Explorations of Non-Supersymmetric Black Holes in Supergravity

by

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Dedication

To my wife, my love, and my everything, Emily.

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Abstract

Finding a quantum theory of gravity and unifying our description of the fundamental forces in the universe has been a long-standing goal of the physics community for the last several decades. Black holes provide a setting where both gravitational and quantum effects are important, and as such they are valuable in probing how such a theory of quantum gravity must behave; any quantum description of a black hole should be consistent with its macroscopic (low-energy) description in general relativity.

The only known examples where the microscopic details of a black hole are understood come from string theory and are heavily reliant on supersymmetry. Little is known about how to extend these descriptions to more realistic, non-supersymmetric settings. In this thesis, we make progress on this front by undergoing a careful exploration of non-supersymmetric black holes in theories of supergravity. In particular, we show how generic Kerr-Newman black holes, when interpreted as solutions to $\mathcal{N} = 2$ supergravity, exhibit a surprising amount of universality in their behavior that points to a great deal of simplicity in their underlying quantum description.

In the first part of this thesis, we embed arbitrary Kerr-Newman black holes into $\mathcal{N} = 2$ supergravity. We then study the quantum fluctuations around these backgrounds and compute their contributions to the black hole entropy. We find that these quantum corrections are protected by a topological quantity and are thus invariant under arbitrary deformations of the black hole away from the BPS limit. The logarithmic corrections to the Bekenstein-Hawking entropy are thus universal and independent of black hole parameters for these black holes.

In the second part of this thesis, we study minimal supergravity with higher derivative corrections and show that Kerr-Newman black holes can be consistently embedded into such theories with no modifications required to the geometry. Moreover, we show that these black holes are continuously connected to a BPS geometry. We also find that the entropy of these black holes is a topological invariant and thus insensitive to whether or not the black hole is a BPS state.

In the final part of this thesis, we explore the symmetries of off-shell $\mathcal{N} = 2$ supergravity and the corresponding constraints these symmetries impose on the theory. In particular, we show that the topological nature of our results follows directly from symplectic duality and supersymmetry. Moreover, these constraints persist even for black hole solutions that break the supersymmetry of the theory. These results establish a previously-unknown level of structure in the underlying microscopic description of these black holes.

Chapter 1

Introduction

One of the longest-standing goals in modern physics is to discover a consistent theory of quantum gravity that can reconcile our current understanding of both gravity and quantum mechanics. There have been steps taken towards solving this puzzle, but these results are heavily rooted in string theory and require a great deal of supersymmetry. In most cases, it is not known how to generalize these results to more realistic cases, particularly those without supersymmetry.

The goal of this thesis is to shed light on these issues and investigate the role of supersymmetry in constraining the dynamics of black holes. In particular, we detail how non-supersymmetric black holes can be embedded into supergravity theories. We then go on to show how many features of these black holes are universal, in the sense that they do not require the black hole to preserve the supersymmetry of the theory. This gives surprising (and previously unknown) insights into general features of quantum gravity without supersymmetry.

In this chapter, we will first give a brief, non-technical overview of our modern understanding of black holes and their underlying microscopic theories. We will then use this to motivate the research presented in the later chapters of this thesis. We end with an overview of the thesis and a summary of our key results.

1.1 Black Holes

Black holes form when very massive astrophysical objects collapse inward at the end of their life cycle. If the matter collapses far enough inward such that it is all within the Schwarzschild radius of the object, a black hole forms. The gravitational field within the Schwarzschild radius is so strong that it warps the spacetime inside such that all particles are forced to travel further and further into the black hole, with no escape back outside the horizon possible.

Throughout this thesis, we will consider four-dimensional, asymptotically flat black holes. In general relativity, such black holes are constrained by the no-hair theorem [1], which says that they are completely characterized by their mass, their angular momentum, and their charges (either electric or magnetic). Any two black holes that have the same set of these parameters are indistinguishable to an outside observer, irrespective of how the black hole was formed. Heuristically, this would seem to imply that black holes have precisely one configuration for any set of black hole parameters and thus have vanishing entropy.

This interpretation is somewhat puzzling, though, because the laws of thermodynamics tell us that the total entropy of a closed system can never decrease. Therefore, if we throw an object with entropy into a black hole, we would expect the entropy of the black hole to increase, despite our previous intuition from the no-hair theorem.

1.1.1 Black Hole Thermodynamics

It would be inaccurate to refer to the puzzles presented above on the entropic nature of black holes as true paradoxes, but they nonetheless motivated further study on the thermodynamic nature of black holes in the 70's. This led to the development of the four laws of black hole mechanics [2], which established a startling similarity between black hole physics and thermodynamics. In particular, it was shown that the area of the black hole horizon cannot decrease as it evolves, in direct parallel to the notion that the thermodynamic entropy of a closed system cannot decrease. As a consequence, it was speculated that the area of the black hole horizon must somehow encode the entropy of the black hole [3].

This notion was made precise by Hawking in his seminal work [4]. He showed that by considering quantum fields propagating on a classical black hole background, you are forced to conclude that black holes are thermodynamic black bodies that radiate with a temperature (known as the Hawking temperature) given by

$$T = \frac{\hbar\kappa}{2\pi} \ , \tag{1.1}$$

where κ is the surface gravity of the black hole. Quantum effects therefore lead to thermal emission from black holes, known as Hawking radiation. Moreover, combining the Hawking temperature with the area law theorem for black holes then unambiguously identifies that black holes have an entropy

$$S = \frac{A}{4\hbar G} \ , \tag{1.2}$$

where A is the area of the black hole horizon.[1] This is the famous Bekenstein-Hawking entropy. These results are intriguing, because they demonstrate a deep connection between gravitational physics and quantum mechanics that is still being probed today.

Importantly, the thermodynamic nature of black holes is purely a quantum effect. In the classical limit where we send $\hbar \to 0$, the black hole temperature vanishes and it ceases to emit radiation. It is only when we consider both general relativity and quantum mechanics simultaneously that black holes become thermal black bodies.

1.1.2 The Information Paradox

The upshot of the above discussion of black hole thermodynamics is that black holes give off thermal radiation. As the black hole radiates over time, it loses energy and correspondingly must shed off some of its mass. Eventually, the black hole will evaporate entirely, leaving nothing but a dilute gas of thermal particles.

This cannot be the full story, however, because it leads to the so-called information paradox. Suppose we take a black hole, throw an object into the black hole, and then let the black hole evaporate. We are left with thermal radiation that is characterized entirely by the temperature of radiation. This thermal radiation no longer contains information about the details of the object we threw into the black hole, and so the process of black hole evaporation destroys information. This leads to a violation of unitarity, meaning that we have a system in which probabilities are no longer conserved.

This consequence would be disastrous for our understanding of physics, and so something must be wrong with this picture. There are many different conjectured resolutions to the information paradox, but most agree that the problem comes from Hawking's semi-classical treatment of the black hole, where the black hole is considered to be a fixed background. This treatment is approximately valid for large black holes with a very small temperature, but not when extrapolated over arbitrarily long periods of time. At some point in the evaporation of the black hole, new physics must come in to explain how the time evolution of the black hole is unitary. This points to our current understanding of gravity as being an effective theory, one that is only valid up to certain energy scales. Beyond those, new dynamics have to come into play. The black hole information paradox (and its related avatars) therefore indicate that we should seek out a "quantum" theory of gravity, one that is valid up to arbitrarily high energies. Such a theory would have to be able to explain the microscopic dynamics of black holes and resolve all of these paradoxes.

[1] In this section, we are using a set of units where the Boltzmann constant k and the speed of light c are both set to one. In all proceeding sections, we will also set $\hbar$ to one for simplicity. It is useful to retain $\hbar$ here in order to take the classical limit.

1.1.3 A Statistical Interpretation of Black Hole Entropy

Another indication that black holes should have microscopic dynamics comes from considering the implications of black hole entropy. In ordinary thermodynamic systems, the entropy is a measure of the internal degrees of freedom of the system. That is, given a set of fixed macroscopic charges, there is an associated degeneracy Ω of microscopic states with those charges. The entropy of system is related to this degeneracy by

$$S = \log \Omega \ . \tag{1.3}$$

If we extend this reasoning to black holes, it seems plausible that the Bekenstein-Hawking law should have a microscopic interpretation that counts the number of microscopic states accessible to the black hole.

A difficulty in this statistical interpretation is that various no-hair and uniqueness theorems tell us that there is a unique black hole geometry that corresponds to any set of fixed macroscopic charges. In this context, a black hole appears to have only one internal configuration, and so we would expect the statistical entropy to vanish. We therefore need to go beyond general relativity in order to test if there really is a statistical underpinning of the Bekenstein-Hawking law.

1.1.4 Microstate Counting in String Theory

One such example of a possible theory of quantum gravity is string theory. At its core, string theory posits that the fundamental degrees of freedom in the universe are one-dimensional strings instead of point-like particles. We will not go into the details in this thesis, but demanding consistency of interactions between these strings leads to an incredibly rich structure in the theory. The important point for us, though, is that string theory automatically gives rise to a gravitational force, with no additional input necessary. Moreover, string theory is an ultraviolet complete theory, which means that it suffers from no inconsistencies at arbitrarily high energies.

String theory led to the first explicit example of a microscopic realization of the Bekenstein-Hawking entropy by Strominger and Vafa in [5]. They studied a particular class of five-dimensional Reissner-Nordström black holes (e.g. non-rotating black holes with electric and magnetic charges) that have an explicit realization in string theory. They used this to compute the degeneracy of the string microstates strictly in terms of the charges of the black hole. In the limit where these charges all become large, they found that this microscopic degeneracy precisely matched the Bekenstein-Hawking entropy. This paved the way for many more black hole microstate counting formulae in string theory [6–11], all of which served to further confirm that microscopic realizations of black hole entropy are possible.

1.2 Microscopic and Macroscopic Black Holes

The explicit examples of microstate counting above indicate that, for any putative theory of quantum gravity, we should be able understand black holes in the theory from either a macroscopic or microscopic perspective. The macroscopic approach (also known as the semi-classical approach) is where you start with some low-energy effective field theory of gravity, possibly coupled to additional matter content. The black hole background is a classical solution to the theory, which you can then quantize by studying quantum fluctuations of the fields in the theory around this background. This approach is only valid for small field fluctuations, and all physical observables are computed perturbatively in a series expansion of these small fluctuations. The microscopic approach, on the other hand, should be a theory of quantum gravity that is valid for arbitrarily large energy scales.

These two different pictures should be compatible with one another, in the sense that they give consistent results for any physical quantities that can be measured at low energies. In particular, we should be able to compute the entropy of the black hole from either the macroscopic or microscopic perspectives and get the same result. And, indeed, this matching has been accomplished in a wide range of examples.

We can go beyond the usual Bekenstein-Hawking formula, though. In the limit where the black hole charges are large and all scale with the size of the black hole, the most general form of the entropy can be expressed as a series expansion in the black hole area as

$$S = \frac{A}{4G} + \alpha \log\left(\frac{A}{G}\right) + \beta + \dots \ , \tag{1.4}$$

for some numerical coefficients α, β, etc. The first term is the now-familiar Bekenstein-Hawking entropy, and is the dominant piece for large black holes. The other terms arise from quantum gravity effects that are sub-leading in this large-charge limit. These corrections have been studied for a number of different supersymmetric black holes in string theory, both on the macroscopic and microscopic sides, and matching has been established for all cases that have microscopic results to compare to [12–15].

The advantage of studying corrections to the Bekenstein-Hawking entropy on the macroscopic side is that the leading order corrections arise from low-energy, infrared physics. These corrections can therefore be computed in an effective field theory without having to know the details of the full ultraviolet completion of the theory. We can view these macroscopic computations as a “laboratory”, in some sense, for testing ideas about quantum gravity. Demanding that the microscopic counting results match the macroscopic results puts stringent constraints on any proposed theory of quantum gravity.

1.2.1 Beyond the BPS Limit

A crucial detail that we have not yet discussed is the role of supersymmetry in understanding black hole microscopics. Most examples in string theory where explicit black hole microstate counting can be done require that the black hole is a BPS black hole, meaning that the black hole is extremal (zero-temperature) and preserves some of the supersymmetry of the theory. Extremality ensures that the black hole is not radiating and hence does not evaporate over time, while supersymmetry dictates that the result for the microstate degeneracy is independent of the string coupling constant.

This last condition is an absolutely crucial component for the microstate counting results mentioned in section 1.1.4. These computations are done in the weak coupling regime, where gravitational interactions become small and the dynamics of string theory become tractable. However, the microstate picture at weak coupling corresponds to a system with a very large curvature near the horizon. Our semi-classical understanding of black holes is only valid in the strong coupling regime, where the black hole is large and hence has a very small curvature near the horizon. At face value, then, it seems like the microscopic and macroscopic computations correspond to very different systems. However, supersymmetry ensures that the microscopic result can be extrapolated to arbitrarily strong coupling, and thus a match is established.

This begs the question: is there hope to understand the microscopic dynamics of non-BPS black holes? Generic black holes do not satisfy the BPS conditions, but they still have a Bekenstein-Hawking entropy that points towards a microscopic interpretation. One hint that non-BPS black holes may in fact have tractable microscopic theories is that black hole microstate counting can also be done for certain classes of black holes that are deformed very slightly away from the BPS limit [16]. There are also arguments (though not full proofs) that certain sectors of string theory allow for black holes where the microstate degeneracy is independent of whether or not the black hole preserves supersymmetry [17–19]. Developing a microscopic description of non-supersymmetric black holes would be a huge breakthrough in our understanding of quantum gravity, and so it is worthwhile to develop macroscopic results in an effort to provide a target for future progress on this issue.

1.3 Challenges for Non-Supersymmetric Black Holes

The central goal of this thesis is to address whether or not we can learn about general features that microscopic theories of non-supersymmetric black holes must have. Specifically, we will consider black holes in four-dimensional Einstein-Maxwell theory, where electromagnetism is minimally coupled to gravity. The only stationary black hole solutions in this theory are Kerr-Newman black holes, parameterized by a mass M, an electric charge Q, and angular momentum J. In this thesis, we will show that when these black holes are interpreted as solutions to supergravity theories, the corresponding subleading corrections to the Bekenstein-Hawking law are universal and insensitive to whether or not the black hole preserves any supersymmetry.

When going beyond the semi-classical analysis of Hawking for these black holes, there are two classes of corrections we can consider: quantum corrections and higher-derivative corrections. We will detail the obstacles present when studying both classes of corrections in the next two sections.

1.3.1 Quantum Corrections

The logarithmic corrections to the Bekenstein-Hawking entropy in (1.4) arise from the leading-order backreaction of the massless fields in the theory on the black hole background [14, 15, 20–23]. In general, they take the form

$$\Delta S = f(M, Q, J) \log A \ . \tag{1.5}$$

The coefficient f in this equation can in principle be a function of all the black hole parameters.

As a concrete example, let's consider a free scalar field propagating on a Kerr-Newman black hole background. We will save the gory details for later, but the backreaction of this scalar field on the black hole background can be understood quantitatively using the heat kernel formalism developed in chapter 2. The net effect is that the leading-order correction to the Bekenstein-Hawking entropy from this scalar field is

$$\begin{aligned}\Delta S = \Bigg[&\frac{1}{45} + \frac{\beta M^2 Q^4 (3r_H^4 M^4 + 2r_H^2 M^2 J^2 + 4J^4)}{1920\pi r_H^3 J^4 (r_H^2 M^2 + J^2)} \\ &- \frac{\beta M Q^4 (r_H M^4 - J^4)}{640\pi r_H^4 J^5} \tan^{-1}\left(\frac{J}{r_H M}\right)\Bigg] \log A \ ,\end{aligned} \tag{1.6}$$

where r_H is the radius of the black hole horizon and β is the inverse temperature of the black hole. This is a messy expression, despite the fact that a free scalar field is the simplest possible field to analyze the dynamics of on our black hole background. Moreover, deriving such an expression from a microscopic perspective would a priori be incredibly difficult, as the result is incredibly sensitive to the precise choice of macroscopic black hole charges.

A way out of this difficulty is to restrict ourselves to the special case of an extremal Reissner-Norström black hole, in which the black hole has no angular momentum and the mass and charge are equal. The logarithmic correction to the black hole entropy due to the scalar field collapses almost entirely in this limit, leaving us with

$$\Delta S = -\frac{1}{180} \log A \ . \tag{1.7}$$

The simplicity of this result is no accident. The extremal Reissner-Nordström black hole is also a black hole solution in compactifications of type II string theory. Moreover, it is a BPS solution in these compactifications, and so prior studies of quantum fluctuations tell us that the logarithmic correction should be topological and independent of the string coupling.

1.3.2 Higher-Derivative Corrections

As we mentioned earlier, Kerr-Newman black holes are solutions to Einstein-Maxwell theory, in which a photon is minimally coupled to gravity. This is only a low-energy effective field theory, though; we expect the dynamics of this theory to be modified from quantum gravity effects when we go to higher energies. In order to account for these effects, we need to introduce an infinite tower of higher-dimension operators into the action of our theory. These operators can be organized by a simple power-counting of how many derivatives appear in the operators. At low energies, the two-derivative operators will dominate the theory, but the higher-derivative operators (e.g. operators with at least four derivatives) become more and more relevant at higher energies. Their dynamics are therefore important when trying to place more precise constraints on microscopic theories.

Higher-derivative corrections to Einstein-Maxwell theory are generically difficult to work with. It is a chore just to determine how these higher-derivative corrections affect the geometry of black holes [24–28], and most approaches rely on finding solutions numerically.

Often, studies of higher-derivative corrections to black holes circumvent these difficulties by restricting their attention to BPS black holes. Demanding that the black hole background preserves supersymmetry greatly constrains the form of the solution; the BPS conditions are first-order differential equations that can often be solved exactly. Checking that such BPS solutions satisfy the equations of motion of the theory is often much simpler than solving the equations of motion directly.

The leading order four-derivative corrections to black holes will modify the Bekenstein-Hawking entropy by an $\mathcal{O}(1)$ correction that can be computed using the Wald entropy formalism [31,32], which we review in chapter 3. This correction will in principle depend intricately on the black hole parameters, especially for non-extremal black holes, making it difficult to to learn about features of the underlying microscopic theory from these higher-derivative corrections alone. However, these corrections simplify drastically for supersymmetric black holes and are determined entirely by the charges of the black hole in a very straightforward manner [33–35], with very little known about the implied interpolation away from the BPS limit.

1.4 A Way Forward: $\mathcal{N} = 2$ Supergravity

At this point, we have motivated the difficulties present when using the macroscopic techniques to gain insight into microscopic physics for non-supersymmetric black holes. These difficulties, coupled with the fact that all known microstate counting formulae require the black hole to be supersymmetric[2], make the prospects of finding a microscopic understanding of black holes in Einstein-Maxwell theory slim.

A natural way to proceed is to uplift these Kerr-Newman black holes into a theory with more symmetry, potentially allowing for a greater degree of control over the dynamics of these black holes. A particularly nice option is to look at theories whose dynamics are constrained by supersymmetry, and then study properties of black holes that break this supersymmetry. This leads us to very naturally consider Kerr-Newman black holes as living in a theory of $\mathcal{N} = 2$ supergravity, as we will discuss in the next few sections.

1.4.1 Supersymmetry and Supergravity

Supersymmetry is a symmetry that pairs up states in a theory that have integer spin (bosons) with states that have half-integer spin (fermions). These states are turned into one another via the action of a fermionic operator Q, referred to as a supercharge. Schematically, the action of this supercharge takes the form

$$Q|\text{boson}\rangle = |\text{fermion}\rangle \ , \quad Q|\text{fermion}\rangle = |\text{boson}\rangle \ . \tag{1.8}$$

[2]Strictly speaking, this is not quite true, because there are microscopic counting formulae for BTZ black holes [36,37], a class of black holes in three-dimensional Anti-de Sitter (AdS) space. These results rely on holographic methods, wherein a gravitational theory in AdS is dual to a conformal field theory that can be used to describe the microscopic details of the theory. These holographic methods will not be the focus of this thesis, as they currently are only valid for extremal black holes with a near-horizon Anti-de Sitter geometry, while we are interested in studying black holes far from extremality.

Since the supercharges are fermionic operators, they square to zero and so acting with the supercharge twice yields zero. For a theory with a single supercharge, then, each bosonic state is paired up with exactly one fermionic state, and vice-versa.

In principle there can be multiple supersymmetry generators in a theory. In a theory with $\mathcal{N}$ supercharges, we can index the supercharges by Q_i for $i = 1, \ldots, \mathcal{N}$. Particles in the theory are then grouped up into supermultiplets that are irreducible representations under the action of these supercharges. In particular, let's consider a massless state $|\lambda\rangle$ in the theory, where λ is the helicity (e.g. the projection of the spin along the direction of travel) of the state. We can then consider all ways to act with the supercharges on this state that yield non-zero states. The result is that we obtain a supermultiplet of massless particles, presented in (1.9), where the degeneracy is the number of ways to create a state of the given form.

$$\begin{array}{r|c|c}
\text{state} & \text{helicity} & \text{degeneracy} \\
\hline
|\lambda\rangle & \lambda & 1 \\
\hline
Q_i^\dagger|\lambda\rangle & \lambda + \frac{1}{2} & \mathcal{N} \\
\hline
Q_i^\dagger Q_j^\dagger|\lambda\rangle & \lambda + 1 & \frac{1}{2}\mathcal{N}(\mathcal{N}-1) \\
\hline
\vdots & \vdots & \vdots \\
\hline
Q_1^\dagger Q_2^\dagger \ldots Q_{\mathcal{N}}^\dagger|\lambda\rangle & \lambda + \frac{1}{2}\mathcal{N} & 1
\end{array} \tag{1.9}$$

Any theory of gravity must contain a spin 2 particle in its spectrum, the dynamics of which determine the geometry of the spacetime. A theory of supergravity is one in which the theory is also supersymmetric. Supergravity theories therefore must have multiplets that contain massless states with helicities of $\lambda = \pm 2$. Additionally, since we are eventually interested in studying black holes that are charged, we also need to couple electromagnetism into the theory. We will therefore also require that the theory has multiplets that contain massless states with helicities of ± 1, since the photon is a spin 1 particle.

Based on our presentation of massless supermultiplets in (1.9), the most natural way to supersymmetrize our Einstein-Maxwell theory is to consider a theory with $\mathcal{N} = 2$ supersymmetry, since we can then fit the graviton and the photon into the same multiplet, referred to as the gravity multiplet. The gravity multiplet also contains two spin 3/2 fermions; such spin 3/2 fields are referred to as gravitino, and they are necessary in any theory of supergravity.

For any solution to Einstein-Maxwell theory, there is a trivial uplift of the solution into minimal $\mathcal{N} = 2$ supergravity, a theory with just the gravity multiplet. First, we identify the metric and photon in Einstein-Maxwell theory as the spin 2 and 1 particles, respectively, in the $\mathcal{N} = 2$ gravity multiplet. We can then account for the gravitinos in this multiplet in a trivial way by simply setting them to zero in our solution. In particular, Kerr-Newman black holes are guaranteed to be solutions in this minimal $\mathcal{N} = 2$ supergravity.

We will go beyond this minimal supergravity theory by coupling the theory to matter in the form of additional multiplets. In particular, we will focus on coupling the theory to an arbitrary number of vector and hyper multiplets. The uplift of Einstein-Maxwell theory to this more general theory of $\mathcal{N} = 2$ supergravity is complicated but nonetheless doable; the details of this are discussed in section 2.3.

1.4.2 Motivation

The main motivation for considering this particular uplift, and not some other theory of supergravity, is that any BPS solution in $\mathcal{N} = 2$ supergravity must be an extremal Reissner-Nordström black hole (or a multi-centered generalization thereof) [38]. In this context, non-extremal Kerr-Newman black holes are interpreted as deformations of these supersymmetric black holes. We can then try to find physical observables that are guaranteed to be simple for the supersymmetric black holes, but are somehow preserved under deformations away from the BPS limit.

This is not just a pipe dream, though. Supersymmetry is a powerful constraint on a theory, and it can lead to some very slick results. For example, the cosmic censorship bound for these black holes can be interpreted as a derived consequence of the supersymmetry of the theory [39], and the electric-magnetic duality of Einstein-Maxwell theory descends directly from the symplectic duality of $\mathcal{N} = 2$ supergravity [38]. Importantly, these results rely on supersymmetry of the theory, but not on the black hole itself.

Another important reason to consider this supergravity theory is that it is a consistent truncation of string theory, compactified down to four spacetime dimensions. This means that $\mathcal{N} = 2$ supergravity has a well-defined microscopic counterpart in string theory that can be constrainted via a macroscopic analysis.

1.4.3 The Main Idea

With all of that said, the goal of this thesis is to study general Kerr-Newman black holes as solutions to $\mathcal{N} = 2$ supergravity. In particular, we will look at both the quantum corrections and the higher-derivative corrections to these black holes in supergravity. As we discussed in section 1.3, these corrections are typically unwieldy and difficult to manage. Contrary to this expectation, though, we find that both classes of corrections are surprisingly manageable. In fact, we find that the corresponding corrections to the Bekenstein-Hawking law are determined entirely by the topology of the black hole. And, since the topology of a black hole is unchanged under smooth deformations of the black hole parameters, we conclude that the leading-order corrections to Bekenstein-Hawking entropy are independent of the supersymmetry of the black hole. Moreover, we prove that the topological natures of our results are guaranteed by the symmetries of the theory and persist even when these symmetries are broken by the background.

These results demonstrate a somewhat unexpected feature: any microscopic description of a black hole in $\mathcal{N} = 2$ supergravity contains a sector whose dynamics are entirely insensitive to whether or not the black hole is a BPS state. This is a novel (and surprising) result, and it points towards the possibility of establishing more general black hole microscopic theories in the future.

1.5 Overview and Summary of Results

The remainder of this thesis is organized as follows:

- In chapter 2, we establish a general procedure for computing one-loop quantum corrections to non-extremal black holes. We then embed arbitrary solutions of Einstein-Maxwell theory into $\mathcal{N} \geq 2$ supergravity and compute the corresponding one-loop quantum corrections. We find that we can organize the quantum fluctuations into $\mathcal{N} = 2$ multiplets, even if the background is not supersymmetric. Moreover, we find that the bosonic and fermionic components of each fluctuation multiplet conspire to make the logarithmic corrections to the black hole entropy entirely topological. These corrections therefore hold arbitrarily far away from extremality.

  Based on: A. M. Charles and F. Larsen, *Universal Corrections to Black Hole Entropy in $\mathcal{N} \geq 2$ Supergravity*, *JHEP* **06** (2015) 200, [`arXiv:1505.01156`].

- In chapter 3, we study the off-shell formulation of $\mathcal{N} = 2$ supergravity in order to consistently introduce higher-derivative corrections into the action. These corrections take the form of a supersymmetrized Weyl invariant that allows for arbitrary Kerr-Newman black holes to be solutions of the theory, with no modifications to their geometry required. We also analyze the BPS conditions in the presence of these higher-derivative corrections to show that the extremal Reissner-Nordström black hole is the unique static and stationary geometry that preserves half of the supersymmetries of the theory. Finally, we show that the supersymmetrized Weyl invariant coincides with the ordinary Euler invariant on-shell, leading to a topological correction to the black hole entropy. We end with a discussion of the implications for an underlying microscopic theory.

Based on: A. M. Charles and F. Larsen, *Kerr-Newman Black Holes with String Corrections*, *JHEP* **10** (2016) 142, [arXiv:1605.07622].

- In chapter 4, we study the symmetries of off-shell $\mathcal{N} = 2$ supergravity and work out how these symmetries constrain the quantum effective action of the theory. In particular, we work out precisely how symplectic duality symmetry and $\mathcal{N} = 2$ supersymmetry constrain the possible four-derivative terms that can appear in the quantum effective action. We then restrict ourselves to Einstein-Maxwell solutions and show that all possible four-derivative terms reduce to the Euler invariant, thus explaining the topological nature of the results in the previous two chapters. We therefore establish a non-renormalization theorem that constrains corrections to the Bekenstein-Hawking entropy for a large class of non-supersymmetric black holes.

  Based on: A. M. Charles, F. Larsen, and Daniel R. Mayerson, *Non-Renormalization for Non-Supersymmetric Black Holes*, *JHEP* **08** (2017) 048, [arXiv:1702.08458].

Chapter 2

Universal Corrections to Black Holes in $\mathcal{N} \geq 2$ Supergravity

2.1 Introduction and Summary

Logarithmic corrections to the area law for black hole entropy are interesting because they are features of the high energy theory that can be computed systematically in the low energy effective theory [12,14,15,23]. In situations far from the supersymmetric limit there is not yet a microscopic theory of black hole entropy so in this setting logarithmic corrections provide a valuable target for future progress. The most promising arenas for such future developments are nonsupersymmetric black hole solutions to theories with a lot of supersymmetry. The goal of this chapter is to compute logarithmic corrections to the entropy of these black holes.

Logarithmic corrections are derived from quantum determinants over quadratic fluctuations around the black hole background [40]. All fields in the theory fluctuate so the results depend on the theory through its matter content and couplings. Concretely, we consider well-known black holes from general relativity in four dimensions such as the Kerr-Newman solutions to the Einstein-Maxwell theory but we embed these solutions into $\mathcal{N} \geq 2$ supergravity. In particular, we focus on theories with a single gravity multiplet, $\mathcal{N} - 2$ gravitino multiplets, an arbitrary number n_V of vector multiplets, and an arbitrary number n_H of hyper multiplets. The matter content is specified by the host supergravity which also determines the nonminimal couplings between the matter and the background.

We find that it is useful to organize the matter in multiplets of $\mathcal{N} = 2$ supergravity even in the presence of a black hole background that breaks supersymmetry completely and also when $\mathcal{N} > 2$. Indeed, this organization diagonalizes the problem in the sense that different $\mathcal{N} = 2$ multiplets decouple. Furthermore, with our embedding the field equations for quadratic fluctuations of such multiplets depend only on the $\mathcal{N}=2$ field content and not on couplings encoded in the prepotential.

The one-loop quantum corrections computed in heat kernel regularization are presented as usual as short distance expansions with coefficients that are invariants formed from the curvature [41, 42]. The four derivative terms that we focus on take the form

$$a_4(x) = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 \ , \tag{2.1}$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor and E_4 is the four-dimensional Gauss-Bonnet term. The values of the coeffiicients c, a are nonstandard because they are for fields with nonminimal couplings specified by $\mathcal{N} \geq 2$ supergravity. Our results for c, a are somewhat complicated for bosons and fermions separately but we find that the sum gives $c = 0$ for any values of n_V, n_H, and $\mathcal{N}$.

The heat kernel coefficient $a_4(x)$ encodes the one-loop effective action which in turn determines the logarithmic correction to the black hole hole entropy in the limit where all parameters with the same length dimension are taken large at the same rate. For BPS black holes there is only one length scale, identified as the scale of the near horizon $\mathrm{AdS}_2 \times S^2$. In this situation there are no dimensionless ratios so the coefficient of the logarithmic correction is a pure number given by [43]

$$\Delta S = \frac{1}{12} \left(23 - 11(\mathcal{N}-2) - n_V + n_H\right) \log A \ . \tag{2.2}$$

Note that this shows that BPS black holes in $\mathcal{N}=4$ supergravity, where $n_V = n_H + 1$, have vanishing logarithmic corrections to their black hole entropy.

Non-extremal black holes are characterized by dimensionless quantities such as the charge-to-mass ratio Q/M and the angular momentum quantum number J/M^2. For such black holes the coefficient in front of $\log A$ is expected to depend on these dimensionless variables. This expectation has proven correct in the case of Kerr-Newman black hole solutions to Einstein-Maxwell theory [21]. The way this comes about is that fluctuations of the metric and vector fields and additional minimally coupled fields all contribute to the c coefficient, and the curvature invariant $W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$ is a complicated function of Q/M and J/M^2 after integration over the black hole geometry.

Our main result is that when the Kerr-Newman black holes are interpreted instead as solutions to $\mathcal{N} \geq 2$ supergravity, the coefficient c vanishes

$$c = 0 \ . \tag{2.3}$$

In this situation the logarithmic correction is much simpler: (2.2) remains valid for *all* of these black holes (modulo integer corrections due to zero modes and ambiguities in the ensemble). There is no dependence on the parameters that deform the black hole off extremality.

This chapter is organized as follows. In section 2.2 we discuss how the heat kernel formalism can be used to compute the one-loop quantum effective action for non-extremal black holes, and then we show how this then gives a logarithmic correction to the entropy of the black hole. In section 2.3 we take solutions to Einstein-Maxwell theory in 4D and embed them into $\mathcal{N} = 2$ supergravity. We determine the quadratic fluctuations around this background and extend these results to general $\mathcal{N} \geq 2$ supergravity. In section 2.4 we compute the first three heat kernel coefficients for each $\mathcal{N} = 2$ multiplet. Finally, in section 2.5 we tabulate the results for the trace anomaly and show that the logarithmic corrections to black hole entropy are independent of black hole parameters. As a concrete example we consider the non-extremal Kerr-Newman black hole.

2.2 One-Loop Quantum Corrections to Non-Extremal Black Holes

In this section we show how to use the heat kernel formalism to evaluate the one-loop quantum effective action of a black hole. We argue that these one-loop contributions are entirely responsible for the logarithmic terms in the quantum effective action. We then go on to compute the corresponding logarithmic corrections to black hole entropy by means of a Laplace transform.

2.2.1 Heat Kernels and the Quantum Effective Action

Consider a Euclidean quantum field theory in $d = 4$ dimensions at a fixed temperature $T = \beta^{-1}$. The partition function Z for the theory is given by a path-integral over all configurations of the fields $\{\phi\}$ in the theory, weighted by the action $\mathcal{S}$ as follows:

$$Z = \int \mathcal{D}\phi \, e^{-\mathcal{S}} \ . \tag{2.4}$$

To a first approximation, we can evaluate this via the saddle-point approximation by simply evaluating the action at a stationary saddle-point that satisfies the classical equations of motion of the theory with appropriate boundary conditions[1]. We can then compute one-loop

[1]If there are multiple saddle points that accomplish this, then we will restrict ourselves to the one that minimizes the on-shell action. This restriction is approximate, though it becomes exact in the classical $\hbar \to 0$

quantum corrections to the partition function by expanding the fields around the classical solution

$$\phi \to \phi_{\rm cl} + \phi \ , \tag{2.5}$$

where $\{\phi\}$ now denotes the quantum field fluctations of the theory. We can then expand the action of the theory to quadratic order in these field fluctations, such that the action now takes the form

$$\mathcal{S} = \mathcal{S}_{\rm cl} - \langle \phi | \Lambda | \phi \rangle \ , \tag{2.6}$$

where Λ is a Hermitian Laplace-type kinetic operator for the field fluctuations. These field fluctuations appear quadratically in the action, and so the path integral simply becomes a Gaussian integral over these field fluctations that can be computed exactly. The result is

$$Z = e^{-\mathcal{S}_{\rm cl}} \frac{1}{\sqrt{\det \Lambda}} \ . \tag{2.7}$$

It is more convenient to re-express this one-loop partition function as

$$\log Z = -\mathcal{S}_{\rm cl} - W \ , \quad W \equiv \frac{1}{2} \log \det \Lambda \ , \tag{2.8}$$

where W is the one-loop quantum effective action. It encapsulates all of the information about the quantum corrections to the classical theory. The problem of understanding quantum corrections to our theory has therefore turned into a problem of how to evaluate this effective action.

We will focus on using the heat kernel method to evaluate W, as first discussed in [40]. We first rewrite the effective action as

$$W = \frac{1}{2} \mathrm{tr}\ \log \Lambda \ . \tag{2.9}$$

For any positive eigenvalue λ of Λ, we can write the "identity"

$$\log \lambda \text{ “ = ” } - \int_0^\infty \frac{ds}{s} e^{-s\lambda} \ . \tag{2.10}$$

Strictly speaking, this identity is only true up to an infinite constant. This constant is independent of λ, though, and so this expression is still useful as long as we carefully keep track of divergences. If we now extend this relation to the entire operator Λ, we find that

$$W = -\frac{1}{2} \int_0^\infty \frac{ds}{s} D(s) \ , \quad D(s) \equiv \mathrm{tr}\, e^{-s\Lambda} \ , \tag{2.11}$$

limit.

where $D(s)$ is known as the heat kernel of the kinetic operator Λ. We refer to the parameter s as the heat kernel time. It is a dimensionful parameter, with units of $(\text{length})^2$. We can also do a spectral decomposition in order to represent the heat kernel as

$$D(s) = \sum_i e^{-s\lambda_i} \ , \tag{2.12}$$

where we denote $\{\lambda_i\}$ as the eigenvalues of the kinetic operator Λ.

2.2.2 Divergences and Cut-Offs

Throughout section 2.2.1, we have ignored all issues and subtleties related to divergences. Many of the expressions we wrote down are either divergent or only really valid up to infinite constants, and so we must be careful to treat these divergences in such a way that we can still extract meaningful physics out of the heat kernel formalism.

The first issue is that the heat kernel representation of the quantum effective action (2.11) is divergent in the $s \to 0$ limit. The divergence is worse for larger eigenvalues of the kinetic operator, and thus we can interpret this as an ultraviolet divergence. This is not unexpected; generic low energy effective field theories coupled to gravity are not ultraviolet-complete, and so we shouldn't expect the semi-classical picture of gravity to be valid at arbitrarily high energy scales. We must therefore must put a cutoff on the theory. In particular, we will enforce that

$$s > \epsilon \ , \quad \epsilon \equiv \frac{1}{\Lambda_{\text{UV}}^2} \ , \tag{2.13}$$

for some ultraviolet cutoff Λ_{UV}. The one-loop quantum effective action therefore takes the form

$$W = -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s} D(s) \ . \tag{2.14}$$

This expression for the quantum effective action is also divergent in the $s \to \infty$ limit for negative and zero eigenvalues of the kinetic operator. This infrared divergence arises in the infinite volume limit, since arbitrarily large values of s are only allowed when the volume of the spacetime is infinite. To rectify this issue, an infrared cutoff must be introduced in order to regulate the infinite volume of the spacetime. We will not worry about this divergence, though, since in the next section we will show how to subtract off the infrared-divergent part of the quantum effective action that comes from the thermal gas the black hole is in equilibrium with.

2.2.3 Black Holes and the Thermal Gas Subtraction

We want to understand quantum corrections to generic non-extremal black holes. However, it is not enough to simply compute the one-loop quantum correction to the partition function given in (2.8); the classical saddle point that we expand around corresponds to a black hole in equilibrium with a thermal gas of all of the particles in the theory. In order to be able to make a connection to the black hole microstate picture, we have to be able to isolate the piece of the partition function that corresponds to the black hole (and not the thermal gas). In this section, we will review the procedure used in [20] to determine the general form of the thermal gas partition function, and then we will use this form to show how to consistently subtract off the thermal gas contribution to the partition function.

Let's first consider a flat four-dimensional Euclidean spacetime where the Euclidean time coordinate has periodicity β and the spatial coordinates are restricted such that they each lie within a length L. The eigenvalues of the scalar d'Alembertian on this spacetime are

$$\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^2 \ , \tag{2.15}$$

where n is an integer that determines the momentum along the compact time direction. Additionally, the spatial momentum $\vec{k}$ must be quantized such that the wavelength of the field modes fit inside the box. In the large volume limit, though, we can approximate the allowed spatial momenta as being continuous instead of discrete, with the density of states

$$d\mu = \frac{V}{(2\pi)^3}\, d^3k \ , \tag{2.16}$$

where $V = L^3$ is the volume of the box the spatial coordinates lie in. Since the spatial component of the eigenvalues are continuous, the discrete sum in (2.12) will become an integral instead, weighted by the density of states. The heat kernel is thus given by

$$D(s) = \sum_{n=-\infty}^{\infty} \int d\mu \, \exp\left(-\frac{4\pi^2 n^2 s}{\beta^2} - s\vec{k}^2 \right) \ . \tag{2.17}$$

We can in principle insert this into (2.11) in order to compute the quantum effective action for this thermal gas. However, the exponential term is finite when s is close to zero, which leads to a divergence in the s-integration for all values of n. A trick we can use to make s-integration easier is to use the Poisson resummation formula to rewrite the sum over the momentum modes along the thermal circle as

$$\sum_{n=-\infty}^{\infty} \exp\left(-\frac{4\pi^2 n^2 s}{\beta^2}\right) = \frac{\beta}{\sqrt{4\pi s}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2\beta^2}{4s}\right) . \tag{2.18}$$

Written in this form, the integrand becomes exponentially suppressed for small values of s when $n \neq 0$. Putting this all together, we find that the quantum effective action for this free gas of non-interacting particles is

$$W = -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\frac{\beta}{\sqrt{4\pi s}} \sum_{n=-\infty}^{\infty} \int d\mu \, \exp\left(-\frac{n^2\beta^2}{4s} - s\vec{k}^2\right) . \tag{2.19}$$

This is in principle straightforward to evaluate. However, there is one subtlety: the $n = 0$ term is ultraviolet divergent and blows up in the $\epsilon \to 0$ limit. The $n \neq 0$ terms in the integrand are exponentially suppressed for small s, though, and thus are ultraviolet finite. We can therefore send $\epsilon \to 0$ freely for these terms. If we now keep track of the ultraviolet cutoff appropriately, we compute the one-loop quantum effective action of the thermal gas to be given by

$$W = -\frac{V\beta}{64\pi^2\epsilon^2} - \frac{\pi^2}{90}\frac{V}{\beta^3} . \tag{2.20}$$

The first term in the thermal gas quantum effective action (2.20) is the ultraviolet divergent term discussed earlier. This term corresponds to the one-loop contribution to the renormalization of the cosmological constant. In the case considered in this chapter of asymptotically flat black holes, the final theory does not have a cosmological constant and so this contribution must be cancelled by a local counterterm. We will therefore drop this divergent piece, keeping in mind that it will eventually be cancelled out.

We have so far ignored the presence of the boundary in our discussion. The details of the boundary affect how the momentum modes are quantized inside the box, which leads to a modification of the density of states. However, these boundary effects will be subleading corrections that involve lower powers of L. The density of states therefore becomes modified such that

$$d\mu = \frac{V}{(2\pi)^3} d^3k + \mathcal{O}(L^2 d^2 k) . \tag{2.21}$$

This will lead to a new boundary contribution to the quantum effective action that can in principle have terms that are finite as well as terms that are divergent in the $\epsilon \to 0$ limit. The divergent parts will be cancelled by appropriate boundary counterterms, while the finite pieces will be independent of the cutoff ϵ and thus depend only on L and β. Since the boundary modification of the density of states is subleading in powers of L, the finite boundary contribution to the quantum effective action must be as well. Therefore, if we include boundary effects while ignoring the divergent pieces that will be cancelled by local counterterms (both in the bulk and on the boundary), the quantum effective action becomes

$$W = -\frac{\pi^2}{90}\frac{V}{\beta^3} + \mathcal{O}\left(\frac{L^2}{\beta^2}\right) \ . \tag{2.22}$$

In the large volume limit $L \gg \beta$ we can neglect the subleading boundary terms and so we simply recover the familiar thermal partition function for an ideal gas.

We have so far only considered the gas propagating on a flat spacetime. However, we now want to consider the case where the thermal gas is on a black hole background. The presence of the black hole causes the light cones of the gas particles to get warped such that the kinetic operator no longer factorizes neatly into time circle modes and spatial modes. The kinetic operator eigenvalues in (2.15) are therefore modified to become much more complicated. However, as discussed in previous work [20, 40] the general arguments presented here still hold for fields propagating on a black hole background. In particular, for a thermal gas in equilibrium with a black hole of inverse temperature β, angular velocity $\vec{\omega}$, and electromagnetic potential $\vec{\mu}$, the quantum effective action takes the form

$$W = V f(\beta, \vec{\omega}, \vec{\mu}) + \mathcal{O}\left(\frac{L^2}{\beta^2}\right) \ , \tag{2.23}$$

where f is a function with homogeneity -3 such that

$$f(\lambda\beta, \lambda\vec{\omega}, \lambda\vec{\mu}) = \lambda^{-3} f(\beta, \vec{\omega}, \vec{\mu}) \ . \tag{2.24}$$

The dominant piece of the thermal gas quantum effective action is therefore invariant under an overall rescaling of the length of the box and the black hole potentials.

Let's now denote the radius of the black hole by a. The upshot of the above arguments is that, in the large volume limit, the dominant contribution to the thermal gas quantum effective action goes like

$$W \sim \left(\frac{L}{a}\right)^3 \ , \tag{2.25}$$

since the black hole potentials β, $\vec{\omega}$, and $\vec{\mu}$ scale linearly with the black hole size for generic non-extremal black holes. With this in mind, consider two different systems:

System 1: a black hole with radius a in a box of size L ,

System 2: a black hole with radius a' in a box of size $L' = L\left(\frac{a'}{a}\right)$.

The thermal gas contribution to the partition function (2.25) is therefore the same for both systems! If we take the difference in the effective actions for system 1 and system 2, the thermal gas contribution cancels out, and we are thus simply left with the difference of the effective actions corresponding solely to the black holes. That is,

$$W_1(\text{BH} + \text{gas}) - W_2(\text{BH} + \text{gas}) = W_1(\text{BH}) - W_2(\text{BH}) \ . \tag{2.26}$$

For notational simplicity, we will now let ΔW denote this difference in black hole quantum effective actions. Using the heat kernel eigenvalue expansion for the quantum effective action, we find that

$$\Delta W = -\frac{1}{2}\int_{\epsilon}^{\infty} \frac{ds}{s} \sum_i \left(e^{-s\lambda_i} - e^{-s\lambda_i'} \right) \ , \tag{2.27}$$

where $\{\lambda_i\}$ are the eigenvalues of the kinetic operator for system 1 and $\{\lambda_i'\}$ are the eigenvalues of the kinetic operator for system 2. Since system 2 can be obtained from system 1 by an overall rigid rescaling of the fields, the kinetic operator eigenvalues are also related by this rigid rescaling. In particular, the eigenvalues of system 2 are given in terms of those of system 1 via

$$\lambda_i' = \lambda_i \left(\frac{a}{a'}\right)^2 \ . \tag{2.28}$$

We can therefore immediately see that, if we define the rescaled UV cutoff

$$\epsilon' = \epsilon \left(\frac{a}{a'}\right)^2 \ , \tag{2.29}$$

then the quantum effective action is given by

$$\Delta W = -\frac{1}{2}\int_{\epsilon}^{\epsilon'} \frac{ds}{s} \sum_i e^{-s\lambda_i} = -\frac{1}{2}\int_{\epsilon}^{\epsilon'} \frac{ds}{s} D(s) \ . \tag{2.30}$$

A by-product of this analysis is that the integration range no longer goes to infinity, and we therefore do not run into any infrared divergences when evaluating the quantum effective action.

