi^2}{8x_0^2 y}\psi^2 \mathrm{d}y}{\int_0^{1/2} (-G_1)\psi^2 \mathrm{d}y} \geq \frac{1}{4a} > 0.$$

Therefore, Ω_1 is the least positive eigenvalue of the BVP (7).

On the other hand, if $G_1(y_1) > 0$ for some $y_1 \in (0, 1/2)$, then G_1 changes sign over $0 < y < 1/2$. It is clear that G_1 and C given by (6) are locally integrable over $0 < y < 1/2$ (i.e. they are integrable over any closed sub-interval of (0,1/2)). Besides, condition (10) ensures that $|G_1| > 0$ a.e over $0 < y < 1/2$. In accordance with standard Sturm-Liouville theory (see, for example, Zettle 2005). The BVP (5) has a least positve eigenvalue Ω_1.

We have so far proved first part in the statement of Theorem 2. Suppose next $K_0(y) = 1 - \exp(-2by)$, where the constant $b > 0$ is arbitrary. Let $h : y \mapsto \frac{yK_{0y}}{K_0} - \frac{2(\gamma-1)}{\gamma}$ be a function with domain $(0, 1/2)$. Let us show that $h' < 0$ for $0 < y < 1/2$. We have

$$h'(y) = \frac{2b}{[\exp(2by) - 1]^2}\left[(1 - 2by)\exp(2by) - 1\right]. \tag{27}$$

Let $k : y \mapsto [(1 - 2by)\exp(2by) - 1]$ be a function with domain $(0, 1/2)$. Since $k'(y) = -4b^2 y \exp(2by) < 0$ for every $0 < y < 1/2$ and $k(0^+) = 0$, we have $k < 0$ for $0 < y < 1/2$. Since h' has the same signature as k for $0 < y < 1/2$, we conclude that $h' < 0$ for $0 < y < 1/2$. Thus, h strictly decreases over $0 < y < 1/2$. Therefore, h has at most one zero over $(0, 1/2)$. Besides, since $h(0^+) = \frac{2-\gamma}{\gamma} > 0$ for every $1 < \gamma < 5/3$, in accordance with the first part of Theorem 2, we conclude the existence of a least positive eigenvalue Ω_1 to the BVP (5) for this case.

Appendix E

Proof of Theorem 3: Let K_0 be a circulation function which is not directly proportional to y, and let $\psi \in H_0^1$ be a non-zero function. Recall that G_1, G_2 and C were defined by (6).

Since K_0, γ, and ψ are given, these coefficients depend on Ma_0 and y only hereafter. Let $l : Ma_0 \mapsto \int_0^{1/2} G_1\psi^2 \mathrm{d}y$ be a function over $0 < Ma_0 < 1$. By virtue of Hardy's inequality (17), l is well-defined and continuous. Hence, we have

$$\lim_{Ma_0 \to 1^-} \int_0^{1/2} G_1\psi^2 \mathrm{d}y = \int_0^{1/2} \lim_{Ma_0 \to 1^-} G_1\psi^2 \mathrm{d}y = -\int_0^{1/2} \frac{K_0}{2y}\left(1 - \frac{\gamma}{2}\right)\left(\frac{K_0}{y}\right)' \psi^2 \mathrm{d}y > 0.$$

By continuity of l, we conclude that $l > 0$ in a vincinity of $Ma_0 = 1^-$. Besides, since $C, G_2 > 0$ in a vincinity of $Ma_0 = 1^-$, the following estimation holds for every $\Omega \geq 0$ in a vincinity of $Ma_0 = 1^-$

$$\int_0^{1/2} \psi_y^2 \mathrm{d}y + \int_0^{1/2} C\psi^2 \mathrm{d}y + \Omega \int_0^{1/2} G_1\psi^2 \mathrm{d}y + \Omega^2 \int_0^{1/2} G_2\psi^2 \mathrm{d}y > 0. \tag{28}$$

Thus, ψ is not an eigenfunction of the BVP (5). Moreover, since ψ was arbitrarily chosen, we conclude that the BVP (5) does not have non-negative eigenvalue in a vincinity of $Ma_0 = 1^-$.

