Teachers’ Encounters with Horizon Content Knowledge
Investigating Knowledge Sensibilities for Teaching Mathematics

by

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Dedication

To *Baba* who saw big dreams for us, to *Aai* and Swati who helped deal with those dreams and to Manu for the constant support, love and bearing with me through the hard times …

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Abstract

Mathematics education researchers have sought to understand the knowledge that teachers need to teach mathematics effectively. Teachers need to know more than merely knowing how to "do the math" at a particular grade level. However, the research community differs on the nature of that knowledge. The construct of "horizon content knowledge" has emerged in the literature as a promising way to characterize advanced mathematical knowledge (AMK) as it relates specifically to teaching practice. Ball, Thames, and Phelps (2008) and Ball and Bass (2009) propose a kind of knowledge that is neither common nor specialized, that is not about curriculum progression, but is more about having a sense of the broader mathematical environment of the discipline. They call this horizon content knowledge (HCK) and argue that knowledge of the mathematical horizon can support teachers in hearing students' mathematical insights, orienting instruction to the discipline, and making judgments about what is mathematically important. However, operationalizing HCK in practice is still under development. Jakobson, Thames, and Ribeiro (2013) offer an overarching definition of HCK, which foregrounds some inherent characteristics of this knowledge.

This dissertation examined cases of teaching and teachers for the purpose of collecting and analyzing examples of HCK in practice, understanding the interaction between teachers' management of what I call "encounters with mathematics at the horizon" and students' learning experiences in the classrooms, and to characterize the knowledge resources that teachers draw upon to make sense of the mathematics at the horizon. I identified and articulated a new domain of knowledge resources that the teachers draw upon, called professional practice knowledge (PPK). I define PPK as a form of mathematical knowledge derived from practice and experience. As PPK is knowledge that is shaped by experience, the culture in school, role of leadership, and kind of students' and parents' involvement impacts PPK. If PPK is the only resource available to the teacher, then teachers' explanations of mathematical deductions are often pseudo-mathematical. Pseudo-mathematical descriptions are generated by the teachers in such ways that they do not explain the concept, term, or formula but instead focus on memorization. These center on the visual patterns or syntactic patterns, use colloquial meanings of the mathematical terms, and often have a cue to remember the term, concept, or formula. These explanations can block mathematical access for the students to investigate or build further. However, if PPK remains rooted in other domains of HCK elaborated in the dissertation, teachers are able to manage encounters with HCK in more meaningful ways.

CHAPTER 1
Introduction

During my researching and teaching years in India, I once saw a documentary called "Young Historians." This film documented a history teacher's and his students' interactions over one whole educational year. The methods that the teacher used were atypical. As part of learning history, the students interviewed families in the neighborhood, visited nearby monuments, read public documents, and annotated old newspapers, all to write various historical accounts of their locality. While doing this, students had to make many decisions, ranging from deciding the authenticity of their accounts to worrying about ways to represent those. The students argued over many things. Some of those are listed here: criteria for truth, consideration of contexts, various kinds of diagrams that would show chronology, and processes and relationships. They also envisioned the tools that would have helped them, like being able to ask many people at the same time, knowing geography to make sense of the elements present in the monuments, etc. By the end of the documentary, each child appears as a historian with her/his traits and strengths of engaging with the historical material. And with this training now, their reading of the school text was more meaningful and relevant from multiple perspectives. This documentary made me question what those students were doing as part of learning history. How would a mathematics class look like if I had to engage my students similarly, in fundamental processes of doing mathematics? And what would it mean for me as a teacher to know and use that knowledge? I have shown this documentary to many teachers, and we together struggled over these questions. What do our students need to do to be "young mathematicians"? And how would I know, what students need to do? What is the knowledge that one needs to identify to facilitate doing mathematics?

The more I thought and heard others' opinions on this the more I realized that there is no single view on "what mathematics is" and hence there exist multiple takes on what it means to do mathematics. During my graduate school years, I came across the book "Proofs and Refutations" by Lakatos. This book focuses on the "methodology of mathematics" and presents a hypothetical conversation between teacher and students who are engaged in creating, refuting and generalizing theories. Lakatos' work in this book, though primarily philosophical in intent, also touched on other domains. Some of those are as follows.

- Sociological: Studying the behavior of mathematicians and partially classifying the recurrent features in their practice of mathematics
- Educational contestation: Considering the impact of the presentation of mathematical developments on students' comprehension of the processes involved, arguing the merits of presentations which retain more of the original structure of the discoveries.

- Mathematical methodology: claiming that mathematicians practice heuristic rather than deductive methods.

Lakatos's challenge to formalism in mathematics opened up a number of ways one can learn to do the mathematics, in addition to traditional ones. Even though the discussion among the students wasn't real and that was not a real class, I could relate some of my students' work to a number of his ideas. For example, when students in my class believed that two even numbers add to an even number based on their trial of adding some even numbers, I knew it proved some individual cases but not a general proof that would hold for all even numbers. Therefore, when I read about local and global proofs in the book, I thought I could use it in my classroom to distinguish what is a proof and what is not. In some sense, this book partially responded to my curiosity about what it might mean to develop "young mathematicians." However, what needs to be done to build such classroom settings and interactions remained a challenge. Some questions remain unanswered like, what are such other encounters where I could use my discipline sensibilities to design pedagogy around topics of school mathematics? And where would I learn about these discipline sensibilities?

Many researchers thought the answer to developing these mathematical sensibilities that affect the pedagogy of the classroom interaction is to know more mathematics; some called it Advanced Mathematical Knowledge (AMK). This knowledge would include grasping topics and concepts taught at the higher level of education and reading of the mathematical text in the early grades from that lens. This notion does not make complete sense, especially AMK as mathematical knowledge for teaching. For example, I related to the idea of local and global proofs as given in the Lakatos's hypothetical classroom because I was familiar with it. My students thought that finding some supporting examples confirmed their theory. There was something I noticed in discussion around advanced mathematics because of my exposure to ways students do mathematics. It was not that I thought what students gave is a proof, but I did not know how to make the connection between what they produced and the proof. Let us take another example from the contemporary context to understand what kind of understanding I am referring to. The most recent standards in the United States, the Common Core State Standards Initiative (CCSSI, 2010), define the rational numbers as numbers that can be expressed as positive and negative fractions. And fractions are defined as numbers that can be expressed in a specific form: "$\frac{a}{b}$ where a is a whole number and b is a positive whole number" (CCSSI, 2010, p. 85). Although these are formal definitions, the way fractions are introduced in the school represent a more concrete perspective. One common way is to define these as some number of parts of all the parts of an equally partitioned whole. However, if one decides to look at this from an advanced mathematics point of view, there are multiple ways in which it can be done. For example, a mathematician or a person trained in mathematics might look at a fraction as an ordered pair (a, b) belonging to $\mathbb{Z} \times (\mathbb{Z}/\{0\})$, where $\mathbb{Z}$ is the set of integers and define all the operations, equivalence and the inverse of fractions within the realm of this set. Even though these comprehensions are beautiful, and there could be many, they are not helping me as a teacher to teach fractions, not directly. For example, when students learn fractions, they learn within the influence of understanding whole numbers. Often teachers observe students seeing a fraction as made up of two numbers, leading to many responses similar to these shown in table 1.1. For students, each digit written in the fraction notation is a number, and therefore, they add, compare these numbers and not fractions.

$$\frac{2}{3}+\frac{3}{4}=\frac{5}{7}$$

$$\frac{3}{2}<\frac{6}{5}\ or\ \frac{1}{5}>\frac{1}{3}$$

Table 1.1: Fractions under whole number influence

One needs a sense of what students are going through to comprehend these responses. Here, for example, figuring out that the problem is not entirely of not understanding the definition of fractions but students trying to associate patterns across the number systems. Knowing these responses would make me sensitive to certain mathematical aspects of representing fractions and the arithmetic of fractions. It is not that you only know more mathematics and that lets you understand problems of teaching mathematics. Therefore, there is this connected and deep mathematics that you learn, because you know certain problems of teaching. It is the knowledge of mathematics at the horizon, but one notices it due to certain pedagogical encounters and familiarity with students' thinking. I specifically ask the question – what such encounters are present in a mathematics classroom, encounters that have leverage in understanding advanced mathematics differently. Further, do teachers notice such encounters in their instruction and can they make use of it? It is complicated territory, as studying classroom practice itself has many facets, and then unpacking it to see what potential encounters are present that productively make use of advanced mathematics. Conversely, suggesting knowledge at the mathematical horizon needed for teaching calls for a meticulous study of it. Before we dive into the details of the study, in the following paragraphs, I first expand on understanding mathematics and its connection with practice. It is a divide that rules the research on mathematics needed for teaching teachers. One side is about how one perceives elementary mathematics through a deep understanding of advanced mathematics, and the other about how mathematics in teaching lets one understand advanced mathematics. Further, I briefly describe the construct of mathematics at the horizon and present the research problem in the form of research questions.

1.1 Two-pointed Understanding of Mathematics in Practice

The contrast between the school mathematics and what could be produced using advanced mathematical knowledge (AMK) is enormous and therefore it is challenging to determine what form of AMK could be relevant in teaching elementary mathematics. On the other hand, using school maths, and problems of teaching and learning around it as a starting point seems more relevant resource for a discussion of knowledge required by teachers. In the last two decades, researchers in mathematics education have produced convincing evidence that the teachers of mathematics require a kind of mathematical knowledge that is different from that of mathematicians. The most striking example is found in Ball's (1988) dissertation where she asked pre-service teachers to design a situation representing "$1\frac{3}{4} \div \frac{1}{2}$". She found that some prospective teachers, who knew that it needs to be inverted and multiplied, could not comprehend what the answer $3\frac{1}{2}$ represents with respect to $1\frac{3}{4}$ and $\frac{1}{2}$. With the understanding of division as sharing or equally distributing as used with whole number division, arriving at the interpretation of how many $\frac{1}{2}$ s are there in $1\frac{3}{4}$ was not easy. Along with this, comprehending how the answer to a division problem can be a bigger number than dividend and divisor, in this case, $3\frac{1}{2}$ was bigger than $1\frac{3}{4}$ and $\frac{1}{2}$, was not straightforward.

Going back to the earlier discussion regarding AMK, interpreting every statement of a division as an inverse multiplication would be sufficient from the perspective of higher mathematics, and one need not engage in the operation of the division at all. However, the mathematics constructed around the knowledge that $a \div b$, also represents how many b's are there in a, and therefore, $\frac{a}{b} \div \frac{c}{d}$ could also be calculated alternatively cannot be overlooked. For example, later when Ma (2010), took up these problems from Ball's (1988) thesis, and presented them to in-service teachers from the U.S. and China, a number of strategies emerged in the Chinese teachers' calculation. (See Table 1.2.) The one on the left is a conditional strategy and works only when the numerators and denominators of the divisor and dividend are divisible. In this case, 7 is divisible by 1 in the numerator, and 4 is divisible by 2 in the denominator. The second strategy initially uses the division distributive law over addition and then exercises the division meaning to find out how many ½ s are there in 1 and how many in ¾.

Alternate strategies for division of fractions $1\frac{3}{4} \div \frac{1}{2}$ by Chinese teachers in Ma (1999)	
$1\frac{3}{4} \div \frac{1}{2}$ $= \frac{7}{4} \div \frac{1}{2}$ $= \frac{7 \div 1}{4 \div 2}$ $= \frac{7}{2}$ $= 3\frac{1}{2}$	$1\frac{3}{4} \div \frac{1}{2}$ $= \left(1 + \frac{3}{4}\right) \div \frac{1}{2}$ $= \left(1 \div \frac{1}{2}\right) + \left(\frac{3}{4} \div \frac{1}{2}\right)$ $= 2 + \left(1\frac{1}{2}\right)$ $= 3\frac{1}{2}$

Table 1.2: Strategies for division of fraction from Ma (2010)

These strategies have emerged from the context of the specific meanings of the division algorithm. Just the definition of fractions need not necessarily give rise to these strategies. The teachers who proposed these strategies suggested that these derivations justified the division algorithm by drawing on another piece of knowledge that students had learned. The approach on the left illustrates the division of numerators by numerator and denominators by denominators. It is similar to the algorithm the students used for multiplying the fractions, and that is what the teacher has utilized here. The derivation on the right uses distributivity. The teacher in Ma's study (Tr. Xie) commented that the derivation (on the right in Table 1.2) appears complicated, but for his students who had experience of using distributivity with whole numbers, it was simpler than the standard computation. These mathematical productions are situated in the practice of teaching, as it involves meaning-making of every step included in the computation concerning the specific context, and not necessarily because of knowledge of advanced mathematics.

1.2 Mathematical Knowledge for Teaching

The findings that Ball (1988) and Ma (2010) brought to light gave rise to many studies that attempted to outline the mathematical knowledge required for teaching, as well as possible conceptualizations of it. These studies invested in both, in distinguishing the nature of such knowledge for teaching as well as ways to improve upon it. The literature review chapter elaborates on the development of mathematical knowledge for teaching mathematics. Here, I briefly describe the construct of mathematics at the horizon to make sense of the research questions of the study.

Ball, Thames, and Phelps (2008) in their seminal work at the University of Michigan, first explicitly identified domains of teacher knowledge as part of Mathematical Knowledge for Teaching (MKT). They defined MKT as the mathematical knowledge, skill, and habits of mind entailed by the work of teaching and proposed a further refinement of subject matter and pedagogical content knowledge, which were earlier introduced by Shulman (1986) into sub-domains. Horizon content knowledge was one of the sub-domains.

Horizon content knowledge (HCK) as defined in Ball, Thames, and Phelps (2008), refers to an orientation to and familiarity with the discipline that contributes to understanding the school subject at hand, providing teachers with a sense of how the content being taught is situated in and connected to the broader disciplinary territory. This knowledge is a resource for managing the fundamental tasks of introducing learners to a vast and highly developed field of mathematics. It includes an awareness of core disciplinary orientations and values, and of essential structures of the discipline. HCK might be a kind of knowledge that is responsible for the ability to make connections across a range of mathematical concepts. This conceptualization of HCK resonated with Ball's (1993) first reference to the term "mathematical horizon" in the context of describing the dilemma and tensions faced by an elementary teacher while dealing with pedagogical and disciplinary mathematics concerns. For example, HCK might include choosing the representation of contexts (e.g. negative numbers) that are consistent with the ideas that the learner encounters in the future (e.g. absolute value) (example used in Ball, 1993). Even though middle-school mathematics might not involve components of a given concept, and students might face them in their later years of mathematics learning, choosing a representation at the primary level cannot be deceiving or so simplistic that it contradicts the advanced form of the concept. More importantly, teaching students so that they learn the potential meanings of the concepts as components of mathematics discipline and the curriculum of the middle-school requires a sense of the discipline of mathematics and its practices. This particular idea, which describes perpetuation of mathematical sensibilities in making decisions about teaching, forms the basis of knowledge of mathematics at the horizon.

1.3 Research Questions and Definition of Terms

In this study, I began analyzing the practice of teaching mathematics through aspects described in the definition of HCK as given in Jakobsen et al. (2013). I explain the definition and its features as presented in Jakobsen et al. in detail in the Conceptual Framework chapter. There are many definitions for HCK, but the horizon of mathematics upon which these definitions are built is often given as a metaphor. This study proposes a definition of mathematics at the horizon and defined encounters with it. The following paragraphs provide these definitions before we understand the research questions of the dissertation.

Drawing upon the definition of Horizon Content Knowledge given by Ball and her colleagues at the University of Michigan, which has been the basis of Jakobsen et al.'s elaborate description of HCK, this study used a more specific definition for the horizon of mathematics. The definition exercised is in alignment with what Ball (1993) and Ball & Bass (2009) envisioned as "mathematics at the horizon." The definition used in this study is specific to the school mathematics curriculum. I define the mathematical horizon as "a projection of mathematical meanings, topics, and structures present in the curriculum into the mathematics extending beyond the support of the curriculum materials concerning a particular location of instruction, such that it enables meaningful learning of mathematics." The curriculum here, is used in a sense Remillard (2015) uses it, "printed, often published resources designed for use by teachers and students during instruction." (p. 213). The meaningful learning of mathematics is defined through two specific frameworks – First, by the instructional triangle as described by Cohen, Raudenbush and Ball (2003) that represents the process of teaching. The instructional triangle positions meaningful listening to students at the core of impactful instruction. The second by mathematical sensibilities of knowledge for teaching that Jakobsen, Thames & Ribeiro (2013) delineate. Their delineation brings disciplinary mathematical norms and values to the center of instruction. I again elaborate these two frameworks in detail in the chapter on the conceptual framework.

An encounter with the horizon of mathematics would be an instance where the meaningful learning of mathematics could take place by taking supports outside the curriculum that are situated in disciplinary sensibilities, norms, and values. The encounter could be initiated by listening to students meaningfully, designing the instruction, developing mathematical explanations or by reading the curriculum. The management of such instances would require supports outside the curriculum, situated in disciplinary sensibilities and norms. At the operational level, an encounter will be considered as "an instance to establish any of the five components listed below to its potential" in a given location in the instruction. If there is an opportunity to

1. establish truth in mathematics using mathematical tools and disciplinary ways in classroom instruction
2. use core disciplinary values and orientation
3. make explicit the knowledge of the ways and tools for knowing the discipline
4. connect with structures in the discipline
5. comprehend kinds of knowledge with its warrants

then that would be considered as an encounter with HCK.

Therefore, for this thesis encounter with HCK did not depend on whether teacher noticed such an encounter or not, but whether the classroom interaction had potential to establish any of the components listed above.

I defined mathematics at the horizon and encounters with the mathematics at the horizon, and yet the research questions below are framed as "encounters with HCK." In this paragraph, I explain the terminology glitch present in the research questions. The encounters with HCK is used to mean encounters with the mathematics at the horizon, which is defined above and where the term "mathematics at the horizon" is in the similar sense that was used by Ball (1993) and Ball & Bass (2009). Ball's use of the term was situated in the practice of teaching mathematics, specifically in the context of dilemmas that a teacher faces while choosing an appropriate representation, representations that do not distort the mathematics and increase the access to learning. However, the phrase "mathematics at the horizon" has been used with multiple meanings by various researchers, mainly to refer to the advanced mathematics that is "out there" at the horizon. This reference makes the mathematics at the horizon not rooted in the practice of teaching mathematics (E.g., Zazkis & Mamolo, 2013), and that is not what this dissertation has built upon, and therefore the term "mathematics at the horizon" is avoided in the research questions. Instead, "Horizon Content Knowledge", i.e., mathematics at the horizon entailed through the practice of teaching is used here.

This dissertation focused on three central questions:

1. What kinds of "encounters with HCK" arise in classroom teaching? And: How do teachers manage them?
2. How are students' opportunities for learning shaped by teachers' encounters with HCK?
3. What kind of mathematical knowledge do teachers exhibit while navigating encounters with HCK?

1.4 Structure of the Dissertation

This dissertation unfolds in six chapters. Appended below is a summary of each chapter. A total 42 of cases of classroom teaching, interviews of 13 teachers of approximately 120 minutes each, and detailed analysis of the curriculum of grade 7 comprise the data for the thesis. Each of these teachers also engaged in solving mathematical problems, which presented ways of doing mathematics from the teachers' point of view. A total of 39 problem solutions by teachers constitute the data for solving mathematical problems.

Chapter 1: Introduction.

This chapter describes the personal journey towards the research problem, situates the problem in the current literature scenario and school settings, proposes the research questions, and defines the terms used in the research questions.

Chapter 2: Conceptual framework.

The chapter on Conceptual Framework provides a detailed account of the evolution and use of the construct HCK within the mathematics education research community and situates my understanding of the construct in the continuum. This chapter also distinguishes the construct of HCK from Advanced Mathematical Knowledge (AMK), which has been used by other researchers and essentially emphasizes a view of school mathematics from the advanced mathematics perspective to see the usefulness of both in school mathematics teaching. I also attempt to unpack the definition of teaching presumed in various conceptions of HCK in the literature and provide my understanding of what is entailed in teaching mathematics before diving into the discussion of knowledge for teaching. The literature analysis is used to elaborate upon how the definitions used in the thesis – of mathematics at the horizon and encounters with mathematics at the horizon are derived from the literature.

Chapter 3: Characterizing encounters with mathematics at the horizon.

The chapter describes the design of the study. It gives a brief account of participants of the study, and describes the researcher's position in understanding the classroom practice, the encounters emerged and their management. Towards the end, it discusses an example of an encounter to characterize and illustrate the analysis of classroom teaching practice. The case is on the congruence of triangles and illustrates how the encounter is initiated, and managed by the concerned teacher.

Chapter 4: Encounters with HCK: Case of teaching algebraic identities.

This chapter first provides a cross-case analysis of 6 cases of teaching algebraic identities (AI). It brings forward three dimensions of teaching AIs: explaining what AIs are, representing them and using them to solve other problems of Mathematics. The chapter provides an overview of the curriculum and analyses teachers' reading and presentation of it in the classroom. In the second part, the analysis is used to make a case for what kinds of encounters were observed in the study, how they were initiated and what resources were used to manage these.

Chapter 5: Encounters with HCK: Case of teaching multiplication and division of fractions.

The first part of the chapter provides a cross-case analysis of cases of teaching Fraction multiplication and division. It brings forward three dimensions of teaching Fraction multiplication and division: Understanding the algorithm, understanding area representation and connecting the two. The chapter provides an overview of the curriculum and analyses teachers' reading and presentation of it in the classroom through encounters observed in their teaching. In the second part, the analysis is used to make a case for what kinds of encounters were seen in the investigation, how they were initiated and what resources were used to manage these.

Chapter 6: Conclusion, implication and limitation.

This chapter summarizes findings from the previous three chapters, phrases these findings in response to the three research questions asked. This chapter also discusses the construct of Professional Practice Knowledge in detail. The thesis ends with a description of implications of the study for research and possible forms of HCK for teacher education.

CHAPTER 2
Conceptual Framework

Concerns about the students' learning across the world have brought researchers' attention to the quality of mathematics instruction and therefore, to teachers' knowledge of mathematics, their beliefs, and practices. In the last two decades, much research in mathematics education has investigated and analyzed teachers' personal beliefs or their knowledge of various kinds. There has been no disagreement among researchers that teachers need a specialized understanding of the subject matter for effective instruction. Still, early studies about mathematics teachers and their content knowledge did not find a strong association between what a teacher knows and what the students learn (Begle, 1972; Eisenberg, 1977). Since then, the notion of content knowledge for teachers has been further conceptualized, e.g., Shulman, 1986 – leading towards a nuanced notion that matches the complexity of teaching. In attempts to characterize, explore, and understand the knowledge of mathematics that may be particularly important for teaching, different domains of teachers' knowledge have been described (e.g., Ball, Thames, & Phelps, 2008; Ma, 2010; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). There also have been distinctions made between knowledge for-, in-, and of-practice (Cochran-Smith & Lytle, 1999).

Meaningfully uniting knowledge demands with the work of teaching has been the prominent effort of researchers seeking to describe useful content knowledge. Ball et al. (2008), conceptualized a specialized content knowledge (SCK) for teaching. This knowledge includes, for example, understanding and appreciating various interpretations of operations, such as how rate, ordered pairs and repeated addition are all models of multiplication that differ mathematically. Such understanding is specific to teaching. Knowledge of these models or their representations is not very useful for a person who wants to find just the answer to a multiplication problem. Therefore, identifying different domains of teachers' knowledge has led to creating more relevant opportunities for teachers to learn content that is important for teaching. However, how understanding the difference between the rate and grouping models of multiplication is utilized in dealing with students' thinking or delivering effective instruction is still to be understood. The question in this regard is, how teachers manage various situation within the instruction, and how do they make use of the specialized knowledge for teaching that they possess? So are there specific norms about teaching or about doing mathematics through which teachers frequently draw on their mathematical knowledge?

The ways that content knowledge gets used often translate into moves or decisions that teachers make in the classroom. In this sense, we might consider actions that draw more explicitly on teachers' mathematical knowledge as a kind of mathematically driven move in teaching. Such moves describe teachers' responses—actions they take to help students learn—that have a significant mathematical component to their nature and encompass strong mathematical consideration. For example, Ball and Bass (2000) described unpacking as central to the work of teaching. They discuss unpacking in two scenarios around mathematics teaching. One in preparation for teaching and another while teaching – what Lampert and Ball (1999, p.38) referred "being able to know things in the situation." The first example involved examining and preparing to teach a mathematics problem. The problem (taken from Gelfand & Shen, 1993) was as follows

Write down a string of 8's. Insert some plus signs at various places so that the resulting sum is 1,000. (p. 90)

Ball and Bass (2000) describe the process of unpacking this problem. They say, on the surface this problem might look trivial and uninteresting. For example, adding 8s, 125 times equals 1000, and even though this is a solution, it doesn't make the problem very interesting. However, realizing that strings of 8, include 888, 88 opens up several possibilities as a solution. Like, 888+88+8+8+8 is a solution, as it follows the condition and sums to 1000. Therefore, a teacher preparing to use such a problem in the classroom would need to work on many aspects – like, what different solutions are there, do those solutions have any patterns, and can these patterns be used to predict a further answer or to record these solutions in a meaningful way. The essential noticing that, using only the addition of 8's to obtain zero at the unit place, requires 8 added five times or in a multiple of 5 times, makes the problem interesting. Ball and Bass (2000) describe the unpacking a teacher would require to work on this problem.

"A teacher preparing to use this task must contemplate: Would this be a good problem for my students? What would it take to figure out the patterns and nuances? Is it worthwhile in terms of what students might learn? At least it would be important to know what the problem is asking, whether it has one or many solutions, how the solutions might be found. How is it (or could be) related to other parts of the curriculum? It seems obvious that the task involves some computation – for example, verifying any one solution – but what is the mathematical potential of the task? Are there important ideas or processes involved in the problem? What would it take to use this task well with students? It would help to know what might make the problem hard, and how students might get stuck, and anticipate what the teacher might do if they did. Would students find this interesting? What might it take to hook them on it?" (p. 92)

An analysis of this sort is central to the work of teaching when a teacher wants to reach to the potential of any mathematics problem, and reveal how much of significant mathematical reasoning is entailed within the teaching of the problem. The second example Ball and Bass (2000) gave for unpacking came from a classroom scenario in Ball's teaching, where students are trying to answer and verify others' solutions. The problem is simple, where the students are expected to figure out the difference between 32 and 16. The students came up with the different strategies, and the authors discuss the complexity that the teacher faces while making a decision about moving forward, building upon responses and remaining on the task. Here, the process of unpacking is live, in action, and involves making various decisions. Some of them are – what the student is saying, what might be the context in which this reasoning is true, what is the mathematical explanation portrayed in the response, is the given explanation enough, how to connect the mathematics in the students' responses, how to make students' see these connections, and many more. Both these examples illustrate the complexity in unpacking mathematics in teaching. Since unpacking mathematics for students in the context of teaching requires significant mathematical attention, this would be considered as an example of a move or action that draws on teachers' mathematical knowledge.

However, one needs to accept that knowledge necessary for effective instruction is closely tied to what is perceived as instruction of mathematics. One of the shifts the reform curricula through Common Core Standards (2009) in the USA and National Curriculum Framework (2005) in India brought forward is a new conception of teaching. These curriculum frameworks focused on students' learning, on how children learn. If we believe that students are passive learners, then the question of teacher knowledge would become trivial. However, we know that students require meaningful understanding and re-discovery to learn new ideas. Primarily, we need to understand what mathematics instruction is with respect to developing citizens of the country. Do we want them to be rational thinkers and accept conceptions that are reasoned, or we want them to be passive listeners? Therefore, gaining a mathematical knowledge that gives autonomy and empowerment and allows them to do mathematics is need of the hour, making the issue of teacher knowledge of primary concern. Therefore, this literature review not only describes the ecology of the construct of teacher knowledge, and the domain in the discussion, horizon content knowledge, but also attempts to unravel the teaching presumed for these knowledge demands.

In the first chapter, I already described the two aspects of AMK and the role of such knowledge in teaching mathematics. The two aspects presented the difference in the relationship between the middle-school mathematics or school mathematics and mathematics as a discipline. One view enables the understanding of school mathematics through the lens of advanced mathematics. According to this view, there is mathematics out there, and the horizon is far from the mathematics in teaching – perhaps consisting of the mathematics from the university courses. The second view, which enables a learner to make sense of advanced mathematics from the school mathematics point of view. This includes various derivations (see example from Ma (2010) in chapter 1, page 8) that would seem redundant from the maths point of view but still are mathematical in nature, and relevant to the learning of mathematics. Based on the bipolar understanding of AMK, four main sections comprise this chapter. The first section starts with an emphasis on AMK in the literature because of its historical relationship to HCK and to lay the groundwork for the conversation I will have about horizon content knowledge. The second section describes theories of teachers' mathematical knowledge and provides a historical framing of how research in this area has evolved to its current state. I then summarize the foundational mathematical knowledge for teaching (MKT) framework of Ball, Thames and Phelps, and her colleagues (Ball, 1993; Ball & Bass, 2009; Ball et al., 2008) and specifically focus on their construct of horizon content knowledge (HCK). This section thus addresses the question: What do teachers need to know to teach effectively? The third section discusses the construct of horizon content knowledge and attempts to delineate it from other domains of knowledge needed for teaching. HCK has emerged a key facet of researcher's attempts to frame mathematical content knowledge for teaching and, in particular, to define the role of AMK in MKT. Following this summary, a description is provided of work that has been done to extend and elaborate on HCK since the introduction of the construct. Building on this literature, I identify important features of HCK that will serve as the foundation for the theoretical framework of the present study, which I will elaborate in the last part of this chapter. In the last section, I discuss how HCK is a promising construct to understand effective and equitable teaching through an example, to connect knowledge of mathematics needed for teaching with what we understand as effective mathematics instruction. I conclude this chapter with the definition and aspect of teachers' HCK, analyzed in the thesis.

2.1 The Ecology of Mathematical Knowledge for Teaching Mathematics

This section presents an ecological development of the research on teachers' content knowledge. The analysis on MKT provides a critical foundation to this dissertation by simultaneously situating the theory of MKT in the more substantial body of research on teacher knowledge and acknowledging the growth in the field. Several studies during the 1970s referred to as the educational production function studies examined the relationships between teacher knowledge and student achievement. These studies, typically used the number of undergraduate or graduate level mathematics courses taken, degrees earned, and performance on quantitative exams as markers of teacher knowledge (Begle, 1976; Goldhaber & Brewer, 2001; Monk, 1994; Rice 2003).

For instance, Begle (1976) examined the role of teacher knowledge in students' performance. He analyzed research conducted over a 16 year period, of how three markers of teachers' qualification– the number of content courses at the level of calculus or beyond; the number of mathematics methods courses; and undergraduate major or minor in mathematics – influenced students' performance. Begle first found that the number of content courses taken at the calculus level or beyond was positively related to student performance in 10% of the studies, and negatively associated with student achievement in 8% of the cases. He also claimed that taking mathematics methods courses yielded positive effects in 24% and negative effects in 6% of the cases. In the third case, having a major or minor in mathematics was positively associated with students' performance in 9% and negatively related in 4% of the studies. The students' performance involved significant observation of students' computational fluency, and not of cognitively demanding skills such as comprehension, application, or analyses. Begle, therefore, concluded that teachers' subject matter knowledge was not as "powerful" (p. 54) as assumed earlier and proposed research that would not focus on teachers' knowledge and their characteristics (Begle, 1979). There were other studies by Monk (1994), and Rice (2003) which studied the connection between the number of mathematics courses taken by a teacher with students' performance. They found similar, yet statistically not many significant results. For instance, Monk(1994), who used data from a longitudinal survey of American youth, found that every additional course in mathematics that teachers took accounted for 1.2% increase in students' performance on standardized assessment developed by National Assessment of Educational Progress. However, the impact of taking undergraduate mathematics methods course was higher than taking content courses in Monk's study.

Overall, the studies mentioned above of production function did not provide a definitive answer for how markers such as the number of courses a teacher take influences student learning (Ball, Lubienski, & Mewborn., 2001; Fennema & Franke, 1992; Hill, Sleep, Lewis, & Ball, 2007). The production function studies incorrectly assumed that the markers were accurate depictions of teacher knowledge.

In addition to the production function studies, other researchers examined how established teaching behaviors influenced students' learning, called the process-product studies. These studies examined a variety of behaviors that occurred inside the classroom. Few examples of these behaviors are – time on task, wait-time, classroom management and organization, curriculum pacing, and question posing (Brophy, 1986; Brophy & Good, 1986). The process-product studies do not explain to what extent teachers' knowledge impacts teachers' enactment of these behaviors.

Shulman's framework for knowledge for teachers.

Shulman (1986) drew attention to the importance of content knowledge in teaching. He identified as the "missing paradigm," (p. 6) the absence of research on teachers' knowledge of content and the role such content knowledge played in instruction (Shulman, 1986). However, the content knowledge he envisioned was not just limited to number of mathematics courses or mathematical performance. Schulman asked three sets of questions:

1. What are the sources of teacher knowledge? What does a teacher know and when did he or she come to know it? How is new knowledge acquired, old knowledge retrieved, and both combined to form a new knowledge base?

2. How does the teacher prepare to teach something never previously learned? How does learning for teaching occur?
3. How do teachers take a piece of text and transform their understanding of it into instruction that their students can comprehend?

While answering these three sets of questions, Shulman proposed a new theoretical framework for teacher knowledge that suggested three domains of content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter content knowledge referred to the amount and organization of knowledge per se in the mind of teacher. Through the construct of pedagogical content knowledge, he distinguished knowing content "for oneself" as different from the special amalgam of content and pedagogy needed to teach the subject. Curricular knowledge refers to familiarity with the lateral and vertical curriculum. The lateral curriculum is the knowledge that underlies the teacher's ability to relate the content of a given course or lesson to topics or issues being discussed simultaneously in other subjects. The vertical curriculum knowledge is familiarity with the topics and issues taught in the same subject area during the preceding and later years in school, and the materials that embody them. These three domains of knowledge together comprise Shulman’s “missing paradigm.” Further, from the teaching teachers point of view, he suggests categories of knowledge that would promote effective instruction.

- Content knowledge
- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter
- Curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers
- Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding
- Knowledge of learners and their characteristics
- Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures
- Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds (Shulman, 1987, p. 8)

Table 2.1 Shulman's categories of Teacher Knowledge

The focus on teachers' content knowledge was then revived and several studies followed Shulman's work.

Among others, the most influential work has been carried out by Ball and her research group at the University of Michigan. Ball and her colleagues strongly suggested the need to understand demands of teaching and therefore mathematical knowledge for teaching was through analyzing the practice of teaching. Ball (1999) describes examining mathematics teaching in their project Mathematics Teaching and Learning to Teach (MTLT) as follows:

We seek to analyze how mathematical and pedagogical issues meet in teaching – at times intertwining, at times mutually supporting, and at times creating conflicts. Through analyses of mathematics in play in the context of teaching, the project extends and challenges existing assumptions of what it is about mathematics that primary teachers need to know and appreciate, and where and how in teaching such understandings and appreciation are needed (p. 28).

This analytical approach also broadened understanding of what the work of teaching is; it included many tasks that come in the context of teaching. And therefore included mathematical demands of planning a lesson, listening to children, coming up with an example or a question, validating, justifying or refuting mathematical claims, etc., everything that might arise in the context of teaching.

Development of MKT framework and its measures.

In 2000, Ball & Bass claimed that "*although conceptions of what is meant by "subject matter knowledge," as well as valid measures thereof, have been developing, we lack an adequate understanding of what and how mathematical knowledge is used in practice*" (p.86). The University of Michigan's Mathematics Teaching and Learning to Teach project (MTLT) spent about 15 years studying mathematics teaching and mathematics used in teaching (Ball & Bass, 2003; Ball et al., 2008). By examining the records of the teaching of an accomplished teacher, the MTLT project investigated the mathematical knowledge demands of teaching from which they developed a number of "testable hypotheses" (p. 390) about the domains of mathematical knowledge for teaching (MKT). The MTLT project then developed measures that assessed these identified knowledge domains. The MKT construct developed in two significant ways. First, the MKT research group studied teaching practice to identify the mathematical entailments of engaging in the work of teaching. The second way involved an in-depth analysis of mathematics education literature that studied the work of teaching. The literature review founded on two main bodies of literature. Shulman's (1986) work inspired the two aspects: "one on teachers' subject matter knowledge and its role in teaching; and the second on the interplay of mathematics and pedagogy in teaching and teachers' learning" (Ball, 1999, p.22). The first body of literature examined teachers' knowledge of specific subjects such as history (Wineburg, 1996), English (Grossman, 1990), mathematics (Wilson, 1988) and science (Carlsen, 1988). The second body of work was more focused on mathematics education (Borko, Eisenhart, Brown, Underhill, et al., 1992; Thompson, 1984).

Practice-based nature of MKT.

Study of teaching practice informs the theory of MKT; in this sense, MKT, as proposed by Ball, is practice-based (Ball & Bass, 2003). To illustrate MKT as an important conception of teacher knowledge Ball and colleagues use an example of a multi-digit multiplication problem: 25 × 41 written in vertical form. They argue that teachers must be able to know how to do the multiplication problem but more importantly, this ability is not sufficient for teaching multi-digit multiplication. For instance, some students might solve the problem in different ways (see Figure 2.1).

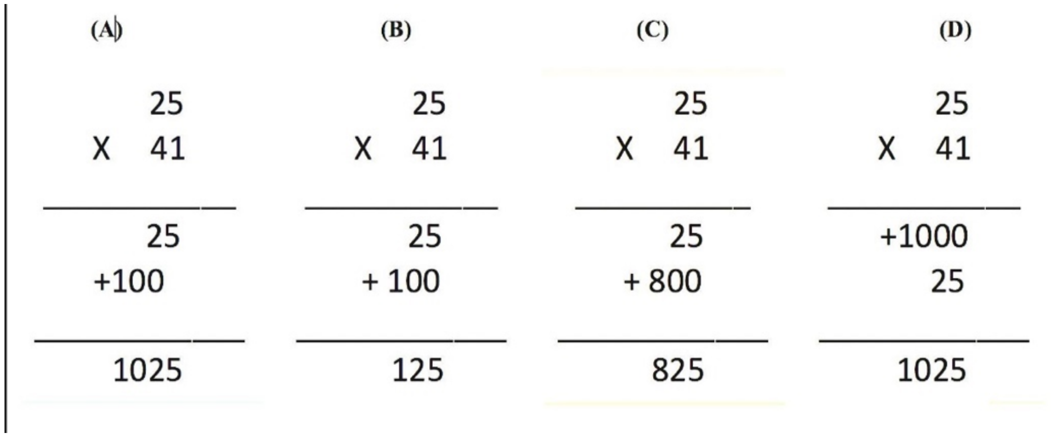


Figure 2.1: Students' ways of solving a multiplication problem

When such responses emerge during classroom interactions, the actual ability to calculate 25×41 is not only what the teacher needs. A cursory analysis of the students' work demands the following from the teacher:

- Identifying the correct responses (responses A and D are correct).
- Understanding the method used to arrive at the correct response, and whether it follows a logic that allows the process to be generalizable (i.e., finding the right answer was not just luck).
- Diagnosing the reason for the emergence of the wrong answer and incorporating the analysis into the pedagogy; for example, a review of response C suggests the student has forgotten to add the carried over number, and in B, there is a problem with place value.
- Figuring out an appropriate response to each student, so that they understand how to move ahead without getting discouraged.

The MKT research group argues that any teacher might be able to identify incorrect solutions. However, "skillful teaching requires being able to size up the source of a mathematical error" (p. 379) and the process of sizing up must be done "in-the-moment" of teaching. The MKT research group argues that error analysis of this nature is unique to the work that teachers do and posit that this kind of mathematical knowledge and reasoning is not likely to be encountered by other adults who are not teachers. Ball and her group also argue that other mathematical tasks of teaching are as crucial as error analysis. Some examples are analyzing students' non-standard approaches to problem-solving, presenting mathematical ideas, linking representations to underlying concepts and other representations, and giving or evaluating mathematical explanations require mathematical knowledge, skills, and habits of mind that are unique to teaching (Ball et al., 2008). The above example illustrates the practice-based nature of MKT; it is both derived from and used in practice.

Other conceptions of teachers' mathematical knowledge.

There are different widely held views about the kind of mathematical knowledge needed for teaching. Senk and colleagues' classified teacher knowledge as curricular knowledge, knowledge of planning for mathematics teaching and learning, and enacting mathematics for teaching and learning (Senk et al., 2012). Blum & Krauss' classified tasks and multiple solution in misconceptions and difficulties, and explanations and representations (Blum and Krauss, 2008). Rowland and his colleagues (2005; 2008), similar to Ball and colleagues, studied videos of pre-service primary mathematics teachers and used a grounded approach to develop their domains of knowledge identified as the

Knowledge Quartet: Foundation, Transformation, Connection, and Contingency. Rowland and colleagues' conception of teacher knowledge includes "beliefs" as a component of foundational knowledge. The inclusion of beliefs distinguishes Rowland and others' conception from Shulman's (1986) theories, which does not include a category for beliefs. Also, they propose contingency as a component of teacher knowledge. They define contingency to mean "a teachers' preparedness to deviate from their teaching agenda" or "the ability to think on one's feet" (Rowland et al., 2005, p. 263). Beliefs and contingency are distinct from Ball and others' MKT. Rowland and others do not report on ways to measures the Knowledge Quartet.