2.2.4 Perturbative Expansion

For small heat kernel times s, the heat kernel can be represented perturbatively as

$$D(s) = \sum_{n=0}^{\infty} s^{n-d/2} a_{2n} \ , \quad a_{2n} = \int d^d x \sqrt{g}\, a_{2n}(x) \ . \tag{2.31}$$

The terms $\{a_{2n}(x)\}$ are known as Seeley-DeWitt coefficients [44], though we will often refer to them as heat kernel coefficients for simplicity.

For kinetic operators that are of the Laplace-type, there are methods that can be used to compute these heat kernel coefficients. We will focus on the covariant perturbative method discussed in [42], which gives an algorithmic way to compute these coefficients for arbitrary background manifolds. To do so, we first note that any Laplace-type kinetic operator Λ acting on the quantum field fluctuations in the theory can be written in the form

$$\Lambda = -\Box I - 2\omega_\mu \nabla^\mu - P \ , \tag{2.32}$$

where ω_μ and P are both matrices constructed from the background fields, and I is the identity operator on the space of field fluctuations. We also denote $\Box \equiv \nabla_\mu \nabla^\mu$ as the covariant d'Alembertian operator on the background manifold. We can complete the square in order to rewrite the kinetic operator as

$$\Lambda = -(\mathcal{D}_\mu \mathcal{D}^\mu) I - E \ , \tag{2.33}$$

where

$$E = P - \omega_\mu \omega^\mu - (\nabla_\mu \omega^\mu) \ , \quad \mathcal{D}_\mu = \nabla_\mu + \omega_\mu \ . \tag{2.34}$$

The matrix E is the effective mass matrix for the quantum field fluctuations, while $\mathcal{D}_\mu$ is the new effective covariant derivative acting on the field fluctuations. Importantly, this new derivative is *not* necessarily torsion-free. The curvature associated with this new effective derivative operator is denoted by

$$\Omega_{\mu\nu} \equiv [\mathcal{D}_\mu, \mathcal{D}_\nu] \ . \tag{2.35}$$

With these definitions, the Seely-DeWitt coefficients can be written entirely in terms of the mass matrix E, the effective curvature $\Omega_{\mu\nu}$, and the background geometry. The first three coefficients are given by

$$\begin{aligned}
(4\pi)^2 a_0(x) &= \mathrm{tr}\, I \ , \\
(4\pi)^2 a_2(x) &= \mathrm{tr}\left(E + \frac{1}{6}R\right) \ , \\
(4\pi)^2 a_4(x) &= \mathrm{tr}\left(\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{6}\Box E + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{30}\Box R \right. \\
&\qquad \left. + \frac{1}{72}R^2 - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \ .
\end{aligned} \tag{2.36}$$

We can also in principle compute the higher-order coefficients $a_6(x)$, $a_8(x)$, etc., but the logarithmic corrections in four dimensions are determined by $a_4(x)$ and so it will be sufficient to compute the heat kernel coefficients up to this order. There are also in general boundary contributions to these coefficients. We have ignored them, however, since we are taking the large volume limit such that all boundary terms are subleading.

It is also important to note that the heat kernel depends only on the integrated heat kernel coefficients $\{a_{2n}\}$ and not on their local densities. Since we are interested in studying asymptotically flat black holes, any total derivative terms in the heat kernel coefficients will simply integrate to zero. We will hence drop all total derivative terms when we compute these heat kernel coefficients in the proceeding work.

2.2.5 Scalings and the Range of Validity

We now have a perturbative realization of the heat kernel (and thus the effective action). This perturbative approach is only valid for small heat kernel times s, though, and fails miserably for large heat kernel times. We must therefore investigate when this small time expansion is valid.

Each heat kernel coefficient $a_{2n}(x)$ contains terms that contain n powers of the Riemann curvature, e.g.

$$a_{2n}(x) \sim R^n \ . \tag{2.37}$$

The curvature of the black hole scales inversely with its length, and we want to integrate these heat kernel coefficients over the whole spacetime. The integrated heat kernel coefficient therefore scales like

$$a_{2n} \sim a^{4-2n} \ , \tag{2.38}$$

where a is the length scale of the black hole. The heat kernel series expansion (2.31) goes like

$$D(s) \sim \sum_{n=0}^{\infty} \left(\frac{s}{a^2}\right)^{n-2} \ . \tag{2.39}$$

In order for this series to be able to converge, each successive term must have a much smaller magnitude than the one before it. This means that the heat kernel series expansion is only valid in the regime where

$$\frac{s}{a^2} \ll 1 \ . \tag{2.40}$$

Said another way, the dominant contribution to the black hole quantum effective action comes from integration range that satisfies

$$\epsilon < s < \epsilon' = \epsilon \left(\frac{a}{a'}\right)^2 \ . \tag{2.41}$$

In order to be able to use the perturbative heat kernel expansion, this integration range must be compatible with the convergence condition (2.40). The lengths a, a' of the relevant black hole solutions must be taken large enough such that

$$\frac{\epsilon}{a^2} \ll 1 \ , \quad \frac{\epsilon}{a'^2} \ll 1 \ . \tag{2.42}$$

Once we have chosen cutoffs and the size of the black hole appropriately such that the perturbative representation of the heat kernel holds over the entire range of integration, the quantum effective action can be expressed in terms of the integrated heat kernel coefficients by

$$\Delta W = -\frac{1}{2}\int_{\epsilon}^{\epsilon'} \frac{ds}{s^3}\left(a_0 + a_2 s + a_4 s^2 + \ldots\right) \ . \tag{2.43}$$

where the dots indicated terms that have higher powers of s. If we now perform the s integration, we find that the quantum effective action is

$$\Delta W = \frac{a_0}{4\epsilon^2}\left(\frac{a'^4}{a^4} - 1\right) + \frac{a_2}{2\epsilon}\left(\frac{a'^2}{a^2} - 1\right) + a_4 \log\frac{a'}{a} + \mathcal{O}(\epsilon) \ . \tag{2.44}$$

The first two terms in this expression have factors of the cutoff ϵ in the denominator and thus diverge in the $\epsilon \to 0$ limit. In particular, the ϵ^{-2} term corresponds to the one-loop contribution to the renormalization of the cosmological constant, while the ϵ^{-1} term corresponds to the one-loop contribution to the renormalization of Newton's constant. All of these divergent terms must be cancelled by local counterterms, so we will drop them. The remaining term listed is independent of ϵ, while the higher-order terms all vanish in the $\epsilon \to 0$ limit. We can thus simply take $\epsilon \to 0$ and ignore these higher-order contributions.

We can now extract the piece of ΔW that is dependent on a and identify it as the quantum effective action W for the black hole of size a. Ignoring divergences and dropping terms that vanish as the cutoff goes to zero, we find that

$$W = -a_4 \log a \ . \tag{2.45}$$

It is therefore the a_4 coefficient that completely determines the quantum corrections to the black hole, in the large volume limit of consideration in this chapter.

2.2.6 Change of Ensembles

At this point, we have a well-defined procedure for computing the one-loop quantum effective action for an arbitrary set of field content propagating on a black hole background.

Importantly, this computation is done in the grand canonical ensemble of the theory, where the inverse temperature β, the angular velocity $\vec{\omega}$, and the electromagnetic potential $\vec{\mu}$ are all fixed and scale linearly with the size of the black hole. The grand canonical partition function can be expressed in terms of the classical and one-loop actions as

$$\log Z(\beta, \vec{\omega}, \vec{\mu}) = -\mathcal{S}_{\text{cl}} - W \ . \tag{2.46}$$

Since we took care to subtract off the contribution to the effective action from the thermal gas that the black hole is in equilibrium with, we can interpret this as the grand canonical partition function of the black hole itself. We now want to use this result to compute the entropy of the black hole. To do this, we need to detail how to change ensembles to the microcanonical ensemble.

From the microscopic perspective, the black hole dynamics are described by a Hilbert space of individual black hole microstates. These microstates each have an associated macroscopic energy E, momentum $\vec{P}$, angular momentum $\vec{J}$, and charge associated with them $\vec{Q}$. In the limit where we take the size of the black hole to be large compared to all other scales in our theory, the energy of each microstate will simply be given by the usual relativistic energy

$$E = M + \frac{\vec{P}}{2M} \ , \tag{2.47}$$

since the interactions between the black hole and the fields in the spacetime will yield subleading corrections to this. The entropy of the black hole counts the number of states in the microcanonical ensemble where the mass, momentum, angular momentum, and charge of the black hole are all fixed. The entropy is a Lorentz-invariant quantiy, and so it cannot depend on the value of the momentum that we have fixed. We can therefore express the microstate degeneracy Ω as

$$\Omega(M, \vec{P}, \vec{J}, \vec{Q}) = e^{S(M, \vec{J}, \vec{Q})} \ . \tag{2.48}$$

The grand canonical partition function can then be expressed as a sum over all black hole microstate numbers, weighted by the microstate degeneracy, such that

$$Z(\beta, \vec{\omega}, \vec{\mu}) = \sum_{M, \vec{P}, \vec{J}, \vec{Q}} e^{S(M, \vec{J}, \vec{Q}) - \beta E - \vec{\omega}\cdot\vec{J} - \vec{\mu}\cdot\vec{Q}} \ . \tag{2.49}$$

We have now arrived at the relationship (2.49) between the grand canonical partition function and the microcanonical entropy of the black hole. This relationship can be formally inverted by means of a Laplace transform, since the summand is strongly peaked around the classical values of M, $\vec{J}$, and $\vec{Q}$ that correspond to the specified potentials β, $\vec{\omega}$, and $\vec{\mu}$.

Moreover, we are interested in black holes with mass large enough such that the logarithmic corrections to the entropy dominate over all other corrections. As argued above, these logarithmic corrections arise from one-loop quantum corrections to the theory. When performing the Laplace transform, it is therefore sufficient to use the saddle-point approximation, as this will capture these one-loop corrections.

This saddle-point inversion is done explicitly in [20]. The result of their calculation is that the entropy of the black hole is given by

$$S(M, \vec{J}, \vec{Q}) = -\mathcal{S}_{\rm cl} + \beta M + \vec{\omega} \cdot \vec{J} + \vec{\mu} \cdot \vec{Q} - W + C_{\rm zm} \log a \ , \tag{2.50}$$

where the potentials are understood to be evaluated on their classical values corresponding to the background saddle-point. Note that we have also included a constant $C_{\rm zm}$ in this expression. This constant is to account for any zero-modes of the theory that we did not correctly incorporate into the one-loop effective action, as well as the integer corrections that arise when performing the saddle-point inversion of (2.49). The computation of this constant are discussed in appendix A.

The first four terms on the right-hand side (2.50) are simply the usual Bekenstein-Hawking entropy, as established from earlier studies of Euclidean quantum gravity [45]. Additionally, we know from (2.45) that the one-loop quantum effective action W is given by $a_4 \log a$, where a_4 is the third Seeley-DeWitt coefficient integrated over all of spacetime. Putting this all together, we find that the one-loop quantum effects in our theory lead to a black hole entropy of

$$S = \frac{A}{4G} + \frac{1}{2} \left(a_4 + C_{\rm zm} \right) \log A \ , \tag{2.51}$$

where A is the area of the black hole horizon. In the limit where the area of the black hole is large, all other corrections will be at most $\mathcal{O}(1)$ corrections to this, and so we can safely neglect them.

We therefore find that the logarithmic corrections to the black hole entropy are completely determined by the heat kernel coefficient $a_4(x)$ for the quantum fluctuations of the theory (modulo the integer corrections detailed in appendix A).

2.3 The Background Solution and its Fluctuations

In this section we embed an arbitrary solution to the $d = 4$ Einstein-Maxwell theory into $\mathcal{N} \geq 2$ supergravity. We then derive the equations of motion for quadratic fluctuations around this background.

2.3.1 Einstein-Maxwell Theory

The starting point is a solution to the standard $d = 4$ Einstein-Maxwell theory

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \ , \tag{2.52}$$

where $\kappa^2 = 8\pi G$. The geometry satisfies the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu} \ , \tag{2.53}$$

where the energy-momentum tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\,(\sqrt{-g}\,\mathcal{L})}{\delta g^{\mu\nu}} = \frac{1}{2\kappa^2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \ . \tag{2.54}$$

The field strength $F_{\mu\nu}$ satisfies Maxwell's equation and the Bianchi identity which we combine into the complex equation

$$\nabla_\mu F^{+\mu\nu} = 0 \ . \tag{2.55}$$

We introduce the self-dual and anti-self-dual parts of the field strength as

$$F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}) \ , \tag{2.56}$$

with

$$\tilde{F}_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \ . \tag{2.57}$$

We will not specify the solution explicitly at this point but it may be useful to have in mind that we will later consider the Kerr-Newman black hole.

2.3.2 Einstein-Maxwell Backgrounds in $\mathcal{N} = 2$ Supergravity

We want to interpret the background of the prior section as a solution to $\mathcal{N} = 2$ supergravity with matter in the form of n_V vector multiplets and n_H hypermultiplets. The most difficult step will be understanding dynamics of the bosonic fields in the gravity and vector multiplets, so in this section we focus on those. We will return to the hyper multiplets and all the fermions in the next section.

The bosonic Lagrangian of $\mathcal{N} = 2$ supergravity coupled to n_V vector fields is

$$\mathcal{L} = \frac{1}{2\kappa^2}R - g_{\alpha\bar{\beta}}\nabla^\mu z^\alpha \nabla_\mu z^{\bar{\beta}} + \frac{1}{2}\mathrm{Im}\left[\mathcal{N}_{IJ}F^{+I}_{\mu\nu}F^{+\mu\nu J}\right] \ . \tag{2.58}$$

The index $\alpha = 1, \ldots, n_V$ enumerates the complex scalar fields z^α in the vector multiplets, while the label $I = 0, \ldots, n_V$ enumerates the field strengths in the theory. Note that $F^0_{\mu\nu}$ is the field strength of the graviphoton in the gravity multiplet, while $F^\alpha_{\mu\nu}$ is one of the vector multiplet field strengths.

The interactions in our theory are determined by a holomorphic prepotential $F(X^I)$ that has homogeneity two with respect to the projective coordinates X^I. Its derivatives are denoted $F_I = \partial_I F$, $F_{IJ} = \partial_I \partial_J F$, etc., and it specifies the coupling between vectors and scalars as

$$\mathcal{N}_{IJ} = \mu_{IJ} + i\nu_{IJ} = \bar{F}_{IJ} + i\frac{N_{IK}X^K N_{JL}X^L}{N_{NM}X^N X^M} \ , \tag{2.59}$$

where $N_{IJ} = 2\,\mathrm{Im}\,F_{IJ}$. The Kähler metric is

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K} \ , \tag{2.60}$$

where the Kähler potential is

$$\mathcal{K} = \log i(X^I \bar{F}_I - F_I \bar{X}^I) \ . \tag{2.61}$$

The Kähler covariant derivatives

$$\begin{aligned} \nabla_\alpha X^I &= (\partial_\alpha + \frac{1}{2}\kappa^2 \partial_\alpha \mathcal{K}) X^I \ , \\ \bar{\nabla}_{\bar{\alpha}} X^I &= (\partial_{\bar{\alpha}} - \frac{1}{2}\kappa^2 \partial_{\bar{\alpha}} \mathcal{K}) X^I = 0 \ , \end{aligned} \tag{2.62}$$

relate the true motion in moduli space to the projective parametrization. The Kähler weights are such that

$$Z^I(z) = e^{-\frac{1}{2}\kappa^2 \mathcal{K}} X^I(z) \tag{2.63}$$

is purely holomorphic $\partial_{\bar{\alpha}} Z^I = 0$. The projective coordinates can be normalized such that

$$N_{IJ} X^I \bar{X}^J = -i(F_J \bar{X}^J - X^J \bar{F}_J) = -\frac{1}{\kappa^2} \ . \tag{2.64}$$

We now want to embed arbitrary background solutions to Einstein-Maxwell theory into $\mathcal{N} = 2$ supergravity. The background solution to Einstein-Maxwell theory specifies the geometry and a single field strength $F_{\mu\nu}$. We claim that the corresponding solution to $\mathcal{N} = 2$ supergravity has the same geometry but the matter fields are given by

$$\begin{aligned} z^\alpha &= \text{const} \ , \\ F^{+I}_{\mu\nu} &= X^I F^+_{\mu\nu} \ . \end{aligned} \tag{2.65}$$

We need to verify that this in fact is a solution to $\mathcal{N}=2$ supergravity.

First, we will look at the Einstein equation for our theory. The scalars are constant so their derivatives do not contribute to the energy-momentum tensor. The non-vanishing part is

$$\begin{aligned} T_{\mu\nu} &= -2\,\mathrm{Im}\left[\mathcal{N}_{IJ}(F^{+I}_{\mu\lambda}F^{-\lambda J}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{+I}_{\lambda\sigma}F^{-J\lambda\sigma})\right] \\ &= -2\nu_{IJ}\left(F^{+I}_{\mu\lambda}F^{-\lambda J}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{+I}_{\lambda\sigma}F^{-J\lambda\sigma}\right) \ . \end{aligned} \tag{2.66}$$

We now note the identity

$$-2\nu_{IJ}X^I\bar{X}^J = i\mathcal{N}_{IJ}X^I\bar{X}^J + \mathrm{c.c} = iF_{IJ}X^I\bar{X}^J + \mathrm{c.c} = i(F_J\bar{X}^J - X^J\bar{F}_J) = \frac{1}{\kappa^2} \ , \tag{2.67}$$

due to (2.59) for $\mathcal{N}_{IJ}$, homogeneity of the prepotential, and the normalization condition (2.64). The energy-momentum tensor (2.66) therefore simplifies to

$$T_{\mu\nu} = \frac{1}{\kappa^2}\left(F^{+}_{\mu\lambda}F^{-\lambda}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{+}_{\lambda\sigma}F^{-\lambda\sigma}\right) = \frac{1}{2\kappa^2}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right) \ . \tag{2.68}$$

This is the same as (2.54) for the Einstein-Maxwell theory so the Einstein equation for $\mathcal{N}=2$ supergravity is satisfied with unchanged geometry.

The combined Maxwell-Bianchi equation in $\mathcal{N}=2$ supergravity,

$$\nabla_\mu\left(\mathcal{N}_{IJ}F^{+I\mu\nu}\right) = 0 \ , \tag{2.69}$$

is automatically satisfied because the background satisfies the Maxwell-Bianchi equation (2.55). The dependence of $\mathcal{N}_{IJ}$ and $F^{I\mu\nu}$ on the scalar fields introduces no spacetime dependence since the scalars are constant.

The scalar field equations are not automatic even though the scalars are constant because the vector fields act as a source unless

$$\frac{\partial\mathcal{N}_{IJ}}{\partial z^\alpha}F^{+I}_{\mu\nu}F^{+\mu\nu J} = \frac{\partial\mathcal{N}_{IJ}}{\partial z^{\bar{\alpha}}}F^{+I}_{\mu\nu}F^{+\mu\nu J} = 0 \ . \tag{2.70}$$

The anti-holomorphic condition

$$\frac{\partial\mathcal{N}_{IJ}}{\partial z^{\bar{\alpha}}}Z^I Z^J = 0 \tag{2.71}$$

is obvious: move the holomorphic coordinates Z^I under the derivative and use $\mathcal{N}_{IJ}Z^IZ^J = F_IZ^I = 2F$ to find an antiholomorphic derivative that vanishes because it acts on the holomorphic prepotential. The holomorphic condition is almost as simple:

$$\frac{\partial \mathcal{N}_{IJ}}{\partial z^\alpha} Z^I Z^J = \partial_\alpha F_I Z^I - F_I \partial_\alpha Z^I = (F_{IJ} Z^I - F_J)\partial_\alpha Z^J = 0 \ . \tag{2.72}$$

We used $\mathcal{N}_{IJ} Z^J = F_I$ again and then $F_{IJ} Z^I = F_J$ from homogeneity of the prepotential.

At this point we have completed the verification that a solution to Einstein-Maxwell remains a solution when embedded in $\mathcal{N} = 2$ supergravity through (2.65). Any additional fields in $\mathcal{N} > 2$ supergravity must all appear quadratically. The further embedding from $\mathcal{N} > 2$ into $\mathcal{N} = 2$ supergravity is therefore automatic.

2.3.3 Quadratic Fluctuations

The quantum corrections to the black hole entropy are determined by the spectrum of quadratic fluctuations around the background. We first consider the general matter equations of motion derived from the action (2.58)

$$\begin{aligned} \nabla^\mu \left(g_{\alpha\bar\beta} \nabla_\mu z^\alpha \right) - \frac{i}{4} \frac{\partial \mathcal{N}_{IJ}}{\partial \bar z^{\bar\beta}} F^{+I}_{\mu\nu} F^{+J\mu\nu} + \frac{i}{4} \frac{\partial \overline{\mathcal{N}}_{IJ}}{\partial \bar z^{\bar\beta}} F^{-I}_{\mu\nu} F^{-J\mu\nu} &= 0 \ , \\ i \nabla_\mu \left(\mathcal{N}_{IJ} F^{+J\mu\nu} - \overline{\mathcal{N}}_{IJ} F^{-J\mu\nu} \right) &= 0 \ , \\ \nabla_\mu \left(F^{+J\mu\nu} - F^{-J\mu\nu} \right) &= 0 \ . \end{aligned} \tag{2.73}$$

The last two lines are the Maxwell-Bianchi equations. Linearizing around the background, these equations become

$$\begin{aligned} \mathcal{N}_{IJ} \nabla_\mu \delta F^{+J\mu\nu} + (\nabla_\mu \delta \mathcal{N}_{IJ}) F^{+J\mu\nu} - \overline{\mathcal{N}}_{IJ} \nabla_\mu \delta F^{-J\mu\nu} - (\nabla_\mu \delta \overline{\mathcal{N}}_{IJ}) F^{-J\mu\nu} &= 0 \ , \\ \nabla_\mu \left(\delta F^{+J\mu\nu} - \delta F^{-J\mu\nu} \right) &= 0 \ , \end{aligned} \tag{2.74}$$

where unvaried fields (without δ) are evaluated on the background. The background fields satisfy (2.65), and so we can show that the second term in the first equation above simplifies:

$$\begin{aligned} (\nabla_\mu \delta \mathcal{N}_{IJ}) F^{+J\mu\nu} &= (\nabla_\alpha \mathcal{N}_{IJ}) X^J \nabla_\mu \delta z^\alpha F^{+\mu\nu} + \nabla_{\bar\alpha} \mathcal{N}_{IJ} X^J \nabla_\mu \delta z^{\bar\alpha} F^{+\mu\nu} \\ &= (\nabla_\alpha F_I - \mathcal{N}_{IJ} \nabla_\alpha X^J) \nabla_\mu \delta z^\alpha F^{+\mu\nu} \\ &= -2i \nu_{IJ} \nabla_\alpha X^J \nabla_\mu \delta z^\alpha F^{+\mu\nu} F^{+\mu\nu} \ . \end{aligned} \tag{2.75}$$

Note that we used symplectic invariance in the form $\nabla_\alpha F_I = \overline{\mathcal{N}}_{IJ} \nabla_\alpha X^J$. Inserting this back into (2.74) and simplifying, we find that

$$\nabla_\mu \left(\delta F^{+I\mu\nu} - \nabla_\alpha X^I \delta z^\alpha F^{+\mu\nu} - \nabla_{\bar\alpha} \bar X^I \delta z^{\bar\alpha} F^{-\mu\nu} \right) = 0 \ . \tag{2.76}$$

This is a complex equation with imaginary part reducing to the Bianchi identity.

After linearizing the scalar equation of motion in (2.73) around the background, the middle term vanishes due to holomorphicity. The (complex conjugate of) the last term simplifies as follows:

$$\begin{aligned}\delta\left(\nabla_\beta \mathcal{N}_{IJ} F^{+I}_{\mu\nu} F^{+J\mu\nu}\right) &= \nabla_\alpha \nabla_\beta \mathcal{N}_{IJ} \delta z^\alpha X^I X^J F^+_{\mu\nu} F^{+\mu\nu} + 2\nabla_\beta \mathcal{N}_{IJ} X^I F^+_{\mu\nu} \delta F^{+J\mu\nu} \\ &= 2\nabla_\beta \mathcal{N}_{IJ} X^I F^+_{\mu\nu} (\delta F^{+J\mu\nu} - \nabla_\alpha X^J \delta z^\alpha F^{+\mu\nu}) \\ &= -4i\nu_{IJ} \nabla_\beta X^I F^+_{\mu\nu} (\delta F^{+J\mu\nu} - \nabla_\alpha X^J \delta z^\alpha F^{+\mu\nu}) \ .\end{aligned} \tag{2.77}$$

Finally, we collect terms and write the linearized scalar equation as

$$g_{\alpha\bar\beta} \nabla^2 \delta z^\alpha - \nu_{IJ} \bar\nabla_{\bar\beta} \bar X^I F^-_{\mu\nu} \left(\delta F^{-J\mu\nu} - \bar\nabla_{\bar\alpha} \bar X^J \delta z^{\bar\alpha} F^{-\mu\nu}\right) = 0 \ . \tag{2.78}$$

The linearized equations of motion for the vectors (2.76) and the scalars (2.78) can both be derived from the single action

$$\mathcal{L} = -g_{\alpha\bar\beta} \nabla_\mu \delta z^\alpha \nabla^\mu \delta z^{\bar\beta} + \frac{1}{2}\nu_{IJ} \left(\delta F^{+I}_{\mu\nu} - \delta X^I F^+_{\mu\nu}\right)\left(\delta F^{+J\mu\nu} - \delta X^J F^{+\mu\nu}\right) + \text{h.c} \ , \tag{2.79}$$

with $\delta X^I = \nabla_\alpha X^I \delta z^\alpha$. This is a consistency check on the manipulations.

2.3.4 Decoupling the Quadratic Fluctuations

The action (2.79) for the quadratic fluctuations is concise but the dependence on the Kähler metric $g_{\alpha\bar\beta}$ and the symplectic metric ν_{IJ} introduces elaborate couplings between the n_V complex scalars z^α and the $n_V + 1$ field strengths $F^I_{\mu\nu}$. We can simplify by expanding the field strengths as

$$F^{+I}_{\mu\nu} = X^I F^+_{\mu\nu} + \bar\nabla_{\bar\alpha} \bar X^I f^{+\bar\alpha}_{\mu\nu} = \begin{pmatrix} X^I & \bar\nabla_{\bar\alpha} \bar X^I \end{pmatrix} \begin{pmatrix} F^+_{\mu\nu} \\ f^{+\bar\alpha}_{\mu\nu} \end{pmatrix} \ . \tag{2.80}$$

This represents the $n_V + 1$ fields $F^I_{\mu\nu}$ as a single graviphoton field $F_{\mu\nu}$ and n_V vector fields $f^\alpha_{\mu\nu}$. The complete basis $\{X^I, \bar\nabla_{\bar\alpha} \bar X^I\}$ is orthogonal with respect to the metric ν_{IJ} in the sense that

$$\begin{pmatrix} \bar X^I \\ \nabla_\alpha X^I \end{pmatrix} \nu_{IJ} \begin{pmatrix} X^I & \bar\nabla_{\bar\beta} \bar X^I \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \kappa^{-2} & 0 \\ 0 & g_{\alpha\bar\beta} \end{pmatrix} \ . \tag{2.81}$$

The component form of the field variations are

$$\delta F^{+I\mu\nu} = X^I \delta F^{+\mu\nu} + \nabla_\alpha X^I \delta z^\alpha F^{+\mu\nu} + \bar\nabla_{\bar\alpha} \bar X^I f^{+\bar\alpha\mu\nu} \ . \tag{2.82}$$

For variations of this form the linearized matter equations (2.76) and (2.78) become

$$\begin{aligned}X^I \nabla_\mu \delta F^{+\mu\nu} + \bar{\nabla}_{\bar{\alpha}} \bar{X}^I \nabla_\mu \left(f^{+\bar{\alpha}\mu\nu} - \delta z^{\bar{\alpha}} F^{-\mu\nu} \right) = 0 \ , \\ \nabla^2 \delta z^\alpha + \frac{1}{2} F^-_{\mu\nu} f^{-\alpha\mu\nu} = 0 \ .\end{aligned} \tag{2.83}$$

Orthogonality forces the two terms in the first equation to vanish separately. Thus $\delta F^{+\mu\nu}$ satisfies the standard Maxwell-Bianchi equations (2.55), even in the presence of a fluctuating scalar. We rename $\delta z^\alpha \to z^\alpha$ and write the remaining equations as

$$\begin{aligned}\nabla_\mu \left(f^{-\alpha\mu\nu} - z^\alpha F^{+\mu\nu} \right) = 0 \ , \\ \nabla^2 z^\alpha + \frac{1}{2} F^-_{\mu\nu} f^{-\alpha\mu\nu} = 0 \ .\end{aligned} \tag{2.84}$$

These matter equations are fully decoupled; there is no interaction between the matter multiplet and the supergravity multiplet (gravity and graviphoton). Additionally, the n_V vector multiplets do not couple to each other so they can be analyzed independently. We will henceforth suppress the index α.

The equations of motion (2.84) are actually misleading as they stand because, according to the first equation, the antisymmetric vector field $f^{\mu\nu}$ does not satisfy the Bianchi identity: the imaginary part of $f^{-\mu\nu}$ has a source. We can remedy this by making the field redefinition

$$f^{-\mu\nu} \to -2i f^{-\mu\nu} + \bar{z} F^{-\mu\nu} \ . \tag{2.85}$$

The transformed field strength $f^{\mu\nu}$ satisfies the Bianchi identity. The equations (2.84) become

$$\begin{aligned}\nabla_\mu \left(f^{\mu\nu} - i z F^{+\mu\nu} + i \bar{z} F^{-\mu\nu} \right) = 0 \ , \\ \nabla^2 z - i F^-_{\mu\nu} f^{-\mu\nu} + \frac{1}{2} \bar{z} F^-_{\mu\nu} F^{-\mu\nu} = 0 \ .\end{aligned} \tag{2.86}$$

This is our final result for the quadratic fluctuations of a $\mathcal{N} = 2$ vector multiplet around a solution to the Einstein-Maxwell theory.

We have not yet analyzed the Einstein equation. Linearizing the energy-momentum tensor (2.66) of the $\mathcal{N} = 2$ supergravity theory around the background, we find that

$$\begin{aligned}\delta \left(\nu_{IJ} F^{+I}_{ac} F^{-Jc}_{b} \right) &= \delta \nu_{IJ} F^{+I}_{ac} F^{-Jc}_{b} + \nu_{IJ} \left(\delta F^{+I}_{ac} F^{-Jc}_{b} + \text{c.c.} \right) \\ &= -\frac{i}{2} \nabla_\alpha \left(\mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ} \right) \delta z^\alpha X^I \bar{X}^J F^+_{ac} F^{-c}_{b} + \nu_{IJ} \delta F^{+I}_{ac} \bar{X}^J F^{-c}_{b} + \text{c.c.} \\ &= \nu_{IJ} \bar{X}^J (\delta F^{+I}_{ac} - \nabla_\alpha X^I \delta z^\alpha F^+_{ac}) F^{-c}_{b} + \text{c.c.} \\ &= -\frac{1}{2\kappa^2} \delta (F^+_{ac} F^{-c}_{b}) \ ,\end{aligned} \tag{2.87}$$

for variations of the form (2.82). Thus fluctuations in the geometry are sourced exclusively by the graviphoton or, equivalently, the Einstein equation respects the decoupling of the $\mathcal{N}=2$ supergravity multiplet from the vector multiplets.

2.3.5 Completing the Fluctuation Multiplets

The full $\mathcal{N}=2$ supergravity theory generally includes many fields that vanish in the background. The actions of such fields can be computed at quadratic order by taking all other fields to have their background value. This process introduces non-minimal couplings because of the background graviphoton. In the following we examine the various $\mathcal{N}=2$ multiplets one by one.

The $\mathcal{N}=2$ supergravity multiplet contains the graviton, two gravitini, and the graviphoton. The dynamics of the bosons are governed by the Einstein-Maxwell action (2.52). The gravitino action is[2]

$$\mathcal{L}_{\text{gravitino}} = -\frac{1}{\kappa^2}\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}\nabla_\nu\psi^i_\rho + \nu_{IJ}\left(F^{-I}_{\mu\nu}Q^{-J\mu\nu} + \text{h.c.}\right) , \tag{2.88}$$

where $i=1,2$ enumerates the two gravitini. The Pauli term depends on $Q^{-J\mu\nu} = X^J\bar{\psi}^\mu_i\psi^\nu_j\varepsilon^{ij}$. In the background (2.65) the normalization condition (2.81) then gives

$$\mathcal{L}_{\text{gravitino}} = -\frac{1}{\kappa^2}\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}\nabla_\nu\psi^i_\rho - \frac{1}{2\kappa^2}\left(F^{-}_{\mu\nu}\bar{\psi}^\mu_i\psi^\nu_j\varepsilon^{ij} + \text{h.c.}\right) . \tag{2.89}$$

The sum of this action and the Einstein-Maxwell action (2.52) is invariant under the $\mathcal{N}=2$ supersymmetry

$$\delta\psi^i_\mu = \nabla_\mu\epsilon^i - \frac{1}{4}\hat{F}\varepsilon^{ij}\gamma^\mu\epsilon_j , \tag{2.90}$$

where $\hat{F} = \frac{1}{2}\gamma_{\mu\nu}F^{\mu\nu}$.

The $\mathcal{N}=2$ vector multiplet has one vector field, two gauginos, and one complex scalar. The bosons satisfy the equations of motion (2.86), as we have shown in detail. The gauginos are subject to Pauli terms that couple them to each other and to the gravitinos. However, these couplings appear in the combination $Q^{-J\mu\nu} = \bar{\nabla}_{\bar{\alpha}}\bar{X}^J\left(\overline{\chi}^{\bar{\alpha}i}\gamma^\mu\psi^{\nu j}\varepsilon_{ij} + \ldots\right)$, and such terms vanish when contracted with a field strength $F^{-I}_{\mu\nu}$ of the background form (2.65) because of orthogonality (2.81). Therefore the gauginos are minimally coupled fermions.

The $\mathcal{N}=2$ hyper multiplet has two Majorana hyper fermions and four real scalars. The scalars are minimally coupled to gravity. The hyper fermion is acted on by a Pauli term

[2] We follow the conventions of [46]. The normalization of the Maxwell field strength in (2.52) is conventional in the gravity literature. The relation between the two conventions for the graviphoton is $F^{(\text{FvP})}_{\mu\nu} = 2F^{(\text{here})}_{\mu\nu}$.

where the metric ν_{IJ} contracts $F^{-I}_{\mu\nu}$ and $Q^{-J\mu\nu} = X^J\left(\frac{1}{2}\kappa^2\bar{\zeta}^A\gamma^{\mu\nu}\zeta^B C_{AB} + \ldots\right)$. The $Q^{-J\mu\nu}$ is proportional to X^J as it was for the gravitino and again orthogonality leads to a simple result for the quadratic action

$$\mathcal{L}_{\text{hyper}} = -\bar{\zeta}_A\gamma^\mu\nabla_\mu\zeta^A - \frac{1}{2}\left(\bar{\zeta}^A\hat{F}\zeta^B\epsilon_{AB} + \text{h.c.}\right) . \tag{2.91}$$

It is sufficient for our purposes to consider each hyper multiplet independently. For a single hyper multiplet the indices A, B take values of either 1 or 2, and so we can without loss of generality let $C_{AB} = \epsilon_{AB}$.

2.3.6 $\mathcal{N} > 2$ Supergravity

In this section, we extend our results for $\mathcal{N} = 2$ supergravity to supergravity theories with $\mathcal{N} > 2$ supersymmetry.

We first embed the background solution to the Maxwell-Einstein theory into supergravity with $\mathcal{N} > 2$. We pick one of the $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ gauge fields in the gravity multiplet and designate it as the graviphoton of an $\mathcal{N} = 2$ theory that is identified with the gauge field of the Maxwell-Einstein theory. This embedding defines the background defined earlier as a solution also to $\mathcal{N} > 2$ supergravity.

We next organize all fluctuating fields in $\mathcal{N} = 2$ multiplets. The $\mathcal{N} > 2$ symmetry constrains the $\mathcal{N} = 2$ matter content. For example:

- An $\mathcal{N} = 4$ theory has $n_V = n_H + 1$ because one $\mathcal{N} = 2$ vector is part of the $\mathcal{N} = 4$ supergravity multiplet while each $\mathcal{N} = 4$ matter multiplet is composed of one $\mathcal{N} = 2$ vector multiplet and one $\mathcal{N} = 2$ hyper multiplet.
- The $\mathcal{N} = 6$ theory: $n_V = 7$ and $n_H = 4$.
- The $\mathcal{N} = 8$ theory: $n_V = 15$ and $n_H = 10$.

The classification under $\mathcal{N} = 2$ supersymmetry takes all matter fields into account except for the $\mathcal{N} - 2$ gravitini and their superpartners. We refer to these as massive gravitini. A massive gravitino multiplet has one gravitino, two vectors, and one gaugino. The two vectors in the massive gravitino multiplet are minimally coupled vector fields. The background graviphoton field couples the remaining fermions:

$$\mathcal{L}_{\text{gravitino}} = -\frac{1}{\kappa^2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \frac{2}{\kappa^2}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda - \frac{1}{2\kappa^2}(\bar{\Psi}_\mu\hat{F}\gamma^\mu\lambda + \text{h.c.}) . \tag{2.92}$$

We found this action by reduction of $\mathcal{N}=8$ supergravity [47] but other approaches give the same result. As a check we verified that in $\text{AdS}_2 \times S^2$ the fermion fields acquire precisely the conformal weights demanded by the superconformal symmetry of the action.

2.4 $\mathcal{N}=2$ Multiplet Heat Kernels

We will now compute the heat kernels of the quadratic fluctuations of the $\mathcal{N}=2$ multiplet field content. The heat kernel coefficient formulae (2.36) are strictly in terms of local invariants constructed from the background fields, so there are no issues in using the classical equations of motion for the background fields to simplify these expressions. We will freely make use of the Ricci-flat Einstein equation

$$R_{\mu\nu} = \frac{1}{2}F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{8}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \ , \tag{2.93}$$

as well as the Schouten identity (given in equation (4.47) of [46])

$$F_{\mu\rho}\tilde{F}_{\nu}{}^{\rho} = \tilde{F}_{\mu\rho}F_{\nu}{}^{\rho} = \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}\tilde{F}^{\rho\sigma} \ . \tag{2.94}$$

We will also use Maxwell's equations and the Bianchi identity in the form

$$\nabla^{\mu}F_{\mu\nu} = 0 \ , \quad \nabla^{\mu}\tilde{F}_{\mu\nu} = 0 \ , \tag{2.95}$$

and the gravitational Bianchi identity

$$R_{\mu[\nu\rho\sigma]} = 0 \ . \tag{2.96}$$

In addition, we will make extensive use of gamma matrix technology in 4D, using the conventions from [46]. In particular, the identity

$$\gamma_{\mu\nu\rho\sigma} = -i\gamma_5\varepsilon_{\mu\nu\rho\sigma} \tag{2.97}$$

will be very useful. Lastly, we can ignore total derivative terms in heat kernel coefficients and so we freely integrate by parts. For example, we find that (up to a total derivative)

$$\begin{aligned}
(\nabla_{\rho}F_{\mu\nu})(\nabla^{\rho}F^{\mu\nu}) &= 2(\nabla_{\rho}F_{\mu\nu})(\nabla^{\nu}F^{\mu\rho}) \\
&= -2(\nabla^{\nu}\nabla_{\rho}F_{\mu\nu})F^{\mu\rho} \\
&= -2([\nabla_{\nu},\nabla_{\rho}]F^{\mu\nu})F_{\mu}{}^{\rho} \ ,
\end{aligned} \tag{2.98}$$

where equality comes from the Bianchi identity, integration by parts, and Maxwell's equations, respectively. The covariant derivative commutator acting on a rank-2 tensor then gives

$$(\nabla_\rho F_{\mu\nu})(\nabla^\rho F^{\mu\nu}) = -2R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \ . \tag{2.99}$$

We note that (2.99) also holds if we replace $F_{\mu\nu}$ with the dual field strength $\tilde{F}_{\mu\nu}$.

2.4.1 Elementary Examples

To see how the heat kernel coefficient calculations work in practice, we will do some explicit calculations for a few elementary examples. The methods used naturally generalize for the more complicated interactions analyzed in the following subsections.

Free Scalar Field

The Lagrangian for a free scalar field with mass m is

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 \ . \tag{2.100}$$

The ordinary derivative ∂_μ is the same as the covariant derivative ∇_μ when acting on scalar fields so we can integrate by parts freely. In the language of section 2.2.4 the differential operator Λ for this theory is

$$\Lambda = -\Box + m^2 \ , \tag{2.101}$$

where Λ acts on the scalar field ϕ. There are no terms in (2.101) linear in derivatives, so the matrices I, ω_μ, and E are

$$I = 1 \ , \quad \omega_\mu = 0 \ , \quad E = -m^2 \ . \tag{2.102}$$

The commutator of two covariant derivatives vanish when acting on scalar fields, and so the curvature is zero

$$\Omega_{\mu\nu} = 0 \ . \tag{2.103}$$

Inserting this data into (2.36) we find the Seeley-DeWitt coefficients for a free scalar field to be

$$\begin{aligned}
(4\pi)^2 a_0(x) &= 1 \ , \\
(4\pi)^2 a_2(x) &= -m^2 \ , \\
(4\pi)^2 a_4(x) &= \frac{1}{2}m^4 + \frac{1}{180}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right) \ .
\end{aligned} \tag{2.104}$$

Free Spinor Field

The Lagrangian for a free Dirac spinor ψ with a (real) mass m in four dimensions is

$$\mathcal{L} = \bar{\psi}\left(\gamma^{\mu}\nabla_{\mu} + m\right)\psi \ . \tag{2.105}$$

The gamma matrices γ_μ satisfy the standard commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{I}_4 \ , \tag{2.106}$$

where $\mathbb{I}_4$ is the identity matrix in our Clifford algebra, which we may suppress for notational simplicity and re-introduce when needed. The action consists of a first-order Dirac-type differential operator $\hat{H} \equiv \gamma^\mu\nabla_\mu + m$ acting on spinors. As is standard procedure [42, 48], we can define the determinant of a Dirac operator $\hat{H}$ as the square root of the determinant of $\hat{H}\hat{H}^\dagger$:

$$\log\det\hat{H} = \log\det\hat{H}^\dagger = \frac{1}{2}\log\det\hat{H}\hat{H}^\dagger \ . \tag{2.107}$$

It is therefore sufficient to compute the heat kernel of $\hat{H}\hat{H}^\dagger$.

Let us now assume that our spacetime is even-dimensional and has Euclidean signature, in which case our gamma matrices are Hermitian, $\gamma_\mu^\dagger = \gamma_\mu$. With this choice, the operator $\gamma^\mu\nabla_\mu$ is anti-Hermitian, $(\gamma^\mu\nabla_\mu)^\dagger = -\gamma^\mu\nabla_\mu$, and hence we find that the relevant second-order differential operator acting on ψ is

$$\Lambda = \hat{H}\hat{H}^\dagger = -\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu + m^2 \ . \tag{2.108}$$

The covariant derivative acts on ψ by

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{4}\gamma_{ab}\omega_\mu^{ab}\psi \ , \tag{2.109}$$

for the spin connection ω_μ^{ab}, where $\mu, \nu, \ldots$ are curved space indices, $a, b, \ldots$ are flat space indices, γ^μ is shorthand for $\gamma^a e_a^\mu$, and e_a^μ is the vierbein. Gamma matrix commutation relations give

$$[\nabla_\mu, \nabla_\nu]\psi = \frac{1}{4}\gamma^{ab}R_{\mu\nu ab}\psi = \frac{1}{4}\gamma^{\rho\sigma}R_{\mu\nu\rho\sigma}\psi \ . \tag{2.110}$$

By breaking up $\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu$ into its symmetric and antisymmetric parts and using (2.110), we find that

$$\begin{aligned}\bar{\psi}\Lambda\psi &= \bar{\psi}\left(-\frac{1}{2}\gamma^\rho\gamma^\sigma\{\nabla_\rho,\nabla_\sigma\} - \frac{1}{2}\gamma^{\rho\sigma}[\nabla_\rho,\nabla_\sigma] + m^2\right)\psi \\ &= \bar{\psi}\left(-\Box - \frac{1}{8}\gamma^{\mu\nu}\gamma^{\rho\sigma}R_{\mu\nu\rho\sigma} + m^2\right)\psi \\ &= \bar{\psi}\left(-\Box + \frac{1}{4}R + m^2\right)\psi \ ,\end{aligned} \tag{2.111}$$

where equality in the last line comes from using gamma matrix commutation relations and the Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$.