Appendix F

Kummer's and Tricomi's functions have the following properties: for each $a, b, z \in \mathbb{C}$ with b not being a non-positive integer, we have the following identities (Slater 1960)

$$\begin{aligned}
{}_1F_1(a;b;z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)2!}z^2 + \cdots, && (29)\\
\exp(-z)\,{}_1F_1(a;b;z) &= {}_1F_1(b-a, b, -z), && (30)\\
{}_1F_1(1+a;b;z) &= {}_1F_1(a;b;z) + \frac{z}{a}\frac{d}{dz}{}_1F_1(a;b;z), && (31)\\
\lim_{a\to\infty} {}_1F_1\left(a, b, -\frac{z}{a}\right) &= \Gamma(b) z^{\frac{1}{2}-\frac{b}{2}} J_{b-1}(2\sqrt{z}) && (32)
\end{aligned}$$

We first prove that the expression $\phi(\eta) \equiv \exp\left(-\frac{\sqrt{\Delta}}{2}\eta\right) {}_1F_1\left(1 - \frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}}\ ;2\ ;\ \eta\sqrt{\Delta}\right)$ is real-valued for each $Ma_0 \geq 0$, $\Omega > 0$ and $0 < x_0 \leq \infty$. We analyze three cases according to the sign of Δ.

Case (I): If $\Delta < 0$, then $\eta\sqrt{\Delta} = i\eta\sqrt{|\Delta|}$ is purely imaginary. Applying (29) and (30) we obtain

$$\phi(\eta) = \exp\left(-\frac{\sqrt{\Delta}}{2}\eta\right) {}_1F_1\left(1 - \frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}} \,; 2\,;\; \eta\sqrt{\Delta}\right)$$
$$= \exp\left(\frac{\sqrt{\Delta}}{2}\eta\right) {}_1F_1\left(1 + \frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}} \,; 2\,;\; -\eta\sqrt{\Delta}\right) = \overline{\phi(\eta)}.$$

This implies that the complex conjugate of ϕ is equal to itself, so ϕ is real-valued in this case.

Case (II): If $\Delta = 0$, then $Ma_0 = 0$ or $Ma_0 = \frac{2\sqrt{\gamma-1}}{\gamma}$. Applying (31) and (32) we obtain

$$\phi(\eta) = \lim_{\Delta\to 0} {}_1F_1\left(-\frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}} \,; 2\,;\; \eta\sqrt{\Delta}\right)$$

$$= \frac{\Gamma(2)}{\sqrt{\eta(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}} J_1\left(2\sqrt{\eta(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}\right)$$
$$= \frac{1}{\sqrt{\eta(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}} J_1\left(2\sqrt{\eta(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}\right),$$

which is also real-valued.

Case (III): If $\Delta > 0$, then from (29) it is clear that ϕ is real-valued.

The following expression

$$\exp\left(\frac{\sqrt{\Delta}}{2}\eta\right) \Psi\left(1 + \frac{(1-Ma_0^2)\left(2-\frac{\pi^2}{8x_0^2\Omega}\right)}{\sqrt{\Delta}} \,;\; 2\,; -\eta\sqrt{\Delta}\right)$$

has a non-zero imaginary part for each real value of Δ and $0 < y \leq 1/2$ (Bateman *et al.* 1955). Hence, by Theorem 3, for the solution (12) to be a real-valued solution of (11), it enforces that $B = 0$. Taking $A = 1$ as a scaling parameter of the linear problem, we obtain (13). Besides, Ω_1 is determined by $\Phi(1/2) = 0$ which leads to (14).

References

[1] Bateman, H., Erdélyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. 1955 *Higher transcendental functions*, Vol. 1, New York, McGraw-Hill.

[2] Benjamin, T. B., 1962 "Theory of the vortex breakdown phenomenon," *J. Fluid Mech.* **14**, 593-629.

[3] Cramer, M. S., 1989 "Negative nonlinearity in selected fluorocarbons," *Physics of Fluids A* **1**, (11), 1894-1897.

[4] Cramer, M. S. and Tarkenton, G. M., 1992 "Transonic flows of Bethe-Zel'dovich-Thompson fluids," *J. Fluid Mech.* **240**, 197-228.

[5] Delery, J. M. 1994 Aspects of Vortex Breakdown. *Progress in Aerospace Sciences*, **30**, 1, 1-59.

[6] Davies, E. B., 1995 Spectral theory and differential operators, Cambridge University Press.

[7] Eisenfeld, J., 1968 "Quadratic eigenvalue problems," *Journal of Mathematical Analysis and Applications* **23**, 58-70.

[8] Hadeler, K. P., 1967 "Variationsprinzipien bei nichtlinearen Eigenwertaufgaben," *Arch. Rat. Mech. Anal.* **30**, 297-307.