In association with the Australian Council for Educational Research, the Teacher Education and Development Study in Mathematics (TEDS-M) research team carried out a study of primary and lower secondary school mathematics teacher preparation in 18 countries to identify the nature and extent of their knowledge for teaching. Senk and her colleagues found the TEDS-M's conceptualization of teacher knowledge be the most closely matched to Ball's MKT. Their "enacting mathematics for teaching and learning" possesses features such as explaining or representing mathematical concepts or procedures, responding to unexpected mathematical issues, and analyzing or evaluating students' mathematical solutions or arguments, which are similar to Ball's conception of Specialized Content Knowledge. In addition, Senk and colleagues (2008) identify conceptual distinctions between content knowledge and pedagogical content knowledge similar to both Ball's and Shulman's conceptualization. There are some basic differences between these two conceptions. (Senk and others with Ball's concpetualization of MKT). First, Senk and others identify teacher knowledge not by a study of the mathematical demands of teaching, but by the content of school mathematics. Second, their school content is based on the Trends in International Mathematics and Science Study (TIMSS), and third, they do not assess the validity of their measures by studying teachers' mathematical quality of instruction.

The COACTIV project under the leadership of Baumert and his colleagues in Germany has focused on the mathematical knowledge of secondary (high) school mathematics teachers. Baumert et al.'s (2010) description of teachers' mathematical knowledge for teaching showed the knowledge domains as distinctly Content Knowledge (CK), and Pedagogical Content Knowledge (PCK). Their conception was based on Shulman's (1986) theory and they did not develop an independent theoretical framework. These scholars describe PCK to include the knowledge of mathematical tasks and tools and the knowledge of student thinking and assessment of mathematical understanding. CK is conceptualized as similar to Ma's (2010) profound mathematical understanding of school mathematics.

The QUANTUM project by Adler and others, addresses a need to contribute to shared-understanding of MKT from a contextual perspective — adapting to African teachers' needs. The project's goal is to develop a method to describe and explain what MKT is and how it is constituted across various instructional practices. Their goals are based on the assumption that teaching and mathematics are *co-constitutive* (Adler & Davis, 2006) — where each shapes and is shaped by the other as they come live in practice.

Ball and her colleagues in the LMT project see the constitution of mathematics and teaching a bit differently. Ball (1988) makes a distinction between the content that one learns for 'oneself' and the one that is a special amalgam of content and pedagogy needed to teach the subject, similar to Shulman's framework. One of her goals appears to be to find empirical evidences for what researchers *think* teachers need. For the last fifteen years, the work of the Mathematics Teaching and Learning to Teach (MTLT) project has focused both on the teaching of mathematics and on the mathematics used in teaching. The aim has been to investigate the demands of teaching. Instead of reasoning from the school curriculum to a list of topics teachers must know, the MTLT group developed an empirical approach to understanding the content knowledge needed for teaching. The first project focused on the *work* that teachers do in teaching mathematics and the second project developed survey measures of *content knowledge for teaching mathematics*.

Silverman and Thompson's work is empirically grounded in a different direction. They focus more on the development of MKT rather investigating what MKT is, grounding their research in mathematics education and learning theories. Their investigation proposes a framework for teacher education subscribing to a constructivist perspective.

Adler and her group in QUANTUM investigated questions regarding constitution and forms of MKT and how these forms relate to pedagogic practice, with focus on pre-service and in-service teacher education. The sites of investigation were teacher education institutes and actual teaching practice in schools. The group does not claim any study indicating change in students' performance, but reports their investigation of what constitutes MKT. Kazima and Adler (2006) pointed out two things. First, they argued that

Ball's framework (Ball, Bass & Hill, 2004) disconnects mathematical terms—linguistic disconnection. For example, students equated disability to probability (Kazima & Adler, 2008), due to a phonetic similarity and the teacher didn't know how to respond to such a situation in the classroom. Second, they argued that Ball's framework does not elaborate learner's intuitions that co-exist with mathematical notions. This challenge of learner's intuition is regarding how teachers can handle students' outside experience and beliefs in the classroom. In one of their studies students believed that it is harder to get six on a dice, even though they experimented and documented a frequency table in the classroom. The teacher's persistence of working on more experimentation was not useful as students concluded it is easier to get six on the dice inside classroom but difficult to get it outside. Although it is not new to understand that students sometimes hold contradictory ideas (Watson and Moritz, 2003), according to Adler and her group it is challenging for a teacher to address such issues without preparation. Building on Ball et al's (2004) eight aspects of mathematical work of teaching that teachers engage with, Adler and her gorup developed six categories. Behind this their aim was to capture everything that the teacher did, irrespective of whether that was correct, appropriate or productive.

Mathematical work in teaching mathematics Ball et al (2004)	Work of Teaching mathematics Kazemi et al (2008)
1. Design mathematically accurate explanations that are comprehensible and useful for learners; 2. Use mathematically appropriate and comprehensible definitions; 3. Represent ideas carefully, mapping between a physical and graphical model, the symbolic notation, and the operation or process;	1. Defining – attempts to provide a defintion 2. Explanations – teachers explain an idea or a procedure 3. Representation – teachers represent ideas and in various ways

4. Interprete and make mathematical and pedogogical judgements ablout learner's questions, solutions, problems and insights (both predictable and unusual); 5. Be able to respond productively to learner's mathematical questions and curiosities; 6. Make judgements about the mathematical quality of instrucitonal material and modify as necessary; 7. Be able to pose mathematical questions and problems that are productive for learners' learning; and 8. Assess learner's mathematics learning and take the next steps (Ball et al., 2004, p. 59)	4. Working with learners' ideas – teachers engage with both expected and unexpected learners' mathematical ideas 5. Restructuring tasks – teachers change set tasks by scaling them up or down 6. Questioning – teacher ask questions to move the lesson on (Kazami et al., 2008, p. 288)

Table 2.2: Tasks of teaching as Ball et al (2004) and Kazima et al (2008)

Adler and her colleagues used these six categories to analyze the cases of teaching and quantify teachers' actions while teaching. The findings of these studies confirmed that the tasks identified by Ball et al. were not only mathematical but took on specific meanings across topics and different approaches to teaching. However, they suggest that each of those tasks needed to be understood more specifically about the topics in mathematics and to particular approaches to teaching, and are working towards it.

Silverman and Thompson (2008), situate their understanding of MKT in students' learning theories.

Our perspective entails a fundamentally different foci than Ball's MKT: rather than focusing on identifying the mathematical reasoning, insight, understanding and skill needed in teaching mathematics, we focus on the mathematical understandings "that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students' reasoning" (p. 1).

They situate their understanding of MKT in practice but in connection with students' work and not as in Ball's study where MKT is associated with "work of teaching". However, the mathematical understandings mentioned by Silverman and Thompson (2008) were never elaborated further nor provided with examples.

It is valuable to look at these theories because they provide the scope of research on teacher knowledge and the extent to which progress has been made in conceptualizing teacher knowledge and developing ways of measuring it. In spite of the growth in the field, none of those above conceptions of teacher knowledge is simultaneously grounded in the practice of teaching, is measurable at scale, and grounded in the discipline of mathematics. Shulman's (1986) seminal work contributes much to work on teacher knowledge but is not rooted in teaching practice or disciplinary mathematics, and does not provide measures to assess his conception of teacher knowledge. Although Rowland and others (2008) conception are grounded in the teaching practice and disciplinary mathematics, they do not provide measures for assessing their understanding of teacher knowledge. Senk and others' (2008) notion grounded in the discipline of mathematics and is measurable, but they do not ground their work in teaching practice. Baumert and his colleagues (2008) do not ground their work in teaching practice or disciplinary mathematics and do not have measures to assess their theory of teacher knowledge. In the following paragraph I describe the framework that has been accepted widely, situated in practice, measurable and grounds the the knowledge demands in mathematics at the horizon.

Domains of MKT.

Ball, Thames, and Phelps (2008) in their seminal work at the University of Michigan, first explicitly identified this knowledge as Mathematical Knowledge for Teaching (MKT). They defined it as the mathematical knowledge, skill, and habits of mind entailed by the work of teaching and proposed a further refinement of subject matter and pedagogical content knowledge into sub-domains, which were earlier introduced by Shulman (1986) (see figure 2.2).

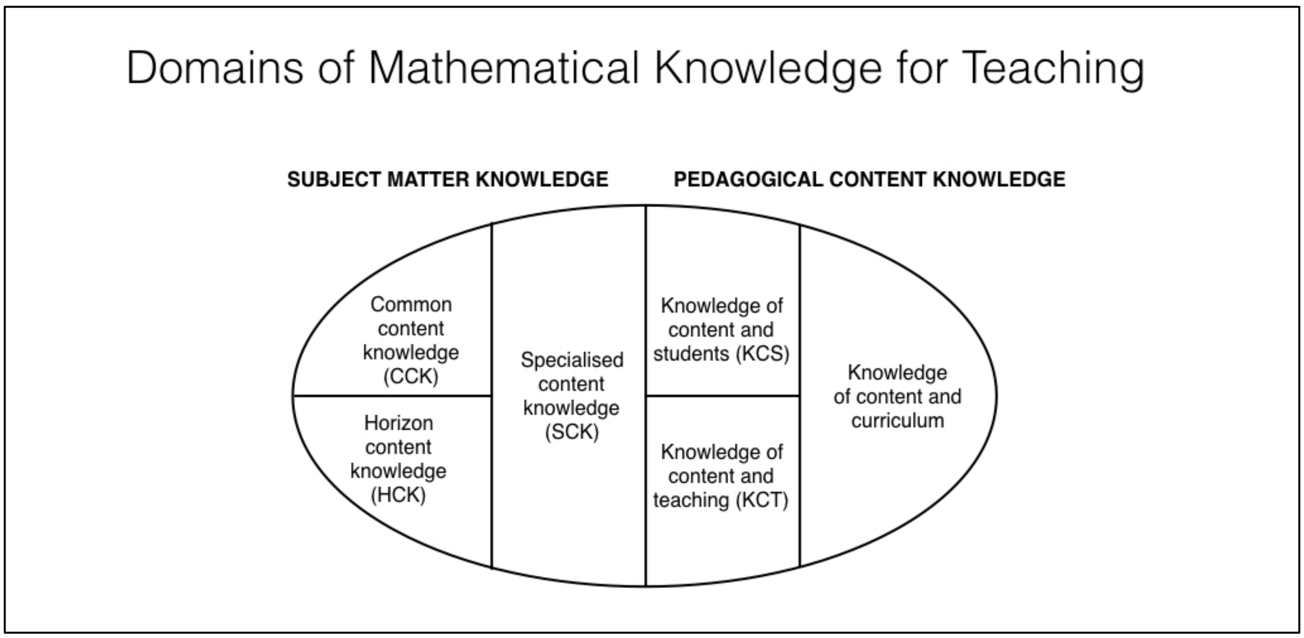


Figure 2.2: MKT framework by Ball, Thames and Phelps (2008)

The three subdomains on the left consist of subject matter knowledge, and the three subdomains on the right comprise pedagogical content knowledge. In subject matter knowledge, common content knowledge refers to the mathematical knowledge and skill possessed by any well-educated adult, while specialized content knowledge is the mathematical knowledge and skill used by teachers in their work but not generally possessed by well-educated adults, such as how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems (Ball et al., 2005). In pedagogical content knowledge, knowledge of content and students involves knowing about both mathematics and students, that is, as content knowledge intertwined with knowledge of what students think about, and how they know or learn a particular content (Hill, Ball, & Schilling, 2008). Knowledge of content and teaching involves knowing about both mathematics and teaching (Delaney, Ball, Hill, Schilling, & Zopf, 2008), as content knowledge intertwined with knowledge of how best to build student mathematical thinking or how to remedy student errors. Knowledge of curriculum denotes all knowledge interconnected within curriculum.

Horizon content knowledge (HCK) refers to an orientation to and familiarity with the discipline that contributes to the school subject at hand. It provides a sense of how the content taught is situated in the discipline of mathematics. This knowledge is a resource for balancing the fundamental tasks of connecting learners to a vast and highly developed field, and also includes an awareness of core disciplinary orientations and values, and of big ideas and essential structures of the discipline. These attributes of HCK suggest that HCK might be a kind of knowledge that is responsible for the ability to make connections across a range of mathematical concepts. This conceptualization of HCK resonated with Ball's (1993) first reference to the term "mathematical horizon" in context of describing the dilemma and tensions faced by an elementary teacher while dealing with pedagogical and disciplinary mathematics concerns. For example, HCK might include choosing representation of contexts (e.g. negative numbers) that are consistent with the ideas that will be encountered in the future (e.g. absolute value) (example used in Ball 1993). What this means is that, even though elementary mathematics might not involve certain components of a given concept, and a student might face them only in their later years of mathematics learning, choosing a representation at primary level cannot be deceiving or so simplistic that it leads to contradictions with the concept in their later years. This particular idea, which describes perpetuation of mathematical sensibilities in making decisions about teaching, forms the basis of knowledge of mathematics at the horizon.

2.2 Delienating and Defining HCK

Developing the idea of HCK further, Ball and Bass (2009) suggested that it might be productive to view HCK as an "elementary perspective on advanced knowledge." In particular, they conceptualized HCK in four key elements:

1. a sense of the mathematical environment surrounding the current "location" in instruction,
2. major disciplinary ideas and structures,
3. key mathematical practices, and
4. core discipline values and sensibilities (p. 6).

Despite these efforts of Ball and Bass (2009) in refining the notion of HCK, it has received relatively little attention in the research community and remains mostly undeveloped. The problem of defining HCK stems from multiple metaphors and from inadequate clarity and consensus, especially regarding HCK's relation to teaching.

Therefore a detailed discussion of what HCK comprises is needed with examples that are embedded in the practice of teaching mathematics. This dissertation will mainly aim to contribute to this aspect of HCK along with bringing forward the connection between equitable access to mathematics and HCK.

Ball and Bass (2009) described HCK as “a kind of mathematical ‘peripheral vision’ needed in teaching, a view of the broader mathematical landscape that teaching requires” (p. 1). Building on ideas of Ball and Bass, Zazkis and Mamolo (2011) use Husserl’s work to propose a conception of “knowledge at the mathematical horizon.” Zazkis and Mamolo elaborate upon inner and outer horizon of school mathematics that are formed due to knowledge of undergraduate mathematics. Their paper was followed by two responses. Foster (2011) proposed “peripheral mathematical knowledge” to refer to mathematics relevant to teaching but not visible to the learner. Figueiras, Ribeiro, Carrillo, Fernández and Deulofeu (2011) point out that the language for HCK needs to be consistent with basic assumptions of the nature and role of teacher content knowledge. They argue for locating the meaning of HCK in the work of teaching instead of conceptualising HCK as advanced knowledge that is then applied to teaching. They specifically say,

> *“Our critique of Zazkis and Mamolo’s paper is much more in terms of their assumptions about the nature of the mathematical knowledge that elementary and secondary teachers need, rather than in terms of their conceptualization of knowledge at the mathematical horizon” (p. 26).*

They are raising the concern whether advanced knowledge would have a bearing on teaching practice or vice versa. Vale, McAndrew, and Krishnan (2011) use the phrase "connecting with the horizon" to describe advanced mathematics in a professional learning context that helped teachers to see more connections and structure among representations and among topics.

Many of these examples of HCK align with Klein's idea of "elementary mathematics from an advanced point of view". In contrast to this Ball and Bass (2009) described HCK as "a kind of mathematical 'peripheral vision' needed in teaching, a view of the broader mathematical landscape that teaching requires" (p. 1). Making this as "advanced mathematics seen from an elementary mathematics point of view." In an attempt to figure out how "advanced mathematics" can be relevant to school mathematics teaching, Jakobsen et al. (2013) developed a definition of HCK:

> *"Horizon Content Knowledge (HCK) is an orientation to, and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how "truth" or validity is established. HCK also includes awareness of core disciplinary orientations and values, and of major structures of the discipline. HCK enables teachers to "hear" students, to make judgments about the importance of particular ideas or questions, and to treat the discipline with integrity, all resources for balancing the fundamental task of connecting learners to a vast and highly developed field" (P. 4642).*

This definition helped distinguish HCK from the rest of the knowledge domains, especially from specialized content knowledge (SCK). For example, SCK is immediately about the content being taught, and HCK is not. From this definition, we see that HCK is neither common nor specialized. It is not about a vertical curriculum, but more about having a sense of the environment of the discipline that is being taught. Thus, when discussing HCK, it is not sufficient to simply consider knowledge about advanced mathematics, in the sense of knowledge beyond the grades that the teacher is teaching. It is also not about knowledge of different topics that students will learn in future. HCK empowers teachers to make sense of what mathematics is present in students' responses. It enables them to make connections with topics that students might or might not face in their future. One key factor of HCK as perceived by Jakobsen et al. (2013) is the ability of the teacher to respond with an understanding of connections to topics that students may or may not meet in the future. (Ball & Bass, 2009, Jakobsen et al., 2013).

I use this definition of HCK and other work that has been produced by Ball and her colleagues at the University of Michigan as a basis for developing a spcific defintion and investigation for encounters with such knowledge in the practice of teaching. In the sections below, I first describe my research questions in light of the definition of HCK as given by Jacobson, Thames, and Ribeiro (2013), along with a description of what I mean by an encounter with HCK.

Research questions.

The definition given by Jakobson, Thames and Ribeiro (2013) brings forward how HCK is neither common nor specialized. It is not limited to only knowing advanced mathematical knowledge, but also makes a call for the knowledge that allows teachers to make additional sense of what students are saying and to act with an awareness of topics that students may or may not meet in the future. A diagrammatic synthesis of what comprises HCK based on the defintion by Jakobson et al. is given below (Figure 2.3). It shows five main components of HCK derived from the definition. That HCK is – explicit knowledge of the ways of and tools for knowing in the discipline; the kinds of knowledge and their warrants; about knowing major structures of the discipline; about knowing how "truth" or validity is established in the discipline; and awareness of core disciplinary orientations and values.

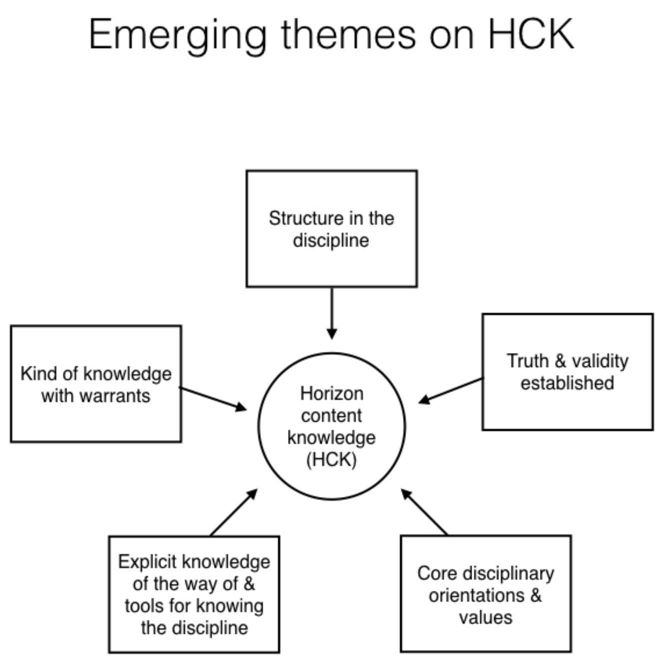


Figure 2.3: Five components of HCK mentioned in Jakobsen et al. (2015)

Each of these components guides certain actions of teaching, beliefs and attitudes that shape knowing of mathematics in the classroom setting.

Drawing upon the definition of Horizon Content Knowledge given by Ball and her colleagues at the University of Michigan, which has been the basis of Jakobsen et al.'s elaborate description of HCK, this study used a more specific definition for the horizon of mathematics. The definition exercised is in alignment with what Ball (1993) and Ball & Bass (2009) envisioned as "mathematics at the horizon." The definition used in this study is specific to the school mathematics curriculum. I define the mathematical horizon as "a projection of mathematical meanings, topics, and structures present in the curriculum into the mathematics extending beyond the support of the curriculum materials concerning a particular location of instruction, such that it enables meaningful learning of mathematics." The curriculum here, is used in a sense Remillard (2015) uses it, "printed, often published resources designed for use by teachers and students during instruction." (p. 213). The meaningful learning of mathematics is defined through two specific frameworks – First, by the instructional triangle as described by Cohen, Raudenbush and Ball (2003) that represents the process of teaching. The instructional triangle positions meaningful listening to students at the core of impactful instruction. The second, mathematical sensibilities of knowledge for teaching that Jakobsen, Thames & Ribeiro (2013) delineates and brings disciplinary mathematical norms and values to the center of instruction. I again elaborate these two frameworks in detail in the chapter on the conceptual framework.

An encounter with the horizon of mathematics would be an instance where the meaningful learning of mathematics could take place by taking supports outside the curriculum and situated in disciplinary sensibilities, norms, and values. The encounter could be initiated by listening to students meaningfully, designing the instruction, developing mathematical explanations or by reading the curriculum. The management of such instances would require supports outside the curriculum, situated in disciplinary sensibilities and norms. At the operational level, an encounter will be considered as "an instance to establish any of the five components listed below to its potential" in a given location in the instruction. If there is an opportunity to

1. establish truth in mathematics using mathematical tools and disciplinary ways in classroom instruction
2. use core disciplinary values and orientation
3. make explicit the knowledge of the ways and tools for knowing the discipline
4. connect with structures in the discipline
5. comprehend kinds of knowledge with its warrants

then that would be considered as an encounter with HCK.

Therefore, for this thesis encounter with HCK did not depend on whether the teacher noticed such an encounter or not, but whether the classroom interaction had potential to establish any component of the components listed above.

I defined mathematics at the horizon and encounters with the mathematics at the horizon, and yet the research questions below are framed as "encounters with HCK." In this paragraph, I explain the terminology glitch present in the research questions. The encounters with HCK is used to mean encounters with the mathematics at the horizon, which is defined above and where the term "mathematics at the horizon" is in the similar sense that was used by Ball (1993) and Ball & Bass (2009). Ball's use of the term was situated in practice of teaching mathematics, specifically in the context of dilemmas that a teacher faces while choosing an appropriate representation, representations that do not distort the mathematics and increase the access to learning. However, the phrase "mathematics at the horizon" has been used with multiple meanings by various researchers, mainly to refer to the advanced mathematics that is "out there" at the horizon. This reference makes the mathematics at the horizon not rooted into the practice of teaching mathematics (E.g., Zazkis & Mamolo, 2013), and that is not what this dissertation has built upon, and therefore the term "mathematics at the horizon" is avoided in the research questions. Instead, the Horizon Content Knowledge, i.e., mathematics at the horizon entailed through the practice of teaching is used here.

Situated in this understanding this dissertation focused on three main questions:

1. What kinds of "encounters with HCK" arise in classroom teaching? And: How do teachers *manage* them?
2. How are students' opportunities for learning shaped by teachers' encounters with HCK?
3. What kind of mathematical knowledge do teachers exhibit while navigating encounters with HCK?

There are specific characteristics of HCK that appear throughout a classroom teaching and are hard to pinpoint as encounters. For example, teachers' beliefs about mathematics and what it means to know mathematics. These will be present in everything that teachers do in the classroom. In one of the cases of teaching, the teacher believed knowing mathematics is about knowing it *speedily*. What he meant by that was, it is not enough if one can solve the problem, or remember formulae, but only if the person can also do it with *speed* does the person know the real mathematics. Now this belief could be about knowing in general, but the teacher expressed it for the case of mathematics. And, in all teaching lessons of this teacher, there was a constant attempt to create opportunities to do mathematics speedily. Now such understanding of the horizon will not be captured through an encounter; rather it would be present in all aspects of the work of teaching. And such beleifs therefore will guide many tasks of teaching as identified by scholars. The representaiton will be chosen such that *speed* of doing mathematics will increase, tasks will be given to complete in time, etc. The variety in answers or ways to solve problems will be superceded by who does faster in the class. In some sense, such strong beliefs manipulate the horizon. The belief has replaced meaningful learning with being able to quickly repeating. Such cases are not counted as encounters.

2.3 Situating HCK in the Literature on AMK

In analyzing MKT frameworks, I take advanced mathematical knowledge (AMK) to refer to both knowledge of tertiary mathematics and knowledge of mathematics of an advanced nature. In the former sense, I adopt Zazkis and Leikin's (2010) definition of AMK as "knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college" (p. 264). In the latter sense, I take AMK to refer to a robust and connected knowledge of mathematical objects, structures, and ideas, such as that given by Liping Ma's (2010) description of teachers' profound understanding of elementary mathematics, or the knowledge of mathematical structure that corresponds to the disciplinary knowledge of an expert in the field (de Groot, 1965; Schoenfeld & Herrmann, 1982; Schwab, 1978).

Researchers have struggled with the relationship between AMK and MKT. Davis and Simmt (2006) proposed that AMK provides the foundation for MKT. Their model of MKT emphasizes the figurative bases that give shape to mathematical structures, including various interpretations, metaphors, images, and applications. That is, in their framework, it is the images and metaphors used to establish mathematical structures that constitute a teacher's knowledge of mathematical structure.

Teachers seem to have little formal experience with such figurative aspects of core principles and as a consequence, their knowledge of these aspects is typically tacit rather than explicit (Davis & Simmt, 2006). For example, a teacher might be able to use an area model to successfully represent the product 12×13 but not explicitly recognize that the property of distribution of multiplication over addition has been invoked in this representation.

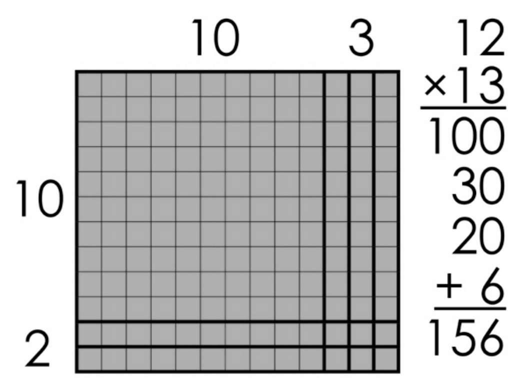


Figure 2.4: Area model for 2-digit multiplication

Teachers' advanced mathematical knowledge, from this perspective, can be viewed as involving the largely tacit knowledge regarding mathematical structures and how those structures are established.

Researchers have argued on the source of this tacit knowledge, with some suggesting that it is not typically derived from tertiary mathematics courses (Davis, 2011) while others propose that the development of this kind of knowledge should be an explicit focus of preservice teachers' mathematical training (Silverman & Thompson, 2008). Davis (2011) argues that teaching largely involves "drawing logical consistency from diverse instantiations, however, teachers' university courses in mathematics typically focus on completed ideas" (p. 1507). That is, the content teachers typically encounter at the tertiary level involve explicit instantiations that may only be useful at that level and do not focus on connections to other instantiations or the evolution of ideas motivating the completed formulation. For example, multiplication can be conceptualized as representing area but it can also be given as an output of a linear function with vertical intercept equal to zero. Teachers may encounter linear functions in a course in college algebra, but that is not enough to make explicit connections to the area model of multiplication used in the school curriculum. According to Davis and Simmt's (2006) perspective then, mathematical content encountered at the tertiary level may not directly contribute to the development of teachers' knowledge of mathematical structures or how they are established. That is, AMK in Davis and Simmt's (2006) MKT framework is not necessarily explicit knowledge acquired in tertiary level mathematics courses but appears to consist primarily of teachers' tacit knowledge of mathematical structure and connections. Davis and Simmt (2006) suggest that, in my two-part construct of AMK, teacher's tacit knowledge is part of the disciplinary knowledge of an expert in the field and not typically developed in tertiary-level courses.

In contrast, Silverman and Thompson (2008) propose a framework for MKT that focuses on the development of MKT continuously throughout teachers' careers beginning with their mathematical learning as students. They argue that teachers gain knowledge of mathematical structures by making connections between, and relating, content knowledge. In particular, they draw on Piaget's concept of reflective abstraction to describe how "new, more advanced conceptions develop out of existing conceptions and involves abstracting properties of action coordinations to develop new cognitive structures" (Silverman & Thompson, 2008, p. 506). In other words, according to Silverman and Thompson (2008), AMK in the structural sense includes the new cognitive structures that are formed by identifying relationships and making connections between mathematical content. Further, unlike the tacit understandings identified by Davis and Simmt (2006), this kind of AMK may be developed at the tertiary level in mathematics courses designed to increase prospective teachers' depth of knowledge relating to elementary level mathematics content. Silverman and Thompson (2008) also caution, however, that these mathematical understandings in isolation do not constitute pedagogical understandings and must undergo further transformation to be useful for teaching. That is, similar to Davis' (2011) observation that completed understandings are not sufficient for teaching, Silverman and Thompson (2008) also recognize that AMK alone requires additional transformations before it is pedagogically useful. As evidenced by the conclusion above that knowledge of mathematical structures alone is not enough for AMK to be useful for teaching, researchers have made efforts to conceptualize AMK in a way that is more specifically oriented to teaching practice. One of such viable constructs is Horizon Content Knowledge, which involves knowledge of structures, but as situated in practice. In the vignette given in Ball and Bass (2009), where students are measuring the handprints on a graph paper, a student comes up with an idea of finer unit for measurement.

> *One child suggested getting the graph paper used by older pupils because the squares were much smaller and they would be able to get a closer count of the area of their handprints. The teacher, who happened to have recently studied integral calculus, heard the comment as reflecting a surprising intuitive grasp of the fundamental idea that finer mesh affords more accurate measurement* (p.1).

Ball and Bass suggest here that with the view of mathematics at the horizon – here the knoweldge of limits, helped the teacher appreciate the students' comment and use it in the class. I would like to add here, that there are other aspects of teacher knowledge involved here which shaped the instruction. And those aspects of knowledge are also part of the mathematics at the horizon. For instance, noticing that the student's suggestion of finer unit for measurement is a mathematically relevant one, many teachers I know would have announced it to the students that using finer unit for measurement is a good idea, as it connects to the advance topic in mathematics such as limits. And that would just indicate the teacher's AMK, but the decision that this teacher took involved other students listening to that idea, deciding whether it makes sense, and actually using a finer mesh to see what happens. These decisions come from a specific understanding of mathematics. The mathematics used here is democratic, and aims to give access to everyone. There are other concerns that arise in this activity which on the surface appear as concerned with Specialized Content Knowledge (SCK). Some of these involve –

- Knowing that Changing the unit of measurement will change the number in the measure, and would that cause any misconception in students that is related to measuring areas
- How would student make connections when the same area measured using different measuring units leads to different measures.
- And when the students all figure out that the measure became more accurate with a smaller mesh, how to talk about accuracy in that context such that it emphasises idea of limit (for e.g. why it became accurate? Operational understanding of accuracy?)

These sensibilities again emphasise that such mathematical thinking is possible because of the particular sense about what mathematics is, and what the teaching is. Definitely this view of what is mathematics is much away from the mathematics in the tests. This is the view that superceded in valuing knowledge for teaching, and that is the sense in which I am talking about Horizon Content Knowledge.

2.4 Situating HCK in Mathematics Instruction

Teaching is a complex phenomena (Freeman, 1996). Therefore, it is difficult to explain it in an elegant and systematic way (Doyle, 1986). This statement about teaching implies that an honest understanding of teaching needs to take account of what teachers and students do in classrooms. In the dynamic situation of teaching, teachers are most immediately confronted with diversity among their students, such as prior knowledge, cultural background, and so on. In this section I present what comprises teaching and therefore what knowledge of mathematics horizon forms the HCK for classroom teaching.

Instruction and its components.

Cohen et al. (2003) explained instruction and resources with a diagram. See figure 2.5.

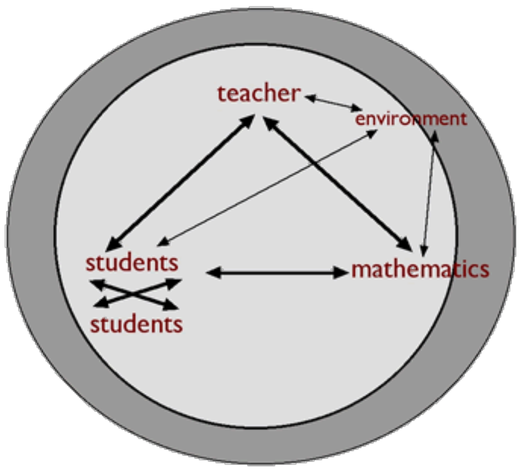


Figure 2.5: Instructional triangle from Cohen, Raudenbush, Ball (2003)

The definition that they provide is around this diagram, and it reads as follows.

Instruction consists of interactions among teachers and students around contents in environment. … "Interaction" refers to no particular form of discourse but to teachers' and students' connected work, extending through, days, weeks, and months. Instruction evolves as tasks develop and lead to others, as students' engagement and understanding waxes and wanes, and organization changes (Lampert, 2001). Instruction is a stream, not an event, and it flows in and draws on environments – including other teachers and *students, school leaders, parents, professions, local districts, state agencies, and test and text publishers (p.122).*

Instruction depends on interactions and evolves as the content progresses with the growth of the learners' understanding. Cohen et al. describe instruction as two way interaction between teacher ←→ student, student ←→ student, student ←→ mathematics (directly or via teacher) and teacher ←→mathematics. The basis on which this thesis builds is that if the mathematics instruction envisioned is similar to what Cohen, et al. suggested, then teacher noticing student actions, thinking and skills becomes the core of effective instruction. There are studies that talk about how teachers improve upon this noticing. For example, Sherin & Han (2004) introduced "professional vision" as a measure to understand practicing teachers' learning in educational research. van Es and Sherin proposed that noticing in teaching involves: (1) identifying what is important in teaching situation, (2) making connections between specific classroom interactions and broader principles of teaching and learning, and (3) using knowledge of teaching contexts (students, school, subject) to reason about a situation (Sherin & Van Es, 2005). The literature in noticing has made major strides in the decade since van Es and Sherin's work. Most of these studies involve (a) facilitating teachers' viewing and discussing video-recorded excerpts of classroom practice in video clubs or similar professional development settings, and (b) analyzing what participants notice and how they reason about it (e.g. Sherin and Han, 2004, Sherin and van Es, 2005 and van Es and Sherin, 2008). As a result, the construct of noticing has received much attention. Noticing within the midst of teaching, noticing details in the work of others and noticing through mediating tools such as video have all become essential foci of teacher education (Sherin, Jacobs & Phillip, 2011). Van Es and Sherin (2008) include analysis of the relationships between teachers' developing professional vision and their teaching activity, emphasizing, "if teachers are to use student thinking to inform pedagogical decisions, they need to "learn to notice" student mathematical thinking" (p. 101). The analysis reported that in the video-club meetings, teachers paid increased attention to students' mathematical thinking. In interviews, these teachers reported having learned about students' mathematical thinking, about the importance of attending to student ideas during instruction, and about their school's mathematics curriculum. Finally, impacts of learning to notice were also uncovered in teachers' instruction. By the end of the year, these teachers increasingly made space for student thinking to emerge in the classroom, probed students' underlying understandings, and learned from their students while teaching.

However, this presents only one side of the story. Teachers being able to notice what students say in the classroom is a two-fold phenomena. One is to be able to listen to what students say/think, to have patience, create publicly safe classrooms so that students share their thinking. The second is to be able to comprehend what the students have said from the point of view of the discipline, and using the practices of the discipline decide what contribution the student thinking makes at that moment of the teaching.

This thesis, as mentioned earlier focuses on the latter – what mathematics do teachers hear in their students' voices, blackboard drawings, choice of examples, as well as what mathematics they chose to value when they give instruction to *do mathematics* in the classroom. I believe that analyzing such encounters will unpack what knowledge of the disciplinary mathematics is at play in our classrooms.

2.5 Assembling the Framework for the Study

This chapter provided a brief review of the literature around mathematical knowledge needed for effective teaching. In the above literature review, I explored the role of advanced mathematical knowledge (AMK) in frameworks for mathematical knowledge for teaching (MKT). In particular, horizon content knowledge (HCK) seems to have emerged during the last decade as a promising construct for investigating the role of AMK specifically involved in the practice of teaching.

As I have demonstrated, the MKT frameworks explored above collectively address the two aspects of AMK about advanced mathematical content acquired in tertiary-level courses and knowledge of connections and mathematical structure. Literature addressing AMK has made further distinctions and connections across these areas to highlight aspects that are important for teaching. In particular, authors have identified a dichotomy, which is often described as a contrast between elementary perspectives of advanced mathematics and advanced aspects of elementary mathematics (Ball and Bass, 2009; Zazkis & Mamolo, 2011).

In addition to distinguishing between elementary perspectives of advanced mathematics and advanced aspects of elementary mathematics, another issue faced by researchers regarding HCK is what belongs in the various categories of knowledge pertaining to a mathematical structure. This presents an especially challenging task given that the content of the elementary curriculum is already quite vast and the possible connections between that content and mathematics on the horizon even more so. Most work has focused directly on generating categories of MKT that can be used to capture or identify specific mathematical content comprising HCK, whether it involves particular structures, broader connections, or a combination (Blackburn, 2011; Ball & Bass, 2009; Ball, et al., 2008; Carrillo, et al., 2012; Zazkis & Mamolo, 2011). Fernández and colleagues (2011) depart from this approach by identifying types of connections present in teaching rather than the mathematical content itself, but their approach is still rooted in the identification of potential HCK and involves knowledge not identified to have a robust presence in teaching practice.

Silverman & Thompson (2008) argue that while the task of identifying particular MKT for teaching specific mathematical content is essential, enumerating such knowledge entirely is challenging because of the vastness of mathematical content making up the K-12 mathematics curriculum. Instead, they propose an MKT framework that shifts away from the goal of developing "particular MKT to developing professional practices that would support teachers' ability to continue developing of MKT throughout their careers" (Silverman & Thompson, 2008, p. 509). In other words, with HCK in particular, it might be helpful to adopt an approach emphasizing the development of knowledge of structure rather than enumerating structural knowledge that teachers should ideally possess. In fact, Davis (2011) challenges the very assumption of the frameworks that emphasize cataloging knowledge, suggesting that particular instantiations may not even operate as fundamentals at all but "appear more to work as agents in an ever-evolving system" (p. 1507). As a consequence, Davis (2011) argues that rather than cataloging a set of basics, which are likely too vast for any individual to have conscious awareness, it might be more productive to focus on developing teachers' ability to identify, investigate, and connect useful instantiations to inform new practice.

I have built upon the perspective of Ball and Bass (2009), where they use the compelling metaphor of a tourist. It suggests that HCK might involve an awareness of the mathematical landscape of present instruction that is more like that of "an experienced and appreciative tourist" rather than that of a "tour guide" (p. 6). Pushing this metaphorical description further, I point out that an experienced tourist does not always have the luxury of knowing what lies in store when they make a decision to explore an unfamiliar path and may not even be able to anticipate where it will lead. The very nature of operating in unfamiliar territory means limits to the kind of explicit knowledge available for navigation will exist and may require drawing on a more tacit and intuitive kind of knowledge to keep moving forward. Therefore, I am motivated by this image of the "tourist" to extend my definition of HCK to include the possibility of accounting for tacit knowledge of mathematical practices and values.

In particular, I draw on the work of Polanyi (2009) who describes tacit knowledge regarding two features of an object: the proximal and the distal. A full elaboration of these terms is beyond the scope of this particular work, but the general sense is that tacit knowing involves the way that individuals "attend from the first term to the second term of the tacit relation" (p. 10). What this means is that in tacit knowledge, an individual is not able to attend to the particulars of an object (the first term) explicitly, but reasons from them to the coherent whole (the second term). Polanyi (2009) illustrates this process by demonstrating how it is possible to know the face of a familiar person by attending from the perception of their distinct features to their face without being able to identify what those features are.

Similarly, tacit knowledge of mathematical discipline would involve the ability to recognize a coherent connection of which a person is not explicitly aware. For example, tacit knowledge could be illustrated as a person using the distributive property of multiplication over addition to calculate multiplication between two numbers. For example, $23\times7=(20+3)\times 7=140+21=161$, is procedurally just understood as multiplying 3 by 7 first and then 2 by 7 leaving a placeholder zero (vertical representation).

On the other hand, an expert can identify when another person has a "certain knowledge that he cannot tell" (Polanyi, 2009, p. 8) when that expert has explicit knowledge of those particulars. Consequently, tacit understanding of mathematical sensibilities, practices, and norms should be identifiable when an expert recognizes that a teacher is engaging in mathematical activity indicating a perception of mathematical structure or practice, even if the teacher is not able to identify this knowledge explicitly herself.

Keeping in mind the desire to define HCK in such a way that it can be used to attend to teachers' tacit knowledge of mathematical sensibilities, practices, and values, I proceed with giving definitions of the mathematical horizon and horizon content knowledge. I propose to view the horizon as a projection of the mathematical topics and structures contained in the school mathematics curriculum in the discipline of mathematics. More specifically, I define the mathematical horizon as the projection of mathematical meanings, topics and structures in the curriculum in the discipline of mathematics to be able to learn the school mathematics meaningfully. By curriculum, I mean what is defined in Remillard (2005), as the “printed, often published resources designed for use by teachers and students during instruction” (p. 213).

