The Kashaev Equation and Related Recurrences

by

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ABSTRACT

The hexahedron recurrence was introduced by R. Kenyon and R. Pemantle in the study of the double-dimer model in statistical mechanics. It describes a relationship among certain minors of a square matrix. This recurrence is closely related to the Kashaev equation, which has its roots in the Ising model and in the study of relations among principal minors of a symmetric matrix. Certain solutions of the hexahedron recurrence restrict to solutions of the Kashaev equation. We characterize the solutions of the Kashaev equation that can be obtained by such a restriction. This characterization leads to new results about principal minors of symmetric matrices. We describe and study other recurrences whose behavior is similar to that of the Kashaev equation and hexahedron recurrence. These include equations that appear in the study of s-holomorphicity, as well as other recurrences which, like the hexahedron recurrence, can be related to cluster algebras.

CHAPTER I

Introduction

The *Kashaev equation* is a polynomial equation involving 8 numbers indexed by the vertices of a cube; this equation is invariant under the symmetries of the cube. It originally appeared in the study of the star-triangle move in the Ising model [3]; it also arises as a relation among principal minors of a symmetric matrix [4].

We say that a $\mathbb{C}$-valued array indexed by $\mathbb{Z}^3$ satisfies the Kashaev equation if for every unit cube C in $\mathbb{Z}^3$, the 8 numbers indexed by the vertices of C satisfy the Kashaev equation. The Kashaev equation is quadratic in each of its variables, so we in general have two choices in solving for one value in terms of the remaining seven. If these seven values are all positive, then both solutions are real, and the larger solution is positive. This leads to a recurrence on positive-valued arrays on $\mathbb{Z}^3$ that we call the *positive Kashaev recurrence*; it expresses the value at the "top vertex" of each unit cube in terms of the 7 values underneath it.

Our first observation is that solutions of this positive recurrence satisfy an additional algebraic constraint not implied by the Kashaev equation alone. This constraint involves the values indexed by the 27 vertices of a $2 \times 2 \times 2$ cube in $\mathbb{Z}^3$. A solution of the Kashaev equation that satisfies this constraint is called *coherent*.

The *hexahedron recurrence* is a birational recurrence satisfied by an array indexed by the vertices and (centers of) two-dimensional faces of the standard tiling of $\mathbb{R}^3$ with unit cubes. This recurrence was introduced by Kenyon and Pemantle [5] in the context of statistical mechanics as a way to count "taut double-dimer configurations" of certain graphs. It also describes a relationship among principal and "almost principal" minors of a square matrix [4].

A key observation of Kenyon and Pemantle [5] was that restricting an array satisfying the hexahedron recurrence to the vertices of the standard tiling of $\mathbb{R}^3$ with cubes (i.e., to $\mathbb{Z}^3$) yields an array satisfying the Kashaev equation. However, not all solutions of the Kashaev equation can be obtained this way. Our main result (Theorem II.22) states that, modulo some natural technical conditions, a solution of the Kashaev equation can be extended to a solution of the hexahedron recurrence if and only if it is coherent.

We then generalize this result to a certain subclass of 3-dimensional cubical complexes. We show that a suitable generalization of Theorem II.22 holds for these complexes (Proposition VIII.3 and Theorem VIII.10), but that the corresponding statement can be false for cubical complexes outside this subclass (Theorem VIII.11).

We use this generalization to study the relations among principal minors of symmetric matrices. Given a symmetric matrix M, we associate principal minors of M to the vertices of a cubical complex, so that the resulting array is a coherent solution of the Kashaev equation. Conversely, for any generic coherent solution of the Kashaev equation, there exists a symmetric matrix whose principal minors appear as the entries of the given array. This leads to Theorem IV.26, which provides a simple test for whether a 2^n-tuple of complex numbers (satisfying certain genericity conditions) arises as a collection of principal minors of an $n \times n$ symmetric matrix. An alternative criterion was given by L. Oeding [9].

Going in another direction, we develop an axiomatic setup for pairs of recurrences whose behavior is similar to that of the Kashaev equation and the hexahedron recurrence, respectively. Theorem X.24 generalizes Theorem II.22 to this class of recurrences.

Among the applications of this generalization, we study a set of equations that appear in the context of *s-holomorphicity* in discrete complex analysis. We introduce an equation (5.1), similar to the Kashaev equation for arrays indexed by $\mathbb{Z}^2$, along with equations (5.12)–(5.14), similar to the hexahedron recurrence for arrays indexed by the edges and vertices of the standard tiling of $\mathbb{R}^2$ with unit squares. The equations (5.13)–(5.14) for the edge values are independent of the values on the vertices, and can be used (with small modifications) to define s-holomorphic functions on the tiling of $\mathbb{R}^2$ with unit squares. While the equations (5.1) and (5.12)–(5.14) have been studied before (cf. [1]), our main novelty is the notion of coherence similar to that for the Kashaev equation.

As another application, we introduce additional recurrences exhibiting hexahedron-like behavior that have their origins in the theory of cluster algebras. Whereas the connections with cluster algebras are to be discussed elsewhere, the definitions of coherence for these recurrences are provided herein.

	Definitions and results	Proofs and generalizations
Kashaev equation in $\mathbb{Z}^3$	Chapter II	Chapter VII
Kashaev equation for cubical complexes	Chapters III, IV	Chapters VIII, IX
Other Kashaev-like recurrences	Chapters V, VI	Chapter X

Table 1.1: General organization of the thesis.

We next review the content of each chapter of the thesis. Chapter II introduces the basic concepts. Its main result is Theorem II.22, which has been discussed above. The results from Chapter II are proved in Chapter VII.

While Chapters II and VII are necessary for the rest of the thesis, Chapters III, IV, VIII, IX are independent of Chapters V, VI, X, and vice versa. In Chapter III, we discuss some combinatorial tools involving cubical complexes and zonotopal tilings that we use in Chapters IV, VIII, and IX. In Chapter IV, we review the background from Kenyon and Pemantle [4] on the use of the hexahedron recurrence and the Kashaev equation in the study of principal and almost principal minors. In that chapter, we also state a version of Theorem II.22 for certain cubical complexes, and then apply this result to the study of principal minors of symmetric matrices. In Chapter VIII, we extend Theorem II.22 to the setting of cubical complexes, and in the process prove some results from Chapter IV. In Chapter IX, we prove the remaining results from Chapter IV.

In Chapter V, we discuss a condition similar to the Kashaev equation that arises in the context of s-holomorphicity. In Chapter VI, we discuss some additional recurrences with behavior similar to the Kashaev equation and hexahedron recurrence, which are related to cluster algebras. Chapters V and VI can be read independently of each other. In Chapter X, we describe an axiomatic setup for equations with properties similar to those of the Kashaev equation, and prove a more general version of Theorem II.22. In the process, we prove all of the results from Chapters V–VI.

CHAPTER II

The Kashaev Equation in $\mathbb{Z}^3$

In this chapter, we introduce the Kashaev equation, the hexahedron recurrence, and the K-hexahedron equations. We then state our main results (Theorems II.22–II.23) about the Kashaev equation for arrays indexed by $\mathbb{Z}^3$.

Definition II.1. Let $z_{000}, \ldots, z_{111} \in \mathbb{C}$ be 8 numbers indexed by the vertices of a cube, as shown in Figure 2.1. We say that these 8 numbers satisfy the *Kashaev equation* if

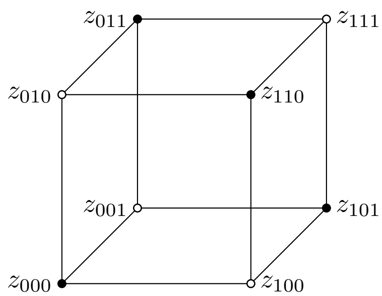
$$2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 + 4(s + t) = 0, \tag{2.1}$$

where a, b, c, d, s, t are the monomials defined in Figure 2.1. Notice that the equation (2.1) is invariant under the symmetries of the cube. Thus, reindexing the 8 values using an isomorphic labeling of the cube does not change the Kashaev equation.

Definition II.2. We say that a 3-dimensional array $\mathbf{x} \in \mathbb{C}^{\mathbb{Z}^3}$ satisfies the Kashaev equation if its components labeled by the vertices of any unit cube in $\mathbb{Z}^3$ satisfy (2.1). More formally, given a unit cube C in $\mathbb{Z}^3$, define $K^C : \mathbb{C}^{\mathbb{Z}^3} \to \mathbb{C}$ by

$$K^C(\mathbf{x}) = 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 - 4(s + t) \tag{2.2}$$

where a, b, c, d, s, t are the monomials in the components of $\mathbf{x}$ at the vertices of C,



$$a = z_{000}z_{111}, \qquad b = z_{100}z_{011}, \qquad c = z_{010}z_{101}, \qquad d = z_{001}z_{110},$$

$$s = z_{000}z_{011}z_{101}z_{110}, \qquad t = z_{111}z_{100}z_{010}z_{001}.$$

Figure 2.1: Notation used in Definition II.1. The quantities a, b, c, and d are the products of the values at opposite vertices of the cube, and s and t are the products corresponding to the two inscribed tetrahedra.

defined as in Figure 2.1. We then say that $\mathbf{x}$ satisfies the Kashaev equation if $K^C(\mathbf{x}) = 0$ for every unit cube C in $\mathbb{Z}^3$.

The Kashaev equation was originally introduced by R. Kashaev [5] in the study of the star-triangle move in the Ising model. It also appears as an identity involving principal minors of a symmetric matrix [4]; this connection is discussed in Chapter IV. Furthermore, up to changes of sign, the Kashaev equation can be interpreted as the vanishing of Cayley's hyperdeterminant of a $2 \times 2 \times 2$ hypermatrix; this connection is also discussed in Chapter IV. The Kashaev equation is also related to the theory of cluster algebras and to Descartes's formula for Apollonian circles, connections that we will explore in later work.

Remark II.3. The left-hand side of equation (2.1) is a quadratic polynomial in each of the variables z_{ijk}. Solving for z_{111} in terms of the other z_{ijk}, we obtain

$$z_{111} = \frac{A \pm 2\sqrt{D}}{z_{000}^2}, \tag{2.3}$$

where

$$\begin{aligned} A &= 2z_{100}z_{010}z_{001} + z_{000}(z_{100}z_{011} + z_{010}z_{101} + z_{001}z_{110}) \\ D &= (z_{000}z_{011} + z_{010}z_{001})(z_{000}z_{101} + z_{100}z_{001})(z_{000}z_{110} + z_{100}z_{010}). \end{aligned} \tag{2.4}$$

and $\sqrt{D}$ denotes any of the two square roots of D. Notice that if all 7 values z_{ijk} contributing to the right-hand side of (2.3) are positive, then $D > 0$, so both solutions for z_{111} in (2.3) are real; moreover, the larger of these two solutions is positive. This observation suggests the following definition.

Definition II.4. We say that a 3-dimensional array $\mathbf{x} \in (\mathbb{R}_{>0})^{\mathbb{Z}^3}$ satisfies the *positive Kashaev recurrence* if for every $(v_1, v_2, v_3) \in \mathbb{Z}^3$, we have

$$z_{111} = \frac{A + 2\sqrt{D}}{z_{000}^2}, \tag{2.5}$$

where z_{ijk} denotes the component of $\mathbf{x}$ at $(v_1 + i, v_2 + j, v_3 + k)$, for $i, j, k \in \{0, 1\}$, and we use the notation introduced in (2.4), with the conventional meaning of the square root.

Remark II.5. By Remark II.3, any solution of the the positive Kashaev recurrence is a positive real solution of the Kashaev equation. However, the converse is false; there exist arrays $\mathbf{x} \in (\mathbb{R}_{>0})^{\mathbb{Z}^3}$ satisfying the Kashaev equation which do not satisfy the positive Kashaev recurrence. (There exist positive z_{ijk} such that both solutions for z_{111} in (2.3) are positive.)

Any solution of the positive Kashaev recurrence must satisfy certain algebraic equations which are not implied by the Kashaev equation.

Definition II.6. Let $\mathbf{x} \in \mathbb{C}^{\mathbb{Z}^3}$. Let v, w be two opposite vertices in a unit cube C in $\mathbb{Z}^3$. We set

$$\begin{aligned} K_v^C(\mathbf{x}) &= \frac{1}{4}\frac{\partial K^C}{\partial x_w}(\mathbf{x}) \\ &= \frac{1}{2}(z_{111}z_{000}^2 - z_{000}(z_{100}z_{011} + z_{010}z_{101} + z_{001}z_{110})) - z_{100}z_{010}z_{001}, \end{aligned} \tag{2.6}$$

where we use a labeling of the components of $\mathbf{x}$ on the vertices of C as in Figure 2.1, with z_{000} corresponding to the component of $\mathbf{x}$ at v.

Definition II.7. Given $v \in \mathbb{Z}^3$ and $i_1, i_2, i_3 \in \{-1, 1\}$, define $C_v(i_1, i_2, i_3)$ to be the unique unit cube containing the vertices v and $v + (i_1, i_2, i_3)$.

Proposition II.8. *Suppose that* $\mathbf{x} = (x_s) \in \mathbb{C}^{\mathbb{Z}^3}$ *satisfies the Kashaev equation. Then for any* $v \in \mathbb{Z}^3$,

$$\left(\prod_{C \ni v} K_v^C(\mathbf{x})\right)^2 = \left(\prod_{S \ni v} (x_v x_{v_2} + x_{v_1} x_{v_3})\right)^2, \tag{2.7}$$

where

- *the first product is over the* 8 *unit cubes* C *incident to the vertex* v,
- *the second product is over the* 12 *unit squares* S *incident to* v *(cf. Figure 2.2), and*
- v, v_1, v_2, v_3 *are the vertices of such a unit square* S *listed in cyclic order.*

Moreover, the following strengthening of (2.7) *holds:*

$$\left(\prod_{\substack{C = C_v(i_1,i_2,i_3) \\ i_1,i_2,i_3 \in \{-1,1\} \\ i_1 i_2 i_3 = 1}} K_v^C(\mathbf{x})\right)^2 = \left(\prod_{\substack{C = C_v(i_1,i_2,i_3) \\ i_1,i_2,i_3 \in \{-1,1\} \\ i_1 i_2 i_3 = -1}} K_v^C(\mathbf{x})\right)^2 = \prod_{S \ni v} (x_v x_{v_2} + x_{v_1} x_{v_3}), \tag{2.8}$$

where the rightmost product is the same as in (2.7).

Theorem II.9. *Suppose that* $\mathbf{x} = (x_s) \in (\mathbb{R}_{>0})^{\mathbb{Z}^3}$ *satisfies the positive Kashaev recurrence. Then for any* $v \in \mathbb{Z}^3$,

$$\prod_{C \ni v} K_v^C(\mathbf{x}) = \prod_{S \ni v} (x_v x_{v_2} + x_{v_1} x_{v_3}), \tag{2.9}$$

where the notational conventions are the same as in equation (2.7).

Proposition II.8 asserts that the expressions being squared in equation (2.7) are equal up to sign; in the case of the positive Kashaev recurrence, Theorem II.9 states that the signs must match.

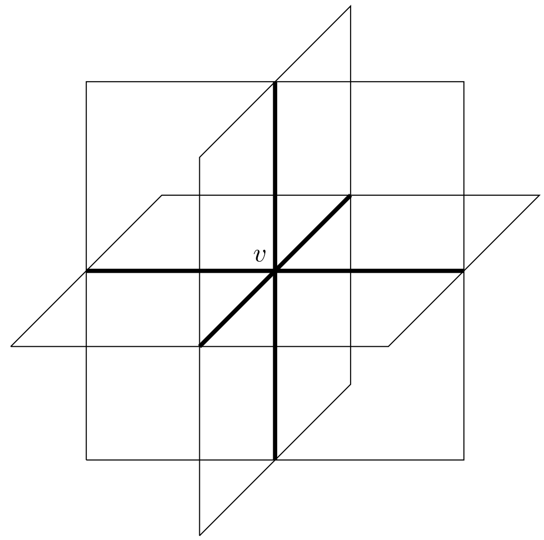


Figure 2.2: The 12 unit squares incident to $v \in \mathbb{Z}^3$.

Definition II.10. We say that a solution $\mathbf{x}$ of the Kashaev equation is *coherent* if it satisfies (2.9) for every $v \in \mathbb{Z}^3$. Equivalently, $\mathbf{x}$ is coherent if

$$\prod_{\substack{C=C_v(i_1,i_2,i_3)\\ i_1,i_2,i_3\in\{-1,1\}\\ i_1i_2i_3=1}} K_v^C(\mathbf{x}) = \prod_{\substack{C=C_v(i_1,i_2,i_3)\\ i_1,i_2,i_3\in\{-1,1\}\\ i_1i_2i_3=-1}} K_v^C(\mathbf{x}) \tag{2.10}$$

(cf. (2.8)).

By Theorem II.9, any solution of the positive Kashaev recurrence is a coherent solution of the Kashaev equation.

Remark II.11. If $\mathbf{x} = (x_s)_{s\in\mathbb{Z}^3}$ is a coherent solution of the Kashaev equation, then for any $v \in \mathbb{Z}^3$, each of the formulas (2.9) and (2.10) represent $x_{v+(1,1,1)}$ as a rational expression in the 26 values $x_{v+(\beta_1,\beta_2,\beta_3)}$ for $(\beta_1,\beta_2,\beta_3) \in \{-1,0,1\}^3 \setminus \{(1,1,1)\}$.

Coherent solutions of the Kashaev equation are closely related to (a special case of) the hexahedron recurrence, introduced and studied by Kenyon and Pemantle [5]. We next discuss this important construction, which plays a central role in this thesis.

Definition II.12. Let L be the subset of $(\frac{1}{2}\,\mathbb{Z})^3$ defined by

$$L = \{(i,j,k) \in \mathbb{R}^3 : 2i, 2j, 2k, i+j+k \in \mathbb{Z}\} \tag{2.11}$$
$$= \mathbb{Z}^3 + \big\{(0,0,0)\,,\big(0,\tfrac{1}{2},\tfrac{1}{2}\big)\,,\big(\tfrac{1}{2},0,\tfrac{1}{2}\big)\,,\big(\tfrac{1}{2},\tfrac{1}{2},0\big)\big\}.$$

Thus, L contains $\mathbb{Z}^3$, together with the centers of unit squares with vertices in $\mathbb{Z}^3$.

Kenyon and Pemantle [5] made the following important observation, which can be verified by direct computation.

Proposition II.13 ([5, Proposition 6.6])**.**

(a) Let $\mathbf{x} = (x_s) \in (\mathbb{R}_{>0})^{\mathbb{Z}^3}$ *satisfy the positive Kashaev recurrence. Extend* $\mathbf{x}$ *to an array* $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{R}_{>0})^L$ *by setting*

$$x_s^2 = x_{v_1}x_{v_3} + x_{v_2}x_{v_4}, \tag{2.12}$$

for all $s \in L - \mathbb{Z}^3$*, where* $v_1, v_2, v_3, v_4 \in \mathbb{Z}^3$ *appear in cyclic order along the unit square corresponding to* s*; see Figure 2.3. In other words, for all* $v \in \mathbb{Z}^3$*,*

$$x_{v+\left(0,\frac{1}{2},\frac{1}{2}\right)} = \sqrt{x_v x_{v+(0,1,1)} + x_{v+(0,1,0)}x_{v+(0,0,1)}}, \tag{2.13}$$

$$x_{v+\left(\frac{1}{2},0,\frac{1}{2}\right)} = \sqrt{x_v x_{v+(1,0,1)} + x_{v+(1,0,0)}x_{v+(0,0,1)}}, \tag{2.14}$$

$$x_{v+\left(\frac{1}{2},\frac{1}{2},0\right)} = \sqrt{x_v x_{v+(1,1,0)} + x_{v+(1,0,0)}x_{v+(0,1,0)}}. \tag{2.15}$$

Then for all $v \in \mathbb{Z}^3$*, we have*

$$z_{1\frac{1}{2}\frac{1}{2}} = \frac{z_{0\frac{1}{2}\frac{1}{2}}z_{\frac{1}{2}0\frac{1}{2}}z_{\frac{1}{2}\frac{1}{2}0} + z_{100}z_{010}z_{001} + z_{000}z_{100}z_{011}}{z_{000}z_{0\frac{1}{2}\frac{1}{2}}}, \tag{2.16}$$

$$z_{\frac{1}{2}1\frac{1}{2}} = \frac{z_{0\frac{1}{2}\frac{1}{2}}z_{\frac{1}{2}0\frac{1}{2}}z_{\frac{1}{2}\frac{1}{2}0} + z_{100}z_{010}z_{001} + z_{000}z_{010}z_{101}}{z_{000}z_{\frac{1}{2}0\frac{1}{2}}}, \tag{2.17}$$

$$z_{\frac{1}{2}\frac{1}{2}1} = \frac{z_{0\frac{1}{2}\frac{1}{2}}z_{\frac{1}{2}0\frac{1}{2}}z_{\frac{1}{2}\frac{1}{2}0} + z_{100}z_{010}z_{001} + z_{000}z_{001}z_{110}}{z_{000}z_{\frac{1}{2}\frac{1}{2}0}}, \tag{2.18}$$

$$z_{111} = \frac{z^2_{0\frac{1}{2}\frac{1}{2}}z^2_{\frac{1}{2}0\frac{1}{2}}z^2_{\frac{1}{2}\frac{1}{2}0} + Az_{0\frac{1}{2}\frac{1}{2}}z_{\frac{1}{2}0\frac{1}{2}}z_{\frac{1}{2}\frac{1}{2}0} + D}{z^2_{000}z_{0\frac{1}{2}\frac{1}{2}}z_{\frac{1}{2}0\frac{1}{2}}z_{\frac{1}{2}\frac{1}{2}0}} \tag{2.19}$$

where z_{ijk} denotes the component of $\tilde{\mathbf{x}}$ *at* $v+(i,j,k)$*, and* A *and* D *are given by* (2.4)*.*

(b) Conversely, suppose $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{R}_{>0})^L$ *satisfies* (2.16)–(2.19) *together with* (2.12)*. Then the restriction of* $\tilde{\mathbf{x}}$ *to* $\mathbb{Z}^3$ *satisfies the positive Kashaev recurrence.*

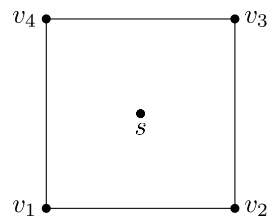


Figure 2.3: The points involved in equation (2.12).

