A Graph and its Complement with Specified Properties. IV. Counting Self-Complementary Blocks

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Dedicated to Robert W. Robinson

ABSTRACT

In this series, we investigate the conditions under which both a graph $G$ and its complement $ar{G}$ possess certain specified properties. We now characterize all the graphs $G$ such that both $G$ and $ar{G}$ have the same number of endpoints, and find that this number can only be 0 or 1 or 2. As a consequence, we are able to enumerate the self-complementary blocks.

1. NOTATIONS AND BACKGROUND

In the first paper [1] in this series, we found all graphs $G$ such that both $G$ and its complement $ar{G}$ have connectivity 1, and other properties. In the second paper [2], we determined the graphs $G$ for which $G$ and $ar{G}$ are obtained from some graph by the same unary operation. More recently [3] we characterized the graphs such that both $G$ and $G$ have the same girth and the same circumference 3 or 4.

An endpoint of a graph has degree 1. We denote the number of endpoints in $G$ by $e = e(G)$ and in $ar{G}$ by $\bar{e}$. We characterize all the graphs $G$ with $e = \bar{e}(\geq 2)$ in the next section, and count the number of self-complementary blocks in the last section.
Following the notation and terminology of [5], we define the join $G_1 + G_2$ of two graphs to be the union of $G_1$ and $G_2$ with the complete bigraph having point sets $V_1$ and $V_2$, and the corona $G \circ H$ of two graphs $G$ with $p$ points $v_i$ and $H$ is obtained from $G$ and $p$ copies of $H$ by joining each point $v_i$ of $G$ with all the points of the $i$th copy of $H$. For our result later we need a ternary operation written $F + G \circ H$ which is defined in [3] as the union of the join $F + G$ with the corona $G \circ H$. Thus this resembles the composition of the path $P_3$ not with just one other graph but with three graphs, one for each point, for example, Figure 1 illustrates the graph $A = K_1 + K_2 \circ K_1$.

2. ENDPOINTS

Let $g_p$ be the number of graphs of order $p$.

Lemma 1. For $n \geq 1$, the mapping $F \rightarrow F + K_n \circ K_1$ which takes graphs $F$ of order $p$ to graphs $G = F + K_n \circ K_1$ of order $p + 2n$ is one-to-one.

Proof. Suppose $G$ can be written in the form $F + K_n \circ K_1$. We will show that $F$ is uniquely recoverable from $G$. Let $S$ be the set of points of $G$ which are adjacent to endpoints. Clearly $S$ is the point set of the distinguished subgraph $K_n$. Let $H$ be the subgraph induced by $V(G) - S$. Then $H$ has at least $n$ isolates, and removing exactly $n$ isolates from $H$ leaves $F$.

Lemma 2. If $G$ has two endpoints, then $\bar{G}$ has at most two endpoints.

Proof. Let $v_0$ and $v_1$ be two endpoints of $G$, adjacent to $u_0$ and $u_1$, respectively. Then obviously the only candidates for endpoints in $\bar{G}$ are $u_0$ and $u_1$.

Theorem 1. A graph $G$ of order $p \geq 4$ has $e = \bar{e} = 2$ iff $G$ is of the form $F + K_2 \circ K_1$, where $F$ is a graph of order $p - 4$.

\[ A = K_1 + K_2 \circ K_1 \]
Proof. If \( e = \bar{e} = 2 \), then \( G \) has exactly two points \( v_0 \) and \( v_1 \) of degree \( p - 2 \) and exactly two points \( u_0 \) and \( u_1 \) of degree 1, where \( u_0, u_1 \) are not adjacent to \( v_0, v_1 \), respectively. Since \( \text{deg} v_0 = \text{deg} v_1 = p - 2 \), \( v_i \) is adjacent to every point other than \( u_i \) for \( i = 0, 1 \). On the other hand, \( u_i \) is not adjacent to any point other than \( v_{1-i} \) for \( i = 0, 1 \), since \( \text{deg} u_0 = \text{deg} u_1 = 1 \). Denote by \( F \) the subgraph of \( G \) induced by the point set \( V(G) - \{v_0, v_1, u_0, u_1\} \). Then in \( G \) any point \( v \) of \( F \) must be adjacent to both \( v_0 \) and \( v_1 \), which are adjacent to each other by the above observations. Thus \( G \) is a graph of the form \( F + K_2 \circ K_1 \).

The converse follows immediately from the proof of Lemma 1.

Corollary 1. The number of graphs of order \( p \) with \( e = \bar{e} = 2 \) is \( g_{p-4} \).

Proof. By Theorem 1, \( G \) is of the form \( F + K_2 \circ K_1 \) where \( F \) has \( p - 4 \) points. Hence by the 1-1 correspondence of Lemma 1, the number of graphs \( G \) with \( e = \bar{e} = 2 \) is \( g_{p-4} \).

