Period Identities of CM Forms on Quaternion Algebras

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in the University of Michigan
2018

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ACKNOWLEDGMENTS

Thank you to the many people whose mathematics has inspired my interests in number theory and representation theory. Firstly, I am deeply grateful to my advisor Kartik Prasanna, for explicit theta lifts and for insight, guidance, support, encouragement, and motivation. Thank you also to Atsushi Ichino, for patient, careful, and always immensely helpful explanations. Thank you to Kartik and Atsushi for making your impeccably written preprints available to me at an early stage—I could not have gotten started on this project otherwise.

I am deeply grateful to my previous advisor Mitya Boyarchenko, for Deligne–Lusztig constructions and for teaching me independence, and to Stephen DeBacker, who continued to foster the representation theorist in me after Mitya left. Thank you also to the many other people who have encouraged my ideas in mathematics, especially Bhargav Bhatt, Tasho Kaletha, Jared Weinstein, Alex Ivanov, Laurent Fargues, Peter Scholze, Elena Mantovan, Gopal Prasad, Mark Reeder, Bill Fulton, Andrew Snowden, Wei Ho. Thank you to Karen Smith, who saw me fail qualifying exams six times, for invaluable mentorship through the years. And thank you to the cognate member of my committee, Jyoti Mazumder, who was my dad's PhD advisor at UIUC.

Thank you to Ari, for math and music and many gchats in between; to Roman, who's still friends with me even after moving to New York; to Patricia, for companionship and low-FODMAPs; to Rohini and Rob, for long walks in the Arb; to KK, for piano-violin duets; to Nina, for playdates; to Julian, for discussions since the beginning. Thank you to Evita, for support from faraway Stanford and later Princeton; to Jessica, for frequent catch-ups at conferences around the world; and to Ila, who will always be my PROMYS counselor. Thank you to my long-time officemates: Derek who had the biggest computer, Harold who (briefly) trained me to run, and Mike who always tried hard on all the boulders.

Thank you to my mom, who is always there for me in every way imaginable; to my dad, who always believed I could get a PhD (minimum requirement!); to my sister, who has the determination and drive and balance and responsibility I have always wished I had.

Thank you to Eloise—for love, for sleeping through the night, and for waiting for me to submit a paper before coming into the world. And to Thomas, my everything.

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This work was partially supported by NSF grants DMS-0943832 and DMS-1160720, and by a Rackham Predoctoral Fellowship. I also wish to thank the Traverwood Branch of the Ann Arbor District Library for excellent working conditions.

TABLE OF CONTENTS

Acknowledgments . . . . **ii**

Abstract . . . . **vi**

Chapter

1 Introduction **1**

2 Automorphic representations **7**

2.1 The Tamagawa measure 7
2.2 Automorphic representations 10
2.3 Hilbert modular forms 11
2.3.1 The Shimura–Maass operator 12
2.4 The Jacquet–Langlands correspondence 12
2.5 Automorphic induction 14
2.6 Conductors 17

3 Weil representations **19**

3.1 Metaplectic groups over local fields 20
3.2 Weil indices and Leray invariants 22
3.3 The doubled Weil representation 23
3.4 Dual reductive pairs and the Howe correspondence 25

4 Waldspurger, Tunnell–Saito, and a pair of quaternion algebras **27**

4.1 The Hilbert symbol 28
4.2 Waldspurger's formula 28
4.3 A pair of quaternion algebras 30
4.4 A seesaw of unitary groups 31

5 Splittings for unitary similitude groups **36**

5.1 Kudla's splitting for split unitary groups 36
5.2 Changing polarizations 38
5.3 Three seesaws of unitary groups 38
5.3.1 Splittings for $\mathrm{G}(\mathrm{U}_E(V_0^\square) \times \mathrm{U}_E(W_0))$ and $\mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0))$. 39
5.3.2 Splittings for $\mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W^\square))$ and $\mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W))$ 41
5.4 Compatibility between the splittings for the three seesaws 42
5.5 Product formula 48
5.6 Two splittings on $E_v^\times \times \mathrm{GL}_2(F_v)$ 50

5.6.1 A splitting $\mathbf{s}$ of $z_{\mathbb{Y}'}$ . . . 52
5.6.2 A splitting $\mathbf{s}'$ of $z_{\mathbb{Y}'}$ . . . 59

6 Global theta lifts . . . 61

6.1 Theta lifts with similitudes . . . 62
6.2 The Rallis inner product formula . . . 66
6.2.1 The Siegel–Weil formula for division quaternion algebras . . . 68
6.2.2 The regularized Siegel–Weil formula for $(E^\times, \mathrm{GL}(2))$ . . . 69
6.3 Local doubling zeta integrals . . . 72
6.4 Unramified local theta lifts from GU(1) to GU(1, 1) . . . 76
6.5 Proof of Theorem 6.1 . . . 82
6.6 Period identities of CM forms . . . 85

7 Interlude . . . 87

8 Special vectors in the Weil representation . . . 88

8.1 Schwartz functions . . . 90
8.1.1 Infinite places . . . 90
8.1.2 Finite nonsplit places . . . 92
8.1.3 Finite split places . . . 100
8.2 Local zeta integrals . . . 102
8.2.1 Infinite nonsplit places . . . 102
8.2.2 Finite nonsplit places . . . 103
8.2.3 Finite split places . . . 104

9 An explicit Rallis inner product formula . . . 107

10 An example: the canonical Hecke character for $\mathbb{Q}(\sqrt{-7})$ . . . 114

10.1 Two quaternion algebras . . . 115
10.2 Torus periods of a weight-$(3+2l)$ CM form . . . 117
10.3 Relation to classical theta series . . . 119
10.4 Nonvanishing torus periods . . . 122

ABSTRACT

A few decades ago, Waldspurger proved a groundbreaking identity between the central value of an L-function and the norm of a torus period. Combining this with the Jacquet–Langlands correspondence gives a relationship between the norm of torus periods arising from different quaternion algebras for automorphic forms attached to Hecke characters. In this setting, the torus and the quaternion algebras can be realized as dual reductive pairs that are compatible in a so-called seesaw. We exploit the theta correspondence to give a direct proof of the identity of the torus periods themselves.

CHAPTER 1

Introduction

One of the central objects of modern number theory is L-functions, which date back to the influential work of Euler, Riemann, Dirichlet, and many others in the 1700s and 1800s. Dirichlet L-functions look deceptively simple: determining the zeroes of the Riemann zeta function

$$L(s, 1) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is still one of the greatest mysteries in mathematics (the Riemann Hypothesis!). Over the last several hundred years, L-functions have established themselves at the center of a rich web spanning ideas from number theory, representation theory, and algebraic geometry.

In the 1980s, Waldspurger established a formula relating L-values to *torus periods*, and this has since inspired an entire industry relating L-functions to periods of automorphic forms, shaped by the Gan–Gross–Prasad conjecture and the Ichino–Ikeda conjecture. These torus periods are weighted averages of automorphic forms: for example, if f is an automorphic form on $\mathrm{GL}_2 = \{\text{invertible } 2 \times 2 \text{ matrices}\}$, the torus period associated to f and a multiplicative function χ on $T = \{\text{diagonal matrices in } \mathrm{GL}_2\}$ is

$$\mathscr{P}(f, \chi) := \int_T f(g) \cdot \chi(g)\, dg.$$

The diagonal T here is associated to the split quadratic extension $E = F \oplus F$, though in general, the torus T can come from any quadratic extension E/F. Waldspurger's formula tells us that the behavior of torus periods for quaternion algebras—a family of algebraic objects similar to GL_2—is governed by two independent inputs:

(loc) Branching rules of *local* representation theory

(glob) Special values of *global* L-functions

Moreover, these local and global conditions uniquely determine a quaternion algebra B.

The global condition implies that the sign associated to the L-function is $\epsilon = +1$. The local condition at infinity is described by the following dichotomy: if f has weight k and χ has infinity-type $(l, 0)$, then Waldspurger's formula has two behaviors:

$k > l$	$k \leq l$
$\epsilon_\infty = -1$	$\epsilon_\infty = +1$
B definite (i.e. $B_\infty^\times \ncong \mathrm{GL}_2(\mathbb{R})$)	B indefinite (i.e. $B_\infty^\times \cong \mathrm{GL}_2(\mathbb{R})$)

The purpose of this thesis is to explain how the two sides of this dichotomy can be related in the special case that the form f comes from automorphic induction.

To this end, we fix a nonsplit torus $T = E^\times$, select two characters χ_1 and χ_2 of T, and consider forms f_1 and f_2 in their automorphic induction. By Waldspurger, considering $\mathscr{P}(f_1, \chi_2)$ and $\mathscr{P}(f_2, \chi_1)$ determines two unique quaternion algebras B_1 and B_2, and in the chart above, if B_1 lies on the left-hand side then B_2 must lie on the right-hand side, and vice versa. Our main theorem is an identity between the these two periods.

We remark that already for GL_2, the study of torus periods $\mathscr{P}(f, \chi)$ has had deep applications in arithmetic geometry: the geometry of modular curves, Iwasawa theory, progress towards the Birch and Swinnerton-Dyer conjecture. Although we do not consider arithmetic applications here, we plan to explore this in future work.

We now state a vague version of our main theorem.

Theorem (Vague version)**.** *Given Hecke characters χ_1 and χ_2, one can explicitly construct a pair of automorphic forms (f_1, f_2) on $B_1^\times$ and $B_2^\times$ such that*

$$\mathscr{P}(f_1, \chi_2) = \mathscr{P}(f_2, \chi_1).$$

In this statement, we have hidden many details. For example, if χ and χ' are not "sufficiently compatible," then both sides of the equation will always be zero. We now explain the arc of the thesis in more detail and address the nuances to the Theorem.

We first give an idea of what Waldspurger's formula looks like. For an irreducible automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, one has an associated automorphic representation π^B called the *Jacquet–Langlands transfer*. Denoting by f^B an automorphic form in π^B, the torus period associated to a Hecke character Ω of a quadratic extension $E^\times$ satisfies an identity of the form

$$|\mathscr{P}(f^B, \Omega)|^2 = * \cdot L(\mathrm{BC}(\pi) \otimes \Omega, \tfrac{1}{2}). \tag{1.1}$$

The asterisk $*$ is comprised of local factors dictated by the local representation theory, and the global L-function satisfies a functional equation centered at $s = \frac{1}{2}$. It is in this sense that

Waldspurger's formula is governed by a *local* input and a *global* input. If $*$ is nonzero, the following *central character condition* must hold:

$$\omega_\pi \cdot \Omega|_{\mathbb{A}_F^\times} = 1,$$

where ω_π denotes the central character of π. In this setting, by work of Tunnell and Saito, there is a unique quaternion algebra B such that the corresponding local factors in Waldspurger's formula are nonzero. We will consider the torus periods arising from two symmetric special cases of this: fixing two Hecke characters χ_1, χ_2 of $E^\times$, consider

(1) $\pi = \pi_{\chi_1}$ and $\Omega = \chi_2$

(2) $\pi = \pi_{\chi_2}$ and $\Omega = \chi_1$

As such, the only automorphic representations of GL_2 we will consider are those that arise as the automorphic induction π_χ of a Hecke character χ. As the central character of π_χ is $\chi|_{\mathbb{A}_F^\times} \cdot \epsilon_{E/F}$, the analogue of the central character condition for both (1) and (2) is:

$$\chi_1|_{\mathbb{A}_F^\times} \cdot \chi_2|_{\mathbb{A}_F^\times} \cdot \epsilon_{E/F} = 1. \tag{1.2}$$

Formally, the Rankin–Selberg L-function for the $(\mathrm{GL}_2 \times \mathrm{GL}_2)$-representation $\pi_{\chi_1} \otimes \pi_{\chi_2}$ satisfies

$$L(\mathrm{BC}(\pi_{\chi_1}) \otimes \chi_2, s) = L(\pi_{\chi_1} \otimes \pi_{\chi_2}, s) = L(\mathrm{BC}(\pi_{\chi_2}) \otimes \chi_1, s).$$

On the other hand, as we see in Equation (1.1), Waldspurger's formula relates (1) to the left-hand side and (2) to the right-hand side, and therefore one obtains a relationship between (the norms of) the torus periods arising from our two symmetric cases.

We will invoke the theta correspondence to construct automorphic forms. To this end, the first key point of our approach to relating these torus periods is that we will construct a *seesaw of dual reductive pairs* that precisely realizes the two quaternion algebras B_1 and B_2 arising from (1) and (2). We then carefully examine the compatibility between the theta correspondences for $B_1^\times$ and $B_2^\times$. After calculating the global theta correspondences representation theoretically, we are able to exploit the seesaw construction to directly establish an identity between the torus periods (1) and (2) themselves (not just between their norms!):

Main Theorem (6.19)**.** *There exist explicitly constructed pairs of automorphic forms* $f_1^{B_1} \in \mathrm{JL}^{B_1^\times}(\pi_{\chi_1})$ *and* $f_2^{B_2} \in \mathrm{JL}^{B_2^\times}(\pi_{\chi_2})$ *such that*

$$\mathscr{P}(f_1^{B_1}, \chi_2) = \mathscr{P}(f_2^{B_2}, \chi_1).$$

We point out an important special case of the Main Theorem. If F is totally real and E is an imaginary quadratic extension of F, then the quaternion algebras B_1 and B_2 have *complementary ramification at infinity.* For example, if $B_1 = M_2(F)$ is the split quaternion algebra, then B_2 is a totally definite quaternion algebra and the main theorem produces a pair (f_1, f_2) of automorphic forms on $\mathrm{GL}_2(\mathbb{A}_F)$ and $B_{2,\mathbb{A}}^\times$. In this setting we have the following theorem:

Theorem (9.1, 9.3)**.** *If F is totally real and $B_1^\times = \mathrm{GL}_2(F)$, one can arrange for f_1 to be any nonzero Hecke eigenform of positive weight and its Petersson inner product can be described explicitly in terms of a special value of an L-function. Furthermore, the corresponding form f_2 on $B_2^\times$ is an explicitly constructed automorphic form on a* definite *quaternion algebra.*

We now give an outline of the present thesis. We begin by establishing some background. In Chapter 2, we recall the construction of the Tamagawa measure, the basic definitions of automorphic forms and representations, and explicitly describe automorphic induction and the Jacquet–Langlands correspondence. These results will be used in Chapter 6 to characterize the global theta lifts representation theoretically.

As our main tool is the theta correspondence, we need to understand the Weil representation, and we spend Chapter 3 recalling these constructions.

In Chapter 4, we give a brief summary of Waldspurger's formula and the ϵ-dichotomy of Tunnell–Saito. In Section 4.3, we give a simple description of the relationship between B_1 and B_2. We then construct dual reductive pairs $(\mathrm{U}_B(V), \mathrm{U}_B(W^*))$ and $(\mathrm{U}_E(\operatorname{Res} V), \mathrm{U}_E(W))$ in Section 4.4 that *both* capture the behavior of $E^\times \subset B_1^\times, B_2^\times$ and *also* map into a shared symplectic group. The goal of the rest of the paper is to study the following seesaw of similitude unitary groups with respect to the theta correspondence:

$$\begin{array}{ccccccc} \mathrm{GU}_E(\operatorname{Res} V) & & \mathrm{GU}_B(W^*) & & B_2^\times & & B_1^\times \\ | & \times & | & \text{“ = ”} & | & \times & | \\ \mathrm{GU}_B(V) & & \mathrm{GU}_E(W) & & E^\times & & E^\times \end{array}$$

In Chapter 5, we use Kudla's splittings for unitary groups and explicitly study their compatibility on $E^\times \times E^\times$. Many of the calculations are similar to the calculations in [IP16b]. From the compatibility statements about the splittings, we can deduce precise information about how the Weil representations on $\mathrm{GU}_B(V) \times \mathrm{GU}_B(W^*)$ and $\mathrm{GU}_E(\operatorname{Res} V) \times \mathrm{GU}_E(W)$ are related.

In Chapter 6, we give a representation theoretic description of the global theta lifts. This requires a careful study of Kudla's splittings at the places v where everything is unramified

(Section 5.6). We prove (Theorem 6.1) that the global theta lifts can be described in terms of automorphic induction and Jacquet–Langlands and that the global theta lift vanishes if and only if the Jacquet–Langlands transfer does not exist. Combining these results with the compatibility results of Chapter 5, we obtain our Main Theorem (Theorem 6.19).

In Chapter 8, we explicitly construct local Schwartz functions which are well behaved under the Weil representation. These Schwartz functions have been considered in various places before. At the finite places, they have appeared for example in [P06, Proposition 2.5.1], [X07, N1]. At the infinite places, our choice is constructed from a confluent hypergeometric function ${}_1F_1(a,b,t)$ of the first type. This is related to the role of hypergeometric functions in matrix coefficients of representations of $\mathrm{SL}_2(\mathbb{R})$ (see for example [X07, Appendix], [VK91, Chapters 6, 7]).

We see in Chapter 9 that in the special case that F is totally real, E is a CM extension of F, and B_1 is split, the theta lifts of these Schwartz functions exactly produce all of the Hecke eigenfunctions of positive weight. We remark that by construction (see Section 6.1), negative-weight Hecke eigenforms are not theta lifts since they are not supported on $\mathrm{GL}_2(F)\,\mathrm{GL}_2(\mathbb{A}_F)^+$. The first step towards showing that the theta lifts give all the Hecke eigenfunctions is seeing that they are nonzero. This is done by analyzing a doubled seesaw of the form

$$\begin{tikzcd}
\mathrm{GU}_E(1)\times\mathrm{GU}_E(1) \arrow[d, dash] \arrow[dr, dash] & \mathrm{GU}_E(4) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{GU}_E(1) & \mathrm{GU}_E(2)\times\mathrm{GU}_E(2)
\end{tikzcd}$$

to obtain a Rallis inner product (Section 6.2), which has the shape

$$\langle \theta_{1,\varphi}(\chi_1), \theta_{1,\varphi}(\chi_1)\rangle = \langle 1, \mathrm{Eis}\rangle.$$

Note that to establish such a formula, one first needs to establish compatibility between the various splittings. Following similar computations in [IP16b], this is done in Section 5.3. Another point of subtlety in the doubling method is due to convergence problems. In the case that B is division, there are no issues, and in the case that B is split, this can be handled by regularizing the theta integral and using the regularized Siegel–Weil formula [GQT] in the second-term range. This gives us a Rallis inner product formula relating Petersson inner product of $\theta_{1,\varphi}(\chi_1)$ to the L-value $L(\widetilde{\chi}_1, 1)$. We calculate the associated doubling zeta integral so that we can completely explicate the formula (Theorem 9.1). We then use Casselman's theorem to show that our theta lifts give all the Hecke eigenforms of positive weight (Theorem 9.3), and we prove an algebraicity result in the case $F = \mathbb{Q}$ using Shimura's algebraicity theorems (Theorem 9.4).

In the final chapter, Chapter 10, we discuss the above construction for the canonical Hecke character χ_{can} of $\mathbb{Q}(\sqrt{-7})$. This is the simplest example of the theorems, and in this setting one can calculate the theta lift directly as well. We do this and compare the theta lift to $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ to a classical theta series. Furthermore, we show that the torus period for the Hecke eigenform on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ of appropriate weight is nonzero. By the Main Theorem (Theorem 6.19), we have an explicitly constructed automorphic form on a *definite* quaternion algebra whose corresponding torus period is nonvanishing.

CHAPTER 2

Automorphic representations

In this chapter, we recall several constructions in the theory of automorphic forms. For a number field F, let $\mathcal{O}$ be the ring of integers of F and $\mathcal{D}$ the different of F over $\mathbb{Q}$. Let r_1 be the number of real embeddings of F and $2r_2$ be the number of complex embeddings of F. For each finite place v of F, let $\mathcal{O}_v$ be the ring of integers of F_v, π_v a uniformizer of $\mathcal{O}_v$, and q_v the cardinality of the residue field $\mathcal{O}_v/\pi_v$. Let $D = D_F$ be the discriminant of F and for each finite place v of F, let d_v be the non-negative integer such that $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{O}_v = \pi_v^{d_v}\mathcal{O}_v$. Set $\delta_v = \pi_v^{-d_v}$. Then $|D| = \prod_{v\nmid\infty} q_v^{d_v}$.

The main groups in this thesis are $\mathbb{A}_E^\times$ and $\mathbb{A}_E^1$ where E/F is a quadratic extension, and $B_{\mathbb{A}}^\times$, where B is a quaternion algebra over F. For shorthand, we write

$$[E^\times] := \mathbb{A}_F^\times E^\times \backslash \mathbb{A}_E^\times, \qquad [E^1] := E^1 \backslash \mathbb{A}_E^1, \qquad [B^\times] := \mathbb{A}_F^\times B^\times \backslash B_{\mathbb{A}}^\times,$$

where in the last definition, we view $\mathbb{A}_F^\times$ as the center of $B_{\mathbb{A}}^\times$.

2.1 The Tamagawa measure

To begin, one must establish what it means to integrate over an adelic group. There is a canonical Haar measure on an adelic group known as the *Tamagawa measure*. We recall this construction here and explicate the Tamagawa measure for a few special cases we will need in Chapters 9 and 10.

Fix an additive character ψ of F. Let $dx = \prod_v dx_v$ be the measure on $\mathbb{A}_F$ that is self-dual with respect to ψ. For a connected reductive group G defined over F that splits over E, let $X(G)$ be the lattice of rational characters on G. Then $X(G) \otimes \mathbb{Q}$ is a $\mathrm{Gal}(E/F)$-module of dimension n which we will denote by ρ_G and we let $L_v(s, G)$ be the v-component of the Artin L-function corresponding to ρ_G. That is,

$$L_v(s, G) = \det(I_n - q_v^{-s}\rho_G(\sigma_v))^{-1},$$

where σ_v is the Frobenius conjugacy class in $\mathrm{Gal}(E/F)$. Following [L80, 1.7], let ω be an F-rational left-invariant nowhere vanishing differential form of highest degree on G. The Tamagawa measure on $G(\mathbb{A}_F)$ is

$$dg = \lim_{s\to 1} \frac{1}{(s-1)^r L(s,G)} \prod_v dg_v, \qquad \text{where } dg_v = \begin{cases} L_v(1,G)|\omega|_v & \text{for finite } v, \\ |\omega|_v & \text{for infinite } v, \end{cases}$$

where r is the rank of $X(G)_F$. This measure is independent of the choice of additive character ψ (which determined the measure on $\mathbb{A}_F$) and the choice of F-rational differential form ω (by the product formula).

Our calculations in Chapters 9 and 10 will require some more explicit information about certain measures in some special cases. We explicate this here now.

2.1.0.1

The *standard additive character* of $F\backslash\mathbb{A}_F$ is $\psi := \psi_0 \circ \mathrm{Tr}_{F/\mathbb{Q}}$, where $\psi_0 = \otimes_v \psi_{0,v}$ is the non-trivial additive character of $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ given by

$$\psi_{0,v}(x) = \begin{cases} e^{2\pi\sqrt{-1}x} & \text{if } v = \infty, \\ e^{-2\pi\sqrt{-1}x} & \text{if } v \nmid \infty. \end{cases}$$

Observe that if v is a finite place of F, then ψ_v is trivial on $\pi_v^{-d_v}\mathcal{O}_{F_v}$ but nontrivial on $\pi_v^{-d_v-1}\mathcal{O}_{F_v}$. The measure dx on $\mathbb{A}_F$ that is self-dual with respect to ψ has the property that:

- If v is finite, then $\mathrm{vol}(\mathcal{O}_{F_v}, dx_v) = q_v^{-d_v/2}$.
- If v is infinite, then dx_v is the Lebesgue measure.

More generally, if ψ' is any additive character of $\mathbb{A}_F$, then for any finite place v, we have $\mathrm{vol}(\mathcal{O}_v, dx_v) = q_v^{c(\psi_v)/2}$, where $c(\psi_v)$ is the smallest integer such that ψ_v is trivial on $\pi_v^{c(\psi_v)}\mathcal{O}_{F_v}$.

2.1.0.2

For any number field k, put

$$\rho_k := \mathrm{Res}_{s=1}\, \zeta_F(x) = \frac{2^{r_1}(2\pi)^{r_2} hR}{|D|^{1/2} w},$$

where r_1 is the number of real places of k, r_2 is the number of complex places of k, $h = h_k$ is the class number of k, $R = R_k$ is the regulator of k, $D = D_k$ is the discriminant of k, and

$w = w_k$ is the number of roots of unity in k. Then the Tamagawa measure of $\mathbb{A}_k^\times$ is

$$d^\times x^{\mathrm{Tam}} = \rho_k^{-1} \cdot \prod_v d^\times x_v^{\mathrm{Tam}},$$

where

$$d^\times x_v^{\mathrm{Tam}} := \begin{cases} (1 - q_v^{-1})^{-1} dx_v/|x|_v & \text{if } v \text{ is finite,} \\ dx_v/|x|_v & \text{if } v \text{ is infinite.} \end{cases}$$

Observe that if v is finite, then $\mathrm{vol}(\mathcal{O}_v^\times, d^\times x_v^{\mathrm{Tam}}) = q_v^{-d_v/2}$. The Tamagawa number of $\mathbb{G}_m$ is 1, i.e. $\mathrm{vol}(k^\times \backslash \mathbb{A}_k^\times, d^\times x^{\mathrm{Tam}}) = 1$.

2.1.0.3

The previous example explicitly describes the Tamagawa measure of $\mathbb{A}_F^\times$ and $\mathbb{A}_E^\times$. For each place v of F, one has a short exact sequence

$$1 \to F_v^\times \to E_v^\times \to E_v^1 \to 1,$$

and hence we may define a local measure $d^1 g_v^{\mathrm{Tam}}$ on E_v^1 as the quotient measure. Then the Tamagawa measure of $E_{\mathbb{A}}^1$ is

$$d^1 g^{\mathrm{Tam}} := \frac{\rho_F}{\rho_E} \cdot \prod_v d^1 x_v^{\mathrm{Tam}}.$$

Observe that if v is a finite place of F, then

$$\mathrm{vol}(E_v^1 \cap \mathcal{O}_{E_v}^\times, d^1 x_v^{\mathrm{Tam}}) = \begin{cases} q_{F_v}^{-1/2} & \text{if } v \text{ ramifies in } E, \\ q_{F_v}^{-d_{F_v}/2} & \text{if } v \text{ is inert or split in } E. \end{cases}$$

Observe that $\mathrm{vol}(E_v^1 \cap \mathcal{O}_{E_v}^\times, d^1 x_v^{\mathrm{Tam}}) = 1$ for all but finitely many places v. If F is totally real and E/F is totally imaginary, then one can show (for example by calculating the measure of an annulus in $\mathbb{C}$ containing the unit circle) that

$$\mathrm{vol}(\mathbb{C}^1, d^1 x_\infty^{\mathrm{Tam}}) = 2\pi.$$

We also have

$$\frac{\rho_F}{\rho_E} \sim \pi^{-1},$$

where we write $\sim$ to denote equality up to an algebraic integer; i.e. $a \sim b$ if $a/b \in \overline{\mathbb{Q}}$.

2.2 Automorphic representations

We briefly recall the definition of an automorphic representation of $G(\mathbb{A}_F)$, where G is an arbitrary reductive group over F.

Definition 2.1. Define $L^2(G(F)\backslash G(\mathbb{A}_F), \psi)$ to be the space of measurable functions $\phi\colon G(\mathbb{A}_F) \to \mathbb{C}$ satisfying the following conditions:

(i) For all $\gamma \in G(F)$,
$$\phi(\gamma g) = \phi(g);$$

(ii) For all $z \in Z_{\mathbb{A}} = Z(G(\mathbb{A}_F))$ and $g \in G(\mathbb{A}_F)$,
$$\phi(gz) = \phi(zg) = \psi(z)\phi(g);$$

(iii) $\displaystyle\int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} |\phi(g)|^2\, dg < \infty.$

Define $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}_F), \psi)$ to be the space of $\phi \in L^2(G(F)\backslash G(\mathbb{A}_F), \psi)$ such that ϕ satisfies the cuspidal condition:
$$\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(ng)\, dx = 0 \qquad \text{for almost every } g \in G_{\mathbb{A}},$$

where N is the unipotent radical of any proper parabolic F-subgroup of G.

Example 2.2. In this thesis, we will only be concerned with automorphic forms and representations of $G_{\mathbb{A}} = B_{\mathbb{A}}^\times$, where B is a quaternion algebra over F. The cuspidality condition has two distinctive cases: If $B = M_2(F)$ (i.e. B is split), we say $\phi \in L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F), \psi)$ if
$$\int_{F\backslash \mathbb{A}_F} \phi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right)\, dx = 0 \qquad \text{for almost every } g \in \mathrm{GL}_2(\mathbb{A}_F).$$
If $B \neq M_2(F)$, then B is a division quaternion algebra and has no proper parabolics. Hence the cuspidality condition is empty and $L^2_{\text{cusp}}(B^\times\backslash B_{\mathbb{A}}^\times, \psi) = L^2(B^\times\backslash B_{\mathbb{A}}^\times, \psi)$.

Definition 2.3 (Automorphic representations and automorphic forms)**.**

- An *automorphic representation* with central character ψ is an irreducible admissible representation of $G(\mathbb{A}_F)$ which is contained in $L^2(G(F)\backslash G(\mathbb{A}_F), \psi)$. A *cuspidal automorphic representation* with central character ψ is an irreducible admissible representation of $G(\mathbb{A}_F)$ which is contained in $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}_F), \psi)$.

- An *automorphic form* is an element of an automorphic representation and a *cusp form* is an element of a cuspidal automorphic representation.

2.3 Hilbert modular forms

In this section we review the relationship between automorphic forms for $\mathrm{GL}_2(\mathbb{A}_F)$, where F is a totally real field, and classical Hilbert modular forms.

Let $h = h_F$ be the narrow class number of F and let $\{t_i\}_{i=1}^h$ be elements of $\mathbb{A}_F$ whose infinity part is 1 and that form a complete set of representatives of the narrow class group. Then

$$\mathrm{GL}_2(\mathbb{A}_F) = \bigsqcup_{i=1}^{h} \mathrm{GL}_2(F) \begin{pmatrix} t_i^{-1} & \\ & 1 \end{pmatrix} \mathrm{GL}_2(F_\infty)^+ K(\mathfrak{n}), \tag{2.1}$$

where

$$K(\mathfrak{n}) := \prod_{v \nmid \infty} K_v(\mathfrak{n}),$$
$$K_v(\mathfrak{n}) := d(\pi_v^{-d_v})^{-1} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : c \in \mathfrak{n}_v \mathcal{O}_{F_v} \right\} d(\pi_v^{-d_v}).$$

Define

$$\Gamma_i(\mathfrak{n}) := \left\{ \begin{pmatrix} a & t_i^{-1} b \\ t_i c & d \end{pmatrix} : a \in \mathcal{O},\, b \in \mathcal{D}^{-1},\, c \in \mathfrak{n}\mathcal{D},\, d \in \mathcal{O} \right\}.$$

Let $\gamma = (\gamma_1, \ldots, \gamma_{r_1}) \in \mathrm{GL}_2(\mathbb{R})^{r_1}$ and write $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ for each $i = 1, \ldots, r_1$. There is a natural action of γ on $\mathfrak{h}^{r_1}$ by

$$\gamma * (z_1, \ldots, z_{r_1}) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_{r_1} z_{r_1} + b_{r_1}}{c_{r_1} z_{r_1} + d_{r_1}} \right).$$

For a function f on $\mathfrak{h}^{r_1}$, an element $\gamma \in \mathrm{GL}_2(\mathbb{R})^n$, and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^{r_1}$, define the slash operator

$$f|_{[\gamma]_k}(z) := \det(\gamma)^{k/2} (cz + d)^{-k} f(\gamma * z).$$

A Hilbert modular form of weight $k = (k_1, \ldots, k_{r_1}) \in \mathbb{Z}^{r_1}$ is a function f on $\mathfrak{h}^{r_1}$ such that for some character ω of $(\mathcal{O}/\mathfrak{n})^\times$,

$$f|_{[\gamma]_k}(z) = \omega(a) f(z), \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_i(\mathfrak{n}).$$

We define the Petersson inner product of two Hilbert modular forms f, g of weight k to be

$$\langle f, g \rangle := \sum_{i=1}^{h} \mu(\Gamma_i \backslash \mathfrak{h}^{r_1})^{-1} \int_{\Gamma_i \backslash \mathfrak{h}^{r_1}} f_\nu(z) \overline{g_\nu(z)} y^k d\mu(z), \tag{2.2}$$

where $d\mu(z) = \prod_{j=1}^{r_1} y_j^{-2} dx_j dy_j$.

Writing $f = (f_1, \ldots, f_{r_1})$ and using (2.1), one can define the associated function $\mathbf{f}$ on $\mathrm{GL}_2(\mathbb{A}_F)$ by

$$\mathbf{f}(\gamma x_i g_\infty k_0) = (f_i|_{[g_\infty]_k})(\mathbf{i})\omega_f(d), \tag{2.3}$$

where $\gamma \in \mathrm{GL}_2(F)$, $g_\infty \in \mathrm{GL}_2^+(F_\infty)$, $k_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K(\mathfrak{n})$, $\mathbf{i} = (i, \ldots, i)$, and ω_f is the finite part of ω.

2.3.1 The Shimura–Maass operator

The Shimura–Maass differential operator

$$\delta_k := \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{k}{z - \overline{z}}\right)$$

maps real analytic modular forms of weight k to real analytic modular forms of weight $k+2$. Define the composite operator

$$\delta_k^l := \delta_{k+2l} \circ \cdots \circ \delta_{k+2} \circ \delta_k$$

mapping real analytic modular forms of weight k to real analytic modular forms of weight $k + 2l$. Applying this to each coordinate $z_i \in \mathfrak{h}$ for a Hilbert modular form on $\mathfrak{h}^{r_1}$, we see that we have an operator mapping

$$\left\{\begin{matrix}\text{real analytic Hilbert modular forms} \\ \text{of weight } k = (k_1, \ldots, k_{r_1})\end{matrix}\right\} \to \left\{\begin{matrix}\text{real analytic Hilbert modular forms} \\ \text{of weight } k + 2l = (k_1 + 2l_n, \ldots, k_{r_1} + 2l_{r_1})\end{matrix}\right\}.$$

2.4 The Jacquet–Langlands correspondence

See [B01], [JL] for more details. Let k be a local field and let D be the unique nonsplit quaternion algebra over k, if it exists. Note here that if $k = \mathbb{C}$, then the only quaternion algebra over k is the split quaternion algebra $M_2(\mathbb{C})$, so the Jacquet–Langlands correspondence is trivial.

We say that $g \in \mathrm{GL}_2(k)$ or $g' \in D^\times$ is *regular semisimple* if its characteristic polynomial has distinct roots over $\bar{k}$. If $g \in \mathrm{GL}_2(k)$ and $g' \in D^\times$ are regular semisimple elements with the same characteristic polynomial, we write $g \sim g'$.

Theorem 2.4 (Local Jacquet–Langlands). *There exists a unique bijection*

$$\left\{\begin{matrix} \textit{irreducible} \\ \textit{representations of } D^\times \end{matrix}\right\} \overset{\mathrm{JL}}{\longleftrightarrow} \left\{\begin{matrix} \textit{irreducible discrete series} \\ \textit{representations of } \mathrm{GL}_2(k) \end{matrix}\right\}$$

such that for any irreducible representation π' of $D^\times$, the central characters of π' and $\mathrm{JL}(\pi')$ coincide and

$$\chi_{\pi'}(g) = \chi_{\mathrm{JL}(\pi')}(g') \qquad \textit{for all } g \sim g'.$$

The behavior of the representation theory of $\mathrm{GL}_2(k)$ can be categorized into two cases: when $k = \mathbb{R}$ and when k is a non-Archimedean local field.

- Let $k = \mathbb{R}$. It is well known that the irreducible representations of $D^\times$ are of the form

  $$\pi'(h) = \mathrm{Nm}(h)^r \rho_k(h),$$

  for some $r \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$. Here, $\mathrm{Nm}\colon D^\times \to k^\times$ is the reduced norm map and $\rho_n \cong \mathrm{Sym}^n(\mathbb{C}^2)$ for the standard representation $\mathbb{C}^2$. Then

  $$\mathrm{JL}(\pi') = \sigma(\mu_1, \mu_2),$$

  where $\mu_1(t) = |t|^{r+k+1/2}$ and $\mu_2(t) = |t|^{r-1/2}\,\mathrm{sgn}(t)^k$. Here, $\sigma(\mu_1, \mu_2)$ is isomorphic to the representation of $\mathrm{GL}_2(\mathbb{R})$ generated by $\{\ldots, \phi_{-k-3}, \phi_{-k-2}, \phi_{k+2}, \phi_{k+4}, \ldots\}$, where for $n \in \mathbb{Z}$,

  $$\phi_n\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\right) = \mu_1(t_1)\mu_2(t_2)\left|\frac{t_1}{t_2}\right|^{1/2} e^{-in\theta}. \tag{2.4}$$

  Note that this defines $\phi_n\colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$ since $\mathrm{GL}_2(\mathbb{R}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\left\{\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\right\}$ by the Iwasawa decomposition. In summary,

  $$\mathrm{JL}(\text{twist of } \mathrm{Sym}^k(\mathbb{C}^2)) = \text{weight-}(k+2) \text{ representation of } \mathrm{GL}_2(\mathbb{R}).$$

- Let k be a non-Archimedean local field. The representation theory of $D^\times$ and $\mathrm{GL}_2(k)$ here is more complicated, and hence the Jacquet–Langlands correspondence cannot be described as explicitly as in the real case. In this case, the set of square-integrable representations of $\mathrm{GL}_2(k)$ consists of twists of Steinberg representations and supercuspidal representations. The Steinberg representation St is the representation $\mathrm{Fun}(\mathbb{P}^1)/\mathbb{C}$, where we view $\mathbb{C}$ as the subspace of $\mathrm{Fun}(\mathbb{P}^1)$ consisting of the constant functions—it is the unique nontrivial quotient of the representation obtained by parabolically inducing from the trivial representation. Supercuspidal representations of $\mathrm{GL}_2(k)$ are the representations that do not arise via parabolic induction.

There is also a global version of the Jacquet–Langlands correspondence. Let B be a quaternion algebra over F and let S be the set of ramified places of B:

$$S := \{v : v \text{ is a place of } F \text{ and } B_v \text{ is nonsplit}\}.$$

Note that S is a finite set of even cardinality.

Theorem 2.5 (Global Jacquet–Langlands)**.** *There is a unique injection*

$$\mathrm{JL}\colon \left\{ \begin{matrix} \textit{irreducible } \dim > 1 \textit{ automorphic} \\ \textit{representations of } D_{\mathbb{A}}^\times \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{irreducible cuspidal automorphic} \\ \textit{representations of } \mathrm{GL}_2(\mathbb{A}_F) \end{matrix} \right\}$$

such that for any $\pi' = \otimes_v \pi'_v$, we have $\mathrm{JL}(\pi')_v \cong \mathrm{JL}_v(\pi'_v)$, where JL_v is the local Jacquet–Langlands correspondence as in Theorem 2.4. The image of JL *consists of the cuspidal automorphic representations $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_v is discrete series for all $v \in S$.*

2.5 Automorphic induction

In [JL], Jacquet and Langlands construct a special class of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ arising from automorphic representations of $\mathbb{A}_E^\times$:

$$\left\{ \begin{matrix} \text{automorphic representations} \\ \text{of } \mathbb{A}_E^\times \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{certain automorphic representations} \\ \text{of } \mathrm{GL}_2(\mathbb{A}_F) \end{matrix} \right\}$$

Let π_χ denote the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to the character $\chi\colon E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$. The representation π_χ enjoys the following property: at almost all places v of F,

$$L(\pi_{\chi,v}, s) = \prod_{w|v} L(\chi_w, s),$$

where the product runs through all places of E dividing v.

There is a (conjectural) notion of automorphic induction as well wherein if E/F is a degree-d extension, each automorphic representation of $\mathrm{GL}_m(\mathbb{A}_E)$ can be assigned an automorphic representation of $\mathrm{GL}_{md}(\mathbb{A}_F)$ satisfying the natural analogue of the above local L-factor condition. Many people have worked on this problem: Jacquet–Langlands [JL] ($m = 1, d = 2$), Clozel [C86], Henniart [H12] (m, d general, E/F cyclic).

We return our focus to automorphic induction to $\mathrm{GL}_2(\mathbb{A}_F)$. The automorphic induction of a Hecke character χ of $E^\times$ can be described at, and is determined by, all but finitely many places.

· Let v be such that $E_v/F_v = \mathbb{C}/\mathbb{R}$. Suppose

$$\chi_v(z) = (z\overline{z})^r z^m \overline{z}^n,$$

where $r \in \mathbb{C}$ and m, n are two integers, one zero and the other positive. Then define

$$\pi_v := \sigma(\mu_1, \mu_2), \qquad \text{where } \mu_1(t) = |t|^r t^{m+n},\ \mu_2(t) = |t|^r.$$

Here, $\sigma(\mu_1, \mu_2)$ is the representation defined in (2.4) of weight $m + n + 1$.

· Let v be a place of F which splits completely in E with divisors w and $\overline{w}$. Since $E_w \cong E_{\overline{w}} \cong F_v$, the characters $\chi_w, \chi_{\overline{w}}$ can be viewed as characters of $F_v^\times$. Define

$$\pi_v := \mathrm{Ind}_B^{\mathrm{GL}_2(F_v)}(\chi_w, \chi_{\overline{w}}).$$

· Let v a place of F which lies under a single prime w of E. Then E_w is a quadratic extension of F_v and χ_w is a character of $E_w^\times$. If χ_w factors through $\mathrm{Nm}\colon E_w^\times \to F_v^\times$, write $\chi_w = \chi_{w,0}(\mathrm{Nm})$ and define

$$\pi_v := \mathrm{Ind}_B^{\mathrm{GL}_2(F_v)}(\chi_{w,0}, \chi_{w,0}\epsilon_{E_v/F_v}).$$

Note that if χ_w is unramified (i.e. trivial on $\mathcal{O}_{E_w}^\times$), then χ_w factors through Nm.

Theorem 2.6 (Jacquet–Langlands)**.** *There exists a unique irreducible automorphic representation $\pi_\chi = \otimes_v \pi_{\chi,v}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that*

$$\pi_{\chi,v} \cong \pi_v$$

for all v such that either v splits completely, or v lies under a single prime w of E and χ_w factors through $\mathrm{Nm}\colon E_w^\times \to F_v^\times$. If v lies under a single prime w and χ_w does not factor through Nm, then $\pi_{\chi,v}$ is a supercuspidal representation of $\mathrm{GL}_2(F_v)$. Furthermore, if χ does not factor through $\mathrm{Nm}\colon \mathbb{A}_E^\times \to \mathbb{A}_F^\times$, then π_χ is cuspidal.

We may now combine automorphic induction with the global Jacquet–Langlands correspondence to obtain a mapping

$$\left\{\begin{matrix}\text{certain automorphic representations}\\ \text{of } \mathbb{A}_E^\times\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{certain automorphic representations}\\ \text{of } B_{\mathbb{A}}^\times\end{matrix}\right\}$$

Note the insertion of the word "certain" in the source: while one could construct an automorphic representation for $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to each Hecke character of E, not every automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ transfers to a representation of $B_{\mathbb{A}}^\times$!

Let B be a quaternion algebra over F containing E and let S_B be the set of places of F where B is ramified. For any Hecke character χ of E, let S_χ denote the set of places of F such that χ_w does not factor through Nm_w. Then composing Theorem 2.6 with Theorem 2.5 gives:

Theorem 2.7. *There exists a correspondence*

$$\left\{\begin{matrix}\textit{characters } \chi \textit{ of } \mathbb{A}_E^\times\\ \textit{with } S_\chi \supset S_B\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\textit{certain automorphic representations}\\ \textit{of } B_{\mathbb{A}}^\times\end{matrix}\right\}$$

given by

$$\chi \mapsto \pi_\chi^B := \mathrm{JL}^B(\pi_\chi).$$

Moreover, π_χ^B is the unique automorphic representation of $B_{\mathbb{A}}^\times$ such that: at every place v satisfying either

a) v splits completely (in which case $B_v^\times \cong \mathrm{GL}_2(F_v)$), or

b) v lies under a single prime w of E and χ_w factors through Nm *(in which case $B_v^\times \cong \mathrm{GL}_2(F_v)$ by the assumption $S_\chi \supset S_B$),*

we must have

$$(\pi_\chi^B)_v \cong \pi_{\chi,v} \cong \pi_v.$$

Remark 2.8. It is interesting to ask what automorphic induction looks like locally. That is, given a non-archimedean local field k and a degree-n extension L of k, one would like to construct a map

$$\left\{\text{characters } \chi \text{ of } L^\times\right\} \longrightarrow \left\{\begin{matrix}\text{irreducible representation of } \mathrm{GL}_m(D)\\ \text{where } D \text{ is a dimension } (n/m)^2 \text{ division algebra over } k\end{matrix}\right\}.$$

This has been done algebraically by work of many people, including Corwin, Howe, Henniart, Herb. As Langlands correspondences typically arise in cohomology (as this is often how one constructs Galois representations), it is interesting to ask whether the above correspondence can be realized geometrically. In the setting that k is a finite field, this is answered by (a special case of) Deligne–Lusztig theory, whose story began in 1976 [DL76]. A few years later, Lusztig [L79] proposed a p-adic analogue of a Deligne–Lusztig variety, though now, 40 years past, still not much is known. It is expected that (a special case of) Lusztig's geometric construction should realize the above correspondence for L/k unramified. Following an approach initiated by Boyarchenko [B12] in 2012, I studied the $m = 1$ case of this correspondence in a series of papers [C16ad] [C15] [C17si]. In forthcoming joint work with A. Ivanov [CI18], we study Lusztig's construction for $\mathrm{GL}_m(D)$ and prove an isomorphism between the varieties from [L79] and certain affine Deligne–Lusztig varieties, which are closely related to Shimura varieties. $\Diamond$

2.6 Conductors

In this section we briefly review the notion of the conductor of an admissible representation. First let k be a non-Archimedean local field with ring of integers $\mathcal{O}_k$ and a fixed uniformizer π. For any integer $N \in \mathbb{Z}_{\geq 0}$, let

$$K_0'(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_k) : c \in \pi^N \mathcal{O}_k \right\}.$$

Theorem 2.9 (Casselman)**.** *Let ρ be an irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(k)$ with central character ω. Let $c(\rho) \in \mathbb{Z}_{\geq 0}$ be the smallest integer such that*

$$\left\{ v \in \rho : \rho(g)v = \omega(a)v \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0'(c(\rho)) \right\} \neq \{0\}.$$

Then this space has dimension one.

We call $c(\rho)$ the *conductor* of ρ. For a smooth character $\chi \colon k^\times \to \mathbb{C}^\times$, define its *conductor* $c(\chi) \in \mathbb{Z}_{\geq 0}$ to be the smallest number such that

$$\chi|_{U_k^{c(\chi)}} = 1, \qquad \text{where } U_k^n = \begin{cases} \mathcal{O}_k^\times & \text{if } n = 0, \\ 1 + \pi^n \mathcal{O}_k & \text{if } n > 0. \end{cases}.$$

It will be useful for us to have an explicit description of $c(\pi_\chi)$ in terms of $c(\chi)$. The next proposition follows from facts about Artin conductors of Galois representations and the fact that conductors of admissible representations of $\mathrm{GL}_2(k)$ are compatible with Artin conductors of Galois representations under the local Langlands correspondence.