My motivation for defining the mathematical horizon as the projection of mathematics in the curriculum to that in the discipline such that meaningful learning of the curriculum materials takes place. This could be seen as similar to what Ben-Peretz (1990) describes as curriculum potential, or the “reinterpretation of curriculum materials leading to classroom uses beyond the scope of developers’ intentions” (p. 49). By defining the horizon concerning this projection, horizon content knowledge can be described in terms of the mathematical knowledge that teachers draw on to navigate mathematical situations that engage them outside of the intentions and supports provided by the curriculum developers/materials.

Further, the motivation to use this definition to analyze teaching practice is to provide insight into teachers’ tacit knowledge if teachers encounter students' reasoning that appears to be using or generating mathematical structures in their work beyond the support of the curriculum that they cannot explicitly identify. The definition of teaching as made in Cohen, Raudenbush, and Ball (2003), makes meaning-making of students' thinking as an essential part of instruction. Therefore, teaching actions can be understood as “skillful and deliberate teacher interpretation of student thinking” rather than an incidental or unintended outcome. Just as the tourist may deliberately choose to explore an unfamiliar path while at the same time while lacking an explicit knowledge of the landscape it traverses or the locations it connects, so the teacher may make curricular decisions despite lacking a precise knowledge of how those choices will impact the mathematical structure of the lesson.

And finally, building on Jakobsen et al. (2015) I define a horizon encounter (or an encounter with the horizon) to be a situation in which a teacher enacts or engages with curriculum and students' thinking reaching to its meaningful potential. Therefore, the conception of HCK in this study involves the mathematical knowledge that teachers draw upon when navigating encounters with the mathematical horizon. At an operational level, an encounter will be considered as "an instance to establish any of the five components to it 's potential that are listed in the diagram." This means if there is an opportunity to

- establish truth in mathematics using mathematical tools and disciplinary ways in classroom instruction
- use core disciplinary values and orientation
- make explicit the knowledge of the ways of and tools for knowing in the discipline
- connect with structures in the discipline
- comprehend kinds of knowledge with its warrants

– then that would be considered as an encounter with HCK. Therefore, for the purpose of this thesis encounter with HCK does not depend on whether teacher noticed it or not, but whether any classroom interaction has potential to establish any component of HCK. An encounter would be an opportunity to construct mathematics using mathematics, its values, and practices.

Using the language of curriculum and instructional triangle, encounters with the mathematical horizon could be described as instances of classroom teaching in which there are differences between the potential mathematical learning and what mathematical gets learned. For example, during one lesson where the textbook suggested having students solve the profit and loss problems numerically, the teacher, Razak sir, instead of simply giving the numbers asked them to situate the story in their context, and then solve it. In this case, the teacher did not omit the original activity but replaced it with something different that expanded it beyond the scope of that suggested by the textbook. However, when students situated it in their contexts and started solving it using day-to-day strategies, the teacher could not accept those strategies as mathematical. This example shows that teacher could draw on his understanding that bringing students‘ context would increase students‘ access to mathematics, however, the mathematics situated in students' practices, went unnoticed. I will analyze such situations to make sense of what mathematical demands undergo making sense of students‘ thinking mathematically.

In this chapter we understood the ecological development of the MKT construct and understood how meaningful understanding of HCK is needed to bind the domains of teacher knowledge for comprehensive teacher education. We saw how HCK itself lacks a definition with consensus and has been divided into two extreme positions. Further, we used the working definition given by Jakobsen et al. (2015) to elicit attributes of HCK, which could be used to study instruction. Then the derivation for the definition of mathematics at the horizon and encounters with such horizon was discussed.

CHAPTER 3

Study Design and Methodology

As described in the introduction section, this study intended to investigate middle-school mathematics teachers' encounters with horizon content knowledge in their classroom practice and understand their views on managing these encounters. The chapter has four sections – Research design, Research Site and Participants, Data sources and data collection methodology, and Data Analysis.

3.1 Research Design

A case study is "an exploration of a 'bounded system'... a program, an event, an activity, or individuals" (Creswell, 1998, p. 61). Often a case study recounts a rare or unusual condition or incident, but a model or exemplar can also be a description of a classic situation.

Multiple–case studies design.

While much case study research focuses on a single case, often chosen because of its unique characteristics, the multiple–case study design allows the researcher to explore the phenomena under study through the use of a replication strategy. Yin (1994) compares the use of the replication strategy to conducting some separate experiments on related topics. Replication is carried out in two stages a literal replication stage, in which cases are selected (as far as possible) to obtain similar results, and a theoretical replication stage, in which cases are chosen to explore and confirm or disprove the patterns identified in the initial cases. According to this model, if all or most of the cases provide similar results, there can be substantial support for the development of a preliminary theory that describes the phenomena (Eisenhardt, 1989).

In the multiple—case studies design, there are no hard-and-fast rules about how many cases are required to satisfy the requirements of the replication strategy. Yin suggests that six to ten cases if the results turn out as predicted, are sufficient to "provide compelling support for the initial set of propositions" (1994, p. 46). Yin goes on to say that since the multiple–case studies approach does not rely on the type of representative sampling logic used in survey research, "the typical criteria regarding sample size are irrelevant" (p. 50). The number of cases required to reach saturation, such that no significant difference in findings, determines the sample size. The sample participants should be selected explicitly to encompass instances in which the phenomena under study are likely to be found. This approach to sample design is consistent with the strategy of homogeneous sampling, in which the desired outcome is the description of some particular subgroup in depth (Patton, 1990).

The multiple case study design or collective case study investigates several cases to gain insight into a central phenomenon (Creswell & Maeitta, 2002; Stake, 2013; Yin, 2003). Here the central event is an encounter with the horizon content knowledge; this study used the cases to gain insight into teachers' encounters with HCK and its management.

In this study, I use a multiple-case study approach with cases drawn from kinds of curriculum and school systems. A multiple-case study enables the researcher to explore differences within and between cases.

3.2 Trustworthiness of the Study

In the following paragraphs, I describe how trustworthiness is taken care of through the four aspects – Credibility, Transferability, Dependability, and Confirmability applicable to a qualitative research study.

Credibility.

The credibility of any qualitative research study speaks to the issue of whether the findings are plausible. It depends upon the steps taken during the process of data collection and analysis. A key factor is to ensure completeness of the data (Yin, 1994). The researcher's field notes and observer's notes complimented the video recordings of the classroom. The video recordings of the lesson consisted of each teacher teaching three units of a mathematics textbooks. A total of 13 teachers taught, amounting to a record of 42 teaching units. Ph.D. candidates for Science and Mathematics Education from Homi Bhabha Centre for Science Education, accompanied the researcher, one every time. They took detailed notes of what they had seen in the classroom. After each lesson, both the researcher and the observer discussed their observations. And I wrote a memo of each such discussion. Along with this, the teacher viewed their lesson videos and presented their side of making decisions in encounters that I chose. The teacher views were also video recorded and analyzed while discussing the encounters and their management.

Transferability.

The transferability of a research study addresses the question of whether the findings are "context-relevant" or subject to non-comparability because of situational uniqueness (Guba, 1981, p. 86). For this particular purpose, the schools have been chosen with the possible different context of – different administration, textbooks, the language of instruction and socio-economic status of the schools. Said this, the study is in the Indian context, and even though the purpose of the investigation was to see mathematics, its practices and sensibilities in teachers' instruction there will be a cultural edge to all of this pemeating in the analysis as well.

Dependability.

Two aspects achieve reliability – one is by replicability, and other by comparing the data analyzed with that by another individual/researcher (Guba, 1981). Replicability is very difficult to achieve in the way Guba defines it. It would mean that I work with one teacher then analyze the data, then form a theory. And then collect the data from another teacher to see whether the theory holds there too. One needs to keep doing this until satisfied. However, for practical reasons, this was not possible. The nature of the data also directed the course of analysis. For example, many teachers taught the same topic, such as algebraic identity, polygons, fractions multiplication and division, and the analysis needed to address these cases as one unit together. For the second aspect, some data of the encounters was initially coded by a research associate at the Homi Bhabha Centre for Science Education, and the difference in the code was discussed and resolved. Although, due to the massive amount of the qualitative data, it was impossible to ask another coder to code more data.

Confirmability.

Confirmability was the most relevant threat for this research, mainly because of the topic of this research. Horizon Content Knowledge is very specialized knowledge. By its very definition, it depends on one's understanding of mathematics, its values, and practices. It also depends on the goals of teaching. Therefore, addressing the researcher's view of mathematics and teaching was necessary. A memo of the researcher's role is given at the end of this chapter, to familiarise the reader and therefore, do the further readings in light of that. This specific step is what Guba (1981, p.87) calls "practicing reflexivity."

3.3 Background of the Study Setting

This study was conducted in Mumbai, India with middle school mathematics teachers. Conducting the study in India had two main reasons. First, based on the research questions it was clear that the study would involve careful examination of teaching practice. Teaching is a cultural activity (Stigler and Heibert, 2009). Teaching, as we saw in the conceptual framework chapter, is very complicated phenomena and would involve practices, notations, and norms that are subtle and specific to a culture. Considering my experience of teaching mathematics and working with mathematics teachers in India, it made more sense to collect the teaching data from India, to be able to make sense of the cultural subtlety. The second reason was to see the diverse classroom settings. India has many kinds of school systems, where many things vary. Like schools with free education too expensive education, with low paid but secured jobs for government teachers to highly paid but no security for private teachers, etc. In some sense such diversity was necessary – as one of the causes for which the HCK is essential is to create meaningful access to mathematics. And it was important to find out whether such meaningful access is available for all the classes and school cadres.

For educational and maintenance purpose, every town in all the districts of India is divided into municipal wards. In Mumbai, there are total 24 wards, which monitor and supervise education, water, and other government provided facilities. This study was conducted in one such ward called M-East ward. (See Figure 3.1). The population in M-East ward is 0.8 million people as per the 2011 Census report. There are 202 government schools in this ward. These run up to grade 7 only.

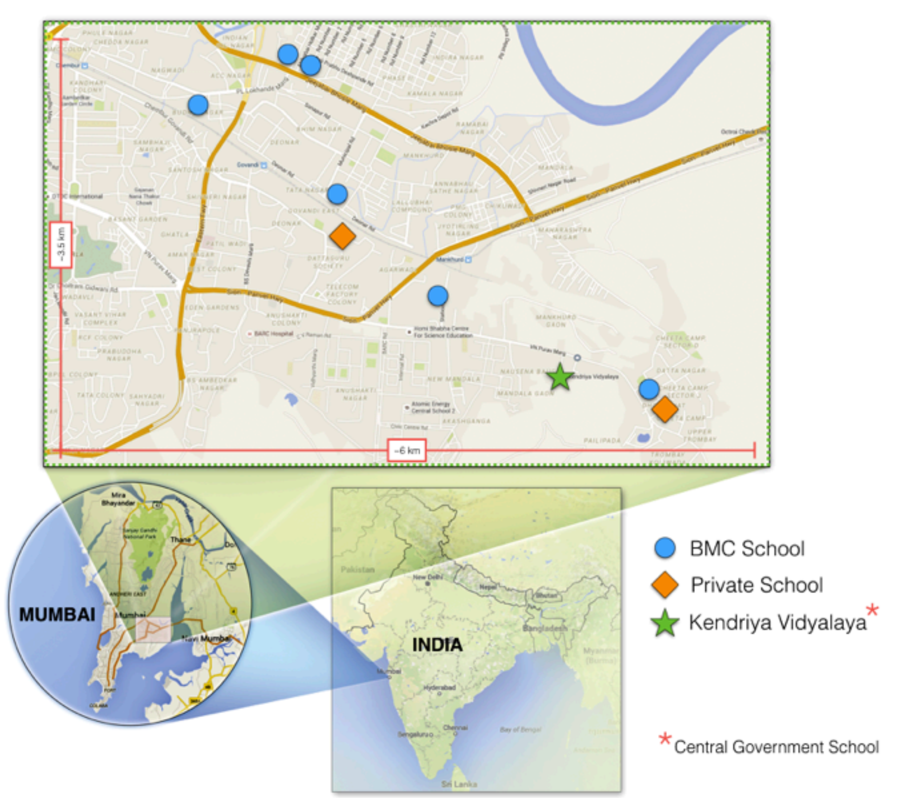


Figure 3.1: Locality of the Schools in the Study

There are three kinds of schools in India: public, private aided, and private un-aided schools. Public schools are often called as government schools. There are two kinds of government schools—ones that are supported by the state (local) government and the other supported by the government in the Centre. These schools are fully funded by the respective government and provide free education to everybody up to higher secondary (K to12). Up to grade 8, these schools also provide school uniforms, mid-day meals, textbooks, notebooks, and minimum supplies. The state public schools are commonly referred by the name of their municipality. For example, a state funded municipal school in Lucknow, will be referred as Lucknow Municipality Corporation School. Similarly a public school in Mumbai will be called as Mumbai Municipal Corporation (MMC) School. The state funded schools follow state curriculum framework and state textbooks. Considering that the education provided is free of cost, one would expect a very high enrollment in these schools. However, doubts about the quality of these schools mean that they struggle to maintain a healthy enrollment.

The government schools supported by the central government operate as a chain of schools across the country. In earlier times, these schools primarily catered to the children of government officers who had moving jobs, like Army, Navy, government administration, etc. These schools across the country follow the same timetable, the same holidays, and the same curriculum in the same order. So even though a child's parents are transferred in the middle of the year to some other state, the child studying in school funded by the government in the centre will face minimum disturbance. However, since Right to

Education Act (2009), these schools are expected to enroll 20% of their children from the families in the local community.

Then there are private schools that are aided by the government. The aid varies from part to full support to the schools. The financial support is provided based on per child calculation and it is not rare to find a school where one of the divisions (e.g., 3rd standard division D) is not at all funded. All the private schools whether they are funded or not by the government, charge a minimal fee to the students. The aid is disbursed based on certain norms; one of the important ones is to follow the state curriculum. The private un-aided schools need to be registered as a school, but do not receive funding from the government. These schools are run by trustee boards and often are more or less experimental in their functioning, depending on the authorities involved. These schools could have their own philosophy of education and they fall into further categories as experimental, religious or international school.

Participants in the study.

The data collected includes 13 teachers' classroom teaching videos. Brief information of these teachers is given in the table below (See Table 3.1). Each teacher's teaching of three mathematical units from the curriculum of grade 7 was observed. All the teachers except two used the same textbook. The state textbooks in India are first written in the language of the state in which they will be used, and in state of Maharashtra they are translated in to 13 mediums of instruction. The textbooks developed under the National council are written in English first and then trasnlated to other languages.

The textbooks used by the teachers in the study were in three main languages – Marathi (local language of the state of Maharashtra), Urdu (language spoken by the Muslim minority) and English (Third official language of the state, and popular among upper social and economic classes). The textbook developed by the State Council of Education, Research and Training (SCERT) for the state of Maharashtra was used by the 11 of the 13 teachers in the study. The remaining two teachers used the textbook developed by the NCERT of the Central Government.

Teacher name (pseudonyms)	Gender	Medium of Instruction	Teaching experience	Education Academic	Education Teaching
Koregaonkar	Female	Marathi	23 years	PhD, MA	B.Ed.
Bhinde	Male	Marathi	22 years	MA	B.Ed.
Tope	Female	Marathi	23 years	BA	B.Ed.
Bhoke	Male	Marathi	18 years	BA	B.Ed.
Ruksat	Female	Urdu	24 years	MSc	B.Ed.
Karim	Male	Urdu	27 years	BA	B.Ed.
Razak	Male	Urdu	19 years	BA	B.Ed.
Asmita	Female	Semi-English	7 years	BA	B.Ed.
Parveen	Female	English	4 years	B.Sc.	B.Ed.
Sangita	Female	English	12 years	B.Sc.	B.Ed.
Meenu	Female	English	16 years	M.Sc.	B.Ed.
Damini	Female	English	23 years	M.Sc.	B.Ed.
Nasir	Male	Urdu	27 years	MA	B.Ed.

Table 3.1: Participants of the Study

There are 8 female teachers and 6 male teachers. All of them have taught different grades ranging from 5th to 7th and during the educational year of the data collection, 2015 they were all teaching grade 7. Seven of these 13 taught in government educational schools and therefore were teaching all the subjects to their respective class, except language and drawing.

3.4 Data

The data for this study were collected in a setting that represents a mixture of a variety of classes, castes and religions. The study was conducted in one of the municipal wards in Mumbai, called M-East ward. This ward has a mix of population and the school going children come from families of the very poor to families of rich business people. Often the government schools that provide free education, food and accessories are considered as lowest in the quality and any families that have a regular income, generally prefer to send their kids to private aided schools. The reason is not that they can easily afford the school but the status associated with the private schools. For example, often uniforms at such schools will involve wearing a tie and polished shoes, which is not the condition in the Mumbai Municipal Corporation (MMC) Schools. The medium of instruction is another variable that dictates the student enrollment rates in school. English-medium schools often get more number of students enrolled than the vernacular (Marathi and Hindi in case of Mumbai) medium. And almost all the private schools provide education in English medium. Therefore, many low income families send their children to the private Englsih medium schools. The government schools have less number of students in the urban setting, who are often from extremely poor families.

This study involves interaction with 13 middle school mathematics teachers. Seven of these come from the MMC Schools, five are from the private aided schools, and two are from the Kendriya Vidyalaya schools supported by the central government. These are all mathematics teachers teaching grade 7.

Official permission was obtained from school officials, superintendents and principals. (Please refer to the consent given in Appendix A). Even though the permission was obtained from higher authorities, a separate consent for participation and documentation (video and audio) was also obtained from each teacher. To each of the teachers, it was explained that the study is investigative in nature and aims to analyze their classroom practice, thoughts on the practice through interviews, and their work on a few mathematics problems.

Classroom recordings.

Three consecutive classes of each teacher were video recorded. All the recordings were done in the same month of the year to achieve maximum overlap in the content across teaching. As this was towards the end of the school year, topics such as algebraic identity, geometric constructions, surface area and volume were observed in almost every teacher's teaching. See table 3.2 for the list of all the topics and their brief description in each class.

	Teacher name (pseudonyms)	Medium of Instruction	Topics taught during the study in order of teaching
1	Asmita	Semi-English	Circle and its properties Construction of quadrilaterals Algebraic identities
2	Bhinde	Marathi	Polygon surface area and volume of cube, cuboid Algebraic Identities Fraction operations – addition, subtraction Fraction Operattions – multiplication, division
3	Bhoke	Marathi	Algebraic identities Polygon surface area and volume of cube, cuboid Circle and its properties
4	Damini	English	Multiplication of Fractions - Pictorial Multiplication of Fractions – Rule based Division of Fractions
5	Jahir	Urdu	Algebraic identities Polygon Polygon surface area and volume of cube, cuboid

			Word problems on profit and loss
6	Koregaonkar	Marathi	Triangle and its properties Triangle Congruence Algebraic Identities
7	Meenu	English	Addition-subtraction of fractions Multiplication of fractions Division of fractions
8	Razak	Urdu	Word problems on profit and loss Polygon surface area and volume Algebraic identities
9	Ruksat	Urdu	Algebraic identities Circle and its properties Polygon surface area and volume of cube, cuboid Fraction multiplication and division
10	Samreen	English	Algebraic identities Polygon, surface area and volume of cube, cuboid Fraction multiplication and division
11	Sonali	Marathi	Algebraic identities Polygon Word problems on profit and loss
12	Tope	Marathi	Polygons Polygon surface area and volume of cube, cuboid Algebraic identities
13	Nasir	Urdu	Algebraic Identities Fraction operations – multiplication, division Polygon surface area and volume

Table 3.2 Topics taught by the teachers

There were three kinds of documentation made for each lesson listed above. The researcher took extensive field notes of the class, a very careful video recording of the class and observer field notes. There was an external observer, mostly a student doing PhD in Science or Maths education in Homi Bhabha Centre for Science Education, a national centre of Tata Institute of Fundamental Research. The observer did not necessarily take the notes of what was happening in the classroom teaching but made note of things that he or she observed. I was present in all the teaching sessions with Ms. Tuba Khan, a project associate, who video recorded all the sesions. The observers were different every day. After every class three of us shared each of our observations and the researcher wrote a memo of this discussion. Given here is a sample paragraph from a memo for one of the days –

> Memo:
>
> In Shinde sir's classroom - it was ~~df~~ interesting to see ~~how~~ who had the authority of deciding what is correct or wrong. It was definitely not with the mathematics - mainly because the mathematics was not accessible. What was discussed was enactment of procedures.
>
> While teaching division of fractions this is how the teacher began - "we have been told to do the division like this". Here either he was referring to the textbook or mathematician who know everything about mathematics. When students were working on the problems they were picking some procedures & were constantly confirming with the teacher. It looked like, the child knows various mantras, but doesn't know which one to read when. So everytime they decide something

The memo above discusses challenges students faced while following a procedure to do the division. The topic came while observer, Tuba and I were discussing what mathematics the students were engaged with, and how were they learning it.

Interview.

The purpose of the interview was threefold – 1. To understand the teacher's journey as a student and teacher of mathematics, 2. To get teachers’ views and opinions on the classroom episodes that were selected by the researcher as vignettes for illustrating encounters with HCK, and 3. To understand teachers' HCK on a non-teaching task. To achieve these three goals the interview was divided into four sections. The meeting with the teacher was on a one-to-one basis and approximately lasted for 80 to 90 minutes. Here are the details of the four parts of the interview.

A. Background information.

This section of the interview was meant to break the ice. Teachers here spoke about where they have come from, how they became interested in mathematics, and in teaching mathematics, what was their teaching experience, and where have they worked for these many years. The protocol also involved asking them about their interest in mathematics and reasons for the same. One note to remember here, that each teacher that was part of this study was nominated as a good teacher by both, their superiors and colleagues. Therefore, the teachers who taught other subjects along with mathematics, like social studies or sciences, were confident about being mathematics teachers. The question of why I got access to teaching of "best mathematics teachers", could be mainly because of the granted video recording access. In my personal experience and the experience of other researchers, it has been very difficult to get video recording of the classroom. Although when superintendent gave this permission, they also made sure they nominated their best maths teachers. Appendix B gives the entire interview protocol of this section.

B. Interview general.

This section involved asking teachers about their teaching and mathematical practices. They were asked something like, how do they decide which students to call, how do they prepare the content, what are the useful resources. For example, the teachers responded to questions involving how they make decisions about writing or making figures on the board. The teachers also spoke about how validity or mathematical truth is decided. This discussion led to what these teachers believe about teaching and learning mathematics. This protocol is given in the Appendix C.

C. Interview specific.

This section involved watching one or two video clips of the teacher's classroom teaching. These clips were what I identified as a vignette of an encounter with HCK. After viewing the clip together, the researcher asked them first to explain what they saw and thought about the clip. Then they answered some questions based on the content of the clip. A list of the questions for one of the vignettes is given as Appendix C. The entire list of the questions with a summary of each clip is presented in Appendix D.

D. Problem solving/analyzing solutions.

In the fourth section the attempt was to focus on other tasks of teaching, such as solving a problem or analyzing a solution. The teachers were given choice of three problems to solve and some students' responses to analyze. Appendix E provides the problems presented to the teacher and Appendix F gives an example of teachers' response with analysis indicating how it relates to the HCK.

Curriculum materials.

As the teaching data was all collected over about in one month, many teachers were teaching the same topic. In this dissertation, I analyze three cases in detail (in chapter 4 and 5) and give examples of some cases in brief (chapter 3). The curriculum used across was of two kinds, and consist of following chapters from the mathematics textbook:

- Identities chapter from Maharashtra State textbook
- Polygon and their surface, volume of cube, cuboid from Maharashtra State textbook
- Geometric construction from Maharashtra State textbook
- Operations on Rational numbers from Maharashtra State Textbook
- Fractions multiplication and division from the Central government NCERT textbook

The data collected is summarised in the figure below.

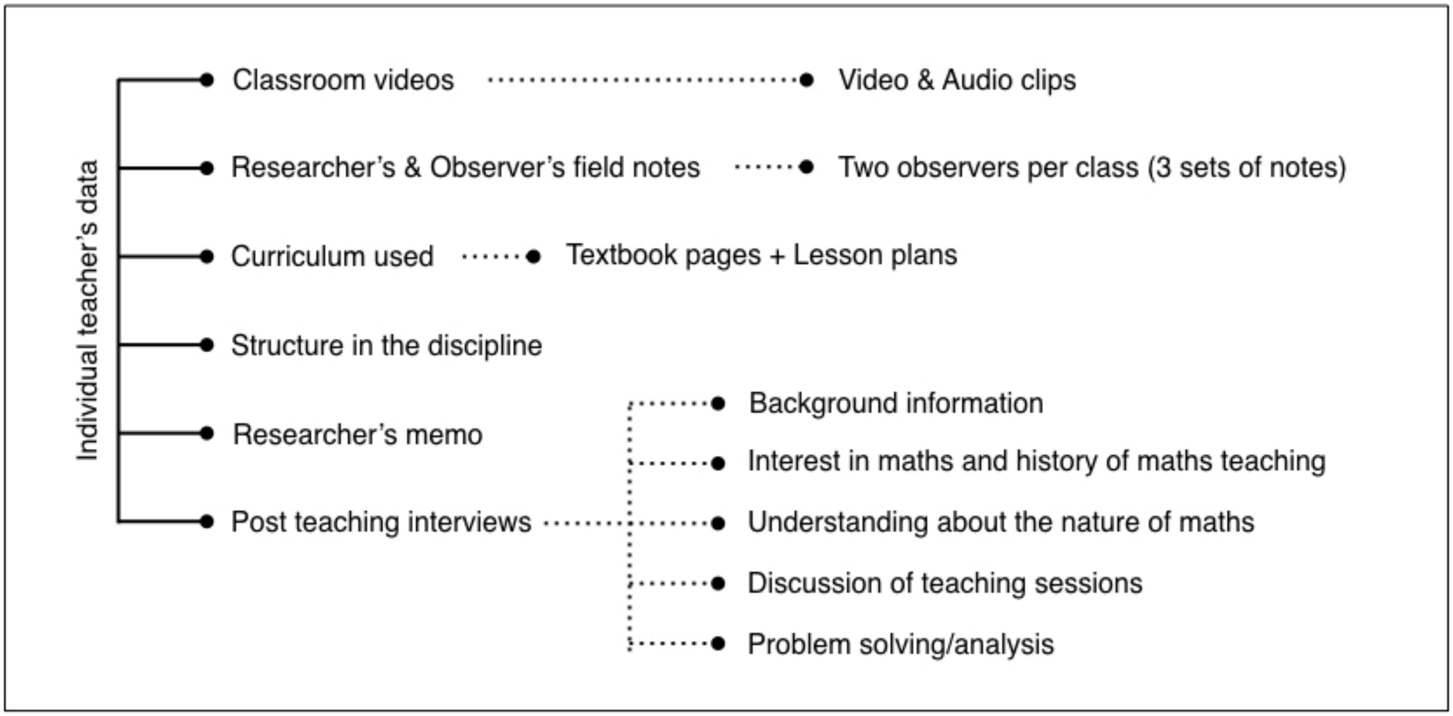


Figure 3.2: Data collected at a glance

3.5 Data Analysis

The data are rich in quality from multiple perspectives. The records comprise an authentic representation of work of teaching. When I approached the school administrations for consent to collect data, the superintendent or superior authority in the administration nominated their "good mathematics teachers." I did not ask for good teachers, rather asked for representative schools of M-East ward and any mathematics teachers in those school. The teachers in the study knew that they were nominated as good mathematics teachers. Due to this, the teachers were confident in what they did and how they spoke to the researcher. They shared how they have been teaching mathematics for so many years and how some of the lesson plans are thorough in their heads. They also mentioned how often when other maths teachers are stuck they come asking for advise. These teachers spoke a lot and shared a lot.

Analysis phase I: Identify encoutners to determine content of the interviews.

The first phase of my investigation began during data collection and involved watching and logging the video recordings of the classroom observations and the selection of interview clips. My field notes included time stamps from the video recordings, descriptions of the class activities, and rough but relatively detailed transcription of the teachers' talk during the enactment of the day's lesson. I additionally made notes regarding the mathematical content of the teaching along with other connected and relevant mathematical ideas from established mathematics. These notes were used to identify the instances in which the teacher had possibilities of engaging students in potentially meaningful learning of mathematics. After the teachers finished teaching the three units, we met for the interview. During the meeting, the teacher saw these clips and presented their point of view about what is happening in the class. We will see some examples in detail.

This initial analysis of identifying the encounters was the most challenging. Identifying knowledge encounters in the practice of teaching was one of the most challenging tasks that researchers have also reported on their work. In this study where the goal was to identify encounters with the knowledge that the teachers need to teach mathematics effectively, began with identifying locations for such encounters within the work of teaching. Based on teaching responsibilities given in Ball & Bass (2009), delineation of HCK as in Jakobsen et al. (2015) and based on my understanding of the teaching of mathematics, I began with the following list of responsibilities of teaching mathematics that would entail examples of encounters with knowledge in classroom practice.

1. Make students intellectually autonomous in mathematics
2. Break the authority of who can decide the truth in mathematics
3. Expose students to fundamental practices for engaging mathematically
4. Make judgments about mathematical importance
5. Hear mathematical significance in what students are saying
6. Highlight and underscore key points
7. Anticipate and make connections
8. Notice and evaluate mathematical opportunities
9. Catching mathematical distortions or misrepresentation

Each of the responsibility listed above, which as per Ball and Bass (2009) also acts, as a dilemma of teaching was further segregated into actions of teaching in a classroom setting. For example in breaking the authority of who can decide the truth in mathematics, teachers' action would depend on various things: initiating a discussion to validate the truth in mathematics by asking questions such as "why it's true," hearing mathematics in students' thinking, etc.

For example, while deciding if a number is prime or not, a student said, "One can start dividing with 2, but only needs to go up to half of that number". This suggestion was in response to the algorithm s/he was following. The algorithm involved dividing a number (say n) by all the numbers between 2 to n – 1 to check its primeness. Now to notice the mathematical significance and then to figure out, what to do with such a response, one requires a combination of understandings.

- Awareness of the possibility that identifying such pattern is a byproduct of the work students were engaged with;
- Knowing why would going till half of the number would work;
- Knowing that checking till half is a good strategy but not the shortest;
- Knowing why $\sqrt{n}$ will give the smallest number of checks for testing the primeness of a number;
- Notice the value of the response and figure out mathematical practices that other students can participate in to decide the validity of this response; and
- Envision the kind of mathematical knowledge students will gain at the end of the discussion on the student's identified pattern.

This set of understandings combined with the responsibilities of teaching leads to teacher actions. In the example above a possible action would be to ask other students "what they think about the strategy?" or "is there a way to check whether the strategy is correct or not" or "why would such strategy work?". Said this, the actions will occur only if the teacher heard the mathematical relevance of the student's response and further judged it in relation to the objectives of the learning of that class. In the findings section, I list the examples of HCK and talk about the core responsibility of teaching from which most of the responsibilities are derived and also how HCK addresses this core responsibility.

Analysis phase II: Analyzing clasrooms and interview.

Each video was transcribed for the encounters that were shown in the interview. There were 53 encounters in total on which the 13 teachers were interviewed. The encounters depended on the instruciton style of the teacher. Some lessons had more encounters that i discussed with the teacher and some lessons had less. Thirty of these 53 encounters were in Marathi or Urdu. These were translated into English. The translations needed special care. English is the dominant language in cross-European projects and publications (Kushner 2003). According to Temple (2008), "language differences have consequences, because concepts in one language may be understood differently in another language." This is relevant for research where classroom and interview data is translated into English. The ramification for the validity of moving across languages has gained considerable attention in many cross-cultural studies (Squires 2009). Therefore, as suggested by Van Nes F, et al. (2010) specific parameters were maintained concerning the validity of the data. Some of those are as follows.

- The original text was used for analysis and only translated when reporting for the readers.
- I made all the translations with the help of professionals whenever needed.
- Avoided one-word translation – like using the dictionary word, etc.

The data in the interview consisted of interview transcriptions and scans of their problem-solving work. The analysis was geared towards the initial list of responsibilities in teaching that would involve examples of encounters with HCK. As the study planned to investigate what encounters are and how they were managed in the teaching, the initial analysis involved looking at work of teaching corresponding to each responsibility of teaching.

The following table lists actions involved in these responsibilities. These actions were then mapped on to the classroom teaching of the 13 teachers who are participants of the study to generate codes for each teaching action.

No	Responsibilities of teaching that involve examples of encounters with HCK	Work of teaching within these encounters
1	Make students intellectually autonomous in mathematics	• (1)Respond to students queries/answers/explanation • (2)References for doing mathematics • (3)Design resources to do mathematics
2	Break the authority of who can decide the truth in mathematics	• (4)Initiating interaction for validating the truth in mathematics • (5)Validation of truth in mathematics • (6)References used to decide the truth
3	Expose students to fundamental practices for engaging mathematically	• (7)Introduction of practices • (8)Valuing practices in doing mathematics
4	Make judgments about mathematical importance	• (9)Valuing a response over the other • (10)Preference for certain strategies
5	Hear mathematical significance in what students are saying	• (11)Hearing students' mathematics • (12)Responding to students' mathematics
6	Anticipate and make connections	• (13)Hear mathematical value in students' responses/ curriculum • (14)Exhibit connections with vertical and horizontal mathematics
7	Notice and evaluate mathematical opportunities	• (15)Identify scope for doing mathematics within student responses and curriculum

		• (16)Negotiate this scope based on the objectives of learning mathematics
8	Catching mathematical distortions or misrepresentation	• (17)Choosing representations/ explanations/ materials that support learning of mathematical concepts • (18)Noticing and addressing the distortions within these

Table 3.3 Teaching responsibiltities and work of teaching

Detailed analysis of classroom teaching.

Before we dive into the coding of the teachers' actions, let us understand more about the structure of these classrooms. There are 8 distinct topics that these teachers taught, namely: (1) Algebraic identities, (2) Introduction to polygons, (3) Surface area of the polygons and volume of cube, cuboid, (4) Triangle and its properties, (5) Circle and its properties, (6) Operations on fractions, (7) Word problems on profit and loss, and (8) Construction of quadrilaterals. The number of lessons for each topic is given in the table below.

No.	Title of the topic	Number of lessons
1	Algebraic identities	11
2	Polygons	3
3	Surface area of the polygons and volume of cubes, cuboids	8
4	Triangle and its properties	1
5	Circle and its properties	2
6	Operations on fractions	11
7	Word problems on profit and loss	4
8	Construction of quadrilaterals	1
		Total teaching sessions: 42

Table 3.4: Number of lessons per topic

Structure of each teaching session was different for the topics but same for each topic across the teachers. This means the way algebraic identities were taught was different than the way triangle and its properties were taught. However, each teacher who taught algebraic identities had a similar class structure. Two of these lessons will be studied in detail in the next chapters.

In the following paragraphs, I explain, how the process of coding evolved and towards the end provide final codes for encounter initiation, their management with kinds of knowledeg resources. Here, is an example of the transcript which shows the initial tagging of the text.

	Tags
RK: [Shows a bangle] What is this? Raise your hand and then everybody answer it at a time. No chorus answer.	Connecting maths with outside world
RK: What is this? [students raise their hands]	
RK: Yes, Speak.	
GS: Circle (*Daira*).	No consideration of dimensions
RK: What is this?	
Ss: Circle.	
RK: What is this? For what purpose it's used?	Connecting maths with outside world
G.S: For wearing.	
RK: Bangle (*Kangan*). For wearing purpose.	
RK: What is this? (Shows a metal ring – used to keep water pot)	Connecting maths with outside world (Concretization) No consideration of the dimensions
RK: Yes, speak Nasreen Anjum.	
Nasreen: This is the loop to place water pot (vessel)	
RK: To place water vessel upon this. Very good.	
RK: What is this? (Shows loop made of thread)	
RK: Muskaan, speak.	
RK: Say it.	
RK: Praveen, tell me, what is this? Where is it used?	
G.Ss: For cloth.	

RK: For cloths. We use this to do embroidery on cloths.	Teacher completed the student’s response
RK: What is this? (Shows a rubber ring used in pressure cooker) Yes. Say Sakeena.	
Sakeena: Rubber from cooker.	
RK: Cooker's rubber. Very good.	
RK: What is this? Boys use to play with this. (Shows a wheel). Say Samiullah.	Gender stereotyping
Not audible.	
RK: Yes. Very good.	
RK: And this? At least everybody knows this? (Shows a CD)	
(Students raised their hands)	
RK: Say, back, Nasreen.	
Nasreen: Cassette.	
RK: Cassette or CD?	
Ss: CD.	
RK: CD. Cassette's CD.	
RK: These are so many things that we saw here. Okay? The shapes of all these things are identical to a shape? Which one?	The concept of *circular* (but not being explicit)
RK: Yes. No, not all together. Say, Saniya.	
Saniya: Same as circle.	
RK: Same as circle. Na? The shapes we saw here are all same, as circle. You will not respond in chorus, not together. Okay?	Use of the term “circle” Definition not given, concrete examples used
RK: So, today we will learn more information about this shape, the circle.	
RK: Now. Have you brought the color-paper, I asked for?	Referred by the teacher as “activity-based”
Ss: Yes.	
RK: Do you have it with you?	
Ss: Yes teacher.	
RK: Take those papers out.	
Ss: Yes teacher, taken out.	
(One student raised her hand to say something – RK did not see her.)	
RK: After taking out those papers, compass...open your geometry box. First, take your compass out. And then, do it the way I'm telling you...The way I'm telling you, you shall make the circle in that way.	“I’m telling you”, the authority is with the teacher.
RK: Take some distance between the compass point and the pencil and draw a circle like this. (Makes a circle on the board using the big compass from the teacher’s math box.)	Why would this procedure give us a circle? Definition?
RK: Like this, you will make it on your color-paper. Okay?	Modeling for what the students need to do (modeling for making records)

(Some students were making circles on the color-paper)	
RK: Now, do this on the color-paper. Done?	
Ss: (one or two students said) Yes teacher.	
RK: Now, take out the scissors and cut the circle. Circle...Cut the circle?	Distorting representation: practical concern (cutting circle using the scissors is difficult)
Few voices: Yes teacher. (Students find cutting circle using scissors difficult and many get a distorted shape)	
RK: Now, show me where have put the tip of the compass. Show me this way (takes up one girl's cut-out and holds it up)	Modeling for what is expected
RK: (looks across) Very good.	What they have made are discs and not circles.
RK: Now, the point at where you have put the tip of the compass, I have given a name 'C' to that point. To that point I have given the name 'C'.	
RK: This 'C', where we put the tip of the compass, that point we called as *markaz* or centre. What it is called?	Operational definition of the centre (what about conceptual definition)

Table 3.5: Sample tagging of the transcript

This round of analysis made me familiar with the happenings in the class at another level than watching the video. It unpacked various levels of exmples, representations, questions, etc. that were not noticed while observing the classroom or watching the video. The first step of the analysis began with reading the transcript carefully, and highlighting the text with tags for its relevance in the current investigation. The tags in the intial coding were used on three more teaching transcripts to understand the pattern in encounters. This process led to the following table of codes that emerged from the classroom teaching data for each teaching actions. These descriptions were obtained by tagging the three classroom transcripts.

Encounter location codes		Management Codes	
Code	Description of the instance	Postponed/ avoided /missed	Addressed

Intellectual autonomy	To make students intellectually autonomous in mathematics	By making judgments about what is mathematically important in the moment	By giving freedom to explore and use the mathematics learned
Mathematical Truth	To break the authority of who can decide the truth in mathematics	The teacher or subset of students decide whether something is true	The teacher and student use mathematics to verify/find the truth
Fundamental mathematical practices	To expose students to fundamental practices for engaging mathematically	Mathematical practices are not mathematics	Explicit discussion on how to *do* mathematics
Students' responses	To hear and understand mathematical significance in students' talk	Acknowledge the responses, and deal with them later	Use the responses to the advancement of teaching or improvise
Mathematical connections	To anticipate and make mathematical/ integrative connections	Make judgment about making connections later	The teaching plan, representations illustrate anticipation of mathematical connection
Mathematical distortion or misrepresentatio n	To chose or find mathematical distortion or misrepresentation	The distorted representation for not the concept in discussion	Representations that use mathematical sense
Mathematical productions	To guide students to produce mathematical objects that are coherent with the horizon	The productions are guided by the pedagogical or managerial constraints	Logical accuracy is considered

Table 3.6: Initial coding for location and management

Analysis of interviews involved two major steps. I describe these analytical steps in this section. The first step of this analysis required processing the video recordings of my class observations and interviews. The second step involved the coding procedures that I employed in my analysis of the processed data that was produced as a result of the work in my first analytical step.

The video recordings I collected during my class observations and interviews required processing before I could conduct my multiple case study analysis (Miles, Huberman, & Saldaña, 2014, p. 71). I transcribed both the classroom observations and interview data using Transana (Version 2.42b-Mac). In this section, I describe the coding methodology that I implemented in my analysis of the data. In addition, I also indicate the selective decisions in my process to determine the final data I considered for my in-depth case study analysis. For my early coding analysis, I utilized a two cycle coding method described by Miles, and colleagues (2014) that involves the initial assignment of codes to data chunks in the first cycle while second cycle coding involves identifying patterns in the first cycle codes. On some occasions, coding categories and definitions determined during the earlier stages in my analysis were adjusted during my in depth reviews of examples for the case study.