Λ defined in (2.111) is of the Laplace-type form required in (2.32), and we identify I, ω_μ, E, and $\Omega_{\mu\nu}$ as

$$I = \mathbb{I}_4 \ , \quad \omega_\mu = 0 \ , \quad E = -m^2\mathbb{I}_4 \ , \quad \Omega_{\mu\nu} = \frac{1}{4}\gamma^{\rho\sigma}R_{\mu\nu\rho\sigma} \ . \tag{2.112}$$

We can use gamma matrix identities to compute the traces of I, E, E^2, and $\Omega_{\mu\nu}\Omega^{\mu\nu}$ needed for our heat kernel coefficients. The result is:

$$\begin{aligned}(4\pi)^2 a_0(x) &= -4 \ , \\ (4\pi)^2 a_2(x) &= 4m^2 \ , \\ (4\pi)^2 a_4(x) &= -\left[2m^4 + \frac{1}{360}\left(-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8R_{\mu\nu}R^{\mu\nu}\right)\right] \ .\end{aligned} \tag{2.113}$$

The overall minus sign on each of these heat kernel coefficients is put in by hand to account for fermion statistics. We also note that this derivation assumed that ψ was a Dirac spinor. Weyl and Majorana spinors have half the degrees of freedom of Dirac spinors, and so we must divide these results by two if we want the heat kernel coefficients for Majorana or Weyl spinors.

This derivation was done in a Euclidean spacetime in order to take advantage of Hermitian gamma matrices. For Lorentzian spacetimes the spinor conjugation includes γ^0, which has the effect of changing the boundary conditions on the conjugate spinor. We are considering manifolds without boundary, so this change in boundary conditions is irrelevant and our results naturally generalize to Lorentzian spacetimes as well [42].

2.4.2 Hyper Multiplet

A single $\mathcal{N} = 2$ hyper multiplet contains two Majorana fermions and four real scalars. The scalars are minimally coupled to gravity and massless, so we can use the free scalar result

(2.104) with $m = 0$:

$$
\begin{aligned}
&(4\pi)^2 a_0^{\mathrm{H},b}(x) = 4 \ , \\
&(4\pi)^2 a_2^{\mathrm{H},b}(x) = 0 \ , \\
&(4\pi)^2 a_4^{\mathrm{H},b}(x) = \frac{1}{45}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right) \ .
\end{aligned}
\tag{2.114}
$$

The Lagrangian for the hyper fermions (2.91) mixes left-handed and right-handed fermions. We want to put this Lagrangian in the form of a diagonal Dirac-type Lagrangian to use the procedure outlined earlier for fermionic heat kernels. We define the spinor

$$
\psi_A \equiv P_R \zeta_A + P_L \zeta^A \ , \tag{2.115}
$$

where $P_L = \frac{1}{2}(1+\gamma_5)$, $P_R = \frac{1}{2}(1-\gamma_5)$. The hyper fermion Lagrangian can then be rewritten as

$$
\mathcal{L}_{\mathrm{hyper}} = \bar{\psi}_A \left(-\gamma^\mu \nabla_\mu \delta_{AB} + \frac{1}{4} F_{\mu\nu}\gamma^{\mu\nu}\epsilon_{AB}\right)\psi_B \ . \tag{2.116}
$$

Though we have lost information about the chirality of the spinors, this Lagrangian is now in the form of (2.105). That is, we can express the Lagrangian as

$$
\mathcal{L}_{\mathrm{hyper}} = \bar{\psi}_A \hat{H}_{AB} \psi_B \ , \tag{2.117}
$$

where $\hat{H}_{AB}$ is a Dirac operator acting on the spinors ψ_A by

$$
\hat{H}_{AB} = -\gamma^\mu \nabla_\mu \delta_{AB} + \frac{1}{4} F_{\mu\nu}\gamma^{\mu\nu}\epsilon_{AB} \ . \tag{2.118}
$$

As with the free spinor field we now continue to Euclidean space, giving us Hermitian gamma matrices $\gamma_\mu^\dagger = \gamma_\mu$ and the spinor conjugate $\bar{\psi}_A = \psi_A^\dagger$. We can also choose the background field $F^{\mu\nu}$ to be real. With these conventions, the Hermitian conjugate of $\hat{H}_{AB}$ is

$$
\hat{H}_{AB}^\dagger = \gamma^\mu \nabla_\mu \delta_{AB} + \frac{1}{4} F_{\mu\nu}\gamma^{\mu\nu}\epsilon_{AB} \ . \tag{2.119}
$$

The relevant Laplace-type operator that we will compute the heat kernel of is

$$
\begin{aligned}
\Lambda_{AB} &= \hat{H}_{AC}\hat{H}^\dagger_{CB} \\
&= \left(-\gamma^\mu\nabla_\mu\delta_{AC} + \frac{1}{4}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_{AC}\right)\left(\gamma^\rho\nabla_\rho\delta_{CB} + \frac{1}{4}F_{\rho\sigma}\gamma^{\rho\sigma}\epsilon_{CB}\right) \\
&= -\left(\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu\delta_{AB} - \frac{1}{4}F_{\mu\nu}\gamma^{\mu\nu}\gamma^\rho\nabla_\rho\epsilon_{AB} + \frac{1}{4}\gamma^\rho\nabla_\rho F_{\mu\nu}\gamma^{\mu\nu}\epsilon_{AB} + \frac{1}{16}F_{\mu\nu}F_{\rho\sigma}\gamma^{\mu\nu}\gamma^{\rho\sigma}\delta_{AB}\right) \\
&= -\left[\Box\delta_{AB} + F_{\rho\mu}\gamma^\mu\nabla^\rho\epsilon_{AB} + \left(\frac{1}{8}F_{\mu\nu}\tilde{F}^{\mu\nu}\gamma_5 - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}\right)\delta_{AB}\right] ,
\end{aligned}
\tag{2.120}
$$

where equality in the last line comes from using (2.111) and (2.97), as well as noting that

$$
\gamma^\rho(\nabla_\rho F_{\mu\nu})\gamma^{\mu\nu} = (\nabla_\rho F_{\mu\nu})\gamma^{\mu\nu}\gamma^\rho = 0 , \tag{2.121}
$$

by the Maxwell-Bianchi equations (2.95).

From the form of Λ in (2.120) we identify the matrices I, ω_μ, and P as

$$
I_{AB} = \mathbb{I}_4\delta_{AB} , \quad (\omega_\mu)_{AB} = \left(\frac{1}{2}F_{\mu\nu}\gamma^\nu\right)\epsilon_{AB} , \quad P_{AB} = \frac{1}{8}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\gamma_5 - F_{\mu\nu}F^{\mu\nu}\right)\delta_{AB} . \tag{2.122}
$$

By Maxwell's equations $\big(D^\mu(\omega_\mu)_{AB}\big) = 0$ so E becomes

$$
E_{AB} = P_{AB} - (\omega_\mu)_{AC}(\omega^\mu)_{CB} = \frac{1}{8}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\gamma_5 + F_{\mu\nu}F^{\mu\nu}\right)\delta_{AB} . \tag{2.123}
$$

The curvature $\Omega_{\mu\nu}$ corresponding to the effective covariant derivative $\mathcal{D}_\mu = \nabla_\mu + \omega_\mu$ is

$$
\begin{aligned}
(\Omega_{\mu\nu})_{AB} &= [(\mathcal{D}_\mu)_{AC}, (\mathcal{D}_\nu)_{CB}] \\
&= [\nabla_\mu, \nabla_\nu]\delta_{AB} + \big(\nabla_\mu(\omega_\nu)_{AB}\big) - \big(\nabla_\nu(\omega_\mu)_{AB}\big) + [(\omega_\mu)_{AC}, (\omega_\nu)_{CB}]
\end{aligned}
\tag{2.124}
$$

We can use our expressions for $[\nabla_\mu, \nabla_\nu]$ and ω_μ from (2.110) and (2.122), respectively, giving

$$
(\Omega_{\mu\nu})_{AB} = \left(\frac{1}{4}R_{\mu\nu\rho\sigma} - \frac{1}{2}F_{\mu\rho}F_{\nu\sigma}\right)\gamma^{\rho\sigma}\,\delta_{AB} + \left(-\frac{1}{2}\gamma^\rho\nabla_\rho F_{\mu\nu}\right)\epsilon_{AB} . \tag{2.125}
$$

We can now compute all of the traces necessary for the Seeley-DeWitt coefficients. These traces are tedious but straightforward to compute, so we will simply quote the results, noting that we use (2.93) and (2.99) to simplify when possible.

$$\begin{aligned} \mathrm{Tr}\ I &= 8\ , \\ \mathrm{Tr}\ E &= F_{\mu\nu}F^{\mu\nu}\ , \\ \mathrm{Tr}\ E^2 &= \frac{1}{8}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{1}{8}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\ , \\ \mathrm{Tr}\ \Omega_{\mu\nu}\Omega^{\mu\nu} &= -R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 16R_{\mu\nu}R^{\mu\nu} - \frac{3}{2}(F_{\mu\nu}F^{\mu\nu})^2\ . \end{aligned} \tag{2.126}$$

We can use these quantities with (2.36) to calculate the Seeley-DeWitt coefficients for the hyper fermions, making sure to add an overall factor of $-1/2$ to account for fermion statistics and the Majorana condition. The result is

$$\begin{aligned} (4\pi)^2 a_0^{\mathrm{H},f}(x) &= -4\ , \\ (4\pi)^2 a_2^{\mathrm{H},f}(x) &= -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\ , \\ (4\pi)^2 a_4^{\mathrm{H},f}(x) &= -\frac{1}{360}\left(-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 232R_{\mu\nu}R^{\mu\nu} - \frac{45}{4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{45}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right)\ . \end{aligned} \tag{2.127}$$

Adding up the bosonic part (2.114) and the fermionic part (2.127), the full hyper multiplet heat kernel coefficients are

$$\begin{aligned} (4\pi)^2 a_0^{\mathrm{H}}(x) &= 0\ , \\ (4\pi)^2 a_2^{\mathrm{H}}(x) &= -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\ , \\ (4\pi)^2 a_4^{\mathrm{H}}(x) &= \frac{1}{24}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 16R_{\mu\nu}R^{\mu\nu} + \frac{3}{4}(F_{\mu\nu}F^{\mu\nu})^2 - \frac{3}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right)\ . \end{aligned} \tag{2.128}$$

The $a_0(x)$ coefficient vanishes because any full multiplet has an equal number of bosonic and fermionic degrees of freedom.

2.4.3 Vector Multiplet

The $\mathcal{N} = 2$ vector multiplet consists of one vector field, two gauginos, and one complex scalar. The gauginos are massless Majorana fermions that couple minimally to gravity, and thus we can use (2.113) to find the vector multiplet fermionic heat kernel coefficients

$$\begin{aligned} (4\pi)^2 a_0^{\mathrm{V},f}(x) &= -4\ , \\ (4\pi)^2 a_2^{\mathrm{V},f}(x) &= 0\ , \\ (4\pi)^2 a_4^{\mathrm{V},f}(x) &= -\frac{1}{360}\left(-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8R_{\mu\nu}R^{\mu\nu}\right)\ . \end{aligned} \tag{2.129}$$

The equations of motion for the bosonic content of the vector multiplet are given in (2.86). We split the complex scalar z into its real and imaginary parts by

$$z = x - iy \ , \tag{2.130}$$

where x is a real pseudoscalar field and y is a real scalar field. The bosonic Lagrangian for these fields consistent with the equations of motion (2.86) is

$$\begin{aligned}\mathcal{L}_b = &-\frac{1}{8}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}(\nabla_\mu y)(\nabla^\mu y) + \frac{1}{4}y f_{\mu\nu}F^{\mu\nu} - \frac{1}{16}y^2 F_{\mu\nu}F^{\mu\nu} \\ &- \frac{1}{4}(\nabla_\mu x)(\nabla^\mu x) + \frac{i}{4}x f_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{16}x^2\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{i}{8}xyF_{\mu\nu}\tilde{F}^{\mu\nu} \ ,\end{aligned} \tag{2.131}$$

where $f_{\mu\nu} = \nabla_\mu a_\nu - \nabla_\nu a_\mu$ is the fluctuation of the background field strength $F_{\mu\nu}$. As a consistency check we note that on $\mathrm{AdS}_2 \times S^2$ (where $F_{\mu\nu}\tilde{F}^{\mu\nu} = 0$) this action is consistent with equation (6.4) of [14].

We choose the Lorenz gauge $\nabla_\mu a^\mu = 0$ by adding a gauge-fixing term to the Lagrangian

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{4}(\nabla_\mu a^\mu)^2 \ . \tag{2.132}$$

This gauge-fixing will introduce two anti-commuting scalar ghosts that will contribute to the heat kernel with an overall minus sign. We denote $\{\phi_m\} = \{y, x, a_\mu\}$ to be the bosonic field fluctuations. Then, we can use Maxwell's equations and the Bianchi identity to rewrite our action in the Hermitian form required in (2.32), up to a total derivative, as

$$\mathcal{S} = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,\phi_n\Lambda^n_m\phi_m \ , \tag{2.133}$$

where

$$\begin{aligned}-\phi_n\Lambda^n_m\phi_m = &\, a_\mu\left(\Box g^{\mu\nu} - R^{\mu\nu}\right)a_\nu + y\left(\Box - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)y + x\left(\Box + \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right)x \\ &+ y\left(F^{\rho\nu}\nabla_\rho\right)a_\nu + a_\mu\left(F^{\mu\rho}\nabla_\rho\right)y + x\left(i\tilde{F}^{\rho\nu}\nabla_\rho\right)a_\nu + a_\mu\left(i\tilde{F}^{\mu\rho}\nabla_\rho\right)x \\ &+ y\left(-\frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right)x + x\left(-\frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right)y \ .\end{aligned} \tag{2.134}$$

From (2.134) we can read off the matrices P and ω_ρ. And, since all of the terms in ω_ρ depend on $F_{\mu\rho}$ or $\tilde{F}_{\mu\rho}$, $(\nabla^\rho\omega_\rho) = 0$ due to Maxwell's equations and the Bianchi identities. E thus becomes

$$\phi_n E^n_m \phi^m = \phi_n \left(P - \omega^\rho \omega_\rho\right)^n_m \phi^m = a_\mu \left(-R^{\mu\nu} + \frac{1}{4} F^{\mu\rho} F^\nu{}_\rho - \frac{1}{4} \tilde{F}^{\mu\rho} \tilde{F}^\nu{}_\rho \right) a_\nu \ . \tag{2.135}$$

The lack of any terms involving x or y in (2.135) was not a priori obvious but a consequence of how terms in the action that coupled x and y to the background conspired to cancel.

There are six off-shell bosonic degrees of freedom for the fields $\{\phi_n\}$: four from the vector a_μ, and two from the scalars x and y, giving Tr $I = 6$. From (2.135) we compute the traces

$$\begin{aligned} \text{Tr } E &= \frac{1}{4}\left(F_{\mu\nu} F^{\mu\nu} - \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right) \ , \\ \text{Tr } E^2 &= R_{\mu\nu} R^{\mu\nu} - \frac{1}{2} R_{\mu\nu} F^{\mu\rho} F^\nu{}_\rho + \frac{1}{2} R_{\mu\nu} \tilde{F}^{\mu\rho} \tilde{F}^\nu{}_\rho \\ &\quad + \frac{1}{16} (F^{\mu\rho} F_{\nu\rho})(F_{\mu\sigma} F^{\nu\sigma}) + \frac{1}{16} (\tilde{F}^{\mu\rho} \tilde{F}_{\nu\rho})(\tilde{F}_{\mu\sigma} \tilde{F}^{\nu\sigma}) - \frac{1}{8} (F^{\mu\rho} F_{\nu\rho})(\tilde{F}_{\mu\sigma} \tilde{F}^{\nu\sigma}) \ . \end{aligned} \tag{2.136}$$

In order to compute the curvature we expand the commutator in (2.35)

$$(\Omega_{\rho\sigma})^n_m = [\nabla_\rho, \nabla_\sigma]^n_m + 2(\nabla_{[\rho} \omega_{\sigma]})^n_m + [\omega_\rho, \omega_\sigma]^n_m \ . \tag{2.137}$$

The covariant derivative commutes when acting on scalars, but not for vectors, and so the first term in (2.137) is

$$\phi_n [\nabla_\rho, \nabla_\sigma]^n_m \phi^m = a_\mu [\nabla_\rho, \nabla_\sigma] a^\mu = a_\mu \left(R^\mu{}_{\nu\rho\sigma} \right) a^\nu \ . \tag{2.138}$$

The second term in (2.137) is calculated by applying the covariant derivative to ω_μ. Using the Maxwell-Bianchi equations to simplify we find that

$$\begin{aligned} \phi_n (\nabla_{[\rho} \omega_{\sigma]})^n_m \phi^m = y &\left(-\frac{1}{4} (\nabla_\nu F_{\rho\sigma}) \right) a^\nu + a_\mu \left(\frac{1}{4} (\nabla^\mu F_{\rho\sigma}) \right) y \\ &+ x \left(-\frac{i}{4} (\nabla_\nu \tilde{F}_{\rho\sigma}) \right) a^\nu + a_\mu \left(\frac{i}{4} (\nabla^\mu \tilde{F}_{\rho\sigma}) \right) y \ . \end{aligned} \tag{2.139}$$

The last term in (2.137) is the product of ω_ρ and ω_σ, antisymmetrized in ρ and σ:

$$\phi_n [\omega_\rho, \omega_\sigma]^n_m \phi^m = a_\mu \left(\frac{1}{4} F^\mu{}_{[\rho} F_{\sigma]\nu} - \frac{1}{4} \tilde{F}^\mu{}_{[\rho} \tilde{F}_{\sigma]\nu} \right) a^\nu \ . \tag{2.140}$$

Adding all of these components up, we find that

$$
\begin{aligned}
\phi_n(\Omega_{\rho\sigma})^n_m\phi^m = a_\mu &\left(R^\mu{}_{\nu\rho\sigma} + \frac{1}{4}F^\mu{}_{[\rho}F_{\sigma]\nu} - \frac{1}{4}\tilde{F}^\mu{}_{[\rho}\tilde{F}_{\sigma]\nu}\right) a^\nu \\
&+ y\left(-\frac{1}{2}(\nabla_\nu F_{\rho\sigma})\right) a^\nu + a_\mu\left(\frac{1}{2}(\nabla^\mu F_{\rho\sigma})\right) y \\
&+ x\left(-\frac{i}{2}(\nabla_\nu \tilde{F}_{\rho\sigma})\right) a^\nu + a_\mu\left(\frac{i}{2}(\nabla^\mu \tilde{F}_{\rho\sigma})\right) y \ .
\end{aligned}
\tag{2.141}
$$

Now that we have all of the components of $\Omega_{\rho\sigma}$, it is straightforward to trace over $\Omega_{\rho\sigma}\Omega^{\rho\sigma}$. We will also simplify by using the Maxwell-Bianchi equations and (2.99). The result, up to a total derivative, is

$$
\begin{aligned}
\mathrm{Tr}\ \Omega_{\rho\sigma}\Omega^{\rho\sigma} = &-R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R_{\mu\nu}\left(F^{\mu\rho}F^\nu{}_\rho - \tilde{F}^{\mu\rho}\tilde{F}^\nu{}_\rho\right) \\
&+ \frac{1}{8}\left((F^{\mu\rho}F_{\nu\rho})(F_{\mu\sigma}F^{\nu\sigma}) + (\tilde{F}^{\mu\rho}\tilde{F}_{\nu\rho})(\tilde{F}_{\mu\sigma}\tilde{F}^{\nu\sigma}) - (F_{\mu\nu}F^{\mu\nu})^2 - (\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})^2\right) \\
&+ \frac{1}{4}\left((F_{\mu\nu}\tilde{F}^{\mu\nu})^2 - (F^{\mu\rho}\tilde{F}_{\nu\rho})(\tilde{F}_{\mu\sigma}F^{\nu\sigma})\right) \ .
\end{aligned}
\tag{2.142}
$$

We now have all of the traces needed to calculate the Seeley-DeWitt coefficients for the bosonic fields. However, our gauge-fixing also introduced two scalar ghosts into our system. These ghosts do not interact with any of the bosonic fields and so their corresponding heat kernels are those for two minimally coupled scalars (2.104). If we insert our traces into the coefficient formulas in (2.36) and subtract off the ghost contribution, we find that:

$$
\begin{aligned}
(4\pi)^2 a_0^{\mathrm{V},b}(x) &= 4 \ , \\
(4\pi)^2 a_2^{\mathrm{V},b}(x) &= \frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu} - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right) \ , \\
(4\pi)^2 a_4^{\mathrm{V},b}(x) &= \frac{1}{180}\Bigg[-11R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 86R_{\mu\nu}R^{\mu\nu} - 30R_{\mu\nu}\left(F^{\mu\rho}F^\nu{}_\rho - \tilde{F}^{\mu\rho}\tilde{F}^\nu{}_\rho\right) \\
&\qquad + \frac{15}{2}\left(F^{\mu\rho}F_{\nu\rho}\right)\left(F_{\mu\sigma}F^{\nu\sigma}\right) + \frac{15}{2}\left(\tilde{F}^{\mu\rho}\tilde{F}_{\nu\rho}\right)\left(\tilde{F}_{\mu\sigma}\tilde{F}^{\nu\sigma}\right) \\
&\qquad - \frac{15}{8}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 - \frac{15}{8}\left(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 \\
&\qquad + \frac{15}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 - \frac{45}{4}\left(F^{\mu\rho}F_{\mu\rho}\right)\left(\tilde{F}_{\mu\sigma}\tilde{F}^{\mu\sigma}\right) - \frac{15}{4}\left(F^{\mu\rho}\tilde{F}_{\nu\rho}\right)\left(\tilde{F}_{\mu\sigma}F^{\nu\sigma}\right)\Bigg] \ .
\end{aligned}
\tag{2.143}
$$

Adding up the fermionic (2.129) and bosonic (2.143) contributions and using the Schouten identity (2.94) to simplify, the full vector multiplet heat kernel is

$$
\begin{aligned}
(4\pi)^2 a_0^{\mathrm{V}}(x) &= 0 \ , \\
(4\pi)^2 a_2^{\mathrm{V}}(x) &= \frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu} - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right) \ , \\
(4\pi)^2 a_4^{\mathrm{V}}(x) &= \frac{1}{24}\Big[- R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 12 R_{\mu\nu}R^{\mu\nu} - 4R_{\mu\nu}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \tilde{F}^{\mu\rho}\tilde{F}^{\nu}{}_{\rho}\right) \\
&\qquad + \left(F^{\mu\rho}F_{\nu\rho}\right)\left(F_{\mu\sigma}F^{\nu\sigma}\right) + \left(\tilde{F}^{\mu\rho}\tilde{F}_{\nu\rho}\right)\left(\tilde{F}_{\mu\sigma}\tilde{F}^{\nu\sigma}\right) \\
&\qquad - \frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 - \frac{1}{4}\left(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 \Big] \ .
\end{aligned}
\tag{2.144}
$$

Our result for $a_2^V(x)$ disagrees with [43] in the special case of BPS black holes. However, $a_2(x)$ determines the quadratic divergences and encodes the renormalization of the Newton constant. These quadratic divergences are scheme-dependent and unphysical. We will record our results for $a_2(x)$ in our heat kernel regularization scheme for the sake of completion.

2.4.4 Gravity Multiplet: Fermions

The gravity multiplet consists of the graviton, two Majorana gravitini, and the graviphoton. We rewrite the Lagrangian for these gravitini (2.89) by using (2.97) to express $\gamma^{\mu\nu\rho\sigma}$ in terms of γ_5 and the Levi-Civita symbol, resulting in

$$
\mathcal{L}_{\text{gravitini}} = -\frac{1}{2\kappa^2}\bar{\Psi}_{A\mu}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_{A\rho} + \frac{1}{4\kappa^2}\bar{\Psi}_{A\mu}\left(F^{\mu\nu} + \gamma_5\tilde{F}^{\mu\nu}\right)\epsilon_{AB}\Psi_{B\nu} \ , \tag{2.145}
$$

where $A, B = 1, 2$ enumerates the two gravitini species. The covariant derivative acts on the gravitino field Ψ_A^ρ by

$$
\nabla_\mu \Psi_A^\rho = \partial_\mu \Psi_A^\rho + \frac{1}{4}\gamma_{ab}\omega_\mu^{ab}\Psi_A^\rho + \Gamma^\rho_{\mu\nu}\Psi_A^\nu \ , \tag{2.146}
$$

for the spin connection ω_μ^{ab} and the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$. The commutator $[\nabla_\mu, \nabla_\nu]$ acting on Ψ_A^ρ will be the sum of the spin and Riemann commutators

$$
\bar{\Psi}_{A\rho}[\nabla_\mu, \nabla_\nu]\Psi_A^\rho = \bar{\Psi}_{A\rho}\left(\frac{1}{4}g_{\rho\sigma}\gamma^{\alpha\beta}R_{\mu\nu\alpha\beta} + R_{\mu\nu}{}^{\rho\sigma}\right)\delta_{AB}\Psi_{B\sigma} \ . \tag{2.147}
$$

Tbe gravitini Lagrangian (2.145) is invariant under the SUSY transformation

$$
\delta\Psi_{A\mu} = \left(\delta_{AB}\nabla_\mu - \frac{1}{8}\epsilon_{AB}\gamma^{\rho\sigma}F_{\rho\sigma}\gamma_\mu\right)\epsilon_B \ , \tag{2.148}
$$

for a spinor ϵ_B. This SUSY transformation acts as a gauge symmetry.

We need the kinetic term of the gravitini to be in Dirac form in order for it to square to a minimal operator. We use the procedure outlined in [49] and gauge-fix our action in such a way that, when paired with a suitable corresponding field redefinition, the kinetic term becomes Dirac-type. In particular, we choose the harmonic gauge for our gravitini $\gamma^\mu \Psi_{A\mu} = 0$ by adding the gauge-fixing term

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{4\kappa^2}(\bar{\Psi}_{A\mu}\gamma^\mu)\gamma^\nu \nabla_\nu (\gamma^\rho \Psi_{A\rho}) \ . \tag{2.149}$$

Then, we consider the field redefinition

$$\Phi_{A\mu} = \Psi_{A\mu} - \frac{1}{2}\gamma_\mu \gamma^\nu \Psi_{A\nu} \ . \tag{2.150}$$

Using gamma matrix identities and (2.97), it is easily verified that

$$\begin{aligned}
\bar{\Phi}_{A\mu}\gamma^\nu \nabla_\nu \Phi^\mu_A &= \bar{\Psi}_{A\mu}\left(\gamma^{\mu\nu\rho}\nabla_\nu - \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho\nabla_\nu\right)\Psi_{A\rho} \ , \\
\bar{\Phi}_{A\mu}F^{\mu\nu}\Phi_{B\nu} &= \frac{1}{2}\bar{\Psi}_{A\mu}\left(F^{\mu\nu} + \gamma_5 \tilde{F}^{\mu\nu} + \frac{1}{2}\gamma^{\rho\sigma}F_{\rho\sigma}g^{\mu\nu}\right)\Psi_{B\nu} \ , \\
\bar{\Phi}_{A\mu}\gamma_5 \tilde{F}^{\mu\nu}\Phi_{B\nu} &= \frac{1}{2}\bar{\Psi}_{A\mu}\left(F^{\mu\nu} + \gamma_5 \tilde{F}^{\mu\nu} - \frac{1}{2}\gamma^{\rho\sigma}F_{\rho\sigma}g^{\mu\nu}\right)\Psi_{B\nu} \ .
\end{aligned} \tag{2.151}$$

Therefore our full action (including gauge-fixing) can be written as

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x \sqrt{-g}\, \bar{\Phi}_{A\mu}\hat{H}^{\mu\nu}_{AB}\Phi_{B\nu} \ , \tag{2.152}$$

where

$$\hat{H}^{\mu\nu}_{AB} = -\gamma^\rho \nabla_\rho g^{\mu\nu}\delta_{AB} + \frac{1}{2}\left(F^{\mu\nu} + \gamma_5 \tilde{F}^{\mu\nu}\right)\epsilon_{AB} \ . \tag{2.153}$$

Our action is now in the Dirac form required for our heat kernel methods. We note that the overall normalization in (2.149) was chosen to enforce this; any other choice would result in an action whose square is non-minimal [50].

As with the hyper fermions we now continue to Euclidean space, giving Hermitian gamma matrices. The gravitino conjugate is $\bar{\Phi}_{A\mu} = \Phi^\dagger_{A\mu}$, and we will again choose $F^{\mu\nu}$ to be real. The Hermitian conjugate of $\hat{H}$ is

$$\hat{H}^{\mu\nu\dagger}_{AB} = \gamma^\rho \nabla_\rho g^{\mu\nu}\delta_{AB} + \frac{1}{2}\left(F^{\mu\nu} - \gamma_5 \tilde{F}^{\mu\nu}\right)\epsilon_{AB} \ . \tag{2.154}$$

The relevant Laplace-type operator that we will calculate the heat kernel of is

$$\begin{aligned}\Lambda^{\mu\nu}_{AB} &= \hat{H}^{\mu\lambda\dagger}_{AC}\hat{H}_{\lambda}{}^{\nu}{}_{CB} \\ &= -\gamma^\rho\gamma^\sigma\nabla_\rho\nabla_\sigma g^{\mu\nu}\delta_{AB} + \frac{1}{4}(F^{\mu\lambda}+\gamma_5\tilde{F}^{\mu\lambda})(F_\lambda{}^\nu - \gamma_5\tilde{F}_\lambda{}^\nu)\delta_{AB} \\ &\quad -\frac{1}{2}\gamma^\rho\nabla_\rho(F^{\mu\nu}-\gamma_5\tilde{F}^{\mu\nu})\epsilon_{AB} + \frac{1}{2}(F^{\mu\nu}+\gamma_5\tilde{F}^{\mu\nu})\gamma^\rho\nabla_\rho\epsilon_{AB}\ .\end{aligned} \tag{2.155}$$

As with the hyper fermions, we can break the two-derivative term $\gamma^\rho\gamma^\sigma\nabla_\rho\nabla_\sigma$ into its symmetric and anti-symmetric parts and use the commutator given in (2.147). We will also use the Schouten identity (2.94) and gamma matrix commutation relations to simplify this expression. The result is

$$\begin{aligned}\Lambda^{\mu\nu}_{AB} = -\bigg(&\Box g^{\mu\nu}\delta_{AB} + \frac{1}{2}\gamma^{\rho\sigma}R_{\rho\sigma}{}^{\mu\nu}\delta_{AB} + \frac{1}{4}(F^{\mu\lambda}F_\lambda{}^\nu - \tilde{F}^{\mu\lambda}\tilde{F}_\lambda{}^\nu)\delta_{AB} \\ &+\frac{1}{2}\gamma^\rho(\nabla_\rho F^{\mu\nu})\epsilon_{AB} - \frac{1}{2}\gamma^\rho\gamma_5(\nabla_\rho\tilde{F}^{\mu\nu})\epsilon_{AB}\bigg)\ .\end{aligned} \tag{2.156}$$

In (2.156) there is no term linear in derivatives. This corresponds to $\omega_\mu = 0$, and so the matrices I and E are

$$\begin{aligned}I^{\mu\nu}_{AB} &= \mathbb{I}_4 g^{\mu\nu}\delta_{AB}\ , \\ E^{\mu\nu}_{AB} &= \left(\frac{1}{2}\gamma^{\rho\sigma}R_{\rho\sigma}{}^{\mu\nu} + \frac{1}{4}F^{\mu\lambda}F_\lambda{}^\nu - \frac{1}{4}\tilde{F}^{\mu\lambda}\tilde{F}_\lambda{}^\nu\right)\delta_{AB} \\ &\quad + \left(\frac{1}{2}\gamma^\rho(\nabla_\rho F^{\mu\nu}) - \frac{1}{2}\gamma^\rho(\nabla_\rho\tilde{F}^{\mu\nu})\gamma_5\right)\epsilon_{AB}\ .\end{aligned} \tag{2.157}$$

Since $\omega_\mu = 0$, the curvature $\Omega_{\mu\nu}$ of the connection $\mathcal{D}_\mu$ is given by the commutator in (2.147)

$$(\Omega_{\rho\sigma})^{\mu\nu}_{AB} = \left(\frac{1}{4}\gamma^{\alpha\beta}R_{\rho\sigma\alpha\beta}g^{\mu\nu} + R_{\rho\sigma}{}^{\mu\nu}\right)\delta_{AB}\ . \tag{2.158}$$

The relevant traces for our heat kernel coefficients are

$$\begin{aligned}\mathrm{Tr}\ I &= 32\ , \\ \mathrm{Tr}\ E &= -2F_{\mu\nu}F^{\mu\nu} + 2\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\ , \\ \mathrm{Tr}\ E^2 &= 4R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{1}{2}(F^{\mu\rho}F_{\nu\rho} - \tilde{F}^{\mu\rho}\tilde{F}_{\nu\rho})(F_{\mu\sigma}F^{\nu\sigma} - \tilde{F}_{\mu\sigma}\tilde{F}^{\nu\sigma}) \\ &\quad + 2(\nabla^\rho F^{\mu\nu})(\nabla_\rho F_{\mu\nu}) - 2(\nabla^\rho\tilde{F}^{\mu\nu})(\nabla_\rho\tilde{F}_{\mu\nu})\ , \\ \mathrm{Tr}\ \Omega_{\rho\sigma}\Omega^{\rho\sigma} &= -12R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\ .\end{aligned} \tag{2.159}$$

We can now calculate the Seeley-DeWitt coefficients (2.36) for the gravitini in the gravity multiplet, making sure to add an overall factor of $-1/2$ to account for fermion statistics and the Majorana condition. We will also simplify the result by using (2.99) to rewrite $(\nabla_\rho F_{\mu\nu})^2$ and $(\nabla_\rho \tilde{F}_{\mu\nu})^2$ in terms of the Riemann tensor and Ricci tensor contracted with these field strengths. We end up with

$$\begin{aligned}
(4\pi)^2 a_0^{\text{gravitini}}(x) &= -16 \ , \\
(4\pi)^2 a_2^{\text{gravitini}}(x) &= F_{\mu\nu}F^{\mu\nu} - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \ , \\
(4\pi)^2 a_4^{\text{gravitini}}(x) &= -\frac{1}{360}\Big(212 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 32 R_{\mu\nu}R^{\mu\nu} - 360 R_{\mu\nu}(F^{\mu\rho}F^{\nu}{}_{\rho} - \tilde{F}^{\mu\rho}\tilde{F}^{\nu}{}_{\rho}) \\
&\qquad +180 R_{\mu\nu\rho\sigma}(F^{\mu\nu}F^{\rho\sigma} - \tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}) \\
&\qquad +45(F^{\mu\rho}F_{\nu\rho} - \tilde{F}^{\mu\rho}\tilde{F}_{\nu\rho})(F_{\mu\sigma}F^{\nu\sigma} - \tilde{F}_{\mu\sigma}\tilde{F}^{\nu\sigma})\Big) \ .
\end{aligned} \tag{2.160}$$

As noted in [14], the particular choice of gauge made in (2.149) induces the ghost Lagrangian

$$\mathcal{L}_{\text{ghost}} = \bar{\tilde{b}}_A \gamma^\mu \nabla_\mu \tilde{c}_A + \bar{\tilde{e}}_A \gamma^\mu \nabla_\mu \tilde{e}_A \ , \tag{2.161}$$

where $\tilde{b}_A$, $\tilde{c}_A$, and $\tilde{e}_A$ are fermionic ghosts, with the same species index $A = 1, 2$ as the gravitinos. Since there are six different species of these minimally coupled Majorana fermions, their contribution to the fermionic heat kernel will be -6 times the free spin-1/2 heat kernel (2.113). The net fermionic heat kernel coefficients, including gauge-fixing and ghosts, are $a_{2n}^{\text{grav},f}(x) = a_{2n}^{\text{gravitini}}(x) - 6a_{2n}^{1/2}(x)$. The final Seeley-DeWitt coefficients for the fermionic content of the gravity multiplet are thus

$$\begin{aligned}
(4\pi)^2 a_0^{\text{grav},f}(x) &= -4 \ , \\
(4\pi)^2 a_2^{\text{grav},f}(x) &= F_{\mu\nu}F^{\mu\nu} - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \ , \\
(4\pi)^2 a_4^{\text{grav},f}(x) &= -\frac{1}{360}\Big(233 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8 R_{\mu\nu}R^{\mu\nu} - 360 R_{\mu\nu}(F^{\mu\rho}F^{\nu}{}_{\rho} - \tilde{F}^{\mu\rho}\tilde{F}^{\nu}{}_{\rho}) \\
&\qquad +180 R_{\mu\nu\rho\sigma}(F^{\mu\nu}F^{\rho\sigma} - \tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}) \\
&\qquad +45(F^{\mu\rho}F_{\nu\rho} - \tilde{F}^{\mu\rho}\tilde{F}_{\nu\rho})(F_{\mu\sigma}F^{\nu\sigma} - \tilde{F}_{\mu\sigma}\tilde{F}^{\nu\sigma})\Big) \ .
\end{aligned} \tag{2.162}$$

2.4.5 Gravity Multiplet: Bosons

As discussed in section 2.3.2, the action for the bosonic content of the gravity multiplet coincides with the Einstein-Maxwell action

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \ , \tag{2.163}$$

where R is the Ricci scalar corresponding to the metric $g_{\mu\nu}$ and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the background graviphoton field strength. We want to consider quadratic fluctuations about the background and then compute the corresponding heat kernel. This calculation has been done for Einstein-Maxwell theory [21], but we find it useful to go through it in detail.

Consider the variations

$$\delta g_{\mu\nu} = h_{\mu\nu} \ , \quad \delta A_\mu = a_\mu \ . \tag{2.164}$$

The fluctuations are the graviton $h_{\mu\nu}$ and the graviphoton a_μ. We will expand the action (2.163) to quadratic order in these field fluctuations. The relevant second-order variations, up to a total derivative, are

$$\begin{aligned}
\delta^2\left(\sqrt{-g}R\right) &= \sqrt{-g}\Big[\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{2}h^\mu{}_\mu\Box h^\rho{}_\rho + h^{\mu\nu}h^{\rho\sigma}R_{\mu\rho\nu\sigma} + h^{\mu\nu}h_{\mu\rho}R^\rho{}_\nu \\
&\quad + \frac{1}{4}(h^\mu{}_\mu)^2 R - h^{\mu\nu}h^\rho{}_\rho R_{\mu\nu} - \frac{1}{2}h^{\mu\nu}h_{\mu\nu}R \\
&\quad + (\nabla^\mu h_{\mu\nu})(\nabla^\rho h_\rho{}^\nu) + (\nabla^\mu\nabla^\nu h_{\mu\nu})h^\rho{}_\rho\Big] \ , \\
\delta^2\left(\sqrt{-g}F_{\mu\nu}F^{\mu\nu}\right) &= \sqrt{-g}\Big[2f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}(h^\mu{}_\mu)^2\right)F_{\rho\sigma}F^{\rho\sigma} \\
&\quad - 8h^{\mu\nu}f_{\mu\rho}F_\nu{}^\rho + 2h^\rho{}_\rho f_{\mu\nu}F^{\mu\nu} + 4h^{\mu\nu}h^\rho{}_\nu F_{\mu\sigma}F_\rho{}^\sigma \\
&\quad + 2h^{\mu\nu}h^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - 2h^{\mu\nu}h^\rho{}_\rho F_{\mu\sigma}F_\nu{}^\sigma\Big] \ ,
\end{aligned} \tag{2.165}$$

where $f_{\mu\nu} = \nabla_\mu a_\nu - \nabla_\nu a_\mu$. We gauge-fix our theory by

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\kappa^2}\left(\nabla^\mu h_{\mu\rho} - \frac{1}{2}\nabla_\rho h^\mu{}_\mu\right)\left(\nabla^\nu h_{\nu\sigma} - \frac{1}{2}\nabla_\sigma h^\nu{}_\nu\right) - \frac{1}{2\kappa^2}(\nabla^\mu a_\mu)^2 \ , \tag{2.166}$$

which picks out the harmonic gauge for the graviton ($\nabla^\mu h_{\mu\rho} - \frac{1}{2}\nabla_\rho h^\mu{}_\mu = 0$) and the Lorenz gauge for the graviphoton ($\nabla^\mu a_\mu = 0$). We use the background Einstein equations to simplify the gauge-fixed quadratic action, which includes setting $R = 0$. Additionally, we let $h_{\mu\nu} \to \sqrt{2}h_{\mu\nu}$ so that the kinetic terms for the graviton and the graviphoton have the same normalization. The resulting action is

$$\begin{aligned}
\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\Big[&h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{2}h^\mu{}_\mu\Box h^\rho{}_\rho + a^\mu\left(\Box g_{\mu\nu} - R_{\mu\nu}\right)a^\nu + 2h^{\mu\nu}h^{\rho\sigma}R_{\mu\rho\nu\sigma} \\
&- 2h^{\mu\nu}h_{\mu\rho}R^\rho{}_\nu - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \frac{1}{8}(h^\mu{}_\mu)^2F_{\rho\sigma}F^{\rho\sigma} - h^{\mu\nu}h^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} \\
&- \frac{1}{\sqrt{2}}h^\rho{}_\rho f_{\mu\nu}F^{\mu\nu} + 2\sqrt{2}h^{\mu\nu}f_{\mu\rho}F_\nu{}^\rho\Big] \ .
\end{aligned} \tag{2.167}$$

We note that (2.167) is not in the required Laplace form needed for our heat kernel analysis, due to the $h^{\mu}{}_{\mu}\Box h^{\rho}{}_{\rho}$ kinetic term. To fix this, we separate the graviton $h_{\mu\nu}$ into its trace h and traceless component $\phi_{\mu\nu}$ by defining

$$h \equiv h^{\mu}{}_{\mu} \ , \tag{2.168}$$

$$\phi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{4} g_{\mu\nu} h \ . \tag{2.169}$$

This decomposition is standard, as the fields h and $\phi_{\mu\nu}$ transform under irreducible representations of $SL(2,\mathbb{C})$ [51–54]. The action becomes

$$\begin{aligned}\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\Big[&\phi^{\mu\nu}\Box\phi_{\mu\nu} - \frac{1}{4}h\Box h + a^{\mu}\left(\Box g_{\mu\nu} - R_{\mu\nu}\right)a^{\nu} + 2\phi^{\mu\nu}\phi^{\rho\sigma}R_{\mu\rho\nu\sigma} \\ &- 2\phi^{\mu\nu}\phi_{\mu\rho}R^{\rho}{}_{\nu} - \frac{1}{4}\phi^{\mu\nu}\phi_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \phi^{\mu\nu}\phi^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - h\phi_{\mu\nu}R^{\mu\nu} + 2\sqrt{2}\phi^{\mu\nu}f_{\mu\rho}F_{\nu}{}^{\rho}\Big] \ .\end{aligned} \tag{2.170}$$

The kinetic term for h has a negative sign. This is the conformal factor problem in gravity, and results in an unbounded path integral for our theory. The resolution to this problem is that the one-loop effective action can be made to converge by performing a conformal rotation that takes the contour of integration for h to be along the imaginary axis [55–57]. We will also simultaneously rescale h to make the normalization of its kinetic term coincide with those for $\phi_{\mu\nu}$ and a_{μ}. Therefore, we let

$$\phi = -\frac{i}{2}h \ , \tag{2.171}$$

and consider the action quadratic in the fields $\{\phi_n\} = \{\phi_{\mu\nu}, a_{\mu}, \phi\}$. The result is

$$\mathcal{S} = -\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\phi_n\Lambda^n_m\phi^m \ , \tag{2.172}$$

where Λ acts on our fields by

$$\begin{aligned}-\phi_n\Lambda^n_m\phi^m = {}&\phi_{\mu\nu}\left(\Box g^{\mu}{}_{\rho}g^{\nu}{}_{\sigma} - 2R^{\mu}{}_{\rho}g^{\nu}{}_{\sigma} + 2R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} - \frac{1}{4}g^{\mu}{}_{\rho}g^{\nu}{}_{\sigma}F_{\lambda\tau}F^{\lambda\tau} - F^{\mu}{}_{\rho}F^{\nu}{}_{\sigma}\right)\phi^{\rho\sigma} \\ &+ a_{\mu}\left(\Box g^{\mu}{}_{\rho} - R^{\mu}{}_{\rho}\right)a^{\rho} + \phi\Box\phi + \phi_{\mu\nu}\left(-iR^{\mu\nu}\right)\phi + \phi\left(-iR_{\rho\sigma}\right)\phi^{\rho\sigma} \\ &+ \phi_{\mu\nu}\left(\frac{\sqrt{2}}{2}(\nabla^{\mu}F_{\rho}{}^{\nu}) + \sqrt{2}(F_{\alpha}{}^{\nu}g^{\mu}{}_{\rho} - F_{\rho}{}^{\nu}g^{\mu}{}_{\alpha})\nabla^{\alpha}\right)a^{\rho} \\ &+ a_{\mu}\left(\frac{\sqrt{2}}{2}(\nabla_{\rho}F^{\mu}{}_{\sigma}) + \sqrt{2}(F^{\mu}{}_{\sigma}g_{\rho\alpha} - F_{\alpha\sigma}g_{\rho}{}^{\mu})\nabla^{\alpha}\right)\phi^{\rho\sigma} \ .\end{aligned} \tag{2.173}$$