Proposition 2.10. *Let L be a degree-2 extension of k. Let χ be a smooth character of $L^\times$.*

(a) If L/k is split, then $\chi = \chi_1 \otimes \chi_2$ and

$$c(\pi_\chi) = c(\chi_1) + c(\chi_2).$$

(b) If L/k is unramified, then

$$c(\pi_\chi) = \mathrm{val}_k(4) + 2c(\chi).$$

(c) If L/k is ramified, then

$$c(\pi_\chi) = 1 + \mathrm{val}_k(4) + c(\chi).$$

CHAPTER 3

Weil representations

Let k be any field. Let $\mathbb{V}$ be a symplectic vector space over k. The Weil representation of $\mathrm{Sp}(\mathbb{V})$ is a representation of a cover of $\mathrm{Sp}(\mathbb{V})$. It arises in a very natural way, which we briefly recall. The symplectic space $\mathbb{V}$ gives rise to a Heisenberg group $H(\mathbb{V})$ which sits inside the short exact sequence

$$0 \to k \to H(\mathbb{V}) \to \mathbb{V} \to 0.$$

The natural action of $\mathrm{Sp}(\mathbb{V})$ on $\mathbb{V}$ extends to an action on $H(\mathbb{V})$ fixing the center $Z(H(\mathbb{V})) = k$. The Stone–von Neumann theorem says that for every nontrivial character ψ of k, there exists a unique irreducible (complex) representation of $H(\mathbb{V})$ with central character ψ. Moreover, given a complete polarization $\mathbb{V} = \mathbb{X} + \mathbb{Y}$, each such irreducible representation of $H(\mathbb{V})$ can be realized on the vector space $\mathcal{S}(\mathbb{X})$ of Schwartz functions. In particular, by Schur's lemma, this means that the action of $g \in \mathrm{Sp}(\mathbb{V})$ on $H(\mathbb{V})$ induces an automorphism ϕ_g of $\mathcal{S}(\mathbb{X})$ that is unique up to scalars. We therefore have a group homomorphism

$$[\omega_\psi] \colon \mathrm{Sp}(\mathbb{V}) \to \mathrm{PGL}(\mathcal{S}(\mathbb{X})), \qquad g \mapsto [\phi_g],$$

where $[\phi_g]$ denotes the image of ϕ_g under the quotient map $\mathrm{GL}(\mathcal{S}(\mathbb{X})) \to \mathrm{PGL}(\mathcal{S}(\mathbb{X}))$. This is the *projective Weil representation* of $\mathrm{Sp}(\mathbb{V})$.

It is natural to try to understand when $[\omega_\psi]$ lifts to a genuine representation of $\mathrm{Sp}(\mathbb{V})$. When $k = \mathbb{F}_q$, there exists a lift, but this isn't the case in general. The assignment $g \mapsto \phi_g$ satisfies

$$\phi_g \phi_h = z_{\mathbb{Y}}(g, h) \phi_{gh}, \qquad \text{for } g, h \in \mathrm{Sp}(\mathbb{V}).$$

It is a straightforward check that $(g, h) \mapsto z_{\mathbb{Y}}(g, h)$ defines a 2-cocycle in $H^2(\mathrm{Sp}(\mathbb{V}), \mathbb{C}^\times)$. The 2-cocycle $z_{\mathbb{Y}}$ corresponds to a central extension $\mathrm{Mp}(\mathbb{V})$ of $\mathrm{Sp}(\mathbb{V})$ and certainly the projective Weil representation of $\mathrm{Sp}(\mathbb{V})$ lifts to a genuine representation of $\mathrm{Mp}(\mathbb{V})$. But we can realize the Weil representation on $\mathrm{Sp}(\mathbb{V})$ itself if and only if $z_{\mathbb{Y}}$ is in fact a 2-coboundary.

In this thesis, we will be interested in the adelic Weil representation, which is comprised of Weil representations of local fields. For the rest of this chapter, let k be a local field of characteristic zero, fix an additive character $\psi\colon k \to \mathbb{C}^\times$, and fix a complete polarization $\mathbb{V} = \mathbb{X} + \mathbb{Y}$.

3.1 Metaplectic groups over local fields

Following [R93, Lemma 3.2], there is an explicit unitary lift $r\colon \operatorname{Sp}(\mathbb{V}) \to \operatorname{GL}(\mathcal{S}(\mathbb{X}))$ (a map of sets) of the projective Weil representation given by

$$(r(\sigma)\varphi)(x) = \int_{\mathbb{Y}/\ker\gamma} f_\sigma(x+y)\varphi(x\alpha + y\gamma)\mu_\sigma(d\bar{y})$$

for any $\varphi \in \mathcal{S}(\mathbb{X})$ and any $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, where:

- μ_σ is a Haar measure on $\mathbb{Y}/\ker\gamma$,
- $\bar{y}$ is the coset $y + \ker\gamma \in \mathbb{Y}/\ker\gamma$,
- $f_\sigma(x+y) = \psi(q_\sigma(x+y))$, where $q_\sigma(x+y) = \frac{1}{2}\langle\langle x\alpha, x\beta\rangle\rangle + \frac{1}{2}\langle\langle y\gamma, y\delta\rangle\rangle + \langle\langle y\gamma, x\beta\rangle\rangle$.

Moreover, this lift is the unique lift satisfying the properties in [R93, Theorem 3.5]. We then define the 2-cocycle $z_{\mathbb{Y}}\colon \operatorname{Sp}(\mathbb{V}) \times \operatorname{Sp}(\mathbb{V}) \to \mathbb{C}^1$ by

$$r(gh) = z_{\mathbb{Y}}(g,h)^{-1} \cdot r(g) \cdot r(h).$$

This represents a class in $H^2(\operatorname{Sp}(\mathbb{V}), \mathbb{C}^1)$ and therefore gives rise to a $\mathbb{C}^1$-extension $\operatorname{Mp}(\mathbb{V})$ of $\operatorname{Sp}(\mathbb{V})$ which we call the *metaplectic group*. Explicitly, this group is the set $\operatorname{Sp}(\mathbb{V}) \times \mathbb{C}^1$ together with the multiplication rule

$$(g,x)\cdot(h,y) = (gh, xy \cdot z_{\mathbb{Y}}(g,h)).$$

We define the Weil representation ω_ψ on the metaplectic group $\operatorname{Mp}(\mathbb{V})$ to be

$$\omega_\psi\colon \operatorname{Mp}(\mathbb{V}) \to \operatorname{GL}(\mathcal{S}(\mathbb{X})), \qquad (g,z) \mapsto z \cdot r(g).$$

Oftentimes, it is easier to work with the following description of ω_ψ:

$$\omega_\psi\left(\begin{pmatrix} a & \\ & (a^\intercal)^{-1}\end{pmatrix}, z\right)\varphi(x) = z\cdot|\det a|^{1/2}\cdot\varphi(xa) \tag{3.1}$$

$$\omega_\psi\left(\begin{pmatrix} \mathbf{1}_n & b \\ & \mathbf{1}_n\end{pmatrix}, z\right)\varphi(x) = z\cdot\psi\left(\frac{1}{2}xb^\intercal x\right)\cdot\varphi(x) \tag{3.2}$$

$$\omega_\psi\left(\begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}, z\right)\varphi(x) = z\cdot\int_{k^n}\varphi(y)\psi(x^\intercal y)\,dy \tag{3.3}$$

for $\varphi\in\mathcal{S}(\mathbb{X})$, $x\in\mathbb{X}\cong k^n$, $a\in\mathrm{GL}(\mathbb{X})\cong\mathrm{GL}_n(k)$, $b\in\mathrm{Hom}(\mathbb{X},\mathbb{Y})\cong\mathrm{M}_n(k)$ with $b^\intercal=b$, and $z\in\mathbb{C}^1$. In (3.3), we take $dy=dy_1\cdots dy_n$, where dy_i is the self-dual Haar measure on k with respect to ψ.

It will later (for example, in Chapter 8) be convenient to understand how changing the additive character ψ affects the Weil representation ω_ψ. Define

$$d(\nu) := \begin{pmatrix} 1 & 0 \\ 0 & \nu\end{pmatrix}, \qquad \text{for } \nu\in k.$$

We have

$$d(\nu)^{-1}\begin{pmatrix} a & b \\ c & d\end{pmatrix}d(\nu) = \begin{pmatrix} a & b\nu \\ c\nu^{-1} & d\end{pmatrix}.$$

By Equations (3.1)-(3.3), we see that

$$\begin{aligned}
\omega_\psi\left(d(\nu)^{-1}\left(\begin{smallmatrix} a & \\ & (a^\intercal)^{-1}\end{smallmatrix}\right)d(\nu), z\right)\varphi(x) &= z\cdot|\det a|^{1/2}\cdot\varphi(xa),\\
\omega_\psi\left(d(\nu)^{-1}\left(\begin{smallmatrix} \mathbf{1}_n & b \\ & \mathbf{1}_n\end{smallmatrix}\right), z\right)\varphi(x) &= z\cdot\psi\left(\nu\cdot\tfrac{1}{2}xb^\intercal x\right)\cdot\varphi(x),\\
\omega_\psi\left(d(\nu)^{-1}\left(\begin{smallmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{smallmatrix}\right)d(\nu), z\right)\varphi(x) &= \omega_\psi\left(\begin{smallmatrix} \nu & \\ & \nu^{-1}\end{smallmatrix}\right)\omega_\psi\left(\left(\begin{smallmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{smallmatrix}\right), z\right)\varphi(x)\\
&= z\cdot|\nu|^{n/2}\cdot\int_{k^n}\varphi(y)\psi(\nu x^\intercal y)\,dy.
\end{aligned}$$

Let dy_ν denote the Haar measure on k^n that is self-dual with respect to $\psi_\nu(x) := \psi(\nu x)$. Then $dy_\nu = |\nu|^{-n/2}dy$, and it follows from the above equations that

$$\omega_\psi(d(\nu)^{-1}gd(\nu), z) = \omega_{\psi_\nu}(g, z). \tag{3.4}$$

If for a subgroup $\iota\colon G\hookrightarrow\mathrm{Sp}(\mathbb{V})$, the restriction of $z_{\mathbb{Y}}$ represents the trivial class in $H^2(G,\mathbb{C}^1)$, then via an explicit trivialization s of $z_{\mathbb{Y}}|_{G\times G}$, we can define the Weil representation

ω_ψ on G as

$$\omega_\psi \colon G \to \mathrm{GL}(\mathcal{S}(\mathbb{X})), \qquad g \mapsto \omega_\psi(g, s(g)).$$

3.2 Weil indices and Leray invariants

One feature that makes the Weil representation computable is the fact that the 2-cocycle $z_{\mathbb{Y}}$ can be expressed in terms of the *Weil index* of the *Leray invariant*. We review these concepts and their basic properties. We follow the exposition in [IP16a, Sections 3.1.1, 3.1.2]. We will use these properties extensively in Chapter 5.

Roughly speaking, the *Weil index* measures the behavior of characters of second degree under Fourier transform (see [R93, Theorem A.1]). Let ψ be a nontrivial additive character of k and let q be a non-degenerate symmetric k-bilinear form. For our purposes, we will only need a list of properties of the Weil index $\gamma_k(\psi \circ q) \in \mu_8$ attached to the character of second degree $x \mapsto \psi(q(x,x))$. In the special case $q(x,y) = xy$, we write $\gamma_k(\psi) := \gamma_k(\psi \circ q)$ and define $\gamma_k(a,\psi) := \gamma_k(a\psi)/\gamma_k(\psi)$, where $a\psi(x) := \psi(ax)$. Then for $a, b \in k^\times$, one has the following list of properties (see [R93, p.367], [IP16a, Section 3.1.1]):

$$\begin{aligned}
\gamma_k(ab^2,\psi) &= \gamma_k(a,\psi), \\
\gamma_k(ab,\psi) &= \gamma_k(a,\psi)\cdot\gamma_F(b,\psi)\cdot(a,b)_F, \\
\gamma_k(a,b\psi) &= \gamma_k(a,\psi)\cdot(a,b)_k, \\
\gamma_k(a,\psi)^2 &= (-1,a)_k, \\
\gamma_k(a,\psi)^4 &= 1, \\
\gamma_k(\psi)^2 &= \gamma_k(-1,\psi)^{-1}, \\
\gamma_k(\psi)^8 &= 1.
\end{aligned}$$

Here, $(\cdot,\cdot)_k$ is the quadratic Hilbert symbol of k (see Section 4.1). Now consider the symmetric k-bilinear form

$$q(x,y) = a_1x_1y_1 + \cdots + a_mx_my_m.$$

Then

$$\gamma_k(\psi\circ q) = \gamma_k(\psi)^m \cdot \gamma_k(\det q, \psi)\cdot h_F(q),$$

where

$$\det q = \prod_{1\le i\le m} a_i, \qquad h_F(q) = \prod_{1\le i<j\le m}(a_i,a_j)_k.$$

The *Leray invariant* attaches a non-degenerate symmetric k-bilinear form $q(\mathbb{Y},\mathbb{Y}',\mathbb{Y}'')$

to an ordered triple $(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}'')$ of maximal isotropic (i.e. Lagrangian) subspaces of $\mathbb{V}$ (see [R93, Definitions 2.4, 2.10]). One first defines the Leray invariant when $\mathbb{Y}, \mathbb{Y}', \mathbb{Y}''$ are pairwise transverse. Let $P_{\mathbb{Y}}$ be the maximal parabolic subgroup of $\mathrm{Sp}(\mathbb{V})$ stabilizing $\mathbb{Y}$ and let $N_{\mathbb{Y}}$ be its unipotent radical. Since $\mathbb{Y}, \mathbb{Y}'$ are transverse, any element of $N_{\mathbb{Y}}$ is of the form $\mathbf{n}(b)$, where $b \in \mathrm{Hom}(\mathbb{Y}', \mathbb{Y})$. There is a unique $\mathbf{n}(b) \in N_{\mathbb{Y}}$ such that $\mathbb{Y}'\mathbf{n}(b) = \mathbb{Y}''$, and the Leray invariant is the non-degenerate symmetric k-bilinear form on $\mathbb{Y}'$ given by

$$q(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}'')(x', y') := \langle\!\langle x', y'b \rangle\!\rangle.$$

In general, the vector spaces

$$\mathbb{Y}_{\mathbb{R}} := (\mathbb{Y} \cap \mathbb{R}^{\perp})/\mathbb{R}, \qquad \mathbb{Y}'_{\mathbb{R}} := (\mathbb{Y}' \cap \mathbb{R}^{\perp})/\mathbb{R}, \qquad \mathbb{Y}''_{\mathbb{R}} := (\mathbb{Y}'' \cap \mathbb{R}^{\perp})/\mathbb{R}$$

are pairwise transverse maximal isotropic subspaces of $\mathbb{V}/\mathbb{R}$, where

$$\mathbb{R} = (\mathbb{Y} \cap \mathbb{Y}') + (\mathbb{Y}' \cap \mathbb{Y}'') \cap (\mathbb{Y} \cap \mathbb{Y}'').$$

We define

$$q(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}'') := q(\mathbb{Y}_{\mathbb{R}}, \mathbb{Y}'_{\mathbb{R}}, \mathbb{Y}''_{\mathbb{R}}).$$

It will also be useful to recall from [R93, Theorem 2.11] that

$$q(\mathbb{Y}g, \mathbb{Y}'g, \mathbb{Y}''g) = q(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}'') \qquad \text{for all } g \in \mathrm{Sp}(\mathbb{V}).$$

3.3 The doubled Weil representation

Now consider the doubled symplectic space $\mathbb{V}^{\square} := \mathbb{V} + \mathbb{V}^{-}$, where $\mathbb{V}^{-}$ has the negated form. Let $\mathbb{X}^{\square} = \mathbb{X} + \mathbb{X}^{-}$ and $\mathbb{Y}^{\square} = \mathbb{Y} + \mathbb{Y}^{-}$. Let $\omega_{\psi}^{\square}$ denote the Weil representation on the metaplectic group $\mathrm{Mp}(\mathbb{V}^{\square})$ with respect to the complete polarization $\mathbb{V}^{\square} = \mathbb{X}^{\square} + \mathbb{Y}^{\square}$. We will also make use of the polarization $\mathbb{V}^{\square} = \mathbb{V}^{\triangle} + \mathbb{V}^{\triangledown}$, where $\mathbb{V}^{\triangle} = \{(v, v) : v \in \mathbb{V}\}$ and $\mathbb{V}^{\triangledown} = \{(v, -v) : v \in \mathbb{V}\}$. (Note that this polarization is intrinsic to $\mathbb{V}$ and that if $\mathbb{V}$ comes from a unitary space, then $\mathbb{V}^{\square} = \mathbb{V}^{\triangle} + \mathbb{V}^{\triangledown}$ comes from a natural splitting of the doubled underlying unitary space.) Identifying $\mathrm{Sp}(\mathbb{V}^{-})$ with $\mathrm{Sp}(\mathbb{V})^{\mathrm{op}}$, we can consider the natural map (*a priori* of sets)

$$\tilde{\iota}\colon \mathrm{Mp}(\mathbb{V}) \times \mathrm{Mp}(\mathbb{V})^{\mathrm{op}} \to \mathrm{Mp}(\mathbb{V}^{\square}), \qquad ((g, z), (h, w)) \mapsto (\mathrm{diag}(g, h^{-1}), zw^{-1}).$$

Lemma 3.1. *$\tilde{\iota}$ is a group homomorphism.*

Proof. We have

$$\begin{aligned}
&\tilde{\iota}((g_1, z_1), (h_1, w_1)) \cdot \tilde{\iota}((g_2, z_2), (h_2, w_2)) \\
&\quad = (\mathrm{diag}(g_1, h_1^{-1}), z_1 w_1^{-1}) \cdot (\mathrm{diag}(g_2, h_2^{-1}), z_2 w_2^{-1}) \\
&\quad = \big(\mathrm{diag}(g_1 g_2, h_1^{-1} h_2^{-1}), z_1 z_2 w_1^{-1} w_2^{-1} z_{\mathbb{Y}^\square}(\mathrm{diag}(g_1, h_1^{-1}), \mathrm{diag}(g_2, h_2^{-1}))\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(g_1, z_1) \cdot (g_2, z_2) &= (g_1 g_2, z_1 z_2 z_{\mathbb{Y}}(g_1, g_2)), \\
(h_1, w_1) \cdot (h_2, w_2) &= (h_2 h_1, w_1 w_2 z_{\mathbb{Y}}(h_2, h_1)),
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\iota}\big(((g_1, z_1) \cdot (g_2, z_2), (h_1, w_1) \cdot (h_2, w_2))\big) \\
&\quad = (\mathrm{diag}(g_1 g_2, (h_2 h_1)^{-1}), z_1 z_2 w_1^{-1} w_2^{-1} z_{\mathbb{Y}}(g_1, g_2) z_{\mathbb{Y}}(h_2, h_1)^{-1}).
\end{aligned}$$

I now claim that

$$z_{\mathbb{Y}^\square}(\mathrm{diag}(g_1, h_1^{-1}), \mathrm{diag}(g_2, h_2^{-1})) = z_{\mathbb{Y}}(g_1, g_2) z_{\mathbb{Y}}(h_2, h_1)^{-1}.$$

By Theorem 4.1(3) of [R93], we have

$$z_{\mathbb{Y}^\square}(\mathrm{diag}(g_1, h_1^{-1}), \mathrm{diag}(g_2, h_2^{-1})) = z_{\mathbb{Y}}(g_1, g_2) \cdot z_{\mathbb{Y}}(h_1^{-1}, h_2^{-1}).$$

By Proposition 3.7 of [R93], $r(1) = 1$, and using this, it is a straightforward chase of definitions to see that

$$z_{\mathbb{Y}}(h_1^{-1}, h_2^{-1}) = z_{\mathbb{Y}}(h_2, h_1)^{-1}. \qquad \square$$

Corollary 3.2. *We have*

$$(\tilde{\iota})^* \omega_\psi = \omega_\psi \otimes \overline{\omega}_\psi.$$

Proof. By Proposition 3.7 and Theorem 4.1(3) of [R93], Lemma 3.1 implies

$$\omega_\psi^\square(\tilde{\iota}(\sigma_1, \sigma_2)) = \omega_\psi(\sigma_1) \otimes \omega_\psi(\sigma_2^{-1}). \qquad \square$$

3.4 Dual reductive pairs and the Howe correspondence

Definition 3.3. A *dual reductive pair* (G, G') in $\mathrm{Sp}(\mathbb{V})$ is a pair of reductive subgroups of $\mathrm{Sp}(\mathbb{V})$ which are mutual centralizers of each other:

$$Z_{\mathrm{Sp}(\mathbb{V})}(G) = G' \qquad \text{and} \qquad Z_{\mathrm{Sp}(\mathbb{V})}(G') = G.$$

In this paper, we will only work with unitary and quaternionic unitary dual pairs, which are two of the three common classes of dual reductive pairs:

1. Let V, W be vector spaces over k endowed with a nondegenerate alternating bilinear form $\langle\cdot,\cdot\rangle$ and a nondegenerate symmetric bilinear form $(\cdot,\cdot)$, respectively. Defining the isometry groups

$$\begin{aligned}\mathrm{Sp}(V) &:= \{g \in \mathrm{GL}_k(V) : \langle v_1 g, v_2 g\rangle = \langle v_1, v_2\rangle \text{ for all } v_1, v_2 \in V\},\\ \mathrm{O}(W) &:= \{g \in \mathrm{GL}_k(W) : (gw_1, gw_2) = (w_1, w_2) \text{ for all } w_1, w_2 \in W\},\end{aligned}$$

   the groups $(\mathrm{Sp}(V), \mathrm{O}(W))$ form a dual reductive pair in $\mathrm{Sp}(V \otimes W)$, where $V \otimes W$ is the vector space endowed with the alternating bilinear form $\langle\!\langle\cdot,\cdot\rangle\!\rangle := (\cdot,\cdot) \otimes \langle\cdot,\cdot\rangle$.

2. Let D be the quaternion algebra of invariant $\frac{1}{2}$ over k. Let V be a right D-space endowed with a nondegenerate Hermitian form $\langle\cdot,\cdot\rangle$ and let W be a left D-space endowed with a nondegenerate skew-Hermitian form $(\cdot,\cdot)$. Defining the quaternionic unitary groups

$$\begin{aligned}\mathrm{U}_D(V) &:= \{g \in \mathrm{GL}_D(V) : \langle v_1 g, v_2 g\rangle = \langle v_1, v_2\rangle \text{ for all } v_1, v_2 \in V\},\\ \mathrm{U}_D(W) &:= \{g \in \mathrm{GL}_D(W) : (gw_1, gw_2) = (w_1, w_2) \text{ for all } w_1, w_2 \in W\},\end{aligned}$$

   the groups $(\mathrm{U}_D(V), \mathrm{U}_D(W))$ form a dual reductive pair in $\mathrm{Sp}(\mathrm{Res}_{D/k}(V \otimes_D W))$, where $\mathrm{Res}_{D/k}(V \otimes_D W)$ is endowed with the alternating bilinear form $\langle\!\langle\cdot,\cdot\rangle\!\rangle = \frac{1}{2}\mathrm{Tr}_{D/k}(\langle\cdot,\cdot\rangle \otimes \overline{(\cdot,\cdot)})$. Here, we denote the involution on D by $a \mapsto \overline{a}$.

3. Let k' be a quadratic extension of k. Let V, W be vector spaces over k' with a nondegenerate skew-Hermitian form $\langle\cdot,\cdot\rangle$ and a nondegenerate Hermitian form $(\cdot,\cdot)$, respectively. Defining the unitary groups

$$\begin{aligned}\mathrm{U}_E(V) &:= \{g \in \mathrm{GL}_{k'}(V) : \langle v_1 g, v_2 g\rangle = \langle v_1, v_2\rangle \text{ for all } v_1, v_2 \in V\},\\ \mathrm{U}_E(W) &:= \{g \in \mathrm{GL}_{k'}(W) : (gw_1, gw_2) = (w_1, w_2) \text{ for all } w_1, w_2 \in W\},\end{aligned}$$

   the groups $(\mathrm{U}_{k'}(V), \mathrm{U}_{k'}(W))$ form a dual reductive pair in $\mathrm{Sp}(\mathrm{Res}_{k'/k}(V \otimes_{k'} W))$, where

$\mathrm{Res}_{k'/k}(V \otimes_{k'} W)$ is endowed with the alternating bilinear form $\langle\langle \cdot, \cdot \rangle\rangle = \frac{1}{2} \mathrm{Tr}_{k'/k}(\langle \cdot, \cdot \rangle \otimes \overline{(\cdot, \cdot)})$. Here, we denote the nontrivial element of $\mathrm{Gal}(k'/k)$ by $a \mapsto \overline{a}$.

Given a dual reductive pair (G, G') of $\mathrm{Sp}(\mathbb{V})$, there is a natural map

$$i \colon G \times G' \to \mathrm{Sp}(\mathbb{V}), \qquad (g, g') \mapsto (v \mapsto g^{-1} v g').$$

If the cocycle $z_{\mathbb{Y}}$ can be trivialized on $i(G \times G') \subset \mathrm{Sp}(\mathbb{V})$, we can define the Weil representation on $i(G \times G')$ and pull back to a Weil representation of $G \times G'$. In Kudla's remarkable paper [K94], he writes down explicit splittings of $z_{\mathbb{Y}}$ in each of the above three classes of dual reductive pairs (except in Case 1 with $\dim W$ odd). We will make use of this work heavily (especially the formulas for unitary groups) in the present paper.

It is very interesting to study the Weil representation ω_ψ of $G \times G'$. If π is an irreducible representation of G, we may consider

$$\mathcal{S}(\pi) := \mathcal{S}(\mathbb{X}) / \bigcap_{\lambda \in \mathrm{Hom}_G(\mathcal{S}(\mathbb{X}), \pi)} \ker(\lambda),$$

the largest quotient of $\mathcal{S}(\mathbb{X})$ such that G acts by π. Then by [MVW, Chapter 2, Lemma III.4], there exists a unique irreducible G'-representation $\Theta(\pi)$ such that

$$\mathcal{S}(\pi) \cong \pi \otimes \Theta(\pi).$$

We call $\Theta(\pi)$ the *local theta lift* of π. In 1979, Howe [H79] conjectured that the assignment $\pi \mapsto \Theta(\pi)$ defines a bijection

$$\left\{ \begin{array}{c} \text{irreducible representations } \pi \text{ of } G \\ \text{such that } \mathrm{Hom}_G(\mathcal{S}(\mathbb{X}), \pi) \neq 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible representations } \pi' \text{ of } G' \\ \text{such that } \mathrm{Hom}_{G'}(\mathcal{S}(\mathbb{X}), \pi') \neq 0 \end{array} \right\}.$$

Howe's conjecture has since been solved by the work of many people: Howe [H89], Kudla [K86], Waldspurger [W90], Gan–Takeda [GT].

CHAPTER 4

Waldspurger, Tunnell–Saito, and a pair of quaternion algebras

For any Hecke character $\chi\colon E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$, let π_χ denote the automorphic induction of χ to an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Now let χ' be another Hecke character of $E^\times$. Recall from Chapter 2 that the central character of π_χ^B is $\chi|_{\mathbb{A}_F^\times} \cdot \epsilon_{E/F}$ so that if χ' is a constituent of π_χ^B viewed as a $\mathbb{A}_E^\times$-representation, then χ' must satisfy

$$\chi|_{\mathbb{A}_F^\times} \cdot \chi'|_{\mathbb{A}_F^\times} \cdot \epsilon_{E/F} = 1, \tag{4.1}$$

where $\epsilon_{E/F}$ is the quadratic character of $\mathbb{A}_F^\times$ associated to the quadratic extension E/F. Explicitly, if we write $E = F(\sqrt{u})$,

$$\epsilon_{E/F}\colon \mathbb{A}_F^\times \to \{\pm 1\}, \qquad (a_v)_v \mapsto (u, a_v)_v,$$

where $(u, a_v)_v$ is the Hilbert symbol. For a quaternion algebra B over F, let π_χ^B denote the Jacquet–Langlands transfer of π_χ to $B_{\mathbb{A}}^\times$. (If π_χ does not transfer to $B_{\mathbb{A}}^\times$, we take $\pi_\chi^B = 0$.)

In this section, we discuss how the work of Tunnell–Saito implies that for any χ' satisfying (4.1), there exists at most one quaternion algebra B such that χ' is a constituent of π_χ^B viewed as a representation of $\mathbb{A}_E^\times$. Combining this with Waldspurger's formula, we see that such a B exists if and only if

$$L(\mathrm{BC}(\pi_\chi) \otimes \chi', \tfrac{1}{2}) \neq 0. \tag{4.2}$$

Since

$$L(\mathrm{BC}(\pi_{\chi'}) \otimes \chi, \tfrac{1}{2}) = L(\mathrm{BC}(\pi_\chi) \otimes \chi', \tfrac{1}{2}) \neq 0,$$

we also see that there exists a unique quaternion algebra B' such that χ is a constituent of $\pi_{\chi'}^{B'}$. The goal of this chapter is to give a simple description relating B and B'.

4.1 The Hilbert symbol

We collect some basic facts about the Hilbert symbol. Given a place v of F, define the Hilbert symbol of $a, b \in F_v$ to be

$$(a,b)_v = \begin{cases} 1 & \text{if there exists a nonzero solution } (x,y,z) \in F_v^{\oplus 3} \text{ to } ax^2+by^2=z^2, \\ -1 & \text{otherwise.} \end{cases}$$

One of the most important properties of the Hilbert symbol is that it is bimultiplicative:

$$\begin{aligned} (a,b_1)_v \cdot (a,b_2)_v &= (a,b_1b_2)_v, && \text{for all } a,b_1,b_2 \in F_v, \\ (a_1,b)_v \cdot (a_2,b)_v &= (a_1a_2,b)_v, && \text{for all } a_1,a_2,b \in F_v. \end{aligned}$$

Lemma 4.1. *For any $a \in F_v^\times$, we have $(u,a)_v = \epsilon_{E_v/F_v}(a)$, where $E_v = F_v(\sqrt{u})$.*

Proof. First observe that if v splits completely in E, then u is a square in $F_v^\times$. Therefore $(u,a)_v = 1$ for all $a \in F_v^\times$, and the conclusion follows.

It remains to prove the lemma in the case when E_v is a field. Since the Hilbert symbol is bimultiplicative, the map $a \mapsto (u,a)_v$ is a homomorphism $F_v^\times \to \{\pm 1\}$. It is straightforward to check that $(u,a)_v = 1$ if and only if $a \in \mathrm{Nm}_{E_v/F_v}(E_v^\times)$. Indeed, if $ux^2+ay^2=z^2$, then $ay^2 = z^2 - ux^2 = \mathrm{Nm}(z + x\sqrt{u})$.

By local class field theory, $\mathrm{Nm}_{E_v/F_v}(E_v^\times)$ is an index-2 subgroup of $F_v^\times$, and therefore there exists $a \in F_v^\times$ such that $(u,a)_v = -1$. This implies that $a \mapsto (u,a)_v$ is nontrivial. Again by local class field theory, there is a unique nontrivial homomorphism $F_v^\times \to \{\pm 1\}$ that is trivial on $\mathrm{Nm}_{E_v/F_v}(E_v^\times)$, and the lemma follows. $\square$

4.2 Waldspurger's formula

Let π be an irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character ω_π that has a nonzero Jacquet–Langlands transfer π^B to $B_{\mathbb{A}}^\times$. Recall that this means that π_v is discrete series at all places v of F such that B_v is ramified. Let Ω be any Hecke character of $E^\times$ such that $\Omega|_{\mathbb{A}_F^\times} = \omega_\pi^{-1}$.

Theorem 4.2 (Waldspurger [W85a])**.** *For any $f \in \pi^B$,*

$$\left| \int_{[E^\times]} f(t)\Omega(t)\,dt \right|^2 = \langle f, f \rangle \cdot \frac{\zeta(2)}{2L_2(\pi,1)} \cdot \prod_v \alpha_v \cdot L\left(\mathrm{BC}(\pi) \otimes \Omega, \tfrac{1}{2}\right),$$

where $\alpha_v := \alpha(f_v, E_v, \Omega_v)$ are local factors that are equal to 1 for almost all v. Precisely,

$$\alpha(f_v, E_v, \Omega_v) = \zeta_v(2)^{-1} \cdot L(\mathrm{BC}(\pi)_v \otimes \Omega_v, \tfrac{1}{2})^{-1} \cdot L(\varepsilon_{E/F,v}, 1) \cdot L_2(\pi_v, 1) \cdot \int_{F_v^\times \backslash E_v^\times} \frac{\langle \pi_v^B(t) f, \pi_v^B(g) f\rangle}{\langle f, f\rangle} \cdot \Omega_v(t)\, dt.$$

Furthermore, Waldspurger proved that the functional

$$\mathscr{P}(\pi^B, \Omega)\colon \pi^B \to \mathbb{C}, \qquad f \mapsto \int_{[E^\times]} f(t)\, \Omega(t)\, dt$$

is nonvanishing if and only if the two obvious local and global obstructions do not occur:

Theorem 4.3 (Waldspurger [W85a]). *There exists a $f \in \pi^B$ such that $\mathscr{P}(\pi^B, \Omega)(f) \neq 0$ if and only if:*

(i) (local) For each place v of F, $\mathrm{Hom}_{E_v^\times}(\pi_v, \Omega_v^{-1}) \neq 0$.

(ii) (global) $L(\mathrm{BC}(\pi) \otimes \Omega, \frac{1}{2}) \neq 0$.

Observe that for any vector $f_v B \in \pi_v^B$, the $E_v^\times$-representation $\pi_v^B(E^\times) f_v := \{\pi_v^B(\alpha) f_v : \alpha \in E_v^\times\}$ is a smooth representation and therefore factors through some compact open subgroup $U \subset E_v^\times$. Since π_v' is irreducible by assumption, $F_v^\times$ acts by a scalar, and we therefore see that $\pi_v^B(E^\times) f_v$ is finite-dimensional. It follows that $\pi_v^B(E^\times) f_v$ is completely decomposable and so $\mathrm{Hom}_{E_v^\times}(\pi_v^B, \Omega_v^{-1}) \neq 0$ if and only if $\mathrm{Hom}_{E_v^\times}(\Omega_v^{-1}, \pi_v^B) \neq 0$.

By appealing to a theorem of Tunnell and Saito, the above rephrasing allows us to give a formulation of the local obstruction in terms of local epsilon factors $\epsilon_v(\mathrm{BC}(\pi) \otimes \Omega)$.

Theorem 4.4 (Tunnell [T83], Saito [S93]). $\mathrm{Hom}_{E_v^\times}(\Omega_v^{-1}, \pi_v^B) \neq 0$ *if and only if*

$$\epsilon_v(\mathrm{BC}(\pi) \otimes \Omega) \cdot \omega_v(-1) = \begin{cases} +1 & \text{if } B_v^\times \cong \mathrm{GL}_2(F_v), \\ -1 & \text{if } B_v^\times \text{ is nonsplit.} \end{cases}$$

Combining the above theorems, we obtain:

Theorem 4.5. *Let π be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character ω_π. If*

$$L(\mathrm{BC}(\pi) \otimes \Omega, \tfrac{1}{2}) \neq 0, \qquad \text{and} \qquad \Omega|_{\mathbb{A}_F^\times} = \omega_\pi^{-1},$$

then there exists a unique quaternion algebra $B = B_{\pi,\Omega}$ over F such that

$$\mathscr{P}(\pi^B, \Omega) \neq 0.$$

Moreover, B is the unique quaternion algebra with ramification set

$$\Sigma_{\pi,\Omega} := \{v : \epsilon_v(\mathrm{BC}(\pi) \otimes \Omega) \cdot \omega_v(-1) = -1\}.$$

Proof. If $L(\mathrm{BC}(\pi) \otimes \Omega, \frac{1}{2}) \neq 0$, then $\epsilon(\mathrm{BC}(\pi) \otimes \Omega) = +1$. Since ω is a Hecke character of $\mathbb{A}^\times$, we must have $\omega(-1) = +1$. Therefore, there must be an *even* number of places v of F such that $\epsilon_v(\mathrm{BC}(\pi) \otimes \Omega) \cdot \omega_v(-1) = -1$, and hence there exists a unique quaternion algebra $B_{\pi,\Omega}$ over F with ramification set $\Sigma_{\pi,\Omega}$, and the conclusion now follows from Waldspurger's formula and the local branching criterion of Tunnell and Saito. $\square$

4.3 A pair of quaternion algebras

We now specialize to the setting where π comes from automorphic induction. Let χ, χ' be Hecke characters of $\mathbb{A}_E^\times$ satisfying Equation (1.2). One has

$$L(\mathrm{BC}(\pi_\chi) \otimes \chi', s) = L(\pi_\chi \otimes \pi_{\chi'}, s) = L(\mathrm{BC}(\pi_{\chi'}) \otimes \chi, s),$$

and let us assume that

$$L(\mathrm{BC}(\pi_\chi) \otimes \chi', \tfrac{1}{2}) = L(\mathrm{BC}(\pi_{\chi'}) \otimes \chi, \tfrac{1}{2}), \tag{4.3}$$

By Theorem 4.5, $B = B_{\pi_\chi,\chi'}$ and $B' = B_{\pi_{\chi'},\chi}$ are the unique quaternion algebras such that

$$\mathscr{P}(\pi_\chi^B, \chi') \neq 0 \qquad \text{and} \qquad \mathscr{P}(\pi_{\chi'}^{B'}, \chi) \neq 0.$$

Proposition 4.6. *Let χ, χ' be Hecke characters of $\mathbb{A}_E^\times$ satisfying Equations* (1.2) *and* (4.3)*, and let $E = F(\mathbf{i})$ with $\mathbf{i}^2 = u$. If $B = B_{\pi_\chi,\chi'}$ is the quaternion algebra that corresponds to the Hilbert symbol (u, J), then $B' = B_{\pi_{\chi'},\chi}$ corresponds to the Hilbert symbol $(u, -J)$.*

Proof. It is a standard computation to show that:

$$\epsilon_v(\mathrm{BC}(\pi_\chi) \otimes \chi') = \epsilon_v(\mathrm{BC}(\pi_{\chi'}) \otimes \chi).$$

By Equation (1.2), we have

$$\omega_{\pi_\chi} \cdot \omega_{\pi_{\chi'}} \cdot \epsilon_{E/F} = 1.$$

Using Theorem 4.5, we see that $\Sigma_{\pi_{\chi'},\chi}$ can be described in terms of $\Sigma_{\pi_\chi,\chi'}$:

$$\Sigma_{\pi_{\chi'},\chi} = \left\{ v : \begin{array}{c} v \in \Sigma_{\pi_\chi,\chi'} \text{ and } \epsilon_{E_v/F_v}(-1) = 1, \text{ or} \\ v \notin \Sigma_{\pi_\chi,\chi'} \text{ and } \epsilon_{E_v/F_v}(-1) = -1. \end{array} \right\}$$

An equivalent way to state this relationship is the following. The quaternion algebra B can be given an F basis $1, \mathbf{i}, \mathbf{j}, \mathbf{ij}$ such that $E = F[\mathbf{i}]$. Write $\mathbf{i}^2 = u$ and $\mathbf{j}^2 = J$ so that B is the quaternion algebra associated to the Hilbert symbol (u, J). That is,

$$(u, J)_v = -1 \qquad \Longleftrightarrow \qquad v \in \Sigma_{\pi_\chi,\chi'}.$$

Then, using the bimultiplicativity of the Hilbert symbol, B' is the quaternion algebra associated to the Hilbert symbol

$$(u, J) \cdot \epsilon_{E/F}(-1) = (u, J) \cdot (u, -1) = (u, -J). \qquad \square$$

4.4 A seesaw of unitary groups

In this section, we introduce the main dual reductive pairs of interest in this paper. We will define a pair of quaternionic unitary similitude groups and a pair of unitary similitude groups such that, roughly speaking, captures the following picture:

$$\begin{tikzcd} B^\times \arrow[d, dash] \arrow[dr, dash] & (B')^\times \arrow[d, dash] \arrow[dl, dash] \\ E^\times & E^\times \end{tikzcd}$$

This allows us to specialize the framework of Chapter 3 to study the torus periods described in earlier sections of the present chapter.

Fix $\mathbf{i} \in E$ with $\mathrm{tr}_{E/F}\, \mathbf{i} = \mathbf{i} + \bar{\mathbf{i}} = 0$. Note that $E = F[\mathbf{i}]$. Let B be a (possibly split) quaternion algebra over F and let $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be a standard basis for B over F.

We consider the following spaces:

- $V = B =$ 1-dimensional right B-space with skew-Hermitian form $\langle x, y \rangle = x^* \mathbf{i} y$
- $W^* = B \otimes_E E =$ 1-dimensional left B-space with Hermitian form $(x, y) = xy^*$
- $\operatorname{Res} V =$ 2-dimensional right E-space with skew-Hermitian form $\langle x, y \rangle = \mathrm{pr}(x^* \mathbf{i} y)$
- $W = E =$ 1-dimensional left E-space with Hermitian form $(a, b) = a\bar{b}$

- $V_0 =$ 1-dimensional right E-space with Hermitian form $\langle a, b\rangle_0 = \overline{a}b$
- $W_0 = B =$ 2-dimensional left E-space with skew-Hermitian form $(x, y)_0 = -\mathbf{i}\,\mathrm{pr}(xy^*)$
- $\mathbb{V} = V \otimes W^* = \operatorname{Res} V \otimes W = V_0 \otimes W_0 =$ 4-dimensional F-space with symplectic form $\frac{1}{2}\operatorname{Tr}_{E/F}(\langle\cdot,\cdot\rangle \otimes \overline{(\cdot,\cdot)})$

Then both pairs $(U_B(V), U_B(W^*))$ and $(U_E(\operatorname{Res} V), U_E(W))$ are irreducible dual reductive pairs (of type 1) in $\mathrm{Sp}(\mathbb{V})$. (See, for example, [P93].) For either of these pairs (G, G'), we have a natural map

$$G \times G' \to \mathrm{Sp}(\mathbb{V}), \qquad (g, h) \mapsto gh^{-1}.$$

It is clear that $U_B(V) \subset U_E(\operatorname{Res} V)$ and that $U_E(W) \subset U_B(W^*)$. Therefore we have the following seesaw of dual reductive pairs

$$\begin{tikzcd}[column sep=small] U_E(\operatorname{Res} V) \arrow[d, dash] \arrow[drr, dash] & & U_B(W^*) \arrow[d, dash] \arrow[dll, dash] & \\ U_B(V) & & U_E(W) \end{tikzcd} = \begin{tikzcd}[column sep=small] (E^1 \times (B')^1)/F^1 \arrow[d, dash] \arrow[drr, dash] & & B^1 \arrow[d, dash] \arrow[dll, dash] \\ E^1 \cup E^{\frac{1}{J}}\mathbf{j} & & E^1 \end{tikzcd}$$

Here, $B' = \left(\frac{\mathbf{i}^2, -\mathbf{j}^2}{F}\right)$ and the superscript $r \in \mathbb{Q}$ picks out the norm-r elements.

We now explicate the above identifications of classical groups.

1. $U_B(V) = E^1 \cup E^{\frac{1}{J}}\mathbf{j}$

   We have

   $$U_B(V) = \{\varphi \in \mathrm{GL}(V) : \varphi \text{ is right } B\text{-linear}, \langle x, y\rangle = \langle \varphi(x), \varphi(y)\rangle\}.$$

   Since V is a one-dimensional right B-space, then all such maps are of the form $\varphi_\alpha \colon v \mapsto \alpha \cdot v$ for some $\alpha \in B$. Write $\alpha = A + B\mathbf{j}$. Then $\varphi_\alpha \in U_B(V)$ if and only if for all $x, y \in B$,

   $$x^*\alpha^*\mathbf{i}\alpha y = x^*\mathbf{i}y.$$

   Equivalently,

   $$\alpha^*\mathbf{i}\alpha = \mathbf{i}.$$

   We have

   $$\alpha^*\mathbf{i}\alpha = (\overline{A} - B\mathbf{j})\mathbf{i}(A + B\mathbf{j}) = (A\overline{A} + B\overline{B}J)\mathbf{i} + 2\overline{A}B\mathbf{ij},$$

   so we obtain

   $$A\overline{A} + B\overline{B}J = 1, \qquad \overline{A}B = 0.$$

The second condition implies that either $A = 0$ or $B = 0$, so the first condition implies $A\overline{A} = 1$ or $B\overline{B} = \frac{1}{J}$. Thus we have an isomorphism $E^1 \cup E^{\frac{1}{J}}\mathbf{j} \to U_B(V)$ given by $\alpha \mapsto \varphi_\alpha$, where $\varphi_\alpha(v) = v \cdot \alpha$.

2. $U_B(W^*) = B^1$

   We have

$$U_B(W^*) = \{\varphi \in \mathrm{GL}(W^*) : \varphi \text{ is left } B\text{-linear, } (x, y) = (\varphi(x), \varphi(y))\}.$$

   All such maps φ are of the form $\varphi_\alpha \colon v \mapsto v \cdot \alpha$ for some $\alpha \in B^\times$. Then $\varphi_\alpha \in U_B(W^*)$ if and only if for all $x, y \in W^*$,

$$x\alpha\alpha^* y^* = xy^*.$$

   This implies $\alpha\alpha^* = 1$ and so $\alpha \in B^1$. We therefore have an isomorphism $B^1 \to U_B(W^*)$ given by $\alpha \mapsto \varphi_\alpha$, where $\varphi_\alpha(w) = w \cdot \alpha$.

3. $U_E(\mathrm{Res}\, V) = (E^1 \times (B')^1)/F^1$

   The right E-space $\mathrm{Res}\, V$ has a natural left-multiplication action by E^1 and a natural right $(B')^\times$ action such whose stabilizer in the product group is an antidiagonal embedding of F^1. These actions preserve the skew-Hermitian form on $\mathrm{Res}\, V$ and we therefore obtain a map $(E^1 \times (B')^1)/F^1 \hookrightarrow U_E(\mathrm{Res}\, V)$ that turns out to be an isomorphism. For more details, see Remark 4.7.

4. $U_E(W) = E^1$

   This identification is easy to see. W is a 1-dimensional E-space and so $U_E(W) \subset \mathrm{GL}_1(E) = E^\times$. It is then easy to see that the Hermitian form on W is preserved by multiplication by $\alpha \in E^\times$ if and only if $\alpha \in E^1$. We therefore obtain the isomorphism $E^1 \to U_E(W)$ via $\alpha \mapsto \varphi_\alpha$, where $\varphi_\alpha(w) = w \cdot \alpha$.

The analogous seesaw with similitudes is

$$\begin{tikzcd}[arrows=dash]
GU_E(\mathrm{Res}\, V) \arrow[d, dotted] \arrow[dr, dotted] & GU_B(W^*) \arrow[d, dotted] \arrow[dl, dotted] & \\
GU_B(V) & GU_E(W)
\end{tikzcd}
=
\begin{tikzcd}[arrows=dash]
(E^\times \times (B')^\times)/F^\times \arrow[d, dotted] \arrow[dr, dotted] & B^\times \arrow[d, dotted] \arrow[dl, dotted] \\
E^\times \cup E^\times \mathbf{j} & E^\times
\end{tikzcd}$$

where the identifications are determined by similar arguments as in the unitary group setting.

The point of introducing the E-spaces V_0 and W_0 is that we have natural maps

$$U_B(V)^0 \cong U_E(V_0), \qquad U_B(W^*) \hookrightarrow U_E(W_0).$$

This will allow us to compute splittings on the quaternionic unitary groups $U_B(V)$ and $U_B(W^*)$ by pulling back splittings on $U_E(V_0)$ and $U_E(W_0)$.