No	Responsibilities and dilemmas of teaching that involve examples of encounters with HCK	Work of teaching within these encounters	Codes for teaching actions
1	Make students intellectually autonomous in mathematics	1. Respond to students queries/answers/explanation 2. References for doing mathematics 3. Resources to do mathematics	Response: Zero – moved onto other student Evaluated – right/wrong Taken up for discussion • Textbook says • Collective understanding of maths

			• Practices referred
2	Break the authority of who can decide the truth in mathematics	1. Initiating interaction for validating the truth in mathematics 2. Validation of truth in mathematics 3. References used to decide the truth	Interaction: Not initiated Initiated • Discussed why its true • Referred – curriculum, textbook, collective understanding or teacher to validate
3	Expose students to fundamental practices for engaging mathematically	7. Introduction of practices 8. Valuing practices in doing mathematics	*Doing* mathematics involves: • Speedy calculations • Accurate drawing • Remembering formulae • Using proper tools • Knowing terms • Accurate answer • Writing *"Steps"*
4	Make judgments about mathematical importance	9. Valuing a response over the other 10. Preference to certain strategies	Students response/answer /explanation is better because • Better strategy • Faster to solve • Given in the textbook • Matches with the textbook answer
5	Hear mathematical significance in what students are saying	11. Hearing students' mathematics 12. Responding to students' mathematics	Students' responses • Forgot the maths/formula • Didn't pay attention

			• Relevant for the next portion • Forgot last year's maths
6	Anticipate and make connections	13. Hear mathematical value in students' responses/ curriculum 14. Exhibit connections with vertical and horizontal mathematics	Students' mathematical responses • Connect it with the mathematics they learned or will learn • Teach shortcuts
7	Notice and evaluate mathematical opportunities	15. Identify scope for doing mathematics within student responses and curriculum 16. Negotiate this scope based on the objectives of learning mathematics	Rearranging the curriculum Student responses noticed but not discussed
8	Catching mathematical distortions or misrepresentation	17. Choosing representations/ explanations/ materials that support learning of mathematical concepts 18. Noticing and addressing the distortions within these	Use of material to illustrate key concepts • Material used with drawbacks

Table 3.7: Teaching actions for the responsibilities

Here is a sample coding of one of episodes in the classroom. Analysing another classroom

Name (pseudonym) and other details of the teacher	Summary of the class	Summary of the clip	Transcript (Yellow – encounters, Blue – management)	Codes	Notes for writing
Razak Sir, Class recorded on 16th February 2015. Day 1 of the recording.	The class focused on the topic of 'Profit and Loss'. In this class, the teacher tried to explain how to use formula of profit and loss. Razak sir teaches in Urdu medium government school. The girls wear *hijab* with their school uniform. The boys wear normal shirt and half-pant. The students did very little writing in their notebooks and often volunteered to write on the board.	This clip covers the part of the class where the teacher gave a problem on finding profit. One of the girl students solved it on the blackboard. The question that the teacher asked is as follows: Saleem bhai bought eight dozens of bananas at the cost of rupees thirty. He sold five dozens of bananas at the cost of rupees forty-five and three dozens of bananas for rupees thirty-five. So, is it a loss or a profit?	[Samreen, the girl student solving given problem on blackboard. She wrote the following on the board] Answer: Cost price = 30 x 8 = 240 rupees Selling price = 45 x 5 = 225 rupees = 35 x 3 = 105 rupees Profit = Selling price - Selling price - Cost price [She chose to subtract the selling price instead of adding those.] 105 = 225 - - 240 [She was not speaking anything but left blank space at the second place. She waited for sometime, other students were asking to interfere, the teacher suggested let her take some more time] Ss: Sir shall I go and do it? T: Let us give some more time to her. Wait. [Samreen was standing still without speaking and staring at the board. After a while, she wrote the following below the line above = 240 - 225 T: Can't do this? Do you think you can do this? Samreen: I am solving. T: If you can't solve then I will call someone else. [Even though the teacher said he would call someone else if she	Encounters (M): mathematics explanation 1. How did the student decide to write formula for profit? 2. Why did she reformulate the formula for the profit? [Profit = Selling price – Cost price] 3. Why did she switched the numbers? 4. Why did she decide to subtract 15 (240-225) from other selling price, i.e. 105? Management: 1. Giving more time to the students to figure out on her own 2. Asking her whether still she thinks she can solve the problem? 3. Asking another students to solve the problem? 4. Not discouraging the student but asking her to stop trying on the board. 5. Not telling whether the work done is right or wrong.	Samreen who wrote earlier 225 – – 240, switched the places of the numbers without saying anything or asking anything. In the beginning when she calculated the cost price and the selling price, she wrote the formula for profit, giving us the indication that somehow she knew that it is going to be a profit. However, when she wrote the expression where two selling price were subtracted from each other, she arrived at 105 = 225 – – 240 . This expression was confusing and indicated that 105 is the profit. She spent some time

			couldn't solve, Samreen was interested in pursuing the problem. She kept on looking at the problem and was not ready to move from the board, she wrote the following as her next step.] = 105 – 15 [Without saying or discussing what is done on the board is right or wrong.] T: Good, good. Now, go and sit. Go. Someone else wants to solve this? Someone here? [Pointing to the boys' sections]	6. But at the same time not discussing it at all.	staring at it and finally wrote this as equal to 240 – 225. So she switched the numbers, why? Because 1. She realized that 225 can not be subtracted from 240 or 2. She realized that it is not profit but the loss situation. She definitely stumbled about use of 105, but her reformulation of the formula indicates that she identified there are two selling prices.

Table 3.8: Sample coding of the transcript

3.6 Researcher Positioning

Scholars agree that an essential aspect of qualitative research methodology involves the role of the researcher as an instrument in both data collection and analysis (Glesne, 2006; Patton, 2002; Marshall & Rossman, 2011; Miles, et al., 2014). In this section, I address my positioning by describing my background, training, and professional activities to establish further the trustworthiness of the interpretations that I have ultimately reported in this final report of my research activities.

I started working at a research institute in India, where I first began to understand the problems of mathematics learning. I began observing teaching in elementary schools, where the teachers were using the research-based curriculum developed by the institute. I had been teaching the students of grade 9th and 10th, since I was in 10th grade, that is almost for 6 years before I joined the research institute. I began teaching to support my family, but I started understanding it as a process when I began observing the teaching. This was the time when I saw parallels in what teachers were doing and what students were experiencing. I used to analyze every students' work through their notebooks in a class of 60. I did a similar exercise for one more school. We were asked to write a description of the analysis, chose the student work that is interesting, and that was the first time I learned there is so much mathematics in students' work that is more complicated than what they were asked to do. This encounter made me decide that I want to continue understanding the process of education. After this particular project, I was part of many projects around students' learning of number sense, operations on numbers, arithmetic-algebra, fractions-ratio, integers, decimals, etc. I also became part of a group of teacher educators who share these research findings with the teachers, and I understood that teachers were not aware of many things that students do or think, like the way I was during my teaching years. Teachers' professional noticing seemed an essential component for learning from teaching. During these years I also pursued my masters degree in mathematics, which gave me an opportunity to see advanced mathematics from the perspective of the work I was doing at that time. I started understanding what it takes to do mathematics successfully. I was struggling to find generic aspects of doing mathematics, that we need to acquire and teach to everyone who wants to study mathematics. It made me constantly dwell on the question of what is mathematics and what it takes to do mathematics. This particular aspect of my inquiry got more exposure during the coursework of the Ph.D. program, where we read about Lakatos, Poincare, Schoenfeld, etc. and learned from reflections how does one do mathematics. This particular sense about mathematics as a discipline, and simultaneously thinking about students‘ work, and goals of doing mathematics from students‘ perspective made me sensitive about components of mathematics instruction. Specifically, about how mathematical ideas can build in the classroom, through students‘ participation. I started feeling that many classrooms that I have seen so far never dealt with mathematics, its structure or the practices that are associated with it. I wanted to understand the scope of this observation, how is it possible that when most schools in India have a maximum number of teaching hours spent on mathematics, they are not doing mathematics. Therefore, I agree that I came with the assumption that the mathematical discourse in most Indian classrooms is inadequate to learn "real mathematics," where real mathematics would mean understanding logic beneath the procedure, justifications for the actions, and the one that empowers the learner to construct mathematics, to mathematize the situations. One more aspect that struck me about mathematics was about its connection to equity in the classroom. I realised that the appropriate way of learning mathematics is the way that creates access to learn more mathematics. The issue of access to mathematics in a school, therefore, is not only concerned with gender or class but more about how a teacher in the classroom creates this access. Consequently, it becomes about the nature of mathematics presented in the class. While working on this dissertation or asking research questions, I have believed that having that particular sense of mathematics plays a more prominent role in mathematics instruction. And this investigation is a product of that specific concern.

3.7 Limitations

I conclude this section by listing some of the limitations of this study that have not already been explicitly identified above. In particular, I recognize limitations of research design and data analysis and how these limitations may impact the extent of validity of my conclusions and their suitability for generalization.

The study uses a multiple case study approach, and that involves, studying cases to gain insight into one particular phenomenon. The phenomenon that I was investigating, which was to see how teachers encounter and handle the mathematical moments within the classroom teaching. Therefore, the study had a direct connection to teachers' knowledge of mathematics. Neither did I examine these teachers for what mathematics they know, nor for how much mathematical knowledge for teaching they exhibit. The argument that I would like to give for this is the purpose was not to see the connection between what these teachers knew and therefore could manifest it in their teaching. Instead, it was to see what are generic possibilities of such encounters and how teachers manage those. This limitation for the research design is more about the perspective one takes towards the data and analysis, and not indeed a limitation.

On the other hand, the second limitation that I think about the data analysis is more concerning as it is about the subjectivities in valuing the mathematics instruction. The study needed a closer examination, and due to massive amount of data, there are numerous things that could be documented. However, this particular noticing in the mathematics teaching is shaped by my understanding of the goals of education and in particular of mathematics education. There are other lenses through which this same data could be analyzed; few examples are – students' out of school mathematical knowledge, the contrast between the daily language of communication and mathematical language, intended and enacted curriculum, etc. All these would respond to the question of teacher knowledge, but not about teaching, its affordances and restraints. The analysis here is guided through my understanding of what teaching mathematics is, and that makes the interpretation valid for certain goals of mathematics education.

3.8 Discussing Encoutners and their Management

Here I narrate two encounters where access to mathematics was denied to students mainly because the teacher could not decipher the mathematics in the students' response. I also provide the teachers' account of why that might have happened and understand how listening to students is filtered through teachers' beliefs about the students and their learning of mathematics.The following vignettes illustrate encounters where the student is denied an opportunity to engage with mathematics.

Vignette 1: Triangle of 13 cm.

The teacher was teaching congruency of triangles. She drew two equilateral triangles of 13 cm dimension on the board, whose actual dimension was more than 13 cm.

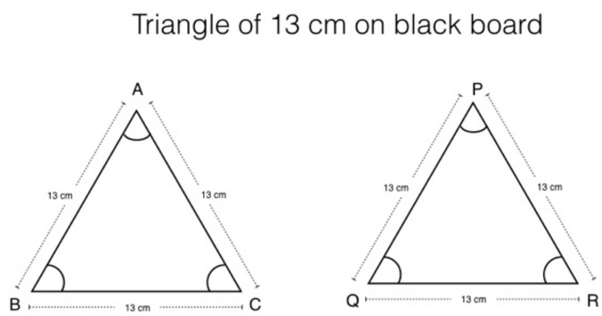


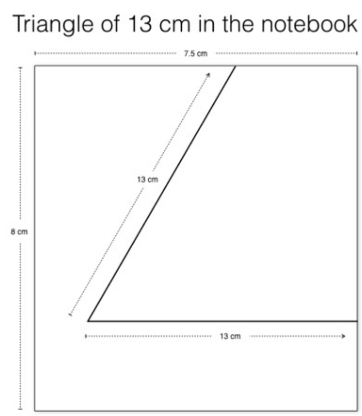
Figure 3.3: Triangle of 13cm on black board

She explained the meaning of congruency as the condition that when two triangles when one superimposed, they match exactly. The students then said the corresponding sides in the given pair of triangles. After this, the teacher asked students to copy the diagrams in their notebooks. She provided a few instructions before the students began. They were to make careful use of the scale, of a pencil that has a sharp point and make the diagram in such a way that the two triangles will be beside each other in their notebooks. All the students began working in their notebooks. The students had a same sized notebook that had a plain brown cover with dimensions 18.5 cm x 14.5 cm. One of the boy students who had a challenge in fitting a 13cm triangle in his notebook had the following discussion with the teacher.

S: See my triangle, it is going outside the page

T: What did I tell you? I told you to make the two triangles beside each other, side-by-side.

Figure 3.4: 13cm triangle on the notebook



S: I only made one, but it is going outside of the notebook area.

T: No, no, that is not the way. Let me show you. [The teacher walks towards the front of the class] Who has finished drawing the two triangles? Can I see anyone's diagram which has two triangles of 13 cm, same as what I drew on the board, side-by-side? Beside each other?[A girl student raises her hand, the teacher walks in her direction, takes her notebook in hand exclaims "haaaa"!]

T: See this. Everyone pay attention here. See how neatly she drew it. [Teacher is holding the notebook up in the air so everyone can see it. In the notebook, there are two equilateral triangles, approximately of the length 4.5 cm and labeled as 13 cm. The triangles are beside each other.]

T: She has drawn both the triangle beside each other as I asked. Very good work! Did everyone get it? You have to draw those two triangles beside each other. S do you get it *now? [S nods] Now everyone finish fast, once you finish drawing, label both the triangles as I have done, ABC, PQR, and also write the lengths. See, how I wrote 13 cm on the sides, write that. [S erases the incomplete triangle that he has drawn]*

The student here had a legitimate question about mathematics. There was no way the two triangles of length 13cm could be fitted in that notebook. However, the teacher failed to understand this question. Following questions were asked on this clip [the questions were presented in the spoken language and norm, for example, q2 below was asked like this "I saw you chose the example of an equilateral triangle of 13cm to talk about congruency, how did you think of that?"]:

1. What did you notice in this clip?
2. How did you decide the example of an equilateral triangle of 13 cm for teaching congruency?
3. What do you think is most critical to understanding congruency of triangles
4. What was the problem that the student S faced?
5. Why was that student facing the difficulty in drawing the figures as you asked?
6. What do you think the students in the class are thinking at the end of this class?

While responding to these questions, the teacher still thought that the student was not paying enough attention and missed listening to her instruction of drawing two triangles beside each other. When probed further about what the student has not understood here, she mentioned about how conditions of students in her class are deplorable and often due to lack of sleep; they miss what has been instructed to do.

Here, teachers understanding of students state of mind goes tangential to what the student is asking. There could have been multiple ways how this question could have been taken up only if the teacher has understood what it meant. For example, if one notices that the length of the notebook doesn't permit triangle of such length what all things one might do:

- Open that question to everyone — are there others who are experiencing the same problem
- Discuss with the class, why two triangles of 13 cm side cannot be fitted in that notebook
- Bring out the real question — would this mean we would never be able to draw triangles of longer length?
- Think about ways of representing the triangle of 13 cm
- Again, can each student represent the equilateral triangle with any dimension they want?
- What implications would that have to the topic at hand — congruency?
- And the broader concept of congruency that a 13 cm equilateral triangle will have the same area and perimeter irrespective of which country or in whose notebook it is drawn

- And if everyone can have their own 13 cm triangle what implication it might have to the real world (landowners' triangle of 13 cm will have more area than the laborers'?)

All these and many more could have been possibilities, but the student who was made to refute his idea lost the opportunity to express his mathematical understanding, and everyone else in the room lost the opportunity to understand the topic of congruency — which was the topic at hand.

Reading this episode, one would think, what can I infer, if the teacher would have noticed the mathematics in the student's question. For example, if she would have noticed that it is the dimension of the triangle that is the issue. Suppose she assumed the student was paying attention and therfore his question is a legitimate question about mathematics in the class, and the line crossing out of his notebook is because 13cm is too long. What I think the teacher would have done here – is to create access to the mathematics at hand. There were two kinds of mathematical understanding at hand:

- She definitely did not draw a 13cm triangle on the board, as that would be not at all visible to anyone. She drew something of the length 30 cm. She knew that, as she measured it using the whole "foot" scale. So there was the understanding of scale in mathematics, which lets one draw very large objects on a very small paper.
- The second understanding was about congruence. Therefore it was not about getting a 13 cm triangle right in one's notebook, but 13cm triangles worldwide being congruent to each other.

The scope of creating mathematical access was huge – concept of scale and deeper sense of congruency. A student noticing that, and then a teacher noticing the student noticing that would have led to access to both these ideas for the entire classroom, and not just the girl who herself understood the rules of the game (concept of scaling – in some sense).

Vignette 2 : Real life contexts in school mathematics.

Punit Sir, is a middle school mathematics teacher and teaches students with diverse background, within a low socioeconomic locality. He was teaching profit and loss to the students on that particular day. After, a rhetorical explanation of how to decide profit or loss in a transaction, he ventured into solving "application" problems. The phrase "application" is commonly used to type of problems, where one uses the mathematics learned to solve real life problems. He wrote the following problem on the board.

"Shaila bought something for Rs. 40 and sold it for Rs. 60, then what happened?"

The students immediately responded in chorus as "profit of Rs. 20". I got impression that, well, the students understood the concept of profit. However, Punit Sir was not happy about the chorus answer, he told everyone to speak one by one. He asked Raima to respond. As soon as she got up he asked her, what is given in the problem. She was confused for a moment, as I think she thought she had to solve the problem. Finally to answer, "what is given?" Raima read the question out-loud that was written on the board. May be because she thought the "given" is nothing but the problem. The teacher accepted it as an answer and re-phrased it as, "yes, what is given to us is the purchase price Rs. 40 and selling price

Rs. 60". Then he called on Shahid, and asked him what the next step would be. Shahid couldn't respond at that moment. What Punit Sir expected from students was, to write down the formula for profit. The work expected to have specific labels – the "steps", a routine to arrive at the answer. This routine includes, writing what is given, what is to be found, writing down the relevant formula and then actually substituting and solving it. We all are exposed to this at least once in our lives. But for the students the problem was already solved. Shaila gained profit of Rs. 20. And hereafter the whole confusion started.

Students at this moment were quiet and listening, maybe they were not sure what was expected since they already gave the answer. The teacher then wrote the formula and "step-by-step" [formula → substitution → calculation → answer] arrived at the answer as "Shaila earned profit of Rs. 20". Students were happy that their answer matched with their teacher's answer. The teacher continued with the next problem that he narrated orally.

The students seem to be fond of their teacher; they listened to him patiently. He had a certain way of speaking which sounded as if he was telling a story. In the next problem, while he was writing on the board, he was also narrating the problem loudly with pauses and change in pitch of his voice, making the narration of the problem much more interesting.

> *"I bought a basket of oranges for Rs. 160. The basket contained 2 dozen oranges. I went to the market to sell these oranges. To the first customer I told that the oranges are for Rs. 100 a dozen. The customer started bargaining and I sold the oranges for Rs. 65 a dozen. Did I earn profit or loss?"*

This problem was not from the textbook and the teacher made it up at that moment. The teacher called Aisha to the board. This time she knew that she was expected to produce the steps similar to the ones written on the other side of the board. She wrote the formula for profit, and then added 65 two times at the corner of the board (See Figure 3.5) and then subtracted the sum from the original purchase price Rs. 160. The teacher did not interfere. He let her finish.

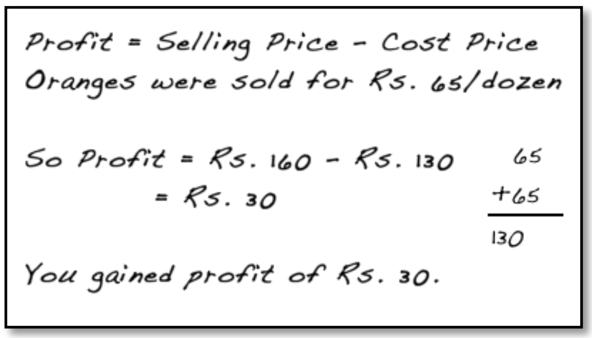


Figure 3.5: Aisha’s work on the board

After Aisha finished writing, Punit Sir asked her to go back and asked if anyone else wants to “correct her answer”. I was not sure whether students understood that the solution written on the board was wrong. The teacher also did not point out that what she actually calculated was the loss, though the formula she wrote was for profit. It seemed to me that the first step of her solution was wrong and therefore there was no discussion on the steps that were followed.

Even though the teacher did not discuss Aisha’s solution, he explained the context of the problem again. This time, making the story juicier. He used a lot of gestures to narrate how hot the day was when he went to market to sell the oranges, and how he decided to go home early and therefore, he sold all the oranges to the very first customer. And, then he asked again, “so what happened?” To which the students responded, “it is actually a loss”. After spending a minute working in their notebooks, most of them started saying that it was a loss of Rs. 30. But the story doesn’t end here. Punit sir insisted on writing the solution of the problem in “steps” and this time the students also needed to notice that they have to use a different formula than what was written on the board. Students were quiet again.

Now what could be the difficulty – the students understood that it was a loss-problem, they even calculated the loss, and they were also convinced by the context of the problem. Still, they were unsure and a little bit confused about writing of the steps. There were two more problems and the scenario remained unchanged. Students were able to understand whether the situation presented leads to loss or profit, but as soon as they were asked to present the solution in steps there was uncertainty and fear to attempt the problem. Some examples that they solved together are given below.

> *Salim bhai bought one bicycle for Rs. 2200. After a year, he sold his bicycle at the cost of Rs. 1800. Find out whether he made profit or loss, and how much?*

> *Reena Auntie bought a TV for Rs. 15400/- and after she decided to move to another city. So she sold it to her neighbor friend for 13000/-. Find out whether she made profit or loss, and how much?*

These contexts due their familiarity made sense to the students. There was even a discussion of how a used TV will have reduced costs and therefore Reena Auntie’s deal made sense. Some students raised a concern about the repairing expenses that Salim Bhai might have incurred on his cycle, but are not part of the problem. The situations in the context were active in students' thinking about the problems, but the procedural part of the solution remained unaffected by this understanding. What were students' ways to reach the answer, and why were they not part of the classroom discourse? And where is the space in school mathematics to account for the students' ways of doing mathematics?

Punit Sir always asked for a solution of a solved problem. "How did you arrive at the answers?" this one question would have given us insight into the students' steps of solving problems. Punit Sir might have his own agenda to push for the steps, but it appeared to me that the students' understanding of profit and loss in each problem was personalized. What I mean by personalized is students' identifying with stories of Shaila, Salim Bhai and Reena Auntie, as it was Shaila's profit for them, and Salim Bhai's loss but not as an application of the general profit and loss formula.

If context made the students understand the problem easily, it might have also played a role in how they solved the problem mentally. Nunes and her colleagues, in their work on coconut sellers brought forward some of these alternate strategies. The coconut seller in her study did not use the school-like strategy, which is , instead used his knowledge of the price of three coconuts i.e. 105 to find the price of 9 coconuts and then added the last 35 for the tenth one. Here is a transcript of their dialogue.

Customer: How much is one coconut?

M: 35

Customer: I'd like 10. How much is that?

M: (Pause) Three will be 105; with three more, that will be 210. (Pause) I need four more. That is…(pause) 315…I think it is 350.

The mathematical work that entails the work done by coconut seller is as follows:

Figure 3.6: Non-formal way of doing multiplication

35×3 (which he might already know)

$105 + 105$

$210 + 105$

$315 + 35$

And also $3 + 3 + 3 + 1$

May be in this case method shown in figure 3.6 would have been the efficient way of solving the problem, however the logical flow in seller's response somehow reveals much more about his understanding of the mathematics. There are steps in this solution as well, but they are specific to this problem. The steps are derived from the inherent nature of the problem. In Punit Sir's class something of similar nature could have been observed, but an emphasis on a specific format was a loss for us.

It is not difficult to be convinced that early emphasis on generalised steps is going to be harmful. In Punit Sir's class with his persistent instruction he made student follow the steps, and on the very next day the students stumbled upon the following problem that needed a little different method from the general method.

Anthony bought 8 dozens of banana at the cost of Rs. 30. He then sold 5 dozens for Rs. 45 a dozen and 3 dozens for Rs. 35 a dozen. Find out whether he gained profit or loss, and how much?

I feel that so much energy of teachers is invested into arriving at a general format and then to teach variations of the generalised form, that there is very little energy and space remained to manage students' mathematical ideas. I would hope that we facilitate all possible ways of doing problems and then abstract the general part of it. Moreover, real-life contexts have the potential to derive different strategies of calculation as part of students' shared knowledge. I believe as teachers we still need to explore this more.

During the interview, when Punit sir was asked what he noticed about what students were doing when they were answering the problems correctly, especially the initial ones. When I asked this question, I was expecting him to say that these students use their own methods, but those won't be useful as the examination requires them to write answers step-wise. However, to my surprise, what he said suggested that he did not know how they got the answers. He also suspected the students might have looked at the answers given at the end of the textbook, and just telling those without actually understanding how one arrives at the answer. It was particularly surprising to know that the teacher excluded any possibility of students' ways of solving the problem.

CHAPTER 4

Horizon Content Knowledge: Case of Teaching Algebraic Identities

The study looks at the mathematics classroom practice in Grade 7 for eliciting and unpacking encounters with the mathematics at the horizon. Forty-two such lessons were analyzed, that were taught by altogether 13 teachers. A total of 53 encounters in these 42 lessons were analyzed. As described in the earlier chapters, the encounters were set based on the interaction between student – teacher – curriculum and classroom material/media interaction. Each encounter identified either involved a question asked by a student, or teachers' choice of a representation, or teachers' explanation of mathematical entities. These encounters were then showed to the teacher, who were asked a set of questions. All the questions for each encounter are given in the Appendix C. Some examples are – what do you notice in this clip, why did you gave that explanation, whay did you use that analogy or that specific metaphor, etc. In this chapter I present the encounters around teaching algebraic identities (AI). Before we dive into the classroom details, I describe the mathematical relevance of the concept of AI and how it is situated in the curriculum. Further we see how students understand these based on the literature and what is the relvance of learning algebraic identities in developing understanding of algebra.

4.1 Algebraic Identities in Mathematics Curriculum

Algebraic identities play an important part in the mathematics curriculum and in mathematics in general. Widespread research has been done to elicit and cognize students' errors of equality and of concepts around equality. Researchers report varied conceptions of the "=" sign and how that functions in students' understanding of other algebraic entities. Algebraic identities are one of the constructs that students experience in the middle grades, where the idea of "always equal" is explicitly introduced for the first time. Said this, not much has been examined around the mathematical knowledge for teaching algebraic identities. In this chapter, I present an analysis of the encounters with mathematics at the horizon while teaching algebraic identities (AI). Out of 13 teachers in the study six teachers taught AI. They each introduced the mathematical concept of AI giving some metaphors, made use of representations, and connected algebraic identities to application. These cases have been analyzed for mathematical explanations, use of representations, coherence with mathematical goals and described applications. The analysis shows how identities have become different mathematical entities in school algebra due to its preached pedagogy, and the explanations and representations used to introduce the concept are situated in the concept of algebraic identities through a pseudo—mathematical logic.

The area of mathematics known as school algebra – and the research base accompanying this branch of mathematics education – has focused on forming and operating on polynomial and rational expressions, as well as representing word problems with algebraic expressions containing variables and unknowns. Freudenthal (1977) characterized school algebra as including not only the solving of linear and quadratic equations but also algebraic thinking, which includes the ability to describe relations and solving problems in a general way. This characterization of school algebra, remains timely today as it covers not only the symbolic aspects of algebraic activity but also the kinds of relational thinking that underlie algebraic reasoning. Algebraic reasoning distinguishes algebra from arithmetic activity, which is computational in nature.

There is considerable research on the learning of algebra since the 1970s. By the time learning of algebra begins, students are already operational with an arithmetical frame of mind, which predisposes them to think in terms of calculating an answer for every problem. Learning of algebra requires thinking toward a perspective where relations, ways of representing relations, and operations involving these representations are the central focus. However, based on the research we know that students have difficulty with conceptualizing certain aspects of school algebra, for example, (a) accepting unclosed expressions such as $x + 3$ or $4x + y$ as answers, and tending to close them as $3x$ or $4xy$; (b) thinking outside of established natural-language-based habits in representing problem situations, such as for every professor in the school there are six students as $P = 6S$; (c) representing and solving of word problems with transformations that are applied to both sides of the equation; and (d) failing to see the power of algebra as a tool for representing the general structure of a situation.

Teaching experiments have been designed to explore developing algebraic frame of mind among students. Some of the approaches that have been found to be successful include (a) generalizing and expressing generality by using patterns, functions, and variables; (b) focus on thinking about equality in a relational way and beginning to understand the transformation first in arithmetic expressions; and (c) make use of technology as it has synonymous symbolic functionality. These teaching approaches have focused on some part of school algebra and respond to students' learning of that specific section.

Given that the main focus in most of the research in school algebra during the 1970s and 1980s was on the learner, and on teaching approaches aimed at improving student learning, not much was revealed about the teacher of algebra. From the few reports available, one could only discern that, just as with teachers of other mathematical subjects, algebra teachers viewed themselves primarily as providers of mathematical information and tended to follow the textbook in their teaching. However, a research interest in the teacher of algebra and the nature of algebra teaching practice took shape in the early 1990s and has continued to this day – research that has begun to deepen our knowledge of this domain. Doerr (2004) has stated that research on teacher of algebra tends to fall into three areas: teachers' subject matter knowledge and pedagogical content knowledge, teachers' conceptualizations of algebra, and teachers learning to become teachers of algebra. However, according to Doerr, progress in teacher-oriented research has been hampered by the lack of development of new methodological and theoretical approaches to effectively investigate the practices of teachers of algebra.

Situated in these challenges, we investigate algebra teaching of six teachers to understand knowledge needed to teach algebra. We analyze these teachers' encounters with situations that call for their knowledge for teaching algebra. In this paper a component of school algebra – teaching "algebraic identities" is discussed at length. All these teachers who were identified as the best mathematics teachers by their superiors, noted teaching algebraic identities as one of the straightforward and easiest units among other units of school algebra. The 13 teachers are from three cadres of schools in India – governed by state or centre and privately run schools.

Algebraic identities appear in grade 7 in the Indian mathematics textbooks. Three identities are introduced to students in this grade, and they are given here.

$$(a+b)^2 = a^2 + 2ab + b^2,$$

$$(a-b)^2 = a^2 - 2ab + b^2,$$

$$(a^2 - b^2) = (a+b)(a-b).$$

In this section we discuss what encounters with knowledge of mathematics these teachers face when they teach algebraic identities, and how they manage these. To understand these knowledge affordances of teaching algebraic identities at the school level, we describe what algebraic identities are and, how they are perceived mathematically. We further understand its emergence and relevance in school algebra. We present our analysis of these cases of teaching with the framework on teachers' knowledge and knowledge of students' errors in school algebra. Towards the end of the paper we narrate knowledge demands for teaching algebraic identities with specifics about knowledge needed of mathematics at the horizon.

4.2 Algebraic Identities as Mathematical Entities

One of the common misconceptions associated with symbolic notation in algebra is the concept of a variable. At present, variable occupies a prominent position in school algebra. In early algebra students are introduced to letters as unknown numbers. The understanding of variable is built on this specific understanding of letters as unknowns. An encounter of a different kind happens when the students are introduced to algebraic identities. Because that is the first time students are exposed to the equations where a and b are not just unknown but could be any numbers from the students' point of view. The variables undergo some operations on both sides where equality is not that explicitly visible. The equations students study prior to identities have a numerical value present on one of the sides of the equation. And that limits the possible values for the variables used in that equations. Table 1 lists the different entities that students learn in school algebra and how each of them uses the letter differently.

Expression	$x + 3$
	Is an algebraic statement consisting of at least one variable and one number. It has no equals sign and so cannot be solved.
Equation	$x + 3 = 10$
	Is an algebraic statement consisting of an expression and a variable (or another expression) separated by an equal symbol. It can be solved to find the particular values of the variable for which it is true.
Formulae	$a^2 + b^2 = c^2$ or $c = 2\pi r$
	Relates one variable (letter) to another. It consists of one variable and an expression separated by an equal symbol.
Identity	$(a + b)^2 = a^2 + 2ab + b^2$

	Is an equation that is always true for any values you choose for the variables.

Table 4.1: Algebraic entities that use letters in school Algebra

Expressions the way they are defined remain a number affected by the numerical value given with in the expression. Formulae actually emphasize the label aspect of the letter, where c stands for circumference.

Given this context, we understand that algebraic identities are the first direct encounter that middle grade students face with true meaning of variables. In identities the equality always holds for any value of the letter numbers. It is doubtful that how much of that is conveyed to the students and how much is understood.

School textbooks are not very consistent in defining "variable" and in learning the concept. Sometimes, a variable is described as a quantity that changes or varies. The mathematical meaning of this statement is vague and obscure. At other times it is asserted that students' understanding of this concept should be beyond recognizing that letters can be used to stand for unknown numbers in equations, but nothing is said about what lies “beyond” this recognition. For example, in the National Research Council volume, Adding It Up (The National Academy Press, 2001), there is a statement that students emerging from elementary school often carry the “perception of letters as representing unknowns but not variables” (p. 270). The difference between “unknowns” and “variables” is unfortunately not clarified. All this deepens the mystery of what a variable really is, and that lack of understanding is apparent while learning algebraic identities.

What are algebraic identities?

The term identity is used in mathematics to indicate that equality is valid for a "large set" of numbers of interest. What "large" means will be indicated in each situation and, is usually clear from context. There is a debate among mathematicians about what the term "identity" precisely means (Wu, 2005). In the following paragraphs we attempt to understand what are the identities in school algebra.

An approximate definition that is commonly accepted is that two given number expressions (where letters such as x, y ... stand for numbers) are equal for every number in a given collection under discussion (such as all whole numbers, all positive numbers, or all numbers) allowing for a small set of exceptions. Wu (2005) emphasizes that an identity is not a precise concept within mathematics, but is no more than a terminology used loosely for convenience. In specific situations, there will be plenty of opportunities to discern what "the given collection under discussion" is and what the "small set of exceptions" means.

The assertion that $ab = ba$ is true for all numbers a and b is an example of an identity, and similarly $\frac{k}{l} \pm \frac{m}{n} = \frac{kn \pm ml}{ln}$ for all integers k, l, m, n provided $l \neq 0$ and $n \neq 0$. The latter identity is one that doesn't hold for $l = 0$ and $n = 0$. The stated equality $\frac{k}{l} \pm \frac{m}{n} = \frac{kn \pm ml}{ln}$ therefore, is true for integers k, l, m, n, as well as for rational numbers provided $l \neq 0$ and $n \neq 0$. There are more examples like this, where equality holds for constrained set of numbers. For example, $\log xy = \log x + \log y$ is an identity for all positive values of *x* and *y.*

The identities under discussion could be understood from arithmetic computations. Consider the computation of the square, 104^2 for example. One can compute it directly, of course. But one can also proceed by appealing to the distributive law, as follows:

$$104^2$$
$$= (100 + 4)^2$$
$$= (100 + 4) \times (100 + 4)$$
$$= \{(100 + 4) \times 100\} + \{(100 + 4) \times 4\}$$
$$= \{100^2 + (4 \times 100)\} + \{(100 \times 4) + 4^2\}$$
$$= 100^2 + 2 \times (100 \times 4) + 4^2$$

At this point, it should be possible to mentally finish the computation as $10000 + 800 + 16 = 10816$. These computations are well defined among the laws of numbers using distributive multiplication over addition. More than a trick, this idea of computing the square of a sum using the distributive law turns out to be almost omnipresent in algebraic manipulations of all kinds. It is a good idea to formalize it once and for all. We therefore have, in an identical fashion:

$(a + b)^2 = a^2 + 2ab + b^2$, for all numbers a and b.

A similar consideration, but worth pointing out in any case, is the computation of the square of 497, for example. We recognize it as $(500 - 3)^2$so arriving at $(500 - 3)^2 = 500^2 - 2 \times (500 \times 3) + 3^2$.

And this computation also leads to:

$(a-b)^2 = a^2 - 2ab + b^2$, for all numbers a and b.

The identity for $(a-b)^2$ can be obtained directly from the identity for $(a+b)^2$. Since the identity $(a+b)^2 = a^2 + 2ab + b^2$ is valid for all numbers, one may replace b by an arbitrary number $-c$ to get

$(a+(-c))^2 = a^2 + 2a(-c) + (-c)^2 = a^2 - 2ac + c^2$

Since $a + (-c) = a - c$ by definition, implies $(a-c)^2 = a^2 - 2ac + c^2$, and since c is arbitrary anyway, we may replace c by b to obtain $(a-b)^2 = a^2 - 2ab + b^2$ for any number b. Thus one retrieves the second identity by way of the first.

The third identity under the discussion could be introduced by a computation of another kind: $409 \times 391 = ?$ We recognize that $409 \times 391 = (400 + 9)(400 - 9)$, so the same reasoning carries over to any two numbers a and b, so that $(a^2 - b^2) = (a+b)(a-b)$. Written in this form, the identity represents factorization of $(a^2 - b^2)$. In this case if $a + b \neq 0$, we can simplify the division as $\frac{(a^2-b^2)}{(a+b)} = (a-b)$.

Historical account of algebraic identities.

The other account of algebraic identities comes from the history of mathematics. Our present ways of doing algebra has only been developed during the last 400 years, and some of the problems we solve and methods we use to solve the problems are ancient.

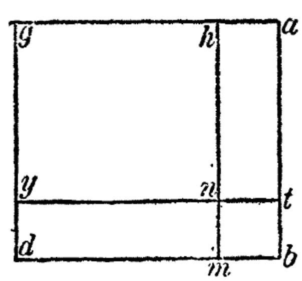


Figure 4.1: Abu Kamil's Solution

By the end of the 9th century, the Egyptian mathematician Abu Kamil had stated and proved the basic laws and identities of algebra. Karpinski (1914) suggests that Kamil used geometric representation to find solutions of quadratic equation. These geometric representations however indicate an implicit understanding of algebraic identities. Kamil's solution of the equation given by Al-Kho-warizmi, $3x + 4 = x^2$, is given in Figure 1.

He assumes the side $\overline{ag}$ as x, and the area of the square $agdb$ as x^2. Taking the side $\overline{ah}$ as one and a half units, he then obtains the square $hnyb$ with the area $4 + (1\frac{1}{2})^2$, concluding $x = 4$. Kamil cites Euclid [1]for the validity of geometric representation and how x^2 is sum of all the areas inside the square $agdb$. Unpacking this further we see that the geometric representation actually uses the fact that $x^2 = \left[m(\overline{gh}) + m(\overline{ha})\right]^2$, and illustrates the expansion geometrically. In all the six types of quadratic equations as suggested by Kamil, he uses this representation and expansion of the identity $(a + b)^2$ to find values for x.

Here, we see that the generality present in the algebraic identities is used in a different manner than what we see its usage in the school algebra. In the data presented below we see a range of interpretations made by teachers for the rational behind algebraic identities and their applications. In statements of equality there is a structural generalization

[1] Euclid's Elements, II, 6, " If a straight line is bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line, together with the square on the half, is equal to the square on the straight line made up of the half and the added straight line."

possible, as well as a numerical generalization. We see how different teachers approach this differently and its affordances to students understanding of school algebra, specifically meaning of variable. We attempt to elicit the implicit meaning of letters taught to students and what parts of these statements of equality in reality are understood as general and what parts as special.

4.3 Students Difficulties in Learning Algebra

For almost four decades, the researchers have analyzed students' understanding of symbols and variables in algebra. Many of these research studies cite students' prior knowledge about language, arithmetic and their exposure to mathematics as the basis for formulation of their ideas about symbols and letters in algebra.

Stacey and MacGregor (1997) point out that teachers often think that students are completely new to algebra. They do not entertain a possibility or consider that students might already have ideas about the use of letters. According to Stacey and MacGregor algebraic thinking in students begins much before their exposure to school algebra. Arithmetic thinking about patterns in number, functions, and equivalence and equations forms the foundation for algebraic thinking (MacGregor & Stacey, 1999). The students' development of relationship within arithmetic procedure, spatial patterns and number sequence all include use of algebraic thinking.

The students have knowledge of numbers and letters when they begin to learn algebra. While learning early algebra students come across letters in place of numbers in the expressions. A sudden use of letter creates conflict among children and that makes them feel confused or uncomfortable to process necessary information to understand algebra.

These letters are also used in many contexts within and outside of mathematics. We understand that students bring their own understanding of mathematical context into their learning. While they learn and educate themselves through formal schooling, they grasp and make few implicit understanding about structures, procedures and concepts. This understanding around specific concepts might become their strengths in performing various tasks or it could lead them to make certain kind of errors, develop misconceptions which might create hindrance to their further understanding of mathematical concepts (Stacey & MacGregor, 1997). These diverse ideas from school cultures are sometimes based on their previous understanding in arithmetic, from various topics that they learn in mathematics and may also stem from languages. Here, we present five prominent categories in students' understanding of “letter numbers” that are relevant to our work.

Confusion with letters and its meaning.

Algebra is a language where presence of letters in an expression denotes restricted meaning associated with it. The letter holds the presence of an unknown value i.e. number which will bring out to equate the given expression. Due to this reason, one can say that algebra has some limited vocabulary and it is a restricted language (Stacey & MacGregor, 1997). The letters cannot be used in an expression to write many things. The structure of algebra is different from structure of language in the use of letters per se. The letters in language can be use to express a lot of things but not in algebra.

Students’ prior experiences.