[9] Herrada, M. A., Pérez-Saborid, M., and Barrero, A., 2003 "Vortex breakdown in compressible flows in pipes," *Physics of Fluids* **15**, (8), 2208-2218.

[10] Malkiel, E., Cohen, J., Rusak, Z. & Wang, S. 1996 Axisymmetric vortex breakdown in a pipe: theoretical and experimental studies. *Proc. 36th Israel Annual Conf. Aerospace Sciences*, (February), 24-34.

[11] Mclelland, G., MacManus, D. & Sheaf C. 2015 The effect of streamwise contraction on the characteristics of a streamwise vortex. *Journal of Fluids Engineering*, **137** (6) 061204.

[12] Melville, R., 1996 ”The role of compressibility in free vortex breakdown,” *AIAA Paper No.* 96-2075.

[13] Mitchell, A. M., & Dlery, J. 2001 Research into vortex breakdown control. Progress in Aerospace Sciences, **37** (4), 385-418.

[14] Peckham, D. H. & Atkinson, S. A. 1957 Preliminary results of low speed wind tunnel tests on a gothic wing of aspect ratio 1.0, *Aeronautical Research Council* TR CP508, London.

[15] Rudin, W., 1991 *Functional analysis*, 2nd ed., McGraw-Hill, New York.

[16] Rusak, Z. 1996 Axisymmetric swirling flow around a vortex breakdown point. *J. Fluid Mech.*, **323**, 79-105.

[17] Rusak, Z. 2000 Review of recent studies on the axisymmetric vortex breakdown phenomenon. *AIAA Fluids 2000 Conference and Exhibit.*

[18] Rusak, Z., Choi, J. J. and Lee, J-H., 2007 ”Bifurcation and stability of near-critical compressible swirling flows.” *Physics of fluids* **19** (11), 114107.

[19] Rusak, Z. & Judd, K.P. 2001 ”The stability of noncolumnar swirling flows in diverging streamtubes.” *Physics of Fluids,* **13**, 10, 2835-2844.

[20] Rusak, Z. & Lamb D. 1999 Prediction of vortex breakdown in leading-edge vortices above slender delta wings. *Journal of Aircraft*, **36**, 4, 659-667.

[21] Rusak, Z. and Lee, J-H., 2002 ”The effect of compressibility on the critical swirl of vortex flows in a pipe,” *J. Fluid Mech.* **461**, 301-319.

[22] Rusak, Z. and Lee, J-H., 2004 ”On the stability of a compressible axisymmetric rotating flow in a pipe,” *J. Fluid Mech.* **501**, 25-42.

[23] Rusak, Z. & Meder C. 2004 Near-critical swirling flow in a slightly contracting pipe. *AIAA J.*, **42** (11), 2284-2293.

[24] Rusak, Z., Seginer, A. & Wasserstrom, E. 1983 Numerical calculation of nonlinear aerodynamics of wing-body configurations. *AIAA J.*, **21**, 7, 929-936.

[25] Rusak, Z., Whiting, C. H., & Wang, S. 1998 Axisymmetric breakdown of a Q-vortex in a pipe. *AIAA J.*, **36**, 10, 1848-1853.

[26] Schnerr, G. H. and Leidner, P., 1993 "Real gas effects on the normal shock behavior near curved walls," *Physics of Fluids A* **5**, (11), 2996-3003.

[27] Slater, L. J., 1960 *Confluent hypergeometric functions*, Cambridge University Press.

[28] Thompson, P. A., 1988 *Compressible fluid dynamics*, McGraw-Hill, New York.

[29] Tosio, K., 1966 *Perturbation theory for linear operators*, Springer-Verlag, New York.

[30] Umeh, Chukwueloka O.U., Rusak, Z., Gutmark, E., Villalva, R., & Cha, D.J. 2010 Experimental and computational study of nonreacting vortex breakdown in a swirl-stabilized combustor. *AIAA J.*, **48**, 11, 2576-2585.

[31] Umeh, C. O., Rusak, Z., & Gutmark, E. 2012 Vortex breakdown in a swirl-stabilized combustor. *Journal of Propulsion and Power*, **28**(5), 1037-1051.

[32] Wang, S. and Rusak, Z., 1996 "On the stability of an axisymmetric rotating flow in a pipe," *Physics of Fluids* **8**, (4), 1007-1016.

[33] Wang, S. and Rusak, Z., 1997 "The dynamics of a swirling flow in a pipe and transition to axisymmetric vortex breakdown," *J. Fluid Mech.* **340**, 177-223.

[34] Zettle, A., 2005 Sturm-Liouville theory, American Mathematical Society.