For any of the pairs (V, W) given by (V, W^*), $(\operatorname{Res} V, W)$, or (V_0, W_0), we take as our convention

$$\mathrm{GL}(V) \times \mathrm{GL}(W) \to \mathrm{GL}(V \otimes W), \qquad (g, h) \mapsto (v \otimes w \mapsto g^{-1}v \otimes wh).$$

This fixes the map from the corresponding dual reductive pairs to $\mathrm{Sp}(\mathbb{V})$.

Remark 4.7. The isomorphism $GU_E(\operatorname{Res} V) \cong (E^\times \times (B')^\times)/F^\times$ can be realized as follows. Write $B' = E \oplus E\mathbf{j}'$. Then there is a natural right action of $(B')^\times$ on $\operatorname{Res} V = B$ defined by

$$1 \cdot \mathbf{j}' = \mathbf{j}, \qquad \mathbf{j} \cdot \mathbf{j}' = -J$$

so that explicitly,

$$(\alpha + \beta\mathbf{j}) \cdot (x + y\mathbf{j}') = (\alpha x - \beta\overline{y}J) + (\alpha y + \beta\overline{x})\mathbf{j}.$$

This is an action:

$$\begin{aligned}
(\alpha + \beta\mathbf{j}) \cdot (x_1 + y_1\mathbf{j}')(x_2 + y_2\mathbf{j}') &= (\alpha + \beta\mathbf{j}) \cdot ((x_1x_2 - y_1\overline{y}_2J) + (x_1y_2 + y_1\overline{x}_2)\mathbf{j}') \\
&= (\alpha(x_1x_2 - y_1\overline{y}_2J) - \beta(\overline{x}_1\overline{y}_2 + \overline{y}_1x_2)J) \\
&\qquad + (\alpha(x_1y_2 + y_1\overline{x}_2) + \beta(\overline{x}_1\overline{x}_2 - \overline{y}_1y_2J)\mathbf{j}) \\
(\alpha + \beta\mathbf{j}) \cdot (x_1 + y_1\mathbf{j}') \cdot (x_2 + y_2\mathbf{j}') &= ((\alpha x_1 - \beta\overline{y}_1J) + (\alpha y_1 + \beta\overline{x}_1)\mathbf{j}) \cdot (x_2 + y_2\mathbf{j}') \\
&= ((\alpha x_1 - \beta\overline{y}_1J)x_2 - (\alpha y_1 + \beta\overline{x}_1)\overline{y}_2J) \\
&\qquad + ((\alpha x_1 - \beta\overline{y}_1J)y_2 + (\alpha y_1 + \beta\overline{x}_1)\overline{x}_2)\mathbf{j}
\end{aligned}$$

Moreover, this action is E-linear and preserves the skew-Hermitian form on $\operatorname{Res} V$ up to a similitude character given by the reduced norm map for B'. Indeed, for $\alpha_i + \beta_i\mathbf{j} \in B$, $i = 1, 2$, and $x + y\mathbf{j}' \in B'$, we have

$$\begin{aligned}
\langle \alpha_1 + \beta_1\mathbf{j}, \alpha_2 + \beta_2\mathbf{j} \rangle_E &= \varphi((\alpha_1 + \beta_1\mathbf{j})(\mathbf{i})(\overline{\alpha}_2 - \beta_2\mathbf{j})) \\
&= \mathbf{i}(\alpha_1\overline{\alpha}_2 + \beta_1\overline{\beta}_2J),
\end{aligned}$$

and

$$\begin{aligned}
&\langle(\alpha_1+\beta_1\mathbf{j})\cdot(x+y\mathbf{j}'),(\alpha_2+\beta_2\mathbf{j})\cdot(x+y\mathbf{j}')\rangle_E \\
&\qquad\qquad = \mathbf{i}((\alpha_1 x-\beta_1\overline{y}J)(\overline{\alpha}_2\overline{x}-\overline{\beta}_2 yJ)+(\alpha_1 y+\beta_1\overline{x})(\overline{\alpha}_2\overline{y}+\overline{\beta}_2 x)J) \\
&\qquad\qquad = (x\overline{x}+y\overline{y}J)(\mathbf{i}(\alpha_1\overline{\alpha}_2+\beta_1\overline{\beta}_2 J)) \\
&\qquad\qquad = N_{B'/F}(x+y\mathbf{j}')\langle\alpha_1+\beta_1\mathbf{j},\alpha_2+\beta_2\mathbf{j}\rangle_E.
\end{aligned}$$

The left-multiplication action of $E^\times$ clearly commutes with the above right action of $(B')^\times$ and therefore we have a natural map

$$E^\times\times(B')^\times\to GU_E(\operatorname{Res}V),\qquad (x,b')\mapsto(v\mapsto x^{-1}\cdot v\cdot b').$$

It is clear that the kernel of this map is exactly the diagonal embedding of $F^\times$. We now prove that this map is surjective. Note that $GU_E(\operatorname{Res}V)$ is a connected Lie group, and therefore the surjectivity of the above map follows from the surjectivity of the induced map on Lie algebras. (A connected Lie group has no nontrivial open subgroups: If $H\subseteq G$ is open, then $\cup_{1\neq g\in G/H}\, g\cdot H$ is open, so H is also closed.) The surjectivity of the induced map on Lie algebras

$$E\oplus B'\to\mathfrak{gu}_E(\operatorname{Res}V),\qquad (X,Y)\mapsto(v\mapsto -X\cdot v+v\cdot Y)$$

is easy to see. With respect to the basis $e_1=1$, $e_2=\mathbf{j}$,

$$\langle\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\cdot v,w\rangle_E=\left\langle v,\left(\begin{smallmatrix}\overline{a}&\overline{c}\\\overline{b}/J&\overline{d}\end{smallmatrix}\right)w\right\rangle_E,$$

and therefore

$$\mathfrak{gu}_E(\operatorname{Res}V)=\left\{\left(\begin{smallmatrix}a\mathbf{i}+\alpha&b\\-\overline{b}/J&d\mathbf{i}+\alpha\end{smallmatrix}\right):\alpha,b\in E;\ a,d\in F\right\}.$$

On the other hand, for $\alpha\in E$ and $a+b\mathbf{j}'\in B'$, the map $v\mapsto-\alpha\cdot v+v\cdot(a+b\mathbf{j}')$ is given by

$$v\mapsto\left(\left(\begin{smallmatrix}-\alpha&0\\0&-\alpha\end{smallmatrix}\right)+\left(\begin{smallmatrix}a&-\overline{b}J\\b&\overline{a}\end{smallmatrix}\right)\right)v,$$

where $v\in\operatorname{Res}V$ is viewed as a column vector. It is now clear that the map on Lie algebras is surjective, finishing the proof that $(E^\times\times(B')^\times)/F^\times\cong GU_E(\operatorname{Res}V)$. $\Diamond$

CHAPTER 5

Splittings for unitary similitude groups

In this section, we define the Weil representation on the dual reductive pairs introduced in Section 4.4 using the explicit splittings of $z_{\mathbb{Y}}$ defined by Kudla [K94]. Throughout this section, we will freely use the properties of the Weil index and the Leray invariant summarized in Section 3.2 without explicitly referring to the exact property in play. We prove that the splittings are compatible with the seesaws constructed in Section 4.4. In Sections 5.1, 5.2, 5.3, and 5.4, we fix a place v of F and suppress v from the notation. In Section 5.5, we combine the local considerations from these sections into the global picture. Many of these calculations (especially in Sections 5.3 and 5.4) are motivated by [IP16a], [IP16b].

In order to describe the global automorphic theta lift from a Hecke character to a quaternion algebra, which we will do later in Chapter 6, we will need to give an explicit description of the local splittings in Section 5.3 in the special case that the quaternion algebra is unramified (i.e. split) at the place in question. We do this in Section 5.6.

5.1 Kudla's splitting for split unitary groups

We first recall Kudla's splitting [K94] of Rao's cocycle [R93] for split unitary groups over E. Let $\mathbf{W} \cong E^{2n}$ (row vectors) be an E-vector space of dimension $2n$ with ϵ-skew Hermitian form

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 \overline{y}_2^{\intercal} - \epsilon y_1 \overline{x}_2^{\intercal},$$

and let $e_1, \ldots, e_n, e_1', \ldots, e_n'$ be the E-basis of $\mathbf{W}$ giving the isomorphism $\mathbf{W} \cong E^{2n}$. Let $\mathbf{V}$ be an E-vector space of dimension m with a non-degenerate ϵ-Hermitian form $(\cdot, \cdot)$. (Here, $\overline{x}$ denotes the image of x under the nontrivial involution of E over F and the superscript $^{\intercal}$ denotes transposition.) Then $(\mathrm{U}_E(\mathbf{V}), \mathrm{U}_E(\mathbf{W}))$ is a dual reductive pair and there is a natural map

$$\iota\colon\ \mathrm{U}_E(\mathbf{V}) \times \mathrm{U}_E(\mathbf{W}) \to \mathrm{Sp}(\mathbf{V} \otimes_E \mathbf{W}), \qquad (h, g) \mapsto (w \otimes v \mapsto h^{-1} w \otimes vg).$$

We denote by $\iota_{\mathbf{W}}\colon \mathrm{U}_E(\mathbf{V}) \to \mathrm{Sp}(\mathbf{V} \otimes_E \mathbf{W})$ and $\iota_{\mathbf{V}}\colon \mathrm{U}_E(\mathbf{W}) \to \mathrm{Sp}(\mathbf{V} \otimes_E \mathbf{W})$ the restrictions of ι to $\mathrm{U}_E(\mathbf{V}) \times \{1\}$ and $\{1\} \times \mathrm{U}_E(\mathbf{W})$, respectively.

For $0 \leq j \leq n$, let $\tau_j \in \mathrm{U}_E(\mathbf{W})$ be the element defined by

$$e_i\tau_j = \begin{cases} -\epsilon e_i' & \text{if } 1 \leq i \leq j, \\ e_i & \text{if } i > j, \end{cases} \qquad \textit{and} \qquad e_i'\tau_j = \begin{cases} e_i & \text{if } 1 \leq i \leq j, \\ e_i' & \text{if } i > j. \end{cases}$$

Then

$$\mathrm{U}_E(\mathbf{W}) = \bigsqcup_{i=0}^{j} P\tau_j P,$$

where $P = P_Y \subset \mathrm{U}_E(\mathbf{W})$ is the parabolic subgroup stabilizing the maximal isotropic subspace $Y := \mathrm{span}_E\{e_1', \ldots, e_n'\}$. If $g = p_1\tau_j p_2 \in P\tau_j P$, then we define

$$j(g) := j, \qquad \text{and} \qquad x(g) := \det(p_1p_2|_Y) \in E^\times.$$

For any E-vector space $\mathbf{V}_0$ endowed with a non-degenerate Hermitian form, define

$$\gamma_F(\tfrac{1}{2}\psi \circ R\mathbf{V}_0) := (u, \det(\mathbf{V}_0))_F \gamma_F(-u, \tfrac{1}{2}\psi)^m \gamma_F(-1, \tfrac{1}{2}\psi)^{-m}.$$

Definition 5.1. Define

$$\beta_{\mathbf{V},\xi}\colon \mathrm{U}_E(\mathbf{W}) \to \mathbb{C}^1, \qquad g \mapsto \begin{cases} \xi(x(g))\gamma_F(\tfrac{1}{2}\psi \circ R\mathbf{V})^{-j(g)} & \text{if } \epsilon = +1, \\ \xi(x(g))\xi(\mathbf{i})^j \gamma_F(\tfrac{1}{2}\psi \circ R\mathbf{V}')^{-j(g)} & \text{if } \epsilon = -1, \end{cases}$$

where $\mathbf{V}'$ is the Hermitian form obtained by scaling the skew-Hermitian form on $\mathbf{V}$ by $\mathbf{i}$.

Theorem 5.2 (Kudla, [K94, Thm 3.1])**.** *Let ξ be a unitary character of $E^\times$ whose restriction to $F^\times$ is $\epsilon_{E/F}^m$, where $\epsilon_{E/F}(x) = (x, u)_F$ is the quadratic character corresponding to the extension E/F. Then for the maximal isotropic subspace $\mathbb{Y} := \mathbf{V} \otimes_E Y$ of $\mathbf{V} \otimes_E \mathbf{W}$,*

$$z_{\mathbb{Y}}(\iota_{\mathbf{V}}(g_1), \iota_{\mathbf{V}}(g_2)) = \beta_{\mathbf{V},\xi}(g_1g_2)\beta_{\mathbf{V},\xi}(g_1)^{-1}\beta_{\mathbf{V},\xi}(g_2)^{-1}.$$

In other words, with respect to the isomorphism $\mathrm{Mp}(\mathbf{V} \otimes_E \mathbf{W}) \cong \mathrm{Sp}(\mathbf{V} \otimes_E \mathbf{W}) \times \mathbb{C}^1$ determined by $z_{\mathbb{Y}}$, the following diagram commutes:

$$\begin{tikzcd} & \mathrm{Mp}(\mathbf{V} \otimes_E \mathbf{W})_{\mathbb{Y}} \arrow[d] \\ \mathrm{U}_E(\mathbf{W}) \arrow[ur, "{(\iota_{\mathbf{V}}, \beta_{\mathbf{V},\xi})}"] \arrow[r, "\iota_{\mathbf{V}}"'] & \mathrm{Sp}(\mathbf{V} \otimes_E \mathbf{W}) \end{tikzcd}$$

5.2 Changing polarizations

Lemma 5.3 (Kudla, [K94, Lemma 4.2])**.** *Let $\mathbb{X}+\mathbb{Y}$ and $\mathbb{X}'+\mathbb{Y}'$ be two polarizations of a symplectic space $\mathbb{V}$. Then*

$$z_{\mathbb{Y}'}(g_1,g_2)=\lambda(g_1g_2)\lambda(g_1)^{-1}\lambda(g_2)^{-1}\cdot z_{\mathbb{Y}}(g_1,g_2),$$

where $\lambda\colon \operatorname{Sp}(\mathbb{V})\to\mathbb{C}^1$ is given by

$$\lambda(g):=\lambda_{\mathbb{Y}\rightsquigarrow\mathbb{Y}'}(g):=\gamma_F(\tfrac{1}{2}\psi\circ q(\mathbb{Y},\mathbb{Y}'g^{-1},\mathbb{Y}'))\cdot\gamma_F(\tfrac{1}{2}\psi\circ q(\mathbb{Y},\mathbb{Y}',\mathbb{Y}g)).$$

In particular, the bijection

$$\operatorname{Mp}(\mathbb{V})_{\mathbb{Y}}\to\operatorname{Mp}(\mathbb{V})_{\mathbb{Y}'},\qquad (g,z)\mapsto(g,z\cdot\lambda(g))$$

is an isomorphism.

5.3 Three seesaws of unitary groups

For any two unitary similitude groups $\mathrm{GU}_E(\mathbf{V})$ and $\mathrm{GU}_E(\mathbf{W})$, we write

$$\mathrm{G}(\mathrm{U}_E(\mathbf{V})\times\mathrm{U}_E(\mathbf{W})):=\{(g,h)\in\mathrm{GU}_E(\mathbf{V})\times\mathrm{GU}_E(\mathbf{W}):\nu(g)=\nu(h)\}.$$

Fix a complete polarization $\mathbb{V}=\mathbb{X}+\mathbb{Y}$. In this section, we define splittings (of $z_{\mathbb{Y}}$ or $z_{\mathbb{Y}^\square}$, depending on context) for the following groups:

(i) $\mathrm{G}(\mathrm{U}_E(V_0^\square)\times\mathrm{U}_E(W_0))$

(ii) $\mathrm{G}(\mathrm{U}_E(V_0)\times\mathrm{U}_E(W_0))$

(iii) $\mathrm{G}(\mathrm{U}_E(\operatorname{Res}V)\times\mathrm{U}_E(W^\square))$

(iv) $\mathrm{G}(\mathrm{U}_E(\operatorname{Res}V)\times\mathrm{U}_E(W))$

These unitary groups fit into seesaw

$$\begin{tikzcd}[arrows=dash] \mathrm{U}_E(\operatorname{Res}V) \arrow[d] \arrow[dr] & \mathrm{U}_E(W_0) \arrow[d] \arrow[dl] \\ \mathrm{U}_E(V_0) & \mathrm{U}_E(W) \end{tikzcd} \tag{5.1}$$

and the two corresponding doubling seesaws:

$$\begin{tikzcd} \mathrm{U}_E(V_0^\square) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}_E(W_0) \times \mathrm{U}_E(W_0) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{U}_E(V_0) \times \mathrm{U}_E(V_0) & \mathrm{U}_E(W_0)^\triangle \end{tikzcd} \qquad \begin{tikzcd} \mathrm{U}_E(\operatorname{Res} V) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}_E(W^\square) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{U}_E(\operatorname{Res} V)^\triangle & \mathrm{U}_E(W) \times \mathrm{U}_E(W) \end{tikzcd} \tag{5.2}$$

5.3.1 Splittings for $\mathrm{G}(\mathrm{U}_E(V_0^\square) \times \mathrm{U}_E(W_0))$ and $\mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0))$

Consider the 2-dimensional E-space $V_0 \otimes_E W_0$ with skew-Hermitian form given by $\overline{(\cdot,\cdot)} \otimes \langle \cdot,\cdot \rangle$. By a straightforward computation, we see that this allows us to identify $V_0 \otimes_E W_0 = W_0$ as E-spaces endowed with skew-Hermitian forms. Define

$$\begin{aligned} i\colon\ \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) &\to \mathrm{U}_E((V_0 \otimes W_0)^\square), \\ (g,h) &\mapsto ((v \otimes w, v^- \otimes w^-) \mapsto (g^{-1}v \otimes wh, v^- \otimes w^-)), \\ i^-\colon\ \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) &\to \mathrm{U}_E((V_0 \otimes W_0)^\square), \\ (g,h) &\mapsto ((v \otimes w, v^- \otimes w^-) \mapsto (v \otimes w, g^{-1}v^- \otimes w^- h)), \\ i^\square\colon\ \mathrm{G}(\mathrm{U}_E(V_0^\square) \times \mathrm{U}_E(W_0)) &\to \mathrm{U}_E(V_0^\square \otimes W_0), \\ (g,h) &\mapsto (v \otimes w \mapsto g^{-1}v \otimes wh). \end{aligned}$$

We may identify $V_0^\square \otimes W_0 = (V_0 \otimes W_0)^\square = W_0^\square$. We have natural embeddings

$$\begin{aligned} &\mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \hookrightarrow \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \times \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \\ &\mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \hookrightarrow \mathrm{G}(\mathrm{U}_E(V_0^\square) \times \mathrm{U}_E(W_0)). \end{aligned}$$

Observe that for $(g_1, g_2, h) \in \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0))$,

$$i(g_1,h)i^-(g_2,h) = i^\square(g_1,g_2,h) \in \mathrm{U}_E(W_0^\square).$$

We identify $\operatorname{Res}_{E/F}(W_0^\square) = \mathbb{V}^\square$ and let

$$\iota\colon\ \mathrm{U}_E(W_0^\square) \to \mathrm{Sp}(\operatorname{Res}_{E/F}(W_0^\square)) = \mathrm{Sp}(\mathbb{V}^\square)$$

be the natural embedding. We will often identify $\mathrm{U}_E(W_0^\square)$ with $\iota(\mathrm{U}_E(W_0^\square))$.

Definition 5.4. Pick a character $\xi\colon E^\times \to \mathbb{C}^1$ such that $\xi|_{F^\times} = \epsilon_{E/F}$. Define

$$\beta\colon\ \mathrm{U}_E(W_0^\square) \to \mathbb{C}^1, \qquad g \mapsto \xi(x(g)) \cdot ((u,-1)_F \gamma_F(u, \tfrac{1}{2}\psi))^{-j(g)}.$$

Define

$$\lambda := \lambda_{V_0 \otimes W_0^{\triangle} \rightsquigarrow \mathbb{Y}^{\square}} \colon \operatorname{Sp}(\mathbb{V}^{\square}) \to \mathbb{C}^1.$$

Define

$$\begin{aligned} &\hat{s} := i^* \beta, && \hat{s}^- := (i^-)^* \beta, && \hat{s}^{\square} := (i^{\square})^* \beta, \\ &s := i^*(\beta\lambda), && s^- := (i^-)^*(\beta\lambda), && s^{\square} := (i^{\square})^*(\beta\lambda). \end{aligned}$$

Lemma 5.5.

(a) $\hat{s}$, $\hat{s}^-$, and $\hat{s}^{\square}$ are splittings of $z_{V_0 \otimes W_0^{\triangle}}$ on the images of i, i^-, and $i^{\square}$, respectively.

(b) s is a splitting of $z_{\mathbb{Y}}$ on the image of i, s^- is a splitting of $z_{\mathbb{Y}}^{-1}$ on the image of i^-, and $s^{\square}$ is a splitting of $z_{\mathbb{Y}^{\square}}$ on the image of $i^{\square}$.

Proof. Observe that $\det(V_0) = 1$ and $\dim(V_0) = 1$ so that

$$\gamma_F(\tfrac{1}{2}\psi \circ RV_0) = (u,1)_F \gamma_F(-u, \tfrac{1}{2}\psi)\gamma_F(-1, \tfrac{1}{2}\psi)^{-1} = (u,-1)_F \gamma_F(u, \tfrac{1}{2}\psi).$$

This implies that $\beta = \beta_{\mathrm{U}_E(V_0),\xi}$ (see Definition 5.1) and hence is a splitting of $z_{V_0 \otimes_E W_0^{\triangle}}$. Since $\hat{s}$, $\hat{s}^-$, and $\hat{s}^{\square}$ are pullbacks of β, they must also be splittings of the same cocycle. $\square$

Lemma 5.6. *For any $(g,h) \in \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0))$,*

$$\hat{s}^-(g,h) = \overline{\hat{s}(g,h)} \cdot \xi(\det(g,h)).$$

Proof. Let $d_{W_0^{\triangle}}(-1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and set

$$j_{W_0^{\triangle}} \colon \mathrm{U}_E(W_0^{\square}) \to \mathrm{U}_E(W_0^{\square}), \qquad g \mapsto d_{W_0^{\triangle}}(-1) g d_{W_0^{\triangle}}(-1).$$

Let $g \in \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0))$. By a straightforward computation, we have

$$x(i^-(g)) = (-1)^{j(g)} x(i(g)), \qquad \text{and} \qquad j(i^-(g)) = j(i(g)).$$

Therefore,

$$\begin{aligned}
\hat{s}^-(g) &= \xi(x(i^-(g)))((u,-1)_F\gamma_F(u,\tfrac{1}{2}\psi))^{-j(i^-(g))} \\
&= \xi(-1)^{j(g)}\xi(x(i(g)))((u,-1)_F\gamma_F(u,\tfrac{1}{2}\psi))^{-j(i^-(g))} \\
&= \xi(x(i(g)))((u,-1)_F\gamma_F(u,\tfrac{1}{2}\psi))^{j(i^+(g))} \\
&= \xi(x(i(g)))^2\overline{\hat{s}(g)} \\
&= \xi(\det(g))\overline{\hat{s}(g)}.
\end{aligned}$$

□

Lemma 5.7. *For* $(g_1, g_2, h) \in \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0))$,

$$s^\square(g_1, g_2, h) = s(g_1, h) \cdot \overline{s(g_2, h)} \cdot \xi(\det(i(g_2, h))).$$

Proof. This is [HKS96, Lemma 1.1]. See also [Lemma D.4, periods2]. □

5.3.2 Splittings for $\mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V)\times\mathrm{U}_E(W^\square))$ and $\mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V)\times\mathrm{U}_E(W))$

This section is completely analogous to Section 5.3.1. The 2-dimensional E-space $\mathrm{Res}\, V \otimes_E W$ with skew-Hermitian form $\overline{(\cdot,\cdot)} \otimes \langle\cdot,\cdot\rangle$ can be identified with $\mathrm{Res}\, V$. Define

$$\begin{aligned}
i'\colon\ \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W)) &\to \mathrm{U}_E(\mathrm{Res}\, V^\square), \\
(g,h) &\mapsto ((v,v^-) \mapsto (g^{-1}vh, v^-)) \\
i^{-\prime}\colon\ \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W)) &\to \mathrm{U}_E(\mathrm{Res}\, V^\square), \\
(g,h) &\mapsto ((v,v^-) \mapsto (v, g^{-1}v^-h) \\
i^{\square\prime}\colon\ \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W^\square)) &\to \mathrm{U}_E(\mathrm{Res}\, V^\square), \\
(g,h) &\mapsto (v \mapsto g^{-1}vh).
\end{aligned}$$

We have natural embeddings

$$\begin{aligned}
&\mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W) \times \mathrm{U}_E(W)) \hookrightarrow \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W)) \times \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W)), \\
&\mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W) \times \mathrm{U}_E(W)) \hookrightarrow \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W^\square)).
\end{aligned}$$

Observe that for $(g, h_1, h_2) \in \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W) \times \mathrm{U}_E(W))$,

$$i'(g, h_1) \cdot i^{-\prime}(g, h_2) = i^{\square\prime}(g, h_1, h_2) \in \mathrm{U}_E(\mathrm{Res}\, V^\square).$$

We identify $\mathrm{Res}_{B/F}(V^\square) = \mathbb{V}^\square$ and let

$$\iota' \colon \mathrm{U}_E(\mathrm{Res}\, V^\square) \to \mathrm{Sp}(\mathbb{V}^\square)$$

be the natural embedding. We will often identify $\mathrm{U}_E(\mathrm{Res}\, V^\square)$ with $\iota(\mathrm{U}_E(\mathrm{Res}\, V^\square))$.

Definition 5.8. Pick a character $\xi' \colon E^\times \to \mathbb{C}^1$ such that $\xi|_{F^\times} = \epsilon_{E/F}$. Define

$$\beta' \colon \mathrm{U}_E(\mathrm{Res}\, V^\square) \to \mathbb{C}^1, \qquad g \mapsto \xi'(x(g)) \cdot ((u,-1)_F \gamma_F(u, \tfrac{1}{2}\psi))^{-j(g)}.$$

Define

$$\lambda' := \lambda_{\mathrm{Res}\, V^\triangle \otimes W \rightsquigarrow \mathbb{Y}^\square} \colon \mathrm{Sp}(\mathbb{V}^\square) \to \mathbb{C}^1.$$

Define

$$\begin{aligned} \hat{s}' &:= (i')^* \beta', & \hat{s}^{-\prime} &:= (i^{-\prime})^* \beta', & \hat{s}^{\square\prime} &:= (i^{\square\prime})^* \beta', \\ s' &:= (i')^*(\beta'\lambda'), & s^{-\prime} &:= (i^{-\prime})^*(\beta'\lambda'), & s^{\square\prime} &:= (i^{\square\prime})^*(\beta'\lambda'). \end{aligned}$$

Lemma 5.9.

(a) $\hat{s}'$, $\hat{s}^{-\prime}$, and $\hat{s}^{\square\prime}$ are splittings of $z_{\mathrm{Res}\, V^\triangle \otimes W}$ on the images of i', $i^{-\prime}$, and $i^{\square\prime}$, respectively.

(b) s' is a splitting of $z_{\mathbb{Y}}$ on the image of i', $s^{-\prime}$ is a splitting of $z_{\mathbb{Y}}^{-1}$ on the image of $i^{-\prime}$, and $s^{\square\prime}$ is a splitting of $z_{\mathbb{Y}^\square}$ on the image of $i^{\square\prime}$.

Lemma 5.10. *For $(g, h_1, h_2) \in \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W) \times \mathrm{U}_E(W))$,*

$$s^{\square\prime}(g, h_1, h_2) = s'(g, h_1) \cdot \overline{s'(g, h_2)} \cdot \xi'(\det(i'(g, h_2))).$$

5.4 Compatibility between the splittings for the three seesaws

In this section, we investigate the compatibility of the splittings of the four pairs of unitary groups relative to the three seesaws presented in (5.1) and (5.2). Compatibility of the splittings in the two doubling seesaws of (5.2) is explicated in Lemmas 5.7 and 5.10. Hence it remains to investigate the compatibility of the splittings

$$s \colon \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \to \mathbb{C}^1 \qquad \text{and} \qquad s' \colon \mathrm{G}(\mathrm{U}_E(\mathrm{Res}\, V) \times \mathrm{U}_E(W)) \to \mathbb{C}^1.$$

Precisely, we would compare s and s' on the subgroup

$$\mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W)) \cong \{(\alpha, \beta) \in E^\times \times E^\times : \mathrm{Nm}(\alpha) = \mathrm{Nm}(\beta)\}.$$

We prove a sequence of lemmas that to break up the long computation that will end in Proposition 5.14.

Let $\alpha, \beta \in E^\times$ with $\mathrm{Nm}(\alpha) = \mathrm{Nm}(\beta)$ so that $(\alpha, \beta) \in \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W))$. Let $g \in \mathrm{U}_E(W_0^\square)$ denote the map $(w, w^-) \mapsto (\alpha^{-1} w \beta, w^-)$ and let $g' \in \mathrm{U}_E(\operatorname{Res} V^\square)$ denote the map $(v, v^-) \mapsto (\alpha^{-1} v \beta, v^-)$. Define:

$$v_1 := \left(-\frac{\mathbf{i}}{2u}, \frac{\mathbf{i}}{2u}\right) \qquad\qquad v_1' := (1, 1)$$
$$v_2 := \left(\frac{\mathbf{ij}}{2uJ}, -\frac{\mathbf{ij}}{2uJ}\right) \qquad\qquad v_2' := (\mathbf{j}, \mathbf{j})$$

This defines an E-basis of $W_0^\square$ and of $\operatorname{Res} V^\square$ with the following property:

$$(v_i, v_j')_0 = \delta_{ij}, \qquad\qquad (v_i, v_j)_0 = (v_i', v_j')_0 = 0,$$
$$\langle v_i, v_j' \rangle = \delta_{ij}, \qquad\qquad \langle v_i, v_j \rangle = \langle v_i', v_j' \rangle = 0.$$

With respect to the basis $\{v_1, v_2, v_1', v_2'\}$,

$$g = \begin{pmatrix} \frac{1+\alpha^{-1}\beta}{2} & 0 & \frac{1-\alpha^{-1}\beta}{4u}\mathbf{i} & 0 \\ 0 & \frac{1+\alpha^{-1}\overline{\beta}}{2} & 0 & -\frac{1-\alpha^{-1}\overline{\beta}}{4uJ}\mathbf{i} \\ (1-\alpha^{-1}\beta)\mathbf{i} & 0 & \frac{1+\alpha^{-1}\beta}{2} & 0 \\ 0 & -(1-\alpha^{-1}\overline{\beta})\mathbf{i}J & 0 & \frac{1+\alpha^{-1}\overline{\beta}}{2} \end{pmatrix} \tag{5.3}$$

$$g' = \begin{pmatrix} \frac{1+\alpha^{-1}\beta}{2} & 0 & \frac{1-\alpha^{-1}\beta}{4u}\mathbf{i} & 0 \\ 0 & \frac{1+\overline{\alpha}^{-1}\beta}{2} & 0 & \frac{1-\overline{\alpha}^{-1}\beta}{4uJ}\mathbf{i} \\ (1-\alpha^{-1}\beta)\mathbf{i} & 0 & \frac{1+\alpha^{-1}\beta}{2} & 0 \\ 0 & (1-\overline{\alpha}^{-1}\beta)\mathbf{i}J & 0 & \frac{1+\overline{\alpha}^{-1}\beta}{2} \end{pmatrix} \tag{5.4}$$

Here, we view each unitary group as a subgroup of $\mathrm{GL}_4(E)$ with $\mathrm{GL}_4(E)$ acting formally by right-multiplication. Note however that $W_0^\square$ is a left E-space, and so we interpret the formal multiplication $v \cdot a$ for $v \in W_0^\square$ and $a \in E$ as av. Throughout this section, we write $\mathbf{g}$ when we want to refer to one of g or g' simultaneously.

Lemma 5.11. *We have*

Conditions	$x(g)$	$x(g')$	$j(\mathbf{g})$
$\alpha^{-1}\beta = 1,\ \alpha^{-1}\overline{\beta} = 1$	1	1	0
$\alpha^{-1}\beta = 1,\ \alpha^{-1}\overline{\beta} \neq 1$	$-(1-\alpha^{-1}\overline{\beta})\mathbf{i}J$	$(1-\overline{\alpha}^{-1}\beta)\mathbf{i}J$	1
$\alpha^{-1}\beta \neq 1,\ \alpha^{-1}\overline{\beta} = 1$	$(1-\alpha^{-1}\beta)\mathbf{i}$	$(1-\alpha^{-1}\beta)\mathbf{i}$	1
$\alpha^{-1}\beta \neq 1,\ \alpha^{-1}\overline{\beta} \neq 1$	$-(1-\alpha^{-1}\beta)(1-\alpha^{-1}\overline{\beta})uJ$	$(1-\alpha^{-1}\beta)(1-\overline{\alpha}^{-1}\beta)uJ$	2

Proof. The proof amounts to giving explicit decompositions

$$\mathbf{g} = p_1 w p_2, \qquad \text{where } p_i \in P_{\mathbf{V}^\triangle} \text{ and } w = \tau_j = \begin{pmatrix} 1_{2-j} & & & \\ & & & -1_j \\ & & 1_{2-j} & \\ & 1_j & & \end{pmatrix}.$$

There are four cases:

(a) If $\alpha^{-1}\beta = 1$ and $\alpha^{-1}\overline{\beta} = 1$, then

$$g = 1, \qquad g' = 1.$$

(b) If $\alpha^{-1}\beta = 1$ and $\alpha^{-1}\overline{\beta} \neq 1$, then $g = p_1\tau_1 p_2$ and $g' = p_1'\tau_1 p_2'$ for

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1+\alpha^{-1}\overline{\beta}}{2(-1+\alpha^{-1}\overline{\beta})\mathbf{i}J} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1+\alpha^{-1}\overline{\beta})\mathbf{i}J & 0 & \frac{1+\alpha^{-1}\overline{\beta}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\alpha^{-1}\overline{\beta}}{(-1+\alpha^{-1}\overline{\beta})\mathbf{i}J} \end{pmatrix}$$

$$p_1' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1+\overline{\alpha}^{-1}\beta}{2(-1+\overline{\alpha}^{-1}\beta)\mathbf{i}J} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(-1+\overline{\alpha}^{-1}\beta)\mathbf{i}J & 0 & \frac{1+\overline{\alpha}^{-1}\beta}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\overline{\alpha}^{-1}\beta}{(-1+\overline{\alpha}^{-1}\beta)\mathbf{i}J} \end{pmatrix}$$

(c) If $\alpha^{-1}\beta \neq 1$ and $\alpha^{-1}\overline{\beta} = 1$, then

$$g = g' = \begin{pmatrix} 0 & 1 & 0 & \frac{1+\alpha^{-1}\beta}{2\mathbf{i}(1-\alpha^{-1}\beta)} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \tau_1 \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ (1-\alpha^{-1}\beta)\mathbf{i} & 0 & \frac{1+\alpha^{-1}\beta}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\alpha^{-1}\beta}{(1-\alpha^{-1}\beta)\mathbf{i}} & 0 \end{pmatrix}.$$

(d) If $\alpha^{-1}\beta \neq 1$ and $\alpha^{-1}\overline{\beta} \neq 1$, then $g = p_1\tau_2 p_2$ and $g' = p_1'\tau_2 p_2'$ for

$$p_1 = \begin{pmatrix} 1 & 0 & \frac{1+\alpha^{-1}\beta}{2(1-\alpha^{-1}\beta)\mathbf{i}} & 0 \\ 0 & 1 & 0 & -\frac{1+\alpha^{-1}\overline{\beta}}{2(1-\alpha^{-1}\overline{\beta})\mathbf{i}J} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} (1-\alpha^{-1}\beta)\mathbf{i} & 0 & \frac{1+\alpha^{-1}\beta}{2} & 0 \\ 0 & -(1-\alpha^{-1}\overline{\beta})\mathbf{i}J & 0 & \frac{1+\alpha^{-1}\overline{\beta}}{2} \\ 0 & 0 & \frac{\alpha^{-1}\beta}{(1-\alpha^{-1}\beta)\mathbf{i}} & 0 \\ 0 & 0 & 0 & -\frac{\alpha^{-1}\overline{\beta}}{(1-\alpha^{-1}\overline{\beta})\mathbf{i}J} \end{pmatrix}$$

$$p_1' = \begin{pmatrix} 1 & 0 & \frac{1+\alpha^{-1}\beta}{2(1-\alpha^{-1}\beta)\mathbf{i}} & 0 \\ 0 & 1 & 0 & \frac{1+\overline{\alpha}^{-1}\beta}{2(1-\overline{\alpha}^{-1}\beta)\mathbf{i}J} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p_2' = \begin{pmatrix} (1-\alpha^{-1}\beta)\mathbf{i} & 0 & \frac{1+\alpha^{-1}\beta}{2} & 0 \\ 0 & (1-\overline{\alpha}^{-1}\beta)\mathbf{i}J & 0 & \frac{1+\overline{\alpha}^{-1}\beta}{2} \\ 0 & 0 & \frac{\alpha^{-1}\beta}{(1-\alpha^{-1}\beta)\mathbf{i}} & 0 \\ 0 & 0 & 0 & \frac{\overline{\alpha}^{-1}\beta}{(1-\overline{\alpha}^{-1}\beta)\mathbf{i}J} \end{pmatrix}$$

From the above decompositions, we can easily read off the desired information. $\square$

Lemma 5.12. *Let $\alpha = a_1 + b_1\mathbf{i}$. Then*

$$\hat{s}(\alpha,\alpha) = \begin{cases} \xi(\alpha^{-1}) \cdot (a_1, u)_F & \text{if } b_1 = 0, \\ \xi(\alpha^{-1}) \cdot (-2b_1uJ, u)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi) \cdot (-1,-u)_F & \text{otherwise.} \end{cases}$$

$$\hat{s}'(\alpha,\alpha) = \begin{cases} \xi'(\overline{\alpha}^{-1}) \cdot (a_1, u)_F & \text{if } b_1 = 0, \\ \xi'(\overline{\alpha}^{-1}) \cdot (-2b_1uJ, u)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi) \cdot (-1,-u)_F & \text{otherwise.} \end{cases}$$

Proof. We use Lemma 5.11 in the two cases where $\alpha^{-1}\beta = 1$. If $\alpha^{-1}\overline{\alpha} = 1$, then $\alpha = \overline{\alpha}$ and so $b_1 = 0$. By Lemma 5.11, we have

$$\hat{s}(\alpha, \alpha) = \hat{s}'(\alpha, \alpha) = 1 = \xi(\alpha^{-1}) \cdot (a_1, u)_F = \xi'(\alpha^{-1}) \cdot (a_1, u)_F.$$

If $\alpha^{-1}\overline{\alpha} \neq 1$, then $b_1 \neq 0$. Note that

$$\begin{aligned} 1 - \alpha^{-1}\overline{\alpha} &= \alpha^{-1}(\alpha - \overline{\alpha}) = \alpha^{-1} \cdot 2b_1\mathbf{i}, \\ 1 - \overline{\alpha}^{-1}\alpha &= \overline{1 - \alpha^{-1}\overline{\alpha}} = -\overline{\alpha}^{-1} \cdot 2b_1\mathbf{i}. \end{aligned}$$

Thus by Lemma 5.11,

$$\begin{aligned} \hat{s}(\alpha, \alpha) &= \xi(\alpha^{-1}) \cdot (-2b_1uJ, u)_F \cdot \gamma_F(u, \frac{1}{2}\psi) \cdot (-1, -u)_F, \\ \hat{s}'(\alpha, \alpha) &= \xi'(\overline{\alpha}^{-1}) \cdot (-2b_1uJ, u)_F \cdot \gamma_F(u, \frac{1}{2}\psi) \cdot (-1, -u)_F. \end{aligned}$$

□

Lemma 5.13. *Let $\zeta = a + b\mathbf{i} \in E^1$. Then*

$$\hat{s}(1, \zeta) = \begin{cases} 1 & \text{if } a = 1, \\ ((2 - 2a)uJ, u)_F & \text{if } a \neq 1. \end{cases}$$

$$\hat{s}'(1, \zeta) = \begin{cases} 1 & \text{if } a = 1, \\ \xi'(\zeta) \cdot ((2 - 2a)uJ, u)_F & \text{if } a \neq 1. \end{cases}$$

Proof. We use Lemma 5.11. If $\zeta = 1$, this corresponds to the case $\alpha^{-1}\beta = 1$, $\alpha^{-1}\overline{\beta} = 1$, and

$$\hat{s}(1, \zeta) = \hat{s}'(1, \zeta) = 1.$$

If $\zeta \neq 1$, this corresponds to the case $\alpha^{-1}\beta \neq 1$, $\alpha^{-1}\overline{\beta} \neq 1$, and

$$\begin{aligned} \hat{s}(1, \zeta) &= \xi(-(1 - \zeta)(1 - \overline{\zeta})uJ) \cdot (-1, u)_F, \\ \hat{s}'(1, \zeta) &= \xi'((1 - \zeta)^2 uJ) \cdot (-1, u)_F. \end{aligned}$$

Now,

$$\begin{aligned} (1 - \zeta)(1 - \overline{\zeta}) &= 2 - 2a \\ (1 - \zeta)^2 &= -\zeta(1 - \zeta)(1 - \overline{\zeta}) = -\zeta(2 - 2a). \end{aligned}$$

Thus

$$\hat{s}(1,\zeta) = ((2-2a)uJ, u)_F,$$
$$\hat{s}'(1,\zeta) = \xi'(\zeta)\cdot((2-2a)uJ, u)_F. \qquad \square$$

Proposition 5.14. *Let* $g \in \mathrm{G}(\mathrm{U}_E(V_0)\times\mathrm{U}_E(W)) \subset \mathrm{G}(\mathrm{U}_E(V_0)\times\mathrm{U}_E(W_0))$ *and* $g' \in \mathrm{G}(\mathrm{U}_E(V_0)\times\mathrm{U}_E(W)) \subset \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V)\times\mathrm{U}_E(W))$ *correspond to* $(\alpha,\beta)\in E^\times\times E^\times$ *with* $\mathrm{Nm}(\alpha)=\mathrm{Nm}(\beta)$. *Then*

$$s'(g') = \xi(\alpha)\xi'(\beta)s(g).$$

Proof. We use the formulas given in Lemma 5.12 and Lemma 5.13 together with Lemma 5.3. Recall that

$$g = g_1\cdot g_2, \qquad g' = g_1'\cdot g_2',$$

where $\mathbf{g}_1$ corresponds to (α,α) and $\mathbf{g}_2$ corresponds to $(1,\beta/\alpha)$.

First notice that under the natural maps

$$i\colon\ \mathrm{U}_E(V_0\otimes W_0)\to\mathrm{Sp}(\mathbb{V}) \qquad i^{\square}\colon\ \mathrm{U}_E(V_0\otimes W_0)\to\mathrm{Sp}(\mathbb{V}^{\square}),$$
$$i'\colon\ \mathrm{U}_E(\operatorname{Res}V\otimes W)\to\mathrm{Sp}(\mathbb{V}) \qquad i^{\square\prime}\colon\ \mathrm{U}_E(\operatorname{Res}V\otimes W)\to\mathrm{Sp}(\mathbb{V}^{\square}),$$

we have

$$i(g_\bullet) = i'(g_\bullet') \in \mathrm{Sp}(\mathbb{V}), \qquad i^{\square}(g_\bullet) = i^{\square\prime}(g_\bullet')\in\mathrm{Sp}(\mathbb{V}^{\square}),$$

where $\mathbf{g}_\bullet$ denotes any of $\mathbf{g}$, $\mathbf{g}_1$, $\mathbf{g}_2$. This implies that for $\lambda := \lambda_{\mathbb{V}^{\triangle}\rightsquigarrow\mathbb{Y}^{\square}}$,

$$\lambda(i^{\square}(g_\bullet)) = \lambda(i^{\square\prime}(g_\bullet')), \qquad \text{and} \qquad z_{\mathbb{Y}}(i(g_1), i(g_2)) = z_{\mathbb{Y}}(i'(g_1'), i'(g_2')).$$

By definition,

$$s(\mathbf{g}) = \hat{s}(\mathbf{g}_1)\cdot\mu(\mathbf{g}_1)\cdot\hat{s}(\mathbf{g}_2)\cdot\mu(\mathbf{g}_2)\cdot z_{\mathbb{Y}}(i(\mathbf{g}_1), i(\mathbf{g}_2)),$$
$$s'(\mathbf{g}) = \hat{s}'(\mathbf{g}_1)\cdot\mu(\mathbf{g}_1)\cdot\hat{s}'(\mathbf{g}_2)\cdot\mu(\mathbf{g}_2)\cdot z_{\mathbb{Y}}(i'(\mathbf{g}_1), i'(\mathbf{g}_2)),$$

Thus we have

$$\boldsymbol{\chi}(\alpha,\beta) = s(g)\cdot s'(g')^{-1} = \hat{s}(g_1)\cdot\hat{s}(g_2)\cdot\hat{s}'(g_1')^{-1}\cdot\hat{s}'(g_2')^{-1}.$$

Now we combine the results of Lemmas 5.12 and 5.13 to compute $\boldsymbol{\chi}(\alpha,\beta)$. Using the fact

$$\overline{\alpha}^{-1}\cdot\beta\cdot\alpha^{-1} = \overline{\beta}^{-1},$$

in the calculation of $\hat{s}'(g_1')\hat{s}'(g_2')$ when $\alpha \neq \beta$, we have:

$$\hat{s}(g_1)\cdot\hat{s}(g_2) = \begin{cases} \xi(\alpha^{-1})\cdot(a_1,u)_F & \alpha \in F^\times,\, \alpha=\beta \\ \xi(\alpha^{-1})\cdot(a_1,u)_F\cdot((2-2a)uJ,u)_F & \alpha \in F^\times,\, \alpha\neq\beta \\ \xi(\alpha^{-1})\cdot(-2b_1uJ,u)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi)\cdot(-1,-u)_F & \alpha \notin F^\times,\, \alpha=\beta \\ \xi(\alpha^{-1})\cdot(-2b_1uJ,u)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi)\cdot(-1,-u)_F & \\ \qquad\qquad\qquad\qquad\cdot((2-2a)uJ,u)_F & \alpha \notin F^\times,\, \alpha\neq\beta \end{cases}$$

$$\hat{s}'(g_1')\cdot\hat{s}'(g_2') = \begin{cases} \xi'(\overline{\alpha}^{-1})\cdot(a_1,u)_F & \alpha \in F^\times,\, \alpha=\beta \\ \xi'(\overline{\beta}^{-1})\cdot(a_1,u)_F\cdot((2-2a)uJ,u)_F & \alpha \in F^\times,\, \alpha\neq\beta \\ \xi'(\overline{\alpha}^{-1})\cdot(-2b_1uJ,u)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi)\cdot(-1,-u)_F & \alpha \notin F,\, \alpha=\beta \\ \xi'(\overline{\beta}^{-1})\cdot(-2b_1uJ,u)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi)\cdot(-1,-u)_F & \\ \qquad\qquad\qquad\qquad\cdot((2-2a)uJ,u)_F & \alpha \notin F^\times,\, \alpha\neq\beta \end{cases}$$

Therefore

$$\begin{aligned} \boldsymbol{\chi}(\alpha,\beta) &= \begin{cases} \xi(\alpha^{-1})\cdot\xi'(\overline{\alpha}) & \alpha \in F^\times,\, \alpha=\beta \\ \xi(\alpha^{-1})\cdot\xi'(\overline{\beta}) & \alpha \in F^\times,\, \alpha\neq\beta \\ \xi(\alpha^{-1})\cdot\xi'(\overline{\alpha}) & \alpha \notin F^\times,\, \alpha=\beta \\ \xi(\alpha^{-1})\cdot\xi'(\overline{\beta}) & \alpha \notin F^\times,\, \alpha\neq\beta \end{cases} \\ &= \xi(\alpha^{-1})\cdot\xi'(\overline{\beta}) = \xi(\alpha^{-1})\cdot\xi'(\beta^{-1})\cdot\xi'(\beta\overline{\beta}) = \xi(\alpha^{-1})\xi'(\beta^{-1}). \end{aligned}$$ □

5.5 Product formula

In this section, we put the local considerations of the Sections 5.1, 5.2, 5.3, and 5.4 into the global picture. Once and for all, pick Hecke characters

$$\xi,\xi'\colon E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^1$$

such that

$$\xi|_{\mathbb{A}_F^\times} = \xi'|_{\mathbb{A}_F^\times} = \epsilon_{E/F}.$$

Note that $\mathrm{U}_E(V_0) \cong E^\times \cong \mathrm{U}_B(V)^0$ and hence we have a natural embeddings

$$\begin{aligned} \mathrm{G}(\mathrm{U}_B(V)^0 \times \mathrm{U}_B(W)) &\hookrightarrow \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \\ \mathrm{G}(\mathrm{U}_B(V^\square)^0 \times \mathrm{U}_B(W)) &\hookrightarrow \mathrm{G}(\mathrm{U}_E(V_0^\square) \times \mathrm{U}_E(W_0)). \end{aligned}$$

Thus functions defined on the unitary spaces pull back to functions on the quaternionic unitary spaces. For each place v of F, by Definition 5.4 and 5.8, we have functions

$$\begin{aligned} s_v &\colon \mathrm{G}(\mathrm{U}_B(V_v) \times \mathrm{U}_B(W_v^*)) \to \mathbb{C}^1, & s_v^\square &\colon \mathrm{G}(\mathrm{U}_B(V_v^\square)^0 \times \mathrm{U}_B(W_v^*)) \to \mathbb{C}^1, \\ s_v' &\colon \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V_v) \times \mathrm{U}_E(W_v)) \to \mathbb{C}^1, & s_v^{\square\prime} &\colon \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V_v) \times \mathrm{U}_E(W_v^\square)) \to \mathbb{C}^1. \end{aligned}$$

Lemma 5.15.