We have adjusted total derivative terms to make Λ Hermitian. From (2.173), the matrices P and ω_α are

$$\begin{aligned}\phi_n P^n_m \phi^m = \phi_{\mu\nu} &\left(-2R^\mu{}_\rho g^\nu{}_\sigma + 2R^\mu{}_\rho{}^\nu{}_\sigma - \frac{1}{4} g^\mu{}_\rho g^\nu{}_\sigma F_{\lambda\tau} F^{\lambda\tau} - F^\mu{}_\rho F^\nu{}_\sigma \right) \phi^{\rho\sigma} \\ &+ a_\mu \left(-R^\mu{}_\rho \right) a^\rho + \phi_{\mu\nu} \left(-iR^{\mu\nu} \right) \phi + \phi \left(-iR_{\rho\sigma} \right) \phi^{\rho\sigma} \\ &+ \phi_{\mu\nu} \left(\frac{\sqrt{2}}{2} (\nabla^\mu F_\rho{}^\nu) \right) a^\rho + a_\mu \left(\frac{\sqrt{2}}{2} (\nabla_\rho F^\mu{}_\sigma) \right) \phi^{\rho\sigma} \ ,\end{aligned} \tag{2.174}$$

$$\phi_n (\omega_\alpha)^n_m \phi^m = \frac{\sqrt{2}}{2} \phi_{\mu\nu} \left(F_\alpha{}^\nu g^\mu{}_\rho - F_\rho{}^\nu g^\mu{}_\alpha \right) a^\rho + \frac{\sqrt{2}}{2} a_\mu \left(F^\mu{}_\sigma g_{\rho\alpha} - F_{\alpha\sigma} g_\rho{}^\mu \right) \phi^{\rho\sigma} \ . \tag{2.175}$$

We now define the operator

$$G^{\mu\nu}_{\rho\sigma} = \frac{1}{2} \left(g^\mu{}_\rho g^\nu{}_\sigma + g^\mu{}_\sigma g^\nu{}_\rho - \frac{1}{2} g^{\mu\nu} g_{\rho\sigma} \right) \ . \tag{2.176}$$

$G^{\mu\nu}_{\rho\sigma}$ projects onto the traceless part of a symmetric tensor. In order to impose that $\phi_{\mu\nu}$ is the traceless part of the graviton, we must use $G^{\mu\nu}_{\rho\sigma}$ to contract pairs of indices for any operator acting on $\phi_{\mu\nu}$. That is, if we have some matrix M acting on our fields such that

$$\phi_n M^n_m \phi^m = \phi_{\mu\nu} M^{\mu\nu}_{\rho\sigma} \phi^{\rho\sigma} \ , \tag{2.177}$$

then M^2 is given by

$$\phi_n (M^2)^n_m \phi^m = \phi_{\mu\nu} M^{\mu\nu}_{\alpha\beta} G^{\alpha\beta}_{\gamma\delta} M^{\gamma\delta}_{\rho\sigma} \phi^{\rho\sigma} \ . \tag{2.178}$$

We must also use $G^{\mu\nu}_{\rho\sigma}$ when taking traces of these operators, i.e.

$$\text{Tr } M = G^{\rho\sigma}_{\mu\nu} M^{\mu\nu}_{\rho\sigma} \ . \tag{2.179}$$

As an example, the identity operator I_g acting on $\phi_{\mu\nu}$ is defined by

$$\phi_n (I_g)^n_m \phi^m = \phi_{\mu\nu} \left(g^\mu{}_\rho g^\nu{}_\sigma \right) \phi^{\rho\sigma} = \phi_{\mu\nu} \phi^{\mu\nu} \ . \tag{2.180}$$

Since $\phi_{\mu\nu}$ is both symmetric and traceless, we expect it to have $10 - 1 = 9$ independent off-shell degrees of freedom. The trace of I_g, using $G^{\mu\nu}_{\rho\sigma}$ to contract indices, is indeed

$$\text{Tr } I_g = G^{\rho\sigma}_{\mu\nu} g^\mu{}_\rho g^\nu{}_\sigma = G^{\mu\nu}_{\mu\nu} = \frac{1}{2} \left(g^\mu{}_\mu g^\nu{}_\nu + \frac{1}{2} g^{\mu\nu} g_{\mu\nu} \right) = 9 \ . \tag{2.181}$$

Using the traceless projection operator (2.176) with our expressions for P and ω_α and the background equations of motion, it follows that $\omega^\alpha\omega_\alpha$ and $\nabla^\alpha\omega_\alpha$ are

$$\begin{aligned}\phi_n(\omega^\alpha\omega_\alpha)^n_m\phi^m &= \phi_{\mu\nu}\left(-F^\mu{}_\rho F^\nu{}_\sigma - 2R^\mu{}_\rho g^\nu{}_\sigma - \frac{1}{4}g^\mu{}_\rho g^\nu{}_\sigma F_{\lambda\tau}F^{\lambda\tau}\right)\phi^{\rho\sigma} \\ &\quad + a_\mu\left(-R^\mu{}_\rho - \frac{3}{8}g^\mu{}_\rho F_{\lambda\tau}F^{\lambda\tau}\right)a^\rho \ ,\end{aligned} \tag{2.182}$$

$$\phi_n(\nabla^\alpha\omega_\alpha)^n_m\phi^m = \phi_{\mu\nu}\left(-\frac{\sqrt{2}}{2}(\nabla^\mu F_\rho{}^\nu)\right)a^\rho + a_\mu\left(\frac{\sqrt{2}}{2}(\nabla_\rho F^\mu{}_\sigma)\right)\phi^{\rho\sigma} \ . \tag{2.183}$$

Using $E = P - \omega^\alpha\omega_\alpha - \nabla^\alpha\omega_\alpha$ and adjusting total derivative terms to make E Hermitian, we find that

$$\begin{aligned}\phi_n E^n_m\phi^m &= \phi_{\mu\nu}\left(2R^\mu{}_\rho{}^\nu{}_\sigma\right)\phi^{\rho\sigma} + a_\mu\left(\frac{3}{8}g^\mu{}_\rho F_{\lambda\tau}F^{\lambda\tau}\right)a^\rho \\ &\quad + \phi_{\mu\nu}(-iR^{\mu\nu})\phi + \phi(-iR_{\rho\sigma})\phi^{\rho\sigma} \\ &\quad + \phi_{\mu\nu}\left(\frac{\sqrt{2}}{2}(\nabla^\mu F_\rho{}^\nu)\right)a^\rho + a_\mu\left(\frac{\sqrt{2}}{2}(\nabla_\rho F^\mu{}_\sigma)\right)\phi^{\rho\sigma} \ .\end{aligned} \tag{2.184}$$

The traceless graviton $\phi_{\mu\nu}$ has nine off-shell degrees of freedom, while the trace ϕ has only one and the graviphoton a_μ has four. Therefore,

$$\mathrm{Tr}\ I = 9 + 1 + 4 = 14 \ . \tag{2.185}$$

From (2.184) it follows that

$$\begin{aligned}\mathrm{Tr}\ E &= \frac{3}{2}F_{\mu\nu}F^{\mu\nu} \ , \\ \mathrm{Tr}\ E^2 &= 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 7R_{\mu\nu}R^{\mu\nu} + \frac{3}{4}R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{9}{16}(F_{\mu\nu}F^{\mu\nu})^2 \ .\end{aligned} \tag{2.186}$$

In order to compute the curvature $\Omega_{\alpha\beta}$ we expand the commutator in (2.35):

$$(\Omega_{\alpha\beta})^n_m = [\nabla_\alpha, \nabla_\beta]^n_m + 2\phi_n(\nabla_{[\alpha}\omega_{\beta]})^n_m\phi^m + [\omega_\alpha, \omega_\beta]^n_m \ . \tag{2.187}$$

The covariant derivative commutes when acting on ϕ but not when acting on a_μ or $\phi_{\mu\nu}$. We also account for the fact that $\phi_{\mu\nu}$ is symmetric. So, the first term in (2.187) is

$$\begin{aligned}\phi_n[\nabla_\alpha, \nabla_\beta]^n_m\phi^m &= \phi_{\mu\nu}[\nabla_\alpha, \nabla_\beta]\phi^{\mu\nu} + a_\mu[\nabla_\alpha, \nabla_\beta]a^\mu \\ &= \phi_{\mu\nu}\left(2R^\mu{}_{\rho\alpha\beta}g^\nu{}_\sigma\right)\phi^{\rho\sigma} + a_\mu(R^\mu{}_{\rho\alpha\beta})a^\rho \ .\end{aligned} \tag{2.188}$$

The second term in (2.187) can be calculated by applying the covariant derivative to ω_α and simplifying with the Bianchi identity:

$$\begin{aligned}\phi_n(\nabla_{[\alpha}\omega_{\beta]})^n_m\phi^m = \frac{\sqrt{2}}{4}\phi_{\mu\nu}\left(-\nabla^\nu F_{\alpha\beta}g^\mu{}_\rho - 2g^\mu{}_{[\beta}\nabla_{\alpha]}F_\rho{}^\nu\right)a^\rho \\ + \frac{\sqrt{2}}{4}a_\mu\left(\nabla_\sigma F_{\alpha\beta}g^\mu{}_\rho + 2g_{\rho[\beta}\nabla_{\alpha]}F^\mu{}_\sigma\right)\phi^{\rho\sigma}\ .\end{aligned} \tag{2.189}$$

Note that the covariant derivative is applied only to the background field strength tensors in the above expression, and not to the fields themselves. The last term in equation (2.187) is obtained by taking a product of ω_α and ω_β, antisymmetrizing, and simplifying with the background equations of motion, giving

$$\begin{aligned}\phi_n[\omega_\alpha,\omega_\beta]^n_m\phi^m = &\frac{1}{2}\phi_{\mu\nu}\big(F^\mu{}_\rho F_\alpha{}^\nu g_{\beta\sigma} - F^\mu{}_\rho F_{\beta\sigma}g_\alpha{}^\nu - F_\alpha{}^\nu F_{\beta\sigma}g^\mu{}_\rho - 2R^\mu{}_\rho g_\alpha{}^\nu g_{\beta\sigma} \\ &- \frac{1}{4}g^\mu{}_\rho g_\alpha{}^\nu g_{\beta\sigma}F_{\lambda\tau}F^{\lambda\tau}\big)\phi^{\rho\sigma} + \frac{1}{2}a_\mu\left(R_\beta{}^\mu g_{\alpha\rho} + R_{\alpha\rho}g_\beta{}^\mu - F^\mu{}_\rho F_{\alpha\beta}\right)a^\rho \\ &- (\alpha\leftrightarrow\beta)\ .\end{aligned} \tag{2.190}$$

We have all of the components of $\Omega_{\mu\nu}$ and so it is straightforward to compute the trace of $\Omega_{\mu\nu}\Omega^{\mu\nu}$ using the background equations of motion and Bianchi identities. The result, up to a total derivative, is

$$\mathrm{Tr}\ \Omega_{\mu\nu}\Omega^{\mu\nu} = -7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 56R_{\mu\nu}R^{\mu\nu} - \frac{9}{2}R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - \frac{27}{8}(F_{\mu\nu}F^{\mu\nu})^2\ . \tag{2.191}$$

The choice of gauge-fixing in (2.166) induces the ghost Lagrangian

$$\mathcal{L}_{\text{ghost}} = 2b_\mu(\Box g^{\mu\nu} + R^{\mu\nu})c_\nu + 2b\Box c - 4bF^{\mu\nu}\nabla_\mu c_\nu\ , \tag{2.192}$$

where b_μ, c_μ are the diffeomorphism ghosts associated with the graviton and b, c are the ghosts associated with the graviphoton. For the purposes of computing the heat kernel coefficients we can treat b_μ, c_μ as vector fields and b, c as scalar fields. In order to make the kinetic term in (2.192) diagonal, we make the change of variables

$$b'_\mu = \frac{i(b_\mu - c_\mu)}{\sqrt{2}}\ ,\quad c'_\mu = \frac{b_\mu + c_\mu}{\sqrt{2}}\ ,\quad b' = \frac{i(b-c)}{\sqrt{2}}\ ,\quad c' = \frac{b+c}{\sqrt{2}}\ . \tag{2.193}$$

If we insert these into (2.192) and adjust the total-derivative terms to make the action

Hermitian, we find that

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\,\Big[c'_\mu(\Box g^{\mu\nu} + R^{\mu\nu})c'_\nu + b'_\mu(\Box g^{\mu\nu} + R^{\mu\nu})b'_\nu + b'\Box b' + c\Box c' \\ - (b'_\mu - ic'_\mu)F^{\mu\nu}\nabla_\nu(b+ic) - (b+ic)F^{\mu\nu}\nabla_\mu(b'_\nu - ic'_\nu)\Big] \;. \tag{2.194}$$

We will now supress the $'$ on these terms for notational simplicity. From this action, we can read off the matrices P and ω_α as

$$\begin{aligned} \phi_n P^n_m \phi^m &= b_\mu(R^\mu{}_\nu)b^\nu + c_\mu(R^\mu{}_\nu)c^\nu \;, \\ \phi_n(\omega_\alpha)^n_m\phi^m &= -\frac{1}{2}(b_\mu - ic_\mu)F^\mu{}_\alpha(b+ic) - \frac{1}{2}(b+ic)F_{\alpha\nu}(b^\nu - ic^\nu) \;. \end{aligned} \tag{2.195}$$

The commutator of two covariant derivatives commutes when acting on the scalar ghosts but not on the vector ghosts, so

$$\phi_n[\nabla_\alpha, \nabla_\beta]^n_m\phi^m = b_\mu(R^\mu{}_{\nu\alpha\beta})b^\nu + c_\mu(R^\mu{}_{\nu\alpha\beta})c^\nu \;. \tag{2.196}$$

Using (2.195) and (2.196) it is straightforward to compute E and $\Omega_{\alpha\beta}$ for the ghosts:

$$\begin{aligned} \phi_n E^n_m\phi^m &= b_\mu(R^\mu{}_\nu)b^\nu + c_\mu(R^\mu{}_\nu)c^\nu \;, \\ \phi_n(\Omega_{\alpha\beta})^n_m\phi^m &= b_\mu(R^\mu{}_{\nu\alpha\beta})b^\nu + c_\mu(R^\mu{}_{\nu\alpha\beta})c^\nu - \frac{1}{2}(b_\mu - ic_\mu)(\nabla^\mu F_{\alpha\beta})(b+ic) \\ &\quad + \frac{1}{2}(b+ic)(\nabla_\nu F_{\alpha\beta})(b^\nu - ic^\nu) \;. \end{aligned} \tag{2.197}$$

Each vector ghost has four degrees of freedom, while the scalars each have one, giving $\mathrm{Tr}\; I = 4+4+1+1 = 10$. The traces of E, E^2, and $\Omega_{\mu\nu}\Omega^{\mu\nu}$ are

$$\mathrm{Tr}\; E = 0 \;, \quad \mathrm{Tr}\; E^2 = 2R_{\mu\nu}R^{\mu\nu} \;, \quad \mathrm{Tr}\; \Omega_{\mu\nu}\Omega^{\mu\nu} = -2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \;. \tag{2.198}$$

The total Seeley-DeWitt coefficients for the bosons in the gravity multiplet are given by inserting the traces in (2.186) and (2.191) (as well as the ghost traces in (2.198) with an overall minus sign) into (2.36). The result is

$$\begin{aligned} (4\pi)^2 a_0^{\text{grav},b}(x) &= 4 \;, \\ (4\pi)^2 a_2^{\text{grav},b}(x) &= \frac{3}{2}F_{\mu\nu}F^{\mu\nu} \;, \\ (4\pi)^2 a_4^{\text{grav},b}(x) &= \frac{1}{180}\left(199R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 26R_{\mu\nu}R^{\mu\nu}\right) \;. \end{aligned} \tag{2.199}$$

$a_4^{\text{grav},b}(x)$ matches exactly with the result for Einstein-Maxwell theory given in [21]. As mentioned there, it has no explicit dependence on the background graviphoton field strength, although we would have obtained a different result if we had ignored the terms involving the field strength in the action.

The full gravity multiplet heat kernel coefficients, with contributions from the graviton, gravitini, and graviphoton fluctuations, are

$$
\begin{aligned}
(4\pi)^2 a_0^{\text{grav}}(x) &= 0\ , \\
(4\pi)^2 a_2^{\text{grav}}(x) &= \frac{5}{2} F_{\mu\nu}F^{\mu\nu} - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\ , \\
(4\pi)^2 a_4^{\text{grav}}(x) &= \frac{1}{24}\Big(11 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 4R_{\mu\nu}R^{\mu\nu} + 24 R_{\mu\nu}(F^{\mu\rho}F^{\nu}{}_{\rho} - \tilde{F}^{\mu\rho}\tilde{F}^{\nu}{}_{\rho}) \\
&\quad -12 R_{\mu\nu\rho\sigma}(F^{\mu\nu}F^{\rho\sigma} - \tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}) \\
&\quad -3(F^{\mu\rho}F_{\nu\rho} - \tilde{F}^{\mu\rho}\tilde{F}_{\nu\rho})(F_{\mu\sigma}F^{\nu\sigma} - \tilde{F}_{\mu\sigma}\tilde{F}^{\nu\sigma})\Big)\ .
\end{aligned}
\tag{2.200}
$$

2.4.6 Gravitino Multiplet

By gravitino multiplet we refer to the additional $\mathcal{N}-2$ gravitini, referred to as massive gravitini, and their superpartners in $\mathcal{N}=2$ supergravity. The $\mathcal{N}=2$ gravitino multiplet consists of a (massive) Majorana gravitino, two vector fields, and a Majorana gaugino. The two vector fields are minimally coupled to gravity, so the heat kernel coefficients (including ghosts resulting from the standard Lorenz gauge-fixing) are well-known [41, 42]:

$$
\begin{aligned}
(4\pi)^2 a_0^{3/2,b}(x) &= 4\ , \\
(4\pi)^2 a_2^{3/2,b}(x) &= 0\ , \\
(4\pi)^2 a_4^{3/2,b}(x) &= \frac{1}{90}\left(-13 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 88 R_{\mu\nu}R^{\mu\nu}\right)\ .
\end{aligned}
\tag{2.201}
$$

The fermions in the gravitino multiplet are coupled together by the background graviphoton field. The Lagrangian describing these interactions is given in (2.92) as

$$
\mathcal{L}_{\text{gravitino}} = -\frac{1}{\kappa^2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \frac{2}{\kappa^2}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda - \frac{1}{2\kappa^2}\left(\bar{\Psi}_\mu\hat{F}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu\hat{F}\Psi_\mu\right)\ , \tag{2.202}
$$

where Ψ_μ is the gravitino field, λ is the gaugino field, and $\hat{F} = \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}$. We will proceed as we did for the gravitini in the gravity multiplet and choose the harmonic gauge $\gamma^\mu\Psi_\mu = 0$ by adding to our Lagrangian the gauge-fixing term

$$
\mathcal{L}_{\text{g.f.}} = \frac{1}{2\kappa^2}(\bar{\Psi}_\mu\gamma^\mu)\gamma^\nu\nabla_\nu(\gamma^\rho\Psi_\rho)\ . \tag{2.203}
$$

We also make the field redefinition

$$\Phi_\mu = \frac{1}{\sqrt{2}}\Psi_\mu - \frac{1}{2\sqrt{2}}\gamma_\mu\gamma^\nu\Psi_\nu \ . \tag{2.204}$$

Let $\{\phi_m\} = \{\Phi_\mu, \lambda\}$ denote our fermionic fields. Then, the total action quadratic in these fields (including gauge-fixing) is

$$\mathcal{S} = \frac{1}{\kappa^2}\int d^4x\,\sqrt{-g}\,\phi_n\hat{H}^n_m\phi^m \ , \tag{2.205}$$

where

$$\phi_n\hat{H}^n_m\phi^m = -\bar{\Phi}_\mu\gamma^\nu\nabla_\nu\Phi^\mu - \bar{\lambda}\gamma^\nu\nabla_\nu\lambda - \frac{\sqrt{2}}{4}\left(\bar{\Phi}_\mu\hat{F}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu\hat{F}\Phi_\mu\right) \ . \tag{2.206}$$

The action (2.205) is in the Dirac form needed to employ our heat kernel methods, where $\hat{H}$ is the Dirac-type operator acting on our fermionic fields.

From here, the story is familiar: we continue to Euclidean space, take $\Lambda = \hat{H}\hat{H}^\dagger$, and compute the heat kernel of Λ using all of our standard tricks. We will also include the ghost contribution (see (2.161)) that results from our choice of gauge-fixing and subtract that from the massive gravitino and gaugino contribution. The resulting heat kernel coefficients are

$$\begin{aligned}
(4\pi^2)a_0^{3/2,f}(x) &= -4 \ , \\
(4\pi^2)a_2^{3/2,f}(x) &= -F_{\mu\nu}F^{\mu\nu} \ , \\
(4\pi^2)a_4^{3/2,f}(x) &= -\frac{1}{360}\bigg[113R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8R_{\mu\nu}R^{\mu\nu} - 15R_{\mu\nu\rho\sigma}\left(F^{\mu\nu}F^{\rho\sigma} - \tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}\right) \\
&\qquad - \frac{45}{4}\left((F_{\mu\nu}F^{\mu\nu})^2 - (F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right)\bigg] \ .
\end{aligned} \tag{2.207}$$

Adding up the bosonic (2.201) and fermionic (2.207) contributions, the net heat kernel coefficients for the massive gravitino multiplet are

$$\begin{aligned}
(4\pi^2)a_0^{3/2}(x) &= 0 \ , \\
(4\pi^2)a_2^{3/2}(x) &= -F_{\mu\nu}F^{\mu\nu} \ , \\
(4\pi^2)a_4^{3/2}(x) &= \frac{1}{24}\bigg[-11R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 24R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}\left(F^{\mu\nu}F^{\rho\sigma} - \tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}\right) \\
&\qquad + \frac{3}{4}\left((F_{\mu\nu}F^{\mu\nu})^2 - (F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right)\bigg] \ .
\end{aligned} \tag{2.208}$$

2.5 Results

In this section we collect our results for the heat kernel coefficients of the theory and simplify their form. We compute the corresponding logarithmic corrections to black hole entropy. We discuss the significance of our results and the implications for Kerr-Newman black holes.

2.5.1 Cancellation of the c-Anomaly

The $a_4(x)$ coefficients derived in each $\mathcal{N}=2$ multiplet are linear combinations of covariant terms that each contain four derivatives

$$a_4(x) = \alpha_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \dots \ , \tag{2.209}$$

for some set of numerical coefficients $\{\alpha_i\}$. We found it useful to keep $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ distinct in section 2.4 but we will now simplify as much as possible by expressing the dual field strength in terms of the Levi-Civita symbol and the field strength. We use the Einstein equation (2.93), the Schouten identity (2.94), and derivatives of the field strength (2.99) to prove the following identities[3]:

$$\begin{aligned}\tilde{F}^{\mu\rho}\tilde{F}^{\nu}{}_{\rho} = -F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{1}{2}g^{\mu\nu}(F_{\rho\sigma}F^{\rho\sigma}) = -2R^{\mu\nu} + \frac{1}{4}g^{\mu\nu}(F_{\rho\sigma}F^{\rho\sigma}) \ ,\\ R_{\mu\nu\rho\sigma}(F^{\mu\nu}F^{\rho\sigma} - \tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}) = 8R_{\mu\nu}R^{\mu\nu} \ ,\\ (F_{\mu\nu}F^{\mu\nu})^2 - (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 = 16R_{\mu\nu}R^{\mu\nu} \ .\end{aligned} \tag{2.210}$$

These three relations are sufficient to rewrite all contractions involving the field strength in our $a_4(x)$ results purely in terms of the Riemann tensor. This simplification is surprising because it would not work for generic four-derivative terms. It was noted previously for the bosonic content of the gravity multiplet [21].

From the argument above, we can write our $a_4(x)$ coefficients as

$$a_4(x) = \frac{c}{16\pi^2}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2}E_4 \ , \tag{2.211}$$

for some constants c and a, where the square of the Weyl tensor $W_{\mu\nu\rho\sigma}$ is

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \ , \tag{2.212}$$

[3] In deriving these we assumed Lorentzian signature. The single time-like direction then gives an extra minus sign when contracting Levi-Civita symbols.

and E_4 is the Euler density (also known as the Gauss-Bonnet term)

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \ . \tag{2.213}$$

In four-dimensional conformal field theories, the heat kernel coefficient $a_4(x)$ determines the conformal anomaly of the theory, and the c, a constants can be identified as the central charges of the theory. The a coefficient is typically interpreted as a measure of the number of degrees of freedom in these theories, as it must satisfy certain monotonicity properties when considering the RG flow of a CFT in the UV down to a CFT in the IR [58–61]. The situation here is different, however, since we have a non-conformal theory that takes dynamical gravity into account. The coefficients c and a that we compute are therefore not subject to CFT constraints, such as the conformal collider bounds [62–64].

We now take the $a_4(x)$ results from section 2.4 and use the identities (2.210) to rewrite them in the form of (2.211). The results for the central charges of our theory (with a single graviton multiplet, $\mathcal{N}-2$ gravitino multiplets, n_V vector multiplets and n_H hyper multiplets) are listed in table 2.1.

Fields	c	a
Bosons	$\frac{1}{60}\left(137 + 12(\mathcal{N}-2) - 3n_V + 2n_H\right)$	$\frac{1}{90}\left(106 + 31(\mathcal{N}-2) + n_V + n_H\right)$
Fermions	$-\frac{1}{60}\left(137 + 12(\mathcal{N}-2) - 3n_V + 2n_H\right)$	$\frac{1}{360}\left(-589 + 41(\mathcal{N}-2) + 11n_V - 19n_H\right)$
Total	0	$\frac{1}{24}\left(-11 + 11(\mathcal{N}-2) + n_V - n_H\right)$

Table 2.1: Central charges c and a for the massless field content of a $\mathcal{N} \geq 2$ supergravity theory minimally coupled to the background gauge field.

As a check on these results we consider the special case of BPS black holes. The near-horizon geometry for these spaces is $\text{AdS}_2 \times S^2$, with non-zero components of the Riemann tensor

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{\ell^2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) \ , \quad R_{ijkl} = \frac{1}{\ell^2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \ , \tag{2.214}$$

where ℓ is the radius of curvature of AdS_2 and S^2. (The indices $\alpha, \beta, \gamma, \delta$ refer to AdS_2 and i, j, k, l refer to S^2.) It is straightforward to compute the curvature invariants

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = 0 \ , \quad E_4 = -\frac{8}{\ell^4} \ . \tag{2.215}$$

If we combine these with the values of c, a found in table 2.1 we reproduce the sum of the bulk and boundary contributions (for bosons and fermions separately) computed in [43, 65] exactly.

The results for c, a in table 2.1 are fairly complicated when considering bosons and fermions separately. However, the bosonic and fermionic values of c for any full $\mathcal{N} = 2$ multiplet exactly cancel, giving $c = 0$. By simultaneously considering quadratic fluctuations of both the bosonic and fermionic fields in our theory, the c-anomaly vanishes for arbitrary $\mathcal{N} = 2$ multiplets. The entire one-loop result depends only on the Euler density E_4:

$$a_4(x) = -\frac{a}{16\pi^2} E_4 \ . \tag{2.216}$$

This cancellation would not be noticed for supersymmetric black holes, since $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = 0$ on $\mathrm{AdS}_2 \times S^2$ (2.215).

The cancellation of the c-anomaly is far from automatic. For example, c does not vanish in pure Einstein-Maxwell theory [21], or equivalently for the bosonic part of our $\mathcal{N} = 2$ supergravity multiplet. c and a have been computed in many theories without dynamical gravity but rarely do these computations yield $c = 0$ [52, 66–68]. For quantum field theories that can be described via the AdS/CFT correspondence the canonical situation is $c = a$ in the large N limit [69–73].

2.5.2 Black Hole Entropy

As discussed in section 2.2.6, the logarithmic dependence of the black hole entropy is governed by

$$\Delta S = \frac{1}{2}\left(a_4 + C_{\mathrm{zm}}\right) \log A \ , \tag{2.217}$$

where a_4 is the integral of the heat kernel coefficient $a_4(x)$ over the whole volume of spacetime, and C_{zm} is an integer we add to account for zero modes [14, 15, 20, 22]. Using our results from the previous subsection, we find that the integrated heat kernel coefficient is

$$a_4 = \int d^4x \sqrt{-g}\, a_4(x) = -2a\chi \ , \tag{2.218}$$

where χ is the four-dimensional Euler characteristic

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{-g}\, E_4 \ . \tag{2.219}$$

If we insert (2.218) into (2.217) and employ the full value of the central charge a from table 2.1, we find the logarithmic correction to the black hole entropy

$$\Delta S = \frac{\chi}{24}\left(11 - 11(\mathcal{N}-2) - n_V + n_H\right)\log A + \frac{1}{2}C_{\text{zm}}\log A\ . \tag{2.220}$$

The logarithmic correction depends only on the Euler characteristic χ (as well as the zero mode correction C_{zm}). This result is important because χ is a pure number that depends only on the topology of the black hole solution and not on any black hole parameters.

2.5.3 Kerr-Newman Black Holes

The metric of a Kerr-Newman black hole parameterized by mass M, angular momentum J, and charge Q is

$$\begin{aligned} ds^2 = &-\frac{r^2 + b^2\cos^2\psi - 2Mr + Q^2}{r^2 + b^2\cos^2\psi}dt^2 + \frac{r^2 + b^2\cos^2\psi}{r^2 + b^2 - 2Mr + Q^2}dr^2 + (r^2 + b^2\cos^2\psi)d\psi^2 \\ &+ \frac{(r^2 + b^2\cos^2\psi)(r^2+b^2) + (2Mr - Q^2)b^2\sin^2\psi}{r^2 + b^2\cos^2\psi}\sin^2\psi\, d\phi^2 \\ &+ \frac{2(Q^2 - 2Mr)b}{r^2 + b^2\cos^2\psi}\sin^2\psi\, dt\, d\phi\ , \end{aligned} \tag{2.221}$$

where $b = J/M$. The horizon is located at

$$r_H = M + \sqrt{M^2 - Q^2 - b^2}\ , \tag{2.222}$$

and the inverse temperature $\beta = \frac{1}{T}$ is

$$\beta = \frac{2\pi M}{\sqrt{M^4 - J^2 - M^2Q^2}}\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2Q^2}\right)\ . \tag{2.223}$$

After Euclidean continuation $t \to -i\tau$ and interpreting τ as a periodic coordinate with period β one can show that [20]

$$\begin{aligned} \int d^4x\,\sqrt{-g}\,W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = 64\pi^2 + \frac{\pi\beta Q^4}{b^5 r_H^4(b^2 + r_H^2)}\Big[&4b^5 r_H + 2b^3 r_H^3 \\ &+ 3(b^2 - r_H^2)(b^2 + r_H^2)^2\tan^{-1}\left(\frac{b}{r_H}\right) + 3br_H^5\Big]\ . \end{aligned} \tag{2.224}$$

This expression can be recast as a complicated function of two dimensionless ratios, e.g. Q/M and J/M^2. In the extremal limit $M^2 = b^2 + Q^2$ the expression depends on only one of these ratios, but still in a very non-trivial way [21]. In contrast, the integral of the Euler density is a pure number

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{-g}\, E_4 = 2 \ , \tag{2.225}$$

for all values of the dimensionless ratios.

For generic field content the coefficient of the logarithmic correction to Kerr-Newman entropy resulting from (2.211) depends on the Weyl invariant and thus on all of the black hole parameters through the complicated expression in (2.224). Our result in (2.220), however, demonstrates that when these black holes are interpreted as solutions to $\mathcal{N} \geq 2$ supergravity there is dependence only on χ and thus the logarithmic corrections to Kerr-Newman entropy are universal:

$$\Delta S = \frac{1}{12}\Big((11 + 6C_{\rm zm}) - 11(\mathcal{N} - 2) - n_V + n_H \Big) \log A \ . \tag{2.226}$$

The correction due to zero modes $C_{\rm zm}$ depends on the setting. Some important examples are:

- **BPS black holes**: $C_{\rm zm} = 2$ [14]. The background is spherically symmetric and preserves supersymmetry, giving rise to translational, rotational, and SUSY zero modes.
- **Extremal rotating Kerr-Newman**: $C_{\rm zm} = -4$ [22]. The angular momentum breaks two of the rotational isometries and the background no longer preserves supersymmetry, leaving translational modes and one rotational mode.
- **Non-extremal rotating Kerr-Newman**: $C_{\rm zm} = -1$ [20]. The zero mode counting is the same as for the extremal case except with an additional correction due to the finite IR volume of integration.

For completeness we review the computation of $C_{\rm zm}$ in Appendix A.

Chapter 3

Kerr-Newman Black Holes with String Corrections

3.1 Introduction and Summary

Most precision studies of black holes in string theory are carried out near the BPS limit where supersymmetry guarantees control. It is thought that various corrections become unwieldy far from this limit. Curiously, most discussions of the black hole information paradox are carried out in the opposite limit of Schwarzschild black holes since their geometries are the simplest. It is thought that this is sufficient to gain universal insights. Few details on the implied interpolation between the BPS and Schwarzchild limits are known.

In this chapter we construct families of black holes that interpolate between these limits while taking certain string corrections into account. We find that the string corrections are surprisingly manageable. The simplifications we report are due to supersymmetry of the theories we consider. Importantly, they persist even though the black holes we construct generally do not preserve any of the supersymmetry.

A convenient starting point for connection with studies that are not motivated by string theory is the 4D Einstein-Maxwell theory

$$\mathcal{L} = -\frac{1}{2\kappa^2}\left(R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{3.1}$$

We primarily consider the standard Kerr-Newman family of solutions that includes BPS black holes and Schwarzchild black holes as special cases.

A simple way to add higher-derivative terms to this theory is to consider the Gauss-Bonnet density

$$\mathcal{L}_{\rm GB} = \alpha E_4 = \alpha\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \ . \tag{3.2}$$

This term is topological so the equations of motion are unchanged and therefore solutions remain the same. Black holes nevertheless have a different entropy in the modified theory because the Wald entropy formula depends on the action [31, 32, 74].

Generally other linear combinations of the curvature invariants are much more complicated. The Weyl invariant

$$\mathcal{L}_{\text{Weyl}} = \gamma W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = \gamma\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) , \tag{3.3}$$

introduces the Bach tensor into the equations of motion which are then difficult to analyze [75, 76]. In this work we are inspired by string theory and consider higher-derivative theories of gravity with $\mathcal{N} = 2$ supersymmetry. In particular, we will study the supersymmetric completion of (3.3) that takes the schematic form (made precise in equation (3.49) below):

$$\mathcal{L}_{\mathcal{N}=2\ \text{Weyl}} = \gamma_1 W^2 + \gamma_2 F^4 + \gamma_3 W F^2 + \dots\ , \tag{3.4}$$

with various contractions of the tensors. In this case the equations of motion are even more complicated and it is not clear from the outset that it is realistic to solve them. We find that, surprisingly, any solution to Einstein-Maxwell theory automatically solves the full theory with $\mathcal{N} = 2$ supersymmetry. This will allow us to study generic non-supersymmetric solutions in the presence of higher-derivative corrections.

These higher-derivative corrections modify the Wald entropy of Kerr-Newman black holes. It turns out that the combined contribution from all the terms in the supersymmetrized Weyl invariant (3.4) is precisely the same as the modification due to the Gauss-Bonnet density (3.2) alone. In particular, the contribution from higher-derivative terms is topological. It is therefore independent of black hole parameters and can be extrapolated arbitrarily far from the BPS limit with no change.

The supersymmetrized Weyl invariant (3.4) commonly appears in low energy effective actions; in particular, it arises when massive string modes are integrated out [35, 77, 78]. The terms we consider are string corrections in this sense. Our result indicates that string corrections are milder than previously suspected.

Massless modes running in virtual loops offer a related quantum mechanism that gives higher-derivative terms at low energy. In chapter 2 we studied the logarithmic corrections to Kerr-Newman entropy due to such effects. In general these logarithmic corrections are very complicated but upon embedding of the Kerr-Newman black hole into a theory with $\mathcal{N} \geq 2$ supersymmetry they greatly simplify and become independent of the black hole parameters.

The two classes of corrections we have considered both show that black hole entropy depends greatly on the setting. In an environment with $\mathcal{N} \geq 2$ supersymmetry there are considerable simplifications even for black holes that do not themselves preserve any supersymmetry. Indeed, several of the corrections to the entropy that have been analyzed precisely in the BPS limit do not depend on black hole parameters at all and so apply far off extremality. This result raises hopes that the entropy of non-supersymmetric black holes can be understood precisely in a microscopic theory.

This chapter is organized as follows. In section 3.2 we present a simplified summary of off-shell $\mathcal{N} = 2$ supergravity. In section 3.3 we study minimal supergravity with higher-derivative corrections in the form of a supersymmetrized Weyl invariant and derive the full equations of motion for the theory. In section 3.4 we embed arbitrary Einstein-Maxwell solutions into our minimal supergravity theory and show that all fields are unchanged, even for solutions that do not preserve supersymmetry. In section 3.5 we show that taking the extremal Reissner-Nördstrom limit of these solutions yields a $\frac{1}{2}$-BPS geometry, even in the presence of higher-derivative corrections to the BPS equations. In section 3.6 we study properties of black holes in our embedding and find that the correction to the black hole entropy is topological and independent of black hole parameters. We then go on to discuss our results and potential implications for microscopic models of Kerr-Newman black holes.

3.2 Higher-Derivative $\mathcal{N} = 2$ Supergravity

Higher-derivative interactions can be consistently introduced into 4D $\mathcal{N} = 2$ supergravity using the off-shell formalism, the details of which have been studied exhaustively [38,79–82]. We review some technical details in appendix B. In this section we present a more elementary and accessible discussion of $\mathcal{N} = 2$ supergravity with higher-derivative corrections.

3.2.1 Field Content

We focus on the bosonic fields in $\mathcal{N} = 2$ supergravity. The physical $\mathcal{N} = 2$ gravity multiplet contains the metric $g_{\mu\nu}$ and a $U(1)$ graviphoton field. We further couple this theory to n_V physical $\mathcal{N} = 2$ vector multiplets, each comprising a $U(1)$ gauge field and a complex scalar. The version of the off-shell formalism we employ realizes this coupling by introducing $n_V + 1$ vectors W^I_μ and $n_V + 1$ complex scalars X^I, where $I = 0, \ldots, n_V$. One of the complex scalars can be removed by symmetries and does not correspond to physical degrees of freedom. Without loss of generality, we can choose the auxiliary scalar to be X^0, and we will index the physical vector multiplets by $a = 1, \ldots, n_V$. The remaining gauge field W^0_μ gets combined with the metric to form the $\mathcal{N} = 2$ gravity multiplet on-shell.

The complete formalism based on realization of superconformal symmetry contains many other auxiliary fields that must be carefully considered. However, for our purposes we can consistently set most of these fields to zero at the level of the action. The only ones we must retain are a $U(1)_R$ vector field A_μ, an anti-self-dual antisymmetric tensor $T^-_{\mu\nu}$ and a scalar D that all belong to an off-shell $\mathcal{N}=2$ Weyl multiplet with the metric.

We summarize this discussion with a list of fields, from both the off-shell and the on-shell perspectives, in table 3.1.

Off-Shell Field Content	
Weyl multiplet:	$g_{\mu\nu}$, A_μ , $T^-_{\mu\nu}$, D
Vector multiplets:	W^I_μ , X^I
$(I=0,\ldots,n_V)$	

Physical Field Content	
Gravity multiplet:	$g_{\mu\nu}$, W^0_μ
Vector multiplets:	W^a_μ , X^a
$(a=1,\ldots,n_V)$	

Auxiliary Fields
X^0 , A_μ , $T^-_{\mu\nu}$, D

Table 3.1: Summary of the field content in the $\mathcal{N}=2$ supergravity theory. The n_V+1 off-shell vector multiplets are indexed by I, while the n_V physical vector multiplets are indexed by a.

3.2.2 Definitions and Notation

We will denote the field strengths of the $U(1)_R$ gauge field A_μ and the n_V+1 vector multiplet gauge fields W^I_μ as

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \quad F^I_{\mu\nu} = \partial_\mu W^I_\nu - \partial_\nu W^I_\mu \ . \tag{3.5}$$

The self-dual and anti-self-dual parts of these field strengths are

$$A^\pm_{\mu\nu} = \frac{1}{2}\left(A_{\mu\nu} \pm \tilde{A}_{\mu\nu}\right) \ , \quad F^{\pm I}_{\mu\nu} = \frac{1}{2}\left(F^I_{\mu\nu} \pm \tilde{F}^I_{\mu\nu}\right) \ , \tag{3.6}$$

where the dual field strengths $\tilde{A}_{\mu\nu}$ and $\tilde{F}^I_{\mu\nu}$ in our conventions are

$$\tilde{A}_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}A^{\rho\sigma} \ , \quad \tilde{F}^I_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma I} \ . \tag{3.7}$$

We denote antisymmetrized and symmetrized indices by

$$[\mu\nu] = \frac{1}{2}(\mu\nu - \nu\mu) \ , \quad (\mu\nu) = \frac{1}{2}(\mu\nu + \nu\mu) \ . \tag{3.8}$$

To make it manageable to present equations in the following work we define the composite fields

$$
\begin{aligned}
\mathcal{F}^{-I}_{\mu\nu} &= F^{-I}_{\mu\nu} - \frac{1}{4}\bar{X}^I T^-_{\mu\nu} \ , \\
\hat{A} &= T^-_{\mu\nu} T^{-\mu\nu} \ , \\
\hat{F}^-_{\mu\nu} &= -16\left(W_{\mu\nu\rho\sigma}T^{-\rho\sigma} + DT^-_{\mu\nu} + 2iA_{\rho[\mu}T^{-\rho}_{\nu]}\right) \ , \\
\hat{C} &= 32\Big(W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + i^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + 6D^2 - 2A_{\mu\nu}A^{\mu\nu} + 2A_{\mu\nu}\tilde{A}^{\mu\nu} \\
&\quad -\frac{1}{2}T^{-\mu\nu}\mathcal{D}_\mu\mathcal{D}^\rho T^+_{\rho\nu} + \frac{1}{4}R^\mu{}_\nu T^-_{\mu\rho}T^{+\nu\rho} + \frac{1}{256}T^-_{\mu\nu}T^{-\mu\nu}T^+_{\rho\sigma}T^{+\rho\sigma}\Big) \ ,
\end{aligned}
\tag{3.9}
$$

where the dual to the Weyl tensor is

$$
{}^*W_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\lambda\tau}W_{\rho\sigma\lambda\tau} \ . \tag{3.10}
$$

The composite fields have significance in the underlying superconformal multiplet calculus. However, in this chapter we take a low-brow attitude (for simplicity) where they represent nothing but notation for combinations of fundamental fields, both physical and auxiliary.

We define the supercovariant derivative $\mathcal{D}^\mu$ which acts on a field ϕ with chiral weight c by

$$
\mathcal{D}^\mu\phi = \left(\nabla^\mu - icA^\mu\right)\phi \ , \tag{3.11}
$$

where ∇^μ is the ordinary covariant derivative. The only (non-composite) fields with non-zero chiral weights are the scalars X^I and the anti-self-dual tensor $T^-_{\mu\nu}$. The fields X^I and $T^-_{\mu\nu}$ have chiral weight $c = -1$, while their Hermitian conjugates $\bar{X}^I$ and $T^+_{\mu\nu}$ have the opposite chiral weight $c = +1$. The supercovariant derivative acts on these fields via

$$
\mathcal{D}^\mu X^I = (\nabla^\mu + iA^\mu)X^I \ , \quad \mathcal{D}^\mu T^-_{\rho\sigma} = (\nabla^\mu + iA^\mu)T^-_{\rho\sigma} \ . \tag{3.12}
$$

The scalar operators $\mathcal{D}^\mu\mathcal{D}_\mu$ and $\nabla^\mu\nabla_\mu$ are both useful. They are distinguished by the notation

$$
\mathcal{D}^2 = \mathcal{D}^\mu\mathcal{D}_\mu \ , \quad \Box = \nabla^\mu\nabla_\mu \ . \tag{3.13}
$$

To summarize, we present all of the fields and their corresponding chiral weight c (which determines how the supercovariant derivative (3.11) acts on the field) in table 3.2. We will need to find the equations of motion for all fundamental fields, both physical and auxiliary, but not the composite fields; those are defined for notational reasons only.

	Fundamental						Composite			
Field	$g_{\mu\nu}$	W_μ^I	X^I	A_μ	$T_{\mu\nu}^-$	D	$\mathcal{F}_{\mu\nu}^-$	$\hat{A}$	$\hat{F}_{\mu\nu}^-$	$\hat{C}$
Chiral weight	0	0	-1	0	-1	0	0	-2	-1	0

Table 3.2: Summary of the fields (and their corresponding chiral weight c) in our theory. The conjugate fields have opposite chiral weights.

3.2.3 Prepotential

The interactions of $\mathcal{N} = 2$ supergravity coupled to vector multiplets can be specified succinctly by a prepotential [83–85]. In the two-derivative theory, the prepotential is a meromorphic function of the complex scalars X^I. A large class of higher-derivative corrections can be incorporated by considering generalized prepotentials that are functions of $\hat{A} = T_{\mu\nu}^- T^{-\mu\nu}$ as well. We will denote the prepotential by

$$F \equiv F(X^I, \hat{A}) \ . \tag{3.14}$$

The derivatives of the prepotential are denoted

$$\frac{\partial F}{\partial X^I} = F_I \ , \quad \frac{\partial F}{\partial \hat{A}} = F_A \ . \tag{3.15}$$

The prepotential is holomorphic, so

$$F_{\bar{I}} = F_{\bar{A}} = 0 \tag{3.16}$$

The prepotential is homogeneous of degree two under weighted Weyl transformations where the scalar fields X^I and $\hat{A} = T_{\mu\nu}^- T^{-\mu\nu}$ have Weyl weight $w = 1$ and $w = 2$, respectively. Thus, the prepotential satisfies the homogeneity relation

$$F_I X^I + 2F_A \hat{A} = 2F \ . \tag{3.17}$$

3.2.4 Action

We can now present the bosonic part of the $\mathcal{N} = 2$ supergravity action as

$$\mathcal{S} = \int d^4x \sqrt{-g}\, \mathcal{L} \ , \tag{3.18}$$

with

$$\begin{aligned} 8\pi\mathcal{L} = &-\frac{i}{2}(F_I\bar{X}^I - \bar{F}_I X^I)R + iD^\mu F_I D_\mu \bar{X}^I + \text{h.c.} \\ &+ \left[\frac{i}{4}F_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-\mu\nu J} - \frac{i}{8}F_I\mathcal{F}^{+I}_{\mu\nu}T^{+\mu\nu} - \frac{i}{32}FT^{+}_{\mu\nu}T^{+\mu\nu}\right. \\ &\left.+ \frac{i}{2}F_{AI}\mathcal{F}^{-I}_{\mu\nu}\hat{F}^{-\mu\nu} + \frac{i}{2}F_A\hat{C} + \frac{i}{4}F_{AA}\hat{F}^{-}_{\mu\nu}\hat{F}^{-\mu\nu}\right] + \text{h.c.}\ , \end{aligned} \tag{3.19}$$

where $\mathcal{F}^{-I}_{\mu\nu}$, $\hat{A}$, $\hat{F}^{-}_{\mu\nu}$ and $\hat{C}$ are the composite fields defined in (3.9), and $F = F(X^I, \hat{A})$ is the prepotential discussed in section 3.2.3. Any solution to the equations of motion of this action must also be subject to the constraint

$$D = -\frac{1}{3}R\ , \tag{3.20}$$

which arises from making sure that the auxiliary D-field equation of motion is consistent with the other equations of motion.