Students' interpretation of letter is often influenced by prior experience. This interpretation not only comes from a particular subject but also from diverse and vast exposure of information from various background like society, cultural use of letters to represent an idea, formal education, etc. Students tend to look at these letters in various forms aligned with their exposure to the use of language, like the use of letters as an abbreviations. For example, sometimes abbreviations stand for specific meanings in various contexts. Students tend to interprete letters as an abbreviation in algebra as well. For example, 'h' is used for height in an expression h + 10. The other idea is to use the letter as place value. Students might think the answer as 18 in an expression h + 10, since 'h' is the eighth letter in alphabetical order, 8 + 10 = 18 (Stacey & MacGregor, 1997).

Use of equal sign.

The restricted and familiar use of 'equal to' sign brings obstacles for students in understanding meanings and values associated with letter. Like in an expression a = 28 + b, students may attach the meaning of equal sign as the next step to solve the expression. It might mean as 'a' is equal to 28 and then add 'b' to it (Stacey & MacGregor, 1997). They might think of equal to sign in an expression as something that “makes” or “gives” the other side. It may also be assumed that equal to sign is used to link other parts of calculation. For example in a study conducted by Naik, et al (2005), students’ response to the following question was studied; 38 + 17 = __ + 16. The initial response of students was 55.

Formal arithmetic education.

The experience of multiple 'step' calculation that students have from arithmetic could also become a barrier. Bell (1988) brought forward some of the students' errors in algebra. One example would be 3 + b = 3b. Many researchers (Bell 1988 ; Booth 1988;

MacGregor & Stacey, 1997) have explained that accepting a "closed answer" is something that students are conditioned during their arithmetic training. An arithmetic expression when simplified gives one number as answer whereas bringing letters as an unknown number could lead to situations where the simplified answer is an expression itself, such as 5 + a. 5 + a does represent a number, a number that 5 more than an unknown number. However, from students' point of view it is still a question to be solved and one answer to be calculated. Hence 5 + a = 5a is a very common error that students do. Banerjee & Subramaniam (2004), further explored the impact of other arithmetic structures on students' work in algebra. They found that some errors such as detachment error, such as -3b + 2b = 5b, where the terms are added by detaching the negative sign, or opening bracket errors, such as 3 (a + b) = 3a + b where multiplicative distributivity over addition is disregarded, were present in students' arithmetic work as well, and not really the errors due to letters in algebra.

Letter could be one, understanding from school mathematics.

Stacey & MacGregor (1997) point out that, "a prior experience of formal mathematics education might play as an obstacle in algebra learning. They said student might assume 'x' could be 1 or 'x' is just like 1, unless specified. This misunderstanding might have come while they learned co-efficients. Often while teaching co-efficients teachers say, "x without a coefficient means 1 like $1x$". This can lead to an interpretation that the letter 'x' by itself is something to do with 1. There are other possible avenues for confusion such as power of 'x' is 1, that is if no index is written then $x = x1$ and that $x0 = 1$" (p. 309). The emphasis of co-efficients has seen much longer reach in even teachers' understanding of mathematics. Naik (2008) reported that many teachers thought $7 \times \frac{2}{8}$ and $7\frac{2}{8}$ as the same fractions. Their explanation was in algebra when we write $2x$ it means 2 into x. Therefore $2.x - 2x = 0$, and this implies $7 \times \frac{2}{8} - 7\frac{2}{8} = 0$.

These study bring out students' conceptions about letter 'as an unknown' while learning algebra and try to highlight students' struggle in making connection from arithmetic to algebra while solving patterns. According to Carpenter, Frank and Levi (2003), the divide between arithmetic and algebra is artificial as they say, “the artificial separation of arithmetic and algebra deprives students of powerful ways of thinking about mathematics in early grades and makes it more difficult for them to learn algebra”. Studies that tried bridging the concept of arithmetic and algebra have seen some success in terms of students learning.

Among other trajectories for teaching algebraic thinking prior to formal introduction to school algebra, Britt and Irwin (2011) proposed a pathway for teaching algebraic thinking through several layers of awareness of generality across all topics of mathematics curriculum. However, there was not much understanding available about the transition from numerical to symbolic computation. One study in the Indian context exhibited a teaching experiment bridging arithmetic and algebra. Banerjee & Subramaniam (2008) built a teaching trajectory focusing on construct of 'term'. In this trajectory the idea of term is brought forward as bridging concept between arithmetic and algebra. The 'term' is defined as a number (later also referred as letter number in their study) with + or – sign preceding it. For example, $+12$ and $+4$ are the two terms in the expression $12 + 4$.

These trajectories that initially start with a proposal of arithmetic expressions and later goes on to the investigation of algebraic expression as a symbol manipulation (National Council of Teachers of Mathematics, 2000, p. 37) are the ones that introduce letter as a variable.

The analysis in this chapter is around teaching algebraic identities and attempts to respond to two specific questions:

- What encounters do teachers face while choosing representations and explanations to teach algebraic identities in school algebra?
- How do these representations and explanations situate understanding of identities as entities of school algebra?

I will first answer these two questions, and then use these descriptions to discuss mathematics at the horizon in the context of AI, and what encounters were observed in these lessons. In the following paragraphs, I first present description of teaching across these six cases. The description involves how teachers opened the topic of algebraic identity, what logic or representation they used to persuade students of the mathematical truth and how teachers understood the role of identities in school mathematics.

4.4 Teaching Descriptions

The textbook of grade seven introduces three identities, namely $(a+b)^2 = a^2 + 2ab + b^2, (a-b)^2 = a^2 - 2ab + b^2,$ and $(a^2 - b^2) = (a+b)(a-b)$. These identities are labeled as "expansion formulas" in the textbook. The expansion is defined as the product of two algebraic expressions. The chapter on identities opens with the definition shown in Figure 4.2.

Expansion formulas

The product of two algebraic expressions is called an expansion. We make formulas in order to write the expansion of certain algebraic expressions directly without multiplying. Let us learn some such formulas.

The square of the sum of two terms

$$(a + b)^2 = (a + b) \times (a + b)$$
$$= a \times (a + b) + b \times (a + b)$$
$$= a^2 + \underline{ab + ab} + b^2$$
$$= a^2 + 2ab + b^2$$

$(a + b)^2 = a^2 + 2ab + b^2$ **This is an expansion formula.**

Figure 4.2: Opening section on algebraic identities in textbook

The entire unit on algebraic identities as given in the textbook, is presented in Appendix G. In the following three sections we unpack six cases to understand – what the teachers mean by algebraic identities, what representations they use to verify or prove the identities and what relevance is made for learning of this concept in school algebra. We do this unpacking by studying their teaching as well as their own views on these episodes of teaching.

Explanations for algebraic identities.

The six teachers opened the topic of identities differently. Some of them opened the class saying that they are going to learn identities that day, whereas some preferred to remind students of arithmetic factorization.

Here are annotations from the transcript of teaching on how each teacher opened the topic of algebraic identities. Instead of the teacher names, a code for each teacher is used here[2].

BoS_D1_0216

"We are going to study identities today and while studying identities, we will first look at the expansion formula. We will work on simple multiplication in expansion formula. Suppose there is a multiplication problem $(a + 4)\ (a + 3)$. *How do we do the multiplication?* (Explains the binomial multiplication, see Figure 4.3.) *Like this only, we will learn a different type of multiplication. First we begin with* $(a + b)\ (a + b)$.*"*

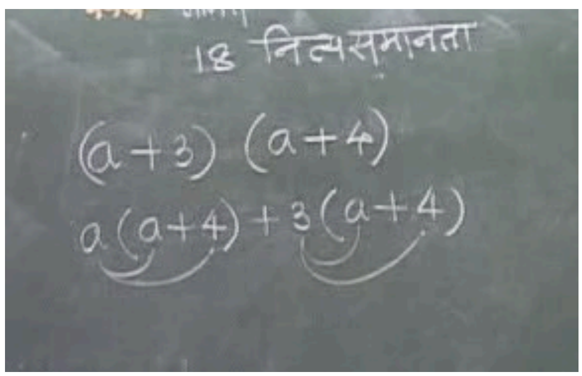


Figure 4.3: Binomial multiplication on board

AM_D2_0318

"What is identity[3], that we will learn at the end of the class today. (Writes 3×7 on the board, and asks what is the operation used. Then takes $8 \times 4, 6 \times 5$, and asks students what is the product. She then highlights that 21 is the answer here.) *What would be factors*

[2] The non-italic text is added in the annotation is to provide extra information to the reader without increasing the text a lot.
[3] Identity in Marathi is referred as *Nitya Samanata, which literally means "always equal"*

of 21*? The factors of* 21 *are* 3 *and* 7 *and not* 3 *into* 7*. And this answer, like 21 here* (pointing on the board*) is called as expansion in mathematics. And how does one get the expansion – by multiplication. What do one has multiply to get the expansion – one has to multiply the factors."*

JS_D2_0213

"We already studied algebraic expressions, and we saw their expansion. In the expansion we saw how to multiply one algebraic expression by another. Today also we will study expansion of the algebraic expressions, let me tell you the only difference that we will do in today's work. For example first expression is $a + b$ *and the second one is also* $a + b$*. We need to do expansion, the multiplication. Isn't it? Do you notice anything special here?"* (The students respond the letters are the same and the teacher suggests the expression is the same and could be written as $(a + b)^2$).

KM_D3_0214

"Today we are going to see page number 120*, the chapter on Identities* (Writes the title on the board. Some students say that they do not have the textbook, and the teachers asks them to adjust among them, she also corrects the page number and says it is 126 not 120.) *In front of you* (Figure 4.4) *there is fill in the blanks exercise. You are going to tell me the answers and I am going to write them."*

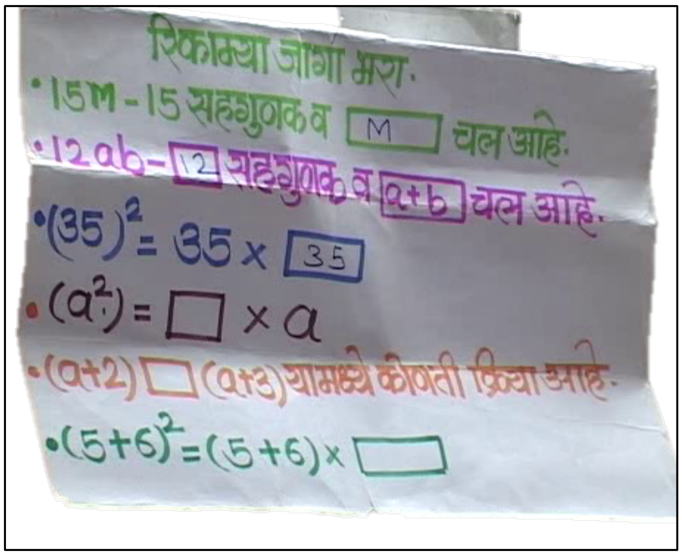


Figure 4.4: Co-efficient and variable exercise

RM_D3_0211

"Listen kids the lesson we are learning today is the 18th lesson (from the textbook) *called as Identity. Can someone tell me what is the meaning of Identity*[4]*?* (A student responds that it is always equal, permanently equal.) *And how can we prove that something is identity? How can we show something on the left is identical with something on the right side? What happens in identity?* (A girl trying to respond, but inaudible to us. Teacher doesn't consider

[4] Identity in Urdu is referred as *Dayani Masaviyat,* which literally means "always equal"

the response). *See what we are going to learn today is a formula; it is called formula for expansion, expansion formula. It is very important because you will use it in your next classes as well as when you go to college."*

SM_D3_0227

"(Writes identity as a title on the board.) *Close books and everything and sit straight. The term that is written over here* (pointing at the board) *is identity. Identity means…*(makes an inconclusive gesture with her both hands). *What do you call as an identity? See you come to this school, isn't it? Then what is your identity, when you come to this school?* (A student said i.d. card.) *Id card? Okay. That is the proof that you are the students of some institute. Okay. That is the meaning of identity. Identity also means, that you can say it's the same. Suppose, if these two terms are same* $(a + b)(a + b)$*, then what will we do. First we will discuss the formula how to multiply the two brackets."*

The six introductions of the concept are different in the metaphors they used, in pedagogical trajectory they choose and in the meanings they project of the identity.

BoS_D1_0216 is not concerned what are identities in relation to expansion formulas, and treats them as one concept after the other. JS_D2_0213 takes exactly the same approach; by recalling the product of binomial expression students did in the earlier class. The interesting difference between the two approaches is that, BoS calls $(a + b)\,(a + b)$ a "different type of multiplication" where as JS_D2_0213 refers to it as a "special case".

AM_D2_0318 on the other hand, prioritizes explanation for the meaning of "expansion" in mathematics. She specifically mentioned that, "the answer 21 here, is called as expansion in mathematics". She believes that not only the students have to understand what expansion means but also in a familiar context – that is the context of numbers and not letters. She defines expansion as the process of multiplying two factors as well as the product obtained by the process. Interestingly, she is taking the idea of process—product duality (Sfard, 1991) inherent to the algebraic symbolism, back to the arithmetic. "21 is called expansion in mathematics" and "one has to multiply factors to get expansion" indicates how expansion is a product as well as a process in algebraic manipulation. She somehow is preparing her students in advance to see $a^2 + 2ab + b^2$as not just something that have to calculate but an answer for the expansion.

KM_D3_0214 begins with a fill in the blanks exercise, as a preparation for the concept of identities. The list of questions suggest us what she considers as the prior knowledge for learning the identities. The first blank to be filled asks for coefficient in $15M$. The second one asks for co-efficient in $12ab$. Interestingly the second blank to be filled states,

- $12ab$ – ___coefficient and $a + b$ variable

Which is mathematically wrong, as the variable is not $a + b$. What she meant here is a and b both are variables, however in the chart the "and" is replaced by plus sign. She doesn't realize this when she discusses the solution in the class.

The questions 3, 4 and 6 revise the concept of square, with the sixth question as a binomial in arithmetic. The fifth one assumes that there is only one answer.

- $(a + 2)$ __ $(a + 3)$ What is the sign between the two

There could be any sign between the two, however the teacher assumes the context of multiplication of binomials and expects multiplication as the only correct answer. Again while discussing this she does not notice the possibility of multiple answers.

RM_D3_0211 is the only teacher who asks the question, "How can we prove that something is an identity". She unpacks this question further as "how can one show that the right side is same as the left side". However she does not begin to prove something is an identity, but describes what they will learn today is nothing but a formula. Formula is an identity in its mathematical sense, but sometimes formulas represent relationship between constructs, such as $A = L \times W$. In this case, if there exists a rectangle with length L and width W, then the area of the rectangle is $L \times W$. Usiskin (1998) notes that in arithmetic when two numbers are multiplied a third number is obtained. However, in school algebra this single operation may leave different senses. The product could be called as – formula ($A = L \times W$), equation ($40 = 5x$), identity ($\sin x = \cos x \cdot \tan x$), a property ($1 = n \times \frac{1}{n}$) and a function of direct variation ($y = kx$). Understanding this delineation among senses of the same phenomenon is not obvious, and clearly was not known to any of these teachers.

The metaphor used by SM_D3_0227 was the most complicated among the others. She tried to connect identity of a person to bring forward something that is the same. Her reference to $(a + b)$ and $(a + b)$ being the same indicates, what remains the same in identity is not obvious. This illustration by SM_D3_0227 opens possibility of different understanding the teachers might have about the identity. The sameness in identity that she refers to is of the expression: $(a + b)$ and $(a + b)$. This also makes us question the understanding demonstrated by JS_D2_0213, who calls it a special case and by BoS_D1_0216 who calls it a special multiplication.

These openings suggest that there are two main structural understanding that the teachers need to develop. The first is the demarcation between formula, equation, identity, property and function, as they all could be represented as product of two quantities. And the second is to understand what does identity really mean? What is it that remains identical – the value on the both sides and not that the two expressions multiplied are identical.

Representations for algebraic identities.

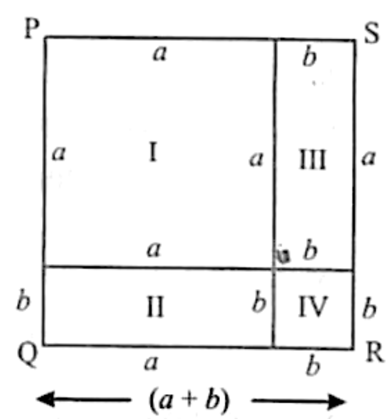
The textbook used by all the teachers, gives the representation given Figure 4.5 as an alternative to arrive at the formula.

This expansion formula can also be worked out with the help of geometrical figures.

(1) In the figure alongside, PQRS is a square and the length of its side is $(a + b)$.

$\therefore$ The area of square PQRS is $(a + b)^2$

(2) As shown in the figure, a segment parallel to PS is drawn at a distance of a from point P. Similarly another segment is drawn parallel to PQ at a distance of a from P. This divides the square PQRS into four parts.



(3) Part I is a square of side a

$\therefore$ the area of part I is a^2

Figure 4.5: Diagramatic justification of the formula

This diagram is similar to the ones recorded in historical write-ups in algebra, and similar to the one we saw in Kamil's work in the sections above. Here, $(a+b)^2$is understood as area of a square whose each side is of the dimension $(a+b)$. By taking $a > b$, the square is further divided into four quadrilaterals. The two of them are of the area a^2 and b^2, and the other two of the area ab. The sum of the areas hence is $a^2 + 2ab + b^2$.

Among the six teachers, only one teacher used this representation in her class. RM_D3_0211 made this representation in her class along with her students. She first calculated $(a+b)^2 = a^2 + 2ab + b^2$using multiplication of two binomials. Then she suggested her students that there is another way to arrive at this formula. She gave all of them a graph paper, and asked them to draw a square of side 8 cm. After all the students made squares, she gave the further instructions.

RM: Here, I have given a name... A, B, C, D. Give a name in whatever way you want – clockwise or anticlockwise. Done? Now, you have learned this, how to find out the area of a square on the graph paper. How do you find?

B.Std: By counting.

RM: By counting? Very good! Tell me how much is it?

RM: What does 'Raqba' mean? It means Area, isn't it? Square A, B, C, D. How much you got? Tell me.

G.Std1: Eight... eight are...

G.Std2: Sixty-four.

G.Std3: Eight, eights are sixty four.

RM: Eight eights are sixty-four. You counted them? Yes. Sixty-four is correct?

RM: Sixty-four. What sixty-four?

Stds: Square cm.

RM: Square cm. Finished this much? Now, in the square you have made, make one vertical and one horizontal line at five cm. Do it. Make horizontal line and vertical line, at five cm. Here (pointing in her model)

(RM showing to different students on their desk)

G.Std: Made it.

RM: Good, very good! Okay.

RM: Yes. Correct. Very good. Now, see... due to this line, in how many parts the square is divided?

Stds: Four.

RM: Four. Don't say it in chorus.

RM: How many parts? Four.

RM: Now, give a number to each part. Number one, number two, number three, and number four. Give it in any form. [RM pointed at the pre-made figure she was holding in her hand].

RM: Now, you have to find out the area of part one. The part one... you have this as part one?

Std: Yes.

RM: What shape is this?

(RM asked them to find the area of each part, and then at the end added $25 + 15 + 15 + 9$ *and showed that to be equal to* 64*)*

In this episode, RM uses the geometric representation, however her justification is based on the numerical values. She discusses the case when $a = 5$ and $b = 3$ and that also not explicitly. She never mentions that a is 5 units in this case. And therefore, the opportunity to extend this representation to a could be any number, and the area of that square will be square of that number, is missed.

Using this representation to reveal the process of expansion and equality requires multi-layer understanding.

- The area of the whole is same as the sum of the areas of its parts.
- The sides of the square a and b are arbitrary, and could take any value.
- In this representation a and b could take only positive values with $a \neq b$.
- If $a = b$ then the representation shows $(a + b)^2 = 4a^2 = 4b^2$.

The work of teaching to make the students understand what identities are – is to bring forward the powerful structure of equality, discuss the idea of variable, and even contrast these with the equations that are not identity.

Therefore, while making the choice for representation one would consider such factors; the factors that unpack the concept of identity and the logic behind its expansion. However, the representation used by the other five teachers, brings forward the visual aspect of the identity.

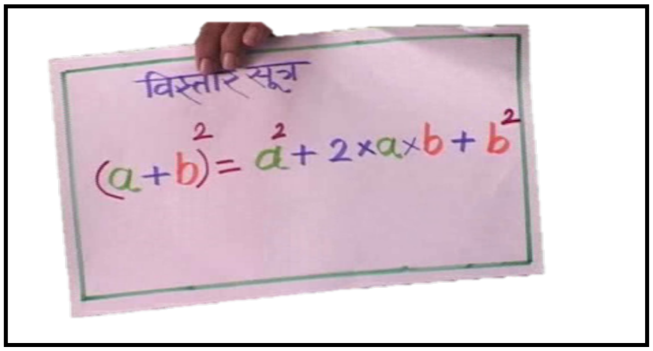


Figure 4.6: Representation of algebraic identity

All the five teachers used what is shown in Figure 4.6 and called it a representation for algebraic identity. In this re-writing of the expression, what they have done is they have used different colors for a and b. According to these teachers what this representation does is to help students remember what to square and what to multiply. This answer given by the teachers contradicts with their earlier answer, where I asked them to describe the difficulties student face in learning the algebraic identities. All of them unanimously mentioned the following two errors that students do.

1. The students confuse the sign, especially in adding ab and ab. When either a or b is negative, they get confused
2. The students find it difficult to apply this formula to something of the form $(ax + by)^2$, where a and b are coefficients.

It is interesting to understand how this knowledge of teachers interferes with their choice of representation. The teachers concluded that for the confusion of the signs the students need to learn the mnemonics, such as + + →+, etc. On further questioning they said the students still keep forgetting. When asked what they do about it, the teachers reiterated how students need to learn the mnemonics.

Two of the teachers, AM_D2_0318 and AM_D2_0318 suggested a solution that they also used in their teaching. According to them the better representation for the algebraic identity than the one shown in Figure 4.6 is as follows.

$$(first\ term\ +\ second\ term)^2$$
$$= (first\ term)^2 + 2\ \times\ first\ term\ \times second\ term + (second\ term)^2$$

They both said that in emphasizing $(a + b)^2$students forget about the co-efficient part of the term. The illustration above highlights the structural part of the equality.

The discussion of teachers' use of representation above indicates, that the teachers perceive some problems of teaching, however they are not aligned with the problems of mathematics. The teachers are concerned about students remembering the formula but not about what they learned in terms of mathematics from the representation. The justification they give for using the representation is not situated within the logic of the concept of identity, rather on how the identity looks. Such support that is not aligned with conceptual structure of the field is what I am referring as a pseudo-mathematical. A support that just looks mathematical. When asked why this structure works, why finding the square of the first term, and the second term and the sum of the product of the two terms gives us the square of the sum of the two terms, all of them said "it has been working for ages" and used numerical examples to show the equality. When they were asked how the multiplication in binomials relates to the multiplication the students learned earlier, BoS_D1_0216 said, *"This is like piece-wise multiplication in arithmetic. There are two terms and we make sure that we multiply each term. This is going on for ages, and that is why also this is true. But sometimes, students think differently. Somebody who is sitting in the back thinking, that we already know 8 x 8 = 64, and then why is our teacher doing this piecewise multiplication, when we can actually find the answer directly."*

The teachers sense the problem of mathematical explanation, either directly or through students. What they are not sure is whether these are the questions of mathematics and could be discussed in the class or they are the question of the curriculum that somebody else needs to discuss.

Applications of algebraic identities.

The textbook suggests that the identity or expansion formula learned could be used to find the square of certain numbers (See Figure 4.7). Except JS_D2_0213 and SM_D3_0227, every teacher used finding square of a number as an application of learning the identity. They treated finding 52^2as another kind of problem. This is what they said precisely.

"Look at this problem carefully. In all the problems you solved so far there was a number with a variable. There used to be some letter a, b, c, or d and then there used to be a number. Here it is just the square of a number. And we can solve it like earlier using the formula."(SM_D3_0227).

We can find the squares of certain numbers using the formula. To do this, write the given number as the sum of two numbers whose squares we can find orally.

Ex. 3. Find the square of 52.

$$52^2 = (50 + 2)^2$$

$$= 50^2 + 2 \times 50 \times 2 + 2^2$$

$$= 2500 + 200 + 4$$

$$52^2 = 2704$$

Figure 4.7: Use of the square identity

In the quote above, SM is phrasing it as a new problem where letters are replaced by the numbers. The part to understand that 52 can be represented as a sum of two numbers and therefore sqaure of 52 as the square of the sum of those two numbers is a presumed equality here. Expanding the number in the binomial form, the students were expected to apply the expansion formula to find the sum of the three products. Two products arrived from the two squares and one from the middle term of the expansion. Although this way of solving involves adding three products, the teachers repeatedly proposed that using the identity was "the easier way" to find the squares. None of them explained to the students why it was easier, but when asked in the interview, they suggested, "finding squares of numbers with unit place as a zero is easier". The "easier way" claim took some extra instructional space, in an interestingly conflicting way. Three of the teachers went on explaining other tricks for finding squares that are easier. The tricks involved finding squares of the numbers with unit place as 5 and with unit place as 0. See Figure 4.7.

BoS_D1_0216 introduced a technique that none of the teachers used. The technique is based on the logic of identity expansion; however, he did not realize it and introduced the technique as yet another easier way of finding the square.

> *"Let us see one more method. Take* 96^2*, how will you do it, you will multiply* 96 *by* 96*. But see here, this is just a fun method, take a look. How are we taught, everywhere? We are taught to multiply* 96 *multiplied by* 96 *(writes on the board simultaenously) and this is how we are taught, but it actually is very simple. Say the square of* 6 *in you mind,* 6*, sixes are* 36 *and nine nines are?* 81 *(students also say* 81 *in chorus. He writes* 8136 *on the board.) Write the product of* 9 *and* 6 *in the middle. (On the board as follows.)*
>
> 8136
>
> 54
>
> 54
>
> *Now add these digits –* 6*, here* 11*,* 1 *carry* 1*,* 12*,* 2 *carry* 1*, and* 8 *plus* 1 *is* 9*. So* 9216*. This has to be the square. It ought to be."*

BoS_D1_0216 was shown this clip of his teaching in the interview. I asked him whether he knows how this technique works. To that he responded, how he has learned many techniques over the years, and how he uses those to help the students. Further when I asked him, whether the logic of this method is similar to something he has taught, he said "not to something that I have taught to these students, but I teach to scholarship students and they learned different Vedic methods, and this is like those only."

What this indicates to us is that the teacher is unaware of the mathematical logic behind this process. As what the method precisely does is find the square of first term, square of the second term and add it to the double of the product of two terms. But apparently that is not the end of the story. The problem of knowing the logic behind this technique becomes more complicated. One evidence of that is seen when he comes back to using the identity for finding the squares of a number after the small detour of "easier techniques", he learns that the students are finding it difficult to add three numbers at a time. His suggestion to solve the problem is very similar to the logic he used in the technique. While finding the square of 42, he introduces the following representation.

"You could make a mistake in adding these numbers, so write them to appropriate places.

1	6	0	0
	1	6	0
			4

Draw lines in between the digits and be careful in adding the digits of the same placevalue."

How different are the two techniques? Just as one adds 54 to 54, one would realize the similarity. It is not likely that the teacher did not notice this because it was difficult, but perhaps because the teacher was not looking for it. The technique works when different numbers are substituted, and that is the dead end of mathematical exploration. The understanding of mathematics does not include – an attempt to unpack the logic of something or connect the two techniques to understand the logic. The teachers are away from mathematical logic in many such small tasks of teaching because they are not looking for it, and because they are not looking for it, they don’t know the logic. The question goes back to square one, whether learning to find the logic is mathematics or knowing the logic is. It appears that school mathematics is not about either. It is about learning to apply the logic embedded in the structure of the identity and not in the concept.

4.5 Encounters, Initiations and Management

I defined mathematics horizon as "a projection of mathematical meanings, topics, and structures present in the curriculum into the mathematics extending beyond the support of the curriculum materials concerning a particular location of instruction, such that it enables meaningful learning of mathematics." The encounter in mathematics is an instance that has potential to reach to this horizon, i.e., achieve access for meaningful learning for everyone in a classroom. What would be meaningful learning of algebraic identities? Would it mean to remember what happens to the first term and the second term? Or would it mean understanding the definition of the algebraic identities? There are many questions, and this cross-case analysis gave answers to some of these.

Definition of algebraic identities.

The textbook (See Figure 4.8) defines identity as “the equality relation in which, the left-hand side and right-hand side remain equal no matter what values are given to the variables is called an identity.”

Thus, no matter what values we give to *a* and *b*, the left hand side remains equal to the right hand side. Hence, we call this an identity.

The equality relation in which the left hand side and right hand side remain equal no matter what values are given to the variables is called an identity.

In this sense, the following equalities are identities.

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$(a + b)(a - b) = a^2 - b^2$

Note that, the equation $x + 5 = 8$ is true only when we give x the value 3. For any other value of x, the two sides of the equation are not equal.

Figure 4.8: Definition of algebraic identity

What did we see in these cases of teaching? We saw that algebraic identities were not given a general understanding, despite this definition in the textbook, the algebraic identity was equated to the identity for a square of a binomial. So the first question is whether mathematical entities need definition and what do teachers think about it? Mathematical definition highlight structures, and describe those with minimum information. Do our mathematics teachers need to know that when we teach mathematical objects, we understand their definition and not just their examples? Recognizing that learning a definition is part of learning the concept is an example of mathematical sensibility – knowledge of norms and nature of the discipline. When the teachers introduced identities, this is what I see:

- Out of the six cases of teaching, in three cases the teachers decided to use the defintion of identity.
- This particular decision presented a challenge in front of them.

- The three teachers who decided not to talk about the definition, introduced identity as a product of two identical binomials and then giving the expansion for $(a + b)^2$, which would be giving an example of the identity.
- These three teachers, never went back to defining the identity, and stayed at the level of examples.
- Soon the examples became formulae.
- The three teachers who wanted to give a generic definition were stuck in their next steps:
  - The teacher who thought identity is like identity card, fumbled upon what is identical in identity, and ended up highlighting how two binomials are identical.
  - The teacher who understood the definition but was stuck at the word "expansion", decided to go back in the domain of multiplication of whole numbers and re-define that multiplication as expansion, and therefore calling 21 as expansion of 3×7.
  - The teachers who told her students that when the left side is same as the right side, it becomes identical, was stuck at how can she show that it remains identical. This is what she said: " *Can someone tell me what is the meaning of Identity? And how can we prove that something is identity? How can we show something on the left is identical with something on the right side?* " She started with all the right questions. But what it means to show equality in the concrete

sense – it meant learning a formula. This is how she continues: *" What happens in identity? See what we are going to learn today is a formula; it is called formula for expansion, expansion formula."*

These instances illustrate two layers of the mathematical horizon. The first was about deciding the role of definition in introducing mathematical objects, and second was once you choose to use the definition, how do you make sense of it and then persuade students to make sense of it. All these teachers used the same textbooks. Each of them had an opportunity to create access for meaningful mathematics, and they tried and fumbled at different locations.

Based on algebra education research, the different openings of the concept suggest two main structural understanding needed for effective instruction. The first is the demarcation between formula, equation, identity, property, and function, as they all could be represented as a product of two quantities. And the second is to understand what does identity really mean? What is it that remains identical? Knowing multiple meanings of a single denotation is part of specialized knowledge needed for teaching mathematics efficiently, but the horizon is formed based on this understanding and further building on this to make sense of equality within the identities. The problem of inequality also becomes the part of this horizon as mathematical norms suggest counterexamples as an extension for a meaningful understanding of the concept.

Noticing variables.

As the definition of the encounters suggests in this study, one possible encounter that the teachers missed was an opportunity to see those letters as variables. RM's question that how do we know that the left and the right side would always remain the same, needed the understanding of those letters as variables. She began with that question, however, turned back to the formula. Again when RM started the geometric representation for the square of a binomial, she couldn't use it as verification. She made it into a problem of finding sums of areas (See RM's transcript, p. 149). The reason she was unable to do it, because she needed the idea of a variable, to understand the diagram generically.

The teachers' use of representation partially illustrated the idea of the variable, but it happened without them noticing it. Look at the following illustration that the teacher gave:

$$(first\ term\ +\ second\ term)^2$$

$$=\ (first\ term)^2 + 2\ \times\ first\ term\ \times second\ term +\ (second\ term)^2$$

One could show this but highlight a different explanation, like why the two sides could remain the same. Asking students to look at the structure of this and raising the question would the left, and the right side will always be equal and why? Interestingly, teachers made these colored writings on chart papers, but to remember the formula. Now, this particular decision is wholly situated in their teaching experience. In the interview – each teacher said, the students forget the formula and therefore they have to emphasize this again and again. When I asked them what do they mean by students forget the formula, how they know that, these are the responses the teachers gave.

$$(2a + 3y)^2 = 2a^2 + 6ay + 3y^2$$

$$(x + y)^2 = x^2 + y^2$$

They also shared identities with a negative sign. The question is why students do it? Is this solely the problem of identities? Or it is a problem carried from arithmetic structures like other errors described in section students' errors in algebra. The point here is that the teachers' notice that the students make such errors, their experience is enabling them to foresee this particular issue around identities. Although, their experience is not helping them why students do this. The teachers' thinking was, students forget, so let us make them remember, but why do they forget, that question was not part of their experiential learning.

I describe in detail, the role of teachers' experience in deciding what to do with varied students responses in the last chapter. I define a resource that teachers make use of and is called Professional Practice Knowledge.

Another perspective through which one could analyze the teachers' actions is the perspective of what is understood as mathematical in identities. The teachers were concerned about students remembering the formula but not about what they learned in terms of mathematics from the representation. The justification they gave for using the representation of the area or writing the terms in visually explicit ways is not situated within the logic of the concept of identity, rather on how the identity looks. The teachers wanted students to learn how the identity looks. When asked why this structure works, why adding the square of the first term, and the second term and the sum of the product of the two terms gives us the square of the sum of the two terms, all of the teachers said “it has been working for ages” and used numerical examples to show the equality. Like what AM did, she decided to understand the idea of whole number multiplication. This move by AM exhibits her understanding of mathematical practice. She tried using the meaning of multiplication to understand why the expansion works. If multiplication is understood as repeated addition, then to understand $(a+b)(a+b)$ one would think that $(a+b)$ needed to be added $(a+b)$ times. And therefore, one could add $(a+b)$, a times and then add $(a+b)$, b times. Adding $(a+b)$, a times would mean $(a+b) \times a$ and adding $(a+b)$, b times would mean $(a+b) \times b$. This same explanation otherwise was given as a heuristic in the instruction. AM's attempt showed interaction with the mathematics at the horizon.

Access to mathematics.

In the last section (p.155-158) we saw the work of teaching needed to understand the logic behind two techniques. How different were those two techniques? The teacher did not notice the logical similarity. Why? When asked he said, "Because one is a technique in mathematics and other is mathematics." The technique works when different numbers are substituted, and that is the dead end of the mathematical exploration for a technique. The question goes back to square one, whether learning to find the logic is mathematics or knowing the reasoning is. It appears the school mathematics is not about either. It is about learning to apply the logic embedded in the structure of the mathematics, where the structure is seen without its conceptual basis.

CHAPTER 5

Encounter with Horizon Content Knowledge: Case of Teaching Multiplication and Division of Fractions

This chapter reports on an analysis of in-service secondary teachers' use of representations for teaching multiplication and division of fractions. Out of 13 teachers who were part of this study 5 teachers taught the topic of fractions division and multiplication. Fractions and its operations is one of the most challenging topics of middle school. The teachers used representations, diagrams to make it understandable for students. In this chapter, I present what the teachers did with those representations, and then discuss the kind of encounters they experienced while teaching operation on fractions.

For introducing fractions multiplication and division, teachers built upon representations given in the Maharashtra state and NCERT textbooks. While doing so, they developed mathematical explanations around these representations so that students can make sense of algorithms involved in these operations on fractions. It was observed that the teachers knew procedures for multiplying and dividing two fractions; however, they encountered a problem with the explanation, especially in connecting procedures with the representations. While using a particular model, context or representation to explain a procedure, teachers seem to be looking for parallels to "steps" within the procedure to actions in the representation or models they use. The dynamism within the representation of multiplication of fractions was conjectured as repeated addition, so the illustration was about adding repeatedly. This created conflicting conclusions for multiplication of two fractions. With representation for the division of fractions, the teachers built two kinds of explanations – partitive and quotitive. However, identifying situations with quotitive meanings was not dealt with an overemphasis on the partitive meaning of division in explanations for division by fractions confounded the real understanding of division of fractions.

I first provide descriptions of the teaching, that is summarised in the paragraph above and then use the descriptions to understand encounters and teachers' knowledge resources in managing those.

5.1 Fractions Division and Multiplication in Mathematics Education

Ma (2010) illustrated that inadequate understanding of the procedure impedes designing of representations. Although, the teachers in the study showed confident knowledge of procedures they still created flawed explanations for the representations. Digging deep into the teachers' mathematical explanations around the representations, their meaning-making of students' responses and choice of teaching trajectory or examples indicate that making sense of dynamism within representations (how actions within representations exhibit steps in a procedure) and seeing its parallel in the algorithm requires a kind of mathematical inquiry that is external to school mathematics.

In the upper elementary grades, the emphasis changes from a focus on the additive to the multiplicative structure of numbers and relationships. Therefore, the students face new kinds of numbers, fractions, and decimals and rely on multiplication for their underlying structure. These numbers are useful in making new kinds of comparisons that rely on two measures (or more) of phenomena. For example, which is cheaper, 5 kg tomatoes for 60 or 8 kg for 90? Simple subtraction will not resolve the issue. A comparison taking into account both the quantity and the prices is required. Having two measures, instead of one, on attributes that we are trying to compare leads us into the world of derived measures (Vergnaud, 1998) that are often per quantities or rates. These new mathematical ideas contain intellectual challenges for the students that are as conceptually difficult as anything anywhere else in the school mathematics curriculum.

Many of these mathematical ideas will not reach their full maturity in the middle years, but it is in the middle grades that the firm foundation for understanding is laid. NCTM (2000) proposes that

> *"...it is here (middle grades) that students have time to experiment, to ponder, to play with mathematical ideas, to seek relationships among ideas and concepts, and to experience the power of mathematics to tackle problem situations that can be mathematized or modeled. It is also here in the middle years that the serious development of the language of mathematics begins."*

We are far far away from achieving this level of learning in the middle grades. The study of mathematics involving quantitative reasoning invariably means reasoning about mathematics in contexts and representations. Part of what makes mathematics so powerful is its science of abstraction from real contexts and symbolic representations. To quote Lynn Steen (p. xxiii, 1997):

> *"The role of context and representations in mathematics poses a dilemma, which is both philosophical and pedagogical. In mathematics itself.... context or representations obscures structure, yet when mathematics connects with the world, context provides meaning. Even though mathematics embedded in representations or contexts often loses the very characteristics of abstraction and deduction that make it useful, when taught without relevant representation it is all but unintelligible to most students. Even the best students have difficulty applying context-free mathematics to problems arising in realistic situations, or applying what they have learned in another context to a new situation."*

Fractions, decimals are topics in the middle grades where often contexts and representations are used to convey meanings. We focus on this aspect of teaching multiplication and division of fractions and analyze teachers mathematical explanations around these representations and contexts.

This study investigates teachers' encounters with mathematics at the horizon — mathematics as a discipline and unpacks teachers' management of such encounters. In this chapter, I report on how teachers make sense of representations used for multiplication and division of fractions, internalize them for pedagogical use and how they actually enact them, what encounters they face while doing so, and how do they manage these. In the following paragraphs, I first summarise students' conception around division and multiplication of fractions and offer a preview of research on teacher knowledge for teaching fraction operations. I present the representations given in the textbook and teachers' mathematical explanations around these representations. I present the encounters with HCK teachers face while choosing these representations and how they manage these. We conclude by eliciting HCK needed for teaching division and multiplication of fractions.

5.2 Students' Challenges in Multiplying and Dividing Fractions

Streefland (1991) proposed that early teaching episodes can and should be built upon realistic mathematics situations and contexts from the students' actual world. The suggestion was based on the thesis that students intertwine fractional ideas as they resolve realistic situations. Later, they can begin to use tools such as pictures, tiles, or symbols to think about and communicate about the fractional situations. Similarly, Kieren (1993) claimed that children build sophisticated knowledge of thought, informal language, and images on their ethnomathematical knowledge before making consistent use of conventional language, notation, or algorithms. However, Kieren suggested that, as needed, students return to prior types of thinking when they encounter unfamiliar or perplexing situations. It is not uncommon for students to flow back and forth between and among ethnomathematical, intuitive, and symbolic types of thinking.

The division of two common fractions remains a problematic aspect of children's middle school experiences with fractions (Capps, 1962; Elashhab, 1978; McMeen, 1962; Warrington, 1997). The two most discussed algorithms for the division of fractions are presented in Table 5.1.

Common-Denominator Method	Invert and Multiply method
$3 \div \frac{1}{4} = \frac{12}{4} \div \frac{1}{4} = \frac{12 \div 1}{4 \div 4} = \frac{12}{1}$ $= 12$	$3 \div \frac{1}{4} = 3 \times \frac{4}{1} = \frac{12}{1} = 12$

Table 5.1: Two algorithms for fraction division

The results from the two algorithms are mathematically equivalent. The common-denominator algorithm is based on a familiar definition for the division, whereas the invert-and-multiply algorithm is based on division's relationship to multiplication. The common denominator algorithm is often accompanied with a diagrammatic representation. See Figure 5.1.