(a) Let $\gamma \in \mathrm{G}(\mathrm{U}_B(V)(F) \times \mathrm{U}_B(W)(F))$. Then

$$s_v(\gamma) = 1 \textit{ for almost all } v \textit{ and } \prod_v s_v(\gamma) = 1.$$

(b) Let $\gamma \in \mathrm{G}(\mathrm{U}_B(V^\square)^0(F) \times \mathrm{U}_B(W)(F))$. Then

$$s_v^\square(\gamma) = 1 \textit{ for almost all } v \textit{ and } \prod_v s_v^\square(\gamma) = 1.$$

(c) Let $\gamma \in \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V)(F) \times \mathrm{U}_E(W)(F))$. Then

$$s_v'(\gamma) = 1 \textit{ for almost all } v \textit{ and } \prod_v s_v'(\gamma) = 1.$$

(d) Let $\gamma \in \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V)(F) \times \mathrm{U}_E(W^\square)(F))$. Then

$$s_v^{\square\prime}(\gamma) = 1 \textit{ for almost all } v \textit{ and } \prod_v s_v^{\square\prime}(\gamma) = 1.$$

Define

$$
\begin{aligned}
s &= \prod_v s_v \colon \ \mathrm{G}(\mathrm{U}_B(V)(\mathbb{A}) \times \mathrm{U}_B(W^*)(\mathbb{A})) \to \mathbb{C}^1, \\
s^\square &= \prod_v s_v^\square \colon \ \mathrm{G}(\mathrm{U}_B(V^\square)(\mathbb{A}) \times \mathrm{U}_B(W^{*\square})(\mathbb{A})) \to \mathbb{C}^1, \\
s' &= \prod_v s'_v \colon \ \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V)(\mathbb{A}) \times \mathrm{U}_E(W)(\mathbb{A})) \to \mathbb{C}^1, \\
s^{\square\prime} &= \prod_v s_v^{\square\prime} \colon \ \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V)(\mathbb{A}) \times \mathrm{U}_E(W^\square)(\mathbb{A})) \to \mathbb{C}^1.
\end{aligned}
$$

Proposition 5.16.

(a) [Lemma 5.7] For $(g_1, g_2, h) \in \mathrm{G}(\mathrm{U}_B(V)^0(\mathbb{A}) \times \mathrm{U}_B(V)^0(\mathbb{A}) \times \mathrm{U}_B(W)(\mathbb{A}))$,

$$s^\square(g_1, g_2, h) = s(g_1, h) \cdot \overline{s(g_2, h)} \cdot \xi(\det(i(g_2, h))).$$

(b) [Lemma 5.10] For $(h, g_1, g_2) \in \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V)(\mathbb{A}) \times \mathrm{U}_E(W)(\mathbb{A}) \times \mathrm{U}_E(W)(\mathbb{A}))$,

$$s^{\square\prime}(h, g_1, g_2) = s'(h, g_1) \cdot \overline{s'(h, g_2)} \cdot \xi'(\det(i'(h, g_2))).$$

(c) [Proposition 5.14] For $\alpha, \beta \in \mathbb{A}_E^\times$ *such that* $\mathrm{Nm}(\alpha) = \mathrm{Nm}(\beta)$,

$$s'(\alpha, \beta) = \xi(\alpha)\xi'(\beta)s(\alpha, \beta).$$

5.6 Two splittings on $E_v^\times \times \mathrm{GL}_2(F_v)$

To calculate the theta lift at all the unramified places, we will have to understand the Weil representation more concretely. In particular, we will need to explicate the local splittings defined in Chapter 5 in the case when $v \notin S_B$ and $v \notin S_{B'}$. These exactly correspond, respectively, to the cases when $W_{0,v}$ and $\operatorname{Res} V_v$ are split Hermitian spaces. For notational convenience, we drop the subscript v in this section.

Consider the group

$$R := \mathrm{G}(E^\times \times \mathrm{GL}_2(F)) = \left\{(\alpha, g) \in E^\times \times \mathrm{GL}_2(F) : \mathrm{Nm}(\alpha) = \det(g)\right\}.$$

Assume that the 2-dimensional E-spaces W_0 and $\operatorname{Res} V$ are hyperbolic planes (i.e. they are split Hermitian spaces). Then we have embeddings

$$R \hookrightarrow \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)), \qquad (\alpha, g) \mapsto (\alpha, g)$$
$$R \hookrightarrow \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V) \times \mathrm{U}_E(W)), \qquad (\alpha, g) \mapsto (g, \alpha).$$

Furthermore, any decomposition of W_0 or $\operatorname{Res} V$ into maximal isotropic subspaces induces a complete polarization

$$\mathbb{V} = \mathbb{X}' + \mathbb{Y}'.$$

Recall that in Chapter 5, we defined functions

$$\hat{s}\colon \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \to \mathbb{C}^1 \qquad \text{such that } z_{V_0 \otimes W_0^{\triangle}} = \partial \hat{s},$$
$$\hat{s}'\colon \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V) \times \mathrm{U}_E(W)) \to \mathbb{C}^1 \qquad \text{such that } z_{\operatorname{Res} V^{\triangle} \otimes W} = \partial \hat{s}'.$$

Recall that these were defined by pulling back Kudla's splitting on split unitary groups along the maps

$$i\colon \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \to \mathrm{U}_E(W_0^{\square}), \qquad i'\colon \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V) \times \mathrm{U}_E(W)) \to \mathrm{U}_E(\operatorname{Res} V^{\square})$$

defined in Sections 5.3.1 and 5.3.2. If we let $\boldsymbol{\lambda}\colon \mathrm{Sp}(\mathbb{V}^{\square}) \to \mathbb{C}^1$ be given by

$$\boldsymbol{\lambda}(g) := \gamma_F(\tfrac{1}{2}\psi \circ q(\mathbb{V}^{\triangle}, \mathbb{Y}'^{\square} g^{-1}, \mathbb{Y}'^{\square})) \cdot \gamma_F(\tfrac{1}{2} \circ q(\mathbb{V}^{\triangle}, \mathbb{Y}'^{\square}, \mathbb{V}^{\triangle} g)),$$

then we may define functions

$$\mathbf{s} := \hat{s} \cdot \boldsymbol{\lambda}\colon \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)) \to \mathbb{C}^1 \qquad \text{such that } z_{\mathbb{Y}'} = \partial \mathbf{s},$$
$$\mathbf{s}' := \hat{s}' \cdot \boldsymbol{\lambda}\colon \mathrm{G}(\mathrm{U}_E(\operatorname{Res} V) \times \mathrm{U}_E(W)) \to \mathbb{C}^1 \qquad \text{such that } z_{\mathbb{Y}'} = \partial \mathbf{s}'.$$

In this section, we explicate the values of $\mathbf{s}$ and $\mathbf{s}'$ on R.

We work with $\mathbf{s}$ first. Let W_1 and W_2 be isotropic subspaces such that $W_0 = W_1 + W_2$. Fix $w_i \in W_i$ so that $\langle w_1, w_2 \rangle = 1$. Make the analogous definitions for $\mathbf{s}'$: let V_1 and V_2 be isotropic subspaces such that $\operatorname{Res} V = V_1 + V_2$ and fix $w_i \in V_i$ such that $\langle w_1, w_2 \rangle = 1$. Define

$$\mathbf{w}_1 = (\tfrac{1}{2}w_1, -\tfrac{1}{2}w_1), \qquad \mathbf{w}_2 = (-\tfrac{1}{2}w_2, \tfrac{1}{2}w_2), \qquad \mathbf{w}_1^* = (w_2, w_2), \qquad \mathbf{w}_2^* = (w_1, w_1)$$

so that we have

$$\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \langle \mathbf{w}_i^*, \mathbf{w}_j^* \rangle, \qquad \langle \mathbf{w}_i, \mathbf{w}_j^* \rangle = \delta_{ij}, \qquad \langle \mathbf{w}_i^*, \mathbf{w}_j \rangle = -\delta_{ij},$$

and

$$
\begin{aligned}
W_0^{\square} &= W_0^{\triangledown} + W_0^{\vartriangle}, && \text{where } W_0^{\triangledown} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\} \text{ and } W_0^{\vartriangle} = \operatorname{span}\{\mathbf{w}_1^*, \mathbf{w}_2^*\},\\
\operatorname{Res} V^{\square} &= \operatorname{Res} V^{\triangledown} + \operatorname{Res} V^{\vartriangle}, && \text{where } \operatorname{Res} V^{\triangledown} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\} \text{ and } \operatorname{Res} V^{\vartriangle} = \operatorname{span}\{\mathbf{w}_1^*, \mathbf{w}_2^*\}.
\end{aligned}
$$

Then a symplectic basis preserving the complete polarization

$$\mathbb{V}^{\square} = \mathbb{V}^{\triangledown} + \mathbb{V}^{\vartriangle}$$

is given by

$$\mathbf{w}_1, \quad \tfrac{-1}{u}\mathbf{i}\mathbf{w}_1, \quad \mathbf{w}_2, \quad \tfrac{-1}{u}\mathbf{i}\mathbf{w}_2, \quad \mathbf{w}_1^*, \quad \mathbf{i}\mathbf{w}_1^*, \quad \mathbf{w}_2^*, \quad \mathbf{i}\mathbf{w}_2^*. \tag{5.5}$$

5.6.1 A splitting s of $z_{\mathbb{Y}'}$

For $a, d \in F^\times$, write $D(a,d) := \operatorname{diag}(a,d)$.

Lemma 5.17. *Let $(\alpha, D(a,d)) \in R$. Then*

$$\mathbf{s}(\alpha, D(a,d)) = \xi(-(\alpha^{-1}a - 1)(\alpha^{-1}d - 1)).$$

In particular, for $a \in F^\times$ and $\alpha \in E^\times$,

$$
\begin{aligned}
\mathbf{s}(1, D(a, a^{-1})) &= (u, a)_F,\\
\mathbf{s}(\alpha, D(1, \operatorname{Nm}(\alpha))) &= \xi(\alpha^{-1}).
\end{aligned}
$$

Proof. We have $(1, D(1,1)) = (1, U(0))$, and this is proved in Lemma 5.18, so we assume that $(\alpha, D(a,d)) \neq (1, D(1,1))$. This assumption will be necessary when we calculate $\hat{s}$.

Recall that $(\alpha, D(a,d))$ sends $w_1 \mapsto \alpha^{-1}aw_1$ and $w_2 \mapsto \alpha^{-1}dw_2$. Recalling that $i\colon \mathrm{U}_E(W_0) \to \mathrm{U}_E(W_0 + W_0^-)$ is defined by $\mathrm{U}_E(W_0)$ acting linearly on W_0 and trivially on W_0^-, it is a straightforward computation to see that the image of $(\alpha, D(a,d))$ in $\mathrm{U}_E(W_0 + W_0^-)$ with respect to the basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_1^*, \mathbf{w}_2^*$ is

$$\begin{pmatrix} \frac{\alpha^{-1}a+1}{2} & 0 & 0 & \frac{\alpha^{-1}a-1}{4} \\ 0 & \frac{\alpha^{-1}d+1}{2} & -\frac{\alpha^{-1}d-1}{4} & 0 \\ 0 & -(\alpha^{-1}d-1) & \frac{\alpha^{-1}d+1}{2} & 0 \\ -(\alpha^{-1}a-1) & 0 & 0 & \frac{\alpha^{-1}a+1}{2} \end{pmatrix}.$$

We have

$$i(\alpha, D(a,d)) = p_1 \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} p_2,$$

where

$$p_1 = \begin{pmatrix} 0 & -\frac{(\alpha^{-1}a+1)^2}{4(\alpha^{-1}a-1)} + \frac{\alpha^{-1}a-1}{4} & -\frac{\alpha^{-1}a+1}{2} & 0 \\ \frac{(\alpha^{-1}d+1)^2}{4(\alpha^{-1}d-1)} - \frac{\alpha^{-1}d-1}{4} & 0 & 0 & -\frac{\alpha^{-1}d+1}{2} \\ 0 & 0 & 0 & (\alpha^{-1}d-1) \\ 0 & 0 & -(\alpha^{-1}a-1) & 0 \end{pmatrix},$$

$$p_2 = \begin{pmatrix} 1 & 0 & 0 & \frac{\alpha^{-1}a+1}{2(\alpha^{-1}a-1)} \\ 0 & 1 & -\frac{\alpha^{-1}d+1}{4(\alpha^{-1}d-1)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies that

$$x(i(\alpha, D(a,d))) = (\alpha^{-1}a - 1)(\alpha^{-1}d - 1), \qquad j(i(\alpha, D(a,d))) = 2,$$

and therefore by Definition 5.4,

$$\hat{s}(i(\alpha, D(a,d))) = \xi((\alpha^{-1}a - 1)(\alpha^{-1}d - 1)) \cdot \gamma_F(u, \tfrac{1}{2}\psi)^{-2} = \xi(-(\alpha^{-1}a - 1)(\alpha^{-1}d - 1)).$$

With respect to the symplectic basis given in (5.5), the image of $i(\alpha, D(a,d))$ in $\mathrm{Sp}(\mathbb{V}^{\square})$ is

$$g = \begin{pmatrix} \frac{xa+1}{2} & -\frac{yau}{2} & & & & & \frac{xa-1}{4} & \frac{ya}{4} \\ -\frac{ya}{2} & \frac{xa+1}{2} & & & & & \frac{ya}{4u} & \frac{xa-1}{4} \\ & & \frac{xd+1}{2} & -\frac{ydu}{2} & -\frac{xd-1}{4} & -\frac{yd}{4} & & \\ & & -\frac{yd}{2} & \frac{xd+1}{2} & -\frac{yd}{4u} & -\frac{xd-1}{4} & & \\ & & -(xd-1) & ydu & \frac{xd+1}{2} & \frac{yd}{2} & & \\ & & yd & -(xd-1) & \frac{yd}{2u} & \frac{xd+1}{2} & & \\ xa-1 & -yau & & & & & \frac{xa+1}{2} & \frac{ya}{2} \\ ya & xa-1 & & & & & \frac{ya}{2u} & \frac{xa+1}{2} \end{pmatrix} \in \mathrm{Sp}(\mathbb{V}^{\square}).$$

By definition,

$$\boldsymbol{\lambda}(\alpha, D(a,d)) = \gamma_F(\tfrac{1}{2}\psi \circ q(\mathbb{V}^{\triangle}, \mathbb{Y}'^{\square}g^{-1}, \mathbb{Y}'^{\square})) \cdot \gamma_F(\tfrac{1}{2}\psi \circ q(\mathbb{V}^{\triangle}, \mathbb{Y}'^{\square}, \mathbb{V}^{\triangle}g)).$$

Since g stabilizes $\mathbb{Y}'^{\square}$,

$$\gamma_F(\tfrac{1}{2}\psi \circ q(\mathbb{V}^{\triangle}, \mathbb{Y}'^{\square}g^{-1}, \mathbb{Y}'^{\square})) = 1.$$

To calculate the second factor, notice that

$$\begin{aligned}
\mathbb{V}^{\triangle} &= \{(0,0,0,0,z_1,z_2,z_3,z_4)\}\\
\mathbb{Y}'^{\square} &= \{(0,0,z_1,z_2,z_3,z_4,0,0)\}\\
\mathbb{V}^{\triangle}g &= \{((xa-1)z_2+yaz_4, -yauz_3+(xa-1)z_4, -(xd-1)z_1+ydz_2, yduz_1-(xd-1)z_2,\\
&\qquad -\tfrac{xd+1}{2}z_1-\tfrac{yd}{2u}z_2, -\tfrac{yd}{2}z_1-\tfrac{xd+1}{2}z_2, \tfrac{xa+1}{2}z_3+\tfrac{ya}{2u}z_4, \tfrac{ya}{2}z_3+\tfrac{xa+1}{2}z_4)\}
\end{aligned}$$

and one can see that this implies that $\mathbb{R}=\{(0,0,*,*,*,*,0,0)\}$ and hence

$$\gamma_F(\tfrac{1}{2}\psi\circ q(\mathbb{V}^{\triangle},\mathbb{Y}'^{\square},\mathbb{V}^{\triangle}g))=1.$$

We therefore have

$$\mathbf{s}(\alpha,D(a,d))=\hat{s}(\alpha,D(a,d))=\xi(-(\alpha^{-1}a-1)(\alpha^{-1}d-1)).$$

This proves the main assertion and the remaining formulas can be deduced as follows: Assuming $a\neq 1$ and $\alpha\neq 1$ (observe that if $\alpha\in E^1$, then $x=1$ if and only if $\alpha=1$),

$$\begin{aligned}
\mathbf{s}(1,D(a,a^{-1})) &= \xi(-(a-1)(a^{-1}-1))=\xi(a-2+a^{-1})\\
&= \xi(a^{-1}(a^2-2a+1))=\xi(a^{-1}(a-1)^2)=\xi(a^{-1})\\
&= \xi(a)=(u,a)_F.
\end{aligned}$$

If $\alpha\in E^{\times}$, then

$$\begin{aligned}
\mathbf{s}(\alpha,D(1,\mathrm{Nm}(\alpha))) &= \xi(-(\alpha^{-1}-1)(\alpha^{-1}\alpha\bar{\alpha}-1))\\
&= \xi(-(\alpha^{-1}-1)(\bar{\alpha}-1))=\xi(\alpha^{-1}(\alpha-1)(\bar{\alpha}-1))\\
&= \xi(\alpha^{-1})\epsilon_{E/F}(\mathrm{Nm}(\alpha-1))=\xi(\alpha^{-1}).
\end{aligned}$$

□

Lemma 5.18. *Let $a\in F$. Then*

$$\mathbf{s}(1,U(a))=1.$$

Proof. The matrix $U(a)$ sends $w_1\mapsto w_1+aw_2$ and $w_2\mapsto w_2$. Recalling that $i\colon \mathrm{U}_E(W_0)\to\mathrm{U}_E(W_0+W_0^-)$ is defined by $\mathrm{U}_E(W_0)$ acting linearly on W_0 and trivially on W_0^-, it is a straightforward computation to see that

$$i(1,U(a))=\begin{pmatrix} 1 & -\frac{a}{2} & \frac{a}{4} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -a & \frac{a}{2} & 1\end{pmatrix}.$$

We have

$$\begin{pmatrix} 1 & \frac{a}{2} & \frac{a}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a & \frac{a}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{2} & \frac{a}{4} & -\frac{1}{2} \\ 0 & -1 & 0 & a^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{a}{2} & -1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & & -1 \\ & & 1 & \\ & 1 & & \end{pmatrix},$$

and therefore

$$x(i(1, U(a))) = -a^{-1}, \qquad j(i(1, U(a))) = 1.$$

By Definition 5.4, we have

$$\hat{s}(1, U(a)) = \begin{cases} 1 & \text{if } a = 0, \\ \xi(-a^{-1}) \cdot (u, -1)_F \cdot \gamma_F(u, \frac{1}{2}\psi)^{-1} = (u, a)_F \cdot \gamma_F(u, \frac{1}{2}\psi)^{-1}, & \text{if } a \in F^\times. \end{cases}$$

We next calculate $\boldsymbol{\lambda}(1, U(a))$. Since $g = (1, U(a))$ stabilizes $\mathbb{Y}'^\square$,

$$\boldsymbol{\lambda}(g) = \gamma_F(\tfrac{1}{2}\psi \circ q), \qquad q := q(\mathbb{V}^\triangle, \mathbb{Y}'^\square, \mathbb{V}^\triangle g).$$

Working in the F-basis given in (5.5)

$$\begin{aligned} \mathbb{V}^\triangle &= \{(0, 0, 0, 0, y_1, y_2, y_3, y_4)\}, \\ \mathbb{Y}'^\square &= \{(0, 0, y_1, y_2, y_3, y_4, 0, 0)\}, \\ \mathbb{V}^\triangle g &= \{(0, 0, -ay_3, \tfrac{a}{u} y_4, y_1 + \tfrac{a}{2} y_3, y_2 - \tfrac{au}{2} y_4, y_3, y_4)\}. \end{aligned}$$

If $a = 0$, then

$$\mathbb{R} = \{(0, 0, 0, 0, *, *, *, *)\}, \qquad \mathbb{R}^\perp = \{(0, 0, 0, 0, *, *, *, *)\},$$

and therefore we must have

$$\boldsymbol{\lambda}(1, U(0)) = 1,$$

and the lemma holds. It remains to prove the assertion for when $a \in F^\times$. Then we have

$$\mathbb{R} = \{(0, 0, 0, 0, *, *, 0, 0)\}, \qquad \mathbb{R}^\perp = \{(0, 0, *, *, *, *, *, *)\}.$$

So:

$$\begin{aligned} (\mathbb{V}^\triangle)_\mathbb{R} &= \{(0, 0, 0, 0, 0, 0, y_1, y_2)\}, \\ (\mathbb{Y}'^\square)_\mathbb{R} &= \{(0, 0, y_1, y_2, 0, 0, 0, 0)\}, \\ (\mathbb{V}^\triangle g)_\mathbb{R} &= \{(0, 0, -ay_1, \tfrac{a}{u} y_2, 0, 0, y_1, y_2)\}. \end{aligned}$$

It is clear from the above equations that

$$(\mathbb{Y}'^{\square})_{\mathbb{R}} \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = (\mathbb{V}^{\triangle} g)_{\mathbb{R}},$$

where

$$\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in P_{(\mathbb{V}^{\triangle})_{\mathbb{R}}} \subset \mathrm{Sp}(\mathbb{R}^{\perp}/\mathbb{R}), \qquad \text{for } b = \begin{pmatrix} -\frac{1}{a} & 0 \\ 0 & \frac{u}{a} \end{pmatrix}.$$

By definition, $q = (\mathbb{Y}'^{\square})_{\mathbb{R}}$ with the symmetric bilinear form given by

$$q((x_1, x_2), (y_1, y_2)) = -\tfrac{1}{a} x_1 y_1 + \tfrac{u}{a} x_2 y_2.$$

Therefore we have

$$\dim q = 2, \qquad \det q = -\tfrac{u}{a^2}, \qquad h_F(q) = (-\tfrac{1}{a}, \tfrac{u}{a})_F.$$

Observe that $(-\frac{1}{a}, \frac{u}{a})_F = (-a, au)_F(-a, a)_F = (-a, u)_F$, and so

$$\begin{aligned} \boldsymbol{\lambda}(1, U(a)) &= \gamma_F(\tfrac{1}{2}\psi)^2 \cdot \gamma_F(-\tfrac{u}{a^2}, \tfrac{1}{2}\psi) \cdot (-\tfrac{1}{a}, \tfrac{u}{a})_F \\ &= \gamma_F(-1, \tfrac{1}{2}\psi)^{-1} \cdot \gamma_F(-u, \tfrac{1}{2}\psi) \cdot (-a, u)_F. \end{aligned}$$

Finally, we have

$$\begin{aligned} \mathbf{s}(1, U(a)) &= (u, a)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi)^{-1} \cdot \gamma_F(-1, \tfrac{1}{2}\psi)^{-1} \cdot \gamma_F(-u, \tfrac{1}{2}\psi) \cdot (-a, u)_F \\ &= (u, a)_F \cdot (-1, u)_F \cdot (-a, u)_F = 1. \end{aligned}$$

□

Lemma 5.19. *We have*

$$\mathbf{s}(1, W) = (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi).$$

In particular, if $\mathrm{ord}(u)$ *is even, then*

$$\mathbf{s}(1, W) = 1.$$

Proof. The matrix W sends $w_1 \mapsto -w_1$ and $w_2 \mapsto -w_2$. Recalling that $i(W)$ acts linearly on W_0 and trivially on W_0^-, it is a straightforward computation to see that

$$i(1, W) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ -1 & 1 & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore we have

$$x(i(1,W)) = \tfrac{1}{8}, \qquad j(i(1,W)) = 2,$$

and by Definition 5.4,

$$\hat{s}(1,W) = \xi(\tfrac{1}{8}) \cdot ((u,-1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi))^{-2} = (u,-2)_F. \tag{5.6}$$

We next calculate $\boldsymbol{\lambda}(1,W)$. With respect to the symplectic basis given in (5.5), the image of $i(1,W)$ in $\mathrm{Sp}(\mathbb{V}^\Box)$ is

$$g = \begin{pmatrix} \frac{1}{2} & & -\frac{1}{2} & & \frac{1}{4} & & -\frac{1}{4} & \\ & \frac{1}{2} & & \frac{1}{2} & & -\frac{u}{4} & & \frac{u}{4} \\ \frac{1}{2} & & \frac{1}{2} & & \frac{1}{4} & & \frac{1}{4} & \\ & \frac{1}{2} & & \frac{1}{2} & & -\frac{u}{4} & & -\frac{u}{4} \\ -1 & & 1 & & \frac{1}{2} & & -\frac{1}{2} & \\ & u & & -u & & \frac{1}{2} & & -\frac{1}{2} \\ -1 & & -1 & & \frac{1}{2} & & \frac{1}{2} & \\ & u & & u & & \frac{1}{2} & & \frac{1}{2} \end{pmatrix} \in \mathrm{Sp}(\mathbb{V}^\Box).$$

By definition,

$$\boldsymbol{\lambda}(g) = \gamma_F(\tfrac{1}{2}\psi \circ q(\mathbb{V}^\triangle, \mathbb{Y}'^\Box g^{-1}, \mathbb{Y}'^\Box)) \cdot \gamma_F(\tfrac{1}{2}\psi \circ q(\mathbb{V}^\triangle, \mathbb{Y}'^\Box, \mathbb{V}^\triangle g)).$$

We have

$$\begin{aligned} \mathbb{V}^\triangle &= \{(0,0,0,0,y_1,y_2,y_3,y_4)\} \\ \mathbb{Y}'^\Box g^{-1} &= \{(y_1,y_2,y_3,y_4,\tfrac{1}{2}y_3,-\tfrac{1}{2u}y_4,\tfrac{1}{2}y_1,-\tfrac{1}{2u}y_2)\} \\ \mathbb{Y}'^\Box &= \{(0,0,y_1,y_2,y_3,y_4,0,0)\} \end{aligned}$$

which implies that $\mathbb{R} = \{(0,0,*,*,*,*,0,0)\}$ and hence

$$\gamma_F(\tfrac{1}{2}\psi \circ q(\mathbb{V}^\triangle, \mathbb{Y}'^\Box g^{-1}, \mathbb{Y}'^\Box)) = 1. \tag{5.7}$$

Now we calculate the second factor of $\boldsymbol{\lambda}(1,W)$. We have

$$\begin{aligned} \mathbb{V}^\triangle &= \{(0,0,0,0,y_1,y_2,y_3,y_4)\}, \\ \mathbb{Y}'^\Box &= \{(0,0,y_1,y_2,y_3,y_4,0,0)\}, \\ \mathbb{V}^\triangle g &= \{(y_1,y_2,y_3,y_4,-\tfrac{1}{2}y_1,\tfrac{1}{2u}y_2,-\tfrac{1}{2}y_3,\tfrac{1}{2u}y_4)\}, \end{aligned}$$

and hence

$$\mathbb{R} = \{(0,0,0,0,*,*,0,0)\}, \qquad \mathbb{R}^\perp = \{(0,0,*,*,*,*,*,*)\}.$$

This implies that

$$\begin{aligned}
(\mathbb{V}^\triangle)_\mathbb{R} &= \{(0,0,0,0,0,0,y_1,y_2)\}\\
(\mathbb{Y}'^\square)_\mathbb{R} &= \{(0,0,y_1,y_2,0,0,0,0)\}\\
(\mathbb{V}^\triangle g)_\mathbb{R} &= \{(0,0,y_1,y_2,0,0,-\tfrac{1}{2}y_1,\tfrac{1}{2u}y_2)\}
\end{aligned}$$

and we have

$$(\mathbb{Y}'^\square)_\mathbb{R}\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) = (\mathbb{V}^\triangle g)_\mathbb{R}, \qquad \text{for } b = \begin{pmatrix} -\frac{1}{2} & \\ & \frac{1}{2u}\end{pmatrix}.$$

It follows that

$$\begin{aligned}
\gamma_F(\tfrac{1}{2}\psi\circ q(\mathbb{V}^\triangle,\mathbb{Y}'^\square,\mathbb{V}^\triangle g)) &= \gamma_F(\tfrac{1}{2}\psi)^2\cdot\gamma_F(-\tfrac{1}{4u},\tfrac{1}{2}\psi)\cdot(-\tfrac{1}{2},\tfrac{1}{2u})_F\\
&= \gamma_F(-1,\tfrac{1}{2}\psi)^{-1}\gamma_F(-u,\tfrac{1}{2}\psi)\cdot(-2,u)_F\\
&= \gamma_F(u,\tfrac{1}{2}\psi)\cdot(2,u)_F. \qquad (5.8)
\end{aligned}$$

Putting together Equations (5.6), (5.7), and (5.8), we have

$$\begin{aligned}
\mathbf{s}(1,W) &= \hat{s}(1,W)\cdot\boldsymbol{\lambda}(1,W)\\
&= (u,-2)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi)\cdot(u,2)_F\\
&= (u,-1)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi).
\end{aligned}$$

To see the final assertion, first observe that if $\mathrm{ord}(u)$ is even, then either E is split or unramified over F. In either case, $(u,-1)_F = 1$. By [R93, Proposition A.11], $\mathrm{ord}(u)$ even implies that $\gamma_F(u,\frac{1}{2}\psi) = 1$. □

Lemma 5.20. *Let $a \in F$. Then*

$$\mathbf{s}(1,D(-1))\mathbf{s}(1,W)\mathbf{s}(1,U(a))\mathbf{s}(1,W) = 1.$$

Proof. We have $\mathbf{s}(1,U(a)) = 1$ and

$$\begin{aligned}
\mathbf{s}(1,D(-1))\mathbf{s}(1,W)^2 &= (u,-1)_F\cdot((u,-1)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi))^2\\
&= (u,-1)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi)^2 = (u,-1)_F\cdot(u,-1)_F = 1.
\end{aligned}$$

□

Lemma 5.21. *If $F = \mathbb{R}$ and $E = \mathbb{C}$, then*

$$\mathbf{s}(\alpha, g) = \xi(\alpha^{-1})$$

for any $(\alpha, g) \in R$.

Proof. Since $(\alpha, D(1, \mathrm{Nm}(\alpha)))$ stabilizes $\mathbb{Y}'$,

$$\mathbf{s}(\alpha, g) = \mathbf{s}(\alpha, D(1, \mathrm{Nm}(\alpha))) \cdot \mathbf{s}(1, D(1, \mathrm{Nm}(\alpha)^{-1})g).$$

By Lemma 5.17, to prove the desired assertion, it remains to show that $\mathbf{s}(1, g) = 1$ for $g \in \mathrm{SL}_2(\mathbb{R})$. But this follows from [R93, Proposition A.10(1)]. $\square$

For convenience, we state the explicated values of $\mathbf{s}$ in the following table:

$(\alpha, g) \in R$	$\mathbf{s}(\alpha, g)$
$(\alpha, D(a, d))$	$\xi(-(\alpha^{-1}a - 1)(\alpha^{-1}d - 1))$
$(1, D(a, a^{-1}))$	$(u, a)_F$
$(\alpha, D(1, \mathrm{Nm}(\alpha)))$	$\xi(\alpha)^{-1}$
$(1, U(a))$	1
$(1, W)$	$(u, -1)_F \cdot \gamma_F(u, \frac{1}{2}\psi)$

5.6.2 A splitting s' of $z_{\mathbb{Y}'}$

The computations in this subsection are very similar to the computations of the preceding subsection. As in the previous subsection, for $a, d \in F^\times$, write $D(a, d) := \mathrm{diag}(a, d)$.

Lemma 5.22. *Let $(D(a, d), \alpha) \in \mathrm{G}(\mathrm{GL}_2(F) \times E^\times)$. Then*

$$\mathbf{s}'(D(a, d), \alpha) = \xi'(-(a^{-1}\alpha - 1)(d^{-1}\alpha - 1)).$$

In particular, for $a \in F^\times$ and $\alpha \in E^\times$,

$$\mathbf{s}'(D(a, a^{-1}), 1) = (u, a)_F,$$
$$\mathbf{s}'(D(1, \mathrm{Nm}(\alpha)), \alpha) = \xi'(\alpha).$$

Proof. The proof is similar to Lemma 5.17 except that $(D(a, d), \alpha)$ sends $w_1 \mapsto a^{-1}\alpha w_1$ and $w_2 \mapsto d^{-1}\alpha w_2$. Thus the image of $(D(a, d), \alpha)$ in $\mathrm{U}_E(\mathrm{Res}\, V + \mathrm{Res}\, V^-)$ with respect to the basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_1^*, \mathbf{w}_2^*$ is

$$\begin{pmatrix} \frac{a^{-1}\alpha+1}{2} & 0 & 0 & \frac{a^{-1}\alpha-1}{4} \\ 0 & \frac{d^{-1}\alpha+1}{2} & -\frac{d^{-1}\alpha-1}{4} & 0 \\ 0 & -(d^{-1}\alpha-1) & \frac{d^{-1}\alpha+1}{2} & 0 \\ -(a^{-1}\alpha-1) & 0 & 0 & \frac{a^{-1}\alpha+1}{2} \end{pmatrix}.$$

To be more precise, this proof is the proof of Lemma 5.17 except with a replaced by a^{-1}, b replaced by b^{-1}, and α^{-1} replaced by α. $\square$

CHAPTER 6

Global theta lifts

In this chapter, we examine the global theta lifts in the similitude seesaw

$$\begin{tikzcd} \mathrm{GU}_E(\mathrm{Res}\,V) \arrow[d, dash] \arrow[dr, dash] & \mathrm{GU}_B(W^*) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{GU}_B(V) & \mathrm{GU}_E(W) \end{tikzcd} \quad = \quad \begin{tikzcd} ((B')^\times \times E^\times)/F^\times \arrow[d, dash] \arrow[dr, dash] & B^\times \arrow[d, dash] \arrow[dl, dash] \\ E^\times \cup E^\times \mathbf{j} & E^\times \end{tikzcd}$$

and their relationship to automorphic induction (see Chapter 2).

Let $\chi\colon E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ be a Hecke character and let π_χ denote its automorphic induction to a representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Recall that π_χ has a Jacquet–Langlands transfer to $B^\times$ if and only if the following condition holds:

(JL) If B_v is ramified, then χ_v does not factor through $\mathrm{Nm}\colon E_v^\times \to F_v^\times$.

We write π_χ^B to denote the Jacquet–Langlands transfer to $B^\times$ if the pair (B,χ) satisfies (JL), and we set $\pi_\chi^B = 0$ otherwise.

The main theorem of this chapter is:

Theorem 6.1. *The theta lifts* $\Theta(\chi\cdot\xi)$ *from* $\mathrm{GU}_B(V)$ *to* $\mathrm{GU}_B(W^*)$ *and* $\Theta'(\overline{\chi'\cdot\xi'^{-1}})$ *from* $\mathrm{GU}_E(W)$ *to* $\mathrm{GU}_E(\mathrm{Res}\,V)$ *can be described in terms of automorphic induction and the Jacquet–Langlands correspondence:*

$$\Theta(\chi\cdot\xi)\cong\pi_\chi^B,\qquad \textit{and}\qquad \Theta'(\overline{\chi'\cdot\xi'^{-1}})^\vee\cong\pi_{\chi'}^{B'}\otimes(\chi'^{-1}\cdot\xi'),$$

where the right-hand side is viewed as a representation of $\mathrm{GU}_E(\mathrm{Res}\,V)$ *descended from* $(B'_{\mathbb{A}})^\times\times\mathbb{A}_E^\times$.

To prove Theorem 6.1, we will need two arguments.

(1) If $\Theta(\chi\cdot\xi)=0$, then $\pi_\chi^B=0$.

(2) If $\Theta(\chi\cdot\xi)\neq 0$, then $\Theta(\chi\cdot\xi)\cong\pi_\chi^B$.

To prove (1), we will need to make use of the theory of doubling zeta integrals. As we will see from Section 6.2, the nonvanishing of the global theta lift $\Theta(\chi \cdot \xi)$ is determined by the nonvanishing of local doubling zeta integrals. Hence the crux of (1) is to establish that the functional determined by a local zeta integral is zero if and only if the corresponding local theta lift is zero. To prove (2), we will need to explicitly calculate the local theta lift from $\mathrm{GU}(1)_v$ to $\mathrm{GU}(2)_v$ at all places where $\mathrm{GU}(2)_v \cong \mathrm{GU}(1,1)_v$. Then, after showing that $\Theta(\chi \cdot \xi)$ must be cuspidal if it is nonzero, we can use Jacquet–Langlands (Theorems 2.5 and 2.6).

6.1 Theta lifts with similitudes

We first recall some general properties of Weil representations. Denote by ω_ψ and $\omega_\psi^\square$ the Weil representations of $\mathrm{Mp}(\mathbb{V})$ on $\mathcal{S}(\mathbb{X})$ and of $\mathrm{Mp}(\mathbb{V}^\square)$ on $\mathcal{S}(\mathbb{X}^\square) = \mathcal{S}(\mathbb{X}) \otimes \mathcal{S}(\mathbb{X})$. We have a natural map

$$\tilde{\iota}\colon \mathrm{Mp}(\mathbb{V}) \times \mathrm{Mp}(\mathbb{V}) \to \mathrm{Mp}(\mathbb{V}^\square)$$

inducing $(z_1, z_2) \mapsto z_1 \overline{z}_2$ on $\mathbb{C}^1$, and the Weil representations ω_ψ, $\omega_\psi^\square$ enjoy the following compatibility:

$$\omega_\psi^\square \circ \tilde{\iota} \cong \omega_\psi \otimes (\omega_\psi \circ \tilde{\mathrm{j}}_{\mathbb{Y}}),$$

where $\tilde{\mathrm{j}}_{\mathbb{Y}}$ is the automorphism of $\mathrm{Mp}(\mathbb{V})_{\mathbb{Y}} = \mathrm{Sp}(\mathbb{V}) \times \mathbb{C}^1$ defined by

$$\tilde{\mathrm{j}}_{\mathbb{Y}}(g, z) = (\mathrm{j}_{\mathbb{Y}}(g), z^{-1}), \qquad \mathrm{j}_{\mathbb{Y}}(g) = d_{\mathbb{Y}}(-1) \cdot g \cdot d_{\mathbb{Y}}(-1).$$

We make the following definitions:

$$\begin{aligned}
G &:= \mathrm{GU}_B(V)^\circ \cong E^\times \cong \mathrm{GU}_E(V_0) \\
H &:= \mathrm{GU}_B(W^*) \cong B^\times \subset \mathrm{GU}_E(W_0) \\
G' &:= \mathrm{GU}_E(W) \\
H' &:= \mathrm{GU}_E(\mathrm{Res}\, V) \cong ((B')^\times \times E^\times)/F^\times \\
G^\square &:= \mathrm{GU}_B(V^\square) \\
G^{\square\prime} &:= \mathrm{GU}_E(W^\square)
\end{aligned}$$

Recall that these groups fit into the following seesaws:

$$\begin{tikzcd}
H' \arrow[d, dash] \arrow[dr, dash] & H \arrow[d, dash] \arrow[dl, dash] \\
G & G'
\end{tikzcd}
\qquad
\begin{tikzcd}
G^\square \arrow[d, dash] \arrow[dr, dash] & H \times H \arrow[d, dash] \arrow[dl, dash] \\
G \times G & H
\end{tikzcd}
\qquad
\begin{tikzcd}
H' \times H' \arrow[d, dash] \arrow[dr, dash] & G^{\square\prime} \arrow[d, dash] \arrow[dl, dash] \\
H' & G' \times G'
\end{tikzcd}$$

Adding a subscript 1 to any of the above groups indicates that we take the kernel of the similitude character. If $G_1, \dots, G_n$ is a collection of unitary similitude groups, we define

$$\mathcal{G}_{G_1 \times \cdots \times G_n} := \{(g_1, \dots, g_n) \in G_1 \times \cdots \times G_n : \nu(g_1) = \cdots = \nu(g_n)\}.$$

We also define

$$Z := F^\times.$$

We will also need to consider:

$$\begin{aligned}
(\mathbb{A}^\times)^+ &:= \nu(G(\mathbb{A})) \cap \nu(H(\mathbb{A})) = \nu(G'(\mathbb{A})) \cap \nu(H'(\mathbb{A})) = \mathrm{Nm}_{E/F}(\mathbb{A}_E^\times) \\
(F^\times)^+ &:= F^\times \cap (\mathbb{A}^\times)^+ \\
\mathcal{C} &:= (\mathbb{A}^\times)^2 (F^\times)^+ \backslash (\mathbb{A}^\times)^+
\end{aligned}$$

Then adding a superscript $+$ to any of the groups G, H, G', H' means we take the preimage of $(\mathbb{A}^\times)^+$ (or $(F^\times)^+$, etc.) under the similitude map.

Lemma 6.2. *The similitude character induces isomorphisms*

$$\begin{aligned}
Z(\mathbb{A})G_1(\mathbb{A})G(F)^+ \backslash G(\mathbb{A})^+ &\cong \mathcal{C}, & Z(\mathbb{A})H_1(\mathbb{A})H(F)^+ \backslash H(\mathbb{A})^+ &\cong \mathcal{C}, \\
Z(\mathbb{A})G_1'(\mathbb{A})G'(F)^+ \backslash G'(\mathbb{A})^+ &\cong \mathcal{C}, & Z(\mathbb{A})H_1'(\mathbb{A})H'(F)^+ \backslash H'(\mathbb{A})^+ &\cong \mathcal{C}.
\end{aligned}$$

Proof. The similitude character induces surjections

$$G(\mathbb{A})^+ \to \mathcal{C}, \qquad H(\mathbb{A})^+ \to \mathcal{C}, \qquad G'(\mathbb{A})^+ \to \mathcal{C}, \qquad H'(\mathbb{A})^+ \to \mathcal{C}.$$

It remains to compute the corresponding kernels. We will do this for G; the computations are completely analogous in the other situations. Recall that $G(\mathbb{A}) \cong \mathbb{A}_E^\times$. Since the norm maps on B and B' restrict to the norm map on E, we have $G(\mathbb{A})^+ = G(\mathbb{A})$. Pick $x \in \mathbb{A}_E^\times$ such that $\nu(x) \in (\mathbb{A}^\times)^2 (F^\times)^+$. By multiplying x by an element of $\mathbb{A}_F$, we may assume that $\nu(x) \in (F^\times)^+$. This condition implies that each place x_v of $x = (x_v)_v$ is a local norm, and thus there exists $z \in E^\times$ such that $\nu(z) = \nu(x)$. Then $x \cdot z^{-1} \in \mathbb{A}_E^1 = G_1(\mathbb{A})$, and so we've shown that

$$\ker(G(\mathbb{A})^+ \to \mathcal{C}) = \mathbb{A}_F^\times \mathbb{A}_E^1 E^\times = Z(\mathbb{A})G_1(\mathbb{A})G(F)^+. \qquad \square$$

Fix sections

$$\mathcal{C} \to G(\mathbb{A})^+, \qquad \mathcal{C} \to H(\mathbb{A})^+, \qquad \mathcal{C} \to G'(\mathbb{A})^+, \qquad \mathcal{C} \to H'(\mathbb{A})^+.$$

We write g_c, h_c, g'_c, h'_c for the images of $c \in \mathcal{C}$ under these sections.

Lemma 6.3. *The similitude character induces isomorphisms*

$$H(\mathbb{A})/(H(F)H(\mathbb{A})^+) \cong H'(\mathbb{A})/(H'(F)H'(\mathbb{A})^+) \cong \mathrm{Gal}(E/F),$$
$$G^\square(\mathbb{A})/(G^\square(F)G^\square(\mathbb{A})^+) \cong G^{\square\prime}(\mathbb{A})/(G^{\square\prime}(F)G^{\square\prime}(\mathbb{A})^+) \cong \mathrm{Gal}(E/F).$$

Proof. The proof is very similar to that of Lemma 6.2. We first argue that

$$\nu^{-1}(F^\times \operatorname{Nm}(\mathbb{A}_E^\times)) = H(F)H(\mathbb{A})^+$$

for $\nu\colon H(\mathbb{A}) \to \mathbb{A}_F^\times$. Indeed, suppose that $h \in \nu^{-1}(F^\times \operatorname{Nm}(\mathbb{A}_E^\times))$. By multiplying h by an element of $\mathbb{A}_E^\times$, we may assume that $\nu(h) \in F^\times$. Since $\nu\colon H(F) \to F^\times$ is surjective this implies that there exists an $h' \in H(F)$ such that $\nu(h') = \nu(h)$. Hence $h = h'h_1$ for some $h_1 \in H_1(\mathbb{A})$. We have hence shown that $\nu^{-1}(F^\times \operatorname{Nm}(\mathbb{A}_E^\times)) = H(F)H(\mathbb{A}_F)^+$.