The coefficient of the Ricci scalar in the action is determined by the Kähler potential

$$e^{-\mathcal{K}} \equiv i\left(F_I\bar{X}^I - \bar{F}_I X^I\right)\ . \tag{3.21}$$

At face value this means the metric is in a non-canonical frame since the Ricci scalar normalization depends on the fields X^I and $\hat{A}$. However, the theory is invariant under a local Weyl symmetry that acts as a gauge symmetry and constrains the scalars X^I such that only n_V of them are independent. In particular, we can gauge-fix our theory and choose one of the scalars such that the Kähler potential is constant. The low-energy action will then reduce to an Einstein-Hilbert action coupled to matter.

3.3 Minimal Supergravity with W^2 Corrections

In this section we specialize to minimal supergravity, where gravity is coupled to a single vector field, with higher-derivative corrections in the form of a supersymmetrized W^2 term. We will present the prepotential and action for the theory and derive the full equations of motion.

3.3.1 Prepotential and Action

Following the discussion in section 3.2.1, the field content for a theory with $n_V = 0$ physical $\mathcal{N} = 2$ vector multiplets is as follows.

There is a Weyl multiplet containing the metric $g_{\mu\nu}$ and a single vector multiplet containing a physical $U(1)$ gauge field W_μ and a complex scalar X. The complex scalar field will eventually be gauge-fixed, leaving no physical scalars. The off-shell formalism reviewed in section 3.2 (and appendix B) further requires that our theory contain the auxiliary $U(1)_R$ vector field A_μ, the auxiliary scalar D and the auxiliary antisymmetric tensor $T^-_{\mu\nu}$. The Lagrangian will be a function of all these fields.

The prepotential in the minimal theory is a function only of the complex scalar X and the composite field $\hat{A} = T^-_{\mu\nu}T^{-\mu\nu}$. In this chapter we focus on four-derivative corrections to minimal supergravity, which corresponds to a term in the prepotential that is linear in $\hat{A}$. Higher powers of $\hat{A}$ will give rise to corrections with at least six derivatives. The homogeneity (3.17) and holomorphicity (3.16) conditions require the prepotential take the form

$$F(X, \hat{A}) = -\frac{i}{2}X^2 - c\hat{A}\ , \quad c = c_1 + ic_2 \in \mathbb{C}\ . \tag{3.22}$$

We can now specialize the full bosonic Lagrangian (3.19) to the minimal supergravity case defined by the prepotential (3.22). Dropping all total derivative terms, we find

$$\begin{aligned}
8\pi\mathcal{L} = &-|X|^2 R + 2\mathcal{D}^\mu X \mathcal{D}_\mu \bar{X} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F_{\mu\nu}\left(XT^{+\mu\nu} + \bar{X}T^{-\mu\nu}\right) \\
&+ \frac{1}{32}\left(X^2 T^+_{\mu\nu}T^{+\mu\nu} + \bar{X}^2 T^-_{\mu\nu}T^{-\mu\nu}\right) + 32c_2\Big(W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + 6D^2 \\
&- 2A_{\mu\nu}A^{\mu\nu} + \frac{1}{2}(\mathcal{D}_\mu T^{-\mu\nu})(\mathcal{D}^\rho T^+_{\rho\nu}) + \frac{1}{4}R^\mu{}_\nu T^-_{\mu\rho}T^{+\nu\rho} \\
&+ \frac{1}{512}T^-_{\mu\nu}T^{-\mu\nu}T^+_{\rho\sigma}T^{+\rho\sigma}\Big)\ .
\end{aligned} \tag{3.23}$$

As we discussed in the general case, any solution is also subject to the constraint equation $D = -\frac{1}{3}R$.

The coefficient of the Ricci scalar is determined by the complex scalar X. As we noted in section 3.2.4, the local Weyl symmetry of the action allows a gauge where X is an arbitrary constant. We will eventually assign it the conventional numerical value but for now we keep X as an independent field.

For $c_2 = 0$ our minimal $\mathcal{N} = 2$ supergravity Lagrangian (3.23) reduces to the standard two-derivative minimal supergravity, albeit presented in a somewhat unfamiliar form. The new terms are collected in the bracket preceded by the factor $32c_2$. They include first of all an explicit $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ term, as we wanted, but there are many other terms as well. We interpret the entire expression proportional to c_2 as the $\mathcal{N} = 2$ supersymmetric completion of $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$.

In the off-shell formalism the auxiliary field $T^-_{\mu\nu}$ is an antisymmetric tensor, a fundamental field. From this point of view the supersymmetric partners of $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ all contain at most two derivatives. This presents a conceptual advantage because it simplifies the initial value problem. On the other hand, in the context of explicit solutions $T^-_{\mu\nu}$ will coincide with a gauge field strength, with one derivative acting on a gauge field. We will additionally take $D^2 = A_{\mu\nu}A^{\mu\nu} = 0$ consistently. Therefore the supersymmetric partners of $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ will all represent four-derivative terms on-shell.

The coefficient c_2 was introduced as the imaginary part of the coupling constant $c = c_1 + ic_2$ in the prepotential (3.22). All dependence on the real part c_1 has dropped out, because c_1 couples only to total-derivative terms such as the Chern-Pontryagin terms ${}^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ and $A_{\mu\nu}\tilde{A}^{\mu\nu}$. We omitted such terms from the Lagrangian since they do not contribute to the equations of motion.

3.3.2 Equations of Motion

Many previous studies focused on BPS solutions that preserve the full $\mathcal{N} = 2$ supersymmetry, or at least $\frac{1}{2}$-BPS solutions that preserve a residual $\mathcal{N} = 1$ supersymmetry. Such solutions are greatly constrained by relatively simple BPS equations and so it is sufficient to consider a small subset of the equations of motion. We are interested in solutions that explicitly break supersymmetry, and so we need to derive and solve the full equations of motion for the Lagrangian (3.23).

The only D-dependence in the Lagrangian is the D^2 term, and so the D-equation of motion forces $D = 0$. When combined with the constraint equation (3.20), this forces us to consider solutions with vanishing Ricci scalar

$$R = 0 \ . \tag{3.24}$$

We compute the equations of motion for the matter fields X, $T^-_{\mu\nu}$, W_μ, and A_μ to be, respectively,

$$\begin{aligned}
0 &= \mathcal{D}^2\bar{X} + \frac{1}{2}\bar{X}R + \frac{1}{8}\left(F^+_{\mu\nu} - \frac{1}{4}XT^+_{\mu\nu}\right)T^{+\mu\nu} \ , \\
0 &= \bar{X}\left(F^-_{\mu\nu} - \frac{1}{4}\bar{X}T^-_{\mu\nu}\right) - \frac{c_2}{2}\left(128\mathcal{D}_{[\mu}\mathcal{D}^\rho T^+_{\nu]\rho} + T^-_{\mu\nu}T^+_{\rho\sigma}T^{+\rho\sigma} - 64R^\rho{}_{[\mu}T^+_{\nu]\rho}\right) \ , \\
0 &= \mathcal{D}^\mu\left(F^+_{\mu\nu} + F^-_{\mu\nu} - \frac{1}{2}XT^+_{\mu\nu} - \frac{1}{2}\bar{X}T^-_{\mu\nu}\right) \ , \\
0 &= X\mathcal{D}^\mu\bar{X} - \bar{X}\mathcal{D}^\mu X + 8c_2\left(T^{-\mu\nu}\mathcal{D}^\rho T^+_{\rho\nu} - T^{+\mu\nu}\mathcal{D}^\rho T^-_{\rho\nu} - 16i\mathcal{D}_\nu A^{\mu\nu}\right) \ .
\end{aligned} \tag{3.25}$$

The field strength $F_{\mu\nu}$ must also satisfy the Bianchi identity $\mathcal{D}^\mu \tilde{F}_{\mu\nu} = 0$ which we express as

$$\mathcal{D}^\mu \left(F^+_{\mu\nu} - F^-_{\mu\nu} \right) = 0 \; . \tag{3.26}$$

In order to derive the Einstein equation, we first rewrite the minimal supergravity Lagrangiann (3.23) as

$$\mathcal{L} = -\frac{1}{8\pi}|X|^2 R + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} \; , \tag{3.27}$$

where $\mathcal{L}^{(2)}$ is the Lagrangian for the two-derivative matter terms

$$\begin{aligned}\mathcal{L}^{(2)} = \frac{1}{8\pi}\Big[& 2\mathcal{D}^\mu X \mathcal{D}_\mu \bar{X} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F_{\mu\nu}\left(X T^{+\mu\nu} + \bar{X} T^{-\mu\nu} \right) \\ & + \frac{1}{32}\left(X^2 T^+_{\mu\nu} T^{+\mu\nu} + \bar{X}^2 T^-_{\mu\nu} T^{-\mu\nu} \right) \Big] \; ,\end{aligned} \tag{3.28}$$

while $\mathcal{L}^{(4)}$ contains all of the four-derivative terms present in the supersymmetrized Weyl invariant

$$\begin{aligned}\mathcal{L}^{(4)} = \frac{4c_2}{\pi}\Big(& W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + 6D^2 - 2A_{\mu\nu}A^{\mu\nu} + \frac{1}{2}(\mathcal{D}_\mu T^{-\mu\nu})(\mathcal{D}^\rho T^+_{\rho\nu}) \\ & + \frac{1}{4} R^\mu{}_\nu T^-_{\mu\rho} T^{+\nu\rho} + \frac{1}{512} T^-_{\mu\nu} T^{-\mu\nu} T^+_{\rho\sigma} T^{+\rho\sigma} \Big) \; .\end{aligned} \tag{3.29}$$

The Einstein equation can now be expressed as

$$\frac{1}{4\pi}|X|^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = T^{(2)}_{\mu\nu} + T^{(4)}_{\mu\nu} \; , \tag{3.30}$$

where $T^{(2)}_{\mu\nu}$ is the energy-momentum tensor for the two-derivative matter

$$\begin{aligned}T^{(2)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}^{(2)}\right)}{\delta g^{\mu\nu}} = \frac{1}{4\pi}\Big[& 2(\mathcal{D}_\mu X)(\mathcal{D}_\nu \bar{X}) - g_{\mu\nu}(\mathcal{D}_\rho X)(\mathcal{D}^\rho \bar{X}) \\ & + F^+_{\mu\rho} F_\nu^{-\rho} - \frac{1}{4}\left(X F^-_{\mu\rho} T_\nu^{+\rho} + \bar{X} F^+_{\mu\rho} T_\nu^{-\rho} \right) \Big] \; ,\end{aligned} \tag{3.31}$$

while $T^{(4)}_{\mu\nu}$ is the energy-momentum tensor for the four-derivative parts of the action

$$
\begin{aligned}
T^{(4)}_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(4)}\right)}{\delta g^{\mu\nu}} \\
&= \frac{8c_2}{\pi}\Bigg(4R_{\mu\rho}R_{\nu}{}^{\rho} - g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - \frac{4}{3}R_{\mu\nu}R + \frac{1}{3}g_{\mu\nu}R^2 - 2\mathcal{D}^2R_{\mu\nu} \\
&\quad + 4\mathcal{D}^\rho\mathcal{D}_\mu R_{\nu\rho} + \frac{1}{3}g_{\mu\nu}\mathcal{D}^2R - \frac{4}{3}\mathcal{D}_\mu\mathcal{D}_\nu R - 4A_{\mu\rho}A_\nu{}^\rho + g_{\mu\nu}A_{\rho\sigma}A^{\rho\sigma} \\
&\quad - \frac{1}{4}g_{\mu\nu}(\mathcal{D}^\rho T^-_{\rho\tau})(\mathcal{D}_\sigma T^{+\sigma\tau}) + \frac{1}{2}(\mathcal{D}_\mu T^-_{\nu\rho})(\mathcal{D}_\sigma T^{+\sigma\rho}) \\
&\quad + \frac{1}{2}(\mathcal{D}_\mu T^+_{\nu\rho})(\mathcal{D}_\sigma T^{-\sigma\rho}) + \frac{1}{2}(\mathcal{D}^\rho T^-_{\rho\mu})(\mathcal{D}^\sigma T^+_{\sigma\nu}) \\
&\quad + \frac{1}{1024}g_{\mu\nu}T^-_{\rho\sigma}T^{-\rho\sigma}T^+_{\tau\lambda}T^{+\tau\lambda} - \frac{1}{8}g_{\mu\nu}R_{\rho\sigma}T^-_{\rho\tau}T^{+\tau}_{\sigma} + \frac{1}{2}R_{\mu\rho}T^-_{\nu\sigma}T^{+\rho\sigma} \\
&\quad + \frac{1}{4}R^{\rho\sigma}T^-_{\mu\rho}T^+_{\nu\sigma} + \frac{1}{4}\mathcal{D}_\rho\mathcal{D}_\mu(T^-_{\nu\sigma}T^{+\rho\sigma}) - \frac{1}{8}\mathcal{D}^2(T^-_{\mu\rho}T^{+\nu\rho}) \\
&\quad - \frac{1}{8}g_{\mu\nu}\mathcal{D}_\rho\mathcal{D}_\sigma(T^{-\rho\tau}T^{+\sigma}{}_{\tau})\Bigg) \ .
\end{aligned}
\tag{3.32}
$$

In summary, we have shown that any solution to our minimal supergravity theory must satisfy the matter field equations of motion (3.25), the Bianchi identity (3.26), the Einstein equation (3.30), and must have a geometry with vanishing Ricci scalar $R = 0$.

3.4 (Non-Supersymmetric) Einstein-Maxwell Solutions

In this section we embed arbitrary solutions to Einstein-Maxwell theory into the minimal $\mathcal{N} = 2$ supergravity theory (with a supersymmetrized W^2 correction) presented in section 3.3. The matter fields of the higher-derivative gravity are specified in terms of the matter in the Einstein-Maxwell theory. The geometry that supports the Einstein-Maxwell solution is unchanged when considered as solution to higher-derivative gravity.

3.4.1 Einstein-Maxwell Theory

The starting point is the standard Einstein-Maxwell theory

$$
\mathcal{L} = -\frac{1}{2\kappa^2}\left(\mathbf{R} + \frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right) \ , \tag{3.33}
$$

where $\kappa^2 = 8\pi G$. We are using boldfaced symbols $\mathbf{g}_{\mu\nu}$, $\mathbf{R}$, and $\mathbf{F}_{\mu\nu}$ for the metric, Ricci scalar, and electromagnetic field strength in Einstein-Maxwell theory in order to avoid any confusion with related quantities in the higher-derivative supergravity Lagrangian (3.23).

Any solution to Einstein-Maxwell theory satisfies the Maxwell equations and the Bianchi identities, which we package together as the Maxwell-Bianchi equations

$$\nabla^\mu \mathbf{F}^{\pm}_{\mu\nu} = 0 \ , \tag{3.34}$$

where the self-dual and anti-self-dual parts of the field strength are defined using the conventions in section 3.2.2. The geometry and the matter fields are related by the Einstein equation

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = -\mathbf{F}^{-}_{\mu\rho}\mathbf{F}^{+\rho}_{\nu} \ . \tag{3.35}$$

We are particularly interested in Kerr-Newman black hole solutions but our embedding will apply to *any* solution of Einstein-Maxwell theory.

3.4.2 Embedding

Starting from a solution to Einstein-Maxwell theory we specify the matter fields in the higher-derivative theory as

$$X = \frac{\sqrt{4\pi}}{\kappa} \ , \quad A_\mu = 0 \ , \quad T^{\pm}_{\mu\nu} = 4\mathbf{F}^{\pm}_{\mu\nu} \ , \quad F^{\pm}_{\mu\nu} = \frac{1}{4}XT^{\pm}_{\mu\nu} = X\mathbf{F}^{\pm}_{\mu\nu} \ . \tag{3.36}$$

As mentioned previously, the geometry is unchanged.

The numerical value of X is such that the Ricci scalar term in the Lagrangian (3.23) is normalized correctly

$$\mathcal{L} = -\frac{1}{2\kappa^2}R + \dots \ . \tag{3.37}$$

By choosing $A_\mu = 0$, the supercovariant derivative operator D^μ reduces to the ordinary covariant derivative operator ∇^μ.

It is rather straightforward to show that all the matter field equations of motion (3.25) are satisfied by the matter (3.36). Since $\mathbf{F}^{\pm}_{\mu\nu}$ is divergence-free by the Maxwell-Bianchi equations (3.34), $T^{\pm}_{\mu\nu}$ must be divergence-free as well

$$\nabla^\mu T^{\pm}_{\mu\nu} = 0 \ . \tag{3.38}$$

Since X is constant and $A_{\mu\nu} = 0$ the final equation in (3.25) follows. We also have $\nabla^\mu F^{\pm}_{\mu\nu} = 0$ (since X is constant) and so the gauge field equations in the third line of (3.25) are satisfied. The scalar equation of motion is satisfied because X is constant, the geometry has $R = 0$, and the matter satisfies

$$\mathcal{F}^{\pm}_{\mu\nu} = F^{\pm}_{\mu\nu} - \frac{1}{4}XT^{\pm}_{\mu\nu} = 0 \ . \tag{3.39}$$

The equation of motion for the antisymmetric tensor $T^-_{\mu\nu}$ is slightly less obvious. It is satisfied due to the following identities for (anti-)self-dual tensors in 4D:

$$T^+_{\mu\nu}T^{-\rho\sigma} + T^{+\rho\sigma}T^-_{\mu\nu} = 4\delta^{[\rho}_{[\mu}T^+_{\nu]\tau}T^{-\sigma]\tau} \ , \quad T^+_{\mu\nu}T^{-\mu\nu} = 0 \ . \tag{3.40}$$

At this point we still need to verify the Einstein equation (3.30). It is important to note that the only dependence on c_2 is in the four-derivative energy-momentum tensor $T^{(4)}_{\mu\nu}$ and not in any of the two-derivative terms. Since we claim the embedding works for any value of the constant c_2, the two-derivative and four-derivative terms must cancel independently. The original Einstein equation (3.30) therefore becomes two separate equations

$$\frac{1}{4\pi}|X|^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = T^{(2)}_{\mu\nu} \quad \text{and} \quad T^{(4)}_{\mu\nu} = 0 \ . \tag{3.41}$$

The energy-momentum tensor $T^{(2)}_{\mu\nu}$, given in (3.31), simplifies greatly due to the embedding (3.36). The two-derivative part of the Einstein equations (3.41) becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\mathbf{F}^+_{\mu\rho}\mathbf{F}^{-\rho}_{\nu} \ . \tag{3.42}$$

We recognize this equation as the original condition on the Einstein-Maxwell geometry (3.35). Taking the trace of this expression yields

$$R = 0 \ , \tag{3.43}$$

as required by the constraint equation (3.24) coming from the auxiliary D-field.

The four-derivative part of the Einstein equations (3.41), with $T^{(4)}_{\mu\nu}$ given in (3.32), becomes

$$\begin{aligned}
0 &= 4R_{\mu\rho}R_\nu{}^\rho - g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - \frac{4}{3}R_{\mu\nu}R + \frac{1}{3}g_{\mu\nu}R^2 - 2\Box R_{\mu\nu} \\
&\quad + 4\nabla^\rho\nabla_\mu R_{\nu\rho} + \frac{1}{3}g_{\mu\nu}\Box R - \frac{4}{3}\nabla_\mu\nabla_\nu R \\
&\quad + \frac{1}{4}g_{\mu\nu}\mathbf{F}^-_{\rho\sigma}\mathbf{F}^{-\rho\sigma}\mathbf{F}^+_{\tau\lambda}\mathbf{F}^{+\tau\lambda} - 2g_{\mu\nu}R_{\rho\sigma}\mathbf{F}^-_{\rho\tau}\mathbf{F}^{+\tau}_{\sigma} + 8R_{\mu\rho}\mathbf{F}^-_{\nu\sigma}\mathbf{F}^{+\rho\sigma} \\
&\quad + 4R^{\rho\sigma}\mathbf{F}^-_{\mu\rho}\mathbf{F}^+_{\nu\sigma} + 4\nabla_\rho\nabla_\mu(\mathbf{F}^-_{\nu\sigma}\mathbf{F}^{+\rho\sigma}) - 2\Box(\mathbf{F}^-_{\mu\rho}\mathbf{F}^{+\nu\rho}) \\
&\quad - 2g_{\mu\nu}\nabla_\rho\nabla_\sigma(\mathbf{F}^{-\rho\tau}\mathbf{F}^{+\sigma}{}_\tau) \ ,
\end{aligned} \tag{3.44}$$

upon insertion of the embedding (3.36). It is not immediately obvious that it is realistic to solve this equation. However, repeated use of $R_{\mu\nu} = -\mathbf{F}^+_{\mu\rho}\mathbf{F}^{-\rho}_{\nu}$ in (3.44) and careful simplification shows that it is in fact satisfied identically.

In summary, we have verified that our embedding (3.36) generates a solution to the higher-derivative theory for each solution to the original Einstein-Maxwell theory. This result relies on supersymmetry of the theory, as the action we consider is far from arbitrary. However, the solutions do not generally preserve any supersymmetry.

As a check on these results, we consider the special case of extremal Reissner-Nordström black holes. We have verified that the BPS equations derived in [81, 82] are satisfied by our embedding (3.36) for extremal Reissner-Nordström geometries. This is expected, since these geometries are known to be $\frac{1}{2}$-BPS domain walls that interpolate between the $\mathcal{N}=2$ supersymmetric $\mathrm{AdS}_2 \times S^2$ geometry at the horizon and the $\mathcal{N}=2$ supersymmetric Minkowski spacetime at infinity.

3.4.3 Simplified Lagrangian

Having showed that the embedding (3.36) satisfies the fairly complicated equations of motion for minimal supergravity with higher-derivative corrections, it is worth understanding why this is the case. We do so by introducing a simplified effective Lagrangian that captures the same dynamics as the original Lagrangian (3.23) within the context of our embedding.

As a first step we can consistently eliminate the auxiliary fields D and A_μ by setting both to zero at the level of the action. We can then use properties of (anti-)self-dual tensors in 4D (3.40) to express the simplified Lagrangian as

$$\begin{aligned} 8\pi\mathcal{L}_{\text{trunc}} = &-|X|^2 R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 2\nabla^\mu X \nabla_\mu \bar{X} + \frac{1}{2}\left(F^+_{\mu\nu} - \frac{1}{4}XT^+_{\mu\nu}\right)^2 + \text{h.c.} \\ &+ 32c_2\bigg(W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{1}{4}R^\mu{}_\nu T^-_{\mu\rho}T^{+\nu\rho} + \frac{1}{512}T^-_{\mu\nu}T^{-\mu\nu}T^+_{\rho\sigma}T^{+\rho\sigma} \\ &+ \frac{1}{2}(\nabla_\mu T^{-\mu\nu})(\nabla^\rho T^+_{\rho\nu})\bigg) \ . \end{aligned} \tag{3.45}$$

We now want to eliminate the auxiliary fields X and $T^-_{\mu\nu}$ from the action by replacing them with their ansatz in the embedding (3.36):

$$X = \frac{\sqrt{4\pi}}{\kappa} \ , \quad T^-_{\mu\nu} = \frac{4}{X}F^-_{\mu\nu} \ , \tag{3.46}$$

at the level of the action. We can see from (3.45) that X is sourced by the Ricci scalar, which vanishes for Einstein-Maxwell backgrounds, and $F^+_{\mu\nu} - \frac{1}{4}XT^+_{\mu\nu}$, which vanishes in (3.46). Similarly, $T^-_{\mu\nu}$ is sourced by $F^-_{\mu\nu} - \frac{1}{4}\bar{X}T^-_{\mu\nu}$ and various other terms that vanish for Einstein-Maxwell backgrounds. The elimination (3.46) is therefore consistent with the X and $T^-_{\mu\nu}$ equations of motion and can be implemented at the level of the action.

To make the normalization simpler we also rescale the vector multiplet field strength by

$$F_{\mu\nu} \to \frac{\sqrt{4\pi}}{\kappa} F_{\mu\nu} \ . \tag{3.47}$$

After these simplifications we find

$$\begin{aligned}\mathcal{L}_{\text{trunc}} = -\frac{1}{2\kappa^2}\left(R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \frac{4c_2}{\pi}\Big(&W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + 4R^{\mu}{}_{\nu}F^{-}_{\mu\rho}F^{+\nu\rho} \\ &+ \frac{1}{2}F^{-}_{\mu\nu}F^{-\mu\nu}F^{+}_{\rho\sigma}F^{+\rho\sigma} + 8(\nabla_{\mu}F^{-\mu\nu})(\nabla^{\rho}F^{+}_{\rho\nu})\Big) \ .\end{aligned} \tag{3.48}$$

This form of the Lagrangian expresses the intuitive notion that our theory is ordinary Einstein-Maxwell theory with addition of a supersymmetrized Weyl invariant that includes mixings between the electromagnetic field strength and the Riemann tensor. Any solution to the truncated theory (3.48) will automatically be a solution to the minimal supergravity theory (3.23).

Our black hole solutions imply that the supersymmetrized Weyl invariant

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=2\ \text{Weyl}} = W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} &+ 4R^{\mu}{}_{\nu}F^{-}_{\mu\rho}F^{+\nu\rho} + \frac{1}{2}F^{-}_{\mu\nu}F^{-\mu\nu}F^{+}_{\rho\sigma}F^{+\rho\sigma} \\ &+ 8(\nabla_{\mu}F^{-\mu\nu})(\nabla^{\rho}F^{+}_{\rho\nu})\end{aligned} \tag{3.49}$$

can be included into the Einstein-Maxwell action without consequence to the geometry or the field strength. To understand this claim we rewrite $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ in terms of the Gauss-Bonnet density E_4 as

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = E_4 + 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 \ , \tag{3.50}$$

and find

$$\mathcal{L}_{\mathcal{N}=2\ \text{Weyl}} = E_4 + 2\left(R_{\mu\nu} + F^{-}_{\mu\rho}F^{+\rho}_{\nu}\right)^2 - \frac{2}{3}R^2 + 8(\nabla_{\mu}F^{-\mu\nu})(\nabla^{\rho}F^{+}_{\rho\nu}) \ . \tag{3.51}$$

The Gauss-Bonnet density E_4 does not contribute to the equations of motion because it is topological. The remaining terms $(R_{\mu\nu} + F^{-}_{\mu\rho}F^{+\rho}_{\nu})^2$, R^2, and $(\nabla_{\mu}F^{-\mu\nu})(\nabla^{\rho}F^{+}_{\rho\nu})$ are all quadratic in expressions that vanish for Einstein-Maxwell backgrounds. That explains why these terms can be introduced in the Einstein-Maxwell action without changing the original solutions.

The simplifications we find are predicated on the precise combination of four-derivative terms appearing in (3.49); any others would lead to complicated corrections of the solutions (see e.g. [26,27]). In our context those coefficients were dictated by the $\mathcal{N}=2$ supersymmetry of the theory. Thus supersymmetry is responsible for substantial simplifications even for solutions that do not preserve any supersymmetry.

It was previously noticed in [34] that the entropy of supersymmetric black holes in heterotic string theory is the same whether one introduces higher-derivative corrections in the form of a supersymmetrized Weyl invariant or an ordinary Gauss-Bonnet term. This led to arguments (see e.g. [14]) that the supersymmetrized Weyl invariant should coincide with the Gauss-Bonnet density on-shell. Our supersymmetrized Weyl invariant (3.51) makes this argument concrete. This is particularly surprising in the near-horizon region of BPS black holes: the $\mathrm{AdS}_2 \times S^2$ geometry has vanishing Weyl tensor, yet the supersymmetrized Weyl invariant is non-zero and matches the Gauss-Bonnet density exactly.

3.5 The BPS Limit

In the previous section, we showed that arbitrary solutions to Einstein-Maxwell theory are also solutions to minimal supergravity, even with higher-derivative corrections present. In particular, arbitrary Kerr-Newman black holes are solutions to the theory with no modifications to their geometry required. These solutions are interesting because they can be thought of as continuous deformations of the extremal Reissner-Nördstrom black hole, which (in the absence of higher-derivative corrections) preserves half of the supersymmetries in $\mathcal{N} = 2$ supergravity. These supersymmetric solutions should have an underlying microscopic description in string theory, and the hope is that such a microscopic theory can be deformed to give a microscopic description of general Kerr-Newman black holes as well.

However, as we discussed in section 3.1, string theory requires an infinite tower of higher-derivative α' corrections that modify the two-derivative $\mathcal{N} = 2$ supergravity action. These α' corrections also lead to modifications of the BPS conditions of the theory. These modified BPS conditions are no longer algebraic and are thus typically hard to solve, as they generically require a non-trivial modification of the original geometry in order to preserve supersymmetry at higher-derivative order [86–90]. This begs the question: do extremal Reissner-Nordström black holes still preserve supersymmetry in our minimal supergravity theory once we account for higher-derivative modifications of the BPS conditions? This is non-trivial to determine, since it requires studying the supersymmetry variations of the fermions in the full off-shell theory with higher-derivative interactions present.

BPS conditions in off-shell higher-derivative $\mathcal{N} = 2$ supergravity have been studied in previous works [81, 82, 91]. We will first review these BPS conditions before specializing to minimal supergravity. We will then show that, even with supersymmetric W^2 corrections turned on, extremal Reissner-Nordström black holes (and their dyonic, multi-centered generalizations) are in fact the only asymptotically flat solutions that preserve half of the supersymmetries of the theory.

3.5.1 BPS Conditions in Off-Shell $\mathcal{N}=2$ Supergravity

We are interested in $\frac{1}{2}$-BPS solutions in off-shell $\mathcal{N}=2$ supergravity when higher-derivative interactions in the form of a supersymmetric W^2 term are present. Note that we are not interested in fully supersymmetric configurations, because these can be recovered from the $\frac{1}{2}$-BPS solutions by approaching the boundary of the theory, either at asymptotic infinity or at the horizon for black hole solutions [38].

$\frac{1}{2}$-BPS configurations can be found by imposing a projection condition relating the two supersymmetry parameters ϵ_i and ϵ^i of the theory, and then finding the conditions under which all fermion variations vanish under the corresponding residual $\mathcal{N}=1$ supersymmetry. That is, we will impose the condition

$$h\epsilon_i = \varepsilon_{ij}\gamma_0\epsilon^j \ , \tag{3.52}$$

where h is some phase factor. Supersymmetry then dictates that the most general stationary, static spacetime that preserves the $\mathcal{N}=1$ supersymmetry corresponding to the projection (3.52) must take the form

$$ds^2 = -e^{2g}dt^2 + e^{-2g}g_{mn}dx^m dx^n \ , \tag{3.53}$$

where the function $g = g(x^m)$ depends only on the spatial coordinates x^m, and the spatial metric g_{mn} must be flat. Supersymmetry also requires that all fermions and fields charged under the $SU(2)_R$ symmetry must vanish, and that the anti-self-dual tensor $T^-_{\mu\nu}$ and the $U(1)_R$ gauge field A_μ are constrained to take the values

$$T^-_{0p} = -4h\nabla_p g \ , \quad A_0 = A_p = 0 \ , \tag{3.54}$$

where $(0,p)$ indicate the tangent space indices. The vector multiplet field strengths are determined by the constraint

$$F^{-I}_{0p} = -e^g\left(\nabla_p(\bar{h}X^I) + (\nabla_p g)h\bar{X}^I\right) \ . \tag{3.55}$$

The only fields that we have not yet constrained are the metric function g and the scalars

X^I. These are related by the stabilization equations

$$e^{-g}\bar{h}\begin{pmatrix} X^I - \bar{X}^I \\ F_I - \bar{F}_I \end{pmatrix} = i\begin{pmatrix} H^I \\ H_I \end{pmatrix} \ , \tag{3.56}$$

where H^I and H_I are harmonic functions given by

$$H^I = h^I + \sum_A \frac{P_A^I}{|\vec{x}-\vec{x}_A|} \ , \quad H_I = h_I + \sum_A \frac{Q_{AI}}{|\vec{x}-\vec{x}_A|} \ . \tag{3.57}$$

These harmonic functions are determined entirely by some constants h^I and h_I, as well as a set of magnetic charges P_A^I and a set of electric charges Q_{AI} that are located at spatial position $\vec{x}_A$. These charges are related to the field strengths by

$$Q_{AI} = \frac{1}{4\pi}\int_{\Sigma_A^2} \mathcal{N}_{IJ} \star F^J \ , \quad P_A^I = \frac{1}{4\pi}\int_{\Sigma_A^2} F^I \ , \tag{3.58}$$

where Σ_A^2 is a two-sphere surrounding the center $\vec{x}_A$ but no other centers. Finally, in order for the solution to be consistent with all auxiliary field equations of motion in the theory, the metric and the scalars must satisfy the extra conditions

$$\begin{aligned} i(F_I\bar{X}^I - \bar{F}_I X^I) + \frac{\chi}{2} &= 128 i e^{3g}\nabla^p\left((\nabla_p e^{-g})(F_A - \bar{F}_A)\right) \ , \\ H^I\nabla_p H_I - H_I\nabla_p H^I &= -256\nabla^q\left((\nabla_{[p} g)\nabla_{q]}(F_A + \bar{F}_A)\right) \ , \end{aligned} \tag{3.59}$$

where χ is some negative constant.

3.5.2 Solving the BPS Conditions in Minimal Supergravity

Now that we have reviewed the BPS conditions in general, we want to study them in the setting of minimal supergravity with W^2 corrections. The prepotential for our theory is

$$F = -\frac{i}{2}X^2 - c\hat{A} \ , \quad c = c_1 + ic_2 \in \mathbb{C} \ , \tag{3.60}$$

for some complex constant c. Plugging this prepotential into the stabilization equations (3.56), we find that

$$e^{-g}\bar{h}\begin{pmatrix} \operatorname{Im} X \\ \operatorname{Re} X \end{pmatrix} = \frac{1}{2}\begin{pmatrix} H_m \\ H_e \end{pmatrix} \ , \tag{3.61}$$

where H_m and H_e are magnetic and electric harmonic functions, respectively, given by

$$H_m = h_m + \sum_A \frac{P_A}{|\vec{x} - \vec{x}_A|} \ , \quad H_e = h_e + \sum_A \frac{Q_A}{|\vec{x} - \vec{x}_A|} \ , \tag{3.62}$$

for some constants h_m and h_e. Note that we have also allowed for the presence of multiple charges and centers in our system, in an effort to be as general as possible in our analysis. The auxiliary conditions (3.59) for our prepotential are

$$\begin{aligned} 2|X|^2 + \frac{\chi}{2} - 256 c_2\, e^{3g} \nabla^p \nabla_p e^{-g} &= 0 \ , \\ H_m \nabla_p H_e - H_e \nabla_p H_m &= 0 \ , \end{aligned} \tag{3.63}$$

where again χ is some as-of-yet undetermined negative constant. To find solutions to these BPS conditions, it will be useful parameterize the complex scalar field X by

$$X = \psi e^{i\phi} \ , \tag{3.64}$$

for some real scalar fields ψ and ϕ. The stabilization equations can then be expressed as

$$e^{-g} = \frac{H_m}{2\bar{h}\psi \sin\phi} = \frac{H_e}{2\bar{h}\psi \cos\phi} \ . \tag{3.65}$$

In particular, this means that the scalar field ϕ determines the ratio of the two harmonic functions by $H_m = H_e \tan\phi$. Plugging this into the second auxiliary condition then tells us that $\nabla_p \phi = 0$, and so we must set the scalar field ϕ to be a constant.

We have thus shown that the harmonic functions H_e and H_m must be related to one another by an overall constant of proportionality. In particular, this means that each the constant terms in each are proportional, as well as the coefficients of each $|\vec{x} - \vec{x}_A|$ term. That is,

$$h_m = h_e \tan\phi \ , \quad P_A = Q_A \tan\phi \ . \tag{3.66}$$

We can now define a new harmonic function H by

$$H = |\vec{h}| + \sum_A \frac{|\vec{Q}_A|}{|\vec{x} - \vec{x}_A|} \ , \tag{3.67}$$

where $|\vec{h}| = \sqrt{h_e^2 + h_m^2}$ and $|\vec{Q}_A| = \sqrt{Q_A^2 + P_A^2}$. Using (3.66), we can see that this new harmonic function is related to the electric and magnetic harmonic functions via

$$H = \frac{H_e}{\cos\phi} = \frac{H_m}{\sin\phi} \ . \tag{3.68}$$

Plugging this new harmonic function back into the stabilization equations (3.65), we find that the metric function g can be expressed as

$$e^{-g} = \frac{H}{2\bar{h}\psi} \ . \tag{3.69}$$

Note that the harmonic function H, the metric factor e^{-g}, and the scalar field ψ are all real. This relation then serves as a reality condition that forces the phase $\bar{h}$ to be real, and thus the only allowed values are $\bar{h} = \pm 1$. We can without loss of generality absorb this constant into our definition of ψ, leaving us with

$$e^{-g} = \frac{H}{2\psi} \ . \tag{3.70}$$

At this point, we have used the stabilization equations and the second auxiliary condition to express e^{-g} in terms of a real scalar field ψ and the harmonic function H. What remains is to determine ψ using the first auxiliary condition, which can be written as

$$2\psi^2 + \frac{\chi}{2} - 256c_2 \left(\frac{H}{2\psi}\right)^{-3} \nabla^p \nabla_p \left(\frac{H}{2\psi}\right) = 0 \ . \tag{3.71}$$

We will solve this equation perturbatively in the coupling constant c_2 of the supersymmetric W^2 corrections to the action by expanding the scalar field as

$$\psi = \psi^{(0)} + c_2 \psi^{(1)} + \dots \ . \tag{3.72}$$

To zeroth order in c_2, the auxiliary condition (3.71) tells us that $\psi^{(0)}$ is a constant, related to the negative constant χ by

$$\chi = -4\left(\psi^{(0)}\right)^2 \ . \tag{3.73}$$

Plugging this back into (3.71), we find that the (to linear order in c_2)

$$2\left(\psi^{(1)}\right)^2 - 256c_2 \left(\frac{H}{2\psi^{(0)}}\right)^{-3} \nabla^p \nabla_p \left(\frac{H}{2\psi^{(0)}}\right) = 0 \ . \tag{3.74}$$

Crucially, since H is a harmonic function and $\psi^{(0)}$ is a constant, the second term in this equation vanishes. We are therefore left with the constraint that $\psi^{(1)} = 0$. Moreover, repeating this procedure shows that $\psi^{(n)}$ vanishes for all $n \geq 1$, to all orders in perturbation theory. We therefore conclude that the most general solution for ψ is to set it to some constant. The metric (3.53) can therefore be written in the form

$$ds^2 = -\frac{4\psi^2}{H^2}dt^2 + \frac{H^2}{4\psi^2}g_{mn}dx^m dx^n \ , \tag{3.75}$$

for some constant ψ. Note that the constant factor $4\psi^2$ is unimportant, though, since it can be absorbed into the coordinates via a simple global rescaling of the coordinates. Additionally, if we impose asymptotically-flat boundary conditions on the background such that the metric approaches the Minkowski metric far away from all centers, we are forced to choose the parameters h_e and h_m such that their norm is

$$|\vec{h}| = \sqrt{h_e^2 + h_m^2} = 1 \ . \tag{3.76}$$

Putting this all together, we find that the most general metric compatible with the BPS conditions can always be put in the form

$$ds^2 = -H^{-2}dt^2 + H^2 g_{mn}dx^m dx^n \ , \quad H = 1 + \sum_A \frac{|\vec{Q}_A|}{|\vec{x} - \vec{x}_A|} \ , \tag{3.77}$$

where g_{mn} is a flat three-dimensional metric. This solution is simply the dyonic, multi-centered generalization of the extremal Reissner-Nordström black hole [92], where the total mass M of the black hole is related to the charges by

$$M = \sum_A |\vec{Q}_A| \ . \tag{3.78}$$

We have therefore shown that the only static, stationary, and asymptotically-flat spacetime that preserves half of the supersymmetries of the theory is simply the ordinary extremal Reissner-Nordström black hole. Although the higher-derivative corrections to our theory make the supersymmetry conditions considerably harder to analyze, at the end of the day they do not affect the geometry of these $\frac{1}{2}$-BPS solutions.

3.6 Properties of Black Holes in Higher-Derivative Gravity

In this section we analyze properties of Kerr-Newman black holes considered as solutions to minimal supergravity with higher-derivative corrections. We show that the black hole entropy simplifies when the theory has $\mathcal{N}=2$ supersymmetry.

3.6.1 Black Hole Entropy

The black hole entropy in the higher-derivative theory is given by the Wald entropy formula [31,32,74]. The entropy is

$$S = 2\pi \int_H \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\sqrt{h}\, d^2x \ , \tag{3.79}$$

where h_{ij} is the induced metric on the black hole horizon H and $\epsilon_{\mu\nu}$ is the (antisymmetric) unit binormal to the horizon, normalized such that $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$. Four-derivative terms in the action give rise to an integrand that includes terms linear in the curvature and terms with two derivatives acting on the matter fields. Each of these terms in the integrand is somewhat intricate and upon integration they will generally give complicated contributions to the entropy.

However, $\mathcal{N}=2$ supersymmetry dictates relations between the coefficients of these contributions such that the four-derivative terms combine into the expression (3.51). Any part of the action that is quadratic in terms that vanish on-shell cannot contribute to the Wald entropy (3.79), since the entropy is determined by a linear variation. For the purposes of computing the Wald entropy it is therefore sufficient to add the Gauss-Bonnet term

$$\mathcal{L}_{\rm GB} = \frac{4c_2}{\pi} E_4 \tag{3.80}$$

to the standard Einstein-Hilbert Lagrangian. This is a considerable simplification.

The Gauss-Bonnet term (3.80) is topological, and so any variation of it with respect to physical fields with produce a total derivative. It can also be expressed in 4D as a total derivative acting on (non-covariant) Christoffel symbols. However, the Wald entropy formalism requires first putting the Lagrangian in a covariant form, e.g. in terms of the metric and the Riemann tensor. The Wald entropy formalism then requires varying the Lagrangian with respect to the Riemann tensor, not a physical field. The contribution to the Wald entropy from a Gauss-Bonnet term in the action is therefore not forced to be zero. This contribution has been studied in detail; it was explicitly shown by Wald and Iyer in [32] that this contribution is proportional to the Euler characteristic of the surface of integration. This result was also independently shown in [93] by use of the Jacobson-Myers functional. The total Wald entropy, including the area law due to the Einstein-Hilbert action, is

$$S = \frac{A}{4G} + 128\pi\chi_{(2)}c_2 \ , \tag{3.81}$$

where $\chi_{(2)}$ is the Euler characteristic of the black hole horizon[1]

$$\chi_{(2)} = -\frac{1}{4\pi}\int_H dA\, R_{(2)} \ . \tag{3.82}$$

For general Kerr-Newman black holes, the Euler characteristic of the horizon is $\chi_{(2)} = 2$, and so the Wald entropy (3.81) becomes

$$S = \frac{A}{4G} + 256\pi c_2 \ . \tag{3.83}$$

This is the entropy of a Kerr-Newman black hole, including the higher-derivative correction in the form of a supersymmetrized Weyl invariant.

In the special case of vanishing charge, the black hole geometry is Ricci flat $R_{\mu\nu} = 0$ and so it is obvious that the Weyl invariant coincides with the Gauss-Bonnet term on-shell. We find that this well-known statement generalizes to Kerr-Newman black holes. That is interesting because this family includes a BPS limit, where the black hole preserves the supersymmetry of the theory and the microscopic description is under control. Previous studies [34, 35, 38, 77, 78, 80, 81, 86, 94–98] have found that higher-derivative corrections in string theory gives rise to a correction of the form (3.83) with a numerical coefficient that can be matched with microscopic considerations.

Our result for the correction to the black hole entropy (3.83) has no dependence whatsoever on the parameters of the black hole. The deformation away from the BPS limit by adding mass and introducing angular momentum does not change the correction due to higher-order derivatives. This is reminiscent of our main result from chapter 2 that quantum corrections to Kerr-Newman black holes are universal and similarly insensitive to deformations off extremality. For both classes of corrections it is significant that the theory preserves $\mathcal{N} = 2$ supersymmetry but it is unimportant whether the black holes preserve the supersymmetry of the theory.

[1]Our curvature conventions are set by the sign on the Ricci scalar in the Einstein-Hilbert action. The curvature of a sphere is negative and the Euler character (3.82) has an unusual overall minus sign.

3.6.2 OSV Conjecture

The correction to the entropy due to the higher-derivative terms is just a constant, independent of the black hole parameters. The value of the constant is therefore captured by the BPS limit and so it can be interpreted in string theory, *e.g.* following the OSV conjecture [99].