Definition II.14 ([5])**.** We say that an array $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$ satisfies the *hexahedron recurrence* if for any $v \in \mathbb{Z}^3$, $\tilde{\mathbf{x}}$ satisfies equations (2.16)–(2.19). Notice that equations (2.16)–(2.19) involve the components of $\tilde{\mathbf{x}}$ at the 14 points in L located at the boundary of the unit cube in $\mathbb{Z}^3$ with the vertices $v + (i, j, k)$, for $i, j, k \in \{0, 1\}$, namely the 8 vertices of the cube, and the 6 centers of its faces.

The hexahedron recurrence was introduced in [5] in the context of statistical mechanics as a way to count "taut double-dimer configurations" of certain graphs. This recurrence also describes a relationship among principal and "almost principal" minors of a square matrix [4], a connection we will discuss in Chapter IV.

Remark II.15. The equations for the hexahedron recurrence, like those for the positive Kashaev recurrence above (and unlike the original Kashaev equation (2.1)), have a "preferred direction," *viz.*, the direction of increase of all three coordinates. While replacing the direction $(1, 1, 1)$ by the opposite direction $(-1, -1, -1)$ does not change these equations, using any of the six remaining directions $(\pm 1, \pm 1, \pm 1)$ yields a different recurrence. See Remark VII.4.

We now extend Proposition II.13 to complex-valued solutions of the hexahedron recurrence.

Theorem II.16.

(a) Let $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ *be a coherent solution of the Kashaev equation, with*

$$x_v x_{v+e_i+e_j} + x_{v+e_i} x_{v+e_j} \neq 0 \tag{2.20}$$

for all $v \in \mathbb{Z}^3$ *and all distinct* $i, j \in \{1, 2, 3\}$. *Then* $\mathbf{x}$ *can be extended to an array* $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{C}^*)^L$ *satisfying the hexahedron recurrence along with* (2.12).

(b) Conversely, suppose $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{C}^*)^L$ *satisfies the hexahedron recurrence along with* (2.12). *Then the restriction of* $\tilde{\mathbf{x}}$ *to* $\mathbb{Z}^3$ *is a coherent solution of the Kashaev equation and satisfies condition* (2.20).

Remark II.17. Theorem II.9 follows from Theorem II.16(b), because a solution of the positive Kashaev recurrence can be extended to a solution of the hexahedron recurrence that satisfies (2.12) (by Proposition II.13).

Remark II.18. If $\mathbf{x}$ doesn't satisfy condition (2.20), and an array $\tilde{\mathbf{x}}$ extending $\mathbf{x} \in \mathbb{C}^L$ satisfies (2.12), then at least one of the face variables for $\tilde{\mathbf{x}}$ equals 0, requiring us to divide by 0 when we apply the hexahedron recurrence. On the other hand, if $\tilde{\mathbf{x}} \in \mathbb{C}^L$ satisfies equations (2.16)–(2.19) with the denominators multiplied out (so that the denominators can equal 0), then the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^3$ doesn't necessarily satisfy the Kashaev equation.

The following statement is straightforward to check.

Proposition II.19. *Let* $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{C}^*)^L$ *be an array satisfying* (2.12), *for any* $s \in L - \mathbb{Z}^3$. *Then the following are equivalent:*

- $\tilde{\mathbf{x}}$ *satisfies the hexahedron recurrence;*

- *for any $v \in \mathbb{Z}^3$, we have*

$$z_{1\frac{1}{2}\frac{1}{2}} = \frac{z_{\frac{1}{2}0\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}0} + z_{0\frac{1}{2}\frac{1}{2}} z_{100}}{z_{000}}, \tag{2.21}$$

$$z_{\frac{1}{2}1\frac{1}{2}} = \frac{z_{0\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}0} + z_{\frac{1}{2}0\frac{1}{2}} z_{010}}{z_{000}}, \tag{2.22}$$

$$z_{\frac{1}{2}\frac{1}{2}1} = \frac{z_{0\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}0\frac{1}{2}} + z_{\frac{1}{2}\frac{1}{2}0} z_{001}}{z_{000}}, \tag{2.23}$$

$$z_{111} = \frac{A + 2z_{0\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}0\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}0}}{z_{000}^2}, \tag{2.24}$$

where, as before, z_{ijk} denotes the component of $\tilde{\mathbf{x}}$ at $v + (i, j, k)$, and A is given by (2.4)*.*

Definition II.20. Let $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$ be an array with $x_s \neq 0$ for $s \in \mathbb{Z}^3$. We say that $\tilde{\mathbf{x}}$ satisfies the *K-hexahedron equations* if $\tilde{\mathbf{x}}$ satisfies equation (2.12) for all $s \in L - \mathbb{Z}^3$, and satisfies equations (2.21)–(2.24) for all $v \in \mathbb{Z}^3$.

Remark II.21. By Proposition II.19, if $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$, i.e., $\tilde{\mathbf{x}}$ has all nonzero components, then the following are equivalent:

- $\tilde{\mathbf{x}}$ satisfies the K-hexahedron equations;
- $\tilde{\mathbf{x}}$ satisfies the hexahedron recurrence, along with equation (2.12) for $s \in L - \mathbb{Z}^3$.

We next restate Theorem II.16 (and slightly strengthen part (b) thereof) using the notion of the K-hexahedron equations.

Theorem II.22.

(a) Any coherent solution $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ of the Kashaev equation satisfying condition (2.20) *can be extended to an array $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$ satisfying the K-hexahedron equations.*

(b) Conversely, suppose that $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$ (with $x_s \neq 0$ for all $s \in \mathbb{Z}^3$) satisfies the K-hexahedron equations. Then the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^3$ is a coherent solution of the Kashaev equation.

The extension from $\mathbf{x}$ to $\tilde{\mathbf{x}}$ in Theorem II.22(a) is not unique. The theorem below clarifies the relationship between different extensions.

Theorem II.23. *Let* $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{C}^*)^L$ *be an array satisfying the K-hexahedron equations.*

(a) Let $\alpha_i, \beta_i, \gamma_i \in \{-1, 1\}$, $i \in \mathbb{Z}$. *Let* $\tilde{\mathbf{y}} = (y_s) \in (\mathbb{C}^*)^L$ *be defined by*

$$y_{(a,b,c)} = x_{(a,b,c)}, \tag{2.25}$$

$$y_{\left(a,b+\frac{1}{2},c+\frac{1}{2}\right)} = \beta_b \gamma_c x_{\left(a,b+\frac{1}{2},c+\frac{1}{2}\right)}, \tag{2.26}$$

$$y_{\left(a+\frac{1}{2},b,c+\frac{1}{2}\right)} = \alpha_a \gamma_c x_{\left(a+\frac{1}{2},b,c+\frac{1}{2}\right)}, \tag{2.27}$$

$$y_{\left(a+\frac{1}{2},b+\frac{1}{2},c\right)} = \alpha_a \beta_b x_{\left(a+\frac{1}{2},b+\frac{1}{2},c\right)}, \tag{2.28}$$

for $(a, b, c) \in \mathbb{Z}^3$. *Then* $\tilde{\mathbf{y}}$ *satisfies the K-hexahedron equations.*

(b) Conversely, suppose $\tilde{\mathbf{y}} = (y_s) \in (\mathbb{C}^*)^L$ *is an array satisfying the K-hexahedron equations that agrees with* $\tilde{\mathbf{x}}$ *on* $\mathbb{Z}^3$ *(cf.* (2.25)*). Then there exist signs* $\alpha_i, \beta_i, \gamma_i \in \{-1, 1\}$, $i \in \mathbb{Z}$, *such that* $\tilde{\mathbf{y}}$ *is given by* (2.25)–(2.28) *for* $(a, b, c) \in \mathbb{Z}^3$.

Theorems II.22–II.23 are proved in Chapter VII.

CHAPTER III

Combinatorial Preliminaries on Cubical Complexes and Zonotopes

In this chapter, we review some standard combinatorial background on cubical complexes and zonotopes. This chapter introduces some unconventional terminology which later chapters will use.

Definition III.1. A *cubical complex* is a polyhedral complex whose cells are cubes of various dimensions, see [6, Definitions 2.39 and 2.42]. We do not require that there exist an embedding of a cubical complex into Euclidean space such that every cell is a polyhedron. A cubical complex $\varkappa$ is *d-dimensional* if the dimension of the largest cube in $\varkappa$ is d; it is *pure of dimension d* if every cube of $\varkappa$ is either dimension d or a face of some d-dimensional cube in $\varkappa$. A *quadrangulation* of a polygon R in $\mathbb{R}^2$ is a realization of R as a (pure, 2-dimensional) cubical complex.

Definition III.2. Let $\varkappa$ be a cubical complex embedded (as a topological space) into a Euclidean space $\mathbb{R}^d$. A point v in $\varkappa$ is called an *interior point* of $\varkappa$ if $\varkappa$ contains a (small) open ball centered at v. (This notion does not depend on the choice of embedding for a fixed d.)

Definition III.3. A *m-dimensional zonotope* $\mathcal{Z}_{v_1,\ldots,v_\ell}$ is the Minkowski sum of line segments $\sum_{j=1}^{\ell}[0, v_j]$ for $v_1, \ldots, v_\ell \in \mathbb{R}^m$ spanning $\mathbb{R}^m$. A *cubical tiling* of $\mathcal{Z}_{v_1,\ldots,v_\ell}$ is a tiling of $\mathcal{Z}_{v_1,\ldots,v_\ell}$ with the translates of the Minkowski sums $\sum_{j\in I}[0, v_j]$ over $I \in \binom{[\ell]}{m}$ such that $\{v_j : j \in I\}$ is linearly independent. Cubical tilings of zonotopes are examples of cubical complexes.

Definition III.4. We denote by $\mathbf{P}_n$ the regular $(2n)$-gon $\mathcal{Z}_{e_1,\ldots,e_n}$ where $e_j = e^{\pi i(j-1)/n} \in \mathbb{C} \cong \mathbb{R}^2$ for $j = 1, \ldots, n$ using the standard identification between $\mathbb{C}$ and $\mathbb{R}^2$ (see Figure 3.1). We denote by v_0 the vertex of $\mathbf{P}_n$ corresponding to the origin. We define a *$\lozenge$-tiling of* $\mathbf{P}_n$ to be a cubical tiling of $\mathcal{Z}_{e_1,\ldots,e_n}$, i.e., a tiling of $\mathbf{P}_n$ with the $\binom{n}{2}$ rhombi given by the translations of the Minkowski sums $[0, e_i] + [0, e_j]$ for $1 \le i < j \le n$ (see Figure 3.2). We label the vertices of a $\lozenge$-tiling of $\mathbf{P}_n$ by subsets of $[n]$ as follows: we label a vertex v by $I \subseteq [n]$ if we can reach v from v_0 by following the edges of the tiling corresponding to the vectors e_j for $j \in I$ (see Figure 3.2).

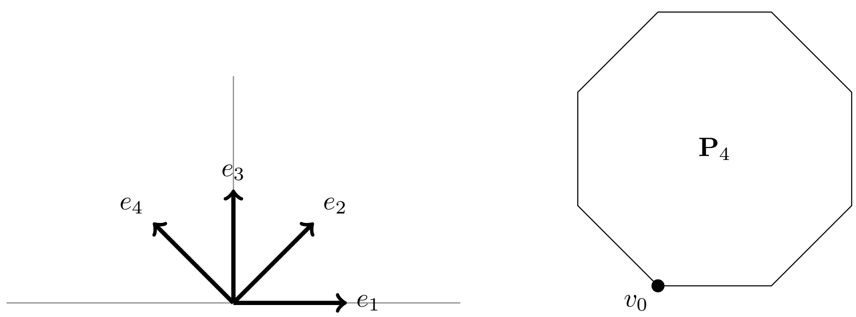


Figure 3.1: On the left, the vectors e_1, e_2, e_3, e_4 from Definition III.4 when $n = 4$. On the right, the regular 8-gon $\mathbf{P}_4$.

Definition III.5. Two quadrangulations T_1, T_2 of a polygon are connected by a *flip* if they are related by a single local move of the form pictured in Figure 3.3. Note that we can picture a flip as placing a cube on top of the hexagon where the flip occurs.

It will be helpful for us to think about quadrangulations through the dual language of divides.

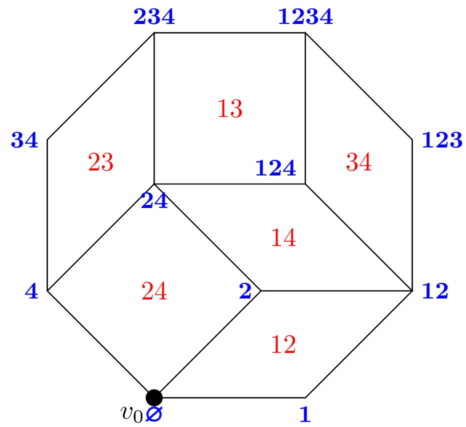


Figure 3.2: A $\Diamond$-tiling of $\mathbf{P}_4$. The vertex v_0 corresponding to the origin is labeled. In red, we label each rhombus that is a translation of the Minkowski sum $[0, e_i] \times [0, e_j]$ by ij. In blue, we label each vertex of the tiling by its corresponding subset of $[4]$.

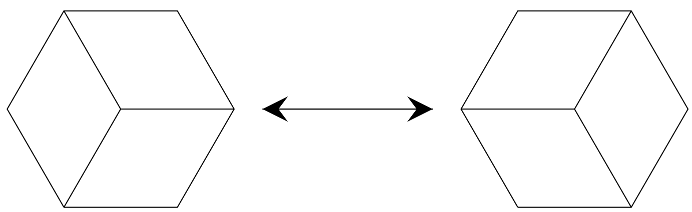


Figure 3.3: A flip.

Definition III.6. A *divide* D in a polygon R in $\mathbb{R}^2$ is an immersion of a finite set of closed intervals and circles, called *branches*, in R, such that

- the immersed circles do not intersect the boundary of R,
- the immersed intervals have pairwise distinct endpoints on the boundary of R, and are otherwise disjoint from the boundary,
- all intersections and self-intersections of the branches are transversal, and
- no three branches intersect at a point,

all considered up to isotopy. For further details, see [2, Definition 2.1]. Given a quadrangulation T of R, the divide associated to T is the divide in R such that for every tile Q in T, branches connect the 2 pairs of opposite edges in Q, and there is a single branch intersection in the interior of Q (see Figure 3.4). (All divides considered in the remainder of this thesis are associated to quadrangulations.) A *braid move* is a local transformation of divides shown in Figure 3.5. A flip in a quadrangulation corresponds to a braid move in its associated divide.

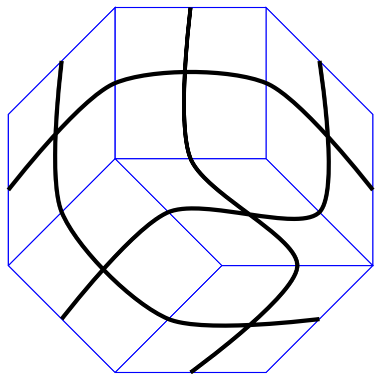


Figure 3.4: The divide (in black) associated to a quadrangulation (in blue).

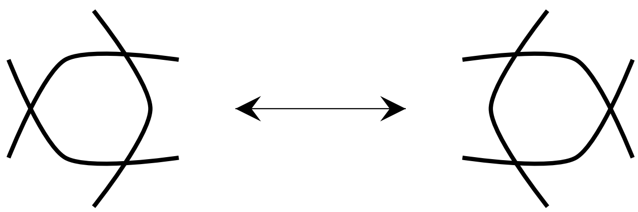


Figure 3.5: A braid move.

Definition III.7. A divide is called a *pseudoline arrangement* if all of its branches are immersed intervals with no self-intersections, and, moreover, each pair of branches intersects at most once. Note that the class of pseudoline arrangements is closed under braid moves.

The following fact is well known.

Proposition III.8. *Let D be a divide in a polygon in $\mathbb{R}^2$. Then the following are equivalent:*

- *D is a pseudoline arrangement in which every pair of branches intersects exactly once;*

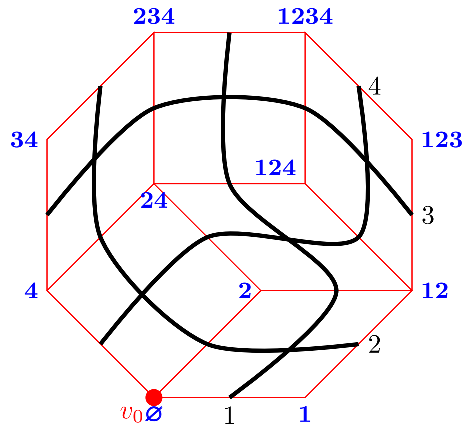


Figure 3.6: The pseudoline arrangement (in black) associated to a $\lozenge$-tiling of $\mathbf{P}_4$ (in red). The branches are labeled (in black) as described in Definition III.10. Note that the label $I \subseteq [n]$ for the vertices of T (in blue) is precisely the set of labels for the branches in between the chamber and v_0.

- *D is topologically equivalent to the divide associated to a $\lozenge$-tiling of $\mathbf{P}_n$.*

Remark III.9. Pseudoline arrangements of n branches, each pair of which intersects exactly once, are in bijection with commutation-equivalence classes of reduced words for the longest element $w_0 \in S_n$ in the symmetric group. A braid move on the pseudoline arrangement corresponds to a braid move on the reduced word.

Definition III.10. Let T be a $\lozenge$-tiling of $\mathbf{P}_n$, and let D be the pseudoline arrangement associated to T. We call the connected components of the complement of D the *chambers* of D. Note that the chambers of D correspond to the vertices of T, and the crossings of D correspond to the tiles of T. Label the branches $1, \ldots, n$ as in Figure 3.6, by starting at v_0 and traveling counterclockwise along the boundary of $\mathbf{P}_n$ (so that branch j intersects the boundary of $\mathbf{P}_n$ at the edges parallel to $e_j = e^{\pi i(j-1)/n}$). Note that the label $I \subseteq [n]$ for a vertex of T is precisely the set of labels for the branches in between the chamber and v_0.

Definition III.11. Let T be a $\lozenge$-tiling of $\mathbf{P}_n$, and let D be the pseudoline arrangement associated to T. Label the branches as in Definition III.10. Given three

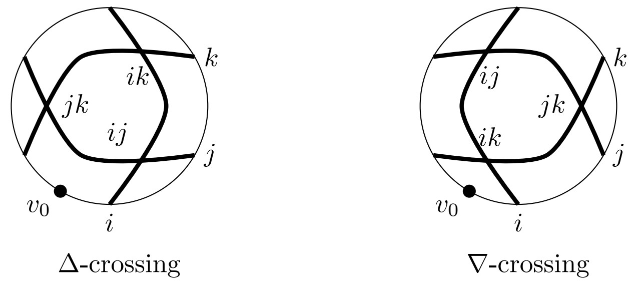


Figure 3.7: A Δ-crossing and a ∇-crossing. Note that in the Δ-crossing, the triangle formed by the 3 intersecting branches points up and away from v_0, while in the ∇-crossing, the triangle formed by the 3 intersecting branches points down and towards v_0.

branches labeled $i < j < k$, we say that i, j, k have a Δ-*crossing* if the pairs of i, j, k intersect in the following counterclockwise order along the boundary of the triangle these branches form: $(i, j), (i, k), (j, k)$. We say that i, j, k have a ∇-*crossing* if the pairs of i, j, k intersect in the following counterclockwise order along the boundary of the triangle these branches form: $(j, k), (i, k), (i, j)$. Note that i, j, k must either have a Δ-crossing or a ∇-crossing. See Figure 3.7 for pictures which make clear the reasoning behind this choice of terminology. Note that when a braid move is performed with i, j, k, the triple i, j, k switches between having a Δ-crossing and having a ∇-crossing.