Now, by class field theory, $\mathbb{A}_F^\times/(F^\times \operatorname{Nm}(\mathbb{A}_E^\times)) \cong \mathrm{Gal}(E/F)$, so that $F^\times \operatorname{Nm}(\mathbb{A}_E^\times)$ is an index-2 subgroup of $\mathbb{A}_F^\times$. Since $\nu\colon H(\mathbb{A}) \to \mathbb{A}_F^\times$ is surjective, then the desired isomorphism follows from the preceding paragraph. This proves the assertion for H, and the proofs for H', $G^\square$, and $G^{\square\prime}$ are completely analogous. $\square$

Recall that in Chapter 5 (see Definitions 5.4 and 5.8), for each place v of F, we defined splittings of $z_{\mathbb{Y}_v}$ and $z_{\mathbb{Y}_v^\square}$ on certain unitary groups. Recall also that the discussion in Section 5.5 allowed us to multiply the local splittings to obtain global splittings of $z_{\mathbb{Y}}$

$$s\colon \mathcal{G}_{G\times H}(\mathbb{A}) \to \mathbb{C}^1, \qquad s'\colon \mathcal{G}_{H'\times G'}(\mathbb{A}) \to \mathbb{C}^1,$$

and global splittings of $z_{\mathbb{Y}^\square}$

$$s^\square\colon \mathcal{G}_{G^\square\times H}(\mathbb{A}) \to \mathbb{C}^1, \qquad s^{\square\prime}\colon \mathcal{G}_{H'\times G^{\square\prime}}(\mathbb{A}) \to \mathbb{C}^1.$$

These allow us to define corresponding Weil representations $\omega_\psi, \omega'_\psi, \omega_\psi^\square, \omega_\psi^{\square\prime}$. By Proposition 5.16,

$$\omega_\psi^\square(g_1, g_2, h) = \omega_\psi(g_1, h) \otimes \xi(\det(g_2, h))\overline{\omega_\psi(g_2, h)}, \qquad (g_1, g_2, h) \in \mathcal{G}_{G\times G\times H}(\mathbb{A}), \tag{6.1}$$
$$\omega_\psi^{\square\prime}(h, g_1, g_2) = \omega'_\psi(h, g_1) \otimes \xi'(\det(h, g_2))\overline{\omega'_\psi(h, g_2)}, \qquad (h, g_1, g_2) \in \mathcal{G}_{H'\times G'\times G'}(\mathbb{A}), \tag{6.2}$$
$$\omega_\psi(g, g') = \xi(g)\xi'(g')\omega'_\psi(g, g'), \qquad (g, g') \in \mathcal{G}_{G\times G'}(\mathbb{A}). \tag{6.3}$$

Define a theta distribution

$$\Theta\colon \mathcal{S}(\mathbb{X}(\mathbb{A})) \to \mathbb{C}, \qquad \varphi \mapsto \sum_{x \in \mathbb{X}(F)} \varphi(x).$$

Let $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$ and let χ be a Hecke character. For $h = h_1 h_c \in H(\mathbb{A})^+$ where $h_1 \in H_1(\mathbb{A})$, define

$$\theta_\varphi(\chi)(h) := \int_{G_1(F)\backslash G_1(\mathbb{A})} \Theta(\omega_\psi(g_1 g_c, h)\varphi)\chi(g_1 g_c)\, dg_1.$$

Here, $dg = \prod_v dg_{1,v}$ is the Tamagawa measure on $G_1(\mathbb{A})$. Note that $\theta_\varphi(\chi)(\gamma h) = \theta_\varphi(f)(\gamma h)$ for $\gamma \in H(F) \cap H(\mathbb{A})^+$ and $h \in H(\mathbb{A})^+$. By declaring

$$\theta_\varphi(\chi)(\gamma h) = \theta_\varphi(\chi)(h), \qquad \text{for all } \gamma \in H(F) \text{ and } h \in H(\mathbb{A})^+,$$

we obtain an automorphic form on the subgroup $H(F)H(\mathbb{A})^+$ of $H(\mathbb{A})$. Let $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$ and let χ' be a Hecke character. For $h' = h'_1 h'_c \in H'(\mathbb{A})^+$ where $h'_1 \in H'_1(\mathbb{A})$, define

$$\theta'_\varphi(\chi')(h') := \int_{G'_1(F)\backslash G'_1(\mathbb{A})} \overline{\Theta(\omega'_\psi(h', g'_1 g'_c)\varphi)}\chi'(g'_1 g'_c)\, dg'_1.$$

Here, $dg'_1 = \prod_v dg'_{1,v}$ is the Tamagawa measure on $G'_1(\mathbb{A})$.

Let $\Theta_+(\chi)$ be the automorphic representation of $H(F)H(\mathbb{A})^+$ generated by $\theta_\varphi(\chi)$ for $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$ and let $\Theta'_+(\chi')$ be the automorphic representation of $H'(F)H'(\mathbb{A})^+$ generated by $\theta'_\varphi(\chi')$ for all $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$. Define

$$\Theta(\chi) := \mathrm{Ind}_{H(F)H(\mathbb{A})^+}^{H(\mathbb{A})}\left(\Theta_+(\chi)\right), \qquad \Theta'(\chi') := \mathrm{Ind}_{H'(F)H'(\mathbb{A})^+}^{H'(\mathbb{A})}\left(\Theta'_+(\chi')\right).$$

By Lemma 6.3, $[H(\mathbb{A}) : H(F)H(\mathbb{A})^+] = 2$, and hence $\theta_\varphi(\chi)$ extends to an automorphic form on $H(\mathbb{A})$ via

$$\theta_\varphi(\chi)(h) := \begin{cases} \theta_\varphi(\chi)(h) & \text{if } h \in H(F)H(\mathbb{A})^+, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $\theta'_\varphi(\chi')$ extends to an automorphic form on $H'(\mathbb{A})$ by setting

$$\theta'_\varphi(\chi')(h') := \begin{cases} \theta'_\varphi(\chi')(h'_+) & \text{if } h' = \gamma h'_+ \text{ for } \gamma \in H'(F) \text{ and } h'_+ \in H'(\mathbb{A})^+, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\theta_\varphi(\chi) \in \Theta(\chi)$ and $\theta_{\varphi'}(\chi') \in \Theta'(\chi')$.

Analogously, define a theta distribution

$$\Theta\colon \mathcal{S}(\mathbb{X}^\square(\mathbb{A})) \to \mathbb{C}, \qquad \varphi \mapsto \sum_{x \in \mathbb{X}^\square(F)} \varphi(x).$$

The corresponding theta lifts for similitude unitary groups are defined completely analogously to above.

6.2 The Rallis inner product formula

In this section, we will write down an equation relating the Petersson inner product of a theta lift to a theta lift to a doubled unitary similitude group. To this end, we will use the doubled seesaw

$$\begin{tikzcd}
G^\square \arrow[d, dash] \arrow[dr, dash] & H \times H \arrow[d, dash] \arrow[dl, dash] \\
G \times G & H
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{GU}_B(V^\square) \arrow[d, dash] \arrow[dr, dash] & \mathrm{GU}_B(W^*) \times \mathrm{GU}_B(W^*) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{GU}_B(V)^\circ \times \mathrm{GU}_B(V)^\circ & \mathrm{GU}_B(W^*)
\end{tikzcd}$$

to write down such a formula for the theta lift $\theta_\varphi(\chi \cdot \xi)$ to $B^\times \cong \mathrm{GU}_B(W^*) \subset \mathrm{GU}_E(W_0)$, and use the doubled seesaw

$$\begin{tikzcd}
H' \times H' \arrow[d, dash] \arrow[dr, dash] & G^{\square\prime} \arrow[d, dash] \arrow[dl, dash] \\
H' & G' \times G'
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{GU}_E(\operatorname{Res} V) \times \mathrm{GU}_E(\operatorname{Res} V) \arrow[d, dash] \arrow[dr, dash] & \mathrm{GU}_E(W^\square) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{GU}_E(\operatorname{Res} V) & \mathrm{GU}_E(W) \times \mathrm{GU}_E(W)
\end{tikzcd}$$

to write down such a formula for the theta lift $\theta'_\varphi(\overline{\chi' \xi'^{-1}})$ to $B'^\times \subset (B'^\times \times E^\times)/F^\times \cong \mathrm{GU}_E(\operatorname{Res} V)$.

For automorphic forms f_1, f_2 on $H(\mathbb{A}) \cong B_\mathbb{A}^\times$ and automorphic forms f_1', f_2' on $H'(\mathbb{A}) \cong (B_\mathbb{A}'^\times \times \mathbb{A}_E^\times)/\mathbb{A}_F^\times$, define

$$\begin{aligned}
\langle f_1, f_2 \rangle_H &:= \int_{[H]} f_1(h) \cdot \overline{f_2(h)}\, dh, \\
\langle f_1', f_2' \rangle_{H'} &:= \int_{[H']} f_1'(h') \cdot \overline{f_2'(h')}\, dh,
\end{aligned}$$

where $dh = \prod_v dh_v$ and $dh' = \prod_v dh_v'$ are the Tamagawa measures of $H(\mathbb{A})$ and $H'(\mathbb{A})$, respectively.

Recall from Proposition 5.16 that the splittings

$$s\colon \mathcal{G}_{G\times H}(\mathbb{A}) \to \mathbb{C}^1, \qquad s^\square\colon \mathcal{G}_{G^\square\times H}(\mathbb{A}) \to \mathbb{C}^1$$

enjoys the property that for $(g_1, g_2, h) \in \mathcal{G}_{G\times G\times H}$,

$$s^\square(g_1, g_2, h) = s(g_1, h) \cdot \overline{s(g_2, h)} \cdot \xi(\det(i(g_2, h))).$$

This compatibility implies that for any $h_1 \in H_1$, $g_1, g_1' \in G_1$, and $(g_c, h_c) \in \mathcal{G}_{G\times H}(\mathbb{A})$,

$$\begin{aligned}\Theta(\omega_\psi(g_1g_c,&h_1h_c)\varphi_1) \cdot \overline{\Theta(\omega_\psi(g_1'g_c, h_1h_c)\varphi_2)}\\ &= \Theta(\omega_\psi^\square((g_1g_c, g_1'g_c), h_1h_c)\varphi_1 \otimes \overline{\varphi}_2) \cdot \xi(\det(h_1h_c))^{-1} \cdot \xi(g_1'g_c)^2.\end{aligned}$$

Hence for $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$ and Hecke characters χ_1, χ_2 of $E^\times$, by formally switching the integrals at the equality, we have

$$\begin{aligned}\langle\theta_{\varphi_1}(\chi_1&\cdot\xi), \theta_{\varphi_2}(\chi_2\cdot\xi)\rangle_H\\ &= \int_{[H]} \theta_{\varphi_1}(\chi_1\cdot\xi)(h) \cdot \overline{\theta_{\varphi_2}(\chi_2\cdot\xi)(h)}\, dh\\ &= \int_{\mathcal{C}}\int_{[H_1]} \theta_{\varphi_1}(\chi_1\cdot\xi)(h_1h_c) \cdot \overline{\theta_{\varphi_2}(\chi_2\cdot\xi)(h_1h_c)}\, dh_1\, dc\\ &= \int_{\mathcal{C}}\int_{[H_1]}\int_{[G_1]}\int_{[G_1]} \Theta(\omega_\psi(g_1g_c, h_1h_c)\varphi_1)(\chi_1\xi)(g_1g_c)\cdot\\ &\qquad\qquad \overline{\Theta(\omega_\psi(g_1'g_c, h_1h_c)\varphi_2)(\chi_2\xi)(g_1'g_c)}\, dg_1\, dg_1'\, dh\, dc\\ &= \int_{\mathcal{C}}\int_{[G_1]}\int_{[G_1]} (\chi_1\xi)(g_cg_c) \cdot (\overline{\chi}_2\xi)(g_1'g_c)\cdot \qquad (6.4)\\ &\quad \int_{[H_1]} \Theta(\omega_\psi^\square((g_1g_c, g_1'g_c), h_1h_c)(\varphi_1 \otimes \overline{\varphi}_2)) \cdot \xi(\det(h_1h_c))^{-1}\, dh_1\, dg_1\, dg_1'\, dc. \qquad (6.5)\end{aligned}$$

The inner integral in Equation (6.5) is the theta lift of $\xi(\det)^{-1}$ to $\mathrm{GU}_B(V^\square)$, but to make actual sense of the above, one must be careful about convergence issues. In the case that B is division, the quotient $B^\times\backslash B_\mathbb{A}^\times$ is compact, and therefore the integral in (6.5) is absolutely convergent. Hence the formal manipulation above is completely justified. In the case that B is split (i.e. $B \cong M_2(F)$), (6.5) does not converge absolutely in general.

The idea of the Siegel–Weil formula is to interpret the integral 6.5 as an Eisenstein series. When (6.5) is absolutely convergent, this dates back to classical work of Siegel that was later extended by Weil in 1965 [W65]. An important idea of Kudla and Rallis in the late 1980s was that one can regularize 6.5 and interpret the resulting absolutely convergent integral as an Eisenstein series, thereby obtaining a regularized Siegel–Weil formula. After the work of many people (Ikeda, Ichino, Yamana, Gan–Qiu–Takeda), the regularized Siegel–Weil formula has now been established for all dual reductive pairs.

The Siegel–Weil formula supplies half the distance between the Petersson inner product of theta lifts and special values of L-functions. The missing ingredient is the theory of doubling zeta integrals, which was initiated by Piatetski-Shapiro and Rallis [PSR87]. They studied the integral (6.4), with the automorphic theta distribution $\Theta(\psi^\square(g,h)\Phi)$ in (6.5) replaced by an Eisenstein series, and proved that it gives rise to standard L-functions. The relation between the Petersson inner product of a theta lift and special values of L-functions is known as the Rallis inner product formula.

6.2.1 The Siegel–Weil formula for division quaternion algebras

In this section, we explain how to obtain a Rallis inner product formula in the case that B is division. For $\varphi \in \mathcal{S}(\mathbb{X}^\triangledown(\mathbb{A}))$, define

$$\mathcal{F}_\varphi(g) := (\omega_\psi^\square(d(\nu(g)^{-1})g)\varphi)(0)$$

and form

$$E(g, \mathcal{F}_\varphi) = \sum_{\gamma \in P(F)\backslash \mathrm{U}(1,1)} \mathcal{F}_\varphi(\gamma g).$$

This is the value of an Eisenstein series at $s = \frac{1}{2}$. In this case, the Siegel–Weil formula states that for $g, g' \in \mathrm{GU}(1)$ such that $\nu(g) = \nu(g')$,

$$E(i(g,g'), \mathcal{F}_\varphi) = \int_{[H_1]} \Theta(\omega_\psi^\square((g,g'),h)(\varphi_1 \otimes \overline{\varphi}_2)) \cdot \xi(\det(h))^{-1}\, dh$$

where $i\colon \mathrm{G}(\mathrm{U}(1)\times \mathrm{U}(1)) \to \mathrm{U}(1,1)$ and $\varphi \in \mathcal{S}(\mathbb{V}^\triangledown(\mathbb{A}))$ is the partial Fourier transform of $\varphi_1 \otimes \overline{\varphi}_2 \in \mathcal{S}(\mathbb{X}^\square(\mathbb{A}))$. We now see that, continuing from (6.4), (6.5), we have

$$\langle \theta_{\varphi_1}(\chi_1\cdot\xi), \theta_{\varphi_2}(\chi_2\cdot\xi)\rangle_H = \int_{\mathcal{C}}\int_{[G_1]}\int_{[G_1]} (\chi_1\xi)(g_1g_c)\cdot(\overline{\chi}_2\xi)(g_1'g_c)\cdot E(i(g_1g_c, g_1'g_c), \mathcal{F}_\varphi)\, dg_1\, dg_1'\, dc.$$

We have $\mathcal{F}_\varphi(i(g_1g_c, g_1'g_c)) = \mathcal{F}(i({g_1'}^{-1}g_1, 1))\overline{\xi}^2(g_1')$, and hence unfolding the above integral and making the substitution $g = g_1g_c$, $g' = {g_1'}^{-1}g_1$ gives

$$= \int_{G_1(\mathbb{A})}\int_{[G]} (\chi_1\xi)(gg')\cdot(\overline{\chi_2\xi})(g)\cdot \mathcal{F}_\varphi(i(g,1))\, dg\, dg'.$$

The Tamagawa measure on $G_1(\mathbb{A})$ can be written as a product of local measures $dg_{1,v}$ on $G_{1,v}$ times a global factor ρ_F/ρ_E (see Section 2.1). Hence if $\chi_1 = \chi_2 = \chi$ and $\varphi_1 = \varphi_2 = \phi = \otimes_v \phi_v$, we have

$$\begin{aligned}\langle \theta_\varphi(\chi \cdot \xi), \theta_\phi(\chi \cdot \xi)\rangle_H &= \int_{G_1(\mathbb{A})} \mathcal{F}_\varphi(i(g,1))\langle (\chi\xi)(g')(\chi\xi), (\chi\xi)\rangle_{[G]}\, dg' \\ &= \frac{\rho_F}{\rho_E} \cdot \prod_v Z(\tfrac{1}{2}, \mathcal{F}_{\varphi_v}, \chi_v),\end{aligned}$$

where

$$Z(\tfrac{1}{2}, \mathcal{F}_{\varphi_v}, \chi_v) := \int_{G_{1,v}} \langle \omega_\psi(g_{1,v})\phi, \phi\rangle \cdot (\chi_v\xi_v)(g_{1,v})\, dg_{1,v}. \tag{6.6}$$

6.2.2 The regularized Siegel–Weil formula for $(E^\times, \mathrm{GL}(2))$

In this section, we follow [GQT] and describe how to make sense of (6.5) and obtain a Rallis inner product formula in the case that B is split. We will need to translate between the quaternionic unitary groups $(\mathrm{GU}_B(V)^\circ, \mathrm{GU}_B(W^*)) \cong (E^\times, \mathrm{GL}_2(F))$ and the dual reductive pair $(\mathrm{GO}(2), \mathrm{GSp}(2)) \cong (E^\times, \mathrm{GL}_2(F))$. In the notation of [GQT], we have $n = m = 2$, $r = 1$, $\epsilon = 1$, which puts us in the second term range since $1 < 2 \leq 2 \cdot 1$.

Recall that we have an embedding

$$\mathrm{G}(\mathrm{U}_B(V)^\circ, \mathrm{U}_B(W^*)) \hookrightarrow \mathrm{G}(\mathrm{U}_E(V_0) \times \mathrm{U}_E(W_0)).$$

When B is split, then there is a decomposition $W_0 = W_1 + W_2$ of the E-space W_0 into isotropic subspaces of dimension 1. Set

$$\mathbb{X}' = \mathrm{Res}_{E/F}(V_0 \otimes W_1), \qquad \mathbb{Y}' = \mathrm{Res}_{E/F}(V_0 \otimes W_2)$$

so that $\mathbb{V} = \mathbb{X}' + \mathbb{Y}'$ forms a complete polarization. In Section 5.6, we explicated a splitting $\mathbf{s}$ of $z_{\mathbb{Y}'}$. Comparing $\mathbf{s}$ to the splitting

$$s_{(\mathrm{O}(2),\mathrm{Sp}(2))}\colon\ \mathrm{G}(\mathrm{O}(2) \times \mathrm{Sp}(2))_{\mathbb{A}} \to \mathbb{C}^1$$

defined in [K94], we see that for $\alpha \in E^\times$, $a \in F^\times$, and $a' \in F$,

$$\begin{aligned}
\mathbf{s}(\alpha, d(\mathrm{Nm}(\alpha))) &= \xi(\alpha)^{-1} \cdot s_{(\mathrm{O}(2),\mathrm{Sp}(2))}(\alpha, d(\mathrm{Nm}(\alpha))),\\
\mathbf{s}\left(1, \mathrm{diag}(a, a^{-1})\right) &= \xi(a)^{-1} \cdot s_{(\mathrm{O}(2),\mathrm{Sp}(2))}\left(1, \mathrm{diag}(a, a^{-1})\right),\\
\mathbf{s}\left(1, \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right)\right) &= s_{(\mathrm{O}(2),\mathrm{Sp}(2))}\left(1, \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right)\right),\\
\mathbf{s}\left(1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) &= s_{(\mathrm{O}(2),\mathrm{Sp}(2))}\left(1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right).
\end{aligned}$$

Now set $V_0^\triangledown := \{(v, -v) : v \in V_0\}$ and $V_0^\vartriangle := \{(v, v) : v \in V_0\}$ so that

$$\mathbb{V}^\triangledown = \mathrm{Res}_{E/F}(V_0^\triangledown \otimes W_0), \qquad \mathbb{V}^\vartriangle = \mathrm{Res}_{E/F}(V_0^\vartriangle \otimes W_0)$$

gives a complete polarization $\mathbb{V}^\square = \mathbb{V}^\triangledown + \mathbb{V}^\vartriangle$ of the doubled symplectic space. Let $\hat{s}_{(\mathrm{O}(2,2),\mathrm{Sp}(2))}$ denote the splitting of $z_{\mathbb{V}^\vartriangle}$ defined in [K94] and define

$$s_{(\mathrm{O}(2,2),\mathrm{Sp}(2))}(h, g) := \hat{s}_{(\mathrm{O}(2,2),\mathrm{Sp}(2))}(h, g) \cdot \lambda^{-1}_{\mathbb{Y}'^\square \rightsquigarrow \mathbb{V}^\vartriangle}(g, h) \qquad \text{for } (g, h) \in \mathrm{G}(\mathrm{O}(2,2), \mathrm{Sp}(2)),$$

where $\lambda := \lambda_{\mathbb{Y}'^\square \rightsquigarrow \mathbb{V}^\vartriangle}$ is the change-of-polarization function defined in Lemma 5.3. Then using Proposition 5.16(a),

$$\begin{aligned}
&\hat{s}(g_1, g_2, h)\\
&\quad= \mathbf{s}^\square(g_1, g_2, 1) \cdot \lambda(g_1, g_2, h)\\
&\quad= \mathbf{s}(g_1, h) \cdot \overline{\mathbf{s}(g_2, h)} \cdot \xi(\det(i(g_2, h))) \cdot \lambda(g_1, g_2, h)\\
&\quad= s_{(\mathrm{O}(2),\mathrm{Sp}(2))}(g_1, h)\xi(g_1)^{-1} \cdot \overline{s_{(\mathrm{O}(2),\mathrm{Sp}(2))}(g_2, h)\xi(g_2)^{-1}} \cdot \xi(g_2)^{-2}\xi(\det(h)) \cdot \lambda(g_1, g_2, h)\\
&\quad= s_{(\mathrm{O}(2),\mathrm{Sp}(2))}(g_1, h) \cdot \overline{s_{(\mathrm{O}(2),\mathrm{Sp}(2))}(g_2, h)} \cdot \xi(g_1)^{-1}\xi(g_2)^{-1}\xi(\det(h)) \cdot \lambda(g_1, g_2, h)\\
&\quad= s_{(\mathrm{O}(2,2),\mathrm{Sp}(2))}(g_1, g_2, h) \cdot \xi(g_1)^{-1}\xi(g_2)^{-1} \cdot \lambda(g_1, g_2, h)\\
&\quad= \hat{s}_{(\mathrm{O}(2,2),\mathrm{Sp}(2))}(g_1, g_2, h) \cdot \xi(g_1)^{-1}\xi(g_2)^{-1}.
\end{aligned} \tag{6.7}$$

Define $P_O \subset \mathrm{GO}(\mathrm{Res}_{E/F} V_0^\square) \cong \mathrm{GO}(2,2)$ to be the stabilizer of the totally isotropic subspace $\mathrm{Res}_{E/F} V_0^\vartriangle$ of $\mathrm{Res}_{E/F} V_0^\square$. For $\phi \in \mathcal{S}(\mathbb{V}^\triangledown(\mathbb{A}))$, define the Siegel–Weil sections

$$\begin{aligned}
\Phi(\phi)(g) &:= (\omega_\psi^\square(g)\phi)(0), && \text{for } g \in \mathrm{GO}(2,2)_\mathbb{A} \subset \mathrm{GU}_E(V_0^\square)_\mathbb{A}\\
\Phi^{\mathrm{O},\mathrm{Sp}}(\phi)(g) &:= (\omega_\psi^{\mathrm{O}(2,2),\mathrm{Sp}(2)}(g)\phi)(0) && \text{for } g \in \mathrm{GO}(2,2)_\mathbb{A}.
\end{aligned}$$

Observe that $\Phi(\phi)(g) = \hat{s}(g) \cdot \hat{s}_{(\mathrm{O}(2,2),\mathrm{Sp}(2))}(g)^{-1} \cdot \Phi^{\mathrm{O},\mathrm{Sp}}(\phi)(g)$. We make the analogous definitions for the local objects $\Phi_v(\phi_v)$ and $\Phi_v^{\mathrm{O},\mathrm{Sp}}(\phi_v)$. The Siegel–Weil section $\Phi^{\mathrm{O},\mathrm{Sp}}(\phi) \in \mathrm{Ind}_{P_O}^{\mathrm{GO}(2,2)}(\det) \cdot |\det|^{1/2}$ determines a standard section $\Phi_s^{\mathrm{O},\mathrm{Sp}}(\phi) \in \mathrm{Ind}_{P_O}^{\mathrm{GO}(2,2)}(\det) \cdot |\det|^s$ and we may form the associated Eisenstein series

$$E(s,\Phi^{\mathrm{O},\mathrm{Sp}}(\phi))(g) := \sum_{\gamma\in P_O(F)\backslash \mathrm{GO}(2,2)} \Phi_s^{\mathrm{O},\mathrm{Sp}}(\gamma g), \qquad \text{for } g\in \mathrm{GO}(2,2)_{\mathbb{A}}.$$

Define

$$Z(s,\Phi,\chi) := \int_{[\mathrm{G}(\mathrm{O}(2)\times\mathrm{O}(2))]} E(s,\Phi)(g_1,g_2)\cdot\chi(g_1)\cdot\overline{\chi}(g_2)\,dg_1\,dg_2.$$

If $\Phi=\otimes_v\Phi_v$, define

$$Z_v(s,\Phi_v,\chi_v) = \int_{E_v^1} \Phi_v(g_v,1)\cdot\chi_v(g_v)\,dg_v.$$

By construction of the Tamagawa measure of $\mathbb{A}_E^1$ (see Section 2.1), one has

$$Z(s,\Phi,\chi) := \frac{\rho_F}{\rho_E}\cdot\prod_v Z_v(s,\Phi_v,\chi_v).$$

Define the partial Fourier transform

$$\delta\colon \mathcal{S}(\mathbb{X}'^{\square}(\mathbb{A}))\to\mathcal{S}(\mathbb{V}^{\triangledown}(\mathbb{A}))$$

by

$$\delta(\varphi)(u) = \int_{((\mathbb{V}^{\vartriangle}\cap\mathbb{Y}'^{\square})\backslash\mathbb{V}^{\vartriangle})(\mathbb{A})} \varphi(x)\psi\left(\tfrac{1}{2}(\langle\!\langle x,y\rangle\!\rangle - \langle\!\langle u,v\rangle\!\rangle)\right)\,dv,$$

where we write $u+v=x+y$ with $u\in\mathbb{V}^{\triangledown}(\mathbb{A})$, $v\in\mathbb{V}^{\vartriangle}(\mathbb{A})$, $x\in\mathbb{X}'^{\square}(\mathbb{A})$, $y\in\mathbb{Y}'^{\square}(\mathbb{A})$, and dv is the Tamagawa measure.

Observe that if $\phi\in\mathcal{S}(\mathbb{V}^{\triangledown}(\mathbb{A}))$ is the partial Fourier transform of $\varphi_1\otimes\overline{\varphi}_2$ for $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{X}'(\mathbb{A}))$, then for the Siegel–Weil section $\Phi=\Phi^{\mathrm{O},\mathrm{Sp}}(\delta(\varphi_1\otimes\overline{\varphi}_2))$, we have

$$\begin{aligned}
Z_v(\tfrac{1}{2},\Phi_v,\chi_v) &= \mathrm{vol}(E_v^1)\int_{E_v^1}\Phi^{\mathrm{O},\mathrm{Sp}}(\delta(\varphi_1\otimes\overline{\varphi}_2))(i(g_{1,v},1))\cdot\chi_v(g_v)\,dg_v\\
&= \mathrm{vol}(E_v^1)\int_{E_v^1}(\omega_\psi^{\mathrm{O}(2,2),\mathrm{Sp}(2)}(g_v,1)\delta(\varphi_1\otimes\overline{\varphi}_2))(0)\cdot\chi_v(g_v)\,dg_v\\
&= \mathrm{vol}(E_v^1)\int_{E_v^1}(\omega_\psi^{\square}(g_v,1)\delta(\varphi_1\otimes\overline{\varphi}_2))(0)\cdot\chi_v(g_v)\cdot\xi_v(g_v)\,dg_v\\
&= \mathrm{vol}(E_v^1)\int_{E_v^1}\langle\omega_\psi(g_v)\varphi_1,\varphi_2\rangle\cdot(\chi_v\xi_v)(g_v)\,dg_v \qquad (6.8)
\end{aligned}$$

Proposition 6.4. *For $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{X}'(\mathbb{A}))$, we have*

$$\langle\theta_{\varphi_1}(\chi\xi),\theta_{\varphi_2}(\chi\xi)\rangle = \frac{\rho_F}{\rho_E}\cdot\prod_v Z_v(\tfrac{1}{2},\Phi_v^{\mathrm{O},\mathrm{Sp}}(\delta(\varphi_1\otimes\overline{\varphi}_2)),\chi_v).$$

Proof. We use (6.7) to translate between our setting and that of [GQT, Proposition 11.1]. We have

$$\begin{aligned}
&\langle \theta_{\varphi_1}(\chi\cdot\xi), \theta_{\varphi_2}(\chi\cdot\xi)\rangle_H \\
&\quad = \int_{\mathcal{C}}\int_{[H_1]} \theta_{\varphi_1}(\chi\cdot\xi)(h_1h_c)\cdot\overline{\theta_{\varphi_2}(\chi\cdot\xi)(h_1h_c)}\,dh_1\,dc \\
&\quad = \int_{\mathcal{C}}\int_{[H_1]}\int_{[G_1]}\int_{[G_1]} \Theta(\omega_\psi(g_1g_c,h_1h_c)\varphi_1)(\chi\xi)(g_1g_c)\cdot \\
&\qquad\qquad \overline{\Theta(\omega_\psi(g_1'g_c,h_1h_c)\varphi_2)(\chi\xi)(g_1'g_c)}\,dg_1\,dg_1'\,dh\,dc \\
&\quad = \int_{\mathcal{C}}\int_{[\mathrm{Sp}(2)]}\int_{[\mathrm{O}(2)]}\int_{[\mathrm{O}(2)]} \Theta(\omega_\psi^{\mathrm{O},\mathrm{Sp}}(g_1g_c,h_1h_c)\varphi_1)(\chi\xi)(g_1g_c)\cdot \\
&\qquad\qquad \overline{\Theta(\omega_\psi^{\mathrm{O},\mathrm{Sp}}(g_1'g_c,h_1h_c)\varphi_2)(\chi\xi)(g_1'g_c)}\cdot\xi^{-1}(g_1)\overline{\xi}^{-1}(g_1')\,dg_1\,dg_1'\,dh\,dc \\
&\quad = \mathrm{Val}_{s=1/2}\int_{\mathcal{C}}\int_{[\mathrm{O}(2)]}\int_{[\mathrm{O}(2)]} E(s,\Phi_{\mathrm{O}(2,2,\,\mathrm{Sp}(2))}(\delta(\varphi_1\otimes\overline{\varphi}_2)))(g_1g_c,g_1'g_c)\cdot \\
&\qquad\qquad \chi(g_1g_c)\cdot\overline{\chi}(g_1'g_c)\,dg_1\,dg_1'\,dc \\
&\quad = \mathrm{Val}_{s=1/2}\,Z(s,\Phi(\delta(\varphi_1\otimes\overline{\varphi}_2)),\chi).
\end{aligned}$$

□

6.3 Local doubling zeta integrals

Let v be a nonsplit place of F. For notational convenience, we drop all subscripts v in this section. We preemptively note that the notation we use to describe the zeta integrals in this section differ from the notation used to describe the same (local) zeta integrals in the rest of the thesis. In this section, we temporarily assume that ξ is unitary.

Consider the Siegel parabolic subgroup

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & (a^*)^{-1} \end{pmatrix} \in \mathrm{GL}_2(E) \right\} \subset \mathrm{U}(1,1),$$

and for any unitary character $\eta\colon \mathrm{U}(1)\to\mathbb{C}^1$, consider the functional

$$Z(s,\eta,\xi^2)\colon I(s,\xi^2)\to\mathbb{C}, \qquad \mathcal{F}\mapsto\int_{E^1}\mathcal{F}(i(g,1))\eta(g)\,dg,$$

where $\iota\colon \mathrm{U}(1)\times\mathrm{U}(1)\to\mathrm{U}(1,1)$ is the natural map and

$$\begin{aligned} I(s,\xi^2) &:= \mathrm{Ind}_P^{\mathrm{U}(1,1)}(\xi^2\cdot|\cdot|^s) \\ &:= \left\{\mathcal{F}\colon\ \mathrm{U}(1,1)\to\mathbb{C}\ \middle|\ \begin{array}{c}\mathcal{F}(pg)=\xi^2(a)|a|_E^{s+1/2}\mathcal{F}(g)\\ \text{for all } g\in\mathrm{U}(1,1)\text{ and } p=\left(\begin{smallmatrix} a & * \\ 0 & \overline{a}^{-1}\end{smallmatrix}\right)\in P\end{array}\right\}\end{aligned}$$

is the normalized principal series representation. One has an intertwining operator

$$M(s,\xi^2)\colon I(s,\xi^2)\to I(-s,\overline{\xi}^{-2})\cong I(-s,\xi^2)$$

given by

$$M(s,\xi^2)\mathcal{F}(g)=\int_{N_P}\mathcal{F}(wng)\,dn,$$

where $w=\mathrm{diag}(1,-1)$ and N_P is the unipotent radical of the parabolic P.

Following Lapid–Rallis (see also Gan–Ichino, Section 10), after normalizing the intertwining operator by some rational function $c_\psi(s,\xi^2)$,

$$M_\psi^{\mathrm{LR}}(s,\xi^2):=c_\psi(s,\xi^2)M(s,\xi)$$

has a functional equation of the shape

$$Z(-s,\eta,\xi^2)(M_\psi^{\mathrm{LR}}(s,\xi^2)\mathcal{F})=*\cdot\gamma\left(s+\tfrac{1}{2},\eta,\overline{\xi},\psi\right)\cdot Z(s,\eta,\xi^2)(\mathcal{F}), \tag{6.9}$$

where $*$ denotes some nonzero factors. In particular, if we understand the behavior of the intertwining operator $M(s,\eta)$ and if $\gamma(s_0+\frac{1}{2},\eta)\neq 0$, the functional equation gives a relation between the nonvanishing of $Z(-s_0,\eta,\xi^2)$ and the nonvanishing of $Z(s_0,\eta,\xi^2)$.

We take a short detour to examine when the local theta lift to the nonsplit unitary group $\mathrm{U}(2)$ vanishes. Define

$$V_n^+:=\mathbb{H}_n,\qquad V_n^-:=D\oplus\mathbb{H}_{n-1},$$

where $\mathbb{H}_n$ is the $2n$-dimensional split Hermitian E-space and D is the nonsplit quaternion algebra over F viewed as a 2-dimensional Hermitian E-space via $\langle x,y\rangle=\mathrm{pr}_E(x^*y)$. For a character $\eta\colon\ \mathrm{U}(1)\cong E^1\to\mathbb{C}^1$, denote its theta lift to $\mathrm{U}(V_n^\pm)$ by $\Theta_{V_n^\pm}(\eta)$. To make tower "compatible" one takes the Weil representation for $\mathrm{U}(1)\times\mathrm{U}(V_n^+)$ to be such that the splitting on $\mathrm{U}(1)$ is given by ξ. In particular, the Weil representation on $\mathrm{U}(1)\times\mathrm{U}(V_0^+)=\mathrm{U}(1)\times\{1\}$ is given by the one-dimensional representation ξ. The *first occurrence* of the theta lift in the towers $\{\mathrm{U}(V_n^+) : n \geq 0\}$, $\{\mathrm{U}(V_n^-) : n \geq 0\}$ is defined to be

$$n^+ = \min\{n : \Theta_{V_n^+}(\eta) \neq 0\}, \qquad n^- = \min\{n : \Theta_{V_n^-}(\eta) \neq 0\}.$$

The following result is a special case of a theorem of Sun–Zhu [SZ15]:

Theorem 6.5 (Sun–Zhu)**.** $n^+(\eta) + n^-(\eta) = 2$.

We can describe the first occurrence in this setting more explicitly. By the compatible choice of splittings in the tower of unitary groups $\mathrm{U}(V_n^+)$, we have that $\Theta_{V_0^+}(\chi\xi) \neq 0$ if and only if χ is the trivial character. Hence we must necessarily be in the setting $n^+(\chi\xi) + n^-(\chi\xi) = 0 + 2$, and in particular, $\Theta_{V_1^-}(\chi\xi) = 0$.

Now suppose that χ is nontrivial. Then by the previous paragraph, $\Theta_{V_0^+}(\chi\xi) = 0$. We now argue that $\Theta_{V_1^+}(\chi\xi) \neq 0$. One explicit way to see this is as follows. Let $V_1^+ = V_1^\bigtriangledown + V_1^\bigtriangleup$ be a decomposition of V_1^+ into totally isotropic E-subspaces. For the Schwartz function $\varphi(x) = \chi(x)\mathbb{1}_{\mathcal{O}_E^\times}(x) \in \mathcal{S}(\mathrm{Res}_{E/F} V_1^\bigtriangledown)$, we have

$$\int_{E^1} (\omega_\psi(g)\varphi)(0) \cdot (\chi\xi)(g)\, dg \neq 0,$$

which proves that there is a nontrivial E^1-equivariant map

$$(\mathcal{S}(\mathrm{Res}_{E/F} V_1^\bigtriangledown), \omega_\psi) \to (\mathbb{C}, \chi\xi).$$

Hence $\Theta_{V_1^+}(\chi\xi) \neq 0$ by definition of the local theta lift. This now implies that we must necessarily be in the setting $n^+(\chi\xi) + n^-(\chi\xi) = 1 + 1$, and $\Theta_{V_1^-}(\chi\xi) \neq 0$.

In summary, the above arguments prove:

Lemma 6.6. *(a)* $\Theta_{V_1^-}(\chi\xi) \neq 0$ *if and only if* $\chi \colon E^1 \to \mathbb{C}^1$ *is nontrivial.*

(b) If $\chi \colon E^1 \to \mathbb{C}^1$ *is nontrivial,* $\Theta_{V_1^+}(\chi\xi) \neq 0$.

We now discuss the relationship between the theory of the doubling zeta integral and the local theta correspondence. Consider the two doubling seesaws for V_1^+ and V_1^-:

$$\begin{tikzcd}
\mathrm{U}(1,1) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(V_1^\pm) \times \mathrm{U}(V_1^\pm) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(1) \times \mathrm{U}(1) & \mathrm{U}(V_1^\pm)
\end{tikzcd}$$

If we have $\mathrm{U}(1,1) = \mathrm{U}(W)$, then one has a decomposition $W = W_1 + W_2$ of W into 1-dimensional isotropic E-spaces, and hence by viewing $V_1^\pm$ as the F-space $\mathrm{Res}_{E/F}(W_1 \otimes_E V_1^\pm) =$

$\mathrm{Res}_{E/F}(V_1^\pm)$, the Weil representation $\omega_\psi^\square$ for $\mathrm{U}(1,1)\times\mathrm{U}(V_1^\pm)$ can then be modeled on the space of Schwartz functions $\mathcal{S}(V_1^\pm)$. Define

$$\mathcal{S}(V_1^\pm)\to I(\tfrac{1}{2},\xi^2),\qquad \varphi\mapsto (g\mapsto(\omega_\psi^\square(i(g,1))\varphi)(0)),$$

where $i\colon \mathrm{U}(1)\times\mathrm{U}(1)\to\mathrm{U}(1,1)$ is the natural map. Let $R(V_1^\pm)$ denote the image of this map. Since $\xi^2|_{F^\times}=1$, there is a unique one-dimensional representation $\widetilde{\xi}^2$ of $\mathrm{U}(1,1)$ extending the representation defined by $\left(\begin{smallmatrix} a & * \\ 0 & \overline{a}^{-1}\end{smallmatrix}\right)\mapsto\xi^2(a)$. For the 0-dimensional Hermitian space V_0^+, we define a map

$$\mathcal{S}(V_0^+)=\mathbb{C}\to I(-\tfrac{1}{2},\xi^2),\qquad z\mapsto(g\mapsto\widetilde{\xi}^2(g)).$$

Let $R(V_0^+)$ denote the image of this map. We say that $\Theta_{V_0^+}(\chi\xi)\neq 0$ if and only if $\mathrm{Hom}_{\mathrm{U}(1)}(\widetilde{\xi}^2,\chi\xi)\neq 0$. Since $\widetilde{\xi}^2$ is one-dimensional, we have $\mathrm{Hom}_{\mathrm{U}(1)}(\widetilde{\xi}^2,\chi\xi)\neq 0$ if and only if $Z(-\frac{1}{2},\chi\xi,\xi^2)|_{R(V_0^+)}\neq 0$. Observe also that $\Theta_{V_0^+}(\chi\xi)\neq 0$ if and only if $\chi=1$.

The goal of the remainder of this section is to prove the following:

Proposition 6.7. *Let $\xi\colon\mathbb{A}_E^\times\to\mathbb{C}^1$ be a character such that $\xi|_{\mathbb{A}_F^\times}=\epsilon_{E/F}$. If $\Theta_{V_1^-}(\chi\xi)\neq 0$, then $Z(\frac{1}{2},\chi\xi,\xi^2)|_{R(V_1^-)}\neq 0$.*

We first remark that the converse of Proposition 6.7 is true and straightforward to see: If $Z(\frac{1}{2},\chi\xi,\xi^2)|_{R(V_1^-)}\neq 0$, then this immediately implies that $\mathrm{Hom}_{\mathrm{U}(1)}(\omega_\psi^\square|_{i(\mathrm{U}(1)\times\{1\})},(\chi\xi)^{-1})\neq 0$. But since $\omega_\psi^\square\cong\omega_\psi\otimes\overline{\omega}_\psi\xi^2$ (see Lemma 5.7) as a representation of $\mathrm{U}(1)\times\mathrm{U}(1)$, we have $\mathrm{Hom}_{\mathrm{U}(1)}(\omega_\psi,(\chi\xi)^{-1})\neq 0$, and so $\Theta_{V_1^-}(\chi\xi)\neq 0$ by definition.

The statement of Proposition 6.7 is actually quite surprising. The nonvanishing of the theta lift $\Theta_{V_1^-}(\chi\xi)$ is equivalent to the existence of a nontrivial element of $\mathrm{Hom}_{\mathrm{U}(1)}(\omega_\psi^\square,(\chi\xi)^{-1})$. But $\omega_\psi^\square$ is an infinite-dimensional representation! What we see from Proposition 6.7 is that the functional $Z(\frac{1}{2},\chi\xi,\xi^2)$ on the image of the Siegel–Weil section can detect the vanishing of the theta lift.

Before we prove Proposition 6.7, we recall a special case of a theorem of Kudla–Sweet:.

Theorem 6.8 (Kudla–Sweet, [KS97, Theorem 1.2(1),(4)])**.**

(i) $R(V_0^+)$ *is the unique irreducible submodule of* $I(-\frac{1}{2},\xi^2)$.

(ii) $I(-\frac{1}{2},\xi^2)/R(0,\xi^2)$ *is an irreducible representation of* $\mathrm{U}(1,1)$.

(iii) $R(V_1^+)=I(\frac{1}{2},\xi^2)$.

(iv) $R(V_1^-)$ *is the unique maximal submodule of* $I(\frac{1}{2},\xi^2)$ *and is irreducible of codimension* 1.

We are now ready to prove the proposition.

Proof of Proposition 6.7. By Lemma 6.6(a), we may assume that $\chi_v \colon E_v^1 \to \mathbb{C}^\times$ is nontrivial. Since $\chi\xi\overline{\xi} = \chi$ and $\overline{\chi\xi}\xi = \overline{\chi}$, by the "Ten Commandments" for γ-factors [LR05, Theorem 4], we have

$$L^S(s,\chi) = \prod_{v\in S} \gamma_v(s,(\chi\xi)_v,\overline{\xi}_v,\psi_v) \cdot L^S(1-s,\overline{\chi}),$$

where S is a finite set of places containing all the archimedean places and all places where χ_v is ramified. Now, since χ is nontrivial, we must have $L^S(0,\chi) \neq 0$ and $L^S(1,\overline{\chi}) \neq 0$, and therefore

$$\gamma_v(0,(\chi\xi)_v,\xi_v,\psi_v) \neq 0.$$

This implies that Equation (6.9) gives

$$Z\left(\tfrac{1}{2},\chi\xi,\xi^2\right)\left(M_\psi^{\mathrm{LR}}(-\tfrac{1}{2},\xi^2)(\mathcal{F})\right) = * \cdot Z\left(-\tfrac{1}{2},\chi\xi,\xi^2\right)(\mathcal{F}), \tag{6.10}$$

where $*$ is nonzero. We now investigate the intertwining operator

$$M_\psi^{\mathrm{LR}}(-\tfrac{1}{2},\xi) \colon I(-\tfrac{1}{2},\xi^2) \to I(\tfrac{1}{2},\xi^2).$$

We refer to Theorem 6.8 for the decomposition of the $\mathrm{U}(1,1)$-representations $I(\pm\frac{1}{2},\xi^2)$. By [KS97, Proposition 6.4],

$$\ker(M_\psi^{\mathrm{LR}}(-\tfrac{1}{2},\xi^2)) = R(0,\xi^2), \qquad \operatorname{im}(M_\psi^{\mathrm{LR}}(-\tfrac{1}{2},\xi^2)) = R(V_1^-).$$

Since χ is nontrivial, $\Theta_{V_0^+}(\chi\xi) = 0$, and therefore $Z(-\frac{1}{2},\chi\xi,\xi^2)|_{R(V_0^+)} = 0$. On the other hand, $Z(-\frac{1}{2},\chi\xi,\xi^2)$ is a nonzero functional, and therefore one can find $\mathcal{F} \in I(-\frac{1}{2},\xi^2)$ such that $M_\psi^{\mathrm{LR}}(-\frac{1}{2},\xi^2)(\mathcal{F}) \neq 0$. By Theorem 6.8(iv), it follows that $Z(\frac{1}{2},\xi\chi,\xi^2)|_{R(V_1^-)} \neq 0$. □

6.4 Unramified local theta lifts from $\mathrm{GU}(1)$ to $\mathrm{GU}(1,1)$

For convenience of notation, in this subsection we drop the subscript v. We denote by $\overline{x}$ the image of $x \in E$ under the nontrivial involution of E/F.