Corollary 2. All graphs with \( e = \bar{e} = 2 \) have diameter 3.

Proof. The maximum distance between two points of \( F + K_2 \circ K_1 \) is 3, as this is the distance between the two endpoints.

3. SELF-COMPLEMENTARY GRAPHS

A graph \( G \) is self-complementary (or briefly, s-c) if it is isomorphic to its complement \( \bar{G} \). The isomorphism between \( G \) and \( \bar{G} \) can be represented as a permutation, \( \alpha \), on \( V(G) \). We will write \( \alpha(G) = \bar{G} \) and call \( \alpha \) a complementing permutation for \( G \) as in Gibbs [6]. We will assume that all permutations are expressed as the product of disjoint cycles. We first state the result obtained independently by Ringel [8] and Sachs [10], which gives the cycle structure of a complementing permutation.

Theorem RS. If \( G \) is s-c of order \( p \) and \( \alpha(G) = \bar{G} \), then if \( p \equiv 0 \pmod{4} \), each cycle of \( \alpha \) has length divisible by 4 and if \( p \equiv 1 \pmod{4} \), \( \alpha \) has exactly one cycle of length 1 and all other cycles have length divisible by 4.

We begin with the result concerning the number of endpoints of a s-c graph, which was communicated to us by R. W. Robinson and proved nicely by one of the referees.

Lemma 3. A self-complementary graph does not have exactly one endpoint.
Proof. Suppose \( G \) is s-c with a unique point of degree 1. Then \( G \) must have a unique point of degree \( p-2 \) and these observations hold for \( \bar{G} \) as well. In \( G \) let \( \deg v_1 = 1 \) and \( \deg v_2 = p-2 \). Hence in \( \bar{G} \), \( \deg v_1 = p-2 \) and \( \deg v_2 = 1 \). But \( v_1 \) and \( v_2 \) are adjacent in exactly one of \( G \) and \( \bar{G} \), a contradiction.

We now characterize all s-c graphs with two endpoints.

**Lemma 4.** All s-c graphs of order \( p+4 \) having two endpoints can be constructed using the ternary operation \( G = F + K_2 \circ K_1 \), where \( F \) is a s-c graph of order \( p \).

**Proof.** Let \( G \) be any s-c graph of order \( p+4 \) having 2 endpoints. Since \( G = G \) and \( G \) has exactly 2 endpoints, we know that \( G \) is of the form \( F + K_2 \circ K_1 \) for some graph \( F \) of order \( p \) by Theorem 1. On the other hand, it is easy to see that \( G = F + K_2 \circ K_1 \) is s-c iff \( F \) is s-c. Thus, \( G \) can be constructed using the ternary operation \( G = F + K_2 \circ K_1 \) for some s-c graph \( F \) of order \( p \).

We denote by \( s_p \) the number of all s-c graphs of order \( p \) and by \( s'_p \) the number of s-c graphs of order \( p \) which have 2 endpoints. Since the ternary operation \( G = F + K_2 \circ K_1 \) is \( 1-1 \) as proved in Lemma 1, we have the following equality from Lemma 4.

**Lemma 5.** For any positive integer \( p \),

\[
s''_{p+4} = s_p.
\]

Recall [5, p. 241] that \( G \) is a block if \( G \) is connected and has no cutpoint. The number of blocks was determined by Robinson [9]. Our object is to derive the number of self-complementary blocks.

**Lemma 6.** If \( G \) is a s-c graph with no endpoints, then \( G \) is a block.

**Proof.** Assume that \( G \) is s-c with no endpoints but has a cutpoint \( v \). The removal of \( v \) from \( G \) results in a subgraph with at least 2 components. Let \( G_1 \) be a component of \( G - v \) and let \( G - v = G_1 \cup G_2 \). Thus \( G - v \) contains a complete spanning bigraph \( B \) whose point sets are \( V(G_1) \) and \( V(G_2) \). The cardinalities of both \( V(G_1) \) and \( V(G_2) \) are at least 2 by the hypothesis that \( G \) has no endpoints. Therefore \( \bar{G} \) is 2-connected and hence \( G = \bar{G} \) cannot have a cutpoint, a contradiction.

Read [7] found a formula for the number of self-complementary graphs \( s_p \). Frucht and Harary [4] derived an alternative equation. We now see how to count s-c blocks in terms of the numbers \( s_p \).

**Theorem 2.** For any positive integer \( p \geq 5 \), the number of s-c blocks of order \( p \) is \( s_p - s_{p-4} \).
Proof. Let $G$ be a self-complementary block of order $p$, so that $p \geq 5$. By Lemmas 3 and 6, the number of $s$-$c$ blocks equals $s_p$ less the number of $s$-$c$ graphs with $e = 2$. But this is $s_{p-4}$ by Lemma 5.

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References