Definition III.12. Define $T_{\min,n}$ to be the unique $\Diamond$-tiling of $\mathbf{P}_n$ in which every vertex is labeled by consecutive subsets $I \subseteq [n]$, and define $T_{\max,n}$ to be the unique $\Diamond$-tiling of $\mathbf{P}_n$ in which every vertex is labeled by a subset $I \subseteq [n]$ whose complement $[n] - I$ is consecutive (see Figure 3.8). Equivalently, $T_{\min,n}$ is the $\Diamond$-tiling of $\mathbf{P}_n$ in which every triple $i < j < k$ has a Δ-crossing in its associated pseudoline arrangement, while $T_{\max,n}$ is the $\Diamond$-tiling of $\mathbf{P}_n$ in which every triple $i < j < k$ has a ∇-crossing in its associated pseudoline arrangement.

Definition III.13. We say that $\mathbf{T} = (T_0, \ldots, T_\ell)$ is a *pile* of quadrangulations of a polygon if T_{i-1} and T_i are connected by a flip for $i = 1, \ldots, \ell$.

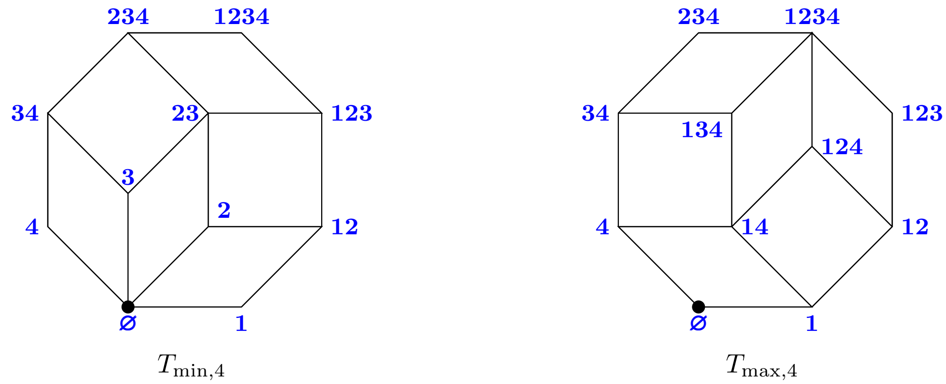


Figure 3.8: The $\lozenge$-tilings $T_{\min,4}$ and $T_{\max,4}$.

Example III.14. Consider the pile $\mathbf{T} = (T_0, \ldots, T_8)$ of $\lozenge$-tilings of $\mathbf{P}_4$ shown in Figure 3.9. Note that $T_0 = T_8$. For each tiling T_i, there are two possible flips that we can perform; applying one gives us T_{i-1}, and applying the other gives us T_{i+1} (with indices taken mod 8).

Definition III.15. Define $\mathcal{C}(n)$ to be the set of all piles $\mathbf{T} = (T_0, \ldots, T_{\binom{n}{3}})$ with $T_0 = T_{\min,n}$ and $T_{\binom{n}{3}} = T_{\max,n}$. Note that the shortest length of a pile starting with $T_{\min,n}$ and ending with $T_{\max,n}$ is $\binom{n}{3}$, corresponding to switching from Δ-crossings to ∇-crossings for each of the $\binom{n}{3}$ triples. Every $\lozenge$-tiling T of $\mathbf{P}_n$ appears in at least one pile in $\mathcal{C}(n)$.

Remark III.16. One can put a poset structure on the set of $\lozenge$-tilings of $\mathbf{P}_n$, called the *second higher Bruhat order* $B(n, 2)$ [8], as follows: given $\lozenge$-tilings T_1, T_2 with associated pseudoline arrangements D_1, D_2, we say that $T_1 \leq T_2$ if i, j, k having a Δ-crossing in T_2 implies that i, j, k have a Δ-crossing in T_1 for all $i < j < k$. Note that $T_{\min,n}$ is the minimum element of $B(n, 2)$, and $T_{\max,n}$ is the maximum element of $B(n, 2)$.

Example III.17. The set $\mathcal{C}(4)$ consists of two piles, namely $(T_0, T_1, T_2, T_3, T_4)$ and $(T_0, T_7, T_6, T_5, T_4)$, where the T_i are the $\lozenge$-tilings of $\mathbf{P}_4$ from Figure 3.9.

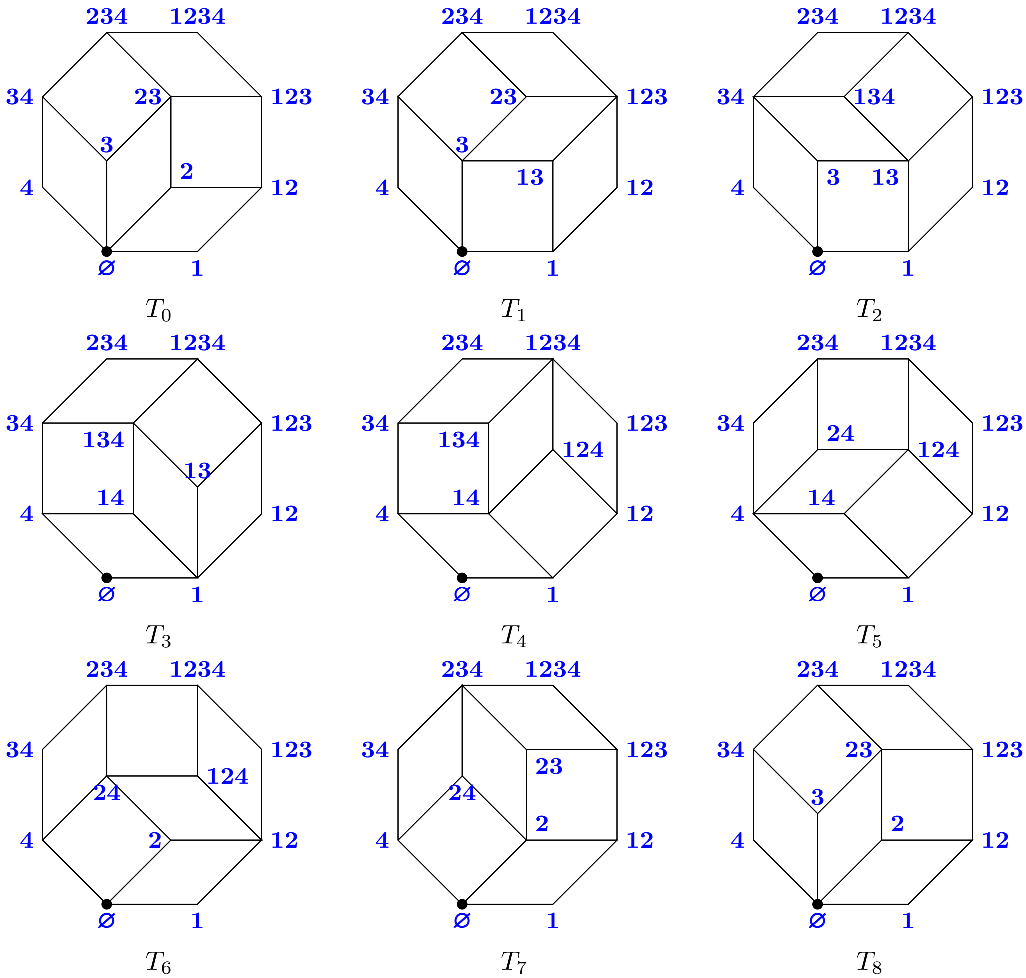
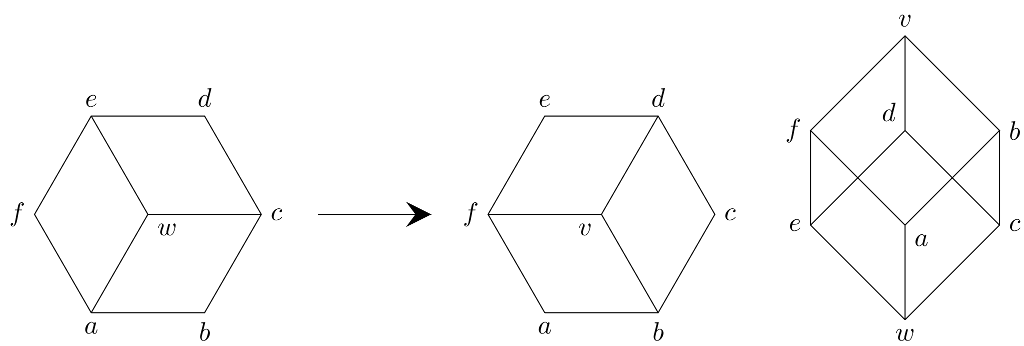


Figure 3.9: The pile of $\Diamond$-tilings of $\mathbf{P}_4$ from Example III.14.

Definition III.18. A *directed cubical complex* is a d-dimensional cubical complex along with a choice of "top" vertex in each d-dimensional cube.

Definition III.19. Given a pile $\mathbf{T} = (T_0, \ldots, T_\ell)$ of quadrangulations of a polygon R, we define an associated 3-dimensional directed cubical complex $\varkappa = \varkappa(\mathbf{T})$ as follows. Start with the 2-dimensional cubical complex corresponding to T_0. For $i = 1, \ldots, \ell$, given T_{i-1} and T_i labeled as in Figure 3.10, add a new vertex v to the complex corresponding to the new vertex in T_i, and add the 3-dimensional cube labeled as in Figure 3.10, with v as its top vertex. Note that a flip cannot be centered at a vertex on the boundary of R, so each vertex on the boundary of R corresponds to a single vertex in $\varkappa$. In the special case where $R = \mathbf{P}_n$ and the quadrangulations are $\Diamond$-tilings, we denote the vertex of $\varkappa$ corresponding to v_0 as v_0, by an abuse of notation.



Flip in a quadrangulation   Corresponding cube

Figure 3.10: A flip in a quadrangulation on the left, and the corresponding cube added to the directed cubical complex.

Definition III.20. Given a cubical complex $\varkappa$, we denote by $\varkappa^i$ the set of i-dimensional faces of $\varkappa$, and set $\varkappa^{02} = \varkappa^0 \cup \varkappa^2$. Similarly, for a pile $\mathbf{T}$ of a quadrangulations, we denote by $\varkappa^i(\mathbf{T})$ the set of i-dimensional faces of $\varkappa(\mathbf{T})$, and $\varkappa^{02}(\mathbf{T}) = \varkappa^0(\mathbf{T}) \cup \varkappa^2(\mathbf{T})$.

Definition III.21. Two 3-dimensional directed cubical complexes $\varkappa_1$ and $\varkappa_2$ are related by a *flip* if there exist piles $\mathbf{T}_1 = (T_{1,0}, \ldots T_{1,\ell})$ and $\mathbf{T}_2 = (T_{2,0}, \ldots T_{2,\ell})$ such that

- $\varkappa_1 = \varkappa(\mathbf{T}_1)$ and $\varkappa_2 = \varkappa(\mathbf{T}_2)$,

and there exists i such that

- $T_{1,j} = T_{2,j}$ for $j = 1, \ldots, i$ and $j = i+4, \ldots, \ell$, and
- $T_{1,j}$ and $T_{2,j}$ for $j = i+1, i+2, i+3$ are related by the local moves shown in Figure 3.11.

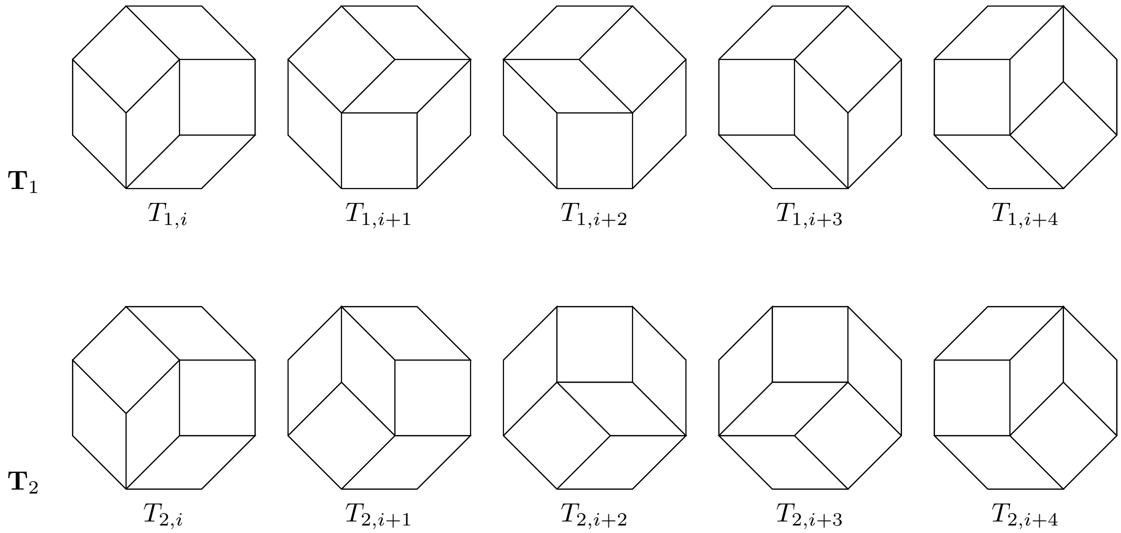


Figure 3.11: The two piles $\mathbf{T}_1$ and $\mathbf{T}_2$ in Definition III.21. The tiles outside the octagon remain in place.

Proposition III.22 ([10, Theorem 4.1])**.** *For any $\mathbf{T}, \mathbf{T}' \in \mathcal{C}(n)$, there exists a sequence of piles $\mathbf{T}_0, \ldots, \mathbf{T}_\ell \in \mathcal{C}(n)$ with $\mathbf{T} = \mathbf{T}_0$ and $\mathbf{T}' = \mathbf{T}_\ell$, such that the directed cubical complexes $\varkappa(\mathbf{T}_{i-1})$ and $\varkappa(\mathbf{T}_i)$ are related by a flip, for $i = 1, \ldots, \ell$.*

Definition III.23. A permutation σ of $\binom{[n]}{k}$ is called *admissible* if for every $I = \{i_1 < \cdots < i_{k+1}\} \in \binom{[n]}{k+1}$, the $k+1$ sets in $\binom{I}{k}$ appear in σ in either

- lexicographic order, i.e., in the order $\{i_1, i_2, \ldots, i_k, \widehat{i_{k+1}}\}$, $\{i_1, i_2, \ldots, \widehat{i_k}, i_{k+1}\}$, $\ldots$, $\{\widehat{i_1}, i_2, \ldots, i_k, i_{k+1}\}$, or
- reversed lexicographic order, i.e., in the order $\{\widehat{i_1}, i_2, \ldots, i_k, i_{k+1}\}$, $\{i_1, \widehat{i_2}, \ldots, i_k, i_{k+1}\}$, $\ldots$, $\{i_1, i_2, \ldots, i_k, \widehat{i_{k+1}}\}$.

Thus, for example, $(\{1,2\}, \{1,3\}, \{2,3\})$ and $(\{2,3\}, \{1,3\}, \{1,2\})$ are the only two admissible permutations of $\binom{[3]}{2}$. The *inversion set* of an admissible permutation σ of $\binom{[n]}{k}$ is the subset of $\binom{[n]}{k+1}$ consisting of those $I \in \binom{[n]}{k+1}$ for which the elements of $\binom{I}{k}$ appear in σ in the reversed lexicographic order.

Definition III.24. Given a pile $\mathbf{T} = \left(T_0, \ldots, T_{\binom{n}{3}}\right) \in \mathcal{C}(n)$, for $i = 1, \ldots, \binom{n}{3}$, let

$\alpha_i \in \binom{[n]}{3}$ be the set of three indices of the edges of the hexagon involved in the flip between T_{i-1} and T_i. Note that $\binom{[n]}{3} = \left\{\alpha_1, \ldots, \alpha_{\binom{n}{3}}\right\}$, as each α_i indexes a triple which is switching from a Δ-crossing to a ∇-crossing. We say that $\left(\alpha_1, \ldots, \alpha_{\binom{n}{3}}\right)$ is the *permutation of* $\binom{[n]}{3}$ *associated to* $\mathbf{T}$. Note that $\mathbf{T} \in \mathcal{C}(n)$ is uniquely determined by its permutation of $\binom{[n]}{3}$.

Theorem III.25 ([8], [10, Definition 2.1, Theorem 4.1])**.**

(a) Let σ be a permutation of $\binom{[n]}{3}$. The following are equivalent:

- *σ is an admissible permutation of $\binom{[n]}{3}$;*
- *there exists a pile $\mathbf{T} \in \mathcal{C}(n)$ whose corresponding permutation of $\binom{[n]}{3}$ is σ.*

(b) Fix $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{C}(n)$. The following are equivalent:

- *$\varkappa(\mathbf{T}_1)$ and $\varkappa(\mathbf{T}_2)$ are isomorphic directed cubical complexes;*
- *the inversion sets of the permutations of $\binom{[n]}{3}$ associated to $\mathbf{T}_1$ and $\mathbf{T}_2$ coincide.*

CHAPTER IV

The Coherence Condition and Principal Minors of Symmetric Matrices

We begin this chapter by reviewing the earlier work of Kenyon and Pemantle [4] concerning the occurence of the hexahedron recurrence as a determinantal identity. We then formulate new criteria for the existence of symmetric matrices with prescribed values of certain principal minors. See in particular Corollary IV.20, Corollary IV.23, and Theorem IV.26. The proofs of these results are given later in Chapters VIII–IX.

We begin by extending the definitions of the Kashaev equation, positive Kashaev recurrence, hexahedron recurrence, and K-hexahedron equations to arrays indexed on directed cubical complexes in the obvious way. For those readers who skipped Chapter III, it may be helpful to review Definitions III.1, III.18, and III.20 before processing the following definition.

Definition IV.1. Fix a directed cubical complex $\varkappa$. An array $\mathbf{x}$ indexed by $\varkappa^0$ satisfies the Kashaev equation if for all 3-dimensional cubes C of $\varkappa$, $\mathbf{x}$ satisfies (2.1) with the components of $\mathbf{x}$ labeled on C as in Figure 2.1. We say that $\mathbf{x}$ satisfies the positive Kashaev equation if the components of $\mathbf{x}$ are all positive and for all 3-dimensional cubes C of $\varkappa$, $\mathbf{x}$ satisfies (2.5) with the components of $\mathbf{x}$ labeled on C as in Figure 2.1 and z_{111} corresponding to the component of $\mathbf{x}$ at the top vertex of C. We say that an array $\tilde{\mathbf{x}}$ indexed by $\varkappa^{02}$ satisfies the hexahedron recurrence (resp., K-hexahedron equations) if for all 3-dimensional cubes C of $\varkappa$, $\tilde{\mathbf{x}}$ satisfies equations (2.16)–(2.19) (resp., equations (2.21)–(2.24), along with equation (2.12) for all $s \in \varkappa^2$) with the components of $\tilde{\mathbf{x}}$ labeled on the vertices of C as in Figure 2.1, labeled on the 2-dimensional faces of C by averaging the indices of the vertices on the boundary, and with z_{111} corresponding to the component of $\tilde{\mathbf{x}}$ on the top vertex of C.

Remark IV.2. As we saw in Chapter II, the Kashaev equation is independent of a choice of direction on each cube, and hence can be defined for arbitrary 3-dimensional cubical complexes. On the other hand, the positive Kashaev recurrence, hexahedron recurrence, and K-hexahedron equations depend on a choice of a pair of opposite distinguished vertices in each cube, and hence are defined on directed cubical complexes.

Definition IV.3. Given an $n \times n$ matrix M and $I, J \subseteq [n]$, we denote by M_I^J the submatrix of M obtained by restricting to rows I and columns J. A *principal minor* of M is the determinant of a submatrix $\det M_I^I$, where $I \subseteq [n]$. We follow the convention that $M_\varnothing^\varnothing = 1$. An *almost-principal minor* of M is the determinant of a submatrix $\det M_{I\cup\{i\}}^{I\cup\{j\}}$ for $I \subset [n]$, and distinct $i, j \notin I$. We say that an almost-principal minor $M_{I\cup\{i\}}^{I\cup\{j\}}$ is *odd* if $(i-j)(-1)^{|I|} > 0$, and *even* if $(i-j)(-1)^{|I|} < 0$.

Before proceeding with the following definition, the reader may want to review Definitions III.4, III.10, III.13, and III.19.