Consider the 2-dimensional E-space $V' = V_1' + V_2'$ with skew-Hermitian form

$$\langle (x_1,x_2),(y_1,y_2)\rangle = \overline{x}_1 y_2 + \overline{x}_2 y_1$$

for $(x_1,x_2),(y_1,y_2) \in V_1' + V_2'$. Then

$$\mathrm{GU}(V') = \mathrm{GU}(1,1) = \left\{ g \in \mathrm{GL}_2(E) : \overline{g}^{\mathsf{T}} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g = \nu(g) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \text{ for some } \nu(g) \in F^\times \right\}.$$

The upper-triangular matrices in $\mathrm{GU}(V')$ form a parabolic subgroup

$$P := \left\{ \left(\begin{smallmatrix} a & \nu' a \\ 0 & \nu a \end{smallmatrix} \right) \in \mathrm{GL}_2(E) : a \in E^\times,\, \nu \in F^\times,\, \nu' \in F \right\}.$$

Let P_F denote the Borel subgroup of $\mathrm{GL}_2(F)$ consisting of upper-triangular matrices in $\mathrm{GL}_2(F)$. Observe that there are natural inclusions $\mathrm{GL}_2(F) \hookrightarrow \mathrm{GU}(V')$ and $E^\times \hookrightarrow \mathrm{GU}(V')$ given by

$$\mathrm{GL}_2(F) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GU}(V') : a, b, c, d \in F \right\}, \qquad E^\times = \left\{ \left(\begin{smallmatrix} a & \\ & a \end{smallmatrix} \right) \in \mathrm{GU}(V') : a \in E^\times \right\}.$$

We have $\mathrm{GU}(V') \cong (\mathrm{GL}_2(F) \times E^\times)/F^\times$ (see Remark 4.7) and $P \cong (P_F \times E^\times)/F^\times$ (an easy direct computation).

Endow E with the Hermitian form

$$(x, y) = x\overline{y}$$

so that

$$\mathrm{GU}(E) = \mathrm{GU}(1) = E^\times.$$

Note that the similitude character on $\mathrm{GU}(E)$, which we also denote by ν, is given by

$$\nu\colon E^\times \to F^\times, \qquad x \mapsto x\overline{x} = \mathrm{Nm}(x).$$

Now consider the group

$$R := \{(h, g) \in E^\times \times \mathrm{GU}(V') : \nu(g) = \nu(h)\}.$$

Endow the 4-dimensional F-space $\mathbb{V}' = \mathrm{Res}_{E/F}(V')$ with the symplectic form

$$\langle\!\langle v, w \rangle\!\rangle = \frac{1}{2} \mathrm{Tr}_{E/F}(\langle v, w \rangle).$$

There is a natural map

$$\iota\colon R \to \mathrm{Sp}(\mathbb{V}), \qquad (h, g) \mapsto (v \mapsto h^{-1} v g).$$

The decomposition $V_1' + V_2'$ of V' into isotropic subspaces induces a polarization of $\mathbb{V}'$ given by

$$\mathbb{V}' = \mathbb{X}' + \mathbb{Y}', \qquad \text{where } \mathbb{X}' = \mathrm{Res}_{E/F}(V_1') \text{ and } \mathbb{Y}' = \mathrm{Res}_{E/F}(V_2').$$

Choose a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_1^*$, $\mathbf{e}_2^*$ of $\mathbb{V}'$ such that

$$\mathbb{X}' = F\mathbf{e}_1 + F\mathbf{e}_2, \qquad \mathbb{Y}' = F\mathbf{e}_1^* + F\mathbf{e}_2^*, \qquad \langle\langle \mathbf{e}_i, \mathbf{e}_j^* \rangle\rangle = \delta_{ij}.$$

The function

$$z_{\mathbb{Y}'} \colon \operatorname{Sp}(\mathbb{V}') \times \operatorname{Sp}(\mathbb{V}') \to \mathbb{C}^1, \qquad (g_1, g_2) \mapsto \gamma_F(\frac{1}{2} \circ q(\mathbb{Y}', \mathbb{Y}' g_2^{-1}, \mathbb{Y}' g_1))$$

defines a 2-cocycle and therefore uniquely determines a $\mathbb{C}^1$-extension of $\operatorname{Sp}(\mathbb{V}')$, which we denote by $\operatorname{Mp}(\mathbb{V}')_{\mathbb{Y}'}$. Explicitly,

$$\operatorname{Mp}(\mathbb{V}')_{\mathbb{Y}'} = \operatorname{Sp}(\mathbb{V}') \times \mathbb{C}^\times$$

with the group law

$$(g_1, z_1) \cdot (g_2, z_2) = (g_1 g_2, z_1 z_2 \cdot z_{\mathbb{Y}'}(g_1, g_2)).$$

Now assume that we have a function $\beta \colon R \to \mathbb{C}^1$ satisfying

$$z_{\mathbb{Y}'}(\iota(g_1), \iota(g_2)) = \beta(g_1 g_2) \beta(g_1)^{-1} \beta(g_2)^{-1}.$$

Then the map

$$R \to \operatorname{Mp}(\mathbb{V}')_{\mathbb{Y}'}, \qquad g \mapsto (\iota(g), \beta(g))$$

is a group homomorphism and the Weil representation ω_ψ on $\operatorname{Mp}(\mathbb{V}')_{\mathbb{Y}'}$ pulls back to a representation of R, which we also denote by ω_ψ.

Abusing notation, define

$$\beta \colon E^\times \to \mathbb{C}^1, \qquad h \mapsto \beta(h, d(\nu(h))).$$

Observe that this defines a character since $\iota(h, d(\nu(h)))$ stabilizes $\mathbb{Y}'$ and therefore

$$z_{\mathbb{Y}'}(\iota(h, d(\nu(h))), \iota(h', d(\nu(h')))) = 1$$

for any $h, h' \in E^\times$. Define

$$L(h)\phi(x) := \omega_\psi(h, d(\nu(h)))\phi(x) = \beta(h)|h|^{-1/2}\phi(xh^{-1})$$

for $h \in E^\times$ and $\phi \in \mathcal{S}(\mathbb{X}')$. Then for any $(h, g) \in R$,

$$\omega_\psi(h, g)\phi(x) = L(h)\omega_\psi(d(\nu(g)^{-1})g)\phi(x) = \beta(h)|h|^{-1/2}(\omega_\psi(d(\nu(g)^{-1})g)\phi)(xh^{-1}). \tag{6.11}$$

Lemma 6.9. *For any $h \in E^\times$ and $g \in \mathrm{U}(V')$,*

$$L(h^{-1})\omega_\psi(g)L(h) = \omega_\psi(d(\nu(h))gd(\nu(h)^{-1})).$$

Proof. For convenience of notation, set $\nu := \nu(h)$. Observe that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$\mathrm{diag}(h^{-1}, \overline{h})g\,\mathrm{diag}(h, \overline{h}^{-1}) = \left(\begin{smallmatrix} a & b\nu \\ c\nu^{-1} & d \end{smallmatrix}\right) = d(\nu)gd(\nu^{-1}).$$

Now, $\mathrm{diag}(h, \overline{h}^{-1})$ and its inverse are elements of $\mathrm{U}(1,1)$ and

$$\omega_\psi(\mathrm{diag}(h^{-1}, \overline{h}))\phi(x) = L(h)\phi(x).$$

It now follows that

$$L(h^{-1})\omega_\psi(g)L(h) = \omega_\psi(\mathrm{diag}(h^{-1}, \overline{h})g\,\mathrm{diag}(h, \overline{h}^{-1})) = \omega_\psi(d(\nu)gd(\nu^{-1})). \qquad \square$$

Consider the semidirect product $E^\times \ltimes \mathrm{U}(V')$ with multiplication

$$(h_1, g_1) * (h_2, g_2) = (h_1h_2, d(\nu(h_2))g_1d(\nu(h_2)^{-1})g_2), \qquad \text{where } h \in E^\times \text{ and } g \in \mathrm{U}(V').$$

This defines a group multiplication since the map d is multiplicative and ν is a group homomorphism to $F^\times$, an abelian group. Lemma 6.9 implies:

Lemma 6.10. *The Weil representation ω_ψ on R extends to a representation of $E^\times \ltimes \mathrm{U}(V')$ defined by*

$$\omega_\psi(h, g) = L(h)\omega_\psi(g), \qquad h \in E^\times,\ g \in \mathrm{U}(V').$$

In particular, the Weil representation on the quotient

$$\Theta^{(1)}(\mathrm{triv}) := \mathcal{S}(\mathbb{X}')/\bigcap_{\alpha \in \mathrm{Hom}_{E^1}(\mathcal{S}(\mathbb{X}'), \mathrm{triv})} \ker(\alpha)$$

extends to a representation of $\mathrm{GU}(V')^+ \cong \{d(\nu) : \nu \in \mathrm{Nm}(E^\times)\} \ltimes \mathrm{U}(V')$ *satisfying*

$$\omega_\psi(d(\nu)) = L(h),$$

where $h \in E^\times$ is any element such that $\nu(h) = \nu$.

Proof. By Lemma 6.9,

$$L(h_1)\omega_\psi(g_1)L(h_2)\omega_\psi(g_2) = L(h_1)L(h_2)\omega_\psi(d(\nu(h_2))g_1d(\nu(h_2)^{-1}))\omega_\psi(g_2).$$

The assertion about $\Theta^{(1)}(\mathrm{triv})$ holds since the Weil representation of $E^\times \ltimes \mathrm{U}(V')$ factors through the norm map $E^\times \to F^\times$. $\square$

Definition 6.11. For any character $\eta_0 \colon F^\times \to \mathbb{C}$ and any $\phi \in \mathcal{S}(\mathbb{X}')$, define

$$\mathcal{F}_{\phi,\eta_0} \colon \mathrm{GU}(V') \to \mathbb{C}^\times, \qquad g \mapsto |\nu(g)|^{-1/2}\eta_0(\nu(g))^{-1}(\omega_\psi(d(\nu(g)^{-1})g)\phi)(0).$$

Lemma 6.12. *For any $p = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{GU}(V')$,*

$$\mathcal{F}_{\phi,\eta_0}(pg) = |a|^{1/2}|d|^{-1/2}\eta_0(\overline{a}d)^{-1}\beta(a)^{-1}\mathcal{F}_{\phi,\eta_0}(g)$$

for all $g \in \mathrm{GU}(V')$ so that

$$\mathcal{F}_{\phi,\eta_0} \in \mathrm{Ind}_P^{\mathrm{GU}(V')}(\widetilde{\eta}_0),$$

where

$$\widetilde{\eta}_0 \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) := \eta_0(\overline{a}d)^{-1}\beta(a)^{-1}.$$

In particular, $\mathcal{F}_{\phi,\eta_0}|_{\mathrm{GSp}(2)}$ is an element of the (normalized) principal series representation

$$\mathrm{Ind}_B^{\mathrm{GSp}(2)}(\eta_0^{-1}\beta^{-1} \otimes \eta_0^{-1}).$$

Proof. First note that $\nu(p) = \overline{a}d \in F^\times$. We have

$$\begin{aligned}
\mathcal{F}_{\phi,\eta_0}(pg) &= |\nu(pg)|^{-1/2}\eta_0(\nu(pg))^{-1}(\omega_\psi(d(\nu(pg)^{-1})pg)\phi)(0) \\
&= |\nu(p)|^{-1/2}\eta_0(\nu(p))^{-1}\beta(a)^{-1}|\nu(a)|^{1/2}\mathcal{F}_{\phi,\eta_0}(g) \\
&= |d|^{-1/2}|a|^{1/2}\eta_0(\overline{a}d)^{-1}\beta(a)^{-1}\mathcal{F}_{\phi,\eta_0}.
\end{aligned}$$

$\square$

Lemma 6.13. *The assignment*

$$\phi \mapsto \mathcal{F}_{\phi,\eta_0}$$

defines a nonzero R-equivariant map

$$(\omega_\psi, \mathcal{S}(\mathbb{X}')) \to \mathrm{Ind}_P^{\mathrm{GU}(V')}(\widetilde{\eta}_0) \otimes (\eta_0(\mathrm{Nm}) \cdot \beta).$$

The right-hand side is irreducible and we have an isomorphism

$$\mathrm{Ind}_P^{\mathrm{GU}(V')}(\widetilde{\eta}_0) \cong \mathrm{Ind}_{P_F}^{\mathrm{GL}_2(F)}(\eta_0^{-1} \otimes (\eta_0 \cdot \beta)^{-1}) \otimes (\eta_0(\mathrm{Nm}) \cdot \beta)^{-1},$$

where the right-hand side is a representation of $\mathrm{GL}_2(F) \times E^\times$ that descends to the quotient $(\mathrm{GL}_2(F) \times E^\times)/F^\times \cong \mathrm{GU}(V')$.

Proof. It is clear by definition that the map is nonzero. To prove R-equivariance of the map $\phi \mapsto \mathcal{F}_{\phi,\eta_0}$, we use Lemma 6.9 to obtain the second equality in:

$$\begin{aligned}
\mathcal{F}_{\omega_\psi(h',g')\phi,\eta_0}(g) &= |\nu(g)|^{-1/2}\eta_0(\nu(g))^{-1}(\omega_\psi(d(\nu(g)^{-1})g)L(h')\omega_\psi(d(\nu(g')^{-1})g')\phi)(0)\\
&= |\nu(g)|^{-1/2}\eta_0(\nu(g))^{-1}(L(h')\omega_\psi(d(\nu(gg')^{-1})gg')\phi)(0)\\
&= |\nu(g)|^{-1/2}\eta_0(\nu(g))^{-1}|\nu(h')|^{-1/2}\beta(h')(\omega_\psi(d(\nu(gg')^{-1})gg')\phi)(0)\\
&= \beta(h')\eta_0(\nu(h'))|\nu(gg')|^{-1/2}\eta_0(\nu(gg'))^{-1}(\omega_\psi(d(\nu(gg')^{-1})gg')\phi)(0)\\
&= \beta(h')\eta_0(\nu(h'))\mathcal{F}_{\phi,\eta_0}(gg').
\end{aligned}$$

The last assertion in the lemma holds since $P \cong (P_F \times E^\times)/F^\times$ and $\mathrm{GU}(V') \cong (\mathrm{GL}_2(F) \times E^\times)/F^\times$. The representation $\mathrm{Ind}_{P_F}^{\mathrm{GL}_2(F)}(\widetilde{\eta}_0)$ is irreducible since the character $\eta_0^{-1}\beta^{-1}\eta_0 = \beta^{-1}$ is not $|\cdot|$ or $|\cdot|^{-1}$. It follows that $\mathrm{Ind}_P^{\mathrm{GU}(V')}(\widetilde{\eta}_0)$ is irreducible. ☐

The map defined in Lemma 6.13 factors through

$$\Theta^{(1)}(\beta) := \mathcal{S}(\mathbb{X}')/\bigcap_{\alpha\in\mathrm{Hom}_{E^1}(\mathcal{S}(\mathbb{X}'),\beta)}\ker\alpha,$$

the largest quotient of $\mathcal{S}(\mathbb{X}')$ such that E^1 acts by β. Note that by construction, $\Theta^{(1)}(\beta)$, as a representation of $\mathrm{U}(V')$, is the local theta lift of β to $\mathrm{U}(V')$.

There are many extensions of $\Theta^{(1)}(\beta)$ to a representation of $E^\times \times \mathrm{GU}(V')^+$, but specifying an action of $E^\times$ determines such an extension. Explicitly, define $\Theta_{\mathrm{ur},\beta}(\beta\cdot\eta_0(\mathrm{Nm}))$ to be the unique representation of $\mathrm{GU}(V')^+$ such that for $g = \left(\begin{smallmatrix}1 & 0\\ 0 & \nu\end{smallmatrix}\right) \in \mathrm{GU}(V')^+$,

$$\Theta_{\mathrm{ur},\beta}(\beta\cdot\eta_0(\mathrm{Nm}))(g) := \eta_0(\mathrm{Nm}(h))^{-1}\cdot\Theta^{(1)}(\beta)(h,g),$$

where $h \in E^\times$ is any element such that $\nu(h) = \nu(g) = \nu$.

Theorem 6.14 (Rallis)**.** *The R-equivariant map in Lemma 6.13 factors through $\Theta_{\mathrm{ur},\beta}(\beta\cdot\eta_0(\mathrm{Nm}))$ and induces an injective map:*

$$\begin{tikzcd}
(\omega_\psi,\mathcal{S}(\mathbb{X}')) \arrow[r] \arrow[d] & \mathrm{Ind}_P^{\mathrm{GU}(V')}(\widetilde{\eta}_0)\\
\Theta_{\mathrm{ur},\beta}(\beta\cdot\eta_0(\mathrm{Nm})) \arrow[ur, hook] &
\end{tikzcd}$$

Moreover,

$$\Theta_{\mathrm{ur},\beta}(\beta\cdot\eta_0(\mathrm{Nm})) \cong \mathrm{Ind}_{P_F}^{\mathrm{GL}_2(F)}(\eta_0^{-1}\epsilon_{E/F}\otimes\eta_0^{-1})\otimes(\eta_0(\mathrm{Nm})^{-1}\cdot\beta^{-1}),$$

where the right-hand side is viewed as a representation of $\mathrm{GL}_2(F) \times E^\times$ *that descends to the quotient* $(\mathrm{GL}_2(F) \times E^\times)/F^\times \cong \mathrm{GU}(V')$.

Proof. This is due to Rallis [R84, Theorem II.1.1]. By the injectivity of

$$\Theta_{\mathrm{ur},\beta}(\beta \cdot \eta_0(\mathrm{Nm})) \hookrightarrow \mathrm{Ind}_P^{\mathrm{GU}(V')}(\widetilde{\eta}_0)$$

and the irreducibility of $\mathrm{Ind}_P^{\mathrm{GU}(V')}(\widetilde{\eta}_0)$, by Lemma 6.13, we have an isomorphism

$$\Theta_{\mathrm{ur},\beta}(\beta \cdot \eta_0(\mathrm{Nm})) \cong \mathrm{Ind}_{P_F}^{\mathrm{GL}_2(F)}(\eta_0^{-1}\beta^{-1} \otimes \eta_0^{-1}) \otimes (\eta_0(\mathrm{Nm})^{-1} \cdot \beta^{-1}).$$

Finally, by Lemma 5.17, the restriction of β to $F^\times$ is exactly the quadratic character $\epsilon_{E/F}$, and this completes the proof. □

6.5 Proof of Theorem 6.1

In this section, we use the calculations in the preceding sections to prove Theorem 6.1, the main theorem of this chapter.

Let χ and χ' be Hecke characters of $E^\times$. Recall from Section 6.1 that for every Schwartz function $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$ we have automorphic forms $\theta_\varphi(\chi)$ and $\theta'_\varphi(\chi')$ on the adelic groups $H(\mathbb{A}) \cong B_{\mathbb{A}}^\times$ and $H'(\mathbb{A}) \cong ((B'_{\mathbb{A}})^\times \times \mathbb{A}_E^\times)/\mathbb{A}_F^\times$, respectively. Let $\Theta(\chi)$ denote the automorphic representation of $H(\mathbb{A})$ generated by $\theta_\varphi(\chi)$ for all $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$ and let $\Theta'(\chi')$ denote the automorphic representation of $H'(\mathbb{A})$ generated by $\theta'_\varphi(\chi')$ for all $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$.

Define

$$\begin{aligned} \widetilde{\xi}\colon \mathbb{A}_E^\times \to \mathbb{C}^\times, &\qquad \alpha \mapsto \mathbf{s}(\alpha, d(\nu(\alpha))), \\ \widetilde{\xi}'\colon \mathbb{A}_E^\times \to \mathbb{C}^\times, &\qquad \alpha \mapsto \mathbf{s}'(d(\nu(\alpha)), \alpha). \end{aligned}$$

Proposition 6.15. *If* $\pi_\chi^B \neq 0$*, then* $\Theta(\chi \cdot \xi) \neq 0$*. Analogously, if* $\pi_{\chi'}^{B'} \neq 0$*, then* $\Theta'(\chi' \cdot \xi') \neq 0$*.*

Proof. Recall from Theorem 2.7 that $\pi_\chi^B \neq 0$ if and only if $\chi_v|_{E_v^1} \neq 1$ for every place v where B_v is nonsplit. Let v such a place, i.e. B_v is nonsplit and $\chi_v|_{E_v^1} \neq 1$. By Lemma 6.6(a), we have $\Theta_v(\chi_v\xi_v) \neq 0$, and by Proposition 6.7, we have $Z_v(\frac{1}{2}, -, \chi_v\xi_v) \neq 0$. Now let v be a place such that B_v is split. By Lemma 6.6(b), we have $\Theta_v(\chi_v\xi_v) \neq 0$, and by Theorem 6.8(c), we have $Z_v(\frac{1}{2}, -, \chi_v\xi_v) \neq 0$. By Rallis inner product formula (Proposition 6.4), $\Theta(\chi \cdot \xi) \neq 0$ if and only if all the local zeta integrals $Z_v(\frac{1}{2}, -, \chi_v\xi_v) \neq 0$, and hence we have shown that $\Theta(\chi \cdot \xi) \neq 0$. □

Lemma 6.16. *If χ, χ' are Hecke characters of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_E^1$ is nontrivial, then $\Theta(\chi \cdot \xi)$ is a cuspidal automorphic representation of $B_{\mathbb{A}}^\times$ and $\Theta'(\chi' \cdot \xi')$ is a cuspidal automorphic representation of ${B'_{\mathbb{A}}}^\times$.*

Proof. If $B \neq M_2(F)$, then the statement holds trivially (see Example 2.2). Now assume $B = M_2(F)$. We would like to prove that for any Schwartz function $\phi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$,

$$\int_{F\backslash\mathbb{A}_F} \theta_\phi(\chi)(\mathbf{n}(b)g)\, db = 0, \qquad \text{where } \mathbf{n}(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \tag{6.12}$$

Observe that if $g \notin \mathrm{GL}_2^+(\mathbb{A}_F)$, then $\mathbf{n}(b)g \notin \mathrm{GL}_2^+(\mathbb{A}_F)$, and hence the integrand in (6.12) is identically zero. Now assume $g \in \mathrm{GL}_2^+(\mathbb{A}_F)$ and pick $\alpha \in \mathbb{A}_E^\times$ with $\mathrm{Nm}(\alpha) = \det(g)$. Then by definition

$$\theta_\phi(\chi)(\mathbf{n}(b)g) = \theta_{\omega_\psi(\alpha,g)\phi}(\chi)(\mathbf{n}(b)),$$

and therefore it remains only to show

$$\int_{F\backslash\mathbb{A}_F} \theta_\phi(\chi)(\mathbf{n}(b))\, db = 0.$$

Recall that if B is split, then the 2-dimensional E-space W_0 is a split Hermitian space and one has a decomposition $W_0 = W_1 + W_2$ into isotropic subspaces of dimension 1. This induces a complete polarization $\mathbb{V} = \mathbb{X}' + \mathbb{Y}'$ given by $\mathbb{X}' = \mathrm{Res}_{E/F}(V_0 \otimes W_1)$ and $\mathbb{Y}' = \mathrm{Res}_{E/F}(V_0 \otimes W_2)$. Then $\mathbb{A}_E^1 \subset \mathrm{U}(V_0)$ stabilizes $\mathbb{X}'$ and $\mathbb{Y}'$, and so for $\alpha \in \mathbb{A}_E^1$, $b \in \mathbb{A}_F$, and $\phi' \in \mathcal{S}(\mathbb{X}'(\mathbb{A}))$,

$$\omega_\psi(\alpha, \mathbf{n}(b))\phi'(x) = \xi^{-1}(\alpha) \cdot \psi\left(\tfrac{1}{2} b x x^\intercal\right) \cdot \phi'(x\alpha).$$

We have

$$\begin{aligned}
\int_{F\backslash\mathbb{A}_F} \theta_\phi(\chi)(\mathbf{n}(b))\, db &= \int_{F\backslash\mathbb{A}_F} \int_{E^1\backslash\mathbb{A}_E^1} \sum_{x \in \mathbb{X}'(F)} (\omega_\psi(\alpha, \mathbf{n}(b)))\phi'(x) \cdot (\chi\xi)(\alpha)\, d\alpha\, db \\
&= \int_{E^1\backslash\mathbb{A}_E^1} \sum_{x \in \mathbb{X}'(F)} \int_{F\backslash\mathbb{A}_F} \xi^{-1}(\alpha) \cdot \psi\left(\tfrac{1}{2} b x x^\intercal\right) \cdot \phi'(x\alpha) \cdot (\chi\xi)(\alpha)\, db\, d\alpha \\
&= \int_{E^1\backslash\mathbb{A}_E^1} \xi^{-1}(\alpha) \cdot \phi'(0) \cdot (\chi\xi)(\alpha)\, d\alpha \\
&= \phi'(0) \int_{E^1\backslash\mathbb{A}_E^1} \chi(\alpha)\, d\alpha = 0.
\end{aligned}$$

This implies that for any $\phi' \in \mathcal{S}(\mathbb{X}'(\mathbb{A}))$, the global theta lift $\theta_{\phi'}(\chi)$ is cuspidal and the desired conclusion follows. $\square$

Theorem 6.17. *Assume that χ and χ' are Hecke characters of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_E^1$ is nontrivial.*

(a) If $\Theta(\chi \cdot \xi)$ is nonzero, then
$$\Theta(\chi \cdot \xi) \cong \pi_\chi^B.$$

(b) If $\Theta'(\overline{\chi' \cdot \xi'^{-1}})$ is nonzero, then
$$\Theta'(\overline{\chi' \cdot \xi'^{-1}})^\vee \cong \pi_{\chi'}^{B'} \otimes (\chi' \cdot \xi'^{-1}),$$

where the right-hand side is viewed as a representation of $H'(\mathbb{A}) \cong ((B'_\mathbb{A})^\times \times \mathbb{A}_E^\times)/\mathbb{A}_F^\times$ descended from the $(B'_\mathbb{A})^\times \times \mathbb{A}_E^\times$ representation written above.

Proof. We prove (a) first. By our normalization (compare the local definition in Section 3.4 to the global definition in Section 6.1), at a place v, the local representation corresponding to the global theta lift of $\chi \cdot \xi$ is the local theta lift of $(\chi_v \cdot \xi_v)^{-1}$. That is,
$$\Theta(\chi \cdot \xi)_v \cong \Theta_v((\chi_v \cdot \xi_v)^{-1}) \cong \Theta_v(\chi_v^{-1} \cdot \xi_v^{-1}).$$

Theorem 6.14 gives a description of the right-hand side for every place v such that

- v splits completely in E, or
- v lies under a single place w of E and $\chi_w \colon E_w^\times \to \mathbb{C}^\times$ factors through $\mathrm{Nm} \colon E_w^\times \to F_v^\times$.

For each such place v, by Lemma 5.17, we have
$$\mathbf{s}(\alpha, d(\nu(\alpha))) = \xi(\alpha)^{-1}, \qquad \text{for all } \alpha \in E_v^\times.$$

Writing $\chi_v = \chi_{v,0}(\mathrm{Nm})$, we have
$$\Theta_v(\chi_v^{-1} \cdot \xi_v^{-1}) \cong \Theta_{\mathrm{ur},\xi_v^{-1}}(\chi_v^{-1}\xi_v^{-1}) \cong \mathrm{Ind}_{P_{F_v}}^{\mathrm{GL}_2(F_v)}(\chi_{v,0}\epsilon_{E_v/F_v} \otimes \chi_{v,0}),$$

and therefore by Theorem 2.7, we have that
$$\Theta(\chi \cdot \beta) \cong \pi_\chi^B.$$

The proof of (b) is very similar. In this case, because we conjugate the theta kernel in the definition of the global theta lift Θ' (see Section 6.1), we have
$$\Theta'(\overline{\chi' \cdot \xi'^{-1}})_v^\vee \cong \Theta_v((\chi'_v \cdot \xi_v'^{-1})^{-1}) = \Theta_v(\chi_v'^{-1} \cdot \xi'_v).$$

At every place v of F where everything is unramified, by Lemma 5.22,

$$\mathbf{s}'(d(\nu(\alpha)),\alpha) = \xi'(\alpha), \qquad \text{for all } \alpha \in E_v^\times.$$

Writing $\chi'_v = \chi'_{v,0}(\mathrm{Nm})$ at each such place, Theorem 6.14 implies

$$\Theta_v({\chi'_v}^{-1} \cdot \xi'_v) \cong \Theta_{\mathrm{ur},\xi'_v}({\chi'_v}^{-1} \cdot \xi'_v) \cong \mathrm{Ind}_{P_{F_v}}^{\mathrm{GL}_2(F_v)}(\chi'_{v,0}\xi'_v \otimes \chi'_{v,0}) \otimes (\chi'_{v,0} \cdot {\xi'_v}^{-1}).$$

By definition, $\xi'_v|_{F_v^\times} = \epsilon_{E_v/F_v}$, and therefore by Theorem 2.7, we have that

$$\Theta'(\overline{\chi' \cdot \xi'^{-1}})^\vee \cong \pi_{\chi'}^{B'} \otimes (\chi'^{-1} \cdot \xi'). \qquad \square$$

Theorem 6.1 now follows from Proposition 6.15 and Theorem 6.17.

Proof of Theorem 6.1. If $\Theta(\chi \cdot \xi) = 0$, then by Proposition 6.15 we must have $\pi_\chi^B = 0$ and therefore $\Theta(\chi \cdot \xi) = \pi_\chi^B$. If $\Theta(\chi \cdot \xi) \neq 0$, then by Theorem 6.17 we must have $\Theta(\chi \cdot \xi) \cong \pi_\chi^B$. The same argument holds to conclude the desired isomorphism for $\Theta'(\overline{\chi' \cdot \xi'^{-1}})$. $\square$

6.6 Period identities of CM forms

We are now ready to prove an identity of toric integrals of automorphic forms in π_χ^B and $\pi_{\chi'}^{B'}$. We use the seesaw

$$\begin{tikzcd}[column sep=small]
H' \arrow[d, dash] \arrow[dr, dash] & H \arrow[d, dash] \arrow[dl, dash] & & \mathrm{GU}_E(\mathrm{Res}\, V) \arrow[d, dash] \arrow[dr, dash] & \mathrm{GU}_B(W^*) \arrow[d, dash] \arrow[dl, dash] & & ((B')^\times \times E^\times)/F^\times \arrow[d, dash] \arrow[dr, dash] & B^\times \arrow[d, dash] \arrow[dl, dash] \\
G & G' & = & \mathrm{GU}_B(V)^\circ & \mathrm{GU}_E(W) & \cong & E^\times & E^\times
\end{tikzcd}$$

Recall from Proposition 5.14 that our choice of splittings

$$s\colon \mathcal{G}_{G\times H}(\mathbb{A}) \to \mathbb{C}^1, \qquad s'\colon \mathcal{G}_{G'\times H'}(\mathbb{A}) \to \mathbb{C}^1$$

enjoys the property that for $(\alpha,\beta) \in \mathcal{G}_{G\times G'}(\mathbb{A})$,

$$s'(\alpha,\beta) = \xi(\alpha) \cdot \xi'(\beta) \cdot s(\alpha,\beta).$$

Theorem 6.18. *For any Hecke characters χ and χ' of E,*

$$\langle \theta_\varphi(\chi\cdot\xi), \overline{\chi'}\rangle_{G'} = \langle \chi, \theta'_\varphi(\overline{\chi'\cdot\xi'^{-1}})\rangle_G.$$

Proof. Unwinding definitions and using Proposition 5.14, we have

$$\begin{aligned}
\langle \theta_\varphi(\chi\cdot\xi), \overline{\chi'}\rangle_{G'} &= \int_{[G']} \theta_\varphi(\chi\cdot\xi)(g')\cdot\chi'(g')\,dg' \\
&= \int_{\mathcal{C}}\int_{[G_1']} \theta_\varphi(\chi\cdot\xi)(g_1'g_c')\cdot\chi'(g_1'g_c')\,dg_1'\,dc \\
&= \int_{\mathcal{C}}\int_{[G_1']}\int_{[G_1]} \Theta(\omega_\psi(g_1g_c, g_1'g_c')\varphi)\cdot\chi(g_1g_c)\cdot\xi(g_1g_c)\cdot\chi'(g_1'g_c')\,dg_1\,dg_1'\,dc \\
&= \int_{\mathcal{C}}\int_{[G_1]}\int_{[G_1']} \chi(g_1g_c)\Theta(\omega_\psi'(g_1g_c, g_1'g_c')\varphi)\cdot\xi'(g_1'g_c')^{-1}\cdot\chi'(g_1'g_c')\,dg_1'\,dg_1\,dc \\
&= \int_{\mathcal{C}}\int_{[G_1]} \chi(g_1g_c)\overline{\theta_\varphi'(\overline{\chi'\cdot\xi'^{-1}})}(g_1g_c)\,dg_1\,dc \\
&= \langle \chi, \theta_\varphi'(\overline{\chi'\cdot\xi'^{-1}})\rangle_G.
\end{aligned}$$

□

Combining Theorems 6.17 and 6.18, we obtain the following result:

Theorem 6.19. *Let χ, χ' be Hecke characters of E and let $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$. Then*

$$f_\chi^B := \theta_\varphi(\chi\cdot\xi) \in \pi_\chi^B, \qquad f_{\chi'}^{B'} := \overline{\theta_\varphi'(\overline{\chi'\cdot\xi'^{-1}})} \in \pi_{\chi'}^{B'},$$

and we have

$$\int_{\mathbb{A}_F^\times E^\times\backslash\mathbb{A}_E^\times} f_\chi(g)\cdot\chi'(g)\,dg = \int_{\mathbb{A}_F^\times E^\times\backslash\mathbb{A}_E^\times} \chi(g)\cdot f_{\chi'}(g)\,dg.$$

CHAPTER 7

Interlude

It may be the case that the choices of B, χ, χ', and $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}))$ make it so that the two sides of the identity in Theorem 6.19 are zero. This is possible at many points: π_χ^B could be zero (see Theorem 2.7), the theta lifts $\theta_\varphi(\chi \cdot \xi)$ or $\theta'_\varphi(\overline{\chi' \cdot \xi'^{-1}})$, $\theta'_\varphi(\overline{\chi' \cdot \xi'^{-1}})$ could themselves be zero, or just the periods could be zero (see Theorem 4.3).

Now start with a totally real number field F and let E be a CM extension of F. Let χ and χ' be two Hecke characters of E and assume that

$$L(\pi_\chi \otimes \pi_{\chi'}, \tfrac{1}{2}) \neq 0.$$

Then there exists a unique quaternion algebra B over F such that the linear functional

$$\pi_\chi^B \to \mathbb{C}, \qquad f_\chi^B \mapsto \int_{[E^\times]} f_\chi^B(g) \cdot \chi'(g)\, dg$$

is nontrivial. Moreover, B' is the unique quaternion algebra over F such that the linear functional

$$\pi_{\chi'}^{B'} \to \mathbb{C}, \qquad f_{\chi'}^{B'} \mapsto \int_{[E^\times]} \chi(g) \cdot f_{\chi'}^{B'}(g)\, dg$$

is nontrivial. In the coming chapters, we will choose a Schwartz function φ for the special case when B is the split quaternion algebra over F. We will see that for our chosen family of Schwartz functions φ_l, the theta lift $\theta_{\varphi_l}(\chi \cdot \xi)$ is a nonzero Hecke eigenform of weight $k + 1 + 2l$ occuring in π_χ (here k is related to the infinity type of χ in a specified way). In certain cases, for example in Chapter 10, where we consider the special setting with χ, χ' being powers of the canonical character of $\mathbb{Q}(\sqrt{-7})$, one can show by hand that the torus period is nonvanishing. The significance of arranging for φ_l to give rise to a Hecke eigenform is that these automorphic forms are exactly the ones in p-adic-limiting families for example in [BDP13]. Examining the theta lift of φ_l on the definite quaternion algebra is the subject of future investigation.

CHAPTER 8

Special vectors in the Weil representation

Recall that F is a totally real field and $E = F(\mathbf{i})$ is a CM extension of F. We choose the trace-free element $\mathbf{i} \in E$ so that $u = \mathbf{i}^2 \in F$ has the property that for any finite place v of F,

$$\mathrm{val}_v(u) = \begin{cases} 0 & \text{if } E_v/F_v \text{ is unramified} \\ 1 & \text{if } E_v/F_v \text{ is ramified.} \end{cases}$$

For the rest of the paper, we take ψ to be the standard additive character of $F\backslash\mathbb{A}_F$ (see Section 2.1). Recall that if v is a finite place of F, then ψ_v is trivial on $\pi_v^{-d_v}\mathcal{O}_v$ but nontrivial on $\pi_v^{-d_v-1}\mathcal{O}_v$. Furthermore, recall that we let dx be the additive Haar measure on $\mathbb{A}_F$ self-dual with respect to ψ and that

$$\mathrm{vol}(\mathcal{O}_{F_v}, dx_v) = q_v^{-d_v/2}.$$

In this chapter, we will specify Schwartz functions ϕ_l' for $l \in \mathbb{Z}_{\geq 0}$ such that if $\chi_\infty(z) = z^{-k}$ on $\mathbb{C}^1$, then the theta lift $\theta_{\phi_l'}(\chi\xi)$ is a Hecke eigenform of weight $|k| + 1 + 2l$. Note that by construction (Section 6.1), negative-weight Hecke eigenforms are not theta lifts since they are not supported on $\mathrm{GL}_2(F)\,\mathrm{GL}_2(\mathbb{A}_F)^+$.

Fix a place v of F. In this section, we work place by place, and drop the subscript v throughout. Let $\mathbf{W}$ be a 2-dimensional E-vector space endowed with the skew-Hermitian form

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \overline{x}_1 y_2 - \overline{x}_2 y_1$$

with respect to a fixed basis w_1, w_2 of $\mathbf{W}$. Let $\mathbf{V}$ be a 1-dimensional E-vector space endowed with the Hermitian form

$$(\alpha, \beta) = \alpha\overline{\beta}.$$

Setting $\mathbf{W}_i = \mathrm{span}_{\mathbb{C}}(w_i)$ for $i = 1, 2$, we have a decomposition $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$ of $\mathbf{W}$ into maximal isotropic subspaces, and this induces a complete polarization of $\mathbb{V}$ given by

$$\mathbb{V} = \mathbb{X}' + \mathbb{Y}', \qquad \mathbb{X}' = \mathbf{V} \otimes \mathbf{W}_1, \qquad \mathbb{Y}' = \mathbf{V} \otimes \mathbf{W}_2.$$

Fix a splitting

$$\mathbf{s}\colon \mathrm{G}(\mathrm{U}(\mathbf{V})\times\mathrm{U}(\mathbf{W}))\to\mathbb{C}^1$$

of the cocycle $z_{\mathbb{Y}'}$ with respect to the map

$$\iota\colon \mathrm{G}(\mathrm{U}(\mathbf{V})\times\mathrm{U}(\mathbf{W}))\to\mathrm{Sp}(\mathbb{V}),\qquad (h,g)\mapsto(v\otimes w\mapsto h^{-1}v\otimes wg).$$

This determines a homomorphism

$$\tilde{\iota}\colon \mathrm{G}(\mathrm{U}(\mathbf{V})\times\mathrm{U}(\mathbf{W}))\to\mathrm{Mp}(\mathbb{V})_{\mathbb{Y}'},\qquad (h,g)\mapsto(\iota(h,g),\mathbf{s}(h,g)).$$

Recall from Equation (6.11) and Lemma 5.17 that for $\phi\in\mathcal{S}(\mathbb{X}')$ and $(h,g)\in\mathrm{G}(\mathrm{U}(\mathbf{V})\times\mathrm{U}(\mathbf{W}))$,

$$\omega_\psi(h,g)\phi(x)=\xi^{-1}(h)|h|^{-1/2}(\omega_\psi(d(\nu(g)^{-1})g)\phi)(xh^{-1}).\tag{8.1}$$

One can choose a basis of $\mathbb{X}'$ and $\mathbb{Y}'$ so that

$$\iota(D(a))=\begin{pmatrix}a&&&\\&a&&\\&&a^{-1}&\\&&&a^{-1}\end{pmatrix},\quad \iota(U(a'))=\begin{pmatrix}1&&a'&\\&1&&a'\\&&1&\\&&&1\end{pmatrix},\quad \iota(W)=\begin{pmatrix}&&1&\\&&&1\\-1&&&\\&-1&&\end{pmatrix}.$$

By the computations of Section 5.6 and Equations (3.1), (3.2), and (3.3),

$$\omega_\psi(1,D(a))\varphi(x)=\xi(a)^{-1}\cdot|\det a|\cdot\varphi(xa)\tag{8.2}$$

$$\omega_\psi(1,U(a'))\varphi(x)=\psi\left(\tfrac{1}{4}\operatorname{Tr}_{E/F}(a'x\overline{x})\right)\cdot\varphi(x)\tag{8.3}$$

$$\omega_\psi(1,W)\varphi(x)=(u,-1)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi)\cdot\int_{F^2}\varphi(y)\psi\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right)\,dy\tag{8.4}$$

If v is a finite place, then recall from Section 2.6 that the conductor of π_χ is given by a simple formula in terms of the conductor of χ:

$$c(\pi_\chi)=\begin{cases}\mathrm{val}_F(4)+2c(\chi) & \text{if } E/F \text{ is unramified,}\\ 1+\mathrm{val}_F(4)+c(\chi) & \text{if } E/F \text{ is ramified,}\\ c(\chi_1)+c(\chi_2) & \text{if } E=F\oplus F \text{ and } \chi=\chi_1\otimes\chi_2.\end{cases}$$

Recall also that we have the subgroup

$$K_0'(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) : c \in \pi^N \mathcal{O}_F \right\},$$

and writing $d(\nu) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \nu \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$ for $\nu \in F^\times$, define

$$K_0(N) := \begin{cases} K_0'(N) & \text{if } F \text{ has odd residue characteristic,} \\ d(2)K_0'(N)d(1/2) & \text{if } F \text{ has even residue characteristic.} \end{cases}$$

8.1 Schwartz functions

8.1.1 Infinite places

In this section, let v be an infinite place of F.

Definition 8.1. For $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$, define

$$\phi'_{k,l}(z) := \begin{cases} {}_1F_1(-l, k+1, 4\pi z\overline{z})\overline{z}^k e^{-2\pi z\overline{z}} & \text{if } k \geq 0, \\ {}_1F_1(-l, -k+1, 4\pi z\overline{z}) z^{-k} e^{-2\pi z\overline{z}} & \text{if } k < 0, \end{cases}$$

where ${}_1F_1(a, b, t)$ is the Kummer confluent hypergeometric function for constants a, b

$${}_1F_1(a, b, t) := \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{1}{j!} t^j,$$

where

$$(a)_0 := 1, \qquad (a)_j := a(a+1)(a+2)\cdots(a+j-1),$$

denotes the rising factorial. Observe that ${}_1F_1(a, b, t)$ is entire in t so long as $b \notin \mathbb{Z}_{\leq 0}$, so that in particular, $\phi'_{k,l}$ is entire for all $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$.

Example 8.2. We give some explicit examples of ${}_1F_1(-l, |k|+1, t)$:

$$\begin{aligned} {}_1F_1(0, 2, t) &= 1 \\ {}_1F_1(-1, 2, t) &= 1 - \tfrac{1}{2}t \\ {}_1F_1(-2, 2, t) &= 1 - t + \tfrac{1}{6}t^2 \\ {}_1F_1(-3, 2, t) &= 1 - \tfrac{3}{2}t + \tfrac{1}{2}t^2 - \tfrac{1}{24}t^3 \end{aligned}$$

Also note that the Laguerre polynomial

$$p_l(t) := \sum_{j=0}^{l} \binom{l}{j} \frac{(-t)^j}{j!}$$

is the function ${}_1F_1(-l, 1, t)$. $\Diamond$

The following lemma is well known.

Lemma 8.3. *(a) The function ${}_1F_1(a, b, t)$ is a solution to the differential equation*

$$tf''(t) + (b-t)f'(t) - af(t) = 0.$$

(b) If $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(c) > 0$, then

$$\int_0^\infty t^{\alpha-1} e^{-ct} {}_1F_1(a, b, -t)\, dt = c^{-\alpha}\Gamma(\alpha) {}_2F_1\left(a, \alpha, b, -\tfrac{1}{c}\right),$$

where

$${}_2F_1(a, \alpha, b, -\tfrac{1}{c}) = \sum_{j=0}^{\infty} \frac{(a)_j (\alpha)_j}{(b)_j} \frac{1}{j!} \left(-\frac{1}{c}\right)^j.$$

Lemma 8.4. *For $\alpha \in \mathbb{C}^1$ and $r(\theta) = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in \mathrm{SO}(2)$,*

$$\omega_\psi(\alpha, r(\theta))\phi'_{k,l} = \xi(\alpha^{-1})\alpha^{-k} e^{i(|k|+1+2l)\theta} \phi'_{k,l}.$$

Proof. We follow a similar proof strategy to [X07, Proposition 2.2.5]. We compute on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. It is well known that

$$\omega_\psi(X_+)\phi = 2\pi i z\overline{z}\phi, \qquad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\omega_\psi(X_-)\phi = -\frac{1}{2\pi i} \frac{\partial}{\partial z}\left(\frac{\partial}{\partial \overline{z}}\phi\right) \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We first handle the case $k \geq 0$. For any doubly differentiable function f satisfying the differential equation

$$tf''(t) + (k+1-t)f'(t) = -lf(t),$$

we have, following from a long calculus computation,

$$\begin{aligned}
&\omega_\psi(X_+ - X_-)(f(4\pi z\overline{z})\overline{z}^k e^{-2\pi z\overline{z}}) \\
&\qquad = i\left[(k+1)f(4\pi z\overline{z}) - 2((k+1-4\pi z\overline{z})f'(4\pi z\overline{z}) + 4\pi z\overline{z}f''(4\pi z\overline{z}))\right]\overline{z}^k e^{-2\pi z\overline{z}} \\
&\qquad = i(k+1+2l)f(4\pi z\overline{z})\overline{z}^k e^{-2\pi z\overline{z}}.
\end{aligned}$$

By Lemma 8.3(a), ${}_1F_1(-l, k+1, t)$ is such an $f(t)$ and hence the desired conclusion follows.

Now assume $k < 0$. For any doubly differentiable function f satisfying the differential equation

$$tf''(t) + (-k+1-t)f'(t) = -lf(t),$$

we have

$$\begin{aligned}
&\omega_\psi(X_+ - X_-)(f(4\pi z\overline{z})z^{-k} e^{-2\pi z\overline{z}}) \\
&\qquad = i\left[(-k+1)f(4\pi z\overline{z}) - 2((-k+1-4\pi z\overline{z})f'(4\pi z\overline{z}) + 4\pi z\overline{z}f''(4\pi z\overline{z}))\right]z^{-k} e^{-2\pi z\overline{z}} \\
&\qquad = i(-k+1+2l)f(4\pi z\overline{z})z^{-k} e^{-2\pi z\overline{z}}.
\end{aligned}$$

By Lemma 8.3(a), ${}_1F_1(-l, -k+1, t)$ is such an $f(t)$, and so the desired conclusion follows.

Finally, it is easy to see that

$$\omega_\psi(\alpha, 1)\phi'_{k,l} = \xi(\alpha^{-1})\alpha^{-k}\phi'_{k,l},$$

and it follows that

$$\omega_\psi(\alpha, r(\theta))\phi'_{k,l} = \xi(\alpha^{-1})\alpha^{-k}e^{-(|k|+1+2l)\theta}\phi'_{k,l}. \qquad \square$$

8.1.2 Finite nonsplit places

In this section, let v be a finite nonsplit place of F lying under a single prime w of E. Then E_w is a field and E_w/F_v is either unramified or ramified. Assume that E_w, F_v have odd residue characteristic. We drop the subscripts w and v throughout this section.