For extremal BPS black holes, the attractor mechanism [33, 100–102] specifies scalars in the horizon $\mathrm{AdS}_2 \times S^2$ geometry in terms of the charges (p^I, q_I) by the attractor equations

$$p^I = \mathrm{Re}[CX^I] \ , \tag{3.84}$$

$$q_I = \mathrm{Re}[CF_I] \ , \tag{3.85}$$

where C is an arbitrary scaling parameter chosen as

$$C^2 \hat{A} = 256 \ , \tag{3.86}$$

with $\hat{A}$ evaluated at the horizon. Expressing the real and imaginary parts of the scalars as

$$CX^I = p^I + \frac{i}{\pi}\phi^I \ , \tag{3.87}$$

the black hole potential is

$$\mathcal{F}(\phi^I, p^I) = -\pi \ \mathrm{Im}[C^2 F(X^I, \hat{A})] \ , \tag{3.88}$$

in a mixed ensemble defined as a microcanonical ensemble of magnetic charges p^I and a canonical ensemble of electric charges q_I with chemical potentials ϕ^I. The black hole entropy, including higher-derivative terms, is then given by the Legendre transform

$$S(q_I, p^I) = \left(1 - \phi^I \frac{\partial}{\partial \phi^I}\right) \mathcal{F}(\phi^I, p^I) \ , \tag{3.89}$$

where the electric potentials ϕ^I have been eliminated in favor of the electric charges q_I through the attractor equation (3.85).

In the case of our minimal prepotential (3.22) the attractor equations are

$$p = \mathrm{Re}[CX] \ , \quad q = \mathrm{Im}[CX] = \frac{1}{\pi}\phi \ , \tag{3.90}$$

and the black hole potential (3.88) becomes

$$\mathcal{F}(\phi, p) = \frac{\pi}{2}p^2 - \frac{1}{2\pi}\phi^2 + 256\pi c_2 \ . \tag{3.91}$$

The Legendre transform of this potential gives the black hole entropy

$$S = \frac{\pi}{2}(q^2 + p^2) + 256\pi c_2 \ . \tag{3.92}$$

The first term agrees with the classical area law for an extremal Reissner-Nordström black hole with dyonic $U(1)$ charge, and the correction agrees with our result (3.83) computed using the Wald entropy formalism.

The OSV conjecture [99] makes connection with microscopic considerations through the relation

$$Z_{\rm BH} = |Z_{\rm top}|^2 \ , \tag{3.93}$$

where $Z_{\rm BH}$ is the supersymmetric partition function

$$Z_{\rm BH}(\phi, p) = \exp\big[\mathcal{F}(\phi, p)\big] \tag{3.94}$$

of a four-dimensional BPS black hole in the mixed ensemble. The partition function of the topological string is similarly

$$Z_{\rm top}(\phi, p) = \exp\big[\mathcal{F}_{\rm top}(\phi, p)\big] \ , \tag{3.95}$$

with

$$\mathcal{F}_{\rm top}(\lambda, X) = \sum_{g=0} \lambda_{\rm top}^{2g-2} F_{{\rm top},g}(X) \ , \tag{3.96}$$

a perturbative expansion in the coupling constant $\lambda_{\rm top} = \frac{4\pi i}{p+iq}$. The correction we consider is charge-independent, corresponding to the torus partition function with genus $g = 1$.

The OSV conjecture and its possible extensions have been subject to much study and debate, including [87,98,103–105]. Since the minimal model we consider has $n_V = 0$ moduli it corresponds to a somewhat singular limit, that of a rigid Calabi-Yau. It would be interesting to study this special case further.

3.6.3 Implications for a Microscopic Model

The motivation for studying Kerr-Newman black holes in string theory is the hope that a precision understanding can be achieved in this setting. We are still far from that goal but we can make some observations in the spirit of phenomenology.

The classical black hole entropy of Kerr-Newman black holes computed from the outer and inner horizons is

$$S_{\pm} = 2\pi \left((M^2 - \frac{1}{2}Q^2) \pm \sqrt{M^2(M^2 - Q^2) - J^2} \right) . \tag{3.97}$$

An appealing (but speculative) interpretation of these formulae identifes the combinations

$$S_R = \frac{1}{2}(S_+ + S_-) \ , \quad S_L = \frac{1}{2}(S_+ - S_-) \ , \tag{3.98}$$

with the entropy of factorized right- and left-moving excitations of an underlying CFT with $(0, 4)$ supersymmetry [19, 106, 107]. This theory would be a generalization of the MSW CFT describing the BPS and near-BPS limits [6]. The assignment of supersymmetry is such that the dependence on the angular momentum quantum number can be entirely accounted for by an $SU(2)_R$ current, arbitrarily far from extremality. This is analogous to the standard BMPV model of rotating BPS black holes in five dimensions [16, 108].

The correction to the black hole entropy due to higher-derivative terms (3.83) is not just independent of black hole parameters; it is the same when computed at the outer and the inner horizons [109]. Therefore, the prescription (3.98) with higher-derivative corrections included identifies the corrections as pertaining to the "Right" sector, with no corrections in the "Left" sector.

The "Left" sector contains the novel excitations, the ones that BPS conditions force into their ground state. These are also the ones that carry the angular momentum of the black hole so the BPS limit is incompatible with rotation. The independence of corrections on black hole parameters suggest that this sector receives no string corrections in the leading approximation. At the level of a phenomenological model this is not unreasonable since, after all, the "Left" sector is subject to $\mathcal{N} = 4$ supersymmetry, albeit spontaneously broken by the state. It would be very interesting to develop such a model in more detail.

Chapter 4

A Non-Renormalization Theorem for Non-Supersymmetric Black Holes

4.1 Introduction and Summary

Precision results for the entropy of BPS black holes give detailed insights into the quantum structure of black holes (see e.g. [5,6,98,99]). The techniques underlying these results involve extrapolation from weak to strong coupling of quantities that are known to be protected by supersymmetry. The physics of black holes with no supersymmetry is much more complicated and it is generally expected that precision results for their entropy is not possible. In this chapter we present evidence that may indicate some precision studies of non-supersymmetric black holes are possible, after all: certain black holes satisfy a non-renormalization theorem when they are embedded in theories with $\mathcal{N} = 2$ supersymmetry even though the black holes themselves do not preserve any supersymmetry, not even an approximate supersymmetry. Moreover, our non-renormalization theorem is protected by a topological invariant.

The objects we study are logarithmic quantum corrections to black hole entropy. The leading order quantum corrections to the Bekenstein-Hawking area law scale with the logarithm of the black hole horizon area. These logarithmic corrections are parameterized by the one-loop quantum effective action of the black hole, which takes the form

$$W = -\frac{\log A}{32\pi^2} \int d^4x \sqrt{-g} \left(aE_4 - cW_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \ldots\right) \ , \tag{4.1}$$

where A is the horizon area, a, c are numerical coefficients, and the dots indicate the presence of other possible four-derivative terms. It is known that these large logarithms offer an infrared window into ultraviolet physics: they are computable in the low energy theory and yield precision data that must be matched by sub-leading terms in the asymptotic density of black hole microstates [14, 20, 21]. Agreement with the microscopic theory has been established in those (highly supersymmetric) cases where precision counting is available [15, 23, 110]. We discuss these logarithms for non-supersymmetric black holes using effective quantum field theory.

The current work is a generalization of the previously-discussed setting in chapter 2. We embed the standard Einstein-Maxwell gauge field $F_{\mu\nu}$ into off-shell $\mathcal{N}=2$ supergravity (with any number of vector multiplets, enumerated by the index I) as

$$F^{+I}_{\mu\nu} = X^I F^{+}_{\mu\nu} \ , \tag{4.2}$$

where X^I and $F^I_{\mu\nu}$ are (respectively) the scalar and vector field strength of a $\mathcal{N}=2$ vector multiplet, and the scalars X^I are taken to be constant (see section 4.2 for more details). In this way, we can obtain non-supersymmetric solutions in $\mathcal{N}=2$ supergravity, such as non-extremal Kerr-Newman black holes. Fluctuations of the $\mathcal{N}=2$ matter exhibit non-minimal couplings in this environment which, by explicit computation, were found to modify the Weyl anomaly coefficients from their standard values such that the total central charge $c=0$ for a complete $\mathcal{N}=2$ multiplet. This chapter complements the explicit computations in chapter 2 by explaining how the null result follows from symmetries, effectively proving a non-renormalization theorem for these non-supersymmetric solutions in $\mathcal{N}=2$ supergravity.

We prove our non-renormalization theorem by exploiting several symmetries which heavily constrain the effective quantum field theory of quantum corrections to black holes. The analysis of each of these symmetries encounters conceptual questions that we address:

- **Duality**: the equations of motion of classical electrodynamics are invariant under electromagnetic duality but the corresponding classical action is not [111]. We show that duality constrains the dependence of the quantum action on the explicit field strength and, in the case of Einstein-Maxwell theory, eliminates it entirely. In this case the dots indicating additional terms in the trace anomaly (4.1) are absent and the effect of matter has been entirely absorbed into the values of the coefficients a, c which then take non-standard values.

- **Supersymmetry**: for black hole solutions to theories with $\mathcal{N}=2$ supersymmetry the quantum effective action is constrained by on-shell supersymmetry. In $d=4$ there are two known distinct four derivative invariants [112, 113]. They complete the two terms written explicitly in (4.1) with particular matter terms and take the schematic form

$$E_4 + \text{SUSY matter} \ , \quad W^2 + \text{SUSY matter} \ , \tag{4.3}$$

in an off-shell formalism. We show that, when evaluated on-shell for our class of solutions (4.2), both $\mathcal{N}=2$ invariants reduce to just the Euler invariant E_4. Thus supersymmetry excludes the second term in the trace anomaly (4.1), so $c=0$.

The significance of this result is that the logarithmic correction to black hole entropy reduces to a topological quantity, independent of the black hole parameters. In particular, it can be deformed from the extremal (supersymmetric) limit to a generic (non-supersymmetric) black hole without any change in value. This property suggests an underlying index theorem, a great surprise in the context of non-supersymmetric black holes.

Our results may superficially appear in conflict with findings obtained in some other areas of inquiry. For example, physical principles require the ratio $c/a \sim 1$ for conformal field theory in a curved background, with precise "conformal collider" bounds easily excluding $c = 0$ [62–64]. Such apparent conflicts are simply due to the additional matter contributions that arise when we take dynamical gravity into account. Our considerations are thus consistent with standard results and complementary to several areas of recent research.

The most obvious generalization of our work would be to understand whether the class of non-supersymmetric solutions (4.2) for which our non-renormalization theorem $c=0$ holds can be broadened and generalized further. In particular, it would be interesting to analyze solutions with non-constant scalars. However, as we discuss, the possible four-derivative corrections to more general backgrounds are expected to involve more (and more complicated) supersymmetric invariants, especially when the scalars are not constant. This will require the introduction of new four-derivative supersymmetric invariants beyond the two we consider.

Our calculations derive a $c=0$ non-renormalization theorem from the symmetries of $\mathcal{N}=2$ supergravity. It would be interesting to understand the $c=0$ result from the different perspective of a (super-)index theorem in the spirit of other gravitational indices (such as e.g. [52]). The strategy employed to establish such theorems involve relating quadratic fluctuations of bosons around the background to those of fermions. When the non-zero modes can be shown to cancel, the only contribution to the quantum corrections comes from the zero modes and is thus topological. There are many examples where this mechanism applies but they generally rely on supersymmetry preserved by the background. It would be novel if index theorems can be generalized to non-supersymmetric backgrounds such as ours. If it is possible it might also help understand how and when one could generalize our non-renormalization theorem to a broader class of solutions.

As stressed in the opening, an important motivation for this work is the potential for a microscopic understanding of black hole entropy and quantum corrections to it. Detailed microscopic models have been established for various types of supersymmetric black holes, using tools inherent to supersymmetry. Analogous microscopic descriptions of non-supersymmetric black holes are typically elusive and, if known, difficult to handle. Our work identifies a family of non-supersymmetric black holes that enjoys a simple and restricted form of one-loop quantum corrections because they are solutions in a theory with supersymmetry. This suggests an underlying structure that may point toward a microscopic description of such non-supersymmetric black holes.

The rest of this chapter is organized as follows. In section 4.2 we briefly summarize the relevant details of our $\mathcal{N} = 2$ supergravity formalism and the particular class of solutions considered. Then, sections 4.3 and 4.4 address duality and supersymmetry, respectively, and show how these symmetries constrain quantum corrections to black hole entropy. Several appendices review further details, especially of the off-shell formalism for $\mathcal{N} = 2$ supergravity.

4.2 Supergravity Formalism and Black Hole Solutions

Our results on duality in section 4.3 and supersymmetry in section 4.4 make extensive use of both the off-shell and on-shell formulations of $\mathcal{N} = 2$ supergravity. In this section, we review the essential parts needed to understand our methods and the relevant class of black hole solutions. More details of off-shell $\mathcal{N} = 2$ supergravity are reviewed in appendix B.

4.2.1 Field Content

The off-shell formalism realizes $\mathcal{N} = 2$ supergravity in four dimensions by imposing constraints on superconformal multiplets whose fields transform under the $\mathcal{N} = 2$ superconformal group. The most important of these multiplets is the Weyl multiplet, which contains the gauge fields associated with each of the superconformal symmetry generators. The independent fields in this Weyl multiplet are

$$\left(e_\mu{}^a \ , \ \psi_\mu^i \ , \ b_\mu \ , \ A_\mu \ , \ \mathcal{V}_\mu{}^i{}_j \ , \ T^-_{\mu\nu} \ , \ \chi^i \ , \ D \right) \ , \tag{4.4}$$

where $e_\mu{}^a$ is the metric vierbein, ψ_μ^i is the gravitino, b_μ is the dilatation generator, A_μ is an auxiliary $U(1)_R$ gauge field, $\mathcal{V}_\mu{}^i{}_j$ is an auxiliary $SU(2)_R$ gauge field, $T^-_{\mu\nu}$ is an auxiliary anti-self-dual tensor, χ^i is an auxiliary $SU(2)$ doublet of Majorana spinors, and D is an auxiliary real scalar field. The Weyl multiplet has $24 + 24$ bosonic and fermionic degrees of freedom off-shell.

We will introduce matter in the form of $n_V + 1$ off-shell $\mathcal{N} = 2$ vector multiplets, denoted by $\mathbf{X}^I$ where $I = 0, \ldots, n_V$. These will reduce down to n_V physical vector multiplets in the on-shell theory. The field content of the vector multiplets is

$$\mathbf{X}^I = \big(X^I \ , \ \Omega_i^I \ , \ W_\mu^I \ , \ Y_{ij}^I \ \big) \ , \tag{4.5}$$

where X^I is a complex scalar, Ω_i^I is an $SU(2)$ doublet of chiral gauginos, W_μ^I is a $U(1)$ vector gauge field, and Y_{ij}^I is an auxiliary $SU(2)$ triplet of real scalars. Each vector multiplet has $8+8$ degrees of freedom off-shell. The scalars X^I have Weyl weight $w=1$ and $U(1)_R$ charge (referred to as a chiral weight) $c=-1$, while their Hermitian conjugates $\bar{X}^I$ have the same Weyl weight and opposite chiral weight. The vector fields W_μ^I are uncharged under the $U(1)_R$ symmetry.

The field strengths of the auxiliary $U(1)_R$ gauge field A_μ and the auxiliary $SU(2)_R$ gauge field $\mathcal{V}_\mu{}^i{}_j$ are (respectively)

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{4.6}$$

$$\mathcal{V}_{\mu\nu}{}^i{}_j \equiv \partial_\mu \mathcal{V}_\nu{}^i{}_j - \partial_\nu \mathcal{V}_\mu{}^i{}_j + \frac{1}{2}\mathcal{V}_\mu{}^i{}_k \mathcal{V}_\nu{}^k{}_j - \frac{1}{2}\mathcal{V}_\nu{}^i{}_k \mathcal{V}_\mu{}^k{}_j \ . \tag{4.7}$$

The field strengths of the vector multiplet gauge fields are

$$F_{\mu\nu}^I \equiv \partial_\mu W_\nu^I - \partial_\nu W_\mu^I \ . \tag{4.8}$$

We will also make use of the supercovariant field strengths

$$\mathcal{F}_{\mu\nu}^{-I} \equiv F_{\mu\nu}^{-I} - \frac{1}{4}\bar{X}^I T_{\mu\nu}^- \ , \quad \mathcal{F}_{\mu\nu}^{+I} \equiv F_{\mu\nu}^{+I} - \frac{1}{4} X^I T_{\mu\nu}^+ \ , \tag{4.9}$$

where $F_{\mu\nu}^{\pm I}$ are the (anti-)self-dual parts of the vector multiplet field strengths.

4.2.2 Two-Derivative Theory

The couplings between the vector multiplets and the Weyl multiplet can be specified succinctly by a prepotential

$$F = F^{(0)}(X^I) \ , \tag{4.10}$$

a meromorphic function of the complex scalars in the vector multiplets. Its derivatives are denoted:

$$F_I \equiv \frac{\partial F}{\partial X^I} \ , \quad F_{\bar{I}} \equiv \frac{\partial F}{\partial \bar{X}^I} = 0 \ , \tag{4.11}$$

where the vanishing of the anti-holomorphic derivative follows from holomorphy. The prepotential is homogeneous with degree two under Weyl transformations. The vector multiplet scalars have Weyl weight one so $F^{(0)}$ must satisfy

$$F^{(0)}(\lambda X^I) = \lambda^2 F^{(0)}(X^I) \ . \tag{4.12}$$

The two-derivative Lagrangian that couples the vector and Weyl multiplets via the prepotential (4.10) is

$$\begin{aligned} 8\pi\mathcal{L}^{(2)} = \Bigg[& i\mathcal{D}^\mu F_I^{(0)} \mathcal{D}_\mu \bar{X}^I - i F_I^{(0)} \bar{X}^I \left(\frac{1}{6}R - D\right) - \frac{i}{8} F_{IJ}^{(0)} Y_{ij}^I Y^{Jij} \\ & + \frac{i}{4} F_{IJ}^{(0)} \mathcal{F}_{\mu\nu}^{-I} \mathcal{F}^{-\mu\nu J} - \frac{i}{8} F_I^{(0)} \mathcal{F}_{\mu\nu}^{+I} T^{+\mu\nu} - \frac{i}{32} F^{(0)} T_{\mu\nu}^{+} T^{+\mu\nu} \Bigg] + \text{h.c.} \\ & + (\text{fermions}) \ . \end{aligned} \tag{4.13}$$

We can reduce the superconformal symmetry to a Poincaré symmetry and further simplify the theory by imposing a consistent truncation

$$b_\mu = Y_{ij}^I = \mathcal{V}_\mu{}^i{}_j = \text{fermions} = 0 \ , \quad D = -\frac{1}{3}R \ , \quad i(F_I^{(0)} \bar{X}^I - \bar{F}_I^{(0)} X^I) = \frac{8\pi}{\kappa^2} \ . \tag{4.14}$$

More details are reviewed in appendix B.7. Under this truncation, the two-derivative Lagrangian (4.13) becomes

$$\begin{aligned} \mathcal{L}^{(2)} = -\frac{1}{2\kappa^2} R + \frac{1}{8\pi} \Bigg[& i\mathcal{D}^\mu F_I^{(0)} \mathcal{D}_\mu \bar{X}^I + \frac{i}{4} F_{IJ}^{(0)} \mathcal{F}_{\mu\nu}^{-I} \mathcal{F}^{-\mu\nu J} \\ & - \frac{i}{8} F_I^{(0)} \mathcal{F}_{\mu\nu}^{+I} T^{+\mu\nu} - \frac{i}{32} F^{(0)} T_{\mu\nu}^{+} T^{+\mu\nu} \Bigg] + \text{h.c.} \ . \end{aligned} \tag{4.15}$$

In the truncation (4.14), the supercovariant derivative acts on the scalar fields by

$$\mathcal{D}_\mu X^I = (\partial_\mu + i A_\mu) X^I \ . \tag{4.16}$$

Thus the auxiliary fields $T_{\mu\nu}^-$ and A_μ both appear algebraically in the Lagrangian. Their equations of motion can be solved, yielding

$$T_{\mu\nu}^- = 4 \frac{N_{IJ} \bar{X}^J F_{\mu\nu}^{-I}}{N_{KL} \bar{X}^K \bar{X}^L} \ , \quad A_\mu = i \frac{N_{IJ} \bar{X}^J \partial_\mu X^I}{N_{KL} \bar{X}^K X^L} \ , \tag{4.17}$$

where we have defined the Hermitian symplectic matrix N_{IJ} as

$$N_{IJ} = 2\text{Im} F_{IJ}^{(0)} \ . \tag{4.18}$$

Eliminating the auxiliary fields $T^-_{\mu\nu}$ and A_μ from the action yields the bosonic terms in the familiar $\mathcal{N}=2$ supergravity Lagrangian

$$\mathcal{L}^{(2)} = -\frac{1}{2\kappa^2}R - \frac{1}{8\pi}\mathcal{M}_{I\bar{J}}\partial^\mu X^I \partial_\mu \bar{X}^J - \frac{i}{32\pi}\mathcal{N}_{IJ}F^{+I}_{\mu\nu}F^{+\mu\nu J} + \text{h.c.} \ , \tag{4.19}$$

where the matrices $\mathcal{M}_{I\bar{J}}$ and $\mathcal{N}_{IJ}$ are defined by

$$\mathcal{M}_{I\bar{J}} = N_{IJ} - \frac{N_{IK}\bar{X}^K N_{JL}X^L}{N_{MN}\bar{X}^M X^N} \ , \quad \mathcal{N}_{IJ} = \bar{F}^{(0)}_{IJ} + i\frac{N_{IK}X^K N_{JL}X^L}{N_{MN}X^M X^N} \ . \tag{4.20}$$

The Einstein, Maxwell, and Bianchi equations of the simplified theory are

$$R_{\mu\nu} = -\frac{\kappa^2}{4\pi}\mathcal{M}_{I\bar{J}}\partial_{(\mu}X^I\partial_{\nu)}\bar{X}^J - \frac{i\kappa^2}{8\pi}\mathcal{N}_{IJ}F^{-I}_{\mu\rho}F^{+\rho J}_{\nu} + \text{h.c.} \ , \tag{4.21}$$

$$0 = \nabla_\mu\left(\mathcal{N}_{IJ}F^{+\mu\nu J} - \bar{\mathcal{N}}_{IJ}F^{-\mu\nu J}\right) \ , \tag{4.22}$$

$$0 = \nabla_\mu\left(F^{+\mu\nu I} - F^{-\mu\nu I}\right) \ . \tag{4.23}$$

These are the equations of motion for $\mathcal{N}=2$ supergravity.

4.2.3 Four-Derivative Theory

Our main interest is to constrain the form of one-loop quantum effective action (4.1), which contains four-derivative terms. We therefore need to introduce higher-derivative corrections to the Lagrangian (4.15).

Higher-derivative terms can be constructed in the off-shell $\mathcal{N}=2$ supergravity formalism by additionally coupling the theory to a chiral multiplet $\hat{\mathbf{A}}$. The field content of the chiral multiplet is

$$\hat{\mathbf{A}} = \Big(\hat{A}\ ,\ \hat{\Psi}_i\ ,\ \hat{B}_{ij}\ ,\ \hat{F}^-_{\mu\nu}\ ,\ \hat{\Lambda}_i\ ,\ \hat{C}\Big) \ , \tag{4.24}$$

where $\hat{A}$ and $\hat{C}$ are complex scalars, $\hat{\Psi}_i$ and $\hat{\Lambda}_i$ are both $SU(2)$ doublets of left-handed fermions, $\hat{B}_{ij}$ is a complex $SU(2)$ triplet of scalars, and $\hat{F}^-_{\mu\nu}$ is an anti-self-dual tensor. A chiral multiplet can have any Weyl weight w from which the Weyl and chiral weights of the component fields can be determined. In particular, the scalars $\hat{A}$ and $\hat{C}$ have Weyl weights w and $w+2$ and chiral weights $-w$ and $-w+2$, respectively.

The chiral multiplet will eventually be realized as a composite of the Weyl and vector multiplets such that four-derivative terms are introduced into the action. The truncation (4.14) can be augmented by setting all fermionic and $SU(2)_R$-charged chiral multiplet fields to zero:

$$\hat{\Psi}_i = \hat{\Lambda}_i = \hat{B}_{ij} = 0 \ . \tag{4.25}$$

The prepotential F still determines all couplings in the theory but, in order to introduce higher-derivative interactions, it must be modified to become a function of the chiral multiplet scalar $\hat{A}$ as well as the vector multiplet scalars X^I . It can be expanded as

$$F(X^I, \hat{A}) = \sum_{n=0}^{\infty} F^{(n)}(X^I)\hat{A}^n \ , \tag{4.26}$$

where each successive power of $\hat{A}$ corresponds to introducing two further derivatives to the Lagrangian, so that $F^{(n)}(X^I)$ controls the $(2+2n)$-derivative terms. We are interested only in two-derivative and four-derivative terms, and so we can truncate this series expansion to obtain

$$F(X^I, \hat{A}) = F^{(0)}(X^I) + F^{(1)}(X^I)\hat{A} \ . \tag{4.27}$$

The new function $F^{(1)}(X^I)$ must be homogenous under rescaling of projective scalars such that

$$F^{(1)}(\lambda X^I) = F^{(1)}(X^I) \ . \tag{4.28}$$

It determines the couplings between the Weyl multiplet, vector multiplets, and chiral multiplet in the four-derivative part of the Lagrangian. This four-derivative Lagrangian, under the truncations (4.14) and (4.25), is

$$\begin{aligned}\mathcal{L}^{(4)} = \frac{1}{8\pi}\Big[& i\mathcal{D}^\mu(F_I^{(1)}\hat{A})\mathcal{D}_\mu\bar{X}^I + \frac{i}{4}F_{IJ}^{(1)}\mathcal{F}_{\mu\nu}^{-I}\mathcal{F}^{-\mu\nu J}\hat{A} - \frac{i}{8}F_I^{(1)}\mathcal{F}_{\mu\nu}^{+I}T^{+\mu\nu}\hat{A} \\ & - \frac{i}{32}F^{(1)}T_{\mu\nu}^{+}T^{+\mu\nu}\hat{A} + \frac{i}{2}F_I^{(1)}\mathcal{F}_{\mu\nu}^{-I}\hat{F}^{-\mu\nu} + \frac{i}{2}F^{(1)}\hat{C}\Big] + \text{h.c.} \ .\end{aligned} \tag{4.29}$$

4.2.4 A Class of Solutions

We are particularly interested in a class of (generally non-supersymmetric) solutions within $\mathcal{N}=2$ supergravity determined by the two conditions:

$$\partial_\mu X^I = 0 \ , \tag{4.30}$$

$$\mathcal{F}_{\mu\nu}^{+I} = 0 \ . \tag{4.31}$$

This also implies the complex conjugate equations $\partial_\mu \bar{X}^I = \mathcal{F}_{\mu\nu}^{-I} = 0$.

The condition (4.31) can be re-written at two-derivative order using the definition (4.9) and the auxiliary equation of motion (4.17) to give

$$\left(\delta_K^I - \frac{X^I N_{KJ} X^J}{X^L N_{LM} X^M}\right) F_{\mu\nu}^{+K} = 0 \ . \tag{4.32}$$

For non-degenerate N_{IJ}, the only non-trivial solution is given by

$$F^{+I}_{\mu\nu} = X^I F^+_{\mu\nu} \ , \tag{4.33}$$

where at this point $F_{\mu\nu}$ is simply an arbitrary anti-symmetric two-tensor (and in particular does not yet need to satisfy a Bianchi identity). Once we also use the condition (4.30) of constant scalars, the field $F_{\mu\nu}$ becomes a genuine Maxwell field, and the resulting effective Lagrangian (at two-derivative order) following from $\mathcal{N} = 2$ supergravity is simply the Einstein-Maxwell Lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2\kappa^2}\left(R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{4.34}$$

For this embedding, we note that (4.17) simplifies to

$$T^+_{\mu\nu} = 4F^+_{\mu\nu} \ , \tag{4.35}$$

so the Weyl multiplet "graviphoton" $T^+_{\mu\nu}$ is proportional to the Maxwell field $F^+_{\mu\nu}$. Additionally, the embedding forces the $U(1)_R$ gauge field A_μ to vanish.

These Einstein-Maxwell solutions are in general not supersymmetric. For example, general Kerr-Newman black holes will break all supersymmetries except in the non-rotating, extremal limit. Interestingly, our Einstein-Maxwell solutions retain a remnant of the supersymmetry of the original theory: the embedding conditions (4.30) and (4.31) are exactly the conditions required for the gaugino supersymmetry variation to vanish, as discussed in [38, 81]. We can think of non-supersymmetric Einstein-Maxwell solutions as continuous deformations of supersymmetric ones such that the relation between scalars and vectors demanded by the SUSY attractor mechanism is maintained. Then the vector multiplet fields force the gaugino variations to vanish (but do not necessarily satisfy any of the other BPS conditions).

To summarize, the conditions (4.30), (4.31) reduce the full $\mathcal{N} = 2$ supergravity equations of motion to the much simpler equations of motion for Einstein-Maxwell theory. Conversely, (4.33) defines an embedding into off-shell $\mathcal{N} = 2$ supergravity of any solution to Einstein-Maxwell theory, independent of the prepotential of the supergravity theory.

4.3 Duality Constraints on Four-Derivative Actions

As discussed in section 4.2, the quantum fluctuations of the massless fields in our theory generate a quantum effective action W for our theory. At one-loop order, this quantum effective action contains only four-derivative terms. In this section we show that duality constraints on possible four-derivative terms can be quite restrictive. For Einstein-Maxwell theory, duality restricts the possible four-derivative terms to purely geometric curvature terms; explicit dependence on the field strength $F_{\mu\nu}$ is not possible. This result is maintained for the embedding of Einstein-Maxwell solutions in $\mathcal{N}=2$ supergravity discussed in section 4.2.4 but $\mathcal{N}=2$ supergravity generally allows more terms.

4.3.1 Einstein-Maxwell Theory and Duality

We first review duality symmetry for Einstein-Maxwell theory and show how it restricts the possible four-derivative terms.

Einstein-Maxwell theory (a $U(1)$ vector field coupled minimally to gravity) has the Lagrangian

$$\mathcal{L} = -\frac{1}{2\kappa^2}\left(R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{4.36}$$

The dual field tensor $G_{\mu\nu}$ is defined through the relation

$$i\tilde{G}_{\mu\nu} = 4\kappa^2 \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} \ , \tag{4.37}$$

so that $G_{\mu\nu} = i\tilde{F}_{\mu\nu}$. Note that the factor of i is due to our definition (B.4) of the dual tensor, since then $\tilde{F}_{\mu\nu}$ is purely imaginary when $F_{\mu\nu}$ is real. The equations of motion and the Bianchi identity can be summarized as

$$\nabla_\mu \begin{pmatrix} F^{\mu\nu} \\ G^{\mu\nu} \end{pmatrix} = 0 \ . \tag{4.38}$$

These equations are invariant under $SO(2,\mathbb{R})$ rotations of the vector $(F_{\mu\nu}, G_{\mu\nu})$ or, equivalently, $U(1)$ transformations of the (anti-)self-dual tensors $F^{\pm}_{\mu\nu}$ of the form

$$F'^{\pm}_{\mu\nu} = e^{\pm i\varphi} F^{\pm}_{\mu\nu} \ , \tag{4.39}$$

for any phase factor $e^{i\varphi}$. Since $F^{+}_{\mu\nu}$ and $F^{-}_{\mu\nu}$ transform under this $U(1)$ symmetry with opposite phases, we can write down an obvious duality-invariant tensor

$$\mathcal{I}_{\mu\nu\rho\sigma} \equiv F^{+}_{\mu\nu} F^{-}_{\rho\sigma} \ . \tag{4.40}$$

All duality invariants can be formed from powers of this tensor. Lorentz invariants can then be formed by appropriate contractions of indices.

The Einstein equations following from (4.36) can be written in a manifestly duality-invariant form

$$R_{\mu\nu} = \mathcal{I}_{(\mu\ \ \nu)\rho}^{\ \ \ \rho} \ . \tag{4.41}$$

The Einstein equation trace condition $R = 0$ follows from the fact that there is no way to form a non-zero Lorentz scalar from a single $\mathcal{I}_{\mu\nu\rho\sigma}$ by contracting all indices.

In section 4.3.3 (and also appendix D) we will show that all four-derivative corrections to the action (4.36) must in fact be invariant under the duality symmetry (4.39) even though, as is well-known, the two-derivative action (4.36) is *not* invariant under duality (4.39), but rather must transform in a very particular way such that the equations of motion respect duality symmetry [111]. In anticipation of this result we proceed to form all possible four-derivative Lorentz invariants from $\mathcal{I}_{\mu\nu\rho\sigma}$ and the Riemann tensor by contraction of Lorentz indices.

It is clear from the equations of motion (4.41) that any four-derivative expression where $\mathcal{I}_{\mu\nu\rho\sigma}$ appear with contracted indices reduces to the geometric invariant $R_{\mu\nu}R^{\mu\nu}$. There are two inequivalent ways to contract indices of two distinct $\mathcal{I}_{\mu\nu\rho\sigma}$'s but one can show using the (anti-)self-duality properties of $F^{\pm}_{\mu\nu}$ that both also reduce to the geometric invariant $R_{\mu\nu}R^{\mu\nu}$:

$$\frac{1}{4}\mathcal{I}_{\mu\nu\rho\sigma}\mathcal{I}^{\mu\nu\rho\sigma} = \mathcal{I}_{\mu\nu\rho\sigma}\mathcal{I}^{\mu\rho\nu\sigma} = \mathcal{I}_{\mu\rho\nu}^{\ \ \ \ \rho}\mathcal{I}^{\mu\sigma\nu}_{\ \ \ \ \sigma} = R_{\mu\nu}R^{\mu\nu} \ . \tag{4.42}$$

We can also form mixed duality invariants by contracting the matter tensor $\mathcal{I}_{\mu\nu\rho\sigma}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$. The two distinct contractions again reduce to geometric invariants

$$\frac{1}{2}\mathcal{I}_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \mathcal{I}_{\mu\nu\rho\sigma}R^{\mu\rho\nu\sigma} = \mathcal{I}_{\mu\rho\nu}^{\ \ \ \ \rho}R^{\mu\nu} = R_{\mu\nu}R^{\mu\nu} \ . \tag{4.43}$$

Thus we find that duality symmetry for Einstein-Maxwell theory restricts *all* the on-shell four-derivative terms to a linear combination of only $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $R_{\mu\nu}R^{\mu\nu}$, with no explicit appearance of the field strength $F_{\mu\nu}$.

It has been noticed before that one loop corrections to Einstein-Maxwell theory reduce to pure geometry in this way [21, 114, 115] and a relation to duality was mentioned in [115], but this is the first explicitly-detailed exposition of this feature.

4.3.2 Symplectic Duality Symmetry

We want to discuss four-derivative duality invariants in a much more general theory of $\mathcal{N} = 2$ supergravity. To get started, we first review the extended symplectic duality symmetry of $\mathcal{N} = 2$ supergravity.

In a theory with $n_V + 1$ $U(1)$ gauge fields (and no explicit sources for the gauge fields) there is a $U(n_V + 1)$ compact duality symmetry that rotates the gauge fields and their dual tensors into each other. When there are also scalars in the theory that transform under duality, such as in $\mathcal{N} = 2$ supergravity, the duality symmetry can further be extended to a non-compact (sub)group of $Sp(2n_V + 2, \mathbb{R})$ [111].[1]

The dual field strengths $G_{I\mu\nu}$ (with $I = 0, \ldots, n_V$) generalizing (4.37) are

$$i\tilde{G}_{I\mu\nu} = \frac{\partial(8\pi\mathcal{L})}{\partial F^{I\mu\nu}} \ . \tag{4.44}$$

In the case of the on-shell two-derivative Lagrangian (4.19), the dual field strengths are given by

$$G^+_{I\mu\nu} = \mathcal{N}_{IJ} F^{+J}_{\mu\nu} \ . \tag{4.45}$$

Under the $Sp(2n_V + 2, \mathbb{R})$ symplectic duality symmetry of $\mathcal{N} = 2$ supergravity, the field strengths $F^I_{\mu\nu}$ and the dual field strengths $G_{I\mu\nu}$ form a symplectic vector

$$\mathbb{F}_{\mu\nu} \equiv (F^I_{\mu\nu} \, , \, G_{I\mu\nu}) \ , \tag{4.46}$$

that transforms under duality as

$$\begin{pmatrix} F^I_{\mu\nu} \\ G_{I\mu\nu} \end{pmatrix} \to \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} F^J_{\mu\nu} \\ G_{J\mu\nu} \end{pmatrix} \ , \tag{4.47}$$

where U, Z, W, and V are real matrices that satisfy

$$\begin{aligned} U^T W - W^T U &= 0 \ , \\ Z^T V - V^T Z &= 0 \ , \\ U^T V - W^T Z &= \mathbb{I} \ . \end{aligned} \tag{4.48}$$

The infinitesimal version of this symplectic transformation is

$$\delta \begin{pmatrix} F^I_{\mu\nu} \\ G_{I\mu\nu} \end{pmatrix} = \begin{pmatrix} A^I{}_J & B^{IJ} \\ C_{IJ} & -(A^T)_I{}^J \end{pmatrix} \begin{pmatrix} F^J_{\mu\nu} \\ G_{J\mu\nu} \end{pmatrix} \ , \tag{4.49}$$

where A is an arbitrary real matrix while B and C are real, symmetric matrices. The vector multiplet scalars X^I and the prepotential derivatives also form a symplectic vector

$$\mathbb{X} \equiv (X^I \, , \, F_I) \ , \tag{4.50}$$

[1] Symplectic duality for $\mathcal{N} = 2$ is discussed in detail in e.g. [116–118], and also reviewed in e.g. [38, 46].

that transforms in the same way as the vector field strengths

$$\delta \begin{pmatrix} X^I \\ F_I \end{pmatrix} = \begin{pmatrix} A^I{}_J & B^{IJ} \\ C_{IJ} & -(A^T)_I{}^J \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix} . \tag{4.51}$$

We can form symplectic scalars by taking the symplectic product of any two symplectic vectors $\mathbb{A}, \mathbb{B}$ via the operation

$$\mathbb{A} \cdot \mathbb{B} \equiv \mathbb{A}\,\Omega\,\mathbb{B} \ , \quad \Omega \equiv \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} . \tag{4.52}$$

Such symplectic scalars generalize the invariant tensor (4.40) from Einstein-Maxwell theory. As in that example, they generally transform under Lorentz symmetry, as the Lorentz indices may be uncontracted at this point.

The prepotential $F = F(X^I)$ is not a symplectic scalar, even though it has no symplectic index. By integrating how the functions F_I change under duality transformations, one can find how the prepotential F transforms. The result is that, for a given prepotential F, the symplectic transformations (4.49) fall into two categories:

1. Transformations that preserve the functional form of the prepotential such that it transforms as $F(X^I) \to F(X^I + \delta X^I)$.

2. Transformations that change the functional form of the prepotential.

The former transformations are true symmetries of the theory for a specified prepotential F, while the latter transformations are not symmetries but rather symplectic reparametrizations that transform the equations of motion of the theory into equivalent but different equations of motion [38, 118].

The generalized prepotential $F = F(X^I, \hat{A})$ needed to introduce four-derivative terms depends on a duality-invariant chiral scalar $\hat{A}$ with Weyl weight two. In this setting, the partial derivative F_A of F with respect to this chiral scalar has no symplectic index I. Crucially, F_A has zero Weyl weight and is always a symplectic scalar [38, 116]. This will be important in the discussion later on, particularly in sections 4.3.3 and 4.4.1.

4.3.3 Duality (In)variance of Four-Derivative Corrections

It is well-known that the two-derivative terms in a Lagrangian of a theory with duality symmetry is not itself invariant under duality transformations; the symmetry is manifest only at the level of the equations of motion. The transformation properties of four-derivative terms under duality symmetry are less familiar. We will show that the on-shell four-derivative terms must be duality invariant already at the Lagrangian level in the situations that we are most interested in, but not in general.

Our claim generalizes a result by Gaillard and Zumino [111]. Consider a Lagrangian $\mathcal{L}$ of scalar fields. They showed that if this Lagrangian depends on a duality-invariant parameter λ and the duality transformations of the scalars do not depend on λ, then $\partial_\lambda \mathcal{L}$ is duality invariant. We can apply this argument by identifying λ with the numerical coefficients that appear in front of four-derivative terms in our effective action, and hence conclude that the four-derivative action $\mathcal{L}^{(4)}$ is duality invariant. This reasoning also applies when there are no scalars at all, such as in Einstein-Maxwell theory. Therefore the four-derivative corrections must be duality invariant in this case, justifying the assumption made in section 4.3.1.

However, for higher-derivative corrections to $\mathcal{N} = 2$ supergravity we can generally *not* choose a λ that the scalar transformations do not depend on: duality acts on coupling constants and, specifically, the couplings encoded in the prepotential (4.27) are not duality invariant. By generalizing the result of Gaillard and Zumino [111] to take dependence of these scalar transformations on the coupling constant λ into account, we find (through a simple calculation spelled out in appendix D)

$$\delta\mathcal{L}^{(4)} = -B^{IJ} F_J^{(1)} \hat{A} \frac{\partial \mathcal{L}^{(2)}}{\partial X^I} - B^{IJ} \partial_\mu \left(F_J^{(1)} \hat{A} \right) \frac{\partial \mathcal{L}^{(2)}}{\partial(\partial_\mu X^I)} \ . \tag{4.53}$$

Thus, in general, the four-derivative corrections to $\mathcal{N} = 2$ supergravity are *not* duality invariant. Fortunately, they are not arbitrary: the transformation properties of the four-derivative Lagrangian $\mathcal{L}^{(4)}$ are completely determined by the two-derivative Lagrangian $\mathcal{L}^{(2)}$.

We are particulatly interested in the class of solutions introduced in section 4.2.4 where the scalars are constant and the superconformal field strength $\mathcal{F}^+_{\mu\nu}$ vanishes. In this case the expression (4.15) of the (off-shell) two-derivative Lagrangian $\mathcal{L}^{(2)}$ gives

$$\left[\frac{\partial \mathcal{L}^{(2)}}{\partial(\partial_\mu X^I)} \right]_{\partial_\mu X^I = 0} = 0 \ , \tag{4.54}$$

and, remembering the dependence of $\mathcal{F}^+_{\mu\nu}$ on the scalars X^I, we have

$$\left[\frac{\partial \mathcal{L}^{(2)}}{\partial X^I} \right]_{\partial_\mu X^I = 0,\, \mathcal{F}^+_{\mu\nu} = 0} = \left[\frac{\partial}{\partial X^I} \left(-\frac{i}{8} F_I^{(0)} \mathcal{F}^{+I}_{\mu\nu} T^{+\mu\nu} - \frac{i}{32} F^{(0)} T^+_{\mu\nu} T^{+\mu\nu} + \text{h.c.} \right) \right]_{\mathcal{F}^+_{\mu\nu} = 0} = 0 \ , \tag{4.55}$$

identically (without using the equation of motion). We conclude that the four-derivative Lagrangian must be duality invariant when the scalars are constant and the supercovariant field strength $\mathcal{F}^+_{\mu\nu}$ vanishes:

$$\left[\delta\mathcal{L}^{(4)}\right]_{\partial_\mu X^I=0,\ \mathcal{F}^+_{\mu\nu}=0} = 0\ . \tag{4.56}$$

Importantly, this result does not in any way depend on supersymmetry, neither of the theory nor of the solution. It comes only from the symplectic duality symmetry of the theory.

In section 4.4.1 below we show that in $\mathcal{N}=2$ supergravity the four-derivative corrections are given by (4.67), an expression that only depends on the symplectic scalar function $F_A = F^{(1)}$ and the (symplectically invariant) components of the Weyl multiplet. The discussion in this section shows that corrections of this form must be invariant under duality at the level of the Lagrangian.

4.3.4 Symplectic Invariants with Constant Scalars

It is interesting to investigate how much we can constrain four-derivative terms using symplectic duality invariance alone. In this section, we restrict ourselves to the case with constant scalars and set all fermions to zero but impose no other restrictions on the bosonic fields. This is a generalization of the discussion at the end of section 4.3.1 where we showed that four-derivative corrections to Einstein-Maxwell theory can always be written in terms of curvature invariants involving only geometry.[2]

As a first step we classify all invariants under duality we can construct using at most two symplectic vectors and at most four covariant derivatives. We do not yet impose Lorentz invariance. We will use the notation we introduced in (4.46) and (4.50) for the symplectic vector of the (anti-)self-dual field strengths $\mathbb{F}^\pm_{\mu\nu}$ and the scalar symplectic vector $\mathbb{X}$ (and its complex conjugate $\overline{\mathbb{X}}$). Our starting point is the on-shell Lagrangian (4.19) where auxiliary fields have been integrated out.

At zero-derivative order, the only symplectic invariants are $\mathbb{X}\cdot\mathbb{X}=0$ and $\mathbb{X}\cdot\overline{\mathbb{X}} = 8\pi i/\kappa^2$ (where we have used (4.14)). At one-derivative order, we have the symplectic invariant

$$T^+_{\mu\nu} = -\frac{i\kappa^2}{2\pi}\mathbb{F}^+_{\mu\nu}\cdot\overline{\mathbb{X}}\ , \tag{4.57}$$

and its complex conjugate, where we have recognized the auxiliary Weyl multiplet field $T^+_{\mu\nu}$ from its two-derivative equation of motion (4.17). The only other possible symplectic

[2] We are not yet setting $\mathcal{F}^{+I}_{\mu\nu}=0$ so, as explained in section 4.3.3, the full four-derivative Lagrangian will generally *not* be a duality-invariant. The duality-invariants we find in this section should therefore be viewed as (at most) *part* of the four-derivative Lagrangian for constant scalars when $\mathcal{F}^{+I}_{\mu\nu}\neq 0$; another (mandatory) part of the Lagrangian must be given by a non-duality-invariant term that transforms according to (4.53).

invariant with one derivative is $\mathbb{F}^+_{\mu\nu}\cdot\mathbb{X}$, which vanishes,

$$\mathbb{F}^+_{\mu\nu}\cdot\mathbb{X} = F^{+I}_{\mu\nu}F_I - G^+_{I\mu\nu}X^I = F^{+I}_{\mu\nu}F_I - \mathcal{N}_{IJ}F^{+J}_{\mu\nu}X^I = F^{+I}_{\mu\nu}F_I - F^{+J}_{\mu\nu}F_J = 0 \;, \tag{4.58}$$

using the explicit form (4.45) for the dual tensor $G^+_{I\mu\nu}$ and the special geometry identity $\mathcal{N}_{IJ}X^I = F_I$.