Definition IV.4. Given a $\Diamond$-tiling T of $\mathbf{P}_n$ and an $n \times n$ complex-valued matrix M, define the array $\mathbf{x}_T(M) = (x_s)_{s\in\varkappa^0(T)}$, where if vertex s of T is labeled by $I \subseteq [n]$,

$$x_s = (-1)^{\lfloor |I|/2 \rfloor} M_I^I. \tag{4.1}$$

Similarly, define the array $\tilde{\mathbf{x}}_T(M) = (x_s)_{s\in\varkappa^{02}(T)}$, where

- if $s \in \varkappa^0(T)$: given that vertex s of T is labeled by $I \subseteq [n]$, set

$$x_s = (-1)^{\lfloor |I|/2 \rfloor} M_I^I. \tag{4.2}$$

- if $s \in \varkappa^2(T)$: given that tile s of T has vertices labeled by $I, I\cup\{i\}, I\cup\{j\}, I\cup\{i,j\} \subseteq [n]$, where i and j are chosen so that $M_{I\cup\{i\}}^{I\cup\{j\}}$ is the odd almost principal minor, set

$$x_s = (-1)^{\lfloor (|I|+1)/2 \rfloor} M_{I\cup\{i\}}^{I\cup\{j\}}. \tag{4.3}$$

More generally, if $\mathbf{T} = (T_0, \ldots, T_\ell)$ is a pile of $\lozenge$-tilings of $\mathbf{P}_n$, define $\mathbf{x}_{\varkappa(T)}(M)$ to be the array indexed by $\varkappa^0(\mathbf{T})$ whose restriction to $\varkappa^0(T_i)$ is $\mathbf{x}_{T_i}(M)$, and define $\tilde{\mathbf{x}}_{\varkappa(T)}(M)$ to be the array indexed by $\varkappa^{02}(\mathbf{T})$ whose restriction to $\varkappa^{02}(T_i)$ is $\tilde{\mathbf{x}}_{T_i}(M)$.

Remark IV.5. For any $\lozenge$-tiling T of $\mathbf{P}_n$, the vertex v_0 is labeled by $\varnothing \subset [n]$. In Definition IV.4, because of the convention that $M_\varnothing^\varnothing = 1$, $x_{v_0} = 1$ independent of the matrix M.

Definition IV.6. Given a $\lozenge$-tiling T of $\mathbf{P}_n$, we say that a complex-valued array $\tilde{\mathbf{x}} = (x_s)_{s\in\varkappa^{02}(T)}$ is *standard* if $x_{v_0} = 1$. Furthermore, given a pile of $\lozenge$-tilings of $\mathbf{P}_n$, $\mathbf{T} = (T_0, \ldots, T_\ell)$, and $\varkappa = \varkappa(\mathbf{T})$, we say that a complex-valued array $\tilde{\mathbf{x}} = (x_s)_{s\in\varkappa^{02}}$ is *standard* with respect to $\mathbf{T}$ if $x_{v_0} = 1$.

Definition IV.7. Given a $\lozenge$-tiling T of $\mathbf{P}_n$, we say that a complex-valued array $\tilde{\mathbf{x}}$ indexed by $\varkappa^{02}(T)$ is *generic* if for any sequence of flips applied to T accompanied by applications of the hexahedron recurrence to $\tilde{\mathbf{x}}$, the resulting coefficients are all nonzero.

Definition IV.8. We say that a square matrix is *generic* if all of its principal minors and odd almost-principal minors are non-zero. Let $M_n^*(\mathbb{C})$ denote the set of $n \times n$ generic complex-valued matrices.

We can now provide some important results of Kenyon and Pemantle [4].

Theorem IV.9 ([4, Theorem 4.2])**.** *Given a $\lozenge$-tiling T of $\mathbf{P}_n$, the map $\tilde{\mathbf{x}}_T(\cdot)$ establishes a bijective correspondence between $M_n^*(\mathbb{C})$ and standard, generic, complex-valued arrays on $\varkappa^{02}(T)$.*

Before proceeding with the following theorem, the reader may want to review Definition III.12.

Theorem IV.10 ([4, Theorem 4.4])**.** *Let $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{C}^*)^{\varkappa^{02}(T_{\min,n})}$ be a standard array. Then there is a unique matrix M such that $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_{\varkappa(T_{\min,n})}(M)$. The entries of this matrix M are Laurent polynomials in the components of $\tilde{\mathbf{x}}$.*

Proposition IV.11 ([4, Lemma 2.1])**.** *Suppose M is an $n \times n$ matrix, and $\mathbf{T}$ is a pile of $\lozenge$-tilings of $\mathbf{P}_n$ such that the components of $\tilde{\mathbf{x}}_{\varkappa(\mathbf{T})}(M)$ are all nonzero. Then $\tilde{\mathbf{x}}_{\varkappa(\mathbf{T})}(M)$ satisfies the hexahedron recurrence.*

Theorem IV.12 ([4, Theorem 5.2])**.** *Let T be a $\lozenge$-tiling of $\mathbf{P}_n$. A matrix $M \in M_n^*(\mathbb{C})$ is symmetric if and only if for every tile s in T with vertices s_1, s_2, s_3, s_4 in cyclic order,*

$$(\tilde{\mathbf{x}}_T(M)_s)^2 = \tilde{\mathbf{x}}_T(M)_{s_1}\tilde{\mathbf{x}}_T(M)_{s_3} + \tilde{\mathbf{x}}_T(M)_{s_2}\tilde{\mathbf{x}}_T(M)_{s_4}. \tag{4.4}$$

Furthermore, if M is any $n \times n$ symmetric matrix, condition (4.4) *holds for every tile of T.*

Remark IV.13. Theorem IV.12 is not stated explicitly in [4], but follows immediately from the cited theorem. The original theorem concerns Hermitian matrices, and a slightly modified version of the hexahedron recurrence in which some complex conjugates are taken.

The next result follows from Proposition IV.11 and Theorem IV.12:

Corollary IV.14. *Let* $\mathbf{T}$ *be a pile of* $\Diamond$*-tilings of* $\mathbf{P}_n$*, with* $\varkappa = \varkappa(\mathbf{T})$*. Let* M *be an* $n \times n$ *symmetric matrix with nonzero principal minors. Then* $\mathbf{x}_{\varkappa(\mathbf{T})}(M)$ *satisfies the K-hexahedron equations.*

The next corollary follows immediately from Theorem IV.12 and the fact that the entries of M are Laurent polynomials in the components of $\tilde{\mathbf{x}}_{\varkappa(T_{\min,n})}(M)$:

Corollary IV.15. *Let* M *be an* $n \times n$ *matrix such that the components of* $\tilde{\mathbf{x}}_{\varkappa(T_{\min,n})}(M)$ *are nonzero. Then* M *is symmetric if and only if condition* (4.4) *holds.*

Hence, the following is immediate from Proposition IV.11 and Corollary IV.15:

Corollary IV.16. *Let* $\mathbf{T}$ *be a pile of* $\Diamond$*-tilings of* $\mathbf{P}_n$ *containing* $T_{\min,n}$*, with* $\varkappa = \varkappa(\mathbf{T})$*. Let* $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\varkappa^0}$ *be a standard array satisfying the property that*

$$x_{v_1}x_{v_3} + x_{v_2}x_{v_4} \neq 0 \tag{4.5}$$

for all 2*-dimensional faces of* $\varkappa$ *with vertices* v_1, v_2, v_3, v_4 *in cyclic order. Then the following are equivalent:*

- $\mathbf{x}$ *can be extended to a standard array indexed by* $\varkappa^{02}$ *satisfying the K-hexahedron equations;*
- *there exists a symmetric matrix* M *such that* $\mathbf{x} = \mathbf{x}_{\varkappa(\mathbf{T})}(M)$*.*

We want a set of equations that tell us whether an array $\mathbf{x}$ indexed by $\varkappa^0(\mathbf{T})$ can be extended to an array indexed by $\varkappa^{02}(\mathbf{T})$ satisfying the K-hexahedron equations. Below, we define a notion of coherence generalizing the notion of coherence from Chapter II.

Definition IV.17. Let $\varkappa$ be a 3-dimensional cubical complex. We say that $\mathbf{x} = (x_s)_{s\in\varkappa}$ is a *coherent* solution of the Kashaev equation if $\mathbf{x}$ satisfies the Kashaev equation (i.e., $K^C(\mathbf{x}) = 0$ for every 3-dimensional cube C of $\varkappa$), and for any interior vertex v of $\varkappa$:

$$\prod_{C\ni v} K_v^C(\mathbf{x}) = \prod_{S\ni v}(x_v x_{v_2} + x_{v_1}x_{v_3}), \tag{4.6}$$

where

- the first product is over 3-dimensional cubes C incident to the vertex v,
- the second product is over 2-dimensional faces S incident to the vertex v, and
- v, v_1, v_2, v_3 are the vertices of such a face S listed in cyclic order.

Remark IV.18. The property of being coherent solution of the Kashaev equation is defined for 3-dimensional cubical complexes, not only for all 3-dimensional directed cubical complexes, as no choice of direction needs to be made in each cube.

Theorem IV.19. *Let $\mathbf{T}$ be a pile of $\lozenge$-tilings of $\mathbf{P}_n$, with $\varkappa = \varkappa(\mathbf{T})$.*
(a) Any coherent solution $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\varkappa^0}$ of the Kashaev equation satisfying the property that

$$x_{v_1}x_{v_3} + x_{v_2}x_{v_4} \neq 0 \tag{4.7}$$

for all faces of $\varkappa$ with vertices v_1, v_2, v_3, v_4 in cyclic order, can be extended to $\tilde{\mathbf{x}} = (x_s)_{s\in\varkappa^{02}(\mathbf{T})}$ satisfying the K-hexahedron equations.
(b) Conversely, suppose that $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{C}^*)^{\varkappa^{02}}$ (with $x_s \neq 0$ for $s \in \varkappa^0$) satisfies the K-hexahedron equations. Then the restriction of $\tilde{\mathbf{x}}$ to $\varkappa^0$ is a coherent solution of the Kashaev equation.

Theorem IV.19 is proved in Chapter VIII, where we obtain results (Proposition VIII.3 and Theorem VIII.10) generalizing both Theorem IV.19 and Theorem II.22.

As an immediate corollary of Corollary IV.16 and Theorem IV.19, we obtain the following:

Corollary IV.20. *Let* $\mathbf{T}$ *be a pile of* $\Diamond$*-tilings of* $\mathbf{P}_n$ *containing* $T_{\min,n}$*, with* $\varkappa = \varkappa(\mathbf{T})$*. Let* $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\varkappa^0}$ *be a standard array satisfying condition* (4.5)*. Then the following are equivalent:*

- $\mathbf{x}$ *is a coherent solution of the Kashaev equation;*
- *there exists a symmetric matrix* M *such that* $\mathbf{x} = \mathbf{x}_{\varkappa(\mathbf{T})}(M)$*.*

Next, we consider the problem of checking whether an array of 2^n numbers could correspond to the principal minors of some symmetric matrix.

Definition IV.21. Given an $n \times n$ symmetric matrix M, let $\bar{\mathbf{x}}(M) = (x_I)_{I \in [n]}$, where

$$x_I = (-1)^{\lfloor |I|/2 \rfloor} M_I^I \tag{4.8}$$

for $I \subseteq [n]$. Given a pile $\mathbf{T}$ of $\Diamond$-tilings of $\mathbf{P}_n$, with $\varkappa = \varkappa(\mathbf{T})$, and an array $\bar{\mathbf{x}} = (x_I)_{I \subseteq [n]}$, let $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}}) = (x_s)_{s \in \varkappa^0}$ where $x_s = x_I$ when vertex s is labeled by $I \subseteq [n]$.

Definition IV.22. Given an array $\bar{\mathbf{x}} = (x_J)_{J \subseteq [n]}$, $I \subseteq [n]$, and distinct $i, j \in [n]$, set

$$L_{I,\{i,j\}}(\bar{\mathbf{x}}) = x_I x_{I\Delta\{i,j\}} + x_{I\Delta\{i\}} x_{I\Delta\{j\}}, \tag{4.9}$$

where Δ denotes the symmetric difference.

Corollary IV.23. *Fix an array* $\bar{\mathbf{x}} = (x_I)_{I \subseteq [n]}$ *with nonzero entries, satisfying the conditions that* $L_{I,\{i,j\}} \neq 0$ *for any* $I \subseteq [n]$ *and distinct* $i, j \in [n]$*, and* $x_\varnothing = 1$*. Then the following are equivalent:*

- *there exists a symmetric matrix M such that $\bar{\mathbf{x}} = \bar{\mathbf{x}}(M)$;*
- *for some pile $\mathbf{T}$ of $\diamondsuit$-tilings of $\mathbf{P}_n$ in which every $I \subseteq [n]$ labels at least one vertex of $\varkappa(\mathbf{T})$, $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation;*
- *for all piles $\mathbf{T}$ of $\diamondsuit$-tilings of $\mathbf{P}_n$, $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation.*

Although Corollary IV.23 can be deduced from Theorem IV.9, Theorem IV.12, Corollary IV.14, and Theorem IV.19, we provide a proof in Chapter IX.

Given a choice of pile $\mathbf{T}$ in which every $I \subseteq [n]$ labels at least one vertex of $\varkappa(\mathbf{T})$, Corollary IV.23 provides a finite set of equations to test whether there exists a symmetric matrix M such that $\bar{\mathbf{x}} = \bar{\mathbf{x}}(M)$. However, in general, the equations (4.6) for different interior vertices v of $\varkappa(\mathbf{T})$ have very different forms.

Definition IV.24. Given an array $\bar{\mathbf{x}} = (x_J)_{J\subseteq[n]}$, $I \subseteq [n]$, and distinct $i, j, k \in [n]$, set

$$K^{I,\{i,j,k\}}(\bar{\mathbf{x}}) = K^C(\mathbf{x}), \tag{4.10}$$

$$K_{I,\{i,j,k\}}(\bar{\mathbf{x}}) = K_v^C(\mathbf{x}), \tag{4.11}$$

where C is a 3-dimensional cube, and $\mathbf{x}$ is the array indexed by the vertices of C shown in Figure 4.1, and v is the vertex of C at which $\mathbf{x}$ has entry x_I.

Let's focus on the $n = 4$ case. Using the $\diamondsuit$-tilings from Figure 3.9, consider the pile $\mathbf{T} = (T_0, T_1, \ldots, T_7, T_0, T_1, T_2, T_3)$. Note that for every $I \subseteq [4]$, there is a tiling in $\mathbf{T}$ where I labels a vertex. (In fact, this property holds for the pile $(T_i, \ldots, T_{i+5})$ for any $i \in [8]$, with indices taken mod 8.) Note $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ satisfies the Kashaev equation if and only if

$$K^{I,\{i,j,k\}}(\bar{\mathbf{x}}) = 0 \tag{4.12}$$

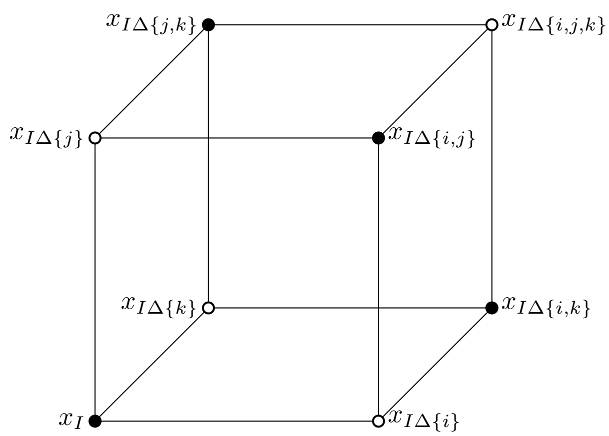


Figure 4.1: The array $\mathbf{x}$ in equations (4.10)–(4.11), where Δ denotes the symmetric difference. In equation (4.11), the vertex v is the lower left vertex, with value x_I.

for all $I \subseteq [4]$ and distinct $i, j, k \in [4]$. Note that every interior vertex of $\varkappa(\mathbf{T})$ is incident with four 3-dimensional cubes, each pair of which shares one 2-dimensional face. Hence, for each $I \subseteq [4]$ labeling an interior vertex of $\varkappa(\mathbf{T})$ (namely, when I is one of $\{1,3\}$, $\{1,4\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2\}$, $\{2,3\}$, $\{2,4\}$, or $\{3\}$), the following condition holds:

$$\prod_{J \in \binom{[4]}{3}} K_{I,J}(\bar{\mathbf{x}}) = \prod_{J \in \binom{[4]}{2}} L_{I,J}(\bar{\mathbf{x}}). \tag{4.13}$$

Furthermore, it is straightforward to check that if $\bar{\mathbf{x}} = \bar{\mathbf{x}}(M)$ for some 4×4 symmetric matrix M, then equation (4.13) holds for all $I \subseteq [4]$. Hence, the result below follows from Corollary IV.23.

Corollary IV.25. *Fix an array $\bar{\mathbf{x}} = (x_I)_{I \subseteq [4]}$, satisfying the conditions that $L_{I,\{i,j\}} \neq 0$ for any $I \subseteq [4]$ and distinct $i, j \in [4]$, and $x_\varnothing = 1$. Then the following are equivalent:*

- *there exists a symmetric matrix M such that $\bar{\mathbf{x}} = \bar{\mathbf{x}}(M)$;*
- *for all piles $\mathbf{T}$ of $\Diamond$-tilings of $\mathbf{P}_4$, $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation;*

- *for all $I \subseteq [4]$ and distinct $i, j, k \in [4]$, equation* (4.12) *holds, and for all $I \subseteq [4]$, equation* (4.13) *holds.*

Equations (4.12)–(4.13) are far more manageable than the coherence conditions that can arise in an arbitrary cubical complex $\varkappa(\mathbf{T})$. The good news is that the only coherence conditions that we need to check for any $n \geq 4$ are of the form of equation (4.13)!

Theorem IV.26. *Fix an array $\bar{\mathbf{x}} = (x_I)_{I \subseteq [n]}$, satisfying the conditions that $L_{I,\{i,j\}} \neq 0$ for any $I \subseteq [n]$ and distinct $i, j \in [n]$, and $x_\varnothing = 1$. Then the following are equivalent:*

- *there exists a symmetric matrix M such that $\bar{\mathbf{x}} = \bar{\mathbf{x}}(M)$;*
- *for all piles $\mathbf{T}$ of $\Diamond$-tilings of $\mathbf{P}_n$, $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation;*
- *for all $I \subseteq [n]$ and distinct $i, j, k \in [n]$, equation* (4.12) *holds, and for all $I \subseteq [n]$ and $A \in \binom{[n]}{4}$,*

$$\prod_{J \in \binom{A}{3}} K_{I,J}(\bar{\mathbf{x}}) = \prod_{J \in \binom{A}{2}} L_{I,J}(\bar{\mathbf{x}}). \tag{4.14}$$

We prove Theorem IV.26 in Chapter IX.

Remark IV.27. We compare Theorem IV.26 to a similar result of Oeding [9, Corollary 1.4]. He considers the natural action of $(\mathrm{SL}_2(\mathbb{C})^{\times n}) \ltimes S_n$ (where S_n is the symmetric group on n elements) on $\mathbb{C}^{2^{[n]}}$, and proves that for $\bar{\mathbf{x}} = (x_I)_{I \subseteq [n]}$ the following are equivalent:

- there exists a symmetric matrix M such that $x_I = M_I^I$ for all $I \subseteq [n]$;
- all images of $\bar{\mathbf{x}}$ under $(\mathrm{SL}_2(\mathbb{C})^{\times n}) \ltimes S_n$ satisfy

$$2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 + 4(s + t) = 0, \tag{4.15}$$

where

$$a = x_\varnothing x_{\{1,2,3\}}, \quad b = x_{\{1\}}x_{\{2,3\}}, \quad c = x_{\{2\}}x_{\{1,3\}}, \quad d = x_{\{3\}}x_{\{1,2\}},$$
$$s = x_\varnothing x_{\{1,2\}}x_{\{1,3\}}x_{\{2,3\}}, \quad t = x_{\{1\}}x_{\{2\}}x_{\{3\}}x_{\{1,2,3\}}. \tag{4.16}$$

The left-hand side of equation (4.15) can be identified as Cayley's $2 \times 2 \times 2$ *hyperdeterminant*. Equivalently, equation (4.15) is just equation (4.12) for $I = \varnothing$ and $\{i, j, k\} = \{1, 2, 3\}$ with the appropriate changes of sign (because we don't put additional signs on the principal minors in this setting). For the above equivalence, Oeding does not impose an assumption of genericity on $\bar{\mathbf{x}}$. Consider the subgroup $H \subseteq \mathrm{SL}_2(\mathbb{C})$ defined by

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathbb{Z}_4\,. \tag{4.17}$$

In this language of [9], our condition that a (signed) array $\bar{\mathbf{x}}$ satisfies equation (4.12) for all $I \subseteq [n]$ and distinct $i, j, k \in [n]$ can be restated as the condition that all images of the "unsigned version" of $\bar{\mathbf{x}}$, under the action of the group $(H^{\times n}) \ltimes S_n$, satisfy equation (4.15). Thus, Theorem IV.26 requires an additional assumption of genericity and an additional equation (equation (4.14)) compared to Oeding's criterion, but uses a weaker version of the second requirement above.

CHAPTER V

S-Holomorphicity in $\mathbb{Z}^2$

In this chapter, we discuss a certain equation (see (5.3)) which shares many properties with the Kashaev equation. We also study a related system of equations (see (5.12)–(5.16)) which plays the role analogous to the K-hexahedron equations. The equations studied herein arise in discrete complex analysis and in the study of the Ising model (see [1] and Remark V.11). The presentation of results in this chapter follows a plan similar to that of Chapter II. The results in this chapter are proved in Chapter X as special cases of a general axiomatic framework.