Definition 8.5. Define

$$\phi'(x) := \begin{cases} \mathbb{1}_{\mathcal{O}_E}(x) & \text{if } \chi \text{ is unramified,} \\ \chi(x)\mathbb{1}_{\mathcal{O}_E^\times}(x) & \text{otherwise.} \end{cases}$$

Lemma 8.6. *Let ψ' be an unramified nontrivial additive character of F. For $h \in \mathcal{O}_E^\times$ and*

$g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0 := K_0(c(\rho_\chi))$ *such that* $\mathrm{Nm}(h) = \det(g)$*, we have*

$$\omega_{\psi'}(h, g)\phi' = (\chi\xi)^{-1}(h) \cdot (\chi\epsilon_{E/F})(a) \cdot \phi'.$$

Proof. By Equation (8.1),

$$\omega_{\psi'}(h, d(\mathrm{Nm}(h)))\phi'(x) = \xi^{-1}(h) \cdot |h|^{1/2} \cdot \phi'(xh^{-1}) = (\xi\chi)^{-1}(h)\phi'(x).$$

It remains to show that for any $g \in K_0 \cap \mathrm{SL}_2(\mathcal{O}_F)$,

$$\omega_{\psi'}(1, g)\phi'(x) = (\chi\epsilon_{E/F})(a) \cdot \phi'(x). \tag{8.5}$$

We divide the calculation into two cases. Note that the Fourier transform $\omega_{\psi'}(1, W)$ is given by integrating against the additive Haar measure dx' on $\mathbb{A}_F$ that is self-dual with respect to ψ' and that in this case, $\mathrm{vol}(\mathcal{O}_F, dx') = 1$.

Case: χ unramified

Assume that E/F is unramified with odd residue characteristic so that $K_0 = \mathrm{GL}_2(\mathcal{O}_F)$. It is well known that $\mathrm{SL}_2(\mathcal{O}_F)$ is generated by the matrices

$$D(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad U(a') := \begin{pmatrix} 1 & a' \\ 0 & 1 \end{pmatrix}, \qquad W := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for $a \in \mathcal{O}_F^\times$ and $a' \in \mathcal{O}_F$. Hence it is sufficient to verify Equation (8.5) for these elements. By equation (8.2), we have

$$\omega_{\psi'}(1, D(a))\phi'(x) = \xi(a)^{-1} \cdot \phi'(xa) = (\chi \cdot \epsilon_{E/F})(a) \cdot \phi'(x),$$

where in the last equality we use the fact that χ is unramified by assumption and $\xi|_{F^\times} = \epsilon_{E/F}$. By Equation (8.3), we have

$$\omega_{\psi'}(1, U(a'))\phi'(x) = \psi'\left(\tfrac{1}{2}a\,\mathrm{Nm}(x)\right) \cdot \phi'(x) = \phi'(x),$$

since by assumption ψ' is trivial on $\mathcal{O}_F$ and F has odd residue characteristic. By Lemma 5.19 and Equation (8.4), we have

$$\omega_{\psi'}(1, W)\phi'(x) = \int_{F^2} \phi'(y)\psi'(xy^\intercal)\,dy' = \int_{\mathcal{O}_E} \psi'(xy^\intercal)\,dy' = \mathbb{1}_{\mathcal{O}_E}(x) = \phi'(x),$$

where we use the assumption that ψ' is trivial on $\mathcal{O}_F$ and nontrivial on $\pi_F^{-1}\mathcal{O}_F$. This verifies Equation (8.5) in the case that $K_0 = \mathrm{GL}_2(\mathcal{O}_F)$.

Assume that E/F is ramified with odd residue characteristic. Now $K_0 \cap \mathrm{SL}_2(\mathcal{O}_F)$ is generated by the matrices

$$D(a), \qquad U(a'), \qquad \text{and} \qquad V(b) := \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = D(-1)WU(-b)W,$$

for $a \in \mathcal{O}_F^\times$, $a' \in \mathcal{O}_F$, and $b \in \pi\mathcal{O}_F$. By Equations (8.2) and (8.3), it is easy to see that

$$\omega_{\psi'}(1, D(a))\phi'(x) = (\chi\epsilon_{E/F})(a) \cdot \phi'(x), \tag{8.6}$$

$$\omega_{\psi'}(1, U(a'))\phi'(x) = \psi\left(\tfrac{1}{4}\operatorname{Tr}_{E/F}(a'x\overline{x})\right) \cdot \mathbb{1}_{\mathcal{O}_E}(x) = \phi'(x). \tag{8.7}$$

We now show that

$$\omega_{\psi'}(1, V(b))\phi'(x) = \phi'(x).$$

We have

$$\begin{aligned} \omega_{\psi'}(1, W)\phi'(x) &= (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right) dy' \\ &= (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \int_{\mathcal{O}_E} \psi'\left(x_1y_1 - ux_2y_2\right) dy' \\ &= (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \mathbb{1}_{\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\pi^{-1}\mathcal{O}_F}(x_2). \end{aligned}$$

Therefore for any $b \in \pi\mathcal{O}_F$,

$$\begin{aligned} \omega_{\psi'}(1, U(-b)W)\phi'(x) &= \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F} bx\overline{x}\right) \cdot \omega_{\psi'}(1, W)\phi'(x) \\ &= \psi'\left(\tfrac{1}{2}b(x_1^2 - ux_2^2)\right) \cdot (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \mathbb{1}_{\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\pi^{-1}\mathcal{O}_F}(x_2) \\ &= \omega_{\psi'}(1, W)\phi'(x). \end{aligned}$$

Hence we have

$$\omega_{\psi'}(1, V(b))\phi'(x) = \omega_{\psi'}(1, D(-1)WU(-b)W)\phi'(x) = \omega_{\psi'}(1, D(-1)W^2)\phi'(x) = \phi'(x).$$

Now assume that F has even residue characteristic. Now $K_0 \cap \mathrm{SL}_2(\mathcal{O}_F)$ is generated by the matrices

$$D(a), \qquad U(a'), \qquad \text{and} \qquad V(b) = D(-1)WU(-b)W,$$

for $a \in \mathcal{O}_F^\times$, $a' \in 2\mathcal{O}_F$, and $b \in \begin{cases} 2\mathcal{O}_F & \text{if } E/F \text{ is unramified} \\ 2\pi\mathcal{O}_F & \text{if } E/F \text{ is ramified} \end{cases}$. Again, by Equations (8.2) and (8.3),

$$\omega_{\psi'}(1, D(a))\phi'(x) = (\chi\epsilon_{E/F})(a) \cdot \phi'(x), \tag{8.8}$$

$$\omega_{\psi'}(1, U(a'))\phi'(x) = \psi\left(\tfrac{1}{4}\operatorname{Tr}_{E/F}(a'x\overline{x})\right) \cdot \mathbb{1}_{\mathcal{O}_E}(x) = \phi'(x). \tag{8.9}$$

We now show that

$$\omega_{\psi'}(1, V(b))\phi'(x) = \phi'(x).$$

We have

$$\begin{aligned}
\omega_{\psi'}(1, W)\phi'(x) &= (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right) dy' \\
&= (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \int_{\mathcal{O}_E} \psi'\left(x_1y_1 - ux_2y_2\right) dy' \\
&= \begin{cases} (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \mathbb{1}_{\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\mathcal{O}_F}(x_2) & E/F \text{ unram} \\ (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \mathbb{1}_{\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\pi^{-1}\mathcal{O}_F}(x_2) & \text{E/F ram.} \end{cases}
\end{aligned}$$

If E/F is unramified, then for $b \in 2\mathcal{O}_F$,

$$\psi'\left(\tfrac{1}{2}b(x_1^2 - ux_2^2)\right) \cdot \mathbb{1}_{\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\mathcal{O}_F}(x_2) = \mathbb{1}_{\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\mathcal{O}_F}(x_2),$$

and if E/F is ramified, then for $b \in 2\pi\mathcal{O}_F$,

$$\psi'\left(\tfrac{1}{2}b(x_1^2 - ux_2^2)\right) \cdot \mathbb{1}_{\pi^{-1}\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\mathcal{O}_F}(x_2) = \mathbb{1}_{\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{\pi^{-1}\mathcal{O}_F}(x_2).$$

Therefore for any $b \in 2\mathcal{O}_F$,

$$\omega_{\psi'}(1, U(-b)W)\phi'(x) = \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F} bx\overline{x}\right)\omega_{\psi'}(1, W)\phi'(x) = \omega_{\psi'}(1, W)\phi'(x).$$

Hence we have

$$\omega_{\psi'}(1, V(b))\phi'(x) = \omega_{\psi'}(1, D(-1)WU(-b)W)\phi'(x) = \omega_{\psi'}(1, D(-1)W^2)\phi'(x) = \phi'(x).$$

Case: χ ramified

We now assume that $c(\chi) > 0$. The calculation will depend on which of the following cases we are handling:

(i) E/F is unramified and $n = c(\chi) > 0$

(ii) E/F is ramified and $n = c(\chi) > 0$

We first treat the case when F has odd residue characteristic. The group $K_0 \cap \mathrm{SL}_2(\mathcal{O}_F)$ is generated by

$$D(a), \qquad U(a'), \qquad \text{and} \qquad V(b) := \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = D(-1)WU(-b)W,$$

for $a \in \mathcal{O}_F^\times$, $a' \in \mathcal{O}_F$, and $b \in \pi^{c(\pi_\chi)}\mathcal{O}_F$. As before, Equations (8.2) and (8.3) reduce to

$$\omega_{\psi'}(1, D(a))\phi'(x) = (\chi\epsilon_{E/F})(a) \cdot \phi'(x), \tag{8.10}$$

$$\omega_{\psi'}(1, U(a'))\phi'(x) = \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F} bx\overline{x}\right)\phi'(x) = \phi'(x). \tag{8.11}$$

It remains to show that

$$\omega_{\psi'}(1, V(b))\phi'(x) = \phi'(x)$$

for $b \in \pi^{c(\pi_\chi)}\mathcal{O}_F$. We have

$$\begin{aligned}
&\omega_{\psi'}(1, W)\phi'(x) \\
&\quad = (u, -1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \int_{\mathcal{O}_E^\times} \chi(y)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right) dy' \\
&\quad = \frac{(u, -1)_F \cdot \gamma_F(u, \frac{1}{2}\psi')}{q_E^{2n}} \cdot \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a) \int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x(\overline{a + \pi_E^n y}))\right) dy' \\
&\quad = \frac{(u, -1)_F \cdot \gamma_F(u, \frac{1}{2}\psi')}{q_E^{2n}} \cdot \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{a})\right) \int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{\pi_E^n y})\right) dy'.
\end{aligned}$$

Write $x = x_1 + \mathbf{i}x_2$ and $y = y_1 + \mathbf{i}y_2$ for $x_1, x_2, y_1, y_2 \in \mathcal{O}_F$ so that

$$x\overline{y} = (x_1y_1 - ux_2y_2) - \mathbf{i}(x_1y_2 - x_2y_1).$$

Then

$$\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{\pi_E^n y}) = \begin{cases} \pi_F^n(x_1y_1 - ux_2y_2) & \text{in Case (i)}, \\ \pi_F^{k+1}(x_1y_2 - x_2y_1) & \text{in Case (ii) with } n = 2k+1, \\ \pi_F^k(x_1y_1 - \pi_F x_2y_2) & \text{in Case (ii) with } n = 2k. \end{cases}$$

This implies that

$$\int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{\pi_E^n y})\right)\,dy = \begin{cases} \mathbb{1}_{\pi_F^{-n}\mathcal{O}_F\oplus\pi_F^{-n}\mathcal{O}_F}(x_1,x_2) & \text{in Case (i),} \\ \mathbb{1}_{\pi_F^{-(k+1)}\mathcal{O}_F\oplus\pi_F^{-(k+1)}\mathcal{O}_F}(x_1,x_2) & \text{in Case (ii) with } n=2k+1, \\ \mathbb{1}_{\pi_F^{-k}\mathcal{O}_F\oplus\pi_F^{-(k+1)}\mathcal{O}_F}(x_1,x_2) & \text{in Case (ii) with } n=2k. \end{cases}$$

Using this explicit calculation together with the fact that

$$\tfrac{1}{4}\operatorname{Tr}_{E/F}(bx\overline{x}) = \tfrac{1}{2}b(x_1^2-ux_2^2),$$

we see that for $b \in \pi_F^{c(\pi_\chi)}\mathcal{O}_F$,

$$\omega_{\psi'}(1,U(-b)W)\phi'(x) = \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F}(bx\overline{x})\right)\omega_{\psi'}(1,W)\phi'(x) = \omega_{\psi'}(1,W)\phi'(x).$$

(Observe at this point that $c(\pi_\chi)$ is the smallest integer such that $\omega_{\psi'}(1,U(-b)W)\phi' = \omega_{\psi'}(1,W)\phi'$!) We can in fact now conclude that

$$\omega_{\psi'}(1,V(b))\phi'(x) = \omega_{\psi'}(1,D(-1)WU(-b)W)\phi'(x) = \omega_{\psi'}(1,D(-1)W^2)\phi'(x) = \phi'(x).$$

It may be useful to see that one can in fact verify this by calculating directly as well. We do this now: we would like to calculate

$$\omega_{\psi'}(1,WU(-b)W)\phi'(x) = (u,-1)_F\cdot\gamma_F(u,\tfrac{1}{2}\psi')\cdot\int_{F^2}\omega_{\psi'}(1,W)\phi'(y)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right)\,dy'$$

in the three cases Case (i), Case (ii) with $n=2k+1$, and Case (ii) with $n=2k$. We record the following easy calculation for reference: for $a=a_1+a_2\mathbf{i}$,

$$\tfrac{1}{2}\operatorname{Tr}_{E/F}(y\overline{a}+x\overline{y}) = (a_1+x_1)y_1-(a_2+x_2)y_2u.$$

In Case (i), we have $q_E^n = q_F^{2n}$ and so

$$\begin{aligned} \omega_{\psi'}(1,&WU(-b)W)\phi'(x) \\ &= \frac{(u,-1)_F}{q_E^n}\cdot\int_{\pi_F^{-n}\mathcal{O}_F\oplus\pi_F^{-n}\mathcal{O}_F}\sum_{a\in\mathcal{O}_E^\times/U_E^n}\chi(a)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(y\overline{a})\right)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right)\,dy' \\ &= (u,-1)_F\cdot\sum_{a\in\mathcal{O}_E^\times/U_E^n}\chi(a)\cdot\mathbb{1}_{-a_1+\pi_F^n\mathcal{O}_F}(x_1)\cdot\mathbb{1}_{-a_2+\pi_F^n\mathcal{O}_F}(x_2) \\ &= (u,-1)_F\cdot\chi(-x)\cdot\mathbb{1}_{\mathcal{O}_E^\times}(x) = (u,-1)_F\cdot\phi'(-x). \end{aligned}$$

In Case (ii) with $n = 2k+1$, we have $q_E^n = q_F^{k+1} \cdot q_F^k$ and so

$$\begin{aligned}
\omega_{\psi'}(1,&WU(-b)W)\phi'(x) \\
&= \frac{(u,-1)_F}{q_E^n} \cdot \int_{\pi_F^{-(k+1)}\mathcal{O}_F \oplus \pi_F^{-(k+1)}\mathcal{O}_F} \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a) \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(y\overline{a})\right) \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right) \, dy' \\
&= (u,-1)_F \cdot \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a) \cdot \mathbb{1}_{-a_1 + \pi_F^{k+1}\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{-a_2 + \pi_F^k \mathcal{O}_F}(x_2) \\
&= (u,-1)_F \cdot \chi(-x) \cdot \mathbb{1}_{\mathcal{O}_E^\times}(x) = (u,-1)_F \cdot \phi'(-x).
\end{aligned}$$

In Case (ii) with $n = 2k$, we have $q_E^n = q_F^{2k}$ and so

$$\begin{aligned}
\omega_{\psi'}(1,&WU(-b)W)\phi'(x) \\
&= \frac{(u,-1)_F}{q_E^n} \cdot \int_{\pi_F^{-k}\mathcal{O}_F \oplus \pi_F^{-(k+1)}\mathcal{O}_F} \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a) \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(y\overline{a})\right) \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right) \, dy' \\
&= (u,-1)_F \cdot \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a) \cdot \mathbb{1}_{-a_1 + \pi_F^{k}\mathcal{O}_F}(x_1) \cdot \mathbb{1}_{-a_2 + \pi_F^k \mathcal{O}_F}(x_2) \\
&= (u,-1)_F \cdot \chi(-x) \cdot \mathbb{1}_{\mathcal{O}_E^\times}(x) = (u,-1)_F \cdot \phi'(-x).
\end{aligned}$$

Thus we see that in all these cases, for $b \in \pi_F^{c(\pi_\chi)}\mathcal{O}_F$,

$$\omega_{\psi'}(1, WU(-b)W)\phi'(x) = (u,-1)_F \cdot \phi'(-x).$$

It finally follows that

$$\begin{aligned}
\omega_{\psi'}(1, V(a))\phi'(x) &= \omega_{\psi'}(1, D(-1))\omega_{\psi'}(1, WU(-a)W)\phi'(x) \\
&= (u,-1)_F \cdot \omega_{\psi'}(1, D(-1))\phi'(-x) = (u,-1)_F^2 \cdot \phi'(x) = \phi'(x),
\end{aligned}$$

and this completes the proof of Equation (8.5) in the odd residue characteristic case.

It remains to show Equation (8.5) when F has even residue characteristic and χ is ramified. Again, this is very similar to the previous calculations, but we include it here in full detail for the sake of completion. The group $K_0 \cap \mathrm{SL}_2(\mathcal{O}_F)$ is generated by

$$D(a), \qquad U(a'), \qquad \text{and} \qquad V(b) := \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = D(-1)WU(-b)W,$$

for $a \in \mathcal{O}_F^\times$, $a' \in 2\mathcal{O}_F$, and $b \in 2\pi^{c(\pi_\chi)}\mathcal{O}_F$. As before, Equations (8.2) and (8.3) reduce to

$$\omega_{\psi'}(1, D(a))\phi'(x) = (\chi\epsilon_{E/F})(a) \cdot \phi'(x), \tag{8.12}$$

$$\omega_{\psi'}(1, U(a'))\phi'(x) = \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F} bx\overline{x}\right)\phi'(x) = \phi'(x). \tag{8.13}$$

It remains to show that

$$\omega_{\psi'}(1, V(b))\phi'(x) = \phi'(x)$$

for $b \in 2\pi^{c(\pi_\chi)}\mathcal{O}_F$. We have

$$\begin{aligned}
&\omega_{\psi'}(1, W)\phi'(x) \\
&\quad = (u,-1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi') \cdot \int_{\mathcal{O}_E^\times} \chi(y)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right) dy' \\
&\quad = \frac{(u,-1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi')}{q_E^{2n}} \cdot \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a) \int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x(\overline{a + \pi_E^n y}))\right) dy' \\
&\quad = \frac{(u,-1)_F \cdot \gamma_F(u, \tfrac{1}{2}\psi')}{q_E^{2n}} \cdot \sum_{a \in \mathcal{O}_E^\times / U_E^n} \chi(a)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{a})\right) \int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{\pi_E^n y})\right) dy'.
\end{aligned}$$

Write $x = x_1 + \mathbf{i}x_2$ and $y = y_1 + \mathbf{i}y_2$ for $x_1, x_2, y_1, y_2 \in \mathcal{O}_F$ so that

$$x\overline{y} = (x_1y_1 - ux_2y_2) - \mathbf{i}(x_1y_2 - x_2y_1).$$

Then

$$\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{\pi_E^n y}) = \begin{cases} \pi_F^n(x_1y_1 - ux_2y_2) & \text{in Case (i)}, \\ \pi_F^{k+1}(x_1y_2 - x_2y_1) & \text{in Case (ii) with } n = 2k+1, \\ \pi_F^k(x_1y_1 - \pi_F x_2y_2) & \text{in Case (ii) with } n = 2k. \end{cases}$$

This implies that

$$\int_{\mathcal{O}_E} \psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{\pi_E^n y})\right) dy = \begin{cases} \mathbb{1}_{\pi_F^{-n}\mathcal{O}_F \oplus \pi_F^{-n}\mathcal{O}_F}(x_1, x_2) & \text{in Case (i)}, \\ \mathbb{1}_{\pi_F^{-(k+1)}\mathcal{O}_F \oplus \pi_F^{-(k+1)}\mathcal{O}_F}(x_1, x_2) & \text{in Case (ii) with } n = 2k+1, \\ \mathbb{1}_{\pi_F^{-k}\mathcal{O}_F \oplus \pi_F^{-(k+1)}\mathcal{O}_F}(x_1, x_2) & \text{in Case (ii) with } n = 2k. \end{cases}$$

Using this explicit calculation together with the fact that

$$\tfrac{1}{4}\operatorname{Tr}_{E/F}(bx\overline{x}) = \tfrac{1}{2}b(x_1^2 - ux_2^2),$$

we see that for $b \in 2\pi_F^{c(\pi_\chi)}\mathcal{O}_F$,

$$\omega_{\psi'}(1, U(-b)W)\phi'(x) = \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F}(bx\overline{x})\right)\omega_{\psi'}(1, W)\phi'(x) = \omega_{\psi'}(1, W)\phi'(x).$$

Hence we have

$$\omega_{\psi'}(1, V(b))\phi'(x) = \omega_{\psi'}(1, D(-1)WU(-b)W)\phi'(x) = \omega_{\psi'}(1, D(-1)W^2)\phi'(x) = \phi'(x).$$

We have now finally completed the proof of Equation (8.5), and the proof of the lemma is done. □

Lemma 8.7. *For $h \in \mathcal{O}_E^\times$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $\operatorname{Nm}(h) = \det(g)$, we have*

$$\omega_\psi(h, d(\delta)^{-1} g d(\delta))\phi' = (\chi\xi)^{-1}(h) \cdot (\chi\epsilon_{E/F})(a) \cdot \phi'.$$

Proof. By construction, the additive character ψ has conductor δ. Therefore the additive character $\psi'(x) := \psi(\delta x)$ is an unramified nontrivial additive character of F. By Equation (3.4) and Lemma 8.6,

$$\omega_\psi(h, d(\delta)^{-1} g d(\delta))\phi' = \omega_{\psi'}(h, g)\phi' = (\chi\xi)^{-1}(h) \cdot (\chi\epsilon_{E/F})(a) \cdot \phi'. \qquad \square$$

8.1.3 Finite split places

In this section we let v be a finite split place of F. Then $E_v \cong F_v \oplus F_v$. We drop the subscript v throughout this section.

Definition 8.8. For a character $\chi = \chi_1 \otimes \chi_2 \colon F^\times \times F^\times \to \mathbb{C}^\times$, define

$$\phi'(x_1, x_2) := \begin{cases} \mathbb{1}_{\mathcal{O}_F}(x_1)\mathbb{1}_{\mathcal{O}_F}(x_2) & \text{if } \chi \text{ is unramified,} \\ \chi(x_1, x_2)\mathbb{1}_{\mathcal{O}_F^\times}(x_1)\mathbb{1}_{\mathcal{O}_F^\times}(x_2) & \text{otherwise.} \end{cases}$$

Lemma 8.9. *Let ψ' be an unramified nontrivial additive character of F. For $h \in \mathcal{O}_F^\times \times \mathcal{O}_F^\times$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0$ with $\operatorname{Nm}(h) = \det(g)$, we have*

$$\omega_{\psi'}(h, g)\phi' = (\chi\xi)^{-1}(h) \cdot \chi_1(a)\chi_2(a) \cdot \phi'.$$

Proof. The proof is very similar to the proof of Lemma 8.7. By Equation (8.1),

$$\omega_{\psi'}(h, d(\operatorname{Nm}(h)))\phi'(x) = \xi^{-1}(h) \cdot |h|^{1/2} \cdot \phi'(xh^{-1}) = (\chi\xi)^{-1}(h)\phi'(x).$$

It remains to show that for any $g \in K_0 \cap \mathrm{SL}_2(\mathcal{O}_F)$,

$$\omega_{\psi'}(1,g)\phi'(x) = (\chi\epsilon_{E/F})(a)\cdot\phi'(x).$$

We give the proof in the case that F has odd residue characteristic. The case when F has residue characteristic 2 is nearly identical (compare the proof of Lemma 8.7 in the odd and even residue characteristic cases).

First assume that $c(\chi)=0$ so that $K_0 = \mathrm{GL}_2(\mathcal{O}_F)$. Since $\mathrm{SL}_2(\mathcal{O}_F)$ is generated by $D(a)$, $U(a)$, and W, it is sufficient to verify the assertion for these elements. We have

$$\begin{aligned}
\omega_{\psi'}(1,D(a))\phi'(x) &= \xi(a)^{-1}\cdot\phi'(xa) = \phi'(xa) = \phi'(x),\\
\omega_{\psi'}(1,U(a))\phi'(x) &= \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F}(ax\overline{x})\right)\cdot\phi'(x) = \phi'(x),\\
\omega_{\psi'}(1,W)\phi'(x) &= \int_{F^2}\phi'(y)\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right)\,dy\\
&= \int_{\mathcal{O}_F\oplus\mathcal{O}_F}\psi'\left(\tfrac{1}{2}\operatorname{Tr}_{E/F}(x\overline{y})\right)\,dy = \mathbb{1}_{\mathcal{O}_F}(x_1)\mathbb{1}_{\mathcal{O}_F}(x_2) = \phi'(x).
\end{aligned}$$

Now assume that $c(\chi)=n>0$ and set $n_1 = c(\chi_1)$, $n_2 = c(\chi_2)$. Then $K_0\cap\mathrm{SL}_2(\mathcal{O}_F)$ is generated by $D(a)$, $U(a')$, and $V(b) = D(-1)WU(-b)W$, where $a\in\mathcal{O}^\times$, $a'\in\mathcal{O}$, and $b\in\pi_F^n\mathcal{O}_F$. We have

$$\begin{aligned}
\omega_{\psi'}(1,D(a))\phi'(x) &= \xi(a)^{-1}\cdot\phi'(xa) = \chi(x_1a,x_2a)\mathbb{1}_{\mathcal{O}_F}(x_1a)\mathbb{1}_{\mathcal{O}_F}(x_2a) = \chi_1(a)\chi_2(a)\phi'(x),\\
\omega_{\psi'}(1,U(a'))\phi'(x) &= \psi'\left(\tfrac{1}{4}\operatorname{Tr}_{E/F}(a'x\overline{x})\right)\cdot\phi'(x) = \psi'\left(\tfrac{1}{2}a'x_1x_2\right)\phi'(x).
\end{aligned}$$

We have

$$\begin{aligned}
\omega_{\psi'}(1,W)\phi'(x) &= \int_{\mathcal{O}_F^\times\oplus\mathcal{O}_F^\times}\chi(y_1,y_2)\psi'\left(\tfrac{1}{2}(x\overline{y})\right)\,dy\\
&= \int_{\mathcal{O}_F^\times\oplus\mathcal{O}_F^\times}\chi_1(y_1)\chi_2(y_2)\psi'(x_1y_2+x_2y_1)\,dy\\
&= \frac{1}{q_F^n}\cdot\sum_{a_i\in\mathcal{O}_F^\times/U_F^{n_i}}\chi_1(a_1)\chi_2(a_2)\int_{\mathcal{O}_F\oplus\mathcal{O}_F}\psi'(x_1(a_2+\pi^{n_2}y_2)+x_2(a_1+\pi^{n_1}y_1))\,dy\\
&= \frac{1}{q_F^n}\cdot\sum_{a_i\in\mathcal{O}_F^\times/U_F^{n_i}}\chi_1(a_1)\chi_2(a_2)\psi'(x_1a_2+x_2a_1)\mathbb{1}_{\pi^{-n_2}\mathcal{O}_F}(x_1)\mathbb{1}_{\pi^{-n_1}\mathcal{O}_F}(x_2).
\end{aligned}$$

If $b\in\pi_F^n\mathcal{O}_F$, then $\psi'(\frac{1}{4}\operatorname{Tr}_{E/F}(bx\overline{x})) = \psi'(\frac{1}{2}bx_1x_2) = 1$ for $x_1\in\pi_F^{-n_2}\mathcal{O}_F$ and $x_2\in\pi_F^{-n_1}\mathcal{O}_F$, so

$$\omega_{\psi'}(1,U(-b)W)\phi'(x) = \omega_{\psi'}(1,W)\phi'(x).$$

Now,

$$\begin{aligned}
&\omega_{\psi'}(1, WU(-b)W)\phi'(x)\\
&\quad= \frac{1}{q_F^n} \cdot \int_{\pi^{-n_1}\mathcal{O}_F \oplus \pi^{-n_2}\mathcal{O}_F} \sum_{a_i \in \mathcal{O}_F^\times / U_F^{n_i}} \chi_1(a_1)\chi_2(a_2)\psi'(y_1a_1 + y_2a_2)\psi'(x_1y_1 + x_2y_2)\, dy\\
&\quad= \frac{1}{q_F^n} \cdot \sum_{a_i \in \mathcal{O}_F^\times / U_F^{n_i}} \chi_1(a_1)\chi_2(a_2) \int_{y_i \in \pi^{-n_i}\mathcal{O}_F} \psi'(y_1(a_1 + x_1))\psi'(y_2(a_2 + x_2))\, dy\\
&\quad= \sum_{a_i \in \mathcal{O}_F^\times / U_F^{n_i}} \chi_1(a_1)\chi_2(a_2)\mathbb{1}_{-a_1+\pi^{n_1}\mathcal{O}_F}(x_1)\mathbb{1}_{-a_2+\pi^{n_2}\mathcal{O}_F}(x_2)\\
&\quad= \chi(x_1, x_2)\mathbb{1}_{\mathcal{O}_F^\times}(-x_1)\mathbb{1}_{\mathcal{O}_F^\times}(-x_2) = \phi'(-x),
\end{aligned}$$

and

$$\omega_{\psi'}(1, D(-1)WU(-b)W)\phi'(x) = \phi'(x). \qquad \square$$

Lemma 8.10. *For $h \in \mathcal{O}_F^\times \times \mathcal{O}_F^\times$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0$ with $\mathrm{Nm}(h) = \det(g)$, we have*

$$\omega_\psi(h, d(\delta)^{-1}gd(\delta))\phi' = (\chi\xi)^{-1}(h) \cdot \chi_1(a)\chi_2(a) \cdot \phi'.$$

Proof. By construction, the additive character ψ has conductor δ. Therefore the additive character $\psi'(x) := \psi(\delta x)$ is an unramified nontrivial additive character of F. By Equation (3.4) and Lemma 8.6,

$$\omega_\psi(h, d(\delta)^{-1}gd(\delta))\phi' = \omega_{\psi'}(h, g)\phi' = (\chi\xi)^{-1}(h) \cdot \chi_1(a)\chi_2(a) \cdot \phi'. \qquad \square$$

8.2 Local zeta integrals

In this section, we calculate the local zeta integrals $Z(\frac{1}{2}, \Phi_v, \chi_v)$ for the Siegel–Weil section $\Phi_v = \Phi_v^{\mathrm{O},\mathrm{Sp}}(\delta(\phi_v' \otimes \overline{\phi_v'}))$, where ϕ_v' is the Schwartz function chosen in Section 8.1.

8.2.1 Infinite nonsplit places

Let v be an infinite nonsplit place. We say that χ_v has infinity type (k_1, k_2) if

$$\chi_v \colon \mathbb{C}^\times \to \mathbb{C}^\times, \qquad z \mapsto z^{-k_1}\overline{z}^{-k_2}.$$

Assume that

$$\chi_v(z) = z^k \qquad \text{for } z \in \mathbb{C}^1,$$

so that either χ_v is of type $(-k+j, j)$ or $(-j, k-j)$ for some integer j. Pick an integer $l \in \mathbb{Z}_{\geq 0}$ and take

$$\phi'_v(z) := \phi'_{k,l}(z) = \begin{cases} {}_1F_1(-l, k+1, 4\pi z\overline{z})\overline{z}^k e^{-2\pi z\overline{z}} & \text{if } k \geq 0, \\ {}_1F_1(-l, -k+1, 4\pi z\overline{z}) z^{-k} e^{-2\pi z\overline{z}} & \text{if } k < 0, \end{cases}$$

Lemma 8.11. *Let v be an infinite nonsplit place. Then*

$$Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) = \mathrm{vol}(\mathbb{C}^1)\langle \phi', \phi' \rangle = \frac{(2\pi)^2}{4^{|k|+1}\pi^{|k|+1}} \cdot \frac{l!(|k|)!^2}{(l+|k|)!}.$$

Proof. By Lemma 8.4,

$$\omega_\psi(\alpha, 1)\phi'_v = \xi(\alpha^{-1})\alpha^{-k}\phi'_v.$$

Thus

$$Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) = \int_{\mathbb{C}^1} \langle \omega_\psi(g, 1)\phi'_v, \phi'_v \rangle (\chi_v \xi_v)(g)\, dg = \mathrm{vol}(\mathbb{C}^1)\langle \phi'_v, \phi'_v \rangle = \pi^{-1}\langle \phi'_v, \phi'_v \rangle.$$

We have

$$\begin{aligned} \langle \phi'_v, \phi'_v \rangle &= \int_{\mathbb{C}} {}_1F_1(-l, |k|+1, 4\pi z\overline{z})^2 \cdot (z\overline{z})^{|k|} \cdot e^{-4\pi z\overline{z}}\, dz\, d\overline{z} \\ &= \int_0^{2\pi} \int_{-\infty}^{\infty} {}_1F_1(-l, |k|+1, 4\pi r^2)^2 \cdot r^{2|k|} \cdot e^{-4\pi r^2}\, r dr\, d\theta \\ &= 2\pi \int_{-\infty}^{\infty} {}_1F_1(-l, |k|+1, 4\pi r^2)^2 \cdot r^{2|k|} \cdot e^{-4\pi r^2}\, r dr \\ &= 2\pi \int_0^{\infty} {}_1F_1(-l, |k|+1, 4\pi s)^2 \cdot s^{|k|} \cdot e^{-4\pi s}\, ds \\ &= \frac{2\pi}{(4\pi)(4\pi)^{|k|}} \int_0^{\infty} {}_1F_1(-l, |k|+1, t)^2 \cdot t^{|k|} \cdot e^{-t}\, dt \\ &= \frac{2\pi}{(4\pi)^{|k|+1}} \frac{l!(|k|)!^2}{(l+|k|)!} = \frac{2\pi}{(4\pi)^{|k|+1}} \frac{(|k|)!}{\binom{l+|k|}{|k|}}. \end{aligned}$$

□

8.2.2 Finite nonsplit places

Recall from Chapter 8 that we set

$$\phi'_v(x) = \begin{cases} \mathbb{1}_{\mathcal{O}_{E_v}}(x) & \text{if } \chi_v \text{ is unramified,} \\ \chi_v(x)\mathbb{1}_{\mathcal{O}_{E_v}^\times}(x) & \text{if } \chi_v \text{ is ramified.} \end{cases}$$

Lemma 8.12. *Let v be a finite nonsplit place. If E_v/F_v is unramified, then*

$$Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) = \begin{cases} q_v^{-d_v/2} & \text{if } \chi_v \text{ is unramified,} \\ q_v^{-d_v/2}(1 - q_v^{-2}) & \text{otherwise.} \end{cases}$$

If E_v/F_v is ramified, then

$$Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) = \begin{cases} q_v^{-1} q_v^{-d_v/2} & \text{if } \chi_v \text{ is unramified,} \\ q_v^{-1} q_v^{-d_v/2}(1 - q_v^{-1}) & \text{otherwise.} \end{cases}$$

Proof. By Lemma 8.7, for $g \in E_v^1$,

$$\omega_\psi(g, 1)\phi' = (\chi_v \xi_v)^{-1}(g) \cdot \phi'.$$

This implies that

$$\begin{aligned} Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) &= \mathrm{vol}(E_v^1, d^1 x_v^{\mathrm{Tam}}) \int_{E_v^1} \langle \omega_\psi(g, 1)\phi', \phi' \rangle (\chi \xi_{\mathbb{Y}'})_v(g)\, dg \\ &= \mathrm{vol}(E_v^1, d^1 x_v^{\mathrm{Tam}})^2 \langle \phi', \phi' \rangle \\ &= \begin{cases} \mathrm{vol}(E_v^1, d^1 x_v^{\mathrm{Tam}})^2 \,\mathrm{vol}(\mathcal{O}_{E_v}, dx_v) & \text{if } \chi_v \text{ is unramified,} \\ \mathrm{vol}(E_v^1, d^1 x_v^{\mathrm{Tam}})^2 \,\mathrm{vol}(\mathcal{O}_{E_v}^\times, dx_v) & \text{otherwise.} \end{cases} \end{aligned}$$

Since

$$\mathrm{vol}(E_v^1, d^1 x_v^{\mathrm{Tam}}) = \begin{cases} 1 & \text{if } E_v/F_v \text{ is unramified} \\ q_v^{-1/2} & \text{if } E_v/F_v \text{ is ramified} \end{cases}, \qquad \mathrm{vol}(\mathcal{O}_{E_v}, dx_v) = q_v^{-d_v/2},$$

the desired conclusion follows. □

8.2.3 Finite split places

Let v be a finite split place and write $\chi_v = \chi_{1,v} \otimes \chi_{2,v} \colon F_v^\times \times F_v^\times \to \mathbb{C}^\times$. Recall that

$$\phi'(x_1, x_2) := \begin{cases} \mathbb{1}_{\mathcal{O}_{F_v}}(x_1)\mathbb{1}_{\mathcal{O}_{F_v}}(x_2) & \text{if } \chi_v \text{ is unramified,} \\ \chi_v(x_1, x_2)\mathbb{1}_{\mathcal{O}_{F_v}^\times}(x_1)\mathbb{1}_{\mathcal{O}_{F_v}^\times}(x_2) & \text{otherwise.} \end{cases}$$

Lemma 8.13. *Let v be a finite split place and assume that χ_v is unramified. Then*

$$Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) = q_v^{-3d_v/2} \cdot \frac{L_v(1, \chi_{1,v} \otimes \chi_{2,v}^{-1}) L_v(1, \chi_{1,v}^{-1} \otimes \chi_{2,v})}{L_v(2, \varepsilon_{E/F})}.$$

Proof. In this setting, $E_v^1 = \{(a, a^{-1}) \in F_v^\times \times F_v^\times\}$. By Lemma 5.17,

$$\begin{aligned}
\omega_\psi((a, a^{-1}), 1)\phi'(x_1, x_2) &= \xi_v(a, a^{-1})^{-1} \phi'(x_1 a^{-1}, x_2 a) \\
&= \xi_v(a, a^{-1})^{-1} \mathbb{1}_{\mathcal{O}_{F_v}}(x_1 a^{-1}) \mathbb{1}_{\mathcal{O}_{F_v}}(x_2 a) \\
&= \xi_v(a, a^{-1})^{-1} \mathbb{1}_{a\mathcal{O}_{F_v}}(x_1) \mathbb{1}_{a^{-1}\mathcal{O}_{F_v}}(x_2).
\end{aligned}$$

Hence

$$\begin{aligned}
\langle \omega_\psi((a, a^{-1}), 1)\phi', \phi' \rangle &= \int_{\mathbb{X}_v'} \xi_v(a, a^{-1})^{-1} \mathbb{1}_{a\mathcal{O}_{F_v}}(x_1) \mathbb{1}_{a^{-1}\mathcal{O}_{F_v}}(x_2) \mathbb{1}_{\mathcal{O}_{F_v}}(x_1) \mathbb{1}_{\mathcal{O}_{F_v}}(x_2)\, dx_1\, dx_2 \\
&= \xi_v(a, a^{-1})^{-1} \operatorname{vol}(a\mathcal{O}_{F_v} \cap \mathcal{O}_{F_v}, dx_v) \operatorname{vol}(a^{-1}\mathcal{O}_{F_v} \cap \mathcal{O}_{F_v}, dx_v) \\
&= \xi_v(a, a^{-1})^{-1} \tfrac{1}{q_v^{|\operatorname{val}(a)|}} \operatorname{vol}(\mathcal{O}_{F_v}, dx_v)^2 = \xi_v(a, a^{-1})^{-1} \tfrac{1}{q_v^{|\operatorname{val}(a)|}} q_v^{-d_v}.
\end{aligned}$$

We therefore have, writing $\pi = \pi_v$ for a uniformizer of F_v,

$$\begin{aligned}
Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) &= \int_{F_v^\times} \langle \omega_\psi(a, a^{-1})\phi_v', \phi_v' \rangle \xi_v(a, a^{-1}) \chi_v(a, a^{-1})\, da \\
&= \sum_{n \in \mathbb{Z}} \int_{\mathcal{O}_{F_v}^\times} \langle \omega_\psi(\pi^n a, \pi^{-n} a^{-1})\phi_v', \phi_v' \rangle \xi_v(\pi^n a, \pi^{-n} a^{-1}) \chi_v(\pi^n a, \pi^{-n} a^{-1})\, da \\
&= q_v^{-3d_v/2} \sum_{n \in \mathbb{Z}} \frac{1}{\chi_v(\pi^{-n}, \pi^n) q_v^{|n|}} \\
&= q_v^{-3d_v/2} \left(\sum_{n=0}^{\infty} \frac{1}{(q_v \chi_v(\pi^{-1}, \pi))^n} + \sum_{n=1}^{\infty} \frac{1}{(q_v \chi_v(\pi, \pi^{-1}))^n} \right) \\
&= q_v^{-3d_v/2} \left(\frac{1}{1 - q_v^{-1} \chi_v(\pi, \pi^{-1})} + \frac{q_v^{-1} \chi_v(\pi^{-1}, \pi)}{1 - q_v^{-1} \chi_v(\pi^{-1}, \pi)} \right) \\
&= q_v^{-3d_v/2} \cdot \frac{1 - q_v^{-2}}{(1 - q_v^{-1} \chi_v(\pi^{-1}, \pi))(1 - q_v^{-1} \chi_v(\pi, \pi^{-1}))} \\
&= q_v^{-3d_v/2} \cdot \frac{L_v(1, \chi_{1,v} \otimes \chi_{2,v}^{-1}) L_v(1, \chi_{1,v}^{-1} \otimes \chi_{2,v})}{L_v(2, \varepsilon_{E/F})}.
\end{aligned}$$

□

Lemma 8.14. *Let v be a finite split place and assume that χ_v is ramified. Then*

$$Z_v(\tfrac{1}{2}, \Phi_v, \chi_v) = q_v^{-3d_v/2} (1 - q_v^{-1})^2.$$

Proof. We have

$$\omega_\psi((a,a^{-1}),1)\phi'(x_1,x_2) = \xi_v(a,a^{-1})^{-1}\chi_v(a,a^{-1})^{-1}\mathbb{1}_{a\mathcal{O}_{F_v}^\times}(x_1)\mathbb{1}_{a^{-1}\mathcal{O}_{F_v}^\times}(x_2).$$

Then

$$\langle\omega_\psi((a,a^{-1}),1)\phi',\phi'\rangle = \xi_v(a,a^{-1})^{-1}\chi_v(a,a^{-1})^{-1}\operatorname{vol}(\mathcal{O}_{F_v}^\times, dx_v)^2\mathbb{1}_{\mathcal{O}_{F_v}^\times}(a),$$

and so

$$\begin{aligned} Z_v(\tfrac{1}{2},\Phi_v,\chi_v) &= \int_{F_v^\times}\langle\omega_\psi(a,a^{-1})\phi',\phi'\rangle\xi_v(a,a^{-1})\chi_v(a,a^{-1})\,da \\ &= \operatorname{vol}(\mathcal{O}_{F_v}^\times, dx_v)^2\operatorname{vol}(\mathcal{O}_{F_v}^\times, d^1x_v^{\mathrm{Tam}}) \\ &= q_v^{-3d_v/2}(1-q_v^{-1})^2. \end{aligned}$$

□

CHAPTER 9

An explicit Rallis inner product formula

In this chapter, we calculate the Rallis inner product formula explicitly for the Schwartz functions chosen in Chapter 8.

Let F be a totally real number field and let E/F be a CM extension. Let $\eta_1, \ldots, \eta_n$ be the real embeddings of F. Let $\chi\colon E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ be a Hecke character of infinity type $(k+j, j)$ where $k = (k_1, \ldots, k_n), j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$. Assume that $B = M_2(F)$ and let $W_0 = \mathrm{Res}_{B/E} B = W_1 + W_2$ be a decomposition of the E-space W_0 into totally isotropic subspaces. Set $\mathbb{X}' = \mathrm{Res}_{E/F}(E \otimes W_1), \mathbb{Y}' = \mathrm{Res}_{E/F}(E \otimes W_2)$, and define a Schwartz function $\phi' = \otimes_v \phi'_v \in \mathcal{S}(\mathbb{X}'(\mathbb{A}))$ as in Chapter 8:

$$\phi'_{l,v}(z) := \begin{cases} {}_1F_1(-l_i, k_i+1, 4\pi z\overline{z})\overline{z}^k e^{-2\pi z\overline{z}} & \text{if } v = \eta_i \mid \infty \text{ and } k \geq 0, \\ {}_1F_1(-l_i, -k_i+1, 4\pi z\overline{z}) z^{-k} e^{-2\pi z\overline{z}} & \text{if } v = \eta_i \mid \infty \text{ and } k < 0, \\ \mathbb{1}_{\mathcal{O}_{E_v}}(z) & \text{if } v \text{ is nonsplit and } \chi_v \text{ is unramified}, \\ \chi_v(z)\mathbb{1}_{\mathcal{O}_{E_v}^\times}(z) & \text{if } v \text{ is nonsplit and } \chi_v \text{ is ramified}, \\ \mathbb{1}_{\mathcal{O}_{F_v}}(z_1)\mathbb{1}_{\mathcal{O}_{F_v}}(z_2) & \text{if } v \text{ splits and } \chi_v \text{ is unramified}, \\ \chi_v(z_1, z_2)^{-1}\mathbb{1}_{\mathcal{O}_{F_v}^\times}(z_1)\mathbb{1}_{\mathcal{O}_{F_v}^\times}(z_2) & \text{if } v \text{ splits and } \chi_v \text{ is ramified}. \end{cases}$$

Define

$$\Sigma_\chi := \{v : \chi_v \text{ is unramified}\},$$
$$\Sigma_{\widetilde{\chi}} := \{v : \widetilde{\chi}_v \text{ is unramified}\}.$$

For each place v of F, define

$$C_v := \begin{cases} \frac{(2\pi)^2}{4^{|k_i|+1}\pi^{|k_i|+1}} \cdot \frac{l_i!(|k_i|)!^2}{(l_i+|k_i|)!} & \text{if } v = \eta_i \mid \infty \\ q_v^{-d_v/2} & \text{if } v \notin \Sigma_\chi,\ v \notin \Sigma_{\widetilde{\chi}},\ v \text{ unram} \\ q_v^{-d_v/2}(1-q_v^{-2}) & \text{if } v \in \Sigma_\chi,\ v \notin \Sigma_{\widetilde{\chi}},\ v \text{ unram} \\ q_v^{-d_v/2} & \text{if } v \in \Sigma_\chi,\ v \in \Sigma_{\widetilde{\chi}},\ v \text{ unram} \\ q_v^{-d_v/2}q_v^{-1}(1-q_v^{-2})^{-1}(1-\widetilde{\chi}_w(\pi_w)q_v^{-1}) & \text{if } v \notin \Sigma_\chi,\ v \notin \Sigma_{\widetilde{\chi}},\ v \text{ ram} \\ q_v^{-d_v/2}q_v^{-1}(1-q_v^{-1})(1-q_v^{-2})^{-1}(1-\widetilde{\chi}_v(\pi_v)q_v^{-1}) & \text{if } v \in \Sigma_\chi,\ v \notin \Sigma_{\widetilde{\chi}},\ v \text{ ram} \\ q_v^{-d_v/2}q_v^{-1}(1-q_v^{-1})(1-q_v^{-2})^{-1} & \text{if } v \in \Sigma_\chi,\ v \in \Sigma_{\widetilde{\chi}},\ v \text{ ram} \\ q_v^{-3d_v/2} & \text{if } v \notin \Sigma_\chi,\ v \notin \Sigma_{\widetilde{\chi}},\ v \text{ split} \\ q_v^{-3d_v/2}\frac{(1-(\chi_{1,v}\chi_{2,v}^{-1})(\pi_v)q_v^{-1})(1-(\chi_{1,v}^{-1}\chi_{2,v})(\pi_v)q_v^{-1})}{(1+q_v^{-1})} & \text{if } v \in \Sigma_\chi,\ v \notin \Sigma_{\widetilde{\chi}},\ v \text{ split} \\ q_v^{-3d_v/2}(1-q_v^{-1})(1+q_v^{-1})^{-1} & \text{if } v \in \Sigma_\chi,\ v \in \Sigma_{\widetilde{\chi}},\ v \text{ split} \end{cases}$$

Theorem 9.1. *The Petersson inner product of the theta lift $\theta_{\phi'}(\chi\xi)$ is*

$$\langle \theta_{\phi'}(\chi\xi), \theta_{\phi'}(\chi\xi) \rangle = \frac{\rho_E}{\rho_F} \cdot \frac{L(1,\widetilde{\chi})}{\zeta(2)} \cdot \prod_v C_v,$$

where $C_v = 1$ at all but finitely many places. In particular, if χ is nontrivial on $\mathbb{A}_E^1$, then $\theta_{\phi'}(\xi\chi) \neq 0$.