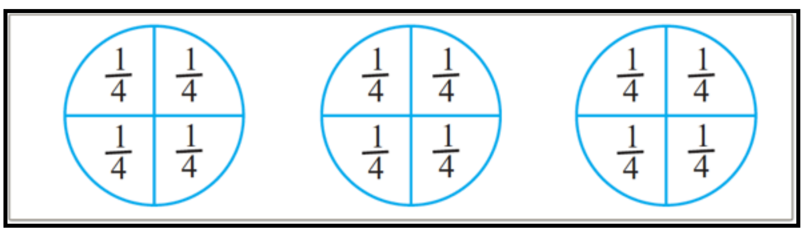


Figure 5.1: Common-denominator methods of division

The repeated-subtraction definition of the division is familiar to most middle-grade students because of their work with whole numbers. To begin the process, students must subtract fractions, which requires a common denominator. Then, they must determine how many subtractions can be made. To make sense of the invert-and-multiply algorithm, students must understand the inverse nature of multiplication and division (an algebraic idea) and use it to rewrite the expression. Proofs of the invert-and-multiply algorithm often include a foray into complex fractions an even deeper algebraic idea. In contrast, students can rely on whole number ideas for the division to understand the least-common-denominator algorithm.

In both these algorithms, I believe that it is complicated for students to comprehend the division aspect of the procedure. What does division mean, and what does it mean when we say division by fractions? Here, we present a brief overview of students' errors on division by fractions.

Procedure based errors.

The most common errors as recorded by Ashlock, 1990 and Barsh and Klein 1996 are around the invert and multiply procedure. As these rules are often rote memorized students invert dividend instead of the divisor, or invert both the dividend and divisor before multiplying numerators and denominators. This also indicates that the algorithm is seen as a meaningless series of steps, and unconnected with operation of division.

Hart (1981) reported that students think that division is commutative, and stated $\frac{1}{2} \div 1 = 1 \div \frac{1}{2}$. When asked how 2 is an answer for $\frac{1}{4} \div \frac{1}{2}$, they responded using invert and multiply $\frac{1}{4} \div \frac{1}{2} = \frac{4}{1} \times \frac{1}{2} = 2$. When counter questioned with $\frac{1}{2} \div \frac{1}{4}$, they said by commutativity it would be 2, and concluded that $\frac{1}{2} \div \frac{1}{4} = \frac{1}{4} \div \frac{1}{2} = 2$.

Interference with division meanings.

These errors occur from intuitions held about division. Research on the conception of operation indicates that students overgeneralize properties of operations in natural numbers to fractions (and rationals). Therefore they tend to use the partitive model of the division. In this model of division, an object or collection of objects is divided into a number of equal parts. For example, 15 packets of biscuits are shared among 5 friends equally. How many packets did each of them get? The three criteria that this model uses are as follows:

- The divisor must be a whole number
- The divisor must be less than the dividend
- The quotient must be less than the dividend

The dominance of this primitive model of the division has limited students' and teachers' ability to respond to division word problems (Fischbein, Deri, Nello & Marino, 1985). Some children argue that it is not possible to divide less among more. (Graeber et al. 1983).

Multiplication of fractions is a deceptively easy skill for students to learn. Often it is seen that students have difficulty with every operation on fractions except multiplication. In a study conducted by the University of Wisconsin, they found that 18 percent of a students' population could find the sum of $\frac{1}{2}$ and $\frac{1}{8}$, where 75 percent could find the product of $\frac{2}{3}$ and $\frac{4}{7}$. The answer, $\frac{8}{21}$, can be calculated without considering the meaning of either of the fractions or of the solution. Therefore, it is more challenging to find what really students understand about multiplication of fractions.

Since students can learn the algorithm for multiplying fractions with little conceptual understanding, the instructional goal for multiplying fractions should emphasize the interpretation of situations involving a product of two fractions, the modeling of those situations physically or pictorially, and the explanation of why the product, for example, $\frac{3}{4}$ and $\frac{2}{3}$ is $\frac{1}{2}$.

5.3 Teachers' Knowledge Needed for Teaching Fraction Multiplication and Division

What knowledge is needed to teach multiplication and division of fractions effectively, is a very challenging question. There are very few studies which actually address this specific concern. In the seminal work by Deborah Ball in her dissertation and its extension study by Liping Ma, pre-service and in-service teachers' understanding of division by fractions was discussed respectively. When asked to make a representation of $1\frac{3}{4} \div \frac{1}{2}$, the common response was to make a representation for $1\frac{3}{4} \div 2$. In both the studies in-service and pre-service, many teachers created a representation of following kind:

If we have 1 and 3/4 pizza left, and we have to equally split between the two, how much of pizza each one would get?

Some teachers also presented pictures for their representation. For example, the teacher Anne explained how she would give two one fourths and two one sixths to each person sharing pizza. See following example from Ball (1988):

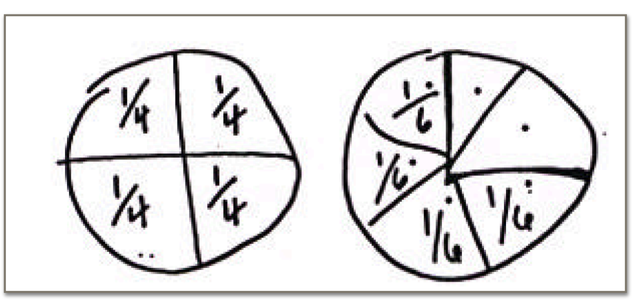


Figure 5.2: A teacher's response from Ball (1988)

Some of the observations these researchers made about teacher knowledge were as follows:

- The teachers saw the question as one about fractions instead of about division.
- The teachers confounded between everyday and mathematical language.
- The prospective teachers tend to confuse dividing *in* half with dividing *by* one-half, and they did not seem to be aware of the difference. This confusion went unnoticed even though the answer they got using algorithm was $\frac{3}{2}$ (double of $1\frac{3}{4}$) and often their representations got share of each person as $\frac{7}{8}$.

In many instances in the context of Indian teachers, I have seen teachers responding similarly. Ma (2010) suggested profound understanding of operations is needed to be able to design and use representations. In alignment with Duckworth (1979), a former student and colleague of Piaget, she equates profoundness with making connections, building multiple perspectives and understanding mathematical ideas longitudinally.

However, an attempt to make connections — for example connecting ideas of division in representation with its algorithm, developing mathematical explanations for the representation and its association with the concept are poorly studied. In this study, I attempt to throw some light on these aspects of knowledge for teaching division by fractions.

Apart from these, some research has been carried out to see how knowledge of students' ideas and errors, can impact teachers' knowledge of teaching division by fractions. (Tirosh, 2000; Fennema et al. 1996; Warrington 1997). However, we also propose that to notice and understand students' alternative thinking, teachers themselves require in-depth knowledge of concepts and its connection with other related topics.

Sub-construct theory of meanings of fractions and fraction operations.

The fractions have five subconstructs as defined by Keiran (1976), which means fractions notation could be understood in five meanings. Those five subconstructs denote fractions as

- a part of a whole
- a ratio
- a measure
- an operator
- a quotient

Each of these subconstructs emphasis different meaning of fraction notation. The part-whole subconstruct is defined as a situation in which a continuous quantity or a set of discrete objects are partitioned into equal size parts (Lamon, 1999; Marshall, 1993). So, the fraction represents a comparison between the number of parts of the partitioned whole to the total number of parts the whole is partitioned. Through part of a whole meaning m/n represents m parts out of equally partitioned whole in n parts. From this definition, the numerator of the fraction must be less or equal than the denominator. The ratio subconstruct of fractions conveys the notion of a comparison between two quantities; therefore, it is considered comparative index, rather than a number (Carraher, 1996). In the measure interpretation of fractions, the fraction is seen as a number formed by repetition of units.

So in $\frac{m}{n}$, the unit is $\frac{1}{n}$ and it is repeated *m* times. In the operator interpretation of fractions, they are regarded as functions applied to some number, object, or set (Behr et al., 1992). So $\frac{m}{n}$ as on operator, will scale up the quantity *m* times and scale it down to *n* times. Within the quotient subconstruct, any fraction can be seen as the result of a division situation. In particular, the fraction $\frac{m}{n}$ indicates the numerical value obtained when is *m* divided by *n*, where *m* and *n* are whole numbers (Kieren, 1993).

One of the speculated reasons for the confounding understanding of division of fractions is the explicit emphasis on the part-whole meaning of fraction notation. Although there has been much research on how to make sense of fractions using different sub-constructs of fractions, there is not much written about how these different meanings impact meanings of operation of fractions.

Guiding questions.

In this chapter, I present the analysis of teachers' use of representations and mathematical explanations in teaching division by fractions. The two specific questions that will be answered are as follows:

1. What encounters did teachers face while chosing representation, giving explanations to teach division and multiplication of fractions?
2. How do these representations and explanations connect with each other and with the algorithms taught to students?

In the following paragraphs, I first present what representations are given in the textbook, how they are used in the classroom and what mathematical explanations are given to the students. Then I use these descriptions to discuss the mathematical horizon and encounters with it.

5.4 Teaching Operation on Fractions

Fractions multiplication and division are difficult topics to teach and learn in the middle school years. The textbook and curriculum reform in India has been partly addressing that. The curricula post-NCF (2005), attempt to be more child-centered and therefore include more representations to facilitate child-centered pedagogy. In this section, we first present the illustrations given in the textbook, that teachers use to build upon. Then we move on to teachers' representations and mathematical explanations around those.

Division and multiplication of fractions in textbook.

As I mentioned earlier, the representations for division of fractions are given through two meanings — (1) repeated subtraction or grouping by finding equivalent form of fractions such that the denominators are same and (2) invert and multiply to find the quotient.

The NCERT textbook, uses the former to introduce division of fractions. For example to divide $3 \div \frac{1}{4}$, following illustration is used and the question asked is how many $\frac{1}{4}$ parts do you see?

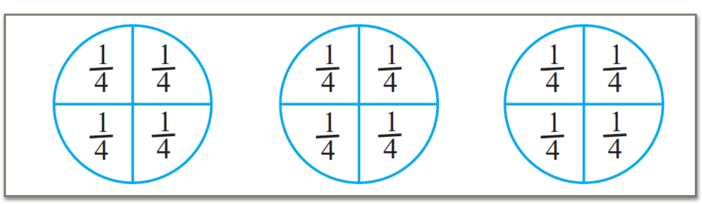


Figure 5.3: Textbook illustration for whole number divided by fractions

Here, the expectation is that the students count and say 12. Once students are done counting, and have arrived at 12, the textbook offers an observation —

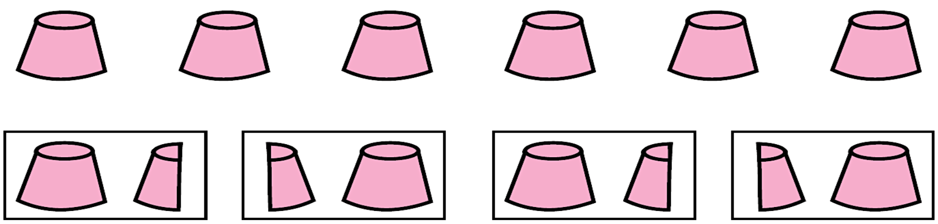
Observe also that, $3\times\frac{4}{1} = 3\times 4 = 12$. Thus, $3\div\frac{1}{4} = 3\times\frac{4}{1} = 12$.
Find in a similar way, $3\div\frac{1}{2}$ and $3\times\frac{2}{1}$.

Table 5.2: Textbook algorithm invert and multiply

What precisely the observation says that $3\times\frac{4}{1} = 3\times 4 = 12$. One needs to note here that the two expressions are equivalent as they lead to same answer, till that part this derivation works fine. However, why in the first place inverting and multiplying gives the same as as equal grouping or repeated subtraction is not discussed here. In the classroom episode reported later in the paper, one of the students suggests, that $6\times 2 = 12$, and can they use that expression to solve this problem. The student's question creates an interesting opportunity for the teacher to clarify why really this specific multiplication matters, but it also indicates how that student has noticed the absurdity of this observation. Later, in the textbook a fact is given — "Dividing by a fraction is the same as multiplying by its reciprocal." This textbook uses the term reciprocal and then defines it separately. No or little attention is given to the meaning of division with fractions and no connections are made between division with fractions and division with whole numbers. Each is treated as a special case. Since division with fractions is most often taught algorithmically, such representations are a strategic site for examining the extent to which teachers understand the meaning of division itself. The Maharashtra or state textbook, uses the term multiplication inverse instead of reciprocal, but other than that the logic of moving from division by making dividend and divisor with equal denominators to invert and multiply is exactly the same. See Figure 5.4.

Example : There are 6 blocks of jaggery, each of one kilogram. If one family requires one and a half kg jaggery every month, for how many families will these blocks suffice?

One and a half is $1 + \frac{1}{2} = \frac{3}{2}$



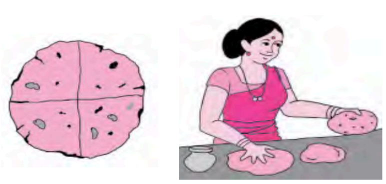
Let us divide to see how many families can share the jaggery.

$6 \div \frac{3}{2} = \frac{6}{1} \div \frac{3}{2} = \frac{6}{1} \times \frac{2}{3} = 4$ Therefore, 6 blocks will suffice for 4 families.

Figure 5.4: Word problem illustration

And some more similar representations as these are given.

Example : Here is one bhakari. If each one is to be given a quarter of it, how many will get a share?



A quarter means $\frac{1}{4}$.

As we can see in the picture, we can get 4 quarters from one bhakari, so it will be enough for four people.

We can write this as $4 \times \frac{1}{4} = 1$.

Now, we shall convert the division of a fraction into a multiplication.

$$1 \div \frac{1}{4} = 4 = 1 \times \frac{4}{1}$$

Figure 5.5: Word problem illustration numerical

The teachers in this study worked with these representations. I have identified certain gaps within the representations, and when teachers made sense of it, they stuck upon these gaps. The following paragraphs describe what mathematical explanations were built around these representations to justify the algorithm for division by fractions.

5.5 Teachers' Use of Representations and Mathematical Explanations.

Out of the five cases of teaching three did not attempt to explain what is the division of fraction, and therefore did not use any representation at all. They provided the rule "invert and multiply" and offered a symbolic justification. During their interviews, they described a rationale for teaching only by rule. Remaining two teachers used the exact same representation as given in the textbook. One teacher encountered a student question that required an explanation for why division by a divisor is the same as multiplication by its reciprocal. Between these two, one teacher showed depth in the understanding division by fractions and hence used a comprehensive pedagogy and explanation to illustrate the same.

Concerns with textbook representations.

During the interview, the teachers were asked about how they make sense of representation in the textbook. The three teachers who did not use any symbolic representations responded they had no specific reason other than "this is the way I learned division of fractions." However, when asked further Samreen teacher thought that the representations given in the textbooks has some limitations, and she describes them here.

"… the representation given in the textbook is confusing. It changes 1 to $\frac{2}{2}$ *or 3 to* $\frac{12}{4}$*, which is confusing for students. Also one can do this only when the first number [dividend] is a*

whole number. Like, students can use this method for $2 \div \frac{1}{3} = \frac{6}{3} \div \frac{1}{3} = \frac{6 \div 1}{3 \div 3} = 6$*, where first number is a whole number. But how can they use this method for say* $\frac{3}{5} \div \frac{1}{4}$*?"*

She has understood the representation given in the textbook but has not internalised the equal denominator aspect of it. The example she cited is possible to represent using this meaning — it requires both the fractions to be converted into equivalent fractions such that they would have the same denominator. The part of the worry would be how to make sense when the numerator is a fraction and not a whole number.

Another teacher was not one of the teachers who actually taught operation on fractions in her class. However, she contacted me after I finished the data collection in her school. She wanted to discuss the unit operation on rational numbers. At that same time I was working with Bhinde sir, so I asked some similar questions to her about representations in the textbook. Asmita, mentioned a different limitation than Samreen, but one connected to that. She understood that making equal denominators is what is suggested in the representation. However, she was not clear why, and also thought that once denominators are made equal and if the numerator of the divisor is not 1, then further simplification is not possible.

"…the representation given in the textbook does not make sense. See we make denominators equal in addition and subtraction of fractions. We don't do that in multiplication. That is what makes them different. So making denominators same for division does not make sense. Also if divisor is not 1, then it is not possible."

When prompted to illustrate her claim using an example — this is what she explains.

"Okay. I need two numbers. Say $3 \div \frac{5}{7}$*. Now I have to get 7 as a denominator for 3, like they have shown in the textbook. So I get, ummm,* $\frac{21}{7} \div \frac{5}{7}$*. Now I can divide 7 by 7, but how to divide 21 by 5. How can the numerator be a fraction?"*

Asmita, has internalized the equal denominator part of the representation, but highlights that $\frac{21/5}{1} = \frac{21}{5}$ would require different kind of convincing with students. Her reference to associating equal denominators to operations such as addition and subtraction was interesting, where operation is not possible as the size of the unit is different. It also indicates that for her the operation of division and multiplication are not connected with operation of addition and subtraction. Asmita is the same teacher who redefined multiplication of whole numbers when encountered with term "expansion" in the context of algebraic identities.

Bhinde sir directly taught invert and multiply as a rule without giving any representation or explanation, he used his own language to state the "invert and multiply" rule. It created some confusion in the classroom.

In the classroom, Bhinde sir said,

"to do division of fractions, you write first number as it is and you write multiplication sign, and you change the numbers in the second number [he used numbers in a number — referring to numerator and denominator in the fractional number] *and write those next to multiplication sign. This is called opposite operation. Then just use the multiplication rule."*

He did not introduce the idea of reciprocal or the term reciprocal like others, he said "change the numbers in the second number". Many students in this class used completely new sets of numbers to replace the divisor and then multiplied with the dividend. This led to a lot of repetition of that rule in the class.

When asked why he said change the numbers, and also did not use the representation, he said,

> *"Maybe I should have used the word interchange, the numbers in the second fraction get interchanged. The representation given in the textbook does not explain this interchange. The explanation for interchanging of numbers comes from opposite operations. I think because division and multiplication are opposite action, we change the numbers and multiply."*

Bhinde sir has sensed that there is something about multiplication and division being opposite operation, but he hasn't been able to pin point why the *interchange* works.

Using textbook representations.

Two teachers used the exact same representation that was given in the textbook. They used the same numbers and mentioned the same observations as presented in the textbook to connect common denominator representation to "multiply by a reciprocal" representation. The descriptions below will illustrate how the two varied in their use and understanding of teaching division by fractions.

Damini was one of them; she used the same observations as in the textbook. (Refer to Figure 5.1 and 5.2). The comment in the textbook mentioned that $3 \times \frac{4}{1} = 3 \times 4 = 12$.

By common denominator method, the answer was 12, by inverting and multiplying too the answer is 12. The logic of the argument goes like this — the two expressions are equivalent as they have the same value, and untill this part, this derivation works fine. However, why inverting and multiplying gives the same value as equal grouping or repeated subtraction (common denominator method) is not discussed here.

In Damini's classroom, one of the students suggested, that as $6 \times 2 = 12$, they could use that expression to solve this problem, he said, "we get the answer 12". The student's question created an interesting opportunity for Damini, where she could clarify why really this specific multiplication matters. She initially responded to that student as one can't use any number that gives the answer 12; and moved on in completing her explanation. But after some time she came back to the question, and this is how she addressed that question.

> *"The answer is 12. And it has come as in number 3 there are 12 pieces of* $\frac{1}{4}$*. And we are solving the problem* $3 \div \frac{1}{4}$*, so we want to find how many* $\frac{1}{4}$ *in 3. Like you do in* $18 \div 9$*. What is the answer here?* [students say 2 in chorus] *Two. Because we are finding how many 9s are there in 18. How many 9s?* [students say 2 in chorus] *Two nines. Similarly when we divide here by fraction we find how many* $\frac{1}{4}$ *in 3. Is that clear to everyone?* [students say yes in chorus] *Clear? See when you move on you will see many complicated fractions for division, and every time you will not be able to find how many of second number in to the first number* [pointing at $\frac{1}{4}$ and 3 respectively]*, so we see that there is pattern. The two problems —* $3 \div \frac{1}{4}$ *and* $3 \times \frac{4}{1}$ *always give the same answer. See we notice*

that now, we notice it early only. So we can use it in the problems. Therefore, we did 3×4*, we did not first decide that we want 12 as an answer, we say that* $3 \div \frac{1}{4}$ *and* $3 \times \frac{4}{1}$ *always give the same answer. That is why. Is that clear?"*

This question from the student did create an encounter where the teacher needed to respond why 3×4. In her interview when asked about this episode she said first she thought the student is doing mistake by giving $6 \times 2 = 12$ as suggestion for answer, but she said, she realised a little later that may be he is confused about why do we multiply by 4. This noticing by Damini[5], provided a hint into how she thinks about the representation as well as about what students say. This indicated that she did doubt the representations, and hence noticed the students' query as not just a wrong answer but as a possibility of thought due to the use of representation. To clarify the meaning of division, Damini used examples in whole numbers first, and also highlighted the limitation of that meaning with fractions.

How many $\frac{1}{6}$ in $\frac{1}{2}$?

Parveen, another of the two teachers, was the only one teacher who presented depth in her inquiry, and designed teaching aid to illustrate her own understanding. She had cut outs of different unit fractions, fractions with denominator one. She is the same teacher who used the paper cut-outs for algebraic identity (referred in chapter 4) and cut-outs for circle (referred in chapter 3). Look at the following representation used by her. (See

[5] Damini was also part of another mathematics education dissertation study in 2009. She was intriduced to meanings of division and multiplication in that study. She was also part of discussions on meanings of fractions.

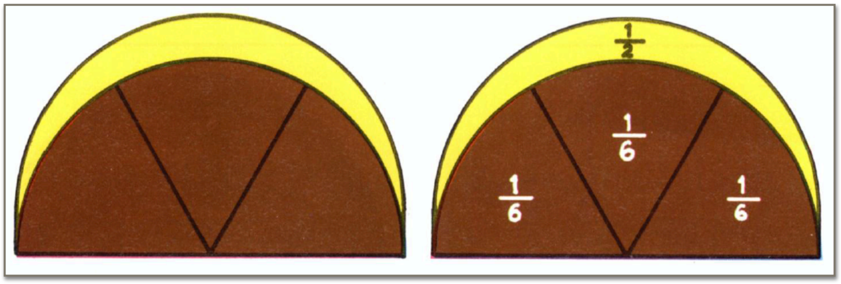


Figure 5.6: Paper cuttinngs for fraction division

She asked the question how many $\frac{1}{6}$ s in $\frac{1}{2}$? Like Damini, but much early in her instruction she pointed out that this question is same as the division problem earlier. She used a whole number division example, just like Damini to connect with the questions of finding how many as a question of division. In her interview when asked why she used this teaching aid, she demonstrated a depth in understanding division overall.

> *"Division is tricky. We can teach the division in two ways. One of the way is forming groups, like forming groups of certain size, like taking a classroom of 56 students and making groups of 4. This is a division question. At the same time, you could ask the question differently, like making parts. You could ask if I made 4 groups in 56 students, how many students would be there in each group? Here you go on dividing…assigning one student to each group, that is your process, but the answers to both the questions are exactly the same."*

What Parveen is referring to, is nothing but the two interpretations of the division — one is measure understanding and other is partitive understanding of the division. As we move on in the number systems we realise that the fractions and rational numbers it is always measure meaning of division — as fractions in itself involve a process of division.

Parveen spoke a lot about making and using teaching aid, and she showed us how she uses various unit fractions to do the same.

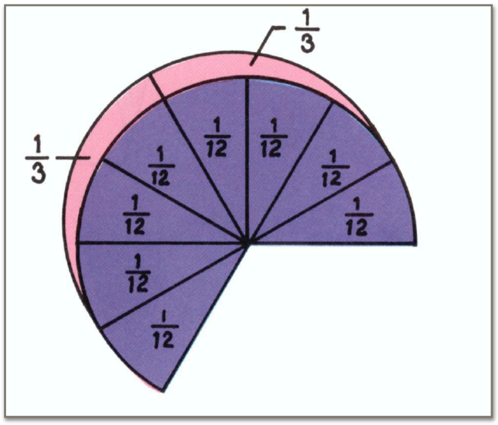


Figure 5.7: Circular discs for fraction division

Here is one illustration where she is measuring how many $\frac{1}{12}$ are there in $\frac{2}{3}$. When asked why does she think the invert and multiply works, this is what she said —

"Multiplication and division are opposite operations. I won't be able to prove why they are exactly the same, but I know fractions involve division in them and that has to do something with it. See if you open the division in fractions you would get something like this, say [thinks for a while and then writes] $\frac{2}{3} \div \frac{3}{4} = (2 \div 3) \div (3 \div 4)$*, now division of division would mean multiplication, I think [smiles]."*

Symbolic derivation.

Tope who did not use the representation given in the textbook, provided some kind of justification why the division of fractions will use "invert and multiply" rule. Presented in Figure 5.8 is a derivation that he used in the interview. He also did these derivations in the classroom mentioning what they learned in the multiplication of fractions. After he finished writing the derivation, he asked students to pay attention to step number 3 in the derivation. Each expression after the first equal to sign is called as a step for solving a problem. This terminology is common to Indian classrooms. He said,

> *"if you look at step 3, it is going to be one, and that is always going to be the case as we are multiplying reciprocal of the denominator fraction or the second fraction in the division problem. That will always be one, and therefore we don't need to write that. We can just write the expression in the top —* $\frac{3}{4} \times \frac{6}{1}$ *to find the answer. So students only need to remember invert and multiply."*

$$
\begin{aligned}
\frac{3}{4} \div \frac{1}{6} &= \frac{\frac{3}{4}}{\frac{1}{6}} \\
&= \frac{\frac{3}{4}}{\frac{1}{6}} \times \frac{\frac{6}{1}}{\frac{6}{1}} \\
&= \frac{\frac{3}{4} \times \frac{6}{1}}{\frac{1}{6} \times \frac{6}{1}} \\
&= \frac{\frac{3}{4} \times \frac{6}{1}}{1}
\end{aligned}
$$

Figure 5.8: Symbolic derivation

I was not sure whether students understood the derivation at all, but he did. When I asked whether he can use algebra to show that this will be true for any two fractions, he immediately said yes, but did not present any algebraic derivation.

5.6 Encounters, Initiation and Management

Each teacher who was part of these cases had given some thought about the division of fractions. I realized during the interview, that fraction division was treated much more seriously than say, algebraic identities. All of them agreed that operation on fractions requires "special teaching." What they meant by this was some preparation from their side is needed before they teach this topic. The representations in the textbook is a recent change. The teachers said earlier there were no pictures for the common-denominator method, and one teacher said the technique of common-denominator was not in the textbooks previously. The teachers showed a lot of interest in talking about fractions. There were some teachers who also contacted me after the data collection was over (like Asmita mam), and my colleague went and observed their classes, after the data collection was over. These efforts from them gave me the impression regarding how serious they are about the topic of division by fractions.

While teachers worked through the teaching and interview, I observe some encounters that they faced and managed. In the following paragraphs, I describe these and illustrate their knowledge resources while managing those.

Making sense and connecting of generic aspects of representation.

The two representations given in the textbook gave the teachers an opportunity to explain the meaning of division. Especially the common denominator method, where the diagram used by the textbook, and various teaching aid designed by Parveen mam involved measuring one fraction with another. The teachers who use it thought it explained the meaning of division to the students, and those who did not use it said that it only added complexity – as each fraction needed to be re-written with its equivalent form. It seemed each of them understood the representation. Said this, I am not sure whether any of them followed the generic aspect of the representation. When Asmita and Samreen pointed out issues with the representation, they identified the symbolic complexity of it, and also a glitch that Asmita pointed out.

> *"Okay. I need two numbers. Say* $3 \div \frac{5}{7}$*. Now I have to get 7 as a denominator for 3, like they have shown in the textbook. So I get, ummm,* $\frac{21}{7} \div \frac{5}{7}$*. Now I can divide 7 by 7, but how to divide 21 by 5. How can the numerator be a fraction? "*

She notices that $\frac{21 \div 5}{1} = \frac{21}{5}$ requires an explanation, and may be she is not aware of that but, she knows that it requires an explanation. However, the process internalized in the representation was not generalized by any. The process demands to measure one fraction by another. Even though the teacher said how many 9s will make 18, and it had the notion of measurement implicitly present in it, they did not quiet generalized the same with fraction division. What does it mean to measure a fraction by another fraction? What if the dividend fraction is smaller than the divisor fraction – what does that mean? So even though they said the common denominator method explains the meaning of division, they relied on whole number examples to illustrate that. An instance like how many halves are there in a quarter was at the horizon of this specific meaning of definition by fractions, but it was never referred or thought about. The generic structure present in the representation is part of the horizon. The teachers' frustration about the representation indicated they did not have access to it and hence gave explanations like this.

"Like you do in $18 \div 9$*. What is the answer here? [students say 2 in chorus] Two. Because we are finding how many 9s are there in 18. How many 9s?* [students say 2 in chorus] *Two nines. Similarly when we divide here by fraction we find how many* $\frac{1}{4}$ *in 3. Is that clear to everyone?* [students say yes in chorus] *Clear?* ***See when you move on you will see many complicated fractions for division, and every time you will not be able to find how many of second number in to the first number*** [pointing at $\frac{1}{4}$ and 3 respectively]***, so we see that there is pattern.****"*

The teacher is valuing the patterns that she observes as mathematical, and seems to be saying that one also gets an answer by "invert and multiply." Using common denominator method is not possible for all kinds of fractions so that the students can use "invert and multiply". So the gist is – the meaning of division is to find how many second number in first, but the meaning cannot be used every time, so there exists an algorithm that gives the same answer as what one would get using the meaning of division. And as it's seen working for number of examples, it should always work. This is where the access ends, what is "invert and multiply," it is a procedure? Does it have a meaning? Why does it work in the division? If it's because of the opposite nature of multiplication and division, would something similar work in multiplying two fractions too? All these questions form the horizon of mathematics for this topic.

Encountering mathematics in students' question.

The student's remark that 12 is also the answer to the problem 6 times 2 was initially disregarded by the teacher (refer to Damini's episode), as "you can't use any numbers." However, later the teacher came back to that query – although not explicitly referring to that student, but she connected the mathematical query that the student raised. She responded that not only in this particular problem but in any problem, the number of times the second number (divisor) is present in the first (dividend) is the same that they get after doing the invert and multiply algorithm. This was not a proof, but the teacher allowed the possibility that the way connection is presented in the textbook – "they have the same answer so the algorithms must be equal" could have contributed to the student's confusion. This sensitivity from the teacher shows that she could manage the encounter more productively than "you can't use any numbers," because, somewhere she has noticed the incomplete nature of the connection between the two representation given in the textbook.

This is an important acknowledgment, where the teacher knows what textbook claims as the explanation of the algorithm is actually not. And she adds into that the pattern seen after doing many examples would justify the algorithm. Interestingly, I can not conclude whether it was that she always noticed the two representations were unconnected – at least are not explanations for each other, or she sensed that when the student made the remark.

CHAPTER 6
Conclusion, Implications and Limitations

Teaching mathematics at the middle grades level is a complex task, and it requires specialized knowledge of mathematics. My motivation for this study was to investigate the role of teachers' mathematical sensibilities in the practice of teaching. In particular, following research questions guided the inquiry behind this dissertation.

1. What kinds of "encounters with HCK" arise in classroom teaching? And: How do teachers manage them?
2. How are students' opportunities for learning shaped by teachers' encounters with HCK?
3. What kind of mathematical knowledge do teachers exhibit while navigating encounters with HCK?

In order to address these questions, I drew on the construct of horizon content knowledge, and defined the mathematical horizon as the projection of the mathematics in the curriculum to the mathematics extending beyond the supports of the curriculum materials for a particular location of instruction. In particular, throughout these case studies, I have attempted to highlight the knowledge that teachers utilized as they navigated the mathematical territory.

In this chapter, I discuss the broader conclusions of my analysis, implications, and recommendations for future research. However, before I begin those discussions, I would be remiss if I did not also honor the teaching practices that my participants skillfully exercised over the duration of this study, particularly during those moments when faced with mathematical ideas in the curriculum or students responses. I begin my discussion of this study's findings, therefore, with some observations about the educational context in India and the participants' noteworthy teaching practices.

6.1 Educational Context in India

Indian students perform particularly low in accepted measures of mathematical skills. In a comparative study on mathematics achievement for 8th grade students among 79 countries (PISA, 2009), India ranked 78th, almost making it to the end of the list. India has struggled with issues such as school enrollment, basic literacy and sustainability in schooling since its independence. Therefore, the Indian government has taken many initiatives to bring every child into the school[6]. According to the 10th Annual Status of Education Report (ASER, 2014) now 96% of the child population between the ages of 6 to14 is entering the schools, but 73.1% of these students can neither perform the calculation as expected in the procedure nor apply the conceptual understanding to any real-life

[6] Different initiatives for quantity in education

problems. These results are significantly similar, and fairly independent of the medium of instruction (vernacular or English) and the setting (rural or urban) of the school. In a way, the resources and focus are exercised more on quantitative aspects of the education but sideline the efforts for quality in education.

For almost a decade now the educational community in India is voicing concerns about the poor quality of Indian education. Documents such as National Curriculum Framework (NCF, 2005) and National Curriculum Framework for Teacher Education (NCFTE, 2010) went through huge democratic iterations, consolidating the voices of teachers, students, teacher educators and parents. These documents show that the existing ways in which the students' learn mathematics constitute a root cause for weak quality of their learning. Out of the six vision statements envisioned for students' learning in the position paper on mathematics education in NCF 2005, three focus on mathematical problem solving and practices. Expectations include that children be able to pose and solve meaningful problems, use abstractions, perceive relationships, see structure, reason, argue the truth or falsity of statements, and understand the basic structure of mathematics. By pointing out the abstraction, relationships, basic structure and connectedness in mathematics, these vision statements are pointing to the nature of mathematics. This is different from the existing focus in mathematics teaching, which is on computation, procedures and remembering formulae. However through the research in reform mathematics education, we know that a reform such as this could be brought in the classroom only when the teachers' knowledge of mathematics for teaching and their views about nature of mathematics are aligned with the reform (Kennedy 2009, Ball & Cohen, 1999; Cohen, 1993)

6.2 Local Precision and Pedagogical Simplicity

Although some problems of teaching are identified, one of the challenges seems to be the ability to understand and transact the new vision in actual teaching. One needs to investigate what our teachers think and believe with respect to these reforms in education. Do they themselves consider posing and solving meaningful problems or using abstractions or perceiving relationships and structure is part of doing mathematics? The research in teacher knowledge is dealing with some of these concerns in order to determine what knowledge of mathematics teachers need to teach mathematics. It is generally agreed that teachers need to know more than simply how to "do the math" at a particular grade level.

The analysis of the classroom teaching demonstrated how prevalence of teaching that emphasizes procedures leads to inconsistent mathematical experiences for students. It was seen that, frequently, teaching of one specific procedure ignored a range of cases, and was disconnected from students' prior experiences. Pedagogical simplicity was achieved, as a particular form of mathematical precision is valued in one context and not in other, at the cost of inconsistency. For e.g., precision for constructing geometrical shapes was the focus when the lesson was on construction, whereas in the lesson on congruency "similar looking triangles" was considered as enough. When students pointed out such inconsistencies, they were asked to focus on the current context. Furthermore, it was seen that the justifications for the procedures were "non-conceptual" and "perceptual". For e.g., often as justification for a computational "shortcut", the teachers said "square of zero is two zeros", which comes from identifying the pattern in squares of numbers [$20^2 = 400, 40^2 = 1600$, etc.], but it does not explain the conceptual part, that is, adding zeroes at the end is actually changing the place value of the digits to the left.

These accounts of teachers' mathematical sensibilities that emerged from analysis of their teaching, points to the choice of perceptual mathematics [patterns, rules, etc.] over the conceptual mathematics [why these patterns/rules work]. The perceptual mathematics with lack of conceptual understanding creates inconsistent learning experiences for students, leading to confusion and a view of mathematics as being arbitrary.

Perceptual vs conceptual maths.

A critical aspect of learning to be an effective mathematics teacher for diverse learners is developing knowledge, dispositions, and practices that support building on children's mathematical thinking, as well as their cultural, linguistic, and community-based knowledge. This kind of research has linked teachers' understanding of children's mathematical thinking (e.g., strategies, conceptual milestones, and common confusions) to productive changes in teachers' knowledge and beliefs, classroom practices, and student learning (Carpenter et al. 1989; Fennema et al. 1996). Other research has argued that teachers need to understand how children's funds of knowledge—the diverse cultural and linguistic knowledge, skills, and experiences found in children's homes and communities—can support children's mathematical learning, particularly for historically underrepresented groups (González et al. 2001; Ladson-Billings 1994).

Attending to younger ones, and listening to them seriously is managed differently in different cultures. However, when it comes to learning in classrooms attending to students is frequently equated with bringing closure to students' queries and doubts. In the Indian context, it is common to see how teachers worry when students ask any question in front of a superior authority. A teacher who opens up that question for other students is a rare occurrence and often seen as "not responding" to students. We complicate these conceptions, unpacking specific practices that teachers acquire in the classroom around listening to students, and illustrate how certain practices are barriers in creating equal access in a mathematics classroom.

Equity in instruction.

By defining the horizon and the encounter with the horizon a call is made for mathematical knowledge needed for equitable instruction. By analyzing classroom teaching and identifying encounters — where students have asked meaningful questions or have made meaningful comments, and are still denied the access to potential mathematics that exploration of those questions could have led to. The denial of access in all these encounters was not because the teacher did not have time to attend to these questions and comments, or not because they were irrelevant to the discourse of the classroom, but because the teacher did not notice the mathematics in those questions.

I analyzed instruction in grade 7 mathematics classes to elaborate on aspects of being able to listen to students. I distinguish hearing from listening, where the latter involves a meaningful hearing. Irrespective of the teacher being gender fair and alert in giving equal opportunities to her students, NOT hearing them constrains students' access to meaningful mathematics. Such denial of access also has dangers of developing deep-rooted pseudo-mathematical ideas.

Teachers' work with students in individual classrooms is at the core of any program designed to attain equity for all students. However, teachers do not and cannot work in isolation to accomplish broadly conceived goals for equity in mathematics education. Administrators, teacher educators, researchers, and policymakers must all understand the issues, take the initiative, and commit to supporting students and teachers in a collaborative effort to address the complex issues involved in achieving mathematics for all.

A definition of equity is an essential starting point and can lead to a different understanding of equitable instruction. My definition begins with the premise that all students, regardless of their class, gender, community or language proficiency, will learn and use mathematics. A second premise is that all the people who are involved with the education of children must become aware of the social, economic, and political contexts of school education that can either hinder or facilitate learning of mathematics for underrepresented students (Apple, 1992). Equity in mathematics education requires: (a) equitable distribution of resources to schools, students, and teachers, (b) equitable quality of instruction, and (c) equitable outcomes for students. Equity is achieved when differences among subgroups of students in these three areas are decreasing or disappearing (Hewson & Kahle, 2001).

National Curriculum Framework (2005) and National Curriculum of Teaching Standards (1989; 2009) both ensure that all students are afforded high-quality instruction. As Allexsaht-Snider and Hart (2001) said, “Teachers’ knowledge of mathematics, their preparation to teach mathematics, and their beliefs about and skills for teaching diverse students are all aspects of equitable instruction" (p. 94). Therefore, there are ways though which one can approach equity in classroom instruction.

In the example of the triangle of 13cm, a girl student who figured out the rules for the game in the class drew two tiny equilateral triangles with a measure of the side as 4.5cm. She labeled them as 13cm. I am not sure whether she had any idea of "scale," but the student who noticed that 13cm triangle could not fit on his tiny notebook was denied this understanding. And every student was denied the broader sense of congruency, where the triangle of 13cm, would be congruent to itself, no matter on which notebook it is drawn or in which country it's drawn.

6.3 Observations from Teaching Practice.

One of the notable aspects of teaching practice demonstrated by all the teachers during their encounters with the mathematical horizon was their focus on steps and procedures involved in obtaining correct answers. Therefore, the horizon encounters I observed in this study were typically present in choices teachers make of examples, in accuracy and in justifications they provided. As researchers attest, teaching involves respecting children as mathematical thinkers (e.g., Ball, 1993; Carpenter et al., 1996). All the teachers in the study made decisions about representations out of their motivation to support their struggling students and to provide them with meaningful and accessible mathematical experiences. Throughout the study, I observed that teachers demonstrated a desire to provide their students with diverse learning experiences. While not explicit in my analysis of the cases described in Chapters 4-5, the teachers encountered the horizon when they left the exact text of the curriculum by making modifications to the lesson in order to provide their students with additional, kinesthetically oriented activities. For example, when Ms. Asmita tried to teach the meaning of algebraic identity to students, she decided to talk about factors and how the product of factors is an expansion similar to the identity. What she was trying to say that, as 3 divides 21, and 7 divides 21, we know that 3 and 7 are factors of 21. Perhaps she also meant that 3 times 7 is equal to 21, it's like a fact. But what she ended up telling the students was that “factors are the same as the expansion.” This is how her board looked.