At two-derivative order, we have the symplectic invariants

$$\mathcal{I}_{\mu\nu\rho\sigma} = \mathbb{F}^+_{\mu\nu}\cdot\mathbb{F}^-_{\rho\sigma}, \quad \nabla_\rho T^+_{\mu\nu} = -\frac{i}{2\pi}\kappa^2\nabla_\rho\mathbb{F}^+_{\mu\nu}\cdot\overline{\mathbb{X}}, \quad R_{\mu\nu\rho\sigma}\;. \tag{4.59}$$

There are two other possible candidates but both vanish identically:

$$\mathbb{F}^+_{\mu\nu}\cdot\mathbb{F}^+_{\rho\sigma} = \nabla_\rho(\mathbb{F}^+_{\mu\nu}\cdot\mathbb{X}) = 0\;, \tag{4.60}$$

using (4.45). Using similar arguments, at three-derivative order we can have

$$\nabla_\lambda\mathbb{F}^+_{\mu\nu}\cdot\mathbb{F}^+_{\rho\sigma}, \quad \nabla_\lambda\mathbb{F}^+_{\mu\nu}\cdot\mathbb{F}^-_{\rho\sigma}, \quad \nabla_\rho\nabla_\sigma\mathbb{F}^+_{\mu\nu}\cdot\overline{\mathbb{X}}\;, \tag{4.61}$$

and their complex conjugates. Finally, at four-derivative order, we can have

$$\nabla_\lambda\mathbb{F}^+_{\mu\nu}\cdot\nabla_\omega\mathbb{F}^+_{\rho\sigma}, \quad \nabla_\lambda\nabla_\omega\mathbb{F}^+_{\mu\nu}\cdot\mathbb{F}^-_{\rho\sigma}, \quad \nabla_\lambda\mathbb{F}^+_{\mu\nu}\cdot\nabla_\omega\mathbb{F}^-_{\rho\sigma}, \quad \nabla_\lambda\nabla_\omega\mathbb{F}^+_{\mu\nu}\cdot\mathbb{F}^-_{\rho\sigma}\;, \tag{4.62}$$

and their complex conjugates.

Having now determined all possible symplectic invariants with at most two symplectic vectors, the next step is to multiply such invariants together and contract Lorentz indices to form four-derivative terms that are invariant under Lorentz symmetry as well as symplectic invariance. There are numerous options but the physically interesting ones are subject to further constraints:

1. Candidate terms for four-derivative corrections to $\mathcal{N}=2$ supergravity must have vanishing $U(1)_R$ charge. This is restrictive since $\mathbb{X}$ is charged under $U(1)_R$.

2. We can use the two-derivative on-shell Einstein equation to trade $\mathbb{F}^+_{\mu\rho}\cdot\mathbb{F}^{-\rho}_{\nu}$ for $R_{\mu\nu}$ (a generalization of (4.41)).

3. We can discard terms that are equivalent up to a total derivative.

Using all of these properties we find (through straightforward but tedious calculations involving the (anti-)self-duality of $\mathbb{F}^\pm_{\mu\nu}$) that there are exactly five independent four-derivative symplectic invariant terms:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ , \ R_{\mu\nu}R^{\mu\nu} \ , \ \nabla_\mu T^{+\mu\nu}\nabla^\rho T^-_{\rho\nu} \ , \ R_{\mu\nu}T^{+\mu}{}_\rho T^{-\nu\rho} \ , \ T^{-\rho}_\mu T^{-\mu\nu}T^{+\sigma}_\nu T^+_{\rho\sigma} \ . \tag{4.63}$$

We spell out more details of the calculation leading to (4.63) in appendix E.

It is interesting that, even with the minimal assumptions made in this subsection, all these terms involve only fields from the Weyl multiplet; all explicit dependence on the vector multiplets has been eliminated using symmetries and equations of motion.

4.3.5 The Einstein-Maxwell Embedding in $\mathcal{N}=2$ Supergravity

We are particularly interested in the embedding of Einstein-Maxwell theory in $\mathcal{N}=2$ supergravity with any number n_V of $\mathcal{N}=2$ vector multiplets. As discussed in section 4.2.4, the embedding sets each of the scalars X^I to be fixed at some constant value and, given those scalars, specifies the $\mathcal{N}=2$ vector fields as (4.33)

$$F^{+I}_{\mu\nu} = X^I F^+_{\mu\nu} \ , \tag{4.64}$$

for some Maxwell gauge field $F_{\mu\nu}$. Since this setting has constant scalars, the results from the previous subsection applies. However, in addition, the Einstein-Maxwell embedding (4.64) demands that the superconformal curvature vanishes $\mathcal{F}^{+I}_{\mu\nu}=0$. In this setting the antisymmetric tensor $T^+_{\mu\nu}$ (4.57) in the Weyl multiplet reduces to the Einstein-Maxwell field strength $F^+_{\mu\nu}$, as noted in (4.35). Then the four-derivative invariants in (4.63) either vanish due to the Maxwell-Bianchi equations for the Einstein-Maxwell field strength $F_{\mu\nu}$ or reduce (using the Einstein equation (4.41) for Einstein-Maxwell theory) to pure geometry. We therefore recover that the four-derivative invariants respect the electric-magnetic duality symmetry of Einstein-Maxwell theory discussed in 4.3.1, and so we are left with the two independent geometric invariants, which we can cast as the W^2 and Euler terms

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} \ , \quad E_4 \ . \tag{4.65}$$

It is interesting to trace the origin of the electric-magnetic duality symmetry of $F_{\mu\nu}$ in the underlying $\mathcal{N}=2$ supergravity theory. Indeed, at first sight this duality symmetry is quite mysterious: since $F^{+I}_{\mu\nu}$ transforms like X^I under the $\mathcal{N}=2$ symplectic duality transformations, the embedded Maxwell field $F^+_{\mu\nu}$ defined in (4.64) is actually invariant under $\mathcal{N}=2$ symplectic duality. Therefore the duality symmetry of the Maxwell theory is not a subset of the $\mathcal{N}=2$ duality symmetry.

We must instead pay attention to the $U(1)_R$ symmetry of $\mathcal{N}=2$ supergravity. $F^{+I}_{\mu\nu}$ is uncharged while X^I is charged under this $U(1)_R$. So, according to (4.64), the embedded Maxwell field $F^+_{\mu\nu}$ must be charged under the $U(1)_R$ symmetry with charge opposite to that of X^I. The Maxwell field transforms as (4.39) under the global $U(1)_R$, and so we conclude that the $U(1)$ duality symmetry of the embedded Maxwell field $F_{\mu\nu}$ is identified with the $U(1)_R$ global symmetry of the $\mathcal{N}=2$ theory. Thus it is ultimately the $U(1)_R$ symmetry of the underlying $\mathcal{N}=2$ supergravity that lets us reduce the five symplectic invariants given in (4.63) to the geometric curvature invariants $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $R_{\mu\nu}R^{\mu\nu}$ or, equivalently, W^2 and E_4.

4.4 Supersymmetry Constraints on Four-Derivative Actions

In section 4.3 we showed that the only four-derivative terms allowed in our Einstein-Maxwell embedding (introduced in section 4.2.4) are the geometric invariants E_4 and W^2. Duality prevents explicit dependence on matter. In this section, we show how supersymmetry further constrains the four-derivative terms such that only the Euler invariant E_4 can appear, hence proving that the c-anomaly vanishes.

This section proceeds as follows. In section 4.4.1 we discuss simplifications of the four-derivative Lagrangian (4.29) due to the form of the Einstein-Maxwell embedding introduced in section 4.2.4 . We will go on to discuss the two known four-derivative chiral multiplets, the $\mathbf{W}^2$ multiplet and the $\mathbb{T}(\log\bar{\mathbf{\Phi}})$ multiplet, in section 4.4.2. We will use use the details of these chiral multiplets to show how we are forced to have $c=0$ in section 4.4.3.

4.4.1 Four-Derivative Action in the Einstein-Maxwell Embedding

The general form of the four-derivative part of the Lagrangian is given in (4.29). In the Einstein-Maxwell embedding (4.33) we set

$$F^{+I}_{\mu\nu} = X^I F^+_{\mu\nu}\ , \quad \partial_\mu X^I = 0\ , \tag{4.66}$$

so the supercovariant field strengths $\mathcal{F}^{\pm I}_{\mu\nu}$ vanish, and then the Lagrangian simplifies to

$$\mathcal{L}^{(4)} = \frac{i}{16\pi}F^{(1)}(X^I)\left(\hat{C} - \frac{1}{16}T^+_{\mu\nu}T^{+\mu\nu}\hat{A}\right) + \text{h.c.}\ . \tag{4.67}$$

We recall from (4.27) that the four-derivative prepotential term $F_A = F^{(1)}$ is a function of the vector multiplet scalars, which are all set to a constant in the Einstein-Maxwell embedding. In this context the four-derivative Lagrangian is therefore given by the supersymmetric invariant

$$\mathcal{L}^{-}_{\hat{\mathbf{A}}} = \frac{1}{64}\left(\hat{C} - \frac{1}{16}T^{+}_{\mu\nu}T^{+\mu\nu}\hat{A}\right) , \tag{4.68}$$

plus its Hermitian conjugate. This shows that, when considering the class of Einstein-Maxwell solutions discussed in section 4.2.4, the only four-derivative Lagrangian that respects supersymmetry is made up entirely of Weyl and chiral multiplet fields; no couplings between the chiral and vector multiplets are allowed when the supercovariant field strengths vanish and the scalars are constant. This is a consequence of the symplectic duality symmetry discussed in the previous section.

This supersymmetric invariant (4.68) matches the chiral multiplet density formula discussed in [119], after the truncation (4.25) has been imposed on the chiral multiplet field content.

4.4.2 Chiral Multiplet Supersymmetric Invariants

As discussed in section 4.2.3, our interest in the chiral multiplet $\hat{\mathbf{A}}$ is to introduce higher-derivative terms into the action. In this context the fields that make up the chiral multiplet are not independent fields, but rather composites of fields that are already introduced as components of other superfields. In order to introduce four-derivative interactions (e.g. R^2, F^4, etc.) the chiral multiplet must have a Weyl weight $w = 2$. This also guarantees that the supersymmetric invariant (4.68) is both symplectically invariant and $U(1)_R$ invariant, as required by the discussion in section 4.3.4.

The only known chiral multiplets that fit these criteria are the $\mathbf{W}^2$ multiplet (introduced in [120] and reviewed in detail in [38,80,81]) and the $\mathbb{T}(\log\bar{\mathbf{\Phi}})$ multiplet (introduced in [112]). In the following, we will discuss the basic structures needed to establish the form of the supersymmetric invariant (4.68) for each of these multiplets. These multiplets are discussed in more detail in appendix B.

We first discuss $\mathbf{W}^2$, the more familiar of the two. Constraints can be imposed on the Weyl multiplet (4.4) such that it forms a reduced chiral multiplet, denoted by $\mathbf{W}_{ab}$. Using standard rules for performing algebraic operations on chiral multiplets, we can take the product of this reduced chiral multiplet with itself to obtain a new chiral multiplet $\mathbf{W}^2 \equiv \mathbf{W}_{ab}\mathbf{W}^{ab}$ that is a Lorentz scalar. The components of $\mathbf{W}^2$ are given in appendix B.5.

The supersymmetrized invariant (4.68) with chiral field $\hat{\mathbf{A}} = \mathbf{W}^2$ is

$$
\begin{aligned}
\mathcal{L}^-_{\mathbf{W}^2} &= \frac{1}{64}\left(C|_{\mathbf{W}^2} - \frac{1}{16} T^+_{\mu\nu} T^{+\mu\nu} A|_{\mathbf{W}^2} \right) \\
&= \frac{1}{2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} + \frac{i}{2} {}^* W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} + \frac{1}{8} R^\mu{}_\nu T^-_{\mu\rho} T^{+\nu\rho} + 3D^2 \\
&\quad + \frac{1}{1024} T^-_{\mu\nu} T^{-\mu\nu} T^+_{\rho\sigma} T^{+\rho\sigma} - \frac{1}{4} T^{-\mu\nu} \mathcal{D}_\mu \mathcal{D}^\rho T^+_{\rho\nu} \\
&\quad - 2 A^-_{\mu\nu} A^{-\mu\nu} + \frac{1}{2} \mathcal{V}^-_{\mu\nu}{}^i{}_j \mathcal{V}^{-\mu\nu j}{}_i + (\text{fermions}) \ .
\end{aligned}
\tag{4.69}
$$

$\mathcal{L}^-_{\mathbf{W}^2}$ thus contains a $W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$ term, in addition to many other terms formed from Weyl multiplet fields. It is the supersymmetric completion of $W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$ denoted schematically in (4.3) as "W^2 + SUSY matter".

Next, we discuss the less familiar $\mathbb{T}(\log \bar{\mathbf{\Phi}})$ multiplet. For an arbitrary chiral multiplet $\mathbf{\Phi}$, we can take its Hermitian conjugate and then (using chiral multiplet algebra rules) take the logarithm of this Hermitian conjugate, resulting in the anti-chiral multiplet $\log \bar{\mathbf{\Phi}}$ with Weyl weight $w = 0$. We can act on this multiplet with the kinetic operator $\mathbb{T}$, which introduces two powers of derivatives in order to make the multiplet kinetic [83]. This new kinetic chiral multiplet has Weyl weight $w = 2$ and is denoted $\mathbb{T}(\log \bar{\mathbf{\Phi}})$ [112]. The field content of the $\mathbb{T}(\log \bar{\mathbf{\Phi}})$ multiplet are discussed in appendix B.5.

As discussed in [112], the supersymmetrized invariant (4.68) derived from the chiral multiplet $\hat{\mathbf{A}} = \mathbb{T}(\log \bar{\mathbf{\Phi}})$ can be written as

$$
\begin{aligned}
\mathcal{L}^-_{\mathbb{T}(\log \bar{\mathbf{\Phi}})} &= \frac{1}{64}\left(C|_{\mathbb{T}(\log \bar{\mathbf{\Phi}})} - \frac{1}{16} T^+_{\mu\nu} T^{+\mu\nu} A|_{\mathbb{T}(\log \bar{\mathbf{\Phi}})} \right) \\
&= -R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 - 3D^2 - \frac{1}{8} R^\mu{}_\nu T^-_{\mu\rho} T^{+\nu\rho} - \frac{1}{1024} T^-_{\mu\nu} T^{-\mu\nu} T^+_{\rho\sigma} T^{+\rho\sigma} \\
&\quad + \frac{1}{4} T^{-\mu\nu} \mathcal{D}_\mu \mathcal{D}^\rho T^+_{\rho\nu} + A_{\mu\nu} A^{\mu\nu} - \frac{1}{2} \mathcal{V}^+_{\mu\nu}{}^i{}_j \mathcal{V}^{+\mu\nu j}{}_i \\
&\quad + \frac{1}{2w} \nabla_a \mathbf{V}^a + (\text{fermions}) \ ,
\end{aligned}
\tag{4.70}
$$

where $\mathbf{V}^a$ is given in terms of $\bar{A}|_{\log \bar{\mathbf{\Phi}}}$, $F^+_{ab}|_{\log \bar{\mathbf{\Phi}}}$, and the Weyl multiplet fields as

$$
\begin{aligned}
\mathbf{V}^a &= 4 \mathcal{D}^a \mathcal{D}^2 \bar{A}|_{\log \bar{\mathbf{\Phi}}} - 8 R^{ab} \mathcal{D}_b \bar{A}|_{\log \bar{\mathbf{\Phi}}} + \frac{8}{3} R \mathcal{D}^a \bar{A}|_{\log \bar{\mathbf{\Phi}}} - 8i A^{ab} \mathcal{D}_b \bar{A}|_{\log \bar{\mathbf{\Phi}}} \\
&\quad - T^{-ac} T^+_{bc} \mathcal{D}^b \bar{A}|_{\log \bar{\mathbf{\Phi}}} + \frac{1}{2} (\mathcal{D}^a T^+_{bc}) F^{+bc}|_{\log \bar{\mathbf{\Phi}}} + 4 T^{+ac} \mathcal{D}^b F^+_{bc}|_{\log \bar{\mathbf{\Phi}}} \\
&\quad + w \left[\frac{2}{3} \mathcal{D}^a R - 4 \mathcal{D}^a D - \frac{1}{2} \mathcal{D}^b (T^{-ac} T^+_{bc}) \right] ,
\end{aligned}
\tag{4.71}
$$

and w is the Weyl weight of the chiral multiplet $\mathbf{\Phi}$. It is important to note that the only dependence in (4.70) on the details of the chiral multiplet $\mathbf{\Phi}$ is in $\mathbf{V}^a$. This means that, no matter what $\mathbf{\Phi}$ is taken as starting point for the construction, the resulting supersymmetric invariants are the same up to a total derivative.

The supersymmetric completion of the Euler invariant (denoted schematically as "E_4 + SUSY matter" in (4.3)) is the sum of the two four derivative terms introduced in this subsection:

$$\begin{aligned}\mathcal{L}_\chi^- &= \mathcal{L}_{\mathbf{W}^2}^- + \mathcal{L}_{\mathbb{T}(\log \bar{\mathbf{\Phi}})}^- \\ &= \frac{1}{2}E_4 + \frac{i}{2}{}^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + A_{\mu\nu}\tilde{A}^{\mu\nu} + \frac{1}{2}\mathcal{V}_{\mu\nu}{}^i{}_j\tilde{\mathcal{V}}^{\mu\nu j}{}_i + \frac{1}{2w}\nabla_a\mathbf{V}^a + (\text{fermions}) \ .\end{aligned} \tag{4.72}$$

As discussed in section 4.2.1, we can consistently truncate the full off-shell supergravity theory down to one with only a subset of the full bosonic content by using the truncation ansatz (4.14). The result of this truncation, when applied to the supersymmetric invariants, is

$$\begin{aligned}\mathcal{L}_{\mathbf{W}^2}^- &= \frac{1}{2}E_4 + \frac{i}{2}{}^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \left(R_{\mu\nu} + \frac{1}{16}T^-_{\mu\rho}T^{+\rho}_{\nu}\right)^2 - \frac{1}{4}T^{-\mu\nu}\mathcal{D}_\mu\mathcal{D}^\rho T^+_{\rho\nu} - 2A^-_{\mu\nu}A^{-\mu\nu} \ , \\ \mathcal{L}_{\mathbb{T}(\log \bar{\mathbf{\Phi}})}^- &= -\left(R_{\mu\nu} + \frac{1}{16}T^-_{\mu\rho}T^{+\rho}_{\nu}\right)^2 + \frac{1}{4}T^{-\mu\nu}\mathcal{D}_\mu\mathcal{D}^\rho T^+_{\rho\nu} + A_{\mu\nu}A^{\mu\nu} + \frac{1}{2w}\nabla_a\mathbf{V}^a \ , \\ \mathcal{L}_\chi^- &= \frac{1}{2}E_4 + \frac{i}{2}{}^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + A_{\mu\nu}\tilde{A}^{\mu\nu} + \frac{1}{2w}\nabla_a\mathbf{V}^a \ .\end{aligned} \tag{4.73}$$

4.4.3 Supersymmetric Invariants in the Einstein-Maxwell Embedding

The final equations for the supersymmetric invariants (4.73) are for any solution that satisfies the consistent truncation (4.14). We now further restrict to Einstein-Maxwell solutions that result from the Einstein-Maxwell embedding (4.33). Then the remaining auxiliary fields are set to

$$T^+_{\mu\nu} = 4F^+_{\mu\nu} \ , \quad A_\mu = 0 \ , \tag{4.74}$$

where $F_{\mu\nu}$ is a $U(1)$ field strength that sources the geometry via an effective Einstein-Maxwell action

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2\kappa^2}\left(R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{4.75}$$

For such backgrounds the supersymmetric invariants (4.73) simplify to

$$\begin{aligned}\mathcal{L}^{-}_{\mathbf{W}^2} &= \frac{1}{2}E_4 + \frac{i}{2}{}^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \big(R_{\mu\nu} + F^{-}_{\mu\rho}F^{+\rho}_{\nu}\big)^2 - \frac{1}{4}F^{-\mu\nu}\nabla_\mu\nabla^\rho F^{+}_{\rho\nu}\ ,\\ \mathcal{L}^{-}_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} &= -\big(R_{\mu\nu} + F^{-}_{\mu\rho}F^{+\rho}_{\nu}\big)^2 + \frac{1}{4}F^{-\mu\nu}\nabla_\mu\nabla^\rho F^{+}_{\rho\nu} + \frac{1}{2w}\nabla_a\mathbf{V}^a\ ,\\ \mathcal{L}^{-}_{\chi} &= \frac{1}{2}E_4 + \frac{i}{2}{}^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{1}{2w}\nabla_a\mathbf{V}^a\ .\end{aligned} \tag{4.76}$$

The Einstein equation (4.21) and the Maxwell-Bianchi equations (4.22), (4.23) for Einstein-Maxwell embedding solutions become

$$R_{\mu\nu} = -F^{-}_{\mu\rho}F^{+\rho}_{\nu}\ ,\quad \nabla_\mu F^{\pm\mu\nu} = 0\ , \tag{4.77}$$

which are just the familiar equations of motion for the effective action (4.75). If we now take the allowed four-derivative Lagrangians in (4.76), put them on-shell by using these Einstein-Maxwell equations of motion (4.77), and drop any total derivative terms in the Lagrangians[3], we find that they collapse almost entirely:

$$\mathcal{L}^{-}_{\mathbf{W}^2} = \mathcal{L}^{-}_{\chi} = \frac{1}{2}E_4\ ,\quad \mathcal{L}^{-}_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} = 0\ . \tag{4.78}$$

We have hence shown that, when considering Einstein-Maxwell solutions in $\mathcal{N}=2$ supergravity, the supersymmetrized Weyl and Euler invariants coincide, while the supersymmetric invariant corresponding to the $\mathbb{T}(\log\bar{\mathbf{\Phi}})$ multiplet becomes trivial.

We first note that all field strength terms have dropped out of the allowed four-derivative Lagrangians (4.78). This was expected, based on how the analysis of section 4.3 showed that electromagnetic duality prohibits such terms. However, the duality analysis allowed for the possibility of independent W^2 and E_4 terms in the four-derivative Lagrangian, since both terms are purely geometric.

What we have shown in (4.78) is that supersymmetry does not allow for a W^2 term in the four-derivative action. Both the supersymmetrized Euler and supersymmetrized Weyl invariants coincide on-shell with the ordinary Euler invariant. Supersymmetry is therefore responsible for drastic simplifications to the four-derivative action, even for solutions that do not preserve any supersymmetries of the theory itself. This generalizes our results from chapter 3, where we showed that the supersymmetrized Weyl invariant coincides with the

[3]Although the Euler invariant E_4 can be written as a total derivative in four dimensions, it is a total derivative acting on (non-covariant) Christoffel symbols that do not fall off to zero at infinity. Its contribution to the one-loop effective action is therefore not automatically zero and is instead proportional to the Euler characteristic of the spacetime. The total derivative terms we drop are ${}^*W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ and $\nabla_a\mathbf{V}^a$, both of which give a vanishing contribution to the effective action.

Gauss-Bonnet invariant for Einstein-Maxwell solutions to minimal $\mathcal{N} = 2$ supergravity.

In summary, we have shown that the c-anomaly must vanish for Einstein-Maxwell solutions embedded in $\mathcal{N} = 2$ supergravity: supersymmetry at the level of the effective action guarantees that no W^2 term can appear. The result applies to each individual $\mathcal{N} = 2$ multiplet by itself and confirms our explicit computations from chapter 2. It applies for any Einstein-Maxwell solutions, including those that are not supersymmetric. As discussed before, the logarithmic corrections to black hole entropy are therefore topological. In particular, they are independent of continuous parameters such as the black hole mass. The coefficient of the logarithmic correction remains the same as we deform a supersymmetric black hole off extremality and break supersymmetry by any amount.

Appendix A

Zero Modes

We initially defined the heat kernel $D(s)$ in (2.12) to include zero modes of Λ, so

$$D(s) = \sum_i e^{-s\lambda_i} = \sum_{\lambda_i \neq 0} e^{-s\lambda_i} + N_{\text{zm}} \ , \tag{A.1}$$

where N_{zm} is the number of zero modes (i.e. the number of distinct eigenvalues of Λ that are zero). The contribution from the zero modes in (A.1) affects only the constant term and not any other terms in $D(s)$.

This contribution from zero modes must be reconsidered carefully. The schematic Euclidean path integral representation of the one-loop effective action (2.4) does not apply to zero modes, as the functional integral over the fields is no longer a Gaussian. Instead, the zero mode piece of the path integral reduces to ordinary integrals over the symmetry groups that give rise to these zero modes. These integrals depend on the scaling dimensions of the symmetry groups. Contributions from zero modes were included in our local expressions but with an incorrect weight of 1, as in (A.1). The correction due to the actual scaling dimension of the zero modes is

$$C_{\text{zm}} = -\sum_{i \in B} (\Delta_i - 1) N_{\text{zm}}^{(i)} + \sum_{i \in F} (2\Delta_i - 1) N_{\text{zm}}^{(i)} \ . \tag{A.2}$$

where Δ is the scaling dimension of the field and N_{zm} is the number of zero modes associated with that field [14, 22, 43]. The fermionic zero modes have the opposite sign as bosonic zero modes to account for fermion spin statistics. The fermionic scaling dimensions also count with double weight due to spin degeneracy.

The correct treatment of zero modes introduces the correction C_{zm} into our expressions for the entropy in chapter 2, specifically in (2.50) and (2.217). As discussed in section 2.2, these expressions describe the logarithmic correction to the entropy in the microcanonical ensemble where M, Q, and J are fixed. In general $C_{\rm zm}$ can depend on how these quantities have been fixed. This correction has been computed in many different cases [14, 15, 20, 22]. We collect these different results and present them compactly as

$$C_{\rm zm} = -(3+K) + 2N_{\rm SUSY} + 3\delta \ , \tag{A.3}$$

where

$$\begin{aligned} K &= \begin{cases} 1 & \text{for } J_3 \text{ fixed with } \vec{J}^2 \text{ arbitrary} \\ 3 & \text{for } J_3 = \vec{J}^2 = 0 \end{cases} \ , \\ N_{\rm SUSY} &= \begin{cases} 4 & \text{for BPS black holes} \\ 0 & \text{for non-BPS black holes} \end{cases} \ , \\ \delta &= \begin{cases} 1 & \text{for non-extremal black holes} \\ 0 & \text{for extremal black holes} \end{cases} \ . \end{aligned} \tag{A.4}$$

Scalars and spin-1/2 fermions have no zero modes. Vector fields have scaling dimension $\Delta_1 = 1$, so there are no corrections due to vector zero modes. All zero modes in the vector and gravitino multiplets are due to vector fields and thus these multiplets do not get corrected. Therefore we only need consider the fields in the gravity multiplet.

The metric has scaling dimension $\Delta_2 = 2$ and $3+K$ zero modes. There are 3 zero modes associated with translational invariance and K zero modes associated with the number of rotational isometries of the black hole solution.

The fermionic zero modes have scaling dimension $\Delta_{3/2} = \frac{3}{2}$. For BPS black hole solutions there are 4 SUSY zero modes, but there are no fermionic zero modes when the background does not preserve SUSY.

Non-extremal black holes have a finite temperature and thus we assume the inverse temperature β scales with the length scale of the black hole, as opposed to the extremal limit where $\beta \to \infty$. We thus have to consider a finite IR volume of integration, which gives a 3δ contribution to (A.3) that exactly cancels the translational zero modes for non-extremal black holes [20].

Appendix B

Off-Shell 4D $\mathcal{N} = 2$ Supergravity

In this section, we summarize some of the important technical details of four-dimensional $\mathcal{N} = 2$ supergravity in the off-shell formalism. These details have been studied extensively in previous works [38, 81, 83–85, 121]. We will first discuss the construction of the relevant supersymmetry multiplets and then go into detail discussing the bosonic part of the $\mathcal{N} = 2$ conformal supergravity action that couples these multiplets together, complete with higher-derivative interactions. We then go on to show how, through appropriate gauge-fixing, we can obtain a Poincaré supergravity action. We conclude with the consistent bosonic truncation that we make use of in this work.

B.1 Notation

The setting is a 4D Lorentzian spacetime, with $(-+++)$ signature, where spacetime indices (also known as curved space indices) are denoted by $\mu, \nu, \ldots$ and flat tangent space indices by $a, b, \ldots$. Many of the fields of consideration will also be charged under an $SU(2)$ gauge group, and we will denote the corresponding $SU(2)$ indices of these fields by $i, j, \ldots$. We denote antisymmetrized and symmetrized indices by

$$[\mu\nu] = \frac{1}{2}(\mu\nu - \nu\mu) , \quad (\mu\nu) = \frac{1}{2}(\mu\nu + \nu\mu) , \tag{B.1}$$

with similar expressions for tangent space indices and $SU(2)$ indices.

The spacetime metric is $g_{\mu\nu}$ and the flat space metric is η_{ab}. The two are related via the vierbein $e_\mu{}^a$, allowing conversion between tangent space indices and curved space indices on any Lorentz tensor. As such, we will be casual about whether we use flat or curved indices. The only time where the distinction is important is in determining how the supercovariant derivative acts, as it acts non-trivially on the vierbein, and thus the supercovariant derivative acts differently on tensors in flat space differently than tensors in curved space.

We will also make extensive use of the Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$, a totally anti-symmetric tensor normalized by

$$\varepsilon_{0123} = \sqrt{-g} \ , \quad \varepsilon^{0123} = -\frac{1}{\sqrt{-g}} \ . \tag{B.2}$$

In flat Minkowski space, the metric determinant is $\sqrt{-g} = 1$, and so this simply becomes the usual Lorentzian Levi-Civita symbol. The Levi-Civita tensor satisfies the contraction identity

$$\varepsilon_{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_p} \varepsilon^{\mu_1 \ldots \mu_n \rho_1 \ldots \rho_p} = -n!\, p!\, \delta^{[\rho_1}_{\nu_1} \ldots \delta^{\rho_p]}_{\nu_p} \ . \tag{B.3}$$

For a $U(1)$ field strength $F_{\mu\nu}$, we will denote the dual field strength by

$$\tilde{F}_{\mu\nu} \equiv -\frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \ . \tag{B.4}$$

We can also express the (anti-)self-dual parts of this field strength as

$$F^{\pm}_{\mu\nu} \equiv \frac{1}{2} \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right) \ . \tag{B.5}$$

B.2 Superconformal Gravity and the Weyl Multiplet

We first want to construct an $\mathcal{N} = 2$ superconformal gauge theory in which all of the generators act as internal symmetries. To do so, we can take the generators of the $\mathcal{N} = 2$ superconformal algebra and introduce a gauge field associated with each generator. These generators and associated gauge fields are given in table B.1.

In principle, we need to define a derivative operator D_μ that is covariant with respect to the full set of $\mathcal{N} = 2$ superconformal symmetries. Acting with the fully supercovariant derivative on fields can in general yield very lengthy and complicated expressions due to the multitude of gauge fields. We can define a new, simpler derivative operator $\mathcal{D}_\mu$ that is covariant with respect to Lorentz transformations, dilatations, R-symmetry transformations, and whatever other internal gauge transformations the field transforms under. For example, if $\phi^{\mu_1 \cdots \mu_n}$ is a bosonic field with a Weyl weight w, a chiral $U(1)_R$ weight c, and no $SU(2)_R$ charge, the covariant derivative $\mathcal{D}_\mu$ acts on $\phi^{\mu_1 \cdots \mu_n}$ by

$$\mathcal{D}_\mu \phi^{\mu_1 \cdots \mu_n} = (\nabla_\mu - w b_\mu - ic A_\mu) \phi^{\mu_1 \cdots \mu_n} \ , \tag{B.6}$$

where ∇_μ is the ordinary covariant derivative in curved space with respect to Lorentz transformations. We will eventually gauge-fix such that we obtain a Poincaré supergravity theory

Transformation	Generator	Gauge Field
Translations	P_a	$e_\mu{}^a$
Lorentz	M_{ab}	ω_μ^{ab}
Dilatations	D	b_μ
Special conformal	K_a	$f_\mu{}^a$
$SU(2)_R$	$V_i{}^j$	$\mathcal{V}_\mu{}^i{}_j$
$U(1)_R$	A	A_μ
Q-supersymmetry	Q^i	ψ_μ^i
S-supersymmetry	S^i	ϕ_μ^i

Table B.1: $\mathcal{N} = 2$ superconformal symmetries and their corresponding generators in the $\mathcal{N} = 2$ superconformal algebra, as well as the gauge fields associated with each transformation.

in section B.6 and then truncate the theory such that all fermions and $SU(2)_R$ charged fields are set to zero in section B.7, all of which will make the covariant derivative (B.6) more useful than the full supercovariant derivative D_μ.

To now obtain a conformal supergravity theory, the superconformal symmetries must be realized as spacetime symmetries instead of internal ones. This leads to the (conventional) constraints that make the fields

$$\omega_\mu^{ab} \ , \quad \phi_\mu^i \ , \quad f_\mu{}^a \ , \tag{B.7}$$

into composite fields. In doing so, we are forced to introduce new auxiliary degrees of freedom in the form of an anti-self-dual tensor T_{ab}^-, an $SU(2)$ doublet of Majorana spinors χ^i, and a real scalar field D[1].

The remaining independent gauge fields, along with these new auxiliary degrees of freedom, form a superconformal gauge multiplet known as the Weyl multiplet. The Weyl multiplet, introduced in (4.4), can be represented as

$$\left(e_\mu{}^a \ , \ \psi_\mu^i \ , \ b_\mu \ , \ A_\mu \ , \ \mathcal{V}_\mu{}^i{}_j \ , \ T_{\mu\nu}^- \ , \ \chi^i \ , \ D \ \right) \ , \tag{B.8}$$

with 24+24 off-shell bosonic and fermionic degrees of freedom.

[1]At this point, we have presented the R-symmetry gauge fields as real, physical fields. However, the $SU(2)_R$ gauge field will eventually be gauge-fixed to zero, and the $U(1)_R$ gauge field does not have a kinetic term at two-derivative order in the action. We can therefore, from the perspective of the on-shell $\mathcal{N} = 2$ supergravity formalism, consider these to be auxiliary fields with no true dynamical degrees of freedom.

B.3 Other $\mathcal{N} = 2$ Superconformal Multiplets

We now want to introduce matter in the form of other superconformal multiplets. In this section, we will detail the field content of the vector, chiral, and non-linear multiplets.

The first multiplet we will consider is the vector multiplet given in (4.5). It is denoted as

$$\mathbf{X}^I = \left(X^I \ , \ \Omega_i^I \ , \ W_\mu^I \ , \ Y_{ij}^I \right) \ , \tag{B.9}$$

with 8+8 off-shell bosonic and fermionic degrees of freedom in the form of a complex scalar X^I, an $SU(2)$ doublet of chiral gauginos Ω_i^I, a vector field W_μ^I, and an auxiliary $SU(2)$ triplet of real scalars Y_{ij}^I. These vector multiplets are indexed by I. We need at least one in the theory in order to have enough degrees of freedom to gauge-fix down to Poincaré supergravity. From the perspective of the on-shell formalism, one of the vector multiplets will get combined with the Weyl multiplet to form a gravity multiplet, while the remaining off-shell vector multiplets will become physical vector multiplets. We therefore let the index I range over

$$I = 0, \ldots, n_V \ , \tag{B.10}$$

where n_V is the number of physical vector multiplets we want to couple to the gravity multiplet.

The next multiplet we will consider is the chiral multiplet, introduced in (4.24). The field content of the chiral multiplet is

$$\hat{\mathbf{A}} = \Big(\hat{A} \ , \ \hat{\Psi}_i \ , \ \hat{B}_{ij} \ , \ \hat{F}_{ab}^- \ , \ \hat{\Lambda}_i \ , \ \hat{C} \Big) \ , \tag{B.11}$$

with 16 + 16 off-shell degrees of freedom in the form of the complex scalars $\hat{A}$ and $\hat{C}$, $SU(2)$ doublets of left-handed fermions $\hat{\Psi}_i$ and $\hat{\Lambda}_i$, an $SU(2)$ triplet of complex scalars $\hat{B}_{ij}$, and an anti-self-dual tensor $\hat{F}_{ab}^-$ that is antisymmetric in its indices[2]. The chiral multiplet can in principle be an independent multiplet, but we will eventually consider it to be a composite function of the Weyl and vector multiplet fields in order to introduce higher-derivative interactions into the action.

The last multiplet we will discuss here is the non-linear multiplet, denoted as

$$\left(\Phi^i{}_\alpha \ , \ \lambda^i \ , \ M^{ij} \ , \ V_a \right) \ . \tag{B.12}$$

The non-linear multiplet consists of an $SU(2)$ matrix scalar fields $\Phi^i{}_\alpha$ (where i is the $SU(2)_R$

[2]We write this anti-self-dual tensor in (4.24) as $\hat{F}_{\mu\nu}^-$ instead of $\hat{F}_{ab}^-$. In doing so, we have implicitly converted from tangent space indices to curved space indices via use of the vierbein $e_\mu{}^a$.

index and $\alpha = 1, 2$ is an additional rigid $SU(2)$ index), a spinor doublet λ^i, an antisymmetric matrix of complex scalars M^{ij}, and a real vector field V_a. The constraint

$$\mathcal{D}^\mu V_\mu - \frac{1}{2} V^\mu V_\mu - \frac{1}{4}|M_{ij}|^2 + \mathcal{D}^\mu \Phi^i{}_\alpha \mathcal{D}_\mu \Phi^\alpha{}_i + \text{(fermions)} = D + \frac{1}{3} R \tag{B.13}$$

must be imposed on the non-linear multiplet fields to assure that the multiplet has the correct $8 + 8$ off-shell degrees of freedom.

B.4 Prepotential and the Action

In the previous section, we constructed superconformal multiplets that each transform under some representation of the full $\mathcal{N} = 2$ superconformal group. In particular, we discussed the Weyl multiplet, vector multiplets, chiral multiplets, and non-linear multiplets. We now want a theory that couples together the Weyl multiplet to $n_V + 1$ vector multiplets and a single chiral multiplet. That is, we would like an action that couples all of these multiplets together such that the $\mathcal{N} = 2$ superconformal symmetry is preserved.

One of the ways to accomplish this is to specify the interactions between the Weyl multiplet and the matter fields in the vector and chiral multiplets by introducing a prepotential $F \equiv F(X^I, \hat{A})$, a meromorphic function of the vector multiplet scalars X^I and the chiral multiplet scalar $\hat{A}$. Derivatives of the prepotential are denoted by

$$\frac{\partial F}{\partial X^I} = F_I \ , \quad \frac{\partial F}{\partial \hat{A}} = F_A \ . \tag{B.14}$$

The prepotential is holomorphic and does not depend on the complex conjugate scalars $\bar{X}^I$ and $\bar{\hat{A}}$, and so $F_{\bar{I}} = F_{\bar{A}} = 0$. The prepotential is also homogeneous of second degree with respect to Weyl-weighted scalings of X^I and $\hat{A}$, so

$$F(\lambda X^I, \lambda^w \hat{A}) = \lambda^2 F(X^I, \hat{A}) \ , \tag{B.15}$$

where w is the Weyl weight of the chiral multiplet scalar $\hat{A}$ and λ is some arbitrary scaling constant.

The action is

$$\mathcal{S} = \int d^4x \sqrt{-g}\, \mathcal{L} \ , \tag{B.16}$$

where $\mathcal{L}$ is the Lagrangian for our off-shell theory that couples the Weyl multiplet, the vector multiplets, and the chiral multiplet via interactions dictated by the prepotential:

$$\begin{aligned}
8\pi\mathcal{L} = \Big[& i\mathcal{D}^\mu F_I \mathcal{D}_\mu \bar{X}^I - iF_I\bar{X}^I\left(\frac{1}{6}R - D\right) - \frac{i}{8}F_{IJ}Y^I_{ij}Y^{Jij} \\
& + \frac{i}{4}F_{IJ}\left(F^{-I}_{\mu\nu} - \frac{1}{4}\bar{X}^I T^-_{\mu\nu}\right)\left(F^{-\mu\nu J} - \frac{1}{4}\bar{X}^J T^{-\mu\nu}\right) \\
& - \frac{i}{8}F_I\left(F^{+I}_{\mu\nu} - \frac{1}{4}X^I T^+_{\mu\nu}\right)T^{+\mu\nu} - \frac{i}{32}FT^+_{\mu\nu}T^{+\mu\nu} \\
& + \frac{i}{2}F_{AI}\left(F^{-I}_{\mu\nu} - \frac{1}{4}\bar{X}^I T^-_{\mu\nu}\right)\hat{F}^{-\mu\nu} - \frac{i}{4}F_{AI}\hat{B}_{ij}Y^{Iij} \\
& + \frac{i}{2}F_A\hat{C} - \frac{i}{8}F_{AA}\hat{B}_{ij}\hat{B}_{kl}\varepsilon^{ik}\varepsilon^{jl} + \frac{i}{4}F_{AA}\hat{F}^-_{\mu\nu}\hat{F}^{-\mu\nu}\Big] + \text{h.c.} \\
& + (\text{fermions}) \; .
\end{aligned} \tag{B.17}$$

We will eventually be interested in purely bosonic backgrounds, so we do not need the details of the fermionic terms. The covariant derivative $\mathcal{D}_\mu$ defined in (B.6) acts on the vector multiplet scalars X^I and the chiral multiplet scalar $\hat{A}$ by

$$\mathcal{D}_\mu X^I = (\partial_\mu - b_\mu + iA_\mu)X^I \; , \quad \mathcal{D}_\mu \hat{A} = (\partial_\mu - wb_\mu + iwA_\mu)\hat{A} \; . \tag{B.18}$$

The Lagrangian (B.17) has a term linear in the auxiliary D field

$$8\pi\mathcal{L} = i(F_I\bar{X}^I - \bar{F}_I X^I)\left(D - \frac{1}{6}R\right) + \ldots \; , \tag{B.19}$$

which leads to inconsistent equations of motion. In order to fix this, we can couple the theory to the non-linear multiplet (B.12) such that all linear terms in D are cancelled. We add the term

$$i(F_I\bar{X}^I - \bar{F}_I X^I)\left(\mathcal{D}^\mu V_\mu - \frac{1}{2}V^\mu V_\mu - \frac{1}{4}|M_{ij}|^2 + \mathcal{D}^\mu \Phi^i{}_\alpha \mathcal{D}_\mu \Phi^\alpha{}_i - D - \frac{1}{3}R\right) \tag{B.20}$$

to the Lagrangian, modulo some fermionic terms. The non-linear multiplet constraint (B.13) makes this vanish, allowing us to consistently add it to the Lagrangian and cancel out all explict D-terms in (B.17). The resulting Lagrangian is

$$\begin{aligned}
8\pi\mathcal{L} = &-\frac{i}{2}(F_I\bar{X}^I - \bar{F}_I X^I)R + \Bigg[i\mathcal{D}^\mu F_I \mathcal{D}_\mu \bar{X}^I - \frac{i}{8}F_{IJ}Y_{ij}^I Y^{Jij} \\
&+ \frac{i}{4}F_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-\mu\nu J} - \frac{i}{8}F_I\mathcal{F}^{+I}_{\mu\nu}T^{+\mu\nu} - \frac{i}{32}FT^+_{\mu\nu}T^{+\mu\nu} + \frac{i}{2}F_{AI}\mathcal{F}^{-I}_{\mu\nu}\hat{F}^{-\mu\nu} \\
&- \frac{i}{4}F_{AI}\hat{B}_{ij}Y^{Iij} + \frac{i}{2}F_A\hat{C} - \frac{i}{8}F_{AA}\hat{B}_{ij}\hat{B}_{kl}\varepsilon^{ik}\varepsilon^{jl} + \frac{i}{4}F_{AA}\hat{F}^-_{\mu\nu}\hat{F}^{-\mu\nu}\Bigg] + \text{h.c.} \\
&+ i(F_I\bar{X}^I - \bar{F}_I X^I)\left(\mathcal{D}^\mu V_\mu - \frac{1}{2}V^\mu V_\mu - \frac{1}{4}|M_{ij}|^2 + \mathcal{D}^\mu \Phi^i{}_\alpha \mathcal{D}_\mu \Phi^\alpha{}_i\right) \\
&+ (\text{fermions}) \ ,
\end{aligned} \tag{B.21}$$

where we have defined the supercovariant field strengths

$$\begin{aligned}
\mathcal{F}^{+I}_{\mu\nu} &= F^{+I}_{\mu\nu} - \frac{1}{4}X^I T^+_{\mu\nu} \ , \\
\mathcal{F}^{-I}_{\mu\nu} &= F^{-I}_{\mu\nu} - \frac{1}{4}\bar{X}^I T^-_{\mu\nu} \ .
\end{aligned} \tag{B.22}$$

B.5 Introducing Higher-Derivative Terms

We are interested in studying higher-derivative interactions in $\mathcal{N} = 2$ supergravity. As discussed in section 4.2.3, we can accomplish this by identifying the chiral multiplet (B.11) as a composite multiplet of other fields. In this section, we will discuss the two known chiral multiplets that introduce four-derivative terms into the action.