Definition V.1. Given a unit square C with vertices in $\mathbb{Z}^2$ and an array $\mathbf{x} \in \mathbb{C}^{\mathbb{Z}^2}$, define

(5.1)

$$Q^C(\mathbf{x}) = z_{00}^2+z_{10}^2+z_{01}^2+z_{11}^2-2(z_{00}z_{10}+z_{10}z_{11}+z_{11}z_{01}+z_{01}z_{00})-6(z_{00}z_{11}+z_{10}z_{01}),$$

where $z_{00}, z_{10}, z_{01}, z_{11}$ denote the components of $\mathbf{x}$ at the vertices of C, as shown in Figure 5.1. Notice that the right-hand side of (5.1) is invariant under the symmetries of the square. In other words, reindexing the 4 values using an isomorphic labeling of the square does not change the definition of $Q^C(\mathbf{x})$.

Definition V.2. Let $\mathbf{x} \in \mathbb{C}^{\mathbb{Z}^2}$. Let v and w be two opposite vertices in a unit square

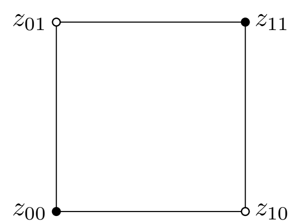


Figure 5.1: Notation used in Definition V.1.

C in $\mathbb{Z}^2$. We set

$$Q_v^C(\mathbf{x}) = \frac{1}{4\sqrt{2}}\frac{\partial Q^C}{\partial x_w}(\mathbf{x}) = \frac{1}{2\sqrt{2}}(z_{11} - z_{10} - z_{01} - 3z_{00}), \tag{5.2}$$

where we use a labeling of the components of $\mathbf{x}$ on the vertices of C as in Figure 5.1, with z_{00} corresponding to the component of $\mathbf{x}$ at v.

Definition V.3. Given $v \in \mathbb{Z}^2$ and $i_1, i_2 \in \{-1, 1\}$, define $C_v(i_1, i_2)$ to be the unique unit square containing the vertices v and $v + (i_1, i_2)$.

Proposition V.4. *Suppose that* $\mathbf{x} = (x_s)_{s\in\mathbb{Z}^2} \in \mathbb{C}^{\mathbb{Z}^2}$ *satisfies*

$$Q^C(\mathbf{x}) = 0 \tag{5.3}$$

for every unit square C in $\mathbb{Z}^2$ (see (5.1)*). Then for any $v \in \mathbb{Z}^2$,*

$$\left(\prod_{C\ni v} Q_v^C(\mathbf{x})\right)^2 = \left(\prod_{S\ni v}(x_v + x_{v_1})\right)^2, \tag{5.4}$$

where

- *the first product is over the 4 unit squares C incident to the vertex v,*
- *the second product is over the 4 edges S incident to v, and*
- *v and v_1 are the vertices of such an edge S.*

Moreover, the following strengthening of (5.4) *holds:*

$$\left(Q_v^{C_v(1,1)}(\mathbf{x})Q_v^{C_v(-1,-1)}(\mathbf{x})\right)^2 = \left(Q_v^{C_v(1,-1)}(\mathbf{x})Q_v^{C_v(-1,1)}(\mathbf{x})\right)^2 = \prod_{S\ni v}(x_v + x_{v_1}), \tag{5.5}$$

where the rightmost product is the same as in (5.4).

Remark V.5. The right-hand side of equation (5.1) is a quadratic polynomial in each of the variables z_{ij}. Setting this expression equal to zero and solving for z_{11} in terms of z_{00}, z_{10}, z_{10}, we obtain

$$z_{11} = 3z_{00} + z_{10} + z_{01} \pm 2\sqrt{2}\sqrt{(z_{00}+z_{10})(z_{00}+z_{01})}, \tag{5.6}$$

where $\sqrt{(z_{00}+z_{10})(z_{00}+z_{01})}$ denotes either of the two square roots. Notice that if $z_{00}, z_{10}, z_{01} > 0$, then both solutions for z_{11} in (5.6) are real; moreover, the larger of these two solutions is positive. However, solving the equation

$$z_{11} = 3z_{00} + z_{10} + z_{01} + 2\sqrt{2}\sqrt{(z_{00}+z_{10})(z_{00}+z_{01})} \tag{5.7}$$

for z_{00}, with $z_{10}, z_{01}, z_{11} > 0$, may result in a unique negative solution. (For example, take $z_{10} = z_{01} = z_{11} = 1$.) Hence, unlike the positive Kashaev recurrence, the equation (5.7) only defines a recurrence on $\mathbb{R}_{>0}$-valued arrays in one direction.

Definition V.6. For $U \subseteq \mathbb{Z}$, let $\mathbb{Z}^2_U$ denote the set

$$\mathbb{Z}^2_U = \{(i,j) \in \mathbb{Z}^2 : i+j \in U\}. \tag{5.8}$$

Theorem V.7. *Suppose that* $\mathbf{x} = (x_s) \in (\mathbb{R}_{>0})^{\mathbb{Z}^2_{\{0,1,2,\dots\}}}$ *satisfies* (5.7) *for all* $v \in \mathbb{Z}^2_{\{0,1,2,\dots\}}$, *where we use the notation* $z_{ij} = x_{v+(i,j)}$. *Then for any* $v \in \mathbb{Z}^2_{\{2,3,4,\dots\}}$, *we have*

$$\prod_{C \ni v} Q_v^C(\mathbf{x}) = -\prod_{S \ni v}(x_v + x_{v_1}), \tag{5.9}$$

where the notational conventions are the same as in equation (5.4).

Definition V.8. Let

$$E = \mathbb{Z}^2 + \left\{\left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right)\right\}. \tag{5.10}$$

In order words, E is the set of centers of edges in the tiling of $\mathbb{R}^2$ with unit squares.

We next state the analogue of Theorem II.22.

Theorem V.9.

(a) Assume that an array $\mathbf{x} = (x_s) \in \mathbb{C}^{\mathbb{Z}^2}$

- *satisfies the equation* $Q^C(\mathbf{x}) = 0$ *for all unit squares* C*,*
- *satisfies equation* (5.9) *for all* $v \in \mathbb{Z}^2$*, and*
- *satisfies*

$$x_v + x_{v+e_i} \neq 0 \tag{5.11}$$

for all $v \in \mathbb{Z}^2$ *and* $i \in \{1,2\}$*.*

Then $\mathbf{x}$ *can be extended to an array* $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^2 \cup E}$ *satisfying the recurrence*

$$z_{11} = 3z_{00} + z_{10} + z_{01} + 2\sqrt{2} z_{\frac{1}{2}0} z_{0\frac{1}{2}}, \tag{5.12}$$

$$z_{\frac{1}{2}1} = z_{\frac{1}{2}0} + \sqrt{2} z_{0\frac{1}{2}}, \tag{5.13}$$

$$z_{1\frac{1}{2}} = z_{0\frac{1}{2}} + \sqrt{2} z_{\frac{1}{2}0}, \tag{5.14}$$

together with the conditions

$$z_{\frac{1}{2}0}^2 = z_{00} + z_{10}, \tag{5.15}$$

$$z_{0\frac{1}{2}}^2 = z_{00} + z_{01}, \tag{5.16}$$

where we use the notation $z_{ij} = x_{v+(i,j)}$*, for all* $v \in \mathbb{Z}^2$*.*

(b) Conversely, suppose $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^2 \cup E}$ *satisfies* (5.12)–(5.16)*. Then the restriction* $\mathbf{x}$ *of* $\tilde{\mathbf{x}}$ *to* $\mathbb{Z}^2$ *satisfies* $Q^C(\mathbf{x}) = 0$ *for all unit squares* C*, and satisfies* (5.9) *for all* $v \in \mathbb{Z}^2$*.*

Remark V.10. Comparing Theorem II.22 to Theorem V.9, we see that equations (5.12)–(5.16) play a role analogous to that of the K-hexahedron equations.

Remark V.11. The equations (5.12)–(5.16) appear in discrete complex analysis in the context of s-holomorphicity [1]. Consider the labeling $\ell : E \to \mathbb{C}$ described in Figure 5.2. If we orient the edge with midpoint $s \in E$ from the even height vertex to the odd height vertex, then $\ell(s)$ is a square root of the complex number associated with the directed edge. An *s-holomorphic function* on the tiling of $\mathbb{R}^2$ with unit squares is a complex-valued function F on the faces of the tiling such that for any two faces f_1, f_2 sharing an edge with midpoint s, we have

$$\operatorname{Re}[\ell(s)F(f_1)] = \operatorname{Re}[\ell(s)F(f_2)]. \tag{5.17}$$

Hence, given an s-holomorphic function F, we can define $\mathbf{x}' = (x_s) \in \mathbb{R}^E$ by setting

$$x_s = \operatorname{Re}[\ell(s)F(f)] \tag{5.18}$$

for either of the faces f using the edge corresponding to s. It is straightforward to check that $\mathbf{x}' = (x_s) \in \mathbb{R}^E$ corresponds to an s-holomorphic function F by (5.18) if and only if $\mathbf{x}'$ satisfies (5.13)–(5.14). If we extend $\mathbf{x}'$ to $\tilde{\mathbf{x}} = (x_s) \in \mathbb{R}^{\mathbb{Z}^2 \cup E}$ satisfying (5.12)–(5.16), the function $H : \mathbb{Z}^2 \to \mathbb{R}$ defined by

$$H(i,j) = \begin{cases} x_{(i,j)} & \text{if } i+j \text{ is even;} \\ -x_{(i,j)} & \text{if } i+j \text{ is odd,} \end{cases} \tag{5.19}$$

corresponds to a certain discrete integral. For more on this recurrence and its connections to discrete complex analysis and the Ising model, see [1].

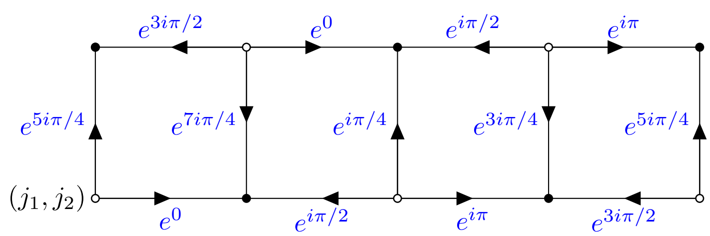


Figure 5.2: Define the labeling $\ell : E \to \mathbb{C}$ invariant under translations by the vectors $(1, 1)$ and $(4, 0)$ as follows. In the vicinity of a point $(j_1, j_2) \in \mathbb{Z}^2$ satisfying $j_1 - j_2 \equiv 0 \pmod 4$, the labeling is given by the values shown in blue in the figure.

CHAPTER VI

Further Generalizations of the Kashaev Equation

In this chapter, we provide two additional examples of equations with behavior similar to the Kashaev equation and its analogue (5.1). In Chapter X, we shall develop a general framework which will allow us to prove all of the results in this chapter (as well as the results in Chapter V).

Both recurrences considered in this chapter come with complex parameters that one can choose arbitrarily. For certain values of these parameters, the corresponding recurrences have cluster algebra-like behavior. We will explore the cluster algebra nature of these recurrences in future work [7].

We begin with the following, relatively simple, one-dimensional example:

Proposition VI.1. *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Let $\mathbf{x} = (x_s) \in \mathbb{C}^{\mathbb{Z}}$ be an array such that*

$$z_0^2 z_3^2 + \alpha_1 z_1^2 z_2^2 + \alpha_2 z_0 z_1 z_2 z_3 + \alpha_3(z_0 z_2^3 + z_1^3 z_3) = 0 \tag{6.1}$$

for all $v \in \mathbb{Z}$, where $z_i = x_{v+i}$. Then

$$(2z_0^2 z_3 + \alpha_2 z_0 z_1 z_2 + \alpha_3 z_1^3)^2 = (2z_{-1} z_2^2 + \alpha_2 z_0 z_1 z_2 + \alpha_3 z_1^3)^2 = D, \tag{6.2}$$

where

$$D = \alpha_3^2 z_1^6 + 2\alpha_2\alpha_3 z_0 z_1^4 z_2 + (\alpha_2^2 - 4\alpha_1) z_0^2 z_1^2 z_2^2 - 4\alpha_3 z_0^3 z_2^3, \tag{6.3}$$

for all $v \in \mathbb{Z}$, where again $z_i = x_{v+i}$.

Remark VI.2. The equation (6.1) involves 4 entries of the array $\mathbf{x}$ indexed by 4 consecutive integers $v, v+1, v+2, v+3$. This equation is invariant under the central symmetry of the line segment $[v, v+3]$. In other words, equation (6.1) is invariant under interchanging z_0 with z_3, and z_1 with z_2. The value D in (6.3) is the discriminant of equation (6.1) viewed as a quadratic equation in z_3. The term squared on the left-hand side of (6.2) is the partial derivative of the left-hand side of (6.1) with respect to z_3. This expression plays the role of K_v^C in Proposition II.8.

Theorem VI.3. *Let* $\alpha_2, \alpha_3 \leq 0$*, and* $\alpha_1 \leq \alpha_2^2/4$*. Let* $\mathbf{x} = (x_s) \in (\mathbb{R}_{>0})^{\mathbb{Z}}$ *satisfy the equation* (6.1) *for all* $v \in \mathbb{Z}$*, where, as before, we denote* $z_i = x_{v+i}$*. Moreover, assume that for all* $v \in \mathbb{Z}$*, the number* $z_3 = x_{v+3}$ *is the larger of the two real solutions of* (6.1)*:*

$$z_3 = \frac{-\alpha_3 z_1^3 - \alpha_2 z_0 z_1 z_2 + \sqrt{D}}{2z_0^2}, \tag{6.4}$$

where D *is given by* (6.3)*. Then*

$$2z_0^2 z_3 + \alpha_2 z_0 z_1 z_2 + \alpha_3 z_1^3 = 2z_{-1} z_2^2 + \alpha_2 z_0 z_1 z_2 + \alpha_3 z_1^3, \tag{6.5}$$

or equivalently,

$$z_0^2 z_3 = z_{-1} z_2^2 \tag{6.6}$$

for all $v \in \mathbb{Z}$*.*

We next show that an array $\mathbf{x}$ satisfying (6.1) satisfies condition (6.6) if and only if it can be extended to an array on a larger index set satisfying conditions resembling the K-hexahedron equations.

Theorem VI.4. *Let* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$*.*

(a) For any array $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\mathbb{Z}}$ *satisfying* (6.1) *and* (6.6), *there exists an array* $\mathbf{y} = (y_s)_{s\in\mathbb{Z}}$ *such that* $\mathbf{x}$ *and* $\mathbf{y}$ *together satisfy the recurrence*

$$z_3 = \frac{-\alpha_3 z_1^3 - \alpha_2 z_0 z_1 z_2 + w_1}{2z_0^2}, \tag{6.7}$$

$$w_2 = \frac{\alpha_3^2 z_1^6 + \alpha_2\alpha_3 z_0 z_1^4 z_2 + 2\alpha_3 z_0^3 z_2^3 + w_1^2 + (-2\alpha_3 z_1^3 - \alpha_2 z_0 z_1 z_2)w_1}{2z_0^3} \tag{6.8}$$

together with the condition

$$w_1^2 = D, \tag{6.9}$$

where D *is given by* (6.3), *and we use the notation* $z_i = x_{v+i}$ *and* $w_i = y_{v+i}$.

(b) Conversely, suppose $\mathbf{x} \in (\mathbb{C}^*)^{\mathbb{Z}}$ *and* $\mathbf{y} \in \mathbb{C}^{\mathbb{Z}}$ *satisfy* (6.7)–(6.9). *Then* $\mathbf{x}$ *satisfies* (6.1) *and* (6.6).

Remark VI.5. The components of $\mathbf{y}$ are most naturally indexed by intervals of length 2 in $\mathbb{Z}$. Here, we index the components of $\mathbf{y}$ by the midpoints of those intervals.

Remark VI.6. If one sets $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, -4)$ or $(\alpha_1, \alpha_2, \alpha_3) = (-3, -6, -4)$, then the pairs of arrays $\mathbf{x}, \mathbf{y}$ satisfying (6.7)–(6.9) are special cases of recurrences that arise from cluster algebras. The $(\alpha_1, \alpha_2, \alpha_3) = (-3, -6, -4)$ case is a special case of the K-hexahedron equations; namely, such pairs $\mathbf{x}, \mathbf{y}$ correspond to isotropic solutions $\tilde{\mathbf{z}} = (z_s) \in \mathbb{C}^L$ of the K-hexahedron equations where

$$z_{(i,j,k)} = x_{i+j+k}, \tag{6.10}$$

$$z_{(i,j,k)+\left(0,\frac{1}{2},\frac{1}{2}\right)} = z_{(i,j,k)+\left(\frac{1}{2},0,\frac{1}{2}\right)} = z_{(i,j,k)+\left(\frac{1}{2},\frac{1}{2},0\right)}, \tag{6.11}$$

$$4z^3_{(i,j,k)+\left(0,\frac{1}{2},\frac{1}{2}\right)} = y_{i+j+k+1}, \tag{6.12}$$

for $(i, j, k) \in \mathbb{Z}^3$. However, the $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, -4)$ case is not a special case of the K-hexahedron equations. We will discuss the related cluster algebra recurrences in later work [7].

Next, we consider the following two-dimensional example:

Proposition VI.7. *Let $\alpha_1, \alpha_2 \in \mathbb{C}$. Let $\mathbf{x} = (x_s) \in (\mathbb{C})^{\mathbb{Z}^2}$ be an array such that*

$$
\begin{aligned}
0 = z_{00}^2 z_{12}^2 + z_{10}^2 z_{02}^2 + \frac{\alpha_2^2 - \alpha_1^2}{4} z_{01}^2 z_{11}^2 - \alpha_1(z_{00}z_{02}z_{11}^2 + z_{10}z_{12}z_{01}^2) \\
-2z_{00}z_{10}z_{02}z_{12} - \alpha_2(z_{00}z_{12}z_{01}z_{11} + z_{10}z_{02}z_{01}z_{11})
\end{aligned}
\tag{6.13}
$$

for all $v \in \mathbb{Z}^2$, where $z_{ij} = x_{v+(i,j)}$. Given a 1×2 rectangle B with vertices in $\mathbb{Z}^2$, a vertex $w \in \mathbb{Z}^2$ at a corner of B, and the components of $\mathbf{x}$ at the 6 points of $\mathbb{Z}^2$ in B labeled as in Figure 6.1, define $R_w^{B,0}(\mathbf{x}) \in \mathbb{C}$ by

$$
R_w^{B,0}(\mathbf{x}) = z_{00}^2 z_{12} - \alpha_1 z_{10} z_{01}^2 - 2z_{00}z_{10}z_{02} - \alpha_2 z_{00}z_{01}z_{11}. \tag{6.14}
$$

Given a 0×2 rectangle (line segment) S with vertices in $\mathbb{Z}^2$, and the components of $\mathbf{x}$ at the 3 points of $\mathbb{Z}^2$ in S labeled as in Figure 6.1, define $R^{S,1}(\mathbf{x}) \in \mathbb{C}$ by

$$
R^{S,1}(\mathbf{x}) = \alpha_1 z_{01}^2 + 4z_{00}z_{02}. \tag{6.15}
$$

Given a 1×1 square C with vertices in $\mathbb{Z}^2$, and the components of $\mathbf{x}$ at the 4 points of $\mathbb{Z}^2$ in C labeled as in Figure 6.1, define $R^{C,2}(\mathbf{x}) \in \mathbb{C}$ by

$$
R^{C,2}(\mathbf{x}) = \alpha_1(z_{00}^2 z_{11}^2 + z_{01}^2 z_{10}^2) + 2\alpha_2 z_{00}z_{01}z_{10}z_{11}. \tag{6.16}
$$

Then for any $v \in \mathbb{Z}^2$,

$$
\left(\prod_{i=1}^{4} R_{w_i}^{B_i,0}(\mathbf{x})\right)^2 = \left(R^{S_1,1}(\mathbf{x}) R^{S_2,1}(\mathbf{x}) R^{C_1,2}(\mathbf{x}) R^{C_2,2}(\mathbf{x})\right)^2, \tag{6.17}
$$

where

- *$(B_1, w_1), (B_2, w_2), (B_3, w_3), (B_4, w_4)$ are the four 1×2 rectangle/corner pairs shown in Figure 6.2,*
- *S_1, S_2 are the two 0×2 rectangles (line segments) shown in Figure 6.2, and*
- *C_1, C_2 are the two 1×1 squares shown in Figure 6.2.*

Moreover, the following strengthening of (6.17) *holds:*

$$\left(R^{B_1,0}_{w_1}(\mathbf{x})R^{B_3,0}_{w_3}(\mathbf{x})\right)^2 = \left(R^{B_2,0}_{w_2}(\mathbf{x})R^{B_4,0}_{w_4}(\mathbf{x})\right)^2. \tag{6.18}$$

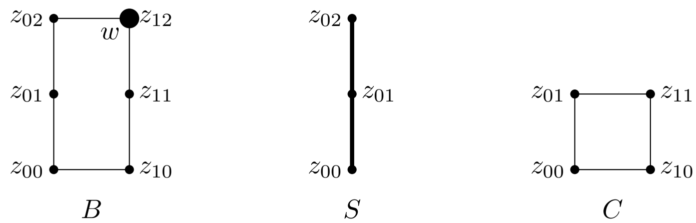


Figure 6.1: The components of $\mathbf{x}$ at a 1×2 rectangle B with distinguished vertex w in (6.14), at a 0×2 rectangle (line segment) S in (6.15), and at a 1×1 square C in (6.16).