Identities

Operation	|=	Multiplication
3×7	|=	21
8×4	|=	32
6×5	|=	30

21 is answer of what two numbers?

3 and 7

∴ so factors of 21 = 3 and 7

The factors we obtained are called expansion in mathematics

Figure 6.1: Factors on blackboard

So the point I am trying to make here is that every occasion which was meant to transform the curriculum in an understandable set of instruction included some encounters with the horizon. It was prejudiced by what teachers assumed students know, but it was more about what structural similarity the teacher observed in the mathematics of that specific topic or definition. The teacher thought the identity is expansion and expansion means multiplication. It is hard to pin point this as right or wrong, but what it elicited was what structure the teachers perceived.

There were encounters that arose due to what they perceived as “student-centered” reading of the curriculum, and there were also encounters that arose due to their “curriculum-centered” reading of students’ responses. For example, in one of the classes, Bhinde sir was teaching about finding the volume of a cube – a straightforward formula. He also explained why do we find the cube of the side to find the volume. The term for volume in Marathi is *ghanaphal*, where ghana means cube, the word literally means “answer when we find cube (raise to 3)”. The explanation for finding the cube of the side in the formula was given like this.

“When we say square we multiply sides 2 times, when we say cube (ghana) what would we do? We find cube of the number?”

This explanation was cyclic in nature. Because volume gives the answer to finding the cube of a number we find the cube of a number. The interesting part was that the students bought the emphasis that the teacher made in the pattern; they understood that area is calculated by finding squares, and volume by finding cubes. One of the students asked so when they find "raise to 4 of a number“, what do they get? What is the word for it?"

This particular encounter is complicated. The reason such encounter arose is two-fold – one because the teacher thought of highlighting the pattern that he observed within the mathematics of measuring cubes, and second because the teacher stuck to what is in the curriculum and therefore just laughed at the student and responded "it is not in your chapter, you will learn that in the next year". What will be learned in the next year was not clear, and during the interview, the teacher discussed that student question as "they always see patterns that we don't want to show them. I was telling them about square and cubes, but they want to talk about something else." The teacher was smiling while saying this, and somewhere understood what the student was asking, but did not see the relevance of it to what he was teaching.

The teachers displayed positive attitudes toward the mathematics they were teaching. They did not display anxiety about the material, even in situations where they faced unfamiliar content. They also modeled reasoning through ideas by appealing to the belief that mathematics should be accurate and consistent and encouraged their students to do the same. For example, Bhoke sir regularly advised his students to "check their answers with what is given at the end of the textbooks." It seems reasonable to assume that modeling these attitudes and beliefs has the potential to impact their students and support the development of one-answer oriented attitudes and beliefs toward mathematics.

6.4 Teachers' Knowledge: Encounters with the Mathematical Horizon

The intent of this study has been to explore the role of mathematical sensibilities that come from the discipline in the practice of teaching school mathematics. In the literature, the construct of horizon content knowledge (HCK), first proposed by Ball and colleagues (2008), has emerged as a helpful tool for analyzing and identifying such advanced knowledge of mathematics for and in teaching practice. Though there is no agreement regarding how HCK should be defined, knowledge of mathematical sensibilities and structure stands out as an integral part of the various treatments of HCK. In this section, I will discuss how my study is positioned with respect to the existing research, how my findings have contributed to the literature, and propose directions for further research.

Ball's (1993) foundational conceptualization of HCK, for example, emphasizes bringing a sense of how mathematical topics are related across the span of mathematics in the curriculum to bear on a particular moment of instruction. In their later work, Ball and Bass (2009) elaborate that HCK might involve an elementary perspective on advanced ideas including an understanding of major disciplinary structures and ideas. Zazkis and Mamolo (2011) propose HCK as including AMK acquired in tertiary level mathematics courses regarding features of mathematical objects and features connected to the object with respect to the disciplinary structure acquired during the course of undergraduate studies in mathematics. In particular, Zazkis and Mamolo (2011) draw on the metaphorical definition of horizon, "where the land appears to meet the sky" (p. 9) to define the mathematical horizon as the "place where advanced mathematical knowledge of a teacher (the sky) appears to meet mathematical knowledge reflected in school mathematical content (the land)" (p. 9).

The above definitions evoke a sense of distance between the mathematical content of the elementary curriculum and advanced mathematical knowledge along dimensions of "space" and "time." By evoking a sense of space, I mean that the above definitions of the mathematical horizon lend themselves to a kind of separation between elementary content and advanced mathematical knowledge. Where the definitions of the mathematical horizon proposed by Ball and colleagues (Ball, 1993; Ball & Bass, 2009; Ball et al., 2008) refer to an awareness of the mathematical landscape, especially with an "eye on the mathematical horizon" (Ball, 1993, p. 394). They seem to imply that the mathematics under consideration are positioned at a continuuam with the elementary content under consideration. But, the sense of "space" is especially powerful in the work of Zazkis and Mamolo (2011), who draw on the metaphorical definition of the horizon to define the mathematical horizon as where advanced mathematical knowledge and the mathematical knowledge reflected in elementary content meet. The metaphorical horizon is an imaginary point that can never be reached, as its location is defined with respect to the position of the observer and is constantly changing because the two mathematical worlds never actually meet in reality. However, this conception also gives rise to the notion of "individual horizons" which would be obstructing to equal access towards mathematics. Also, defining the horizon as an intersection of AMK obtained in tertiary level mathematics courses and the mathematical knowledge reflected in the elementary level content could suggest that their intersection, assuming the two do meet, is located far from the context of the elementary classroom. In other words, this definition of the mathematical horizon, though it represents an intersection of advanced and elementary mathematical content, seems to establish a kind of distance or separation between the mathematics under consideration in the classroom and mathematics at the horizon.

One of the challenges I have encountered in this study is that the space between the content under consideration in the elementary classroom and advanced ideas is not actually very great at all. For example, most teacher participants in this study spoke about learning through activities. Where activites are portrayed as the one that are liked by students and because they have less concentration of the content. Working on an activity, like in Ruksat's class, the student used paper cutting and assembling to make sense of how diferent areas -- ab, a^2, b^2 together formed the area of $(a+b)^2$. Considering this activity as easier because it dilutes the content is a belief that suggests the mathematics here is much distant from the discipine mathematics. Here, the mathematics at the horizon is understood as located far from the curriculum, but understanding space on paper and how the space gets added, how dimensions for the sides are obtained, have much deeper scope of learning fundamental mathematics. Where as take the episode of "Sean Numbers" described in several of Ball's discussions of the the horizon (Ball, 1993; Ball & Bass, 2009). It is an example of work in the classroom that was initiated and led by students. In that case, the teacher (Ball) facilitated a class discussion involving concepts arising in number theory, such as modular arithmetic. My findings, and Ball's example, demonstrate that school mathematics content is rarely more than a "stone's throw" away from advanced mathematical content and the boundary between them, artificial. Further, my findings indicate that time does not appear to be necessary either to encounter those advanced ideas or to work with them effectively to a large degree. These metaphors additionally convey a sense of distance between elementary mathematical content and the mathematical horizon along a time dimension. By conveying a sense of distance along a time dimension, I refer to the time between the elementary content under consideration and the timeframe in which students or teachers will encounter more advanced topics. The challenges related to distance along the dimension of time are multifaceted. One challenge relates to the ability of differentiating HCK from other subdomains of content knowledge. Another challenge relates to the potential HCK that might relate to a particular area of elementary content. In terms of the challenge relating to differentiating between HCK and other subdomains of MKT, Ball and colleagues (Ball, 1993; Ball & Bass, 2009; Ball, et al., 2008, Jakobson, et al., 2015), for example, emphasize the anticipation of mathematics at the horizon. That is, anticipating or attending to mathematical content that students will encounter in the future, or drawing on mathematical content that students have encountered in the past. For example, evidence of HCK might involve recognizing that a whole number comparison strategy that depends on the length of a number will not work in the future for decimal comparisons. One of the challenges this perspective poses is whether or not HCK can be differentiated from simply the knowledge of number comparison strategies (and possibly their development and relationship to student learning). The more profound challenge that seems to arise when considering distance between elementary content and the mathematical horizon along the dimension of time involves the potential HCK that might relate to that particular area of elementary content. This is especially true for characterizations of HCK involving the acquisition of AMK in tertiary level mathematics classes. For example, proportional relationships, which appear in the elementary curriculum from the time students begin to generate multiples of whole numbers, are foundational to the study of linear relationships, which appear as prominent concepts in tertiary level courses in mathematics from calculus to linear algebra. In other words, as time passes for an individual studying mathematics, more connections between mathematical ideas can be generated to the point that all the potential connections between a particular area of elementary content and the mathematical horizon simply cannot be enumerated. This poses two related concerns: 1) No one person can be expected to acquire all possible HCK if it is too vast to enumerate and 2) Individuals acquiring a substantial amount of any such knowledge are likely to require a significant investment of time as well.

In my analysis of the classroom data, I drew extensively on my own mathematical training and understanding of matehmatics teaching – more than a decade of mathematics education study – which allowed me to see how classroom interactions and encounters could contribute valuable insights into many aspects of the curriculum design for teacher education. In particular, the teachers in my study seemed to be making sense of the body of mathematics from a perspective that was entirely different from both the worlds – that of a mathematician's and that of a specialist in mathematics education.

These challenges have prompted me to propose that teachers appear to draw on an entirely new kind of HCK that has not yet been identified in the literature. In the next section I elaborate on a new organizational perspective for viewing HCK to better describe how this new mathematical knowledge is related to the knowledge identified in the current conceptions of HCK in the literature.

6.5 Organizing Content Knowledge from a Horizon Perspective

My findings have indicated the presence of a kind of HCK not adequately captured by descriptions in the current literature. In this section, I propose a way of organizing teachers' mathematical content knowledge that I have found helpful as I have reflected on what I have learned about teachers' HCK in this study. The purpose of this framework is to highlight different areas of teachers' content knowledge that seemed to contribute to their ability to use and generate mathematical representations, explanations and contexts in their practice.

I initially embraced Silverman and Thompson's (2008) observation that the content of a teacher's knowledge for teaching mathematics is so vast that it cannot simply be enumerated. In particular, defining horizon knowledge as a vast understanding of mathematical structure or setting the horizon far away from the elementary curriculum makes it challenging to identify what kind of knowledge is important and how it can be used to help in teaching. Ball and Bass (2009) also admit that they are uncertain as to "how to estimate how far out or in what direction the pedagogically relevant and useful horizon extends" (p. 11) or "the level of detail that is needed for horizon knowledge to be useful" (p. 11). These challenges have led me to further this exploration of HCK by developing an alternative operationalization of the concept to those in the current literature. For this study, I defined the mathematical horizon as the projection of mathematics in the curriculum to the enacted mathematics that extends beyond curriculum materials. I have defined HCK as the mathematical knowledge teachers draw on when they encounter such horizons. These definitions have enabled me to explore the vast potential HCK in terms of teachers' knowledge resources thus distinguishing this HCK from the majority of researchers who focus on identifying particular mathematical content and notions.

In the paragraphs below, as I describe my model of HCK, I will elaborate on these distinctions. By drawing on the existing literature on MKT and the results of the present study, I propose a way of organizing teachers' HCK. I have identified four overlapping domains of knowledge and visually represented them in Figure 6.2.

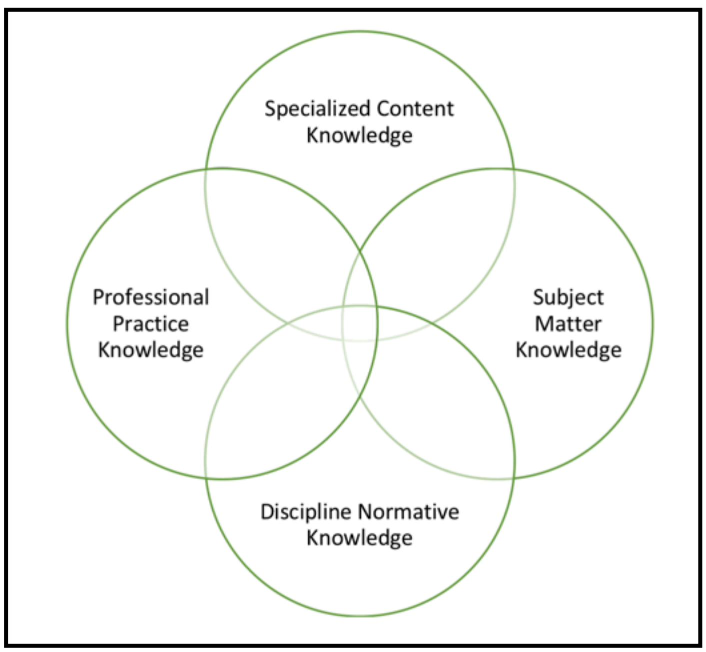


Figure 6.2: Horizon Content Knowledge

Two of these domains, specialised content knoweldge (SCK) and subject matter knowledge (SMK), are based on the current literature on MKT. The third domain, knowledge of discpline sensibilities and norms is also defined in the research recently as HCK, and situates in the definition given by Jakobson et al. The knowledge of discipine sensibilities and norms will be referred as disciplinary normative knowledge (DNK). The fourth category is a new one, called professional practice knowledge (PPK), represents an additional kind of knowledge of mathematics developed through teaching and re-teaching over time. The evidence of this knowledge is pertinent in more experienced teachers than the novice teachers with in the study. This knowledge is not always strictly mathematical in nature but informs teachers' professional judgment regarding their interpretations and generation of mathematical notions in their teaching. Because of the way in which the teachers in this study used this knowledge as a resource for using and generating mathematical notions, repersentations, explanations, etc., even though I do not consider it to be mathematical in nature, I do propose that it is a kind of content knowledge of mathematics. PPK seems to capture the kind of knowledge I observed in use by the teachers of this study as they endeavored to make sense of mathematics in curriculum, in students' responses and make decisions to make instruction effective. I will elaborate on this domain as the discussion continues.

By defining overlapping regions of these four domains of knowledge, I am able to remove the trouble encountered by defining the mathematical horizon in a way that artificially creates distance between elementary and advanced mathematical knowledge along the dimensions of space and time.

Experienced teachers' horizon content knowledge includes a situated, professional practice knowledge. Several of the approaches to the study of HCK that I identified in the literature have drawn heavily on the knowledge of teaching experts. Fernández and colleagues (2011), for example, explored HCK by attending to the insights of teachers with experience at both the primary and secondary levels. I argue that the knowledge of mathematics that middle school teachers require in order to teach may be qualitatively different from that of a tertiary or high school teacher.

Experienced teachers' PPK also appeared to have become self-sufficient with roots detached from the SMK and DNK. For example, many teachers in this study explained reasoning for why square of 40 is 1600. They often said square of 4 is 16 and square of 0 is two zeroes. They are obviously right if one decided to generalize a pattern here, like seeing square of 40, 20, 30 one would observe that pattern. But, can we count this as a mathematical explanation? The teachers often said, this is the explanation that help students remember how to find the square of a number with 0 at the unit place. This knowledge is a resource for handling many encounters – the knowledge that is developed over the years, has some benefit for students, but has detached from the other three domains of the HCK.

Basically I saw evidence of each kind of PPK, where connection or overlap with one or the other domains is lost. Here is an example of PPK that has roots with SCK. One of the teachers who was teaching volume of a cube, spent extensive time on teaching how to draw a cube. She suggested that there are many students who don't know how to draw a cube, so she showed them 3 techniques for drawing a cube. It was an interesting point, as during the data collection, in other schools, I found one of the teachers not being able to draw a cube on the board.

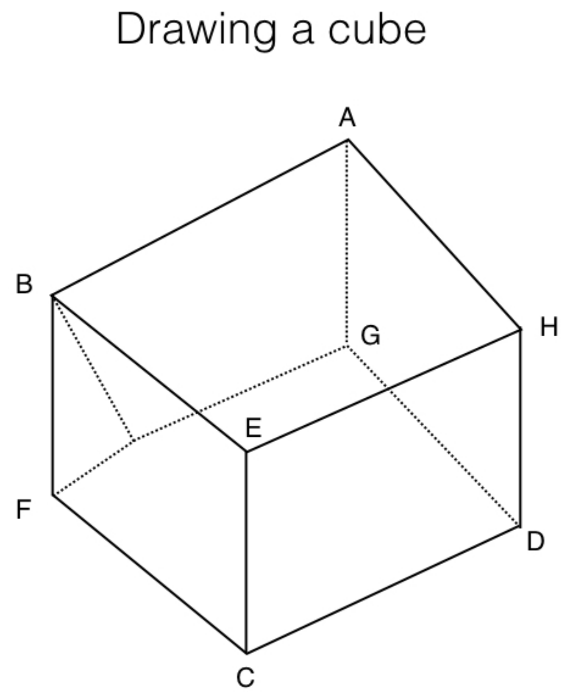


Figure 6.3: Drawing a cube

Being able to draw a cube on the board is definitely a specialized understanding for mathematics teaching. However, deciding to make it part of instruction and figuring out multiple ways of doing the same is a professional knowledge that the teachers develop over time.

Another example is from teacher's DNK who considers speed as an important criteria for learning mathematics. In the class where Bhoke sir made students recite everything very quickly. If the students said a formula right, he insisted that they repeat it faster and again still faster.

PPK is knowledge that is inevitable and is shaped by culture in school, role of leadership to kinds of students and parents' involvement, as each of these are part of teachers' expereince. But as long as PPK remains rooted in other domains of HCK, the teachers would be able to manage encounters with HCK in more meaningful ways. If PPK is the only knowledge resource available to the teacher then teachers' explanations of mathematical deductions are pseudo-mathematical. Pseudo-mathematical explanations are generated by the teachers not to explain the concept/term/formula but to aid the memorization of the concept/term/formula. These explanations call heavily upon the visual patterns or syntactic patterns, use colloquial meanings of the mathematical terms and often has a cue that could be used as a key to remember the term/concept or a formula. Being pseudo-mathematical in nature these explanations block access for the students to investigate or build further.

Risks of only PPK.

In the thesis, we saw many vignettes from different cases of teaching, and saw how teachers are present in the classroom, hearing the students but NOT really listening to them. The students in all these classes where PPK dominated without being rooted into other knowledge were denied opportunity to engage with mathematics. Here, are some reasons given by teachers for such management of these encounters:

- Students from poor background have lack of attention based on their backgrounds.

For example, not much effort was made to understand the students' questions imagining that this poor student, due to lack of resources might not have slept well and therefore may not be paying any attention. Hence repeating the instruction was the action taken — not really thinking that it could be a fundamental question about mathematics.

- Students require techniques.

Most of the time the classroom teaching is spent on teaching techniques, and therefore conceptual explanation took secondary importance. Students who remembered techniques were considered as good student of mathematics.

- Doing speedily is the only way one learns mathematics.

It was common to see students were asked to recite expansion formula, that too very fast. Therefore, a child who was speaking slow with less speed was labeled as a student who doesn't know the formula.

Forms of HCK for teacher education.

Lesh and Zawojewski (2007) also observed that the users of mathematics in professions requiring heavy use of substantive mathematics "tend to organize their mathematical ways of thinking around situations and problem contexts" (p. 787). Further, the situations encountered in these environments "often require users to draw on powerful industry-based mathematical models and procedures learned in professional settings" because the problems they encounter "frequently do not fit the assumptions underlying the relevant conventional mathematical approaches and therefore are actively adapted or discarded and recreated to meet challenging and novel conditions" (p. 787). I argue that in a similar way, knowledge for teaching school mathematics is a situated and professional practice knowledge that differs from knowledge of school mathematics. Further, this professional practice knowledge is situated in the practice of teaching, not in the discipline of mathematics. Therefore, teachers' efforts to interpret and use curriculum materials are not typically conducted from the perspective of a mathematical disciplinary task but as a task situated in the teaching profession. That is, the decisions made by the teachers in this study, resulting in their encounters with the horizon and requiring them to engage with mathematics beyond the curriculum, seemed to have been the result of sense making that utilized a situated, professional teaching knowledge.

In summary, my findings suggest that experienced teachers seem to draw on a kind of situated, professional practice knowledge to make sense of mathematics in the curriculum and in students' responses. In particular, this knowledge impacts the mathematical notions, representations and contexts that teachers may use and generate as they interpret and enact mathematical content in and beyond the curriculum.

Reading curriculum same as problem solving.

In their handbook chapter on problem solving and modeling, Lesh and Zawojewski (2007) have observed that the definition of problem solving in mathematics education is moving away from traditional views of problem solving and "towards a view of complex mathematical activity and the mathematics of modeling complex systems" (p. 782). They additionally propose that a modeling perspective toward problem solving is one of the distinguishing characteristics of a mathematician's perspective that differentiates their expert knowledge from that of novices, emphasizing that mathematics is the study of structure and that a models-and-modeling perspective leans toward producing mathematical descriptions and explanations that focus on structural characteristics of a given situation. Rather than viewing curriculum interpretation and implementation as a problem solving activity in which mathematics specialists are defined as experts, I propose that experienced teachers possess a kind of PPK that defines them as experts in the profession of teaching. I would argue that my findings suggest that these teachers view mathematical notions from a modeling perspective in the teaching discipline and not mathematics as a discipline.

Teachers rely on their reasoning and are guided by concerns for students and real-life considerations. The interesting aspect is that the concerns for students also shaped by their PPK, and therefore may not reflect real challenges that students face. During data collection for this study, it was clear that the teachers in my study often preferred to rely on their own reasoning to make sense of both the mathematics in the curriculum and the mathematics they engaged with during their encounters with the horizon.

My findings also suggest that these teachers' reliance on their reasoning was also something of a double-edged sword. For example, consider the case where Punit makes changes in the lesson to make it closer to students' real life. On one hand, the problems became relevant for students, but on the other Punit sir had difficult time understanding their solutions in the context of curriculum.

Teachers' situated, professional practice knowledge: A complete departure from AMK.

The present study suggests that teachers' thinking about and interpretations of mathematical representations may be a productive area of research for such a model of teachers' mathematical thinking. For example, the case of Parveen and the representation for fractions multiplication suggests that she was drawing on her knowledge of procedure of multplying two numbers. I am thus prompted to ask if this interpretation is unique to Parveen or if there might be a collection of common interpretations that teachers would predictably produce for this situation. Further, choosing one representation to highlight specific variables will often decrease the emphasis of others. Therefore, making simplifying assumptions and allowing some relationships to be represented "untruthfully" is more a matter of decreasing the emphasis on certain aspects that might be recovered if another representation is chosen in order to highlight it.

In conclusion, I propose that teachers' modeling perspective may be complemented by involving their recognition that data loss occurs when information from real-life contexts are represented mathematically in order to support their ability to make simplifying assumptions. In particular, this combination of knowledge may be helpful in supporting teachers' ability to choose representations that highlight only the aspects of the real-life situation that they wish to emphasize and accept the decreased emphasis on other variables and relationships that are more mathematical in nature. I also propose that this kind of reasoning could be potentially valuable in courses for preservice teachers in which the content involves emphasis on mathematical representations.

6.6 Concluding Remarks

Teaching middle school mathematics is a complex task requiring specialized knowledge. The goal of this study was to investigate the role of mathematical sensibilities in teaching. The existing work investigating teachers' mathematical knowledge for teaching incorporated advanced mathematical knowledge into perspectives regarding knowledge of mathematical conceptions, explanations and representations by using the construct of horizon content knowledge (HCK).

There is no consensus in the literature regarding the definition of the mathematical horizon or what kind of knowledge comprises HCK. By defining the horizon as the projection of the curriculum to established mathematics beyond curricular supports, I have been able to identify and classify the kind of knowledge that teachers use when they encounter the horizon and engage in mathematics that is present in the curriculum and students' responses. In studying these teachers' encounters with the horizon, I identified that experienced teachers seem to exhibit a situated professional teaching knowledge to make sense of and navigate mathematical territory. This knowledge both greatly supported teachers' ability to use and generate mathematical repersentations, explanations, etc., that aligned with established mathematics while in other cases added pseudo-mathematical nature to their work and limited their ability to move forward.

Rather than requiring an enumeration of specific mathematical knowledge with regard to concepts, I propose that teachers' mathematical knowledge might be thought of in terms of four overlapping domains that decrease in specificity depending on the extent to which all teachers can be expected to obtain explicit understandings of the mathematical knowledge in that domain. The domain of PPK consists of teachers' situated, professional teaching knowledge encompassing the whole of their professional understandings across content domains and developed through practice, along with their tacit knowledge of mathematical topics. This study reveals that teachers draw on this professional teaching knowledge to navigate unfamiliar mathematical territory and may include realworld and student considerations, existing knowledge of mathematical representations, assumptions about the nature of mathematics, and reasoning about correspondences between real-world factors and mathematical representations of those factors.

Further, I have recommended that additional research should be conducted to identify if teachers' professional knowledge results in common interpretations of mathematical representations in well-defined content areas. If so, this information could be valuable in guiding mathematics educators, and other experts, in their work to support the development of teachers' formal knowledge of mathematics by using teachers' mathematical thinking as a basis.

Appendices

Appendix A
Interview Part I

1. Name:
2. Age:
3. Educational and professional qualifications (Mention subject specialization and qualifying year):
4. Places/cities where you received your education:
5. Teaching experience (mention which classes and subjects you have taught and for how many years):
6. Number of years in service:
7. Present position:
8. Have you conducted workshop sessions for mathematics teachers?
9. If yes, write some details (roughly how many, on what subjects/topics, teachers at which level, etc.)
10. If there is anything else that you would like to mention about yourself, please do so here:

Appendix B

Interview Part II

- How did you decide to be a mathematics teacher?
- Can you narrate me your experience of being a mathematics teacher so far?
    - This is an opening-up question; it helped me get the continuum of their experience as a teacher. There is a chance that that there will be few open ends here – and which might tempt me to ask further questions, but I will take up only if they are relevant to the research questions.
- Please describe any mathematical experience from your school or college days that you remember distinctly. Why do you think it is special or distinct?
- What according to you are the most crucial things for anybody to be good in mathematics?
    - This gave me idea of their views about mathematics and what it takes to be good in mathematics. As they had to talk about someone

else here, the accountability was less and therefore they describe their opinion clearly and freely.

- When you write on the board, what do you make sure you do? Are there specific things you aim for in asking your pupils to write in their notebooks? What is important when writing mathematics?
    - As I had been to their classes, I know that good percentage of the class time is spent in writing on the board or copying in the notebook or writing in the notebook. Therefore, it was important to find out what are their views about writing mathematics and what kind of writing they value. The question was geared towards finding their understanding of precision and doing mathematics.
    - A sub-question could be asked here as How do you think writing in mathematics helps in learning mathematics?
- When you draw on the board, what do you make sure you do? Are there specific things you aim for in asking your pupils when they draw in their notebooks? What is important in drawing in mathematics?
    - Same as above, copying geometric figures was another big chunk of the class on geometry and therefore, what mathematics was learned or exercised could be extrapolated from this question.
- How do you think precision in drawing helps in learning mathematics?

- How do we know that things mentioned in the textbooks are true? [See whether the teacher use some example, if don't use one of the examples from their teaching]
    - Here is one example that was asked based to a teacher. For example, how do we know that square of 40 is 1600, in one class the teacher said square of 4 is 16 and square of zero is two zeros – how do you respond to such teacher or why do you gave this shortcut, how it works; how do we know that volume is written in cubic cm [the teacher said because the cube has three sides when we write volume of the cube we write cm^3]

Appendix C
Interview Part III

The interview in the third section involved teachers watching their teaching clips, as selcted by the researchers and describing their side of decision making in that particular episode of teaching.

Clip 1: Day 1

1:30 to 2:30

- What did you notice in this clip?
- What according to you is difficult in learning algebraic identities?
- When you explained the multiplication of two binomials, you gave a specific explanation, why do you think this bracket multiplication actually works?

Day 1

39:30 to 40:10

- What did you notice in this clip?
- What according to you is difficult in learning squaring of the numbers?

- When you explained this short method, you were very confident that it has to be the square of 96, how do you know that shortcut will give you the correct answer?

Day 2

33:10 to 36:21

- What did you notice in this clip?
- Why were you asking students to make questions?
- What does making questions has to do with learning of mathematics?

Day 3

8:53 to 10:15

- What did you notice in this clip?
- What do you think is difficult in learning volume of the 3d shapes?
- What did you think of the definition that student said and you gave?

Appendix D
Problems Solved by the Teachers

Problem 1

A printer only prints the page numbers of a book. While doing so printer uses 999 digits. What is the last page number of the numbered pages?

Problem 2

I have a barrel of sharbat, and you have a cup of tea. I put a teaspoon of my sharabat into your cup of tea. Then you take a teaspoon of the mixture from your teacup, and put it back into my sharabat barrel. Is there now more sharabat in the teacup than there is tea in the sharabat barrel, or is it the other way around?

Problem 3

The following picture shows a 3-step staircase made up of cubes. How many cubes are needed to make an n-step staircase?

Appendix E

Maharashtra State Textbook Pages

Expansion formulas

The product of two algebraic expressions is called an expansion. We make formulas in order to write the expansion of certain algebraic expressions directly without multiplying. Let us learn some such formulas.

The square of the sum of two terms

$$(a + b)^2 = (a + b) \times (a + b)$$

$$= a \times (a + b) + b \times (a + b)$$

$$= a^2 + \underline{ab + ab} + b^2$$

$$= a^2 + 2ab + b^2$$

$(a + b)^2 = a^2 + 2ab + b^2$ **This is an expansion formula.**

Here, the left hand side is the square of a binomial (here, the sum of two terms). The first term is 'a' and the second is 'b'. We can write this formula in words as follows.

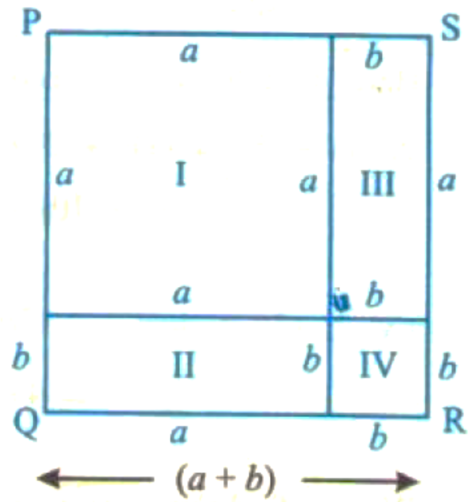
This expansion formula can also be worked out with the help of geometrical figures.

(1) In the figure alongside, PQRS is a square and the length of its side is

$(a + b)$.

$\therefore$ The area of square PQRS is $(a + b)^2$

(2) As shown in the figure, a segment parallel to PS is drawn at a distance of a from point P. Similarly another segment is drawn parallel to PQ at a distance of a from P. This divides the square PQRS into four parts.



(3) Part I is a square of side a

$\therefore$ the area of part I is a^2

(4) Part II and part III are rectangles having length a and breadth b.

$\therefore$ The area of part II = ab

and area of part III = ab

(5) Part IV is a square of side b

$\therefore$ the area of part IV is b^2

Now, the area of square PQRS = area of part I + area of part II + area of part III + area of part IV.

Substituting the actual values of each we get,

$$(a + b)^2 = a^2 + \underline{ab + ab} + b^2$$
$$= a^2 + 2ab + b^2$$

Expansion of $(a - b)^2$

$$
\begin{aligned}
(a - b)^2 &= (a - b) \times (a - b) \\
&= a \times (a - b) - b \times (a - b) \\
&= a^2 - \underline{ab - ab} + b^2 \\
&= a^2 - 2ab + b^2
\end{aligned}
$$

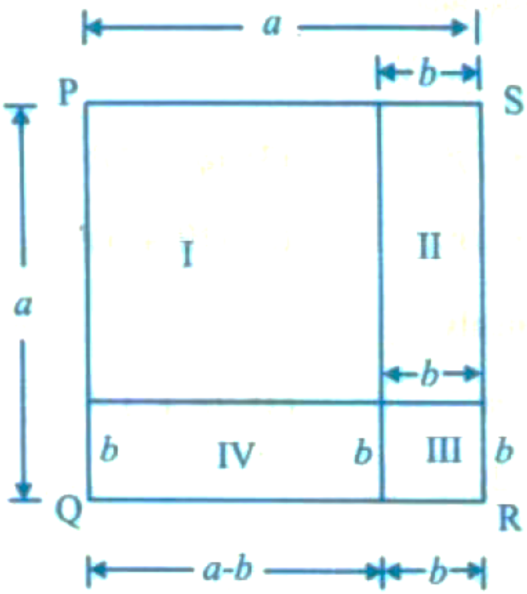
$(a - b)^2 = a^2 - 2ab + b^2$ **This is an expansion formula.**

The left hand side of this expansion formula is the square of the binomial $(a - b)$. (That is, the difference of two terms.) The first term is a and the second is b. The formula can be expressed in words as follows.

(First term − Second term)^2

= (First term)^2 − 2 × (First term) × (Second term) + (Second term)^2

This expansion formula can also be worked out with the help of geometrical figures.



(1) □ PQRS is a square. the length of its sides is a.

The area of square PQRS is a^2

(2) As shown in the figure, a segment is drawn parallel to seg PS at a distance b from point Q. Similarly another segment is drawn parallel to seg PQ at a distance b from point S. This divides square PQRS into four parts.

Part I and part III are squares, parts II and IV are rectangles.

(3) Area of part I $= (a - b)^2$

Area of part II $= b(a - b)$

Area of part III $= b^2$

Area of part IV $= b(a - b)$

(4) Area of part I = Area of PQRS – (area of part II + area of part III + area of part IV).

$$\therefore (a - b)^2 = a^2 - [b(a - b) + b^2 + b(a - b)]$$

$$= a^2 - [ab - b^2 + b^2 + ab - b^2]$$

$$= a^2 - [2ab - b^2]$$

$$= a^2 - 2ab + b^2$$

The expansion of $(a + b)(a - b)$

$$(a + b) \times (a - b) = a \times (a - b) + b \times (a - b)$$
$$= a^2 \underline{- ab + ab} - b^2$$
$$= a^2 - b^2$$

Thus, $(a + b)(a - b) = a^2 - b^2$ is the expansion formula.

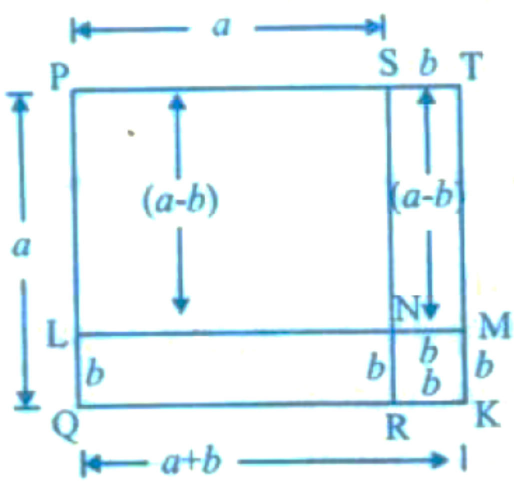
This formula has been made by multiplying two binomials. Of these, one binomial is the sum of two terms while the other is the difference of the same two terms. $(a + b)(a - b) = a^2 - b^2$

This formula is written in words as follows.

(First term + Second term) (First term − Second term)

= $(\text{First term})^2 - (\text{Second term})^2$

This expansion formula too can be worked out with the help of geometric figures.



(1) □ PQRS is a square with side a.

$\therefore$ Area of square PQRS = a^2

(2) Seg TK is drawn parallel to seg SR from the point T at a distance b from point S, on the line that contains seg PS. Seg LM is drawn parallel to seg PS from the point L on seg PQ, at a distance b from Q.

Identity

$(a + b)^2 = a^2 + 2ab + b^2$

Take several different values of a and b in the expansion above. Verify that every time, the left hand side and the right hand side are equal.

In, $(a + b)^2 = a^2 + 2ab + b^2$

Substitute the values $a = 2$ and $b = 5$

Left hand side $= (a + b)^2$

Right hand side $= (2 + 5)^2 = 7^2 = 49$

Right hand side $= a^2 + 2ab + b^2$

$= 2^2 + 2 \times 2 \times 5 + 5^2$

$= 4 + 20 + 25 = 49$

Left hand side = Right hand side

In the same expansion formula, let us substitute the values $a = 4$ and $b = 7$

Left hand side $= (a + b)^2$

$= (4 + 7)^2 = 11^2 = 121$

Right hand side $= a^2 + 2ab + b^2$

$= 4^2 + 2 \times 4 \times 7 + 7^2$

$= 16 + 56 + 49 = 121$

Left hand side = Right hand side

Appendix F

NCERT Textbook Pages

2.3 Multiplication of Fractions

You know how to find the area of a rectangle. It is equal to length × breadth. If the length and breadth of a rectangle are 7 cm and 4 cm respectively, then what will be its area? Its area would be $7 \times 4 = 28$ cm^2.

What will be the area of the rectangle if its length and breadth are $7\frac{1}{2}$ cm and $3\frac{1}{2}$ cm respectively? You will say it will be $7\frac{1}{2} \times 3\frac{1}{2} = \frac{15}{2} \times \frac{7}{2}$ cm^2. The numbers $\frac{15}{2}$ and $\frac{7}{2}$ are fractions. To calculate the area of the given rectangle, we need to know how to multiply fractions. We shall learn that now.

2.3.1 Multiplication of a Fraction by a Whole Number

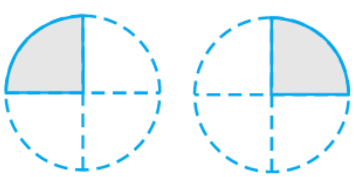


Fig 2.1

Observe the pictures at the left (Fig 2.1). Each shaded part is $\frac{1}{4}$ part of a circle. How much will the two shaded parts represent together? They will represent $\frac{1}{4} + \frac{1}{4} = 2 \times \frac{1}{4}$.

Combining the two shaded parts, we get Fig 2.2. What part of a circle does the shaded part in Fig 2.2 represent? It represents $\frac{2}{4}$ part of a circle.

Fig 2.2

The shaded portions in Fig 2.1 taken together are the same as the shaded portion in Fig 2.2, i.e., we get Fig 2.3.

Fig 2.3

or $$2\times\frac{1}{4}=\frac{2}{4}\ .$$

Can you now tell what this picture will represent? (Fig 2.4)

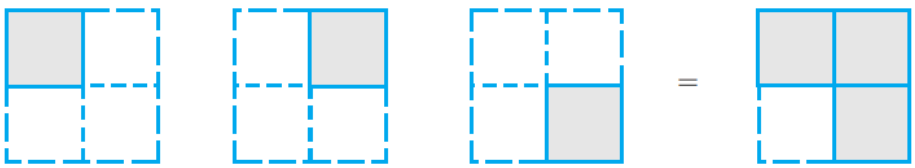


Fig 2.4

And this? (Fig 2.5)

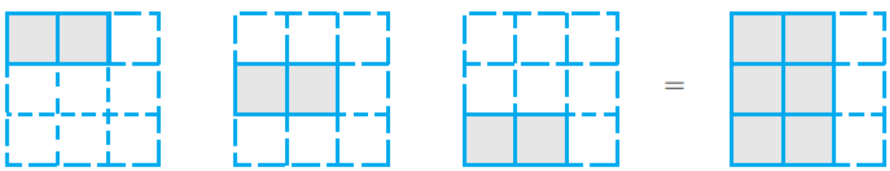


Fig 2.5

Let us now find $3\times\frac{1}{2}$.

We have $$3\times\frac{1}{2}=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$$

We also have $$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{1+1+1}{2}=\frac{3\times1}{2}=\frac{3}{2}$$

So $$3\times\frac{1}{2}=\frac{3\times1}{2}=\frac{3}{2}$$

Similarly $$\frac{2}{3}\times5=\frac{2\times5}{3}=?$$

Can you tell $$3\times\frac{2}{7}=?\qquad 4\times\frac{3}{5}=?$$

The fractions that we considered till now, i.e., $\frac{1}{2},\frac{2}{3},\frac{2}{7}$ and $\frac{3}{5}$ were proper fractions.

For improper fractions also we have,

$$2 \times \frac{5}{3} = \frac{2 \times 5}{3} = \frac{10}{3}$$

Try, $3 \times \frac{8}{7} = ?$ $\quad 4 \times \frac{7}{5} = ?$

Thus, *to multiply a whole number with a proper or an improper fraction, we multiply the whole number with the numerator of the fraction, keeping the denominator same.*

TRY THESE

1. Find: (a) $\frac{2}{7} \times 3$ (b) $\frac{9}{7} \times 6$ (c) $3 \times \frac{1}{8}$ (d) $\frac{13}{11} \times 6$

   If the product is an improper fraction express it as a mixed fraction.

2. Represent pictorially : $2 \times \frac{2}{5} = \frac{4}{5}$

TRY THESE

Find: (i) $5 \times 2\frac{3}{7}$

(ii) $1\frac{4}{9} \times 6$

To multiply a mixed fraction to a whole number, first convert the mixed fraction to an improper fraction and then multiply.

Therefore, $3 \times 2\frac{5}{7} = 3 \times \frac{19}{7} = \frac{57}{7} = 8\frac{1}{7}$.

Similarly, $2 \times 4\frac{2}{5} = 2 \times \frac{22}{5} = ?$

Fraction as an operator 'of'

Observe these figures (Fig 2.6)

The two squares are exactly similar.

Each shaded portion represents $\frac{1}{2}$ of 1.

So, both the shaded portions together will represent $\frac{1}{2}$ of 2.