B.5.1 $\mathbf{W}^2$ Multiplet

The fields in the Weyl multiplet can be also be fit into a chiral multiplet, denoted as

$$\mathbf{W}_{ab} = \left(A_{ab}\, ,\, \Psi_{abi}\, ,\, B_{abij}\, ,\, (F^-_{ab})_{cd}\, ,\, \Lambda_{abi}\, ,\, C_{ab}\right) \ , \tag{B.23}$$

of which the bosonic components are

$$\begin{aligned}
A_{ab}|_{\mathbf{W}_{ab}} &= T^-_{ab} \ , \\
B_{abij}|_{\mathbf{W}_{ab}} &= -8\varepsilon_{k(i}\mathcal{V}^-_{ab}{}^k{}_{j)} \ , \\
(F^-_{ab})^{cd}|_{\mathbf{W}_{ab}} &= -8W^-_{ab}{}^{cd} - 4\left(\delta_{[a}{}^c\delta_{b]}{}^d + \frac{i}{2}\varepsilon_{ab}{}^{cd}\right)D + 16iA^{-[c}_{[a}\delta_{b]}{}^{d]} \ , \\
C_{ab}|_{\mathbf{W}_{ab}} &= 4D_{[a}D^c T^+_{b]c} + 4D^c D_{[a}T^+_{b]c} + 2\Box_c T^+_{ab} \ .
\end{aligned} \tag{B.24}$$

where we have defined

$$W_{ab}^{-\ cd} = \frac{1}{2}\left(W_{ab}{}^{cd} + i^{*}W_{ab}{}^{cd}\right) \ , \quad {}^{*}W_{ab}{}^{cd} = \frac{1}{2}\varepsilon_{ab}{}^{ef}W_{ef}{}^{cd} \ . \tag{B.25}$$

We can then obtain the chiral multiplet $\mathbf{W}^2$ by squaring $\mathbf{W}_{ab}$, i.e.

$$\mathbf{W}^2 = \mathbf{W}_{ab}\mathbf{W}^{ab} \ , \tag{B.26}$$

where chiral multiplets are multiplied using superconformal calculus rules discussed in [119]. The bosonic components of $\mathbf{W}^2$ are

$$\begin{aligned}
A|_{\mathbf{W}^2} &= T_{ab}^{-}T^{-ab} \ , \\
B_{ij}|_{\mathbf{W}^2} &= -16\varepsilon_{k(i}\mathcal{V}_{ab}{}^{k}{}_{j)}T^{-ab} \ , \\
F_{ab}^{-}|_{\mathbf{W}^2} &= -16\left(W_{abcd}T^{-cd} + DT_{ab}^{-} + 2iA_{c[a}T_{b]}^{-c}\right) \ , \\
C|_{\mathbf{W}^2} &= 32\Big(W_{abcd}W^{abcd} + i^{*}W_{abcd}W^{abcd} + 6D^2 - 2A_{ab}A^{ab} \\
&\quad + 2A_{ab}\tilde{A}^{ab} - \frac{1}{2}T^{-ab}\mathcal{D}_a\mathcal{D}^cT_{cb}^{+} + \frac{1}{4}R^a{}_bT_{ac}^{-}T^{+bc} \\
&\quad + \frac{1}{256}T_{ab}^{-}T^{-ab}T_{cd}^{+}T^{+cd} + \frac{1}{2}\mathcal{V}_{ab}{}^{i}{}_{j}\mathcal{V}^{abj}{}_{i} - \frac{1}{2}\mathcal{V}_{ab}{}^{i}{}_{j}\tilde{\mathcal{V}}^{abj}{}_{i}\Big) \ .
\end{aligned} \tag{B.27}$$

The scalar $C|_{\mathbf{W}^2}$ in the $\mathbf{W}^2$ multiplet has (Weyl)^2-type terms in it. This introduces four-derivative terms into the Lagrangian (B.21), making the $\mathbf{W}^2$ chiral multiplet one way to introduce higher-derivative terms into $\mathcal{N} = 2$ supergravity.

B.5.2 $\mathbb{T}(\log\bar{\mathbf{\Phi}})$ Multiplet

Let $\mathbf{\Phi}$ be an arbitrary chiral multiplet, denoted by

$$\mathbf{\Phi} = \left(A\,,\,\Psi_i\,,\,B_{ij}\,,\,F_{ab}^{-}\,,\,\Lambda_i\,,\,C\right) \ , \tag{B.28}$$

The Hermitian conjugate of $\mathbf{\Phi}$ is the anti-chiral multiplet $\bar{\mathbf{\Phi}}$, denoted by

$$\bar{\mathbf{\Phi}} = \left(\bar{A}\,,\,\Psi^i\,,\,B^{ij}\,,\,F_{ab}^{+}\,,\,\Lambda^i\,,\,\bar{C}\right) \ . \tag{B.29}$$

From the chiral multiplet $\mathbf{\Phi}$, we can also construct the chiral multiplet $\log\mathbf{\Phi}$. Ignoring all fermions, the bosonic components of $\log\mathbf{\Phi}$ are related those of $\mathbf{\Phi}$ by

$$\begin{aligned}
A|_{\log\mathbf{\Phi}} &= \log A|_{\mathbf{\Phi}}\ ,\\
B_{ij}|_{\log\mathbf{\Phi}} &= \frac{B_{ij}}{A}\bigg|_{\mathbf{\Phi}}\ ,\\
F^-_{ab}|_{\log\mathbf{\Phi}} &= \frac{F^-_{ab}}{A}\bigg|_{\mathbf{\Phi}}\ ,\\
C|_{\log\mathbf{\Phi}} &= \left(\frac{C}{A}+\frac{1}{4A^2}\left(\varepsilon^{ik}\varepsilon^{jl}B_{ij}B_{kl}-2F^-_{ab}F^{-ab}\right)\right)\bigg|_{\mathbf{\Phi}}\ .
\end{aligned}\tag{B.30}$$

We can also take the Hermitian conjugate of this multiplet to obtain the anti-chiral multiplet $\log\bar{\mathbf{\Phi}}$. We can then construct the chiral kinetic multiplet $\mathbb{T}(\log\bar{\mathbf{\Phi}})$ whose bosonic components are related to the components of $\log\bar{\mathbf{\Phi}}$ by

$$\begin{aligned}
A|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} &= \bar{C}|_{\log\bar{\mathbf{\Phi}}}\ ,\\
B_{ij}|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} &= \left(-2\varepsilon_{ik}\varepsilon_{jl}(\Box_c+3D)B^{kl}-2\varepsilon_{jk}\mathcal{V}^{ab\ k}{}_{i}F^+_{ab}\right)|_{\log\bar{\mathbf{\Phi}}}\ ,\\
F^-_{ab}|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} &= \Big(T^-_{ab}\Box_c\bar{A}-\varepsilon_{ij}\mathcal{V}^-_{ab}{}^{i}{}_{k}B^{jk}+\frac{1}{16}T^-_{ab}T^+_{cd}F^{+cd}\\
&\quad-\Pi^-_{ab}{}^{cd}\left(4D_cD^eF^+_{ed}+(D_cT^-_{de})D^e\bar{A}+(D^eT^-_{ed})D_c\bar{A}-wD_cD^eT^-_{ed}\right)\Big)\bigg|_{\log\bar{\mathbf{\Phi}}}\ ,\\
C|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} &= \Big(4(\Box_c+3D)\Box_c\bar{A}+6(D_aD)D^a\bar{A}-16D^a\left(R(D)^+_{ab}D^b\bar{A}\right)\\
&\quad-\frac{1}{2}D^a(T^+_{ab}T^{-cb}D_c\bar{A})-\frac{1}{4}D^a(T^+_{ab}T^{-cb})D_c\bar{A}+\frac{1}{16}T^+_{ab}T^{+ab}\bar{C}\\
&\quad+\frac{1}{2}\Box_c(T^+_{bc}F^{+bc})+2D_a\left((D^bT^+_{bc})F^{+ac}+T^{+ac}D^bF^+_{bc}\right)-w\mathcal{V}^+_{ab}{}^{i}{}_{j}\mathcal{V}^{+ab\ j}{}_{i}\\
&\quad-8wR(D)^+_{ab}R(D)^{+ab}-\frac{w}{2}D^aT^+_{ab}D_cT^{-cb}-\frac{w}{2}D^a(T^+_{ab}D_cT^{-cb})\Big)\bigg|_{\log\bar{\mathbf{\Phi}}}\ ,
\end{aligned}\tag{B.31}$$

where w is the Weyl weight of the $\mathbf{\Phi}$ multiplet, $\Pi^-_{ab}{}^{cd}$ is the anti-self-dual projection operator

$$\Pi^-_{ab}{}^{cd}=\delta_a{}^{[c}\delta_b{}^{d]}+\frac{i}{2}\varepsilon_{ab}{}^{cd}\ ,\tag{B.32}$$

and $R(D)^+_{ab}$ is the self-dual part of the connection $R(D)_{ab}$ defined by

$$R(D)_{\mu\nu}=2\partial_{[\mu}b_{\nu]}+\frac{i}{2}\tilde{A}_{\mu\nu}\ .\tag{B.33}$$

Note that the derivative operator D_μ appearing in (B.31) is the fully superconformally covariant derivative discussed in section B.2, and the operator $\Box_c \equiv D_\mu D^\mu$ is the superconformal d'Alembertian. These can be expressed in terms of the covariant derivative $\mathcal{D}_\mu$ and its square $\mathcal{D}^2$ [119]. For our purposes, though, we will only need the particular linear combination of these fields appearing in (4.70), which simplifies in such a way that no explicit occurences of the superconformal derivative appear.

B.5.3 Higher-Derivative Action

The Poincaré supergravity Lagrangian (B.21) couples an arbitrary chiral multiplet $\hat{\mathbf{A}}$ to the Weyl and vector multiplets. By identifying this chiral multiplet with a linear combination of $\mathbf{W}^2$ and $\mathbb{T}(\log\bar{\Phi})$, both of which contain four-derivative terms, the Lagrangian will contain (at least) four-derivative terms in it. That is, we will set

$$\hat{\mathbf{A}} = a_1 \mathbf{W}^2 + a_2 \mathbb{T}(\log\bar{\mathbf{\Phi}}) \ , \tag{B.34}$$

for some constants a_1, a_2. This sets the bosonic components of $\hat{\mathbf{A}}$ to be

$$\begin{aligned} \hat{A} &= a_1 A|_{\mathbf{W}^2} \quad + a_2 A|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} \\ \hat{B}_{ij} &= a_1 B_{ij}|_{\mathbf{W}^2} + a_2 B_{ij}|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} \\ \hat{F}^-_{ab} &= a_1 F^-_{ab}|_{\mathbf{W}^2} + a_2 F^-_{ab}|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} \\ \hat{C} &= a_1 C|_{\mathbf{W}^2} \quad + a_2 C|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})} \end{aligned} \tag{B.35}$$

Under this identification, the prepotential $F(X^I, \hat{A})$ is a function of the scalars X^I and $\hat{A} = a_1 A|_{\mathbf{W}^2} + a_2 A|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})}$. Both $A|_{\mathbf{W}^2}$ and $A|_{\mathbb{T}(\log\bar{\mathbf{\Phi}})}$ have Weyl weight $w = 2$, and so the homogeneity relation of the prepotential (B.15) tells us that

$$F_I X^I + 2F_A \hat{A} = 2F \ . \tag{B.36}$$

The Lagrangian (B.21), subject to the identification (B.35), can now contain higher-derivative terms, with the derivative order depending on the form of the prepotential. We will represent the prepotential perturbatively as

$$\begin{aligned} F(X^I, \hat{A}) &= \sum_{n=0}^{\infty} F^{(n)}(X^I)\hat{A}^n \\ &= F^{(0)}(X^I) + F^{(1)}(X^I)\hat{A} + \dots \ , \end{aligned} \tag{B.37}$$

for some functions $F^{(n)}(X^I)$. The zeroth-order function $F^{(0)}(X^I)$ dictates the two-derivative terms in the Lagrangian, the first order function $F^{(1)}(X^I)$ dictates the four-derivative terms in the Lagrangian, and so on. As discussed in section 4.2.3, we truncate the prepotential to finite order:

$$F(X^I,\hat{A}) = F^{(0)}(X^I) + F^{(1)}(X^I)\hat{A}\ , \tag{B.38}$$

in order to have only two- and four-derivative interactions. The bosonic two-derivative part of the Lagrangian is

$$\begin{aligned}8\pi\mathcal{L}^{(2)} = &-\frac{i}{2}(F_I^{(0)}\bar{X}^I - \bar{F}_I^{(0)}X^I)R + \Big[i\mathcal{D}^\mu F_I^{(0)}\mathcal{D}_\mu\bar{X}^I - \frac{i}{8}F_{IJ}^{(0)}Y_{ij}^I Y^{Jij} \\ &+ \frac{i}{4}F_{IJ}^{(0)}\mathcal{F}_{\mu\nu}^{-I}\mathcal{F}^{-\mu\nu J} - \frac{i}{8}F_I^{(0)}\mathcal{F}_{\mu\nu}^{+I}T^{+\mu\nu} - \frac{i}{32}F^{(0)}T_{\mu\nu}^{+}T^{+\mu\nu}\Big] + \text{h.c.} \\ &+ i(F_I^{(0)}\bar{X}^I - \bar{F}_I^{(0)}X^I)\left(\mathcal{D}^\mu V_\mu - \frac{1}{2}V^\mu V_\mu - \frac{1}{4}|M_{ij}|^2 + \mathcal{D}^\mu\Phi^i{}_\alpha\mathcal{D}_\mu\Phi^\alpha{}_i\right)\ ,\end{aligned} \tag{B.39}$$

while the bosonic four-derivative part is

$$\begin{aligned}8\pi\mathcal{L}^{(4)} = &-\frac{i}{2}(F_I^{(1)}\bar{X}^I\hat{A} - \bar{F}_I^{(1)}X^I\bar{\hat{A}})R + \Big[i\mathcal{D}^\mu(F_I^{(1)}\hat{A})\mathcal{D}_\mu\bar{X}^I - \frac{i}{8}F_{IJ}^{(1)}Y_{ij}^I Y^{Jij}\hat{A} \\ &+ \frac{i}{4}F_{IJ}^{(1)}\mathcal{F}_{\mu\nu}^{-I}\mathcal{F}^{-\mu\nu J}\hat{A} - \frac{i}{8}F_I^{(1)}\mathcal{F}_{\mu\nu}^{+I}T^{+\mu\nu}\hat{A} - \frac{i}{32}F^{(1)}T_{\mu\nu}^{+}T^{+\mu\nu}\hat{A} \\ &+ \frac{i}{2}F_I^{(1)}\mathcal{F}_{\mu\nu}^{-I}\hat{F}^{-\mu\nu} + \frac{i}{2}F^{(1)}\hat{C} - \frac{i}{4}F_I^{(1)}\hat{B}_{ij}Y^{Iij}\Big] + \text{h.c.} \\ &+ i(F_I^{(1)}\bar{X}^I\hat{A} - \bar{F}_I^{(1)}X^I\bar{\hat{A}})\left(\mathcal{D}^\mu V_\mu - \frac{1}{2}V^\mu V_\mu - \frac{1}{4}|M_{ij}|^2 + \mathcal{D}^\mu\Phi^i{}_\alpha\mathcal{D}_\mu\Phi^\alpha{}_i\right)\ ,\end{aligned} \tag{B.40}$$

subject to the identifications (B.35) for the chiral multiplet fields.

B.6 Gauge-Fixing Down to Poincaré

The Lagrangian (B.21) has an $\mathcal{N}=2$ superconformal symmetry that acts as an internal symmetry. To obtain an $\mathcal{N}=2$ Poincaré supergravity theory, we must gauge-fix the extra symmetries of the superconformal theory, including special conformal transformations, dilatations, and a local chiral $SU(2)_R\times U(1)_R$ symmetry. We gauge-fix the special conformal symmetry by choosing the K-gauge

$$b_\mu = 0\ . \tag{B.41}$$

To gauge-fix the dilatational symmetry, we choose the D-gauge that sets the Kähler potential to be constant:

$$e^{-\mathcal{K}} \equiv i(F_I \bar{X}^I - \bar{F}_I X^I) = \frac{8\pi}{\kappa^2} \ , \tag{B.42}$$

with the value of the constant chosen to reproduce the standard normalization of the Einstein-Hilbert term in the action. The $SU(2)_R$ symmetry can be gauge-fixed by imposing the V-gauge

$$\Phi^i{}_\alpha = \delta^i{}_\alpha \ , \tag{B.43}$$

on the non-linear multiplet, while the $U(1)_R$ symmetry can be gauge-fixed via the A-gauge condition

$$X^0 = \bar{X}^0 \ . \tag{B.44}$$

The D-gauge (B.42) and A-gauge (B.44) remove two degrees degree of freedom from the vector multiplet scalars, and thus the Poincaré supergravity theory has only n_V independent scalars, as expected.

The gauge choices made here are by no means unique. And, since physical observables should not depend on the choice of gauge, different sets of gauge-fixing conditions can be useful for different types of problems. The gauge choices presented here are typical and useful in a broad class of applications.

B.7 Consistent Truncation

We now have a theory of Poincaré $\mathcal{N} = 2$ supergravity with higher-derivative interactions, introduced by gauging the superconformal symmetries and making particular choices for the chiral multiplet coupled to our theory. We are in principle at the point where we can solve the full set of equations of motion to our theory and investigate particular solutions. However, the action presented thus far is fairly complicated and includes implicit dependence on a great number of fields, some physical and some auxiliary. This makes finding solutions difficult. We will therefore look at how to consistently truncate our theory down to a more manageable set of fields and interactions. We will do this by eliminating auxiliary fields from our theory wherever possible.

We are primarily interested in purely bosonic backgrounds, and so we will turn off all fermions. These backgrounds will still capture the most salient features of our theory, including the structure of black hole entropy corrections. The fields Y^I_{ij} and $\mathcal{V}_\mu{}^i{}_j$ and their derivatives couple either to fermionic terms, or appear at least quadratically with one another. This is true even when higher-derivative terms are present. It is therefore consistent to set them both to zero

$$Y^I_{ij} = 0 \ , \quad \mathcal{V}_\mu{}^i{}_j = 0 \ , \tag{B.45}$$

at the level of the action. Note that this sets the $SU(2)_R$-charged chiral multiplet field $\hat{B}_{ij} = 0$, and so we can ignore all such terms in the action.

Next, we want to eliminate the non-linear multiplet fields V_μ, M_{ij}, and $\Phi^i{}_\alpha$ from our theory. The scalar fields $\Phi^i{}_\alpha$ are easy to eliminate: the V-gauge condition (B.43), combined with setting the $SU(2)_R$ gauge field to zero, makes the derivative $\mathcal{D}_\mu \Phi^i{}_\alpha$ vanish. The remaining non-linear multiplet fields can be eliminated by noticing that they interact with the other matter fields only through the Kähler potential $e^{-\mathcal{K}} = i(F_I \bar{X}^I - \bar{F}_I X^I)$, which is set to a constant via the D-gauge condition (B.42). The non-linear multiplet fields effectively decouple from the rest of our theory, and so we can study their equations of motion independently from the others. We find that we can choose

$$V_\mu = 0 \ , \quad M_{ij} = 0 \ , \tag{B.46}$$

at the level of the action. Now that we have eliminated all of the non-linear multiplet fields from the theory, the non-linear multiplet constraint (B.13) forces the background value of D to satisfy

$$D = -\frac{1}{3} R \ . \tag{B.47}$$

The only remaining unconstrained auxiliary fields in our theory are the anti-self-dual tensor $T^-_{\mu\nu}$ and the $U(1)_R$ gauge field A_μ. In principle, we should find their respective equations of motion, solve for these auxiliary fields in terms of physical ones, and then replace them with their on-shell values at the level of the action. However, this procedure only works when the fields are pure Lagrange multiplier fields with no kinetic terms. This is spoiled by the higher-derivative interactions introduced in section B.5, which include terms like $T^{-\mu\nu} \mathcal{D}_\mu \mathcal{D}^\rho T^+_{\rho\nu}$ and $A^-_{\mu\nu} A^{-\mu\nu}$. The equations of motion for these fields are therefore no longer algebraic, and so these auxiliary fields cannot be eliminated in closed form. However, we take the view that the action is an effective action valid at energy scales well below the UV scale. We will therefore treat the higher-derivative terms as perturbative corrections to the two-derivative theory, and thus we will still always be able to eliminate all auxiliary fields.

The result of the preceeding discussion, when combined with section B.6, is that we can eliminate almost all of the fields from our theory. This is done by imposing the Poincaré gauge-fixing conditions

$$b_\mu = 0 \ , \quad \Phi^i{}_\alpha = \delta^i{}_\alpha \ , \quad i(F_I \bar{X}^I - \bar{F}_I X^I) = \frac{8\pi}{\kappa^2} \ , \tag{B.48}$$

and then consistently setting

$$Y^I_{ij} = \mathcal{V}_\mu{}^i{}_j = V_\mu = M_{ij} = \text{fermions} = 0 \ , \tag{B.49}$$

at the level of the action. The Lagrangian (B.21) therefore becomes

$$\begin{aligned}\mathcal{L} = -\frac{1}{2\kappa^2} R + \frac{1}{8\pi} \bigg[& i\mathcal{D}^\mu F_I \mathcal{D}_\mu \bar{X}^I + \frac{i}{4} F_{IJ} \mathcal{F}^{-I}_{\mu\nu} \mathcal{F}^{-\mu\nu J} - \frac{i}{8} F_I \mathcal{F}^{+I}_{\mu\nu} T^{+\mu\nu} \\ & - \frac{i}{32} F T^+_{\mu\nu} T^{+\mu\nu} + \frac{i}{2} F_{AI} \mathcal{F}^{-I}_{\mu\nu} \hat{F}^{-\mu\nu} + \frac{i}{2} F_A \hat{C} + \frac{i}{4} F_{AA} \hat{F}^-_{\mu\nu} \hat{F}^{-\mu\nu} \bigg] + \text{h.c.} \ .\end{aligned} \tag{B.50}$$

Additionally, any solution to the equations of motion of this Lagrangian must satisfy the constraints

$$X^0 = \bar{X}^0 \ , \quad D = -\frac{1}{3} R \ . \tag{B.51}$$

We can also look at the two-derivative and four-derivative parts of the Lagrangian (B.50) by using the two-term prepotential (B.38). These are given, respectively, by

$$\begin{aligned}\mathcal{L}^{(2)} &= -\frac{1}{2\kappa^2} R + \frac{1}{8\pi} \bigg[i\mathcal{D}^\mu F^{(0)}_I \mathcal{D}_\mu \bar{X}^I + \frac{i}{4} F^{(0)}_{IJ} \mathcal{F}^{-I}_{\mu\nu} \mathcal{F}^{-\mu\nu J} - \frac{i}{8} F^{(0)}_I \mathcal{F}^{+I}_{\mu\nu} T^{+\mu\nu} \\ &\qquad - \frac{i}{32} F^{(0)} T^+_{\mu\nu} T^{+\mu\nu} \bigg] + \text{h.c.} \ , \end{aligned} \tag{B.52}$$

$$\begin{aligned}\mathcal{L}^{(4)} &= \frac{1}{8\pi} \bigg[i\mathcal{D}^\mu (F^{(1)}_I \hat{A}) \mathcal{D}_\mu \bar{X}^I + \frac{i}{4} F^{(1)}_{IJ} \mathcal{F}^{-I}_{\mu\nu} \mathcal{F}^{-\mu\nu J} \hat{A} - \frac{i}{8} F^{(1)}_I \mathcal{F}^{+I}_{\mu\nu} T^{+\mu\nu} \hat{A} \\ &\qquad - \frac{i}{32} F^{(1)} T^+_{\mu\nu} T^{+\mu\nu} \hat{A} + \frac{i}{2} F^{(1)}_I \mathcal{F}^{-I}_{\mu\nu} \hat{F}^{-\mu\nu} + \frac{i}{2} F^{(1)} \hat{C} \bigg] + \text{h.c.} \ , \end{aligned} \tag{B.53}$$

which are precisely the Lagrangians presented in (4.15) and (4.29).

Appendix C

Gaugino Variations

In chapter 4, we studied an embedding of Einstein-Maxwell solutions into off-shell $\mathcal{N}=2$ supergravity. And, as we mentioned in section 4.2.4, these solutions are generically non-supersymmetric but they turn out to all satisfy a subset of the BPS conditions: namely, the supersymmetry variations of the vector multiplet gauginos all vanish. In this appendix, we will show this explicitly.

In the off-shell approach to $\mathcal{N}=2$ supergravity, we couple the Weyl multiplet to n_V+1 vector multiplets $\mathbf{X}^I$. The vector multiplet components are

$$\mathbf{X}^I = \big(X^I \ , \ \Omega_i^I \ , \ W_\mu^I \ , \ Y_{ij}^I \big) \ , \tag{C.1}$$

where X^I is a complex scalar, Ω_i^I is an $SU(2)$ doublet of chiral gauginos, W_μ^I is a $U(1)$ gauge field, and Y_{ij}^I is an auxiliary $SU(2)$ triplet of scalars. The components of the vector multiplet transform under supersymmetry as follows:

$$\begin{aligned}
\delta X^I &= \bar{\epsilon}^i \Omega_i^I \ , \\
\delta \Omega_i^I &= 2 \not{D} X^I \epsilon_i + \frac{1}{2} \mathcal{F}_{ab}^{-I} \gamma^{ab} \varepsilon_{ij} \epsilon^j + Y_{ij}^I \epsilon^j + 2 X^I \eta_i \ , \\
\delta W_\mu^I &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^I + 2 \varepsilon_{ij} \bar{\epsilon}^i \bar{X}^I \psi_\mu^j + \text{h.c.} \ , \\
\delta Y_{ij}^I &= 2 \bar{\epsilon}_{(i} \not{D} \Omega_{j)}^I + 2 \varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \not{D} \Omega^{l)I} \ .
\end{aligned} \tag{C.2}$$

In the truncation discussed in appendix B.7, we set all fermions and all fields charged under the $SU(2)_R$ symmetry to zero. The variations of all the vector multiplet fields therefore vanish entirely, except for that of the gaugino. After imposing the truncation and simplifying, we are left with

$$\delta \Omega_i^I = 2 \not{\mathcal{D}} X^I \varepsilon_i + \frac{1}{2} \mathcal{F}_{ab}^{-I} \gamma^{ab} \varepsilon_{ij} \epsilon^j + 2 X^I \eta_i \ . \tag{C.3}$$

Importantly, the gaugino transform under both Q- and S-supersymmetries, with corresponding parameters ϵ^i and η_i, respectively. Since we eventually gauge-fix the superconformal symmetries of our theory to end up with a theory of Poincaré supergravity, demanding that our theory is both Q- and S-supersymmetric is too restrictive. Instead, we need to consider linear combinations of fields that are invariant under S-supersymmetry, and then impose that their Q-supersymmetry variation vanishes.

In order to find S-invariant spinors, we first recall that the Kähler potential $\mathcal{K}$ of our theory is defined by

$$e^{-\mathcal{K}} = i(F_I \bar{X}^I - \bar{F}_I X^I) \ . \tag{C.4}$$

The derivative of this potential with respect to the vector multiplet scalars is

$$\mathcal{K}_I \equiv \frac{\partial \mathcal{K}}{\partial X^I} = -ie^{\mathcal{K}}(F_{IJ}\bar{X}^J - \bar{F}_I) \ . \tag{C.5}$$

Using this, we can now define the compensating spinor ζ_i by

$$\zeta_i = \mathcal{K}_I \Omega_i^I \ , \tag{C.6}$$

such that it depends on all of the vector multiplets and the coupling between them (e.g. the prepotential). This compensating spinor transforms under supersymmetry variations as

$$\delta\zeta_i = 2\mathcal{K}_I \not{D} X^I \epsilon_i + \frac{1}{2}\mathcal{K}_I \mathcal{F}_{ab}^{-I}\gamma^{ab}\varepsilon_{ij}\epsilon^j - 2\eta_i \ . \tag{C.7}$$

Notice that the compensating spinor transforms linearly with the S-variation parameter η_i under S-transformations. This means that, given any spinor that transforms under S-transformations, we can take a suitable linear combination of that spinor with the compensating spinor in order to obtain an S-invariant spinor. In particular, the S-invariant version of each gaugino field is

$$\Omega_i^I + X^I \zeta_i \ , \tag{C.8}$$

the supersymmetry variation of which is

$$\delta\left(\Omega_i^I + X^I\zeta_i\right) = 2\left(\delta_J^I + X^I\mathcal{K}_J\right)\not{\partial} X^J \epsilon_i + \frac{1}{2}\left(\delta_J^I + X^I\mathcal{K}_J\right)\mathcal{F}_{ab}^{-J}\gamma^{ab}\varepsilon_{ij}\epsilon^j \ . \tag{C.9}$$

For any field configuration that preserves the full $\mathcal{N} = 2$ supersymmetry of the theory, this variation must vanish. Moreover, since the supersymmtry parameters ϵ_i and ϵ^i are independent of one another, the two terms in this variation must cancel independently. This can be accomplished in a prepotential-independent manner by requiring that

$$\partial_a X^I = \mathcal{F}^{-I}_{ab} = 0 \ . \tag{C.10}$$

These are precisely the two conditions discussed in section 4.2.4 required to embed Einstein-Maxwell solutions into off-shell $\mathcal{N} = 2$ supergravity.

We have so far just looked at the variations of the vector multiplet fields in our theory. In general, we must also look at the variations of the Weyl multiplet fields in order to determine if a solution is fully supersymmetric. As it turns out, $\mathcal{N} = 2$ supersymmetric solutions require that the background is conformally flat with zero Weyl tensor, and that the mass coincides with the central charge of the theory [38, 81]. The only Einstein-Maxwell solutions that satisfy these conditions extremal Reissner-Nördstrom black holes, and even then only when we consider the near-horizon $\mathrm{AdS}_2 \times S^2$ geometry or the asymptotic Minkowski geometry alone. The full Einstein-Maxwell solutions in the bulk of the spacetime will not satisfy the Weyl multiplet BPS conditions. The fact that these solutions do satisfy the vector multiplet BPS conditions is therefore surprising. It is possible that this "residual" supersymmetry is responsible for the simplifications we find in this work. We will leave further investigations of this to future work.

Appendix D

Duality Transformations of Four-Derivative Actions

In this appendix, we will derive how a four-derivative Lagrangian in a theory with symplectic invariance must transform under duality transformations. We will then specialize to $\mathcal{N}=2$ supergravity, leading to (4.53) in D. We will also discuss higher-derivative correction to Maxwell theory in D.1, explaining along the way how our results are consistent with earlier work regarding the duality-invariant deformations of Maxwell theory with higher order terms, such as [122–124].

The derivation leading to (4.53) is a slight generalization of the derivation in appendix B of [111]. Using the notation of [111], let us consider a theory with gauge field strengths $F^I_{\mu\nu}$, scalars χ^i, and duality transformation functions of the scalars $\delta\chi^i = \xi^i(\chi)$. The duality transformation of the Lagrangian of this theory is given by [111]:[1]

$$\delta\mathcal{L} = \left(\xi^i \frac{\partial}{\partial\chi^i} + \partial_\mu\xi^i \frac{\partial}{\partial(\partial_\mu\chi^i)} + (A^I{}_J F^J_{\mu\nu} + B^{IJ} G_{J\mu\nu})\frac{\partial}{\partial F^I_{\mu\nu}}\right)\mathcal{L}\ , \tag{D.1}$$

where $G_{I\mu\nu}$ is the dual field strength to $F^I_{\mu\nu}$. If in addition the theory is duality-invariant, then the above must reduce to [111]

$$\delta\mathcal{L} = \frac{i}{4}\left(C_{IJ} F^I_{\mu\nu}\tilde{F}^{J\mu\nu} + B^{IJ} G_{I\mu\nu}\tilde{G}^{\mu\nu}_J\right)\ . \tag{D.2}$$

Now, let's assume that $\mathcal{L}$ depends on a duality-invariant parameter λ. We can then take the

[1] This can easily be generalized to allow for higher derivative terms involving the scalars and gauge fields.

derivative of both sides of (D.1) with respect to λ; after some rewriting, we get

$$\delta\left(\frac{\partial\mathcal{L}}{\partial\lambda}\right)=\frac{\partial}{\partial\lambda}\left(\delta\mathcal{L}-\frac{i}{4}B^{IJ}G_{I\mu\nu}\tilde{G}_J^{\mu\nu}\right)-\frac{\partial\xi^i}{\partial\lambda}\frac{\partial\mathcal{L}}{\partial\chi^i}-\partial_\mu\left(\frac{\partial\xi^i}{\partial\lambda}\right)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi^i)}\ . \tag{D.3}$$

If duality-invariance is preserved, we can use (D.2) as well as the fact that the field strengths $F^I_{\mu\nu}$ are independent of λ to conclude that

$$\delta\left(\frac{\partial\mathcal{L}}{\partial\lambda}\right)=-\frac{\partial\xi^i}{\partial\lambda}\frac{\partial\mathcal{L}}{\partial\chi^i}-\partial_\mu\left(\frac{\partial\xi^i}{\partial\lambda}\right)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi^i)}\ . \tag{D.4}$$

This equation generalizes (B.3) in [111] to the case where the ξ^i are allowed to depend on λ.

Now, to use (D.4) in the context of four-derivative corrections, assume we have a Lagrangian of the form

$$\mathcal{L}=\mathcal{L}^{(2)}+\lambda\mathcal{L}^{(4)}+\mathcal{O}(\lambda^2)\ , \tag{D.5}$$

so that $\partial_\lambda\mathcal{L}=\mathcal{L}^{(4)}+\mathcal{O}(\lambda)$. That is, we are assuming that the duality-invariant parameter λ is a small parameter governing the size of higher-derivative terms in the action. All functions of the fields should be viewed as having a perturbative series in λ, so the functions ξ^i can be written as

$$\xi^i=\xi^{(2)i}+\lambda\xi^{(4)i}+\mathcal{O}(\lambda^2)\ . \tag{D.6}$$

Then, it easily follows that (D.4) can be rewritten as

$$\delta\mathcal{L}^{(4)}=-\xi^{(4)i}\frac{\partial\mathcal{L}^{(2)}}{\partial\chi^i}-\partial_\mu\left(\xi^{(4)i}\right)\frac{\partial\mathcal{L}^{(2)}}{\partial(\partial_\mu\chi^i)}\ , \tag{D.7}$$

to leading order in λ. In other words, the duality transformation properties of the subleading piece $\mathcal{L}^{(4)}$ are completely determined (to leading order in λ) by the leading piece $\mathcal{L}^{(2)}$ and the subleading piece of the duality transformation functions of the scalars $\xi^{(4)i}(\chi)$.

Finally, to arrive at (4.53) for the four-derivative corrections of $\mathcal{N}=2$ supergravity, we note that the scalars transform as

$$\delta X^I=A^I{}_JX^J+B^{IJ}F_J\ . \tag{D.8}$$

The prepotential has an expansion (4.27) in higher-derivative terms, and so we can break this variation up into two-derivative and four-derivative pieces as

$$\delta X^{I(2)}=A^I{}_JX^J+B^{IJ}F_J^{(0)}\ ,\quad \delta X^{I(4)}=B^{IJ}F_J^{(1)}\hat{A}\ . \tag{D.9}$$

Combining this with (D.7) yields the variation (4.53) presented in section 4.3.3.

D.1 Example: Einstein-Maxwell

As mentioned in section 4.3.1, Maxwell theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{D.10}$$

has a duality symmetry of $SO(2,\mathbb{R})$ rotations of the vector $(F_{\mu\nu}, G_{\mu\nu})$, where $G_{\mu\nu}$ is the dual field strength defined by (4.37). If we wish to deform the Lagrangian by adding terms that are higher-order in $F_{\mu\nu}$, the dual field strength $G_{\mu\nu}$ will change and thus the duality vector $(F_{\mu\nu}, G_{\mu\nu})$ receives corrections. Since the form of the (altered) duality transformations themselves depend on the higher-order terms added to the Lagrangian, it is in principle highly non-trivial to determine what can be added to the Lagrangian while keeping duality invariance of the theory. In fact, it can be proven that if we add any $\mathcal{O}(F^4)$ terms to the Lagrangian, to ensure duality invariance we would also need to add an infinite amount of higher order terms F^{2n} for all $n > 2$ [122–124]. One possible way of doing so is Born-Infeld theory

$$\mathcal{L}_{\rm BI} = \frac{1}{g^2}\left(1 - \sqrt{1 + \frac{g^2}{2}F_{\mu\nu}F^{\mu\nu} - \frac{g^4}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}\right) \ , \tag{D.11}$$

where g is a coupling constant with units of (length)^2. When g is small compared to some cut-off scale in the theory, we can expand the Born-Infeld action as

$$\mathcal{L}_{\rm BI} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2}{32}\left((F_{\mu\nu}F^{\mu\nu})^2 - (F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right) + \dots \ . \tag{D.12}$$

In the $g \to 0$ limit clearly gives back the Maxwell action (D.10). There are also other non-equivalent ways to deform the Maxwell Lagrangian in a way consistent with duality symmetry [123].

As we have mentioned many times in this work, the point of view we are adapting for the $\mathcal{O}(F^4)$ terms in section 4.3 (and in particular section 4.3.3) is that these terms are a perturbative correction to the two-derivative Lagrangian, and so we can express the full Lagrangian as

$$\mathcal{L} = \mathcal{L}^{(2)} + g^2\mathcal{L}^{(4)} + \mathcal{O}(g^4) \ , \tag{D.13}$$

so that all relevant quantities must also be expanded consistently in orders of g. Thus, to demand duality invariance of our theory at four-derivative order is demanding duality invariance up to order $\mathcal{O}(g^2)$ only, and not fully non-linear in g. For example, keeping only the $\mathcal{O}(g^2)$ terms in the Born-Infeld action (D.11), we see the unique four-derivative duality-invariant term $\mathcal{I}_{\mu\rho\nu}{}^{\rho}\mathcal{I}^{\mu\sigma\nu}{}_{\sigma}$ (see (4.42)) appearing at four-derivative order. This is consistent with our discussion in section 4.3.3 and above in appendix D, which demonstrates that in a theory without scalars, the four-derivative corrections must be invariant under duality transformations, $\delta\mathcal{L}^{(4)} = 0$, in order for the theory to respect duality symmetry.

Appendix E

Four-Derivative Symplectic Invariants with Constant Scalars

In this appendix, we elaborate on the discussion in section 4.3.4 on determining the full set of four-derivative, duality invariant terms allowed in off-shell $\mathcal{N}=2$ supergravity when the scalars are all constant. We will show that there are only five such independent terms on-shell, and these are precisely the terms given in (4.63).

In section 4.3.4, it was explained that the only duality-invariants containing at most two symplectic vectors and at most four derivatives are given by

$$T^{+}_{\mu\nu} = -\frac{i\kappa^2}{2\pi}\mathbb{F}^{+}_{\mu\nu}\cdot\overline{\mathbb{X}}\ , \tag{E.1}$$

$$\mathcal{I}_{\mu\nu\rho\sigma} = \mathbb{F}^{+}_{\mu\nu}\cdot\mathbb{F}^{-}_{\rho\sigma}\ , \quad \nabla_\rho T^{+}_{\mu\nu} = -\frac{i\kappa^2}{2\pi}\nabla_\rho\mathbb{F}^{+}_{\mu\nu}\cdot\overline{\mathbb{X}}, \quad R_{\mu\nu\rho\sigma}\ , \tag{E.2}$$

$$\nabla_\lambda\mathbb{F}^{+}_{\mu\nu}\cdot\mathbb{F}^{+}_{\rho\sigma}\ , \quad \nabla_\lambda\mathbb{F}^{+}_{\mu\nu}\cdot\mathbb{F}^{-}_{\rho\sigma}\ , \quad \nabla_\rho\nabla_\sigma\mathbb{F}^{+}_{\mu\nu}\cdot\overline{\mathbb{X}}\ , \tag{E.3}$$

$$\nabla_\lambda\mathbb{F}^{+}_{\mu\nu}\cdot\nabla_\omega\mathbb{F}^{+}_{\rho\sigma}\ , \quad \nabla_\lambda\nabla_\omega\mathbb{F}^{+}_{\mu\nu}\cdot\mathbb{F}^{-}_{\rho\sigma}\ , \quad \nabla_\lambda\mathbb{F}^{+}_{\mu\nu}\cdot\nabla_\omega\mathbb{F}^{-}_{\rho\sigma}\ , \quad \nabla_\lambda\nabla_\omega\mathbb{F}^{+}_{\mu\nu}\cdot\mathbb{F}^{-}_{\rho\sigma}\ , \tag{E.4}$$

and their complex conjugates. Note that each line gives the invariants at a given order in derivatives (from one to four derivatives). We have left out the zero-derivative symplectic invariant $\mathbb{X}\cdot\overline{\mathbb{X}}$, as it is proportional to the Kähler potential which is set to be a constant on-shell.

To now find allowed four-derivative terms on-shell, we should multiply the above terms together in such a way that we get a four-derivative term, and then contract Lorentz indices to form Lorentz scalars. We will use the following principles to determine such terms that are allowed and independent:

1. *Respecting $U(1)_R$ symmetry:* Under the $\mathcal{N}=2$ global $U(1)_R$ symmetry, $\mathbb{X}$ and $\overline{\mathbb{X}}$ carry

opposite charges, while $\mathbb{F}^{\pm}_{\mu\nu}$ is uncharged. Since the four-derivative Lagrangian should respect the $U(1)_R$ symmetry, any allowed term should have vanishing total $U(1)_R$ charge.

2. *Discarding total derivatives:* If two allowed terms are related by a total derivative, then we consider them equivalent and thus discard one of the two terms. This is because we are interested in the independent terms that can appear in a Lagrangian, and so we are allowed to integrate by parts at will.

3. *Using two-derivative equations of motion:* When the scalars are constant, the two-derivative Einstein equations (4.21) set $R_{\mu\nu} = \mathcal{I}_{\lambda(\mu\nu)}{}^{\lambda}$, so we will freely interchange the two and use the relation to eliminate any terms containing $\mathcal{I}_{\lambda(\mu\nu)}{}^{\lambda}$ in favor of $R_{\mu\nu}$. A consequence of this is also that $R = 0$. We will also use the two-derivative Bianchi identity and equations of motion (4.22), (4.23) for the vectors, which allows us to set $\nabla_\mu \mathbb{F}^{+\mu\nu} = \nabla_\mu \mathbb{F}^{-\mu\nu}$.

4. *(Anti-)self-duality of $\mathbb{F}^{\pm}_{\mu\nu}$:* Finally, when contracting Lorentz indices to form a Lorentz scalar, we will use the (anti-)self-duality properties of $\mathbb{F}^{\pm}_{\mu\nu}$ intensively to relate different ways of contracting Lorentz indices to each other. This will drastically reduce the number of independent four-derivative Lorentz scalars we can construct, as many different contractions of Lorentz indices can often be shown to be equal using these (anti-)self-duality properties. We will also allow ourselves to keep in mind the explicit form of $G^{+}_{I\mu\nu}$ given in (4.45) in terms of $F^{+I}_{\mu\nu}$.

We now proceed systematically to investigate all possible four-derivative Lorentz scalar terms that we can write down using the above principles:

- We can take a single term from the four quantities in line (E.4) and contract Lorentz indices to obtain a four-derivative Lorentz scalar. First of all, it is obvious that we can ignore the second and fourth terms in (E.4) as they are equivalent to the third and first term, respectively, via integration by parts. Using self-duality of $\mathbb{F}^{+}_{\mu\nu}$ and the explicit form (4.45) of $G^{+}_{I\mu\nu}$ in the vector $\mathbb{F}^{+}_{\mu\nu}$, it can be shown that there are actually no non-zero contractions of the first term in (E.4). Finally, there is only one independent, potentially non-zero contraction of the third term, given by $(\nabla_\mu \mathbb{F}^{+\mu\nu})(\nabla_\rho \mathbb{F}^{-\rho}{}_{\mu})$, but we can use the Bianchi identity and equations of motion to relate $\nabla_\rho \mathbb{F}^{-\rho}{}_{\mu} = \nabla_\rho \mathbb{F}^{+\rho}{}_{\mu}$, so that this term will also vanish.

- We can take a quantity from line (E.3) and multiply it by (E.1). However, we can use total derivatives to relate any such resulting term to a term that is a product of two quantities from (E.2), so there are no such independent terms.

- We can take two quantities from line (E.2) and multiply them together, contracting Lorentz indices. The second quantity in (E.2) is charged under $U(1)_R$ and can be multiplied by its complex conjugate to obtain a $U(1)_R$ invariant term. There is one independent way of forming a Lorentz scalar in this way:

$$\nabla_\mu T^{+\mu\nu} \nabla^\rho T^-_{\rho\nu} \ . \tag{E.5}$$

  We can also multiply the first or third quantities from (E.2) amongst themselves. Using the Einstein equations of motion and (anti-)self-duality properties, we can see that there are only two independent such terms:

$$R_{\mu\nu} R^{\mu\nu} \ , \quad R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \ . \tag{E.6}$$

- We can take a quantity from line (E.2) and multiply it twice with (E.1). We must take care that the resulting term is $U(1)_R$ invariant. Then, again using the Einstein equations of motion and (anti-)self-duality properties, we can conclude there is only one independent such term:

$$R_{\mu\nu} T^{+\mu}{}_{\rho} T^{-\nu\rho} \ . \tag{E.7}$$

- Finally, we can multiply (E.1) or its complex conjugate with itself four times. We must take a $U(1)_R$ invariant term, of course, and (anti-)self-duality properties tell us there is only one such term:

$$T^{-}_{\mu}{}^{\rho} T^{-\mu\nu} T^{+\sigma}_{\nu} T^{+}_{\rho\sigma} \ . \tag{E.8}$$

Putting everything together, we see we have obtained a total of five independent terms, as given in (4.63).

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