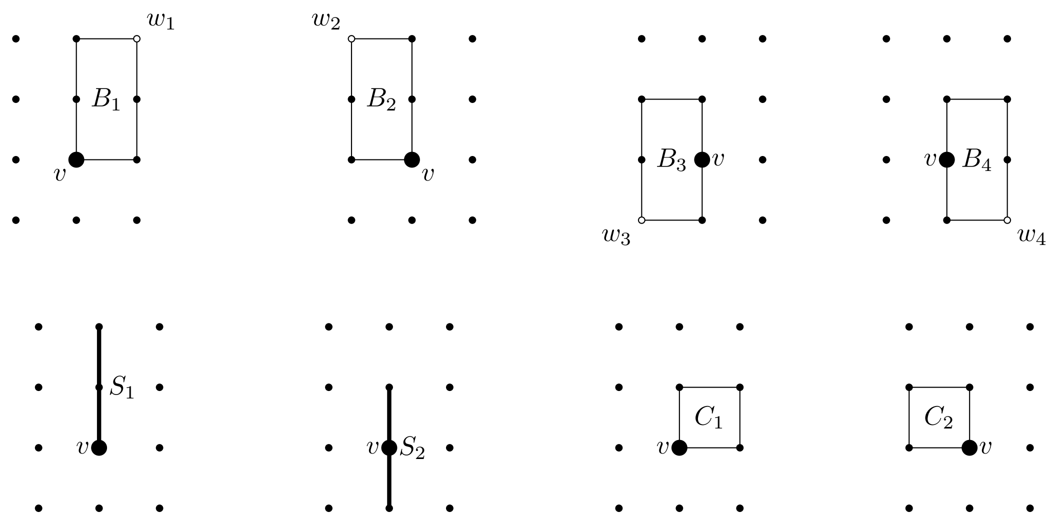


Figure 6.2: On the top row, the rectangle/corner pairs (B_i, w_i) for $i = 1, \ldots, 4$, and on the bottom row, the line segments S_1, S_2 and the squares C_1, C_2 that appear in (6.17).

Remark VI.8. Let $A = \{v, v + (0, 1)\}$. In equation (6.17), B_1, B_2, B_3, B_4 are the four 1×2 rectangles with vertices in $\mathbb{Z}^2$ containing A, S_1, S_2 are the two 0×2 rectangles (line segments) with vertices in $\mathbb{Z}^2$ containing A, and C_1, C_2 are the two 1×1 unit squares with vertices in $\mathbb{Z}^2$ containing A. Each w_i is the farthest vertex from A in B_i.

Theorem VI.9. *Let* $\alpha_1, \alpha_2 \geq 0$. *Define* $R^{B,0}_w$, $R^{S,1}$, *and* $R^{C,2}$ *as in* (6.14)–(6.16). *Let* $\mathbf{x} = (x_s) \in (\mathbb{R}_{>0})^{\mathbb{Z}^2}$ *satisfy the equation* (6.13), *where, as before, we denote*

$z_{ij} = x_{v+(i,j)}$. *Moreover, assume that for all* $v \in \mathbb{Z}^2$, *the number* $z_{12} = x_{v+(1,2)}$ *is the larger of the two real solutions of* (6.13)*:*

$$z_{12} = \frac{\alpha_1 z_{10} z_{01}^2 + 2z_{00} z_{10} z_{02} + \alpha_2 z_{00} z_{01} z_{11} + \sqrt{D}}{2z_{00}^2} \tag{6.19}$$

where

$$\begin{aligned} D &= (\alpha_1 z_{01}^2 + 4z_{00} z_{02})(\alpha_1(z_{00}^2 z_{11}^2 + z_{01}^2 z_{10}^2) + 2\alpha_2 z_{00} z_{01} z_{10} z_{11}) \\ &= R^{S,1}(\mathbf{x}) R^{C,2}(\mathbf{x}), \end{aligned} \tag{6.20}$$

and $S = v + (\{0\} \times \{0,1,2\})$, $C = v + \{0,1\}^2$. *Then*

$$\prod_{i=1}^{4} R_{w_i}^{B_i,0}(\mathbf{x}) = R^{S_1,1}(\mathbf{x}) R^{S_2,1}(\mathbf{x}) R^{C_1,2}(\mathbf{x}) R^{C_2,2}(\mathbf{x}), \tag{6.21}$$

and

$$R_{w_1}^{B_1,0}(\mathbf{x}) R_{w_3}^{B_3,0}(\mathbf{x}) = R_{w_2}^{B_2,0}(\mathbf{x}) R_{w_4}^{B_4,0}(\mathbf{x}), \tag{6.22}$$

for all $v \in \mathbb{Z}^2$, *using the same notation as in equation* (6.17)*.*

Theorem VI.10. *Let* $\alpha_1, \alpha_2 \in \mathbb{C}$.

(a) Suppose $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\mathbb{Z}^2}$ *satisfies* (6.13) *and* (6.21)*, and* $R^{S,1}(\mathbf{x}) \neq 0$ *and* $R^{C,2}(\mathbf{x}) \neq 0$ *for any* 0×2 *rectangle* S *with vertices in* $\mathbb{Z}^2$ *and* 1×1 *square* C *with vertices in* $\mathbb{Z}^2$. *Then there exist arrays* $\mathbf{y}_1 = (y_s)_{s \in \mathbb{Z}^2}$ *and* $\mathbf{y}_2 = (y_s)_{s \in \left(\mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right)}$ *such that* $\mathbf{x}$, $\mathbf{y}_1$, *and* $\mathbf{y}_2$ *together satisfy the recurrence*

$$z_{12} = \frac{\alpha_1 z_{10} z_{01}^2 + 2z_{00} z_{10} z_{02} + \alpha_2 z_{00} z_{01} z_{11} + w_{01} w_{\frac{1}{2}\frac{1}{2}}}{2z_{00}^2}, \tag{6.23}$$

$$w_{11} = \frac{z_{10} w_{01} + w_{\frac{1}{2}\frac{1}{2}}}{z_{00}}, \tag{6.24}$$

$$w_{\frac{3}{2}\frac{1}{2}} = \frac{z_{01}(\alpha_1 z_{01} z_{10} + \alpha_2 z_{00} z_{11}) w_{01} + (\alpha_1 z_{01}^2 + 2z_{00} z_{02}) w_{\frac{1}{2}\frac{1}{2}}}{2z_{00}^2}, \tag{6.25}$$

together with the conditions

$$w_{01}^2 = \alpha_1 z_{01}^2 + 4z_{00} z_{02}, \tag{6.26}$$

$$w_{\frac{1}{2}\frac{1}{2}}^2 = \alpha_1(z_{00}^2 z_{11}^2 + z_{01}^2 z_{10}^2) + 2\alpha_2 z_{00} z_{01} z_{10} z_{11}, \tag{6.27}$$

where $z_{ij} = x_{v+(i,j)}$ *and* $w_{ij} = y_{v+(i,j)}$.

(b) *Conversely, suppose* $\mathbf{x} \in (\mathbb{C}^*)^{\mathbb{Z}^2}$, $\mathbf{y}_1 \in \mathbb{C}^{\mathbb{Z}^2}$, *and* $\mathbf{y}_2 \in \mathbb{C}^{\mathbb{Z}^2+\left(\frac{1}{2},\frac{1}{2}\right)}$ *satisfy* (6.23)–(6.27). *Then* $\mathbf{x}$ *satisfies* (6.13) *and* (6.21).

Remark VI.11. In Theorem VI.10, the components of $\mathbf{y}_1$ are most naturally associated to 0×2 rectangles (line segments) with vertices in $\mathbb{Z}^2$ (although we index it by the center of the line segment in the theorem), and the components of $\mathbf{y}_2$ are most naturally associated to 1×1 unit squares with vertices in $\mathbb{Z}^2$ (although we index it by the center of the strip in the theorem). If we think about the recurrence (6.23)–(6.25) in this way, each step of the recurrence uses the six vertices, two 0×2 rectangles, and two 1×1 unit squares contained in the 1×2 rectangle $v + \{0, 1\} \times \{0, 1, 2\}$, as is pictured in Figure 6.3.

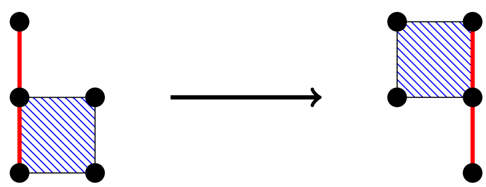


Figure 6.3: The vertices/line segments/squares indexing the values in a step of the recurrence (6.23)–(6.25). The black vertices index the values of $\mathbf{x}$, the red line segments index the values of $\mathbf{y}_1$, and the blue squares index the values of $\mathbf{y}_2$.

Remark VI.12. If one sets $(\alpha_1, \alpha_2) = (4, 4)$ or $(\alpha_1, \alpha_2) = (4, 0)$, then the arrays $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2$ satisfying (6.23)–(6.27) are special cases of recurrences that arise from cluster algebras. The $(\alpha_1, \alpha_2) = (4, 4)$ case is a special case of the K-hexahedron equations; namely, such $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2$ correspond to isotropic solutions $\tilde{\mathbf{z}} = (z_s) \in \mathbb{C}^L$ of the

K-hexahedron equations where

$$z_{(i,j,k)} = x_{(i,j+k)}, \tag{6.28}$$

$$2z_{(i,j,k)} = y_{(i,j+k+1)}, \tag{6.29}$$

$$z_{(i,j,k)+\left(\frac{1}{2},0,\frac{1}{2}\right)} = z_{(i,j,k)+\left(\frac{1}{2},\frac{1}{2},0\right)}, \tag{6.30}$$

$$2z^2_{(i,j,k)+\left(\frac{1}{2},0,\frac{1}{2}\right)} = y_{\left(i+\frac{1}{2},j+k+\frac{1}{2}\right)} \tag{6.31}$$

for $(i,j,k) \in \mathbb{Z}^3$. However, the $(\alpha_1, \alpha_2) = (4,0)$ case is not a special case of the K-hexahedron equations. We will discuss the related cluster algebra recurrences in future work [7]. Using machinery from cluster algebras, we can prove Laurentness results for these recurrences similar to the one given by Kenyon and Pemantle in [5, Theorem 6.8].

CHAPTER VII

Proofs of Results from Chapter II

In this chapter, we prove Proposition II.8 and Theorems II.22–II.23 (of which all other results in Chapter II are corollaries).

Lemma VII.1. *Let C be a cube with vertices $V(C)$ labeled as in Figure 7.1, and let $\mathbf{x} = (x_s) \in \mathbb{C}^{V(C)}$. Then*

$$(K_v^C(\mathbf{x}))^2 = \frac{x_v^2}{4} K^C(\mathbf{x}) + (x_v x_d + x_b x_c)(x_v x_e + x_a x_c)(x_v x_f + x_a x_b). \tag{7.1}$$

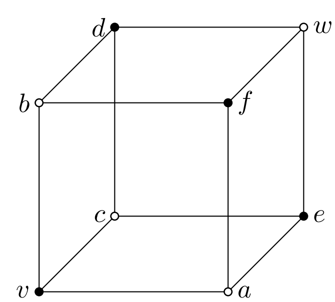


Figure 7.1: Labels for the vertices of a cube C.

Proof. The proof follows from a straightforward computation. □

Proof of Proposition II.8. Let $\mathbf{x} = (x_s) \in \mathbb{C}^{\mathbb{Z}^3}$ satisfy the Kashaev equation. Given a unit cube C of $\mathbb{Z}^3$ labeled as in Figure 7.1, by Lemma VII.1, we have

$$\begin{aligned}(K_v^C(\mathbf{x}))^2 &= \frac{x_v^2}{4}K^C(\mathbf{x}) + (x_vx_d + x_bx_c)(x_vx_e + x_ax_c)(x_vx_f + x_ax_b) \\ &= (x_vx_d + x_bx_c)(x_vx_e + x_ax_c)(x_vx_f + x_ax_b).\end{aligned} \tag{7.2}$$

Taking the product over unit cubes C of $\mathbb{Z}^3$ containing v, we obtain

$$\left(\prod_{C\ni v} K_v^C(\mathbf{x})\right)^2 = \prod_{C\ni v}\prod_{C\supset S\ni v}(x_vx_{v_2} + x_{v_1}x_{v_3}) = \left(\prod_{S\ni v}(x_vx_{v_2} + x_{v_1}x_{v_3})\right)^2; \tag{7.3}$$

here we use that the double product counts each unit square containing v twice.

Similarly,

$$\begin{aligned}\left(\prod_{\substack{C=C_v(i_1,i_2,i_3)\\ i_1,i_2,i_3\in\{-1,1\}\\ i_1i_2i_3=1}} K_v^C(\mathbf{x})\right)^2 &= \prod_{\substack{C=C_v(i_1,i_2,i_3)\\ i_1,i_2,i_3\in\{-1,1\}\\ i_1i_2i_3=1}}\prod_{C\supset S\ni v}(x_vx_{v_2} + x_{v_1}x_{v_3}) \\ &= \prod_{S\ni v}(x_vx_{v_2} + x_{v_1}x_{v_3}) \\ &= \prod_{\substack{C=C_v(i_1,i_2,i_3)\\ i_1,i_2,i_3\in\{-1,1\}\\ i_1i_2i_3=-1}}\prod_{C\supset S\ni v}(x_vx_{v_2} + x_{v_1}x_{v_3}) \\ &= \left(\prod_{\substack{C=C_v(i_1,i_2,i_3)\\ i_1,i_2,i_3\in\{-1,1\}\\ i_1i_2i_3=-1}} K_v^C(\mathbf{x})\right)^2,\end{aligned} \tag{7.4}$$

because each double product counts each unit square containing v twice. $\square$

For the proof of Theorem II.22(b), we will need the following lemma.

Lemma VII.2. *Suppose $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$ (with $x_s \neq 0$ for $s \in \mathbb{Z}^3$) satisfies the K-hexahedron equations. Let $\mathbf{x} \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ be the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^3$. Let $v \in \mathbb{Z}^3$ and $\mathbf{i} = (i_1, i_2, i_3) \in \{-1,1\}^3$, and let C be the unique unit cube in $\mathbb{Z}^3$ containing vertices v and $v + \mathbf{i}$. Then*

$$K_v^C(\mathbf{x}) = \pm x_{v+\frac{1}{2}(0,i_2,i_3)}x_{v+\frac{1}{2}(i_1,0,i_3)}x_{v+\frac{1}{2}(i_1,i_2,0)}, \tag{7.5}$$

where the sign is positive if $\mathbf{i} \in \{\pm(1,1,1)\}$*, and negative otherwise.*

Proof. Let $R = \mathbb{C}[z_{ijk} : 0 \leq i,j,k \leq 1, (i,j,k) \in L]$. With the equations (2.21)–(2.24) in mind, we define $p_0, p_1, p_2, p_3 \in R$ by

$$\begin{aligned} p_1 &= z_{1\frac{1}{2}\frac{1}{2}} z_{000} - z_{\frac{1}{2}0\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}0} - z_{0\frac{1}{2}\frac{1}{2}} z_{100}, \\ p_2 &= z_{\frac{1}{2}1\frac{1}{2}} z_{000} - z_{0\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}0} - z_{\frac{1}{2}0\frac{1}{2}} z_{010}, \\ p_3 &= z_{\frac{1}{2}\frac{1}{2}1} z_{000} - z_{0\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}0\frac{1}{2}} - z_{\frac{1}{2}\frac{1}{2}0} z_{001}, \\ p_4 &= z_{111} z_{000}^2 - A - 2 z_{0\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}0\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}0}, \end{aligned} \tag{7.6}$$

where A is the expression given in (2.4). With (2.12) in mind, we define $q_1, \ldots, q_6 \in R$ by

$$\begin{aligned} q_1 &= z_{0\frac{1}{2}\frac{1}{2}}^2 - z_{011} z_{000} - z_{010} z_{001}, & q_4 &= z_{1\frac{1}{2}\frac{1}{2}}^2 - z_{111} z_{100} - z_{110} z_{101}, \\ q_2 &= z_{\frac{1}{2}0\frac{1}{2}}^2 - z_{101} z_{000} - z_{100} z_{001}, & q_5 &= z_{\frac{1}{2}1\frac{1}{2}}^2 - z_{111} z_{010} - z_{110} z_{011}, \\ q_3 &= z_{\frac{1}{2}\frac{1}{2}0}^2 - z_{110} z_{000} - z_{100} z_{010}, & q_6 &= z_{\frac{1}{2}\frac{1}{2}1}^2 - z_{111} z_{001} - z_{101} z_{011}. \end{aligned} \tag{7.7}$$

Let $I = \langle p_0, p_1, p_2, p_3, q_1, \ldots, q_6 \rangle$ be the ideal in R generated by these elements. Let v' be the "bottom" vertex in the cube C, i.e., $v' = v + (\min(0,i_1), \min(0,i_2), \min(0,i_3))$. Because $\tilde{\mathbf{x}}$ satisfies the K-hexahedron equations, it follows that if $g \in I$, then specializing $z_{ijk} = x_{v'+(i,j,k)}$ in g yields 0. Given $j_\ell = \min(i_\ell, 0) + 1$ and $k_\ell = 1 - j_\ell$ for $\ell = 1,2,3$, let

$$\begin{aligned} K = \frac{1}{2}&(z_{k_1k_2k_3} z_{j_1j_2j_3}^2 - z_{j_1j_2j_3}(z_{k_1j_2j_3} z_{j_1k_2k_3} + z_{j_1k_2j_3} z_{k_1j_2k_3} + z_{j_1j_2k_3} z_{k_1k_2j_3})) \\ &- z_{k_1j_2j_3} z_{j_1k_2j_3} z_{j_1j_2k_3}. \end{aligned} \tag{7.8}$$

Note that specializing $z_{ijk} = x_{v'+(i,j,k)}$ in K yields $K_v^C(\mathbf{x})$. It can be checked that

$$z_{000}^3 \left(K - z_{j_1\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}j_2\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}j_3} \right) \in I \tag{7.9}$$

if $\mathbf{i} \in \{\pm(1,1,1)\}$, and that

$$z_{000}^2 \left(K + z_{j_1\frac{1}{2}\frac{1}{2}} z_{\frac{1}{2}j_2\frac{1}{2}} z_{\frac{1}{2}\frac{1}{2}j_3} \right) \in I \tag{7.10}$$

otherwise. Because $x_{v'} \neq 0$, the lemma follows. □

Proof of Theorem II.22(b). Let $\mathbf{x} \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ be the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^3$. Applying Lemma VII.2 for the 8 cubes incident to a vertex $v \in \mathbb{Z}^3$, we get:

$$\begin{aligned}\prod_{C \ni v} K_v^C(\mathbf{x}) &= (-1)^6 \prod_{(i_1,i_2,i_3)\in\{-1,1\}^3} x_{v+\frac{1}{2}(0,i_2,i_3)} x_{v+\frac{1}{2}(i_1,0,i_3)} x_{v+\frac{1}{2}(i_1,i_2,0)} \\ &= \prod_{S \ni v} x_S^2 \\ &= \prod_{S \ni v} (x_v x_{v_2} + x_{v_1} x_{v_3}),\end{aligned} \tag{7.11}$$

so $\mathbf{x}$ is a coherent solution of the Kashaev equation. □

Proposition VII.3. *Let $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$, with $x_s \neq 0$ for $s \in \mathbb{Z}^3$. The following are equivalent:*

- *$\tilde{\mathbf{x}}$ satisfies the K-hexahedron equations;*
- *the array $(x_{-s})_{s \in L}$ satisfies the K-hexahedron equations.*

Proof. The proof follows from a straightforward computation. □

Proposition VII.3 enables us to run the K-hexahedron equations "in reverse." We note that the property that we show for the K-hexahedron equations in Proposition VII.3 holds for the original hexahedron recurrence.

Remark VII.4. Here, we address the comments in Remark II.15. Let $\tilde{\mathbf{x}}$ be an array of 14 numbers indexed by the vertices and 2-dimensional faces of a cube C with a distinguished "top" vertex v, where the components of $\tilde{\mathbf{x}}$ indexed by the 8 vertices of the cube are nonzero. Suppose $\tilde{\mathbf{x}}$ satisfies the K-hexahedron equations. Proposition VII.3 tells us that $\tilde{\mathbf{x}}$ would satisfy the K-hexahedron equations if we took the vertex w opposite v to be the "top" vertex of C. On the other hand, Lemma VII.2 tells us that if the components of $\tilde{\mathbf{x}}$ indexed by the faces are nonzero, and w is a vertex of C other than v or the vertex opposite it, then $\tilde{\mathbf{x}}$ would not satisfy the K-hexahedron equations if we place the vertex w at the "top" of C. This argument also implies the analogous statement for the hexahedron recurrence.