Combine the 2 shaded $\frac{1}{2}$ parts. It represents 1.

So, we say $\frac{1}{2}$ of 2 is 1. We can also get it as $\frac{1}{2} \times 2 = 1$.

Thus, $\frac{1}{2}$ of $2 = \frac{1}{2} \times 2 = 1$

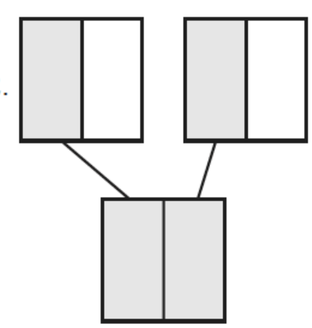


Fig 2.6

John cuts another strip of length 6 cm into smaller strips of length $\frac{3}{2}$ cm each. How many strips will he get now? He will get $6 \div \frac{3}{2}$ strips.

A paper strip of length $\frac{15}{2}$ cm can be cut into smaller strips of length $\frac{3}{2}$ cm each to give $\frac{15}{2} \div \frac{3}{2}$ pieces.

So, we are required to divide a whole number by a fraction or a fraction by another fraction. Let us see how to do that.

2.4.1 Division of Whole Number by a Fraction

Let us find $1 \div \frac{1}{2}$.

We divide a whole into a number of equal parts such that each part is half of the whole. The number of such half ($\frac{1}{2}$) parts would be $1 \div \frac{1}{2}$. Observe the figure (Fig 2.11). How many half parts do you see?

There are two half parts.

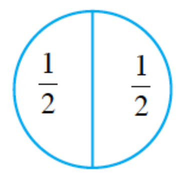


Fig 2.11

So, $1 \div \frac{1}{2} = 2$. Also, $1 \times \frac{2}{1} = 1 \times 2 = 2$. Thus, $1 \div \frac{1}{2} = 1 \times \frac{2}{1}$

Similarly, $3 \div \frac{1}{4}$ = number of $\frac{1}{4}$ parts obtained when each of the 3 whole, are divided into $\frac{1}{4}$ equal parts = 12 (From Fig 2.12)

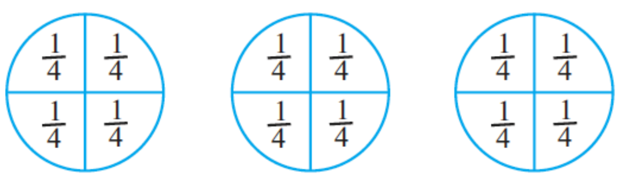


Fig 2.12

Observe also that, $3 \times \frac{4}{1} = 3 \times 4 = 12$. Thus, $3 \div \frac{1}{4} = 3 \times \frac{4}{1} = 12$.

Find in a similar way, $3 \div \frac{1}{2}$ and $3 \times \frac{2}{1}$.

Reciprocal of a fraction

The number $\frac{2}{1}$ can be obtained by interchanging the numerator and denominator of $\frac{1}{2}$ or by inverting $\frac{1}{2}$. Similarly, $\frac{3}{1}$ is obtained by inverting $\frac{1}{3}$.

Let us first see about the inverting of such numbers.

Observe these products and fill in the blanks :

$7 \times \frac{1}{7} = 1$	$\frac{5}{4} \times \frac{4}{5} =$ ---------
$\frac{1}{9} \times 9 =$ ------	$\frac{2}{7} \times$ ------- $= 1$
$\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = \frac{6}{6} = 1$	------ $\times \frac{5}{9} = 1$

Multiply five more such pairs.

The non-zero numbers whose product with each other is 1, are called the reciprocals of each other. So reciprocal of $\frac{5}{9}$ is $\frac{9}{5}$ and the reciprocal of $\frac{9}{5}$ is $\frac{5}{9}$. What is the receiprocal of $\frac{1}{9}$? $\frac{2}{7}$?

You will see that the reciprocal of $\frac{2}{3}$ is obtained by inverting it. You get $\frac{3}{2}$.

Think, Discuss and Write

(i) Will the reciprocal of a proper fraction be again a proper fraction?

(ii) Will the reciprocal of an improper fraction be again an improper fraction?

Therefore, we can say that

$$1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 1 \times \text{ reciprocal of } \frac{1}{2}.$$

$$3 \div \frac{1}{4} = 3 \times \frac{4}{1} = 3 \times \text{ reciprocal of } \frac{1}{4}.$$

$$3 \div \frac{1}{2} = \text{------} = \text{----------------------}.$$

$$\text{So, } 2 \div \frac{3}{4} = 2 \times \text{ reciprocal of } \frac{3}{4} = 2 \times \frac{4}{3}.$$

$$5 \div \frac{2}{9} = 5 \times \text{ ------------------ } = 5 \times \text{ ------------}$$

Thus, *to divide a whole number by any fraction, multiply that whole number by the reciprocal of that fraction*.

TRY THESE

Find: (i) $7 \div \frac{2}{5}$ (ii) $6 \div \frac{4}{7}$ (iii) $2 \div \frac{8}{9}$

- *While dividing a whole number by a mixed fraction, first convert the mixed fraction into improper fraction and then solve it.*

Thus, $4 \div 2\frac{2}{5} = 4 \div \frac{12}{5} = ?$ Also, $5 \div 3\frac{1}{3} = 3 \div \frac{10}{3} = ?$

TRY THESE

Find: (i) $6 \div 5\frac{1}{3}$

(ii) $7 \div 2\frac{4}{7}$

2.4.2 Division of a Fraction by a Whole Number

- What will be $\frac{3}{4} \div 3$?

Based on our earlier observations we have: $\frac{3}{4} \div 3 = \frac{3}{4} \div \frac{3}{1} = \frac{3}{4} \times \frac{1}{3} = \frac{3}{12} = \frac{1}{4}$

So, $\frac{2}{3} \div 7 = \frac{2}{3} \times \frac{1}{7} = ?$ What is $\frac{5}{7} \div 6$, $\frac{2}{7} \div 8$?

- While dividing mixed fractions by whole numbers, convert the mixed fractions into improper fractions. That is,

$2\frac{2}{3} \div 5 = \frac{8}{3} \div 5 =$ ------ ; $4\frac{2}{5} \div 3 =$ ------ $=$ ------; $2\frac{3}{5} \div 2 =$ ------ $=$ ------

2.4.3 Division of a Fraction by Another Fraction

We can now find $\frac{1}{3} \div \frac{5}{6}$.

$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \times$ reciprocal of $\frac{5}{6} = \frac{1}{3} \times \frac{5}{6} = \frac{2}{5}$.

Similarly, $\frac{8}{5} \div \frac{2}{3} = \frac{8}{5} \times$ reciprocal of $\frac{2}{3} = ?$ and, $\frac{1}{2} \div \frac{3}{4} = ?$

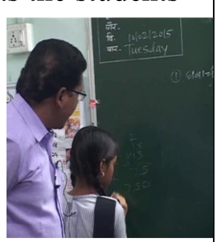
TRY THESE

Find: (i) $\frac{3}{5} \div \frac{1}{2}$ (ii) $\frac{1}{2} \div \frac{3}{5}$ (iii) $2\frac{1}{2} \div \frac{3}{5}$ (iv) $5\frac{1}{6} \div \frac{9}{2}$

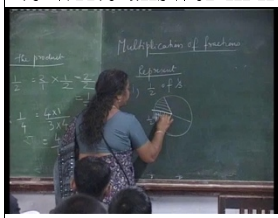
Appendix G

Encounters Summary Table

Summary of the chunks	Questions asked during the interview
After a student drew one of the radii, the teacher asks everyone, what is the length of that line. All the students notice that it is one of the radii and say that the length of the radius is 3 cm. Then the teacher asks if radius is 3 cm and the diameter is 6 cm what is the relationship between the two. Students come up with various relations between the two except that it is double the radius. Different things that the students said were – the diameter is more than the radius, the measure is more, etc. When the teacher gave different measures for radius the students gave correct answer for the diameter by doubling the number. When one of the students responds that the diameter is double the radius, the teacher becomes happy.	What did you notice in this clip? Why you think student were finding difficult to answer the relationship between radius and diameter? Why is it important to understand this relationship?
The teacher revises the topic that they studied the earlier day, i.e. on Circle sectors. In this clip she asks a question that if I draw a circle what one should do to get two circle segments. She doesn't ask who will come and draw a chord, rather asks to get circle segments.	What did you notice in this clip? What is important to understand circle sectors? Why do you ask questions as – what will we do that will make the two parts in the circle area? Why didn't you ask who will make the sector? Why such questioning is important in learning mathematics?
In this clip the teacher begins with simple multiplication facts to highlight that those are the factors of a product and then attempts to connect it with Algebraic identities. This connection is not that explicit, so she faces some challenges while doing so.	What did you notice in this clip? What is more difficult to understand in learning identities? This class was on identities, why did you decide to talk about factors and products?

In this clip the teacher is discussing the expansion of binomials. She emphasized that one of the meanings of multiplication is to expand and she highlights some of the steps in the process of multiplication as the important ones.	What did you notice in this clip? What is important to understand in expansion of binomials? You taught the students to multiply one term at a time while expanding the binomials; why does that technique work? Can you explain?
In this clip the teacher is explaining the vertices, edges and sides of the cube. He makes a wrong drawing of a cube and realizes only when he is counting vertices. He makes 9 vertices. He doesn't tell students what happened but he erases the diagram. This time he holds the textbook in front of him and then draws the cube. Then he discusses the corners of the video. He reads all the vertices and then says that the vertices are called as edges.	What did you notice in this clip? What is important in drawing a cube? Why was there confusion before? What instruction would you give someone to be able to construct a cube? Why we should draw mathematical objects?
In this clip the teacher asks students to make questions for opposite side, he also tells what they should be asking. Like, "Ask question about opposite side, ask about which side is opposite." Similarly he asked them to make questions for opposite angles.	What did you notice in this clip? Why were you asking students to make questions? What does making questions has to do with learning of mathematics?
This clip begins with a definition of what is volume of an object. The teacher first gives idea of space, the space around and then highlights that it is the space occupied by that object. He also shows that these objects occupy the space by taking examples of objects keeping in the box or water spilling when the rod is inserted.	What did you notice in this clip? What do you think is difficult in learning volume of the 3d shapes? What did you think of the definition that student said and you gave?
In this clip the teacher is explaining how formula for finding volume of a cube is hidden in the word "volume" (*ghanphal)* itself. *Ghanphal* is a word for volume and *ghan* also means cube of a number. He is using this similarity to mention that taking the cube of the side is implicit in the word *ghanaphal* or volume.	What did you notice in this clip? What is most crucial concept in understanding of volume of a cube? What is volume of a cube? Do we use volume of a cube in any other topics of mathematics?
This clip also has an example of SCK, as the students are attempting to do the  multiplication $15 \times 15 \times 15$. When they multiplied 15×15, using the  traditional steps, where they multiplied by 5 first and	What that girl has done? Why do you think it is wrong?

then by 1, the teacher did not recognize what students were doing, the problem the students were facing was in multiplying 1 by 1 and the teacher thought the problem is in 15 x 15, the students are not using table of 15 correctly. See in the pictures below work by two students:	
6 x (7.5) the teacher refers to 7.5 as 75 and says we do not the square of 75. Then we he gets the answer as 56.25, the students refer to it as 5 thousand six hundred twenty five.	What did you notice in this clip? Can we read the number 56.25 as fifty six point twenty five? Why?
This whole section is on calculating volumes of cube in the real contexts – such as finding the pain price for cubic shaped safe, etc. This particular chunk has example for everything – SCK, PCK, CCK and HCK In this clip the teacher is teaching the multiplication of rational numbers. He explains the procedure of multiplying and then talks about simplifying the numbers to "small numbers". His reference to small numbers is problematic as it indicates that the reduced form of the number is smaller than the actual form. So equivalency is not understood.	What did you notice in this clip? What according to you is difficult in learning multiplication of fractions? Why did you go on simplification of the fractions? Why simplification is difficult to learn? Do you think any student in this class faced any challenge in understanding multiplication of fractions? If so, what is it? What do you think the students in the class are thinking at the end of this class?
In this clip the teacher is explaining the expansion of $(a+b)(a+b)$. He describes that to do such multiplication first multiply by a and then multiply by b to the whole bracket $(a+b)$.	What did you notice in this clip? What according to you is difficult in learning algebraic identities? When you explained the multiplication of two binomials, you gave a specific explanation, why do you think this bracket multiplication actually works?
The teacher in this clip demonstrates a method for finding square of 96. He describes that generally the way all of them have been taught is to multiply the same number twice. But the method he shows, involves squaring last digit, placing square of the first digit in front of it and adding product of the two digits twice in the middle of this four digit number. Then he says, "it has to work" – multiple times. He doesn't explain why, but he says it has to work, as if it is magic. Square of 96 is calculated as 8136 +0540+0540	What did you notice in this clip? What according to you is difficult in learning squaring of the numbers? When you explained this short method, you were very confident that it has to be the square of 96, how do you know that shortcut will give you the correct answer?
The teacher asks what is the meaning of $2 \times \frac{1}{2}$. She first shows a pictorial presentation of it. She calls a student to do the work on the board. The student does not speak, but draws two wholes (not at least looking equal), shades half of each of the whole and then gives	When you explained the multiplication using diagrams, you showed that $2 \times \frac{1}{2}$ could be understood as taking 2 halves, or as you concluded, two times repeated addition of ½. What other meaning the

equal to sign and draws another whole whose both parts are shaded and marks each part as ½. Later he writes a mathematical sentence as ½ + ½ = 2/2, which teacher says is equal to 1. The teacher explains that 2 times ½ is same as ½ taken two times. And therefore, when you shade ½ of one and again ½ of one, in total it becomes two parts shaded out of 2. Then she suggests that she will give a diagram like this and the students will have to figure out what is the mathematical expression for it. She draws two circles shaded 2/8 each and asks a girl student to come to the board. The girl student draws another circle and marks all the four pieces together in it. The teacher asks her to write answer in number, and further to reduce it.	expression $2 \times \frac{1}{2}$ has, other than the repeated addition meaning? How would you use the repeated addition meaning of multiplication of fractions in the following problem – $\frac{3}{5} \times \frac{1}{2}$ What are the most crucial things one needs to understand in multiplication of fractions?
 In this clip the teacher is making pictorial representation for half of one third. She drew one circle on the board and then made three equal parts of it. Asked the entire class whether they know how to find half of this one third. The students were busy drawing the same figure in their notebook and therefore did not respond. The teacher then repeated the question, and answered herself, by drawing a line on the shaded 1/3rd portion of the figure. She cross—shaded one of the two parts. As shown in the image here. She labeled that part as ½ of 1/3. Then asked, what is ½ of 1/3 (and writes on the board ½ x 1/3 =). Students respond something that is not audible, then one of the students said, that it is 1x 1 upon 2 x 3. Someone said 2/6 and then all of them changed the answer to 1/6. The teacher wrote on the board. She explains this answer further as "if at all you are making six equal parts of the circle then this one part will be 1/6". She further explained that we only made it in one part, but the each part is 1/3 (labeled them simultaneously). This one part we divided into two and that is ½ of 1/3. And concludes that ½ of 1/3 is 1/6. And she concluded the solution with "again the answer is product of the numerators and product of the denominators."	What did you notice in this clip? How can one show the answer is $\frac{1}{6}$ even in the diagram explanation?
In this clip the teacher has given the problem 2/5 x ½. What students did was to make 5 parts of an half. And they called as 2/5. The teacher said, that won't be correct and she attempted to answer how 1/5 that the students drew is of ½. First she removed the other	What do you notice here? What are you trying to convince the students? Why do you choose this explanation?

semicircle and highlighted how the 1/5 they drew is of only half. Then she drew a full circle and drew 1/5 in it and showed that is 1/5 of the complete circle. And then she also showed that another way for 2/5 would be to draw two circles, and shade 1/5 in each circle. Then to explain that 1/5 of ½ will be 1/10 and 2/5 of ½ will be 2/10 she used the multiplication algorithm as a justification.	
In this class the teacher kept on practicing on the problems of the kind $a \times \frac{b}{c}$, $\frac{a}{b} \times \frac{c}{d}$. In the first kind students need to write that the denominator of a is nothing but 1 and then use that in the algorithm for multiplication numerator x numerator divided by denominator x denominator. There is no particular clip here, as she mainly took different exercise of the form *a/b* × *c/d* in the class and while doing that she used various explanations around it. The questions are on those. Remind her of the topic of that day.	When you took an example ¼ x ¼, the students calculated its answer as $\frac{1}{16}$. Even though they did multiply the answer they got $\frac{1}{16}$ is smaller than ¼. Why do you think this happened? You taught some rules like, "product of proper fraction is smaller", "product of improper fraction is greater", etc. Why did you teach this? Can you provide a proof for any one of these rules? Can you prove that they are true?
The teacher worked on the word problems that involved multiplication as a repeated addition with fractions. Therefore the problem types were $a \times \frac{b}{c}$. The problems involved contexts of distance, reading number of pages of book, number of days for certain works, etc. This involved many other procedural steps such as converting mixed fraction to improper fractions before/ during solving the problem and again converting improper fraction to mixed fraction for the final answer if needed.	On this day many word problems were solved that used multiplication as repeated addition. Design a word problem based on what you taught these three days. Design a word problem for the problem $\frac{3}{4} \div \frac{1}{2}$ Where do you think division and multiplication of fractions is used in the curriculum after 7^{th} grade?
The teacher here is discussing how to multiply two binomials. The example chosen is $(a + 2)(a + 3)$. After the first step that a multiplies the bracket $(a + 3)$ the teacher now is unpacking the multiplication with the bracket. The student after some discussion have come up that $a \times a$ is a^2 and now the discussion is going on whether $a \times 3$ becomes $a3$ or $3a$. Jahir sir writes the two options on the board and asks which one of these is correct. A boy student responds that we could use any of these. The teacher uses example of 2×3 or 3×2 to show how the product of the two will remain same and hence the product of $a \times 3$ will as well remain the same. He further says, however	What do you notice here? What are you trying to convince the students? Why do you choose this explanation?

when you work with algebraic expressions, the number is usually written before the letter, and that is why we will write this product as $3a$ and not $a3$. Then proceeding the expansion for $2 \times (a+3)$ another student responded the first product as $a2$. The teacher reminded him of the discussion just happened and asked another boy student who responded $2a$. Interestingly when they discuss 2×3 a girl student gave the answer 5. He reminds her that the operation they are expected to do here is multiplication. He then himself says the product is 6 and writes that beside $2a$, without writing a plus sign before that. Students start saying it is wrong and ask the teacher to put a plus sign. The teacher repeats their answer and emphasizes the importance of putting plus sing before the product.	
While finding the square of a binomial $(2x+y)$ a girl student wrote, $2x^2$as square of the first term. The teacher stopped her and corrected her answer. Then he corrects her error of $+2xy+2xy$. Initially he asked her but he does not wait for her answer and then dictated her steps – as first write 2 then write the multiplication sign, then the first term $2x$, and then the second term y. And then he says, not this is correct.	What did you notice in this clip? What challenges the girl is facing here?
The teacher has finished explaining the expansion of an algebraic identity $(a-b)$. Now they are reaching to a reduced form. So the discussion is about simplifying $-ab-\ ab$. The first response is $-ab^2$ and then the next response is -2ab. The teacher writes both the responses and then discusses with them which one is correct. Most say -2ab, and then the teacher asks why? Then the teacher uses the metaphor of chalks to explain how two chalks together will make a set of two chalks. He then explains how two – quantities get combined together to form a new minus quantity*.	What did you notice in this clip? What is difficult in expansion of binomials? Why can't we add ab and ba? Why did the girl made that mistake of $ab + ab = ab^2$
The teacher begins with the problem of finding the square of a binomial $(a+b)$. She first explains the meaning of square of binomial. She says it is nothing but square of first term and second term together*. She then describes what is a square, and she mentions a square is same as doubling. Then she explains how to proceed. She describes that they have to keep the first binomial expression same, and they need to break the second. If they will do that then only they get the "true terms". Then she explains that they need to multiply by each of the term. She doesn't talk much about	What did you notice in this clip? What do you mean by true terms? What is important to understand in expansion of binomials? You explained that they need to multiply the whole bracket term by one term at a time, why does that technique work? Can you explain?

adding $ab + ba$, rather makes students repeat the expansion of it.	
In this clip the teacher has worked on the problem of expansion $(m+4)^2$. They all have arrived at the answer $m^2 + 8m + 16$ and the teacher asks them the question, how did they find the answer. She doesn't ask that question to students but answer herself as due to the expansion formula that they learned of $(a+b)^2$. Therefore she emphasizes that it is important to remember the formula. And she read the entire formula and asks student to read it again. However students reading does not match with each other due to different speeds of talking, so she asks them again to read it together. She specifically says, that as you are not speaking together it is appearing as chaos, so just repeat after me. After this she reads the expansion formula in a rhythm and students repeat after her for 2 times. *	What did you notice in this clip? What is important to say these loudly? Do you find student forgetting this formula is the main problem in learning to do expansion of algebraic identities?
The teacher has drawn the two triangles on the board. One equilateral triangle of $13cm$ and another a scalene triangle of dimension $12cm$, $17cm$ and $11.5cm$. Here on the scale she is measuring by inches and writing as centimeters. While she is measuring one of the lengths, the students make a guess for it as $12\ cm$, she says no, it is not $12\ cm$, and it is actually $11.5cm$. She further insists that when you measure the lengths, you need to write the exact measurement. You can't write anything that you think, you need to write the exact measurement. Then she says that she is going to draw exactly same triangle as the one there, the $13cm$ triangle. Then with a scale she draws $13inch$ segment, holds the scale on one of its corner and draws another $13inch$ segment. The she doesn't actually measure the third side and join the ends of two segments. Then she labels each side as $13cm$. And then she names the triangle as $\triangle PQR$.	What did you notice in this clip? How did you make the drawing of that triangle? Why did you choose that way to draw?
In this clip the teacher is asking students how do they simplify $\frac{10}{4} + \frac{7}{6}$. She asks all the students what they will do. Then she asks, "you will take LCM of?" The students respond to this by saying 4 and 6. Then she asks what is the LCM of 4 and 6. Many students say that it is 24. Meanwhile the teacher writes the title of the class as "Fractions and Decimals". After that she again asks what is the LCM and this time students loudly say, "it is 24". She asks the question, "is it 24?" and in the background starts checking it. She get the	What do you notice in this clip? What could have happened if you would have used 24 instead of 12 as suggested by the students? Why 24 is not LCM of 4 and 6? Why do we need to find LCM when we equal the denominators?

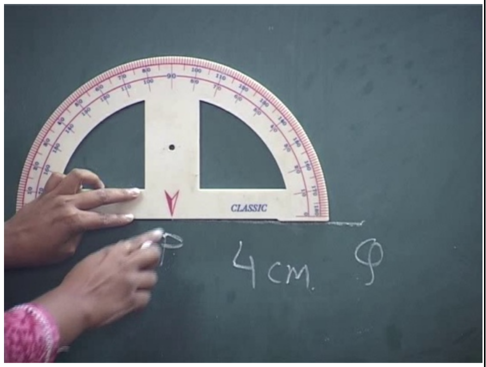
answer 12 on the board and then she asks "how much?" Then some already start saying that it is 12. Then she asks one student to get up and he answers that its 12. She says after this, "it is 12. You don't have to just multiply the two numbers but find proper LCM of it. You find out before you decide (pointing to her method on board for finding LCM), don't just multiply numbers.	
In this class the teacher tells the general rule for multiplying fractions. She tells them that whatever number of numerators are there, they need to multiply them together and same with denominators. She covers multiplying by whole numbers as the fractions with denominator 1. She also asks them to reduce the answer to the lowest form. She defines the lowest form as "the fraction where there is no common factor between numerator and denominator". And lastly when the answer is in the reduced form, she asks them to convert to the mixed fractions from. To find the mixed fraction she says that they need to divide numerator by denominator. And shows how to use quotient, remainder and divisor. At one incidence a student who makes the mistake of converting the fraction $\frac{21}{5}$ as $4\frac{21}{5}$, is asked how come he didn't know how to write mix fractions. And the student responds that he has "forgotten it". This cycle keeps on happening for various examples – multiply, reduce to lowest form and find the mixed form of it if possible.	There is no particular clip here, as she mainly took different exercise of the form $a \times c/d$ in the class and while doing that she used various explanations around it. The questions are on those. Remind her of the topic of that day. What are the most crucial things one needs to understand in multiplication of fractions? At one place a student wrote $4\frac{21}{5}$ as a mixed fraction for $\frac{21}{5}$, instead of $4\frac{1}{5}$. How could you prove that the correct mixed fraction is $4\frac{1}{5}$ and not $4\frac{21}{5}$? What is the difference and relationship between lowest fraction and a mixed fraction?
In this clip the teacher is going to talk about division of fractions. She takes the first category of the division problem as whole number divided by a fraction. She takes the problem $12 \div \frac{3}{4}$. She then says that they have to make this as a multiplication problem and multiply by the reciprocal. She asks a girl student to say the reciprocal of $\frac{3}{4}$. She gives the correct answer as $\frac{4}{3}$. She says, "so division what we do is change to multiplication by changing it by reciprocal of the fraction". The teacher then calls a student to the board who solves the problem $12 \times \frac{4}{3}$ by canceling 3 and 12, with replacing 12 as 4. The teacher says yes they can cancel numerator and denominators if they have any common factor. What is not allowed is cancel something across numerators. The teacher then writes the answer as 16. Then she summarizes the division	What did you notice in this clip? Why did you change the division problem in to a multiplication one? Why did you take reciprocal of ¾? Why not of 12? When you calculated the answer you got 16, which is bigger than 12 and ¾, how is that possible? How can one get a bigger answer by division? Can you make a word problem that would require division of a whole number by a fraction?

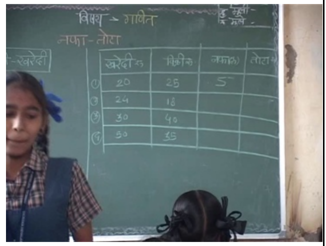
process again as multiplication by the reciprocal of the second number. Then she moves on to taking another example.	
In this class the teacher is working on profit and loss word problems. [Samreen, the girl student solving given problem on blackboard. She wrote the following on the board] Answer: Cost price = 30 x 8 = 240 rupees Selling price = 45 x 5 = 225 rupees = 35 x 3 = 105 rupees Profit = Selling price - Selling price - Cost price [She chose to subtract the selling price instead of adding those.] 105 = 225 - - 240 [She was not speaking anything but left blank space at the second place. She waited for sometime, other students were asking to interfere, the teacher suggested let her take some more time] Ss: Sir shall I go and do it? T: Let us give some more time to her. Wait. [Samreen was standing still without speaking and staring at the board. After a while, she wrote the following below the line above = 240 - 225 T: Can't do this? Do you think you can do this? Samreen: I am solving. T: If you can't solve then I will call someone else. [Even though the teacher said he would call someone else if she couldn't solve, Samreen was interested in perusing the problem. She kept on looking at the problem and was not ready to move from the board, she wrote the following as her next step.] = 105 – 15 [Without saying or discussing what is done on the board is right or wrong.] T: Good, good. Now, go and sit. Go. Someone else wants to solve this? Someone here? [Pointing to the boys' sections] Further the teacher himself solves the problem.	What did you notice in this clip? What is crucial in understanding profit and loss? What mathematics that girl student know and what she don't know?
The class is working on solving word problems on concept of area. A girl has come to the board to solve the problem. She asks is this problem same as the one we solved before, and the teacher responds – yes, it is exactly the same, only the numbers are different. The	What did you notice in this clip? Why students were writing so much to find the area or rectangle? Is there any other way to understand area of the rectangle?

teacher asks her to write bigger on the board, the girl tries but falls back to writing in small letters. All other students are watching her solve on the board. After she finishes the writing, the teacher asks, is she right? Everyone nods as yes. He says, area of rectangle is length into breadth. But she has written it too small. So the length is 15 and the width is 5, and she has multiplied 15 by 5. And then he asks how much is 15 times 5, everyone in chorus says, its 75. The teacher then says, as she has solved it correctly we will clap for her and everyone claps. The teacher then concludes that this is how you solve the problem of finding area of a rectangle when both sides are given. He further adds that if suppose the area is given and one of the dimension is given then how would one find the other dimension, and they will solve such problem after this point.	What would be the definition of area of rectangle? How is that definition related with this formula?
In this class the teacher talks about variable and teaches the students about how to solve algebraic statement. He writes an algebraic statement on the board, as $x + 4 = 9$. Then he says that the letter x here is a variable. He describes them what is a variable. He says that variable always changes. It increases or decreases. He explains that in a market the price of things keeps on changing, it either increases or decreases. He then says that something that is variable is not a constant. The he says that they will try different values for x and writes on the board that for x = 0, 1, 2, 3. He calls these 1, 2, 3 as constant numbers. Then he asks students that they will try each of this value and see whether they achieve equality. He then asks, how equality is achieved; and responds himself that it could be achieved only when both sides are equal in value. He elaborates this further – when the left side is equal to the right side then only it can be called as equality. "And we need to prove that the left side is same as the right side and then only we can call it as equality".	What did you notice in this clip? What is an algebraic expression? What is an algebraic equation? Why x + 4 = 9 would have more than one solution? Why the values of x would be ascending or descending? When x = 0 does not satisfy the equation, does one need to check for other values to see whether the given equation is equality or not? What would be an example of equality?
In this clip the teacher is doing an activity. She along with everyone has a circular cut-outs in their hands. She is also making diagrams on the board that corresponds to the actions she is making on the paper. She first asks every one to fold the circular paper such that two halves superimpose on each other. Then she asks to draw a line on the fold. While asking students to draw lines on their paper, she herself draws lines on	What did you notice in this clip? What is important in constructing circle? Why didn't you teach the definition of circle? Why understanding the relationship between radius and diameter is important?

the board. She then points to half of the diameter – assuming that it would be, she said if this is half of the diameter, what should we call it. The all agree to the name half-diameter – this is how radius is called in Urdu. Then she makes a chord by folding the paper. Her speed of doing things is too fast for many kids and they are still drawing lines on their paper. Still the teacher continues and finishes what she has planned to do. Then while making the chord she suddenly asks how many radius (half-diameter) can they make. While asking she points to the figure on the board. One of students says "unlimited" and repeats the answer as "unlimited". She then adds to that, that see these lines could be numerous; again pointing to the figure on the board.	How do we know that there are infinite radii?
The teacher is discussing construction of square. She first gives the name of square and writes it down on the board. The she writes the dimension of the board. She asks everyone that for every construction – there are three stages, and what are those three stages. Different students respond as – rough diagram, fair diagram and check the authenticity. She agrees with the students and repeats names of the three stages. Then she does the first step of making the rough diagram. Then to move on to the next step that is making a fair diagram – she asks the students to make a segment of 8 cm. She then tells them that she will be using a scale and reminds them that on the scale we always start with zero. She marks a point for zero and then counts up to 16, for 8 units – she doesn't tell this to students but she is using two units as one unit (may be for visibility).	What did you notice in this clip? What is crucial in constructing? What is a rough figure? Why it is important in mathematics?
In this clip the teacher is finding square of a binomial (a+b). She asks a student to come to the board and do the expansion. Other students are working with her simultaneously. The girl finds the answer as square of a + 2ab + square of b, the teacher reads them and says this is the expansion of the formula.	What did you notice in this clip? What is important to understand in expansion of binomials? You explained that, we need to multiply by one term at a time, why does that technique work? Can you explain?
In this clip the teacher is giving geometric verification of the algebraic expansion. At this moment the students have made a square with dimension 8, and following the instruction by the teacher. She asks them to draw lines at 5 cm on two sides, and this makes the 4 parts of the square. Then she asks them to label each part of the square as 1, 2, 3 and 4. Further she asks	What did you notice in this clip? You showed the geometric representation of (a + b) whole square, what is the meaning of that representation?

them to find area of each of the part, however, she doesn't give them any time and starts asking sub questions as what is the shape of this part. The students give numerical answer for the area, for the bigger square they give answer 25. Then asks them to find area of each part, and at last ask them to add all the areas. She points out that the sum of the area of all the parts is same as the area of the big square. Then she draws another square and names it as EFGH. And this time she makes the partition as a and b, and derives area of each part, and then sums it.	Why does the geometric representation get the same areas as the ones in the algebraic identities? What was "a" and what was "b" in your diagram? Why to make lines on 5 cm?
The teacher is teaching the topics indices. A very simple exercise is been conducted where the teacher writes; index with power and the students have to write what is base and what is index. While doing so, the teacher herself adds an example (-1)^2 and asks how would they write base and index in this case. She begins the discussion as, "when the base is negative how do you write a base", and then in the example of square of (-1), she says, "the base is negative and you have to write it as a negative". Unlike other problems where she has stopped at writing what is base and index, here she mentions that if they do not write the minus sign there will be problem. To explain what that problem could be, she writes the expansion $(-1)^2 = -1 \times -1$. Then she asks everyone, what is 1×1, and they all say 1. Then she asks what is $- \times -$ and responds herself that two negative always make it positive. Next she takes example of $(-2)^5$and calculates the answer for it using the logic she just gave above, two minus signs make it plus, and one minus sign alone is minus.	What did you notice in this clip? Why it is so important to learn the language of index and base? When do you first teach them, two minus become plus or alone minus is minus? How did you teach back then? What is the explanation for two minus to be plus and one alone minus to be minus?

In this clip the teacher is teaching construction of square, when the side is given. She first draws a segment and then uses a compass to mark a point at ninety degrees measuring at the end points of the segment. She emphasizes on how to measure the angle, as the students make more mistakes in that. She shows how to measure the angle; interestingly she herself doesn't know how to  use her instrument correctly. It is not that she doesn't know how to use the compass, however the compass she is using has specifically made for the classroom demo and therefore has a particular structure to it. See the figure here. Therefore she ends up using it wrongly and measures angle more than 90 degrees.	What did you notice in this clip? What is crucial in constructing angles for a given measurement? Why do you think the protractor does not have straight base?
In this clip the teacher has written (a + b)(a+b) on the board and asks students about what they see here. Without actually waiting to find out what students noticed she starts asking sub-questions – whether they see the same thing twice here, is there a variable and then asks what is a variable? She then tells them about variable. She provides the definition of variable as "the letter keeps changing, then it is called a variable". She further adds that "in different expressions that are given in the textbook, there are different letters, we do not use a and b again and again and therefore we call them variables".	What did you notice in this clip? What is important to understand in expansion of binomials? You explained that we need to multiply by one term at a time, why does that technique work? Can you explain?
The teacher often teaches a procedure or a rule on the board. Most of the time she is clear but she is a little fast also, she is less bilingual than the other teachers in her school. And therefore there are many students who are still blank after she has finished teaching, and then she reaches out to them and teaches the same thing on the black board.	I often observed that you did a lot of Re-teaching after the instruction. Can you tell us what is happening in your class that you have to do a lot of re-teaching?

 She is rhythmic teacher, she teaches like a poem. In this clip the teacher has given a table, with buying price and selling price. The values used are two digit numbers and easy to calculate the difference. For each pair the student have to find which is more selling or buying price and then decide whether it is profit or loss. She calls different students and asks them to write the answer in the tabular form. See the picture below.	What did you notice in this clip? What is critical in understanding profit and loss? What is the benefit of such tabular exercise? How did you choose these values?
 In this clip the teacher is discussing construction of a square when the length of the side is given. Again this teacher is using the instrument that has a special cut—for a pedagogic purpose so that one can match the line and also show students the alignment with the line of zero. However this teacher again does the similar error and measures the angle that is more than 90 degrees.	What did you notice in this clip? What is important in constructing a square? Why did you choose 3 cm as a side? The one you drew on the board is not 3 cm, why did not you tell this to students? Do you know why the protractor does not have straight edge? Why is naming of geometric figure is important?
This class the teacher keeps on giving problems of area, she begins with the definition of area, however she gives very simple examples such as 3cm side, not clear what was her goal in this class. She emphasizes a lot on writing the steps in the problem.	You taught area of square and rectangle, and used very simple examples, as the side is 3 and 10 cm or 4 and 2 cm. why were you taking so simple example? What was your goal of this class?
The teacher in this clip is teaching how to draw a 3-dimensional figure. This is how she teaches – first draw a square then find the mid-point of the figure, and draw square from that point, now join these two squares with horizontal lines. She repeats the instruction and then asks student to name the figure, reminds them how to be clock or anticlockwise they need to be.	What did you notice in this clip? How did you make the drawing of that triangle? Why did you choose that way to draw? Why do you think it is important to teach student to draw the cube?
The teacher holds a cube in her hand and says that there are three sides that we will use to find the volume, (in Marathi the term for volume and for cube has similarity). Then she says that because we use three sides to multiply we will write unit as cubic.	How do we know that volume is written in cubic cm [the teacher said because the cube has three sides when we write volume of the cube we write cm^3 What do we write cube in the unit, because there are three sides?
The teacher has brought various objects in the class that have the shape cube, cuboid and then she is	You taught surface area and volume of cube and cuboid. What is important in

discussing the total surface area formula for each of the shape	learning this? What should one learn to understand this? What mistakes student make?

Table A-G.1: Summary of the encounters

Appendix H

School and Teacher Consents

Consent to Participate in a Research Study

Title of the Project: Teachers' Encounters with Horizon Content Knowledge: Investigating the Knowledge Sensibilities for Teaching Mathematics
Principal Investigator: Shweta Shripad Naik, Doctoral Candidate, School of Education University of Michigan
Faculty Advisor: Prof. Deborah Ball, The Dean of School of Education University of Michigan

Invitation to Participate in a Research Study

I invite you to be part of a research study about teachers' encounters with horizon content knowledge. This study analyses practice of teaching closely to understand the encounters that teachers face with mathematical knowledge at the horizon and traces management of such encounters.

Description of Your Involvement

If you agree to be part of the research study, we will ask for your consent to use your classroom teaching videos and your interview audios, which we recorded with your and your school head's permission. We require the consent to analyze your teaching for identifying the encounters you face with the mathematics at the horizon.

Benefits of Participation

Although you may not directly benefit from being in this study, other than getting an opportunity to see your own teaching (in the forms of video record), prospective teachers would benefit from this study as the analysis will identify patterns of opportunity in teaching practice that would facilitate quality of instruction. The analysis will not only inform the practices that leverage the quality of instruction but also contribute to understanding of mathematical knowledge that teachers of mathematics need.

Risks and Discomforts of Participation

I do not believe that there are any risks or discomforts from participating in this research.

Confidentiality

I plan to publish the results of this study. I will not include any information that would identify you. Your privacy will be protected and your research records will be confidential.

It is possible that other people may need to see the information you give us as part of the study, such as organizations responsible for making sure the research is done safely and properly like the University of Michigan, government offices. The video or audio data of your teaching will never be showed to any of your peers, administrators or anyone related to your school system. The data will not be seen by anyone other than the researchers engaged with the research issue.

Storage and Future Use of Data

Your classroom teaching videos and interview audios are stored for the analysis of encounters and for the further research studies. Your name, and any other identifying information is not attached with any of the actual data. The data is encrypted and will be stored with password protection to maintain the confidentiality. Only Ms. Shweta Naik, the principal investigator would have information of your name and other details. During the analysis phase the data around your teaching will be shared with other researchers but utmost care will be taken such that it will not contain any information that could identify you.

Voluntary Nature of the Study

Participating in this study is completely voluntary. Even if you decide to participate now, you may change your mind and stop at any time. You do not have to answer a question you do not want to answer. Just tell me and I will go to the next question. If you decide to withdraw your consent before the study ends or anytime in future, you can do so. The contact details are provided in further section.

Contact Information for the Study Team

If you have questions about this research, including questions about your participation, you may contact

Shweta Shripad Naik (principal investigator)
315 Catherine Street, Apt. No. 5, Ann Arbor, MI
Contact: 7342772642, 919757180111(India),
shwetan@umich.edu

Prof. Deborah Loewenberg Ball (faculty advisor), William H. Payne Collegiate Professor and Arthur F. Thurnau Professor in Education; Dean, School of Education
University of Michigan
Ann Arbor, MI 48109-1259,
734.647.1637 (office phone)

If you have questions about your rights as a research participant, or wish to obtain information, ask questions or discuss any concerns about this study with someone other than the researcher(s), please contact the:

University of Michigan Health
Sciences and Behavioral Sciences
Institutional Review Board
2800 Plymouth Road
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Phone: +1 (734) 936-0933
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Mumbai-400 088, India
Tel : 091-22 2557 0813 (Direct)
Email: subra@hbcse.tifr.res.in,

Consent

By signing this document, you are agreeing to be in the study. I will give you a copy of this document for your records. I will keep one copy with the study records. Be sure that I have answered any questions you have about the study and that you understand what you are being asked to do. You may contact the researcher if you think of a question later.

I agree to participate in the study.

____

Participant's Name Signature Date

____

Researcher's Name Signature Date

Please provide a separate consent on the way this classroom teaching data was collected and how it should be used further in research.

Consent to be video recorded

I agreed to be video recorded for my classroom teaching.
YES__________ NO__________

Consent to be audio recorded

I agreed to be audio recorded for my interviews.
YES__________ NO__________

Consent to use data/specimens in future research

I agree that my data/specimens may be used in future research.
YES__________ NO__________

Consent to be contacted for participation in future research

I agree to be contacted for participation in future research.
YES__________ NO__________

Name ______________________ Signature _____________________

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