Proof. We first recall that the local L-factor for a character η on a non-Archimedean local field k with fixed uniformizer π and residue field of size q is

$$L(s,\eta) = \begin{cases} (1-\eta(\pi)q^{-s})^{-1} & \text{if } \eta \text{ is unramified,} \\ 1 & \text{if } \eta \text{ is ramified.} \end{cases}$$

Now let η be a Hecke character of $E^\times$. For each place v of F, define

$$L(s,\eta) = \prod_v L_v(s,\eta_v), \quad \text{where } L_v(s,\eta_v) = \begin{cases} L(s,\eta_v) & \text{if } v \text{ is nonsplit in } E, \\ L(s,\eta_w)L(s,\eta_{\overline{w}}) & \text{if } v = w\overline{w} \text{ splits in } E. \end{cases}$$

Let q_v be the size of the residue field of F_v, let π_v be a uniformizer of F_v. If a place v of F

lies under a single place w of E, let π_w be a uniformizer of E_w. We therefore have

$$L_v(1,\widetilde{\chi}) = \begin{cases} 1 & \text{if } v \mid \infty \\ (1-q_v^{-2})^{-1} & \text{if } \widetilde{\chi} \text{ is unram and } E_v \text{ is unram} \\ 1 & \text{if } \widetilde{\chi} \text{ is ram and } E_v \text{ is unram} \\ (1-\widetilde{\chi}_v(\pi_w)q_v^{-1})^{-1} & \text{if } \widetilde{\chi} \text{ is unram and } E_v \text{ is ram} \\ 1 & \text{if } \widetilde{\chi} \text{ is ram and } E_v \text{ is ram} \\ (1-(\chi_{1,v}\chi_{2,v}^{-1})(\pi_v)q_v^{-1})^{-1}(1-(\chi_{1,v}^{-1}\chi_{2,v})(\pi_v)q_v^{-1})^{-1} & \text{if } \widetilde{\chi} \text{ is unram and } E_v \text{ is split} \\ 1 & \text{if } \widetilde{\chi} \text{ is ram and } E_v \text{ is split} \end{cases}$$

$$\zeta_v(2) = \begin{cases} 1 & \text{if } v \mid \infty \\ (1-q_v^{-2})^{-1} & \text{if } v \nmid \infty \end{cases}$$

and

$$\frac{L_v(1,\widetilde{\chi})}{\zeta_v(2)} = \begin{cases} 1 & \text{if } v \mid \infty \\ 1 & \text{if } \widetilde{\chi} \text{ is unram and } E_v \text{ is unram} \\ (1-q_v^{-2}) & \text{if } \widetilde{\chi} \text{ is ram and } E_v \text{ is unram} \\ (1-q_v^{-2})(1-(\widetilde{\chi}_w)(\pi_v)q_v^{-1})^{-1} & \text{if } \widetilde{\chi} \text{ is unram and } E_v \text{ is ram} \\ (1-q_v^{-2}) & \text{if } \widetilde{\chi} \text{ is ram and } E_v \text{ is ram} \\ \frac{(1-q_v^{-2})}{(1-(\chi_{1,v}\chi_{2,v}^{-1})(\pi_v)q_v^{-1})(1-(\chi_{1,v}^{-1}\chi_{2,v})(\pi_v)q_v^{-1})} & \text{if } \widetilde{\chi} \text{ is unram and } E_v \text{ is split} \\ (1-q_v^{-2}) & \text{if } \widetilde{\chi} \text{ is ram and } E_v \text{ is split} \end{cases}$$

Recall from the computations of Section 8.2 that

$$Z(\tfrac{1}{2},\Phi_v,\chi_v) = \begin{cases} \frac{(2\pi)^2}{4^{|k_i|+1}\pi^{|k_i|+1}} \cdot \frac{l_i!(|k_i|)!^2}{(l+|k_i|)!} & \text{if } v = \eta_i \mid \infty \\ q_v^{-d_v/2} & \text{if } \chi \text{ is unram and } E_v \text{ is unram} \\ q_v^{-d_v/2}(1-q_v^{-2}) & \text{if } \chi \text{ is ram and } E_v \text{ is unram} \\ q_v^{-1}q_v^{-d_v/2} & \text{if } \chi \text{ is unram and } E_v \text{ is ram} \\ q_v^{-1}q_v^{-d_v/2}(1-q_v^{-1}) & \text{if } \chi \text{ is ram and } E_v \text{ is ram} \\ q_v^{-3d_v/2} \cdot \frac{(1-q_v^{-2})}{(1-(\chi_{1,v}\chi_{2,v}^{-1})(\pi_v)q_v^{-1})(1-(\chi_{1,v}^{-1}\chi_{2,v})(\pi_v)q_v^{-1})} & \text{if } \chi \text{ is unram and } E_v \text{ is split} \\ q_v^{-3d_v/2}(1-q_v^{-1})^2 & \text{if } \chi \text{ is ram and } E_v \text{ is split} \end{cases}$$

Then for all places v of F,

$$Z(\tfrac{1}{2}, \Phi_v, \chi_v) = C_v \cdot \frac{L_v(1, \widetilde{\chi})}{\zeta_v(2)},$$

where C_v is as in the theorem statement. Since all but finitely many places simultaneously satisfy the conditions $d_v = 0$, $v \notin \Sigma_\chi$, $v \notin \Sigma_{\widetilde{\chi}}$, and v is split or unramified, we see that $C_v = 1$ for all but finitely many places, and the desired equation follows from the doubling method. Observe that the factor ρ_F/ρ_E comes from the fact definition of the Tamagawa measure on $\mathbb{A}_E^1$ and the local measures on E_v^1 (Section 2.1).

Finally, since $C_v \neq 0$ for all v, it follows that $\theta_{\phi'}(\chi\xi) \neq 0$ if and only if $L(1, \widetilde{\chi}) \neq 0$. But $L(1, \widetilde{\chi}) \neq 0$ if and only if χ is trivial on $\mathbb{A}_E^1$, so the final assertion holds. □

Let f_χ be the normalized newform of weight $|k|+1 = (|k_1|+1, \ldots, |k_n|+1)$ corresponding π_χ. For $l = (l_1, \ldots, l_n)$, let F_χ^l denote the automorphic form on $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to the Hilbert modular form $\delta_{|k|+1}^l f_\chi$.

Proposition 9.2. *Let ξ and ξ' be two Hecke characters of $E^\times$ whose restriction to $\mathbb{A}_E^\times$ is the quadratic character $\epsilon_{E/F}$. Then for any $\phi' \in \mathcal{S}(\mathbb{X}'(\mathbb{A}))$,*

$$\theta_{\phi'}^{\xi}(\chi\xi)(g) = \theta_{\phi'}^{\xi'}(\chi\xi')(g) \qquad \text{for all } g \in \mathrm{GL}_2(\mathbb{A}_F),$$

where $\theta_{\phi'}^{\xi}$ and $\theta_{\phi'}^{\xi'}$ denote the theta lifts correspond to the splitting characters ξ and ξ'.

Proof. Let ω_ψ^ξ and $\omega_\psi^{\xi'}$ denote the Weil representations corresponding to the splitting characters ξ and ξ'. Then by Equations (8.2)-(8.4), we have

$$\omega_\psi^\xi(1, g) = \omega_\psi^{\xi'}(1, g) \qquad \text{for all } g \in \mathrm{SL}_2(\mathbb{A}_F),$$

and by Lemma 5.17,

$$\omega_\psi^\xi(h, d(\nu(h))) = \xi'(h)\xi^{-1}(h)\omega_\psi^{\xi'}(h, d(\nu(h))).$$

The desired equality now follows by construction of the similitude theta lift (Chapter 6). □

Theorem 9.3. *If $L(1, \widetilde{\chi}) \neq 0$, we have*

$$\theta_{\phi_l'}(\chi\xi) = D_l \cdot F_\chi^l, \qquad \text{for some } D_l \neq 0.$$

Proof. First recall that by Theorem 6.17(a), the theta lift $\theta_{\phi'}(\chi\xi)$ is an automorphic form in the automorphic induction π_χ to $\mathrm{GL}_2(\mathbb{A}_F)$. If f is a Hecke eigenform of weight $|k|+1+2l$

in π_χ, then it must satisfy that for all $r(\theta) := r(\theta_1)\cdots r(\theta_n)$ with $r(\theta_j) \in \mathrm{SO}(2)$ and $k_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0 := \prod_{v\nmid\infty} K_{0,v}$ with $\det(k_0) = 1$, we have

$$f(gr(\theta)d(\mathfrak{d})^{-1}k_0 d(\mathfrak{d})) = \prod_{j=1}^{n} e^{i(|k_j|+1+2l_j)\theta_j}(\chi\epsilon_{E/F})(a)f(g) \qquad \text{for all } g \in \mathrm{GL}_2(\mathbb{A}_F). \tag{9.1}$$

By Casselman's theorem [C73, Theorem 1], the dimension of automorphic forms satisfying (9.1) must have dimension 1. Therefore to see that $\theta_{\phi'}(\chi\xi)$ is a (possibly zero!) multiple of F_χ^l, we need only see that it satisfies (9.1).

We first recall the definition of the theta lift $\theta_{\phi'}(\chi\xi)$ on $\mathrm{GL}_2(\mathbb{A}_F)$. If $g \in \mathrm{GL}_2(\mathbb{A}_F)^+ := \{g \in \mathrm{GL}_2(\mathbb{A}_F) : \det(g) \in \mathrm{Nm}(\mathbb{A}_E^\times)\}$, then for any $h \in \mathbb{A}_E^\times$ such that $\det(g) = \mathrm{Nm}(h)$,

$$\theta_{\phi'}(\chi\xi)(g) = \int_{[E^1]} \Theta(\omega_\psi(hh_1, g)\phi') \cdot (\chi\xi)(hh_1)\, dh_1.$$

We define $\theta_{\phi'}(\chi\xi)$ on $\mathrm{GL}_2(F)\,\mathrm{GL}_2(\mathbb{A}_F)^+$ by

$$\theta_{\phi'}(\chi\xi)(\gamma g) = \theta_{\phi'}(\chi\xi)(g), \qquad \text{for } \gamma \in \mathrm{GL}_2(F), g \in \mathrm{GL}_2(\mathbb{A}_F)^+.$$

Note that

$$\mathrm{GL}_2(F)\,\mathrm{GL}_2(\mathbb{A}_F)^+ = \left\{g \in \mathrm{GL}_2(\mathbb{A}_F) : \det(g) \in F^\times\,\mathrm{Nm}(\mathbb{A}_E^\times)\right\}$$

is an index-2 subgroup of $\mathrm{GL}_2(\mathbb{A}_F)$. We define $\theta_{\phi'}(\chi\xi)$ on $\mathrm{GL}_2(\mathbb{A}_F)$ by extending by 0 outside $\mathrm{GL}_2(F)\,\mathrm{GL}_2(\mathbb{A}_F)$. Define $K_0 := \prod_v K_{0,v}$, where $K_{0,v} \subset \mathrm{GL}_2(\mathcal{O}_{F_v})$ as defined in Chapter 8. Note that $K_0 \subset \mathrm{GL}_2(F)\,\mathrm{GL}_2(\mathbb{A}_F)^+$. By Lemmas 8.4, 8.7, and 8.10, for $r(\theta) = r(\theta_1)\cdots r(\theta_n)$ with $r(\theta_j) \in \mathrm{SO}(2)$ and $k_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0 \cap \mathrm{GL}_2(\mathbb{A}_F)^+$,

$$\omega_\psi(h_0, r(\theta)d(\mathfrak{d})^{-1}k_0 d(\mathfrak{d}))\phi_l' = \prod_{j=1}^{n} e^{i(|k_j|+1+2l_j)\theta_j}(\chi\xi)^{-1}(hh_0)(\chi\epsilon_{E/F})(a)\phi_l',$$

where $h_0 \in \mathbb{A}_E^\times$ is such that $\mathrm{Nm}(h_0) = \det(k_0)$. This implies that for any $g \in \mathrm{GL}_2(\mathbb{A}_F)^+$ and any $h \in \mathbb{A}_E^\times$ with $\mathrm{Nm}(h) = \det(g)$,

$$\begin{aligned}
&\theta_{\phi_l'}(\chi\xi)(gr(\theta)d(\mathfrak{d})^{-1}k_0 d(\mathfrak{d})) \\
&\quad = \int_{[E^1]} \Theta(\omega_\psi(hh_1h_0, gr(\theta)d(\mathfrak{d})^{-1}k_0d(\mathfrak{d}))\phi_l') \cdot (\chi\xi)(hh_1h_0)\, dh_1 \\
&\quad = \int_{[E^1]} \Theta(\omega_\psi(hh_1, g)\omega_\psi(h_0, r(\theta)d(\mathfrak{d})^{-1}k_0d(\mathfrak{d}))\phi_l') \cdot (\chi\xi)(hh_1h_0)\, dh_1 \\
&\quad = \prod_{j=1}^n \int_{[E^1]} \Theta(\omega_\psi(hh_1, g)\phi_l') \cdot e^{i(|k_j|+1+2l_j)\theta_j} \cdot (\chi\xi)^{-1}(h_0) \cdot (\chi\epsilon_{E/F})(a) \cdot (\chi\xi)(hh_1h_0)\, dh_1 \\
&\quad = \prod_{j=1}^n e^{i(|k_j|+1+2l_j)\theta_j}(\chi\epsilon_{E/F})(a) \cdot \int_{[E^1]} \Theta(\omega_\psi(hh_1, g)\phi_l') \cdot (\chi\xi)(hh_1)\, dh_1 \\
&\quad = \prod_{j=1}^n e^{i(|k_j|+1+2l_j)\theta_j}(\chi\epsilon_{E/F})(a) \cdot \theta_{\phi_l'}(\chi\xi)(g).
\end{aligned}$$

This shows that the theta lift $\theta_{\phi'}(\chi\xi)$ satisfies (9.1) for $g \in \mathrm{SL}_2(\mathbb{A}_F)$. Therefore

$$\theta_{\phi_l'}(\chi\xi) = D_l \cdot F_\chi^l. \qquad \square$$

Theorem 9.4. *If $F = \mathbb{Q}$ and $k \geq 0$, then $\theta_{\phi_0'}(\chi\xi)$ is an algebraic holomorphic Hecke eigenform of weight $k+1$ and level $c(\chi)$, and*

$$|D_l| \sim \pi^l.$$

Proof. We retain the notation as in Theorem 9.3. First observe that $\theta_{\phi_0'}(\chi\xi)$ is an algebraic holomorphic Hecke eigenform of weight $k+1$ and level $c(\chi)$ by Theorem 9.3. We now examine the algebraicity of D_l. Observe that if χ has infinity type $(k+j, j)$, then $\widetilde{\chi}$ has infinity type $(k, -k)$. Hence the character $\eta := \widetilde{\chi} \cdot || \cdot ||^k$ has the property that as a character on ideals,

$$\eta((a)) = a^{2k} \quad \text{for } a \equiv 1 \pmod{\mathfrak{c}},$$

where the ideal $\mathfrak{c}$ is the conductor of χ. By definition, $L(s, \widetilde{\chi}) = L(s+k, \eta)$ and hence by Shimura's algebraicity theorem [S76, Proposition 5], we then have

$$L(1, \widetilde{\chi}) = L(k+1, \eta) \sim \pi^{k+1}\Omega^{2k}.$$

To apply Shimura's algebraicity theorem [S76, Proposition 5] to the Petersson inner product $\langle F_\chi^l, F_\chi^l \rangle$, one must first translate between the inner product of the automorphic form and the inner product of the classical form. Following [IP16a, Lemmas 6.1, 6.2], we have

$$\langle F_\chi^l, F_\chi^l\rangle \sim \pi^{-1}\int_{\Gamma_1(c(\chi))\backslash\mathfrak{h}} \delta_{k+1}^l f_\chi(z)\overline{\delta_{k+1}^l f_\chi(z)}y^k\,\frac{dx\,dy}{y^2} \sim \langle \delta_{k+1}^l f_\chi, \delta_{k+1}^l f_\chi\rangle,$$

where the Petersson inner product $\langle f, g\rangle$ is normalized as in Equation (2.2). (This is the same normalization as in [S76].) By Theorem 9.1 and again applying Shimura's algebraicity theorem [S76, Proposition 5],

$$\begin{aligned}\langle \theta_{\phi_l'}(\chi\xi), \theta_{\phi_l'}(\chi\xi)\rangle &\sim \pi^{-1}\pi^{-k+1}L(1,\widetilde{\chi})\zeta(2)^{-1} \sim \pi^{-1}\pi^{-k-1}\pi^{k+1}\Omega^{2k} \sim \pi^{-1}\Omega^{2k},\\ \langle F_\chi^l, F_\chi^l\rangle &\sim \langle \delta_{k+1}^l f_\chi, \delta_{k+1}^l f_\chi\rangle \sim \pi^{-2l}\langle f_\chi, f_\chi\rangle \sim \pi^{-2l-1}\Omega^{2k},\end{aligned}$$

and therefore

$$|D_l|^2 \sim \pi^{2l}.$$

□

CHAPTER 10

An example: the canonical Hecke character for $\mathbb{Q}(\sqrt{-7})$

Let $F = \mathbb{Q}$ and let $E = \mathbb{Q}(\sqrt{-7})$. Then E has class number 1 and there is a unique *canonical character* χ_{can} in the sense of Rohrlich [Ro80]. (See page 52 of Tonghai Yang's thesis [Ya] for an exposition.) Explicitly, χ_{can} can be described as follows. First consider the character

$$\epsilon\colon \mathcal{O}_E/(\sqrt{-7}) \cong \mathbb{Z}/7\mathbb{Z} \xrightarrow{\left(\frac{\cdot}{7}\right)} \{\pm 1\}.$$

Then $\epsilon(-1) = -1$ and hence the map on principal ideals

$$P(\sqrt{-7}) = \{\alpha\mathcal{O}_E : \alpha \in E^\times \text{ is relatively prime to } 7\} \to E^\times, \qquad \alpha\mathcal{O}_E \mapsto \epsilon(\alpha)\alpha$$

is a well-defined homomorphism. Since E has class number 1, then $P(\sqrt{-7}) = I(\sqrt{-7})$, and the above defines a Hecke character of $E^\times$. It's easy to see that for any positive integer n, the character χ_{can}^n has the following properties:

(a) It has ∞-type $(n, 0)$.

(b) It has conductor $\sqrt{-7}\mathcal{O}_E$ if n is odd and conductor $\mathcal{O}_E$ if n is even.

Idelically, we have $\chi_{\mathrm{can}} = \prod_v \chi_{\mathrm{can},v}$, where

- $\chi_{\mathrm{can},\infty}(z) = z^{-1}$.
- If $l \nmid 7$ is inert, then $\chi_{\mathrm{can},l}$ is the unramified character determined by $\chi_{\mathrm{can},l}(l) = -l$.
- If $l \nmid 7$ splits, write $l = v\bar{v}$, and $\chi_{\mathrm{can},v}$ is the unramified character of $\mathbb{Q}_l^\times$ determined by $\chi_{\mathrm{can},v}(l) = v$.
- $\chi_{\mathrm{can},7}$ is a character of level 1 on $E_7^\times$, the multiplicative group of a ramified extension of $\mathbb{Q}_7$. We have $\chi_{\mathrm{can},7}(\sqrt{-7}) = \sqrt{-7}$ and $\chi_{\mathrm{can},7}(-1) = -1$.

One can normalize χ_{can} to a unitary character by multiplying by an appropriate power of the norm character $||\cdot||_{\mathbb{A}_K}$. We explicate the norm character:

- If v is a finite place, define $|\pi_v|_v := 1/\#k_v$.
- If v is a real place, define $|\cdot|_v$ to be the norm such that $|2| = 2$.
- If v is a complex place, define $|\cdot|_v$ to be the norm such that $|2| = 4$.

Now the character $\chi'_{\mathrm{can}} := \chi_{\mathrm{can}} \cdot ||\cdot||_{\mathbb{A}_K}^{1/2}$ is unitary.

- Since K is an imaginary quadratic field, the infinite place is complex, and $\chi'_\infty(z) = |z|^{1/2}/z$.
- If $l \nmid 7$ is inert, then $\chi'_l(l) = -l/\sqrt{l^2} = -1$.
- If $l \nmid 7$ is split and v is a place above l, then $\chi'_v(l) = v/\sqrt{l}$ and $\chi'_l(l) = \chi'_v(l)\chi'_{\bar{v}}(l) = v\bar{v}/l = 1$.
- We have $\chi'_7(\sqrt{-7}) = \sqrt{-7}/\sqrt{7} = \sqrt{-1}$ and $\chi'_7(-1) = -1$.

10.1 Two quaternion algebras

We'll now consider the automorphic induction $\pi_{\chi^n_{\mathrm{can}}}$ of χ^n_{can} to GL_2 and compute the local epsilon factors $\epsilon_v(\mathrm{BC}(\pi_{\chi^n_{\mathrm{can}}}) \otimes \chi^m_{\mathrm{can}})$. At $v = \infty$, this calculation depends on whether $n < m$ or $n \geq m$. At the local places, this can be calculated by specializing [T83, Section 1] to our setting. The interesting place finite place is $v = 7$.

(a) Momentarily let v be a real place of a number field F, take f to be any automorphic form of GL_2 of weight k at v and let Ω be a Hecke character of E such that $\Omega_v(z) = z^{l_1}\overline{z}^{l_2}$. Then

$$\epsilon_v(f, \Omega) \cdot \omega_v(-1) = \begin{cases} +1 & \text{if } k \leq l_1 - l_2, \\ -1 & \text{if } k > l_1 - l_2. \end{cases}$$

Since $\pi_{\chi^n_{\mathrm{can}}}$ has weight $n+1$, this implies that

$$\epsilon_\infty(\mathrm{BC}(\pi_{\chi^n_{\mathrm{can}}}) \otimes \chi^m_{\mathrm{can}}) \cdot \omega_\infty(-1) = \begin{cases} +1 & \text{if } n+1 \leq m, \\ -1 & \text{if } n+1 > m. \end{cases}$$

(b) Since $\chi_{\mathrm{can},v}$ factors through Nm for all $v \nmid 7$, the representation $\mathrm{Ind}_{\mathcal{W}_{E_v}}^{\mathcal{W}_{F_v}}(\chi_{\mathrm{can},v})$ is decomposable. By [T83, Proposition 1.6], for any Hecke character Ω, we have

$$\epsilon_v(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}}) \otimes \Omega) \cdot \omega_v(-1) = +1 \qquad \text{for all } v \nmid 7.$$

(c) First observe that $\mathrm{Res}_{\mathcal{W}_E} \mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\chi) = \chi \oplus \chi^\tau$ for any character χ of $\mathcal{W}_E$. Since base change on the GL_2 side corresponds to restriction on the Galois side, we have

$$\epsilon_7(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}}) \otimes \Omega) = \epsilon_7(\mathrm{Res}_{\mathcal{W}_E} \mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\chi) \otimes \Omega) = \epsilon_7(\chi_{\mathrm{can}}\Omega)\epsilon_7(\chi_{\mathrm{can}}^\tau\Omega),$$

where the last equality holds because local ϵ-factors change direct sums to products. By [Ya, Lemma 3.2], we have

$$\epsilon_7(\chi_{\mathrm{can}}\Omega) = -\left(\tfrac{2}{7}\right)\sqrt{-1} = \epsilon_7(\chi_{\mathrm{can}}^\tau\Omega).$$

Since $\chi_{\mathrm{can}}|_{F^\times} = \varepsilon_{E/F}$, the automorphic representation π_{can} has trivial central character and hence the above calculation shows $\epsilon_7(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}}) \otimes \Omega)\omega_7(-1) = -1$. By the above argument,

$$\epsilon_7(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}^n}) \otimes \chi_{\mathrm{can}}^m) \cdot \omega_7(-1) = \begin{cases} +1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

We can now discuss the possibilities for the quaternion algebra determined by the pair of Hecke characters χ_{can}^n and χ_{can}^m. First observe that the central character condition $\chi_{\mathrm{can}}^n\chi_{\mathrm{can}}^m\varepsilon_{E/F} = 1$ on $\mathbb{A}^\times$ implies that n and m must have different parity. We now have two cases:

(i) If n is odd, then $\epsilon_v(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}^n}) \otimes \chi_{\mathrm{can}}^m) = -1$ if and only if $v = 7$. This implies that if $L(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}^n}) \otimes \chi_{\mathrm{can}}^m, \frac{1}{2}) \neq 0$, then necessarily $n+1 > m$ so that $\epsilon_\infty(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}^n}) \otimes \chi_{\mathrm{can}}^m) = -1$ and hence

$$S_{\pi_{\chi_{\mathrm{can}}^n}, \chi_{\mathrm{can}}^m} = \{7, \infty\}.$$

(ii) If n is even, then $\epsilon_v(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}^n}) \otimes \chi_{\mathrm{can}}^m) = +1$ for all finite v. This implies that if $L(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}^n}) \otimes \chi_{\mathrm{can}}^m, \frac{1}{2}) \neq 0$, then necessarily $n+1 \leq m$ so that $\epsilon_\infty(\mathrm{BC}(\pi_{\chi_{\mathrm{can}}^n}) \otimes \chi_{\mathrm{can}}^m) = +1$ and hence

$$S_{\pi_{\chi_{\mathrm{can}}^n}, \chi_{\mathrm{can}}^m} = \varnothing.$$

Summarizing, take n, m to have opposite parity, we have the chart

	$m < n+1$ $\epsilon_\infty = -1$	$m \geq n+1$ $\epsilon_\infty = +1$
$\epsilon = +1$	n odd $\epsilon_7 = -1$ (definite)	n even $\epsilon_7 = +1$ (indefinite—in fact, split!)
$\epsilon = -1$	n even $\epsilon_7 = +1$	n odd $\epsilon_7 = -1$

The main theorem (Theorem 6.19) gives an identity between the first two boxes. As we see above, if we start with the top right box, then we are in the setting that $B = M_2(F)$ and $B' = B_{\{7,\infty\}}$. In Sections 8 and 9, we constructed a family of Schwartz functions such that their theta lifts realize all the Hecke eigenforms of positive weight. In the next section, we recall this construction.

10.2 Torus periods of a weight-$(3+2l)$ CM form

Take the special case $n = 2$. First let $m = 3$. In this case, we take $\phi'_0 := \otimes_v \phi'_{0,v}$ where

$$\phi'_{0,v}(z) = \begin{cases} {}_1F_1(0, 3, 4\pi z\overline{z}) z^2 e^{-2\pi z\overline{z}} = z^2 e^{-2\pi z\overline{z}} & \text{if } v \mid \infty, \\ \mathbb{1}_{\mathcal{O}_{F_v}}(z_1) \cdot \mathbb{1}_{\mathcal{O}_{F_v}}(z_2) & \text{if } v \nmid \infty. \end{cases}$$

Then

$$C_v = \begin{cases} \frac{(2\pi)^2}{4^3\pi^4} = \frac{1}{16\pi^2} & \text{if } v \mid \infty, \\ 1 & \text{if } v \neq 7, \\ \frac{1}{7}(1 - \frac{1}{49})^{-1}(1 - \frac{1}{7}) = \frac{1}{8} & \text{if } v = 7, \end{cases}$$

so that by Theorem 6.17(b) and Theorem 9.3, the theta lift $\theta_{\phi'_0}(\chi\xi)$ a Hecke eigenform on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ in π_χ. Furthermore, by Theorem 9.1,

$$\langle \theta_{\phi'_0}(\chi\xi), \theta_{\phi'_0}(\chi\xi) \rangle = \rho_\mathbb{Q} \cdot \rho_E^{-1} \cdot \frac{1}{8 \cdot 16 \cdot \pi^2} \cdot \frac{L(1, \widetilde{\chi})}{\zeta(2)} = \left(\frac{2\pi}{\sqrt{7} \cdot 2} \right)^{-1} \cdot \frac{1}{128\pi^2} \cdot \frac{L(1, \widetilde{\chi})}{\zeta(2)}.$$

By Theorem 6.18,

$$\int_{[E^\times]} \theta_{\phi'_0}(\chi \cdot \xi)(g) \cdot \chi^3_{\mathrm{can}}(g)\, dg = \int_{[E^\times]} \chi(g) \cdot \overline{\theta'_{\phi'_0}(\overline{\chi^3_{\mathrm{can}} \cdot \xi'^{-1}})(g)}\, dg,$$

where by Theorem 6.17(b) the theta lift $\theta'_{\phi'_0}(\overline{\chi_{\mathrm{can}}^3 \cdot \xi'^{-1}})$ is an automorphic form in $\pi^{B'}_{\chi^3_{\mathrm{can}}}$, where B' is the quaternion algebra ramified at 7 and ∞.

Now let $m = 3 + 2l$, where $l \geq 0$. We take $\phi'_l := \otimes_v \phi'_{l,v}$ where

$$\phi'_{l,v}(z) = \begin{cases} {}_1F_1(-l, 3, 4\pi z\overline{z}) z^2 e^{-2\pi z\overline{z}} & \text{if } v \mid \infty, \\ \mathbb{1}_{\mathcal{O}_{F_v}}(z_1) \cdot \mathbb{1}_{\mathcal{O}_{F_v}}(z_2) & \text{if } v \nmid \infty. \end{cases}$$

If we set $\xi = \chi_{\mathrm{can}}$,

$$C_v = \begin{cases} \frac{(2\pi)^2}{4^3\pi^4} \cdot \frac{l! \cdot 4}{(l+2)!} = \frac{1}{2(l+2)(l+1)\pi^2} & \text{if } v \mid \infty, \\ 1 & \text{if } v \neq 7, \\ \frac{1}{7}(1 - 49^{-1})^{-1}(1 - 7^{-1}) = \frac{1}{8} & \text{if } v = 7, \end{cases}$$

so that by Theorem 6.17(b) and Theorem 9.3, the theta lift $\theta_{\phi'_l}(\chi\xi)$ is a Hecke eigenform on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ in π_χ. Furthermore, again by Theorem 9.1,

$$\langle \theta_{\phi'_l}(\chi\xi), \theta_{\phi'_l}(\chi\xi) \rangle = \left(\frac{2\pi}{\sqrt{7} \cdot 2} \right)^{-1} \cdot \frac{1}{16 \cdot (l+2) \cdot (l+1) \cdot \pi^2} \cdot \frac{L(1, \widetilde{\chi})}{\zeta(2)}.$$

And as before, by Theorem 6.18,

$$\int_{[E^\times]} \theta_{\phi'_l}(\chi \cdot \xi)(g) \cdot \chi_{\mathrm{can}}^{3+2l}(g)\, dg = \int_{[E^\times]} \chi(g) \cdot \overline{\theta'_{\phi'_l}(\overline{\chi_{\mathrm{can}}^{3+2l} \cdot \xi'^{-1}})}(g)\, dg,$$

where by Theorem 6.17(b) the theta lift $\overline{\theta'_{\phi'_l}(\overline{\chi_{\mathrm{can}}^{3+2l} \cdot \xi'^{-1}})}$ is an automorphic form in $\pi^{B'}_{\chi^{3+2l}_{\mathrm{can}}}$, where B' is the quaternion algebra ramified at 7 and ∞.

Let f_χ denote the normalized newform of weight 3 in π_χ. Then by the definition of the Shimura–Maass operator δ_3^l (see Section 2.3.1) and by Shimura's algebraicity theorem [S76, Proposition 5(ii)],

$$\langle \delta_3^l f_\chi, \delta_3^l f_\chi \rangle \sim \pi^{2l-1}\Omega^4,$$

where $\sim$ denotes equality up to an algebraic number. Combining Theorem 9.1 with Shimura's algebraicity theorem [S76, Proposition 5(i)] and Euler's algebraicity theorem [Z],

$$\langle \theta_{\phi'_l}(\chi\xi), \theta_{\phi'_l}(\chi\xi) \rangle \sim \pi^{-3}\zeta(2)^{-1}L(1, \widetilde{\chi}) \sim \pi^{-3}\pi^{-2}\pi^4\Omega^4 \sim \pi^{-1}\Omega^4.$$

By Theorem 9.3 and Casselman's theorem [C73, Theorem 1],

$$\theta_{\phi'_l}(\chi\xi) = D_l \cdot F_\chi^l, \qquad \text{where } |D_l| \sim \pi^l.$$

10.3 Relation to classical theta series

Consider the theta lift $\theta_{\phi'_l}(\chi^2_{\mathrm{can}}\xi)$ on $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$. In this section we will give a direct proof of Theorem 9.4 in this setting by unfolding the integral defining the theta lift and relating this form to a classical modular form. See [H11, Chapter 4] for a detailed exposition on unfolding the theta lift from an orthogonal group to $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$.

By definition, for any $g \in \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$,

$$\theta_{\phi'_0}(\chi^2_{\mathrm{can}}\xi)(g) = \int_{[E^1]} \Theta(\omega_\psi(h,g)\phi'_0) \cdot (\chi^2_{\mathrm{can}}\xi)(h)\, dh.$$

Now, since $E = \mathbb{Q}(\sqrt{-7})$ has class number 1, we have the decomposition

$$\mathbb{A}^1_E = E^1 K_{\mathbb{A}}, \qquad \text{where } K_{\mathbb{A}} = \prod_v K_v,\ K_v = \begin{cases} E^1_v & \text{if } v \text{ is nonsplit,} \\ \mathcal{O}^\times_{F_v} & \text{if } v \text{ is split.} \end{cases}$$

Therefore

$$\theta_{\phi'_0}(\chi^2_{\mathrm{can}}\xi)(g) = \int_{K_{\mathbb{A}}} \Theta(\omega_\psi(h,g)\phi'_0) \cdot (\chi^2_{\mathrm{can}}\xi)(h)\, dh.$$

By strong approximation for SL_2, we know that

$$\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{SL}_2(\mathbb{R}) \prod_{p<\infty} K_p,$$

where $K_p = \mathrm{SL}_2(\mathbb{Z}_p)$ for all but finitely many p. In this setting, since the conductor of $\pi_{\chi^2_{\mathrm{can}}}$ is 7, we take

$$K_p = \begin{cases} \mathrm{SL}_2(\mathbb{Z}_p) & \text{if } p \neq 7, \\ \{g \in \mathrm{SL}_2(\mathbb{Z}_7) : g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod 7\} & \text{if } p = 7. \end{cases}$$

Write $g = \gamma \cdot g_\infty \cdot k$ for $\gamma \in \mathrm{SL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{SL}_2(\mathbb{R})$, and $k \in \prod_{p<\infty} K_p$. By the calculations of Chapter 8, the action of $k_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p)$ on $\phi'_{0,p}(x_1,x_2) = \mathbb{1}_{\mathbb{Z}_p}(x_1)\mathbb{1}_{\mathbb{Z}_p}(x_2)$ is

$$\omega_\psi(k_p)\phi'_{0,p}(x) = \chi^2_{\mathrm{can}}(a_p)\epsilon_{E/F}(a_p)\phi'_{0,p}(x) = \begin{cases} \phi'_{0,p}(x) & \text{if } p \neq 7, \\ \epsilon_{E/F}(a_7)\phi'_{0,7}(x) & \text{if } p = 7. \end{cases}$$

Therefore

$$\omega_\psi(h,g)\phi'_0 = \epsilon_{E/F}(a_7) \cdot (\chi^2_{\mathrm{can}}\xi)^{-1}(h) \cdot \omega_\psi(1,g_\infty)\phi'_0,$$

and the integral simplifies to

$$\epsilon_{E/F}(a_7)\int_{K_{\mathbb{A}}}\sum_{v\in\mathbb{X}'(F)}(\omega_\psi(1,g_\infty)\phi_0')(v)\,dh = \epsilon_{E/F}(a_7)\,\mathrm{vol}(K_{\mathbb{A}})\sum_{v\in\mathcal{O}_E}(\omega_\psi(1,g_\infty)\phi_{0,\infty}')(v).$$

Observe that if

$$g_\infty = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & \sqrt{y}^{-1}\end{pmatrix}$$

for $y>0$, then

$$g_\infty\cdot i = x+iy\in\mathfrak{h} := \{z\in\mathbb{C} : \Im(z)>0\}.$$

For this g_∞, we have

$$\begin{aligned}
\omega_\psi(1,g_\infty)\phi_{0,\infty}'(v) &= \omega_\psi\left(1,\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\right)\omega_\psi\left(1,\left(\begin{smallmatrix}\sqrt{y} & 0\\ 0 & \sqrt{y}^{-1}\end{smallmatrix}\right)\right)\phi_{0,\infty}'(v)\\
&= \sqrt{y}\,\omega_\psi\left(1,\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\right)\phi_{0,\infty}'(v\sqrt{y})\\
&= \sqrt{y}\,\psi(xv\overline{v})\phi_{0,\infty}'(v\sqrt{y})\\
&= \sqrt{y}\,e^{2\pi ixv\overline{v}}(v\sqrt{y})^2e^{-2\pi yv\overline{v}}\\
&= y^{3/2}v^2e^{2\pi i(x+iy)v\overline{v}}.
\end{aligned}$$

Therefore

$$\theta_{\phi_0'}(\chi_{\mathrm{can}}^2\xi)(g) = \epsilon_{E/F}(a_7)\,\mathrm{vol}(K_{\mathbb{A}})\sum_{v\in\mathcal{O}_E}y^{3/2}v^2e^{2\pi izv\overline{v}},$$

where $z=x+iy\in\mathfrak{h}$. Recall that the classical modular form associated to this automorphic form of $\mathrm{GL}_2(\mathbb{A}_F)$ is the weight-3 form

$$f_{\chi_{\mathrm{can}}^2}(z) = \mathrm{vol}(K_{\mathbb{A}})\cdot\sum_{v\in\mathcal{O}_E}v^2e^{2\pi zv\overline{v}}.$$

Now,

$$\mathrm{vol}(K_{\mathbb{A}}) = \rho_F\cdot\rho_E^{-1}\cdot 7^{-1/2}\cdot(2\pi) = \left(\frac{2\pi}{7^{1/2}\cdot 2}\right)^{-1}\cdot 7^{-1/2}\cdot(2\pi) = 2.$$

The space of holomorphic modular forms of level 7, weight 3, with nebentypus $\left(\frac{\cdot}{7}\right)$ has dimension 1 and (by SAGE!) is generated by the modular form with q-expansion

$$f(z) = q-3q^2+5q^4-7q^7-3q^8+9q^9-6q^{11}+21q^{14}-11q^{16}-27q^{18}+O(q^{20}). \tag{10.1}$$

Using the fact that $\mathcal{O}_E = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$, it is an straightforward calculation to show that

$$\sum_{v \in \mathcal{O}_E,\, v\overline{v}=n} v^2 = 2a_n,$$

where a_n is the nth Fourier coefficient of the normalized newform. Therefore

$$f_{\chi^2_{\mathrm{can}}}(z) = 4\left(\text{normalized newform of level 7, weight 3, with nebentypus } \left(\frac{\cdot}{7}\right)\right),$$

and this shows that

$$D_0 = 4.$$

We also have the following table. Write

$$\sum_{v \in \mathcal{O}_E} {}_1F_1(-l, 3, 2\pi v\overline{v}(z-\overline{z}))(v\sqrt{y})^2 e^{2\pi i z v\overline{v}} = d_l \delta^l(f),$$

where

$$\delta^l = \delta_{k+2l-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k, \qquad \delta_k := \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{k}{z-\overline{z}}\right),$$

is the Maass–Shimura operator which raises the weight of the newform f in (10.1) by $2l$. Then calculating directly,

l	0	1	2	3	4	5	6
D_l	4	$8\pi/3$	$4\pi^2/3$	$8\pi^3/15$	$8\pi^4/45$	$16\pi^5/315$	$4\pi^6/315$

That is,

$$D_l \sim \pi^l.$$

We have hence given a direct proof of (a more precise algebracity statement than) Theorem 9.4 in this setting.

Remark 10.1. By performing the unfolding of the theta lift as in this section, one can explicitly calculate the q-expansion of the classical modular form associated to $\theta_{\phi_0'}(\chi^m_{\mathrm{can}}\xi)$.

(i) If m is even, then $\theta_{\phi_0'}(\chi^m_{\mathrm{can}}\xi)$ has weight $m+1$ and level 7, and is a multiple of the automorphic form associated to the newform

$$\frac{1}{2}\sum_{v \in \mathcal{O}_E} v^m e^{2\pi z v\overline{v}}.$$

In this case, this is a classical theta series arising from the lattice $\mathcal{O}_E$, which has rank 2 over $\mathbb{Z}$.

(ii) If m is odd, then $\theta_{\phi_0'}(\chi_{\mathrm{can}}^m\xi)$ has weight $m+1$ and level 49, and is a multiple of the automorphic form associated to the newform

$$\frac{1}{2}\sum_{v\in\mathcal{O}_E\smallsetminus\sqrt{-7}\mathcal{O}_E}\epsilon(v)v^m e^{2\pi z v\overline{v}},$$

where $\epsilon\colon \mathcal{O}_E/(\sqrt{-7})\cong\mathbb{Z}/7\mathbb{Z}\to\{\pm1\}$ is the character given by $\left(\frac{\cdot}{7}\right)$. (See the beginning of this section.) $\Diamond$

10.4 Nonvanishing torus periods

Using the same unfolding argument as in the preceding section, we can show explicitly that

$$\int_{[E^1]}\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g)\cdot\chi_{\mathrm{can}}^{3+2l}(g)\,dg\neq 0.$$

As before, setting

$$K_v=\begin{cases}E_v^1 & \text{if } v \text{ is nonsplit,}\\ \mathcal{O}_{F_v}^\times & \text{if } v \text{ is split,}\end{cases}$$

we have

$$\int_{[E^1]}\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g)\cdot\chi_{\mathrm{can}}^{3+2l}(g)\,dg=\frac{\rho_F}{\rho_E}\prod_v\int_{K_v}\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g_v)\cdot\chi_{\mathrm{can}}^{3+2l}(g_v)\,dg_v,$$

where dg_v is the Tamagawa measure as in Section 2.1. Recall that χ_{can}^2 is unramified at every place v and that χ_{can} is unramified at every place $v\nmid 7$. We now proceed place-by-place:

(i) If $v\nmid\infty$ is unramified, then both χ_{can} and χ_{can}^2 are unramified. Writing $g_v=a_v+b_v\mathbf{i}$, by Lemma 8.7,

$$\begin{aligned}\int_{E_v^1}\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g_v)\cdot\chi_{\mathrm{can}}^{3+2l}(g_v)\,dg_v&=\int_{E_v^1}\chi_{\mathrm{can}}^2(a_v)\epsilon_{E/F}(a_v)\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1)\cdot\chi_{\mathrm{can}}^{3+2l}(g_v)\,dg_v\\&=\mathrm{vol}(E_v^1)\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1).\end{aligned}$$

(ii) If $v\nmid\infty$ is ramified, then $v=7$, and so χ_{can}^2 is unramified, but $\chi_{\mathrm{can}}=\epsilon_{E_v/F_v}$ has conductor 1. Writing $g_v = a_v + b_v\mathbf{i}$, by Lemma 8.7, we have

$$\begin{aligned}\int_{E_v^1} \theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g_v)\cdot\chi_{\mathrm{can}}^{3+2l}(g_v)\,dg_v &= \int_{E_v^1} \chi_{\mathrm{can}}^2(a_v)\epsilon_{E/F}(a_v)\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1)\cdot\chi_{\mathrm{can}}^{3+2l}(g_v)\,dg_v \\ &= \int_{E_v^1} \epsilon_{E/F}(a_v)\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1)\cdot\epsilon_{E/F}(g_v)\,dg_v \\ &= \mathrm{vol}(E_v^1)\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1).\end{aligned}$$

(iii) If $v \nmid \infty$ is split, then both χ_{can} and χ_{can}^2 are unramified and by Lemma 8.7, we have

$$\int_{\mathcal{O}_{F_v}^\times} \theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g_v)\cdot\chi_{\mathrm{can}}^{3+2l}(g_v)\,dg_v = \mathrm{vol}(\mathcal{O}_{F_v}^\times)\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1).$$

(iv) If $v \mid \infty$, then

$$\begin{aligned}\int_{\mathbb{C}^1} \theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g_v)\cdot\chi_{\mathrm{can}}^{3+2l}(g_v)\,dg_v &= \int_{\mathbb{C}^1} g_v^{3+2l}\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1)\cdot g_v^{-(3+2l)}\,dg_v \\ &= \mathrm{vol}(\mathbb{C}^1)\cdot\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1).\end{aligned}$$

Then we see that

$$\int_{[E^1]} \theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(g)\cdot\chi_{\mathrm{can}}^{3+2l}(g)\,dg \neq 0 \qquad \iff \qquad \theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1) \neq 0.$$

On the other hand, if $\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)(1) = 0$, then necessarily $\theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi)$ is identically zero, which contradicts Theorem 9.1. Combining the above with Theorems 6.19, we obtain:

Corollary 10.2. *Let $B' = B_{7,\infty}$ denote the definite quaternion algebra over $\mathbb{Q}$ ramified at exactly 7 and ∞. Define*

$$f_{\chi_{\mathrm{can}}^2}^{(l)} := \theta_{\phi_l'}(\chi_{\mathrm{can}}^2\xi), \qquad f_{\chi_{\mathrm{can}}^{3+2l}}^{B'} := \overline{\theta_{\phi_l'}'(\overline{\chi_{\mathrm{can}}^{3+2l}\xi'})}.$$

Then:

(a) $f_{\chi_{\mathrm{can}}^{3+2l}}^{B'}$ is an automorphic form in the Jacquet–Langlands transfer $\pi_{\chi_{\mathrm{can}}^{3+2l}}^{B'}$,

(b) there is an identity of nonzero *torus periods*

$$0 \neq \int_{[E^\times]} f_{\chi_{\mathrm{can}}^2}^{(l)}(g)\cdot\chi_{\mathrm{can}}^2(g)\,dg = \int_{[E^\times]} \chi_{\mathrm{can}}^2(g)\cdot f_{\chi_{\mathrm{can}}^{3+2l}}^{B'}(g)\,dg.$$

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