We next work toward a proof of Theorem II.22(a).

Lemma VII.5. *Fix $v \in \mathbb{Z}^3$ and $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$ satisfying the equations* (2.21)–(2.24), *with $x_s \neq 0$ for $s \in \mathbb{Z}^3$. Then the following are equivalent:*

- *the following equations hold:*

$$x^2_{v+\left(0,\frac{1}{2},\frac{1}{2}\right)} = x_v x_{v+(0,1,1)} + x_{v+(0,1,0)} x_{v+(0,0,1)}; \tag{7.12}$$

$$x^2_{v+\left(\frac{1}{2},0,\frac{1}{2}\right)} = x_v x_{v+(1,0,1)} + x_{v+(1,0,0)} x_{v+(0,0,1)}; \tag{7.13}$$

$$x^2_{v+\left(\frac{1}{2},\frac{1}{2},0\right)} = x_v x_{v+(1,1,0)} + x_{v+(1,0,0)} x_{v+(0,1,0)}. \tag{7.14}$$

- *the following equations hold:*

$$x^2_{v+\left(1,\frac{1}{2},\frac{1}{2}\right)} = x_{v+(1,0,0)} x_{v+(1,1,1)} + x_{v+(1,1,0)} x_{v+(1,0,1)}; \tag{7.15}$$

$$x^2_{v+\left(\frac{1}{2},1,\frac{1}{2}\right)} = x_{v+(0,1,0)} x_{v+(1,1,1)} + x_{v+(1,1,0)} x_{v+(0,1,1)}; \tag{7.16}$$

$$x^2_{v+\left(\frac{1}{2},\frac{1}{2},1\right)} = x_{v+(0,0,1)} x_{v+(1,1,1)} + x_{v+(1,0,1)} x_{v+(0,1,1)}. \tag{7.17}$$

Proof. This is a straightforward verification. □

Definition VII.6. For $U \subseteq \mathbb{Z}$, we denote

$$\mathbb{Z}^3_U = \{(i,j,k) \in \mathbb{Z}^3 : i+j+k \in U\}. \tag{7.18}$$

For $U, V \subseteq \mathbb{Z}$, we denote

$$L_{U,V} = \mathbb{Z}^3_U \cup \left\{(i,j,k) \in L - \mathbb{Z}^3 : i+j+k \in V\right\}. \tag{7.19}$$

In other words, $L_{U,V}$ contains the integer points at heights in U, and the half-integer points of L at heights in V. In particular, we will be interested in $\mathbb{Z}^3_{\text{init}} = \mathbb{Z}^3_{\{0,1,2\}}$ and $L_{\text{init}} = L_{\{0,1,2\},\{1\}}$.

Corollary VII.7. *Suppose that* $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$ *(with* $x_s \neq 0$ *for* $s \in \mathbb{Z}^3$*) satisfies the recurrence* (2.21)–(2.24)*. If* $\tilde{\mathbf{x}}$ *satisfies* (2.12) *for all* $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$*, then* $\tilde{\mathbf{x}}$ *satisfies* (2.12) *for all* $s \in L - \mathbb{Z}^3$*; in other words,* $\tilde{\mathbf{x}}$ *satisfies the K-hexahedron equations.*

Proof. This follows directly from Lemma VII.5. □

Remark VII.8. By Proposition VII.3, given $\tilde{\mathbf{x}}_{\text{init}} = (x_s) \in \mathbb{C}^{L_{\text{init}}}$ with $x_s \neq 0$ for all $s \in \mathbb{Z}^3_{\text{init}}$, there exists at most one extension $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^L$ of $\tilde{\mathbf{x}}_{\text{init}}$ to L (with $x_s \neq 0$ for all $s \in \mathbb{Z}^3$) satisfying the K-hexahedron equations. We say "at most one" instead of "one" because in the course of running the recurrence (2.21)–(2.24), we might get a zero value at an integer point.

Definition VII.9. We say that an array $\tilde{\mathbf{x}}_{\text{init}}$ indexed by L_{init} that satisfies equation (2.12) for all $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$ is *generic* if $\tilde{\mathbf{x}}_{\text{init}}$ can be extended to an array $\tilde{\mathbf{x}}$ indexed by L satisfying the K-hexahedron equations, with all components of $\tilde{\mathbf{x}}$ nonzero. Similarly, we say that an array $\mathbf{x}_{\text{init}}$ indexed by $\mathbb{Z}^3_{\text{init}}$ is generic if every extension of $\mathbf{x}_{\text{init}}$ to an array $\tilde{\mathbf{x}}_{\text{init}}$ indexed by L_{init} satisfying equation (2.12) for all $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$ is generic.

Definition VII.10. Let $\tilde{\mathbf{x}}_{\text{init}}$ be a generic array indexed by L_{init} that satisfies equation (2.12) for $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$. We denote by $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}$ the unique extension of $\tilde{\mathbf{x}}_{\text{init}}$ to L satisfying the K-hexahedron equations.

Lemma VII.11. *Let* $C = [0,1]^3$ *be a unit cube. Fix values* $t_{(0,\frac{1}{2},\frac{1}{2})}, t_{v+(\frac{1}{2},0,\frac{1}{2})}, t_{v+(\frac{1}{2},\frac{1}{2},0)} \in \{-1,1\}$*. Suppose* $\tilde{\mathbf{x}} = (x_s)_{s \in C \cap L}$ *and* $\tilde{\mathbf{y}} = (y_s)_{s \in C \cap L}$ *are arrays of* 14 *complex numbers indexed by the vertices and faces of* C *such that*

- $\tilde{\mathbf{x}}$ *and* $\tilde{\mathbf{y}}$ *both satisfy the K-hexahedron equations,*
- $x_s, y_s \neq 0$ *for* $s \in \{0,1\}^3$

- *$y_s = x_s$ for all $s \in \{0,1\}^3 \setminus \{(1,1,1)\}$,*
- *$y_s = t_s x_s$ for all $s \in \left\{\left(0,\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right\}$, and*
- *$t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} t_{v+\left(\frac{1}{2},0,\frac{1}{2}\right)} t_{v+\left(\frac{1}{2},\frac{1}{2},0\right)} = 1$.*

Then the following equations hold:

$$y_{\left(1,\frac{1}{2},\frac{1}{2}\right)} = t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} x_{\left(1,\frac{1}{2},\frac{1}{2}\right)}; \tag{7.20}$$

$$y_{\left(\frac{1}{2},1,\frac{1}{2}\right)} = t_{\left(\frac{1}{2},0,\frac{1}{2}\right)} x_{\left(\frac{1}{2},1,\frac{1}{2}\right)}; \tag{7.21}$$

$$y_{\left(\frac{1}{2},\frac{1}{2},1\right)} = t_{\left(\frac{1}{2},\frac{1}{2},0\right)} x_{\left(\frac{1}{2},\frac{1}{2},1\right)}; \tag{7.22}$$

$$y_{(1,1,1)} = x_{(1,1,1)}. \tag{7.23}$$

Proof. Note that

$$y_{\left(1,\frac{1}{2},\frac{1}{2}\right)} = \frac{t_{\left(\frac{1}{2},0,\frac{1}{2}\right)} t_{\left(\frac{1}{2},\frac{1}{2},0\right)} x_{\left(\frac{1}{2},0,\frac{1}{2}\right)} x_{\left(\frac{1}{2},\frac{1}{2},0\right)} + t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} x_{\left(0,\frac{1}{2},\frac{1}{2}\right)} x_{(1,0,0)}}{x^2_{(0,0,0)}} \tag{7.24}$$

Either $t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} = 1$ and $t_{\left(\frac{1}{2},0,\frac{1}{2}\right)} t_{\left(\frac{1}{2},\frac{1}{2},0\right)} = 1$, or $t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} = -1$ and $t_{\left(\frac{1}{2},0,\frac{1}{2}\right)} t_{\left(\frac{1}{2},\frac{1}{2},0\right)} = -1$. By equation (7.24), $y_{\left(1,\frac{1}{2},\frac{1}{2}\right)} = t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} x_{\left(1,\frac{1}{2},\frac{1}{2}\right)}$ in either case. Equations (7.21)–(7.22) hold by the same argument. Furthermore, note that

$$y_{(1,1,1)} - x_{(1,1,1)} = 2\left(t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} t_{v+\left(\frac{1}{2},0,\frac{1}{2}\right)} t_{v+\left(\frac{1}{2},\frac{1}{2},0\right)} - 1\right) \frac{x_{\left(0,\frac{1}{2},\frac{1}{2}\right)} x_{\left(0,\frac{1}{2},\frac{1}{2}\right)} x_{\left(0,\frac{1}{2},\frac{1}{2}\right)}}{x^2_{(0,0,0)}} = 0, \tag{7.25}$$

so equation (7.23) holds. □

The idea behind Lemma VII.11 is that if $t_{\left(0,\frac{1}{2},\frac{1}{2}\right)} t_{v+\left(\frac{1}{2},0,\frac{1}{2}\right)} t_{v+\left(\frac{1}{2},\frac{1}{2},0\right)} = 1$, then each sign t_s propagates to the face of C opposite s. With this in mind, we make the following definition.

Definition VII.12. Define an equivalence relation on $L - \mathbb{Z}^3$ by setting $s_1 \sim s_2$ if and only if

- $s_1 = \left(a_1, b+\frac{1}{2}, c+\frac{1}{2}\right)$ and $s_2 = \left(a_2, b+\frac{1}{2}, c+\frac{1}{2}\right)$,
- $s_1 = \left(a+\frac{1}{2}, b_1, c+\frac{1}{2}\right)$ and $s_2 = \left(a+\frac{1}{2}, b_2, c+\frac{1}{2}\right)$, or
- $s_1 = \left(a+\frac{1}{2}, b+\frac{1}{2}, c_1\right)$ and $s_2 = \left(a+\frac{1}{2}, b+\frac{1}{2}, c_2\right)$.

Let $\mathbb{Z}^3_{\boxdot}$ denote the set of equivalence classes under this equivalence relation. Denote by $[s] \in \mathbb{Z}^3_{\boxdot}$ the equivalence class of $s \in L - \mathbb{Z}^3$. If we think of $L - \mathbb{Z}^3$ as the set of unit squares in $\mathbb{Z}^3$, then the equivalence relation $\sim$ is generated by the equivalences $s_1 \sim s_2$ where s_1 and s_2 are opposite faces of a unit cube in $\mathbb{Z}^3$. Geometrically, we can think of an element $[s] \in \mathbb{Z}^3_{\boxdot}$ as the line through the point s which is perpendicular to the corresponding unit square. Note that $L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$ is a set of representatives of $\mathbb{Z}^3_{\boxdot}$.

Definition VII.13. Define an action of $\{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$ on arrays indexed by L_{init} as follows: given $\mathbf{t} = (t_s) \in \{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$ and $\tilde{\mathbf{x}}_{\text{init}} = (x_s)_{s \in L_{\text{init}}}$, set $\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}} = (\tilde{x}_s)_{s \in L_{\text{init}}}$, where

$$\tilde{x}_s = \begin{cases} x_s & \text{if } s \in \mathbb{Z}^3_{\text{init}} \\ t_{[s]} x_s & \text{if } L_{\text{init}} - \mathbb{Z}^3_{\text{init}}. \end{cases} \tag{7.26}$$

Remark VII.14. Assume that $\mathbf{x}_{\text{init}}$ is a generic array indexed by $\mathbb{Z}^3_{\text{init}}$. Let $\tilde{\mathbf{x}}_{\text{init}}$ be any (generic) array indexed by L_{init} that restricts to $\mathbf{x}_{\text{init}}$ and satisfies equation (2.12) for $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$. Note that

$$\left\{ (\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}})^{\uparrow L} : \mathbf{t} \in \{-1,1\}^{\mathbb{Z}^3_{\boxdot}} \right\} \tag{7.27}$$

is the set of arrays indexed by L which satisfy the K-hexahedron equations and restrict to $\mathbf{x}_{\text{init}}$.

Definition VII.15. For $\mathbf{t} = (t_s) \in \{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$, define $\psi(\mathbf{t}) = (u_s) \in \{-1,1\}^{\mathbb{Z}^3 + \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$

by setting

$$u_{s+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = t_{\left[s+\left(0,\frac{1}{2},\frac{1}{2}\right)\right]} t_{\left[s+\left(\frac{1}{2},0,\frac{1}{2}\right)\right]} t_{\left[s+\left(\frac{1}{2},\frac{1}{2},0\right)\right]} \tag{7.28}$$

for $s \in \mathbb{Z}^3$. If we think of the elements of $\mathbb{Z}^3_{\boxdot}$ as lines in $\mathbb{R}^3$, then u_s is the product of the components of $\mathbf{t}$ indexed by the 3 lines in $\mathbb{Z}^3_{\boxdot}$ passing through the point s. If we think of $\mathbb{Z}^3_{\boxdot}$ as equivalence classes of unit squares in $\mathbb{Z}^3$, then u_s is the product of the components of $\mathbf{t}$ indexed by the 3 equivalence classes of 2-dimensional faces of the unit cube centered at s.

Proposition VII.16. *An array* $\mathbf{t} = (t_s) \in \{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$ *is in the kernel of* ψ *(i.e., formula* (7.28) *yields* 1 *for all* s*) if and only if there exist signs* $\alpha_i, \beta_i, \gamma_i \in \{-1,1\}$, $i \in \mathbb{Z}$, *such that*

$$t_{\left[\left(a,b+\frac{1}{2},c+\frac{1}{2}\right)\right]} = \beta_b \gamma_c \tag{7.29}$$

$$t_{\left[\left(a+\frac{1}{2},b,c+\frac{1}{2}\right)\right]} = \alpha_a \gamma_c \tag{7.30}$$

$$t_{\left[\left(a+\frac{1}{2},b+\frac{1}{2},c\right)\right]} = \alpha_a \beta_b \tag{7.31}$$

for all $(a,b,c) \in \mathbb{Z}^3$.

Proof. Suppose there exist constants $\alpha_i, \beta_i, \gamma_i \in \{-1,1\}$ for $i \in \mathbb{Z}$ such that $\mathbf{t}$ satisfies equations (7.29)–(7.31). Let $\mathbf{u} = (u_s) = \psi(\mathbf{t})$. Then for any $(a,b,c) \in \mathbb{Z}^3$, $u_{(a,b,c)+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = \alpha_a^2 \beta_b^2 \gamma_c^2 = 1$, as desired.

Next, suppose that $\mathbf{t}$ is in the kernel of ψ. It is straightforward to check that the following identities for $\mathbf{t}$ for all $(a,b,c) \in \mathbb{Z}^3$:

$$t_{\left[\left(a,b+\frac{1}{2},c+\frac{1}{2}\right)\right]} = t_{\left[\left(a+\frac{1}{2},0,c+\frac{1}{2}\right)\right]} t_{\left[\left(a+\frac{1}{2},b+\frac{1}{2},0\right)\right]}; \tag{7.32}$$

$$t_{\left[\left(a+\frac{1}{2},b,c+\frac{1}{2}\right)\right]} = t_{\left[\left(\frac{1}{2},\frac{1}{2},0\right)\right]} t_{\left[\left(a+\frac{1}{2},\frac{1}{2},0\right)\right]} t_{\left[\left(\frac{1}{2},b+\frac{1}{2},0\right)\right]}; \tag{7.33}$$

$$t_{\left[\left(a+\frac{1}{2},b+\frac{1}{2},c\right)\right]} = t_{\left[\left(\frac{1}{2},\frac{1}{2},0\right)\right]} t_{\left[\left(a+\frac{1}{2},\frac{1}{2},0\right)\right]} t_{\left[\left(\frac{1}{2},b+\frac{1}{2},0\right)\right]}. \tag{7.34}$$

Hence, setting

$$\alpha_a = t_{\left[\left(\frac{1}{2},0,\frac{1}{2}\right)\right]}\alpha_a = t_{\left[\left(a+\frac{1}{2},0,\frac{1}{2}\right)\right]}, \tag{7.35}$$

$$\beta_b = t_{\left[\left(\frac{1}{2},b+\frac{1}{2},0\right)\right]}, \tag{7.36}$$

$$\gamma_c = t_{\left[\left(\frac{1}{2},0,c+\frac{1}{2}\right)\right]}, \tag{7.37}$$

it follows that $\mathbf{t}$ satisfies (7.29)–(7.31). $\square$

Definition VII.17. For $(a_1,\dots,a_d),(b_1,\dots,b_d)\in\mathbb{R}^d$, the notation $(a_1,\dots,a_d)\le(b_1,\dots,b_d)$ will mean that $a_i\le b_i$ for $i=1,\dots,d$, and $(a_1,\dots,a_d)<(b_1,\dots,b_d)$ will mean that $(a_1,\dots,a_d)\le(b_1,\dots,b_d)$ but $(a_1,\dots,a_d)\ne(b_1,\dots,b_d)$.

Lemma VII.18. *Let $\tilde{\mathbf{x}}_{\text{init}}$ be a generic array indexed by L_{init} satisfying equation* (2.12) *for $s\in L_{\text{init}}-\mathbb{Z}^3_{\text{init}}$. Let $\mathbf{t}\in\{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$, and $\mathbf{u}=(u_s)_{s\in\mathbb{Z}^3+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}=\psi(\mathbf{t})$. Denote $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}=(x_s)_{s\in L}$, and $(\mathbf{t}\cdot\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}=(y_s)_{s\in L}$. Suppose $v\in\mathbb{Z}^3_{\{3,4,5,\dots\}}$ satisfies the condition that $u_{w-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}=1$ for all $w\in\mathbb{Z}^3_{\{3,4,5,\dots\}}$ with $w\le v$. Then:*

(a) $y_v=x_v$;

(b) $y_{v-s}=t_{[v-s]}x_{v-s}$ for $s\in\left\{\left(0,\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right\}$.

Proof. We prove statements (a) and (b) together for $w\in\mathbb{Z}^3_{\{3,4,5,\dots\}}$ with $w\le v$ by induction. Assume by induction that we have proved statements (a) and (b) for all $w\in\mathbb{Z}^3_{\{3,4,5,\dots\}}$ with $w<v$. By construction, $x_w=y_w$ for all $w\in\mathbb{Z}^3_{\text{init}}$ and statement (b) holds for $w\in\mathbb{Z}^3_{\{2\}}$. Hence, $y_{v-s'}=x_{v-s'}$ for $s'\in\{0,1\}^3\setminus\{(0,0,0)\}$, and $y_{v-s}=t_{[v-s]}x_{v-s}$ for $s\in\left\{\left(1,\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},1,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},1\right)\right\}$. Because $u_{v-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}=1$, statements (a) and (b) follow from Lemma VII.11. $\square$

Lemma VII.19. *An array $\mathbf{u}=(u_s)\in\{-1,1\}^{\mathbb{Z}^3+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$ is in the image of ψ (see Definition VII.15) if and only if for every $v\in\mathbb{Z}^3$,*

$$\prod_{a_1,a_2,a_3\in\{-1,1\}} u_{v+(a_1,a_2,a_3)/2}=1. \tag{7.38}$$

Remark VII.20. Using the interpretation of $\mathbb{Z}^3 + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ as the set of unit cubes of $\mathbb{Z}^3$, the product on the left-hand side of (7.38) is a product over unit cubes incident with v.

Proof of Lemma VII.19. First, suppose $\mathbf{u} = \psi(\mathbf{t})$, where $\mathbf{t} = (t_s) \in \{-1, 1\}^{\mathbb{Z}^3_{\boxdot}}$. Consider the cube $C = v + \left[-\frac{1}{2}, \frac{1}{2}\right]^3$. Note that $v + (a_1, a_2, a_3)/2$ for $a_1, a_2, a_3 \in \{-1, 1\}$ are the vertices of C. Using the interpretation of $\mathbb{Z}^3_{\boxdot}$ as lines in $\mathbb{R}^3$, the edges of C are contained in the lines in $\mathbb{Z}^3_{\boxdot}$. If s_1, s_2, s_3 are the 3 lines in $\mathbb{Z}^3_{\boxdot}$ that intersect at $v + (a_1, a_2, a_3)/2$, then $u_{v+(a_1,a_2,a_3)/2} = t_{s_1} t_{s_2} t_{s_3}$. Each edge of C is incident with 2 vertices of C. Hence, expanding the left-hand side of (7.38) in terms of components of $\mathbf{t}$, we obtain

$$\prod_{a_1,a_2,a_3 \in \{-1,1\}} u_{v+(a_1,a_2,a_3)/2} = \prod_s t_s^2 = 1, \tag{7.39}$$

where the second product is over the lines $s \in \mathbb{Z}^3_{\boxdot}$ determined by the edges of C.

Next, suppose that condition (7.38) holds. It is clear that $\mathbf{u}$ is uniquely determined by its components at $S = \left\{\left(v_1 + \frac{1}{2}, v_2 + \frac{1}{2}, v_3 + \frac{1}{2}\right) \in \mathbb{Z}^3 + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) : v_1 v_2 v_3 = 0\right\}$ and condition (7.38). For $(v_1, v_2, v_3) \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$, set

$$t_{[(v_1,v_2,v_3)]} = \begin{cases} u_{\left(\frac{1}{2}, v_2, v_3\right)} & \text{if } v_1 \in \mathbb{Z} \\ u_{\left(v_1, \frac{1}{2}, v_3\right)} u_{\left(\frac{1}{2}, \frac{1}{2}, v_3\right)} & \text{if } v_2 \in \mathbb{Z} \text{ and } v_1 \neq \frac{1}{2} \\ u_{\left(v_1, v_2, \frac{1}{2}\right)} u_{\left(v_1, \frac{1}{2}, \frac{1}{2}\right)} u_{\left(\frac{1}{2}, v_2, \frac{1}{2}\right)} & \text{if } v_3 \in \mathbb{Z} \text{ and } v_1, v_2 \neq \frac{1}{2} \\ 1 & \text{if } v_2 \in \mathbb{Z} \text{ and } v_1 = \frac{1}{2} \\ 1 & \text{if } v_3 \in \mathbb{Z} \text{ and either } v_1 = \frac{1}{2} \text{ or } v_2 = \frac{1}{2}. \end{cases} \tag{7.40}$$

Set $\mathbf{t} = (t_s) \in \{-1, 1\}^{\mathbb{Z}^3_{\boxdot}}$. It is straightforward to check that $\psi(\mathbf{t})$ agrees with $\mathbf{u}$ at S. Hence, because $\psi(\mathbf{t})$ and $\mathbf{u}$ both satisfy condition (7.38), it follows that $\mathbf{u} = \psi(\mathbf{t})$. $\square$

Lemma VII.21. *Let $\hat{\mathbf{x}}$ be an array indexed by $\mathbb{Z}^3_{\{0,1,2,3,4,5\}}$. Assume that $\hat{\mathbf{x}}$ satisfies the Kashaev equation, and, moreover, its restriction to $\mathbb{Z}^3_{\text{init}}$ is generic. Then there exists an array $\tilde{\mathbf{x}}$ indexed by L satisfying the K-hexahedron equations and extending $\hat{\mathbf{x}}$.*

Proof. For $i = 3, 4, 5$, we will show by induction on i that there exists an array $\tilde{\mathbf{x}}_{\text{init}}$ indexed by L_{init} satisfying equation (2.12) for $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$ such that $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}$ agrees with $\hat{\mathbf{x}} = (x_s)_{s\in\mathbb{Z}^3_{\{0,1,2,3,4,5\}}}$ on $\mathbb{Z}^3_{\{0,1,\dots,i\}}$. Let $\tilde{\mathbf{x}}'_{\text{init}}$ be an array indexed by L_{init} satisfying equation (2.12) for $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$ such that $(\tilde{\mathbf{x}}'_{\text{init}})^{\uparrow L} = (y_s)_{s\in L}$ agrees with $\hat{\mathbf{x}}$ on $\mathbb{Z}^3_{\{0,1,\dots,i-1\}}$. (For $i = 3$, we can obtain $\tilde{\mathbf{x}}'_{\text{init}}$ by taking an arbitrary extension of $\mathbf{x}_{\text{init}}$ to L_{init} satisfying equation (2.12) for $s \in L_{\text{init}} - \mathbb{Z}^3_{\text{init}}$. For $i = 4, 5$, we have shown that $\tilde{\mathbf{x}}'_{\text{init}}$ exists by induction.) Choose $\tilde{\mathbf{u}} = (u_s) \in \{-1,1\}^{\mathbb{Z}^3_{\{3,4,5\}}-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$ so that

- $u_{s-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = 1$ if $s \in \mathbb{Z}^3_{\{i\}}$ and $y_s = x_s$;
- $u_{s-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = -1$ if $s \in \mathbb{Z}^3_{\{i\}}$ and $y_s \neq x_s$;
- $u_{s-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = 1$ if $i = 4$ and $s \in \mathbb{Z}^3_{\{3\}}$, or $i = 5$ and $s \in \mathbb{Z}^3_{\{3,4\}}$.

Extend $\tilde{\mathbf{u}}$ to $\mathbf{u} \in \{-1,1\}^{\mathbb{Z}^3+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}$ by condition (7.38). By Lemma VII.19, there exists $\mathbf{t} \in \{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$ such that $\mathbf{u} = \psi(\mathbf{t})$. Set $\tilde{\mathbf{x}}_{\text{init}} = \mathbf{t}\cdot\tilde{\mathbf{x}}'_{\text{init}}$. Then, by Lemma VII.18, $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}$ agrees with $\hat{\mathbf{x}}$ on $\mathbb{Z}^3_{\{0,1,\dots,i\}}$, as desired. □

We can now prove a weaker version of Theorem II.22(a), under the additional constraint of genericity.

Corollary VII.22. *Let $\mathbf{x} \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ be a coherent solution of the Kashaev equation, whose restriction to $\mathbb{Z}^3_{\text{init}}$ is generic. Then $\mathbf{x}$ can be extended to $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$ satisfying the K-hexahedron equations.*

Proof. Let $\mathbf{x} \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ be a coherent solution of the Kashaev equation, whose restriction to $\mathbb{Z}^3_{\text{init}}$ is generic. By Lemma VII.21, there exists an array $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$ satisfying the K-hexahedron equations that agrees with $\mathbf{x}$ on $\mathbb{Z}^3_{\{0,1,2,3,4,5\}}$. Let $\mathbf{x}'$ be the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^3$. By Theorem II.22(b), $\mathbf{x}'$ is a coherent solution of the Kashaev equation. There is a unique coherent solution of the Kashaev equation agreeing with $\mathbf{x}$ at $\mathbb{Z}^3_{\{0,1,2,3,4,5\}}$, as condition (2.9) gives the remaining values as rational expressions in the values at $\mathbb{Z}^3_{\{0,1,2,3,4,5\}}$ (see Remark II.11). (As $\mathbf{x}$ is generic, $\mathbf{x}$ must satisfy condition (2.20), and so $K_v^C(\mathbf{x}) \neq 0$ for all unit cubes C in $\mathbb{Z}^3$ and vertices $v \in C$. Hence, the denominator of this rational expression is nonzero.) Hence, $\mathbf{x}' = \mathbf{x}$, as desired. □

Proof of Theorem II.22(a). We need to loosen the genericity condition in Corollary VII.22 to the conditions that $\mathbf{x}$ satisfies (2.20) and has nonzero entries.

Let $\mathbf{x} \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ be a coherent solution of the Kashaev equation satisfying (2.20). Let $A_j = [-j, j]^3 \cap \mathbb{Z}^3$ and $B_j = [-j, j]^3 \cap L$ for $j \in \mathbb{Z}_{\geq 0}$. We claim that if there exist $\tilde{\mathbf{x}}_j \in (\mathbb{C}^*)^{B_j}$ satisfying the K-hexahedron equations that agree with $\mathbf{x}$ on A_j for all j, then there exists $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$ satisfying the K-hexahedron equations that agrees with $\mathbf{x}$ on $\mathbb{Z}^3$. Construct an infinite tree T as follows:

- The vertices of T are solutions of the K-hexahedron equation indexed by B_j that agree with $\mathbf{x}$ on A_j (over $j \in \mathbb{Z}_{\geq 0}$).
- Add an edge between $\tilde{\mathbf{x}}_j \in (\mathbb{C}^*)^{B_j}$ and $\tilde{\mathbf{x}}_{j+1} \in (\mathbb{C}^*)^{B_{j+1}}$ if $\tilde{\mathbf{x}}_{j+1}$ restricts to $\tilde{\mathbf{x}}_j$.

Thus, T is an infinite tree in which every vertex has finite degree. By König's infinity lemma, there exists an infinite path $\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots$ in T with $\tilde{\mathbf{x}}_j \in (\mathbb{C}^*)^{B_j}$. Thus, there exists $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$ restricting to $\tilde{\mathbf{x}}_j$ for all $j \in \mathbb{Z}_{\geq 0}$, so $\tilde{\mathbf{x}}$ is a solution of the K-hexahedron equations that agrees with $\mathbf{x}$ on $\mathbb{Z}^3$.

Given $j \in \mathbb{Z}_{\geq 0}$, we claim that there exists $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^{B_j}$ satisfying the K-hexahedron equations that agree with $\mathbf{x}$ on A_j. It is straightforward to show that there exists a sequence $\mathbf{x}_1, \mathbf{x}_2, \cdots \in (\mathbb{C}^*)^{\mathbb{Z}^3}$ of coherent solutions of the Kashaev equation that converge pointwise to $\mathbf{x}$ whose restrictions to $\mathbb{Z}^3_{\text{init}}$ are generic. By Corollary VII.22, there exist $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \cdots \in (\mathbb{C}^*)^L$ satisfying the K-hexahedron equations such that $\tilde{\mathbf{x}}_i$ restricts to $\mathbf{x}_i$. However, the sequence $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots$ does not necessarily converge (see Proposition II.23 below). Let $\tilde{\mathbf{x}}'_1, \tilde{\mathbf{x}}'_2, \cdots \in (\mathbb{C}^*)^{B_j}$ be the restrictions of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots$ to B_j. There exists a subsequence of $\tilde{\mathbf{x}}'_1, \tilde{\mathbf{x}}'_2, \ldots$ that converges to some $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^{B_j}$. (For each $s \in B_j \setminus A_j$, we can partition the sequence $\tilde{\mathbf{x}}'_1, \tilde{\mathbf{x}}'_2, \ldots$ into two sequences, each of which converges at s. Because B_j is finite, the claim follows.) The array $\tilde{\mathbf{x}}$ must satisfy the K-hexahedron equations and agree with $\mathbf{x}$ on A_j, so we are done. $\square$

We shall now work towards a proof of Theorem II.23.

Lemma VII.23. *Let $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$ be a solution of the K-hexahedron equations. Let $\tilde{\mathbf{x}}_{\text{init}} \in (\mathbb{C}^*)^{L_{\text{init}}}$ denote the restriction of $\tilde{\mathbf{x}}$ to L_{init}. Let $\mathbf{t} = (t_s) \in \{-1, 1\}^{\mathbb{Z}^3_{\boxdot}}$ be in the kernel of ψ. Then $(\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}})^{\uparrow L} = (y_s)_{s \in L}$, where*

$$y_s = \begin{cases} x_s & \text{if } s \in \mathbb{Z}^3 \\ t_{[s]} x_s & \text{if } s \in L - \mathbb{Z}^3 \,. \end{cases} \tag{7.41}$$

Proof. This follows from Lemma VII.18 and Proposition VII.3. $\square$

Lemma VII.24. *Let $\tilde{\mathbf{x}} \in (\mathbb{C}^*)^L$ be a solution of the K-hexahedron equations. Let $\tilde{\mathbf{x}}_{\text{init}} \in (\mathbb{C}^*)^{L_{\text{init}}}$ denote the restriction of $\tilde{\mathbf{x}}$ to L_{init}. For $\mathbf{t} \in \{-1, 1\}^{\mathbb{Z}^3_{\boxdot}}$, the following are equivalent:*

- *$\tilde{\mathbf{x}}$ and $(\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}$ agree on $\mathbb{Z}^3$;*
- *$\mathbf{t}$ is in the kernel of ψ (see Proposition VII.16).*

Proof. If $\mathbf{t}$ is in the kernel of ψ, then $\tilde{\mathbf{x}}$ and $(\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}$ agree on $\mathbb{Z}^3$ by Lemma VII.23.

If $\mathbf{t}$ is not in the kernel of ψ, let $\mathbf{u} = (u_s)_{s\in\mathbb{Z}^3+\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = \psi(\mathbf{t})$. Write $\tilde{\mathbf{x}} = (x_s)_{s\in L}$ and $(\mathbf{t}\cdot\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L} = (y_s)_{s\in L}$. Choose $v \in \mathbb{Z}^3_{\{3,4,5,\ldots\}}$ such that $u_{v-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = -1$ and $u_{w-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = 1$ for all $w \in \mathbb{Z}^3_{\{3,4,5,\ldots\}}$ with $w < v$. (Such a choice of v exists because if $u_{v-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = 1$ for all $v \in \mathbb{Z}^3_{\{3,4,5,\ldots,\}}$, then $u_{v-\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)} = 1$ for all $v \in \mathbb{Z}^3$ because $\mathbf{u}$ must satisfy equation (7.38) for all $v \in \mathbb{Z}^3$.) Then by Lemma VII.18, $y_{v-s} = x_{v-s}$ for $s \in \{0,1\}^3 - \{(0,0,0)\}$, and $y_{v-s} = t_{[v-s]}x_{v-s}$ for $s \in \left\{\left(1,\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},1,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},1\right)\right\}$. Hence,

$$y_v - x_v = -4\frac{x_{v-\left(1,\frac{1}{2},\frac{1}{2}\right)}x_{v-\left(\frac{1}{2},1,\frac{1}{2}\right)}x_{v-\left(\frac{1}{2},\frac{1}{2},1\right)}}{x^2_{v-(1,1,1)}} \neq 0, \tag{7.42}$$

so $y_v \neq x_v$. $\square$

Proof of Theorem II.23. Let $\tilde{\mathbf{x}}_{\text{init}} \in (\mathbb{C}^*)^{L_{\text{init}}}$ denote the restriction of $\tilde{\mathbf{x}}$ to L_{init}.

Suppose that for some signs $\alpha_i, \beta_i, \gamma_i \in \{-1,1\}$, $i \in \mathbb{Z}$, $\tilde{\mathbf{y}} \in (\mathbb{C}^*)^L$ satisfies equations (2.25)–(2.28) for $(a,b,c) \in \mathbb{Z}^3$. Define $\mathbf{t} = (t_s) \in \{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$ by equations (7.29)–(7.31), so $\mathbf{t}$ is in the kernel of ψ by Proposition VII.16. Hence, by Lemma VII.23, $\tilde{\mathbf{y}} = (\mathbf{t}\cdot\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}$, so $\tilde{\mathbf{y}}$ satisfies the K-hexahedron equations, proving part (a).

Next, if $\tilde{\mathbf{y}} \in (\mathbb{C}^*)^L$ satisfies the K-hexahedron equations, then $\tilde{\mathbf{y}} = (\mathbf{t}\cdot\tilde{\mathbf{x}}_{\text{init}})^{\uparrow L}$ for some $\mathbf{t} = (t_s) \in \{-1,1\}^{\mathbb{Z}^3_{\boxdot}}$. By Lemma VII.24, $\mathbf{t}$ is in the kernel of ψ. Hence, by Proposition VII.16, there exist signs $\alpha_i, \beta_i, \gamma_i \in \{-1,1\}$, $i \in \mathbb{Z}$, such that $\mathbf{t}$ is given by equations (7.29)–(7.31). Hence, by Lemma VII.23, $\tilde{\mathbf{y}}$ satisfies equations (2.25)–(2.28) for $(a,b,c) \in \mathbb{Z}^3$, proving part (b). $\square$

CHAPTER VIII

Coherence for Cubical Complexes

In this chapter, we generalize Proposition II.8 and Theorem II.22 from $\mathbb{Z}^3$ to certain classes of 3-dimensional cubical complexes. Proposition II.8 generalizes to arbitrary 3-dimensional cubical complexes embedded in $\mathbb{R}^3$ (see Proposition VIII.1), while Theorem II.22(b) generalizes to directed cubical complexes corresponding to piles of quadrangulations of a polygon (see Proposition VIII.3). Theorem II.22(a) does not hold for arbitrary directed cubical complexes corresponding to piles of quadrangulations of a polygon. It turns out that an additional property of a cubical complex is required, which we call *comfortable-ness*. This property is satisfied by the standard tiling of $\mathbb{R}^3$ with unit cubes, as well as by cubical complexes corresponding to piles of $\Diamond$-tilings of $\mathbf{P}_n$ (see Proposition VIII.8). Let $\varkappa$ be the directed cubical complex corresponding to a pile of quadrangulations of a polygon. In Theorems VIII.10–VIII.11, we show that Theorem II.22(a) holds for $\varkappa$ if and only if $\varkappa$ is comfortable. The proof of Theorem VIII.10 is nearly identical to the proof of Theorem II.22(a) in Chapter VII.

First, we note that Proposition II.8 generalizes to arbitrary 3-dimensional cubical complexes embedded in $\mathbb{R}^3$ as follows:

Proposition VIII.1. *Let $\varkappa$ be a 3-dimensional cubical complex embedded in $\mathbb{R}^3$.*

Suppose that $\mathbf{x} = (x_s)_{s \in \varkappa^0}$ *satisfies the Kashaev equation. Then for any interior vertex* $v \in \varkappa^0$ *(see Definition III.2),*

$$\left(\prod_{C \ni v} K_v^C(\mathbf{x}) \right)^2 = \left(\prod_{S \ni v} (x_v x_{v_2} + x_{v_1} x_{v_3}) \right)^2, \tag{8.1}$$

where

- *the first product is over* 3*-dimensional cubes* C *incident to the vertex* v*,*
- *the second product is over* 2*-dimensional faces* S *incident to the vertex* v*, and*
- v, v_1, v_2, v_3 *are the vertices of such a face* S *listed in cyclic order.*

Proof. The proof is almost identical to the proof of Proposition II.8 in Chapter VII.

□

Remark VIII.2. With Proposition VIII.1 in mind, we can think of the notion of coherence from Definition IV.17 as follows. Let $\mathbf{T} = (T_0, \ldots, T_\ell)$ be a pile of quadrangulations of a polygon with $\varkappa = \varkappa(\mathbf{T})$. Start with an arbitrary array $\mathbf{x}_{\text{init}}$ indexed by $\varkappa^0(T_0)$ whose entries are "sufficiently generic." We want to extend $\mathbf{x}_{\text{init}}$ to an array $\mathbf{x}$ indexed by $\varkappa^0$ that is a coherent solution of the Kashaev equation. Building $\mathbf{x}$ inductively, suppose we have defined the values of $\mathbf{x}$ at $\varkappa^0(T_0, \ldots, T_{i-1})$, and we need to define the value x_w of $\mathbf{x}$ at the new vertex w in T_i. Let $C \in \varkappa^3$ be the cube corresponding to the flip between T_{i-1} and T_i, and let v be the bottom vertex of C, i.e., let v be the unique vertex in T_{i-1} but not T_i. In order that $\mathbf{x}$ continue to satisfy the Kashaev equation, there are 2 possible values for x_w, say a and b, so that $K^C(\mathbf{x}) = 0$. If the vertex v is in T_0, i.e., v is not an interior vertex of $\varkappa$, then we can either set $x_w = a$ or $x_w = b$, and $\mathbf{x}$ will continue to be a coherent solution of the Kashaev equation. Now, suppose v is not in T_0, i.e., v is an interior vertex of $\varkappa$. Because we have chosen $\mathbf{x}_{\text{init}}$ to be "sufficiently generic," the value of $\prod_{C \ni v} K_v^C(\mathbf{x})$

depends on whether we set $x_w = a$ or $x_w = b$. Proposition VIII.1 tells us that for one of the 2 possible values, say $x_w = a$, equation (4.6) holds, while for the other value, $x_w = b$, the following equation holds:

$$\prod_{C \ni v} K_v^C(\mathbf{x}) = -\prod_{S \ni v}(x_v x_{v_2} + x_{v_1} x_{v_3}). \tag{8.2}$$

Hence, the condition of coherence tells us which of the 2 solutions is the "correct" one when v is an interior vertex of $\varkappa$.

We now prove the following generalization of Theorem II.22(b).

Proposition VIII.3. *Let* $\mathbf{T}$ *be a pile of quadrangulations of a polygon. Let* $\tilde{\mathbf{x}} = (x_s)_{s \in \varkappa^{02}(\mathbf{T})}$ *be an array (with* $x_s \neq 0$ *for all* $s \in \varkappa^0(\mathbf{T})$*) satisfying the K-hexahedron equations. Then the restriction of* $\tilde{\mathbf{x}}$ *to* $\varkappa^0(\mathbf{T})$ *is a coherent solution of the Kashaev equation.*

Proof. The proof follows almost exactly the same as the proof of Theorem II.22(b). For an interior vertex v of $\varkappa$, there is exactly one cube C for which v is the top vertex, and exactly one cube C for which v is the bottom vertex. Let $\mathbf{x}$ be the restriction of $\tilde{\mathbf{x}}$ to $\varkappa^0$. By Lemma VII.2, taking the product over the cubes incident to v,

$$\prod_{C \ni v} K_v^C(\mathbf{x}) = (-1)^2 \prod_{S \in \varkappa^2 : S \ni v} x_S = \prod_{S \ni v} x_S^2 = \prod_{S \ni v}(x_v x_{v_2} + x_{v_1} x_{v_3}), \tag{8.3}$$

so the restriction of $\tilde{\mathbf{x}}$ to $\mathbf{x}$ is a coherent solution of the Kashaev equation. □

The following statement generalizes Theorem II.9:

Corollary VIII.4. *Let* $\mathbf{T}$ *be a pile of quadrangulations of a polygon. Let* $\mathbf{x} = (x_s)_{s \in \varkappa^0(\mathbf{T})}$ *be an array satisfying the positive Kashaev recurrence. Then* $\mathbf{x}$ *is a coherent solution of the Kashaev equation.*

Proof. This follows immediately from Proposition VIII.3 because $\mathbf{x}$ can be extended to an array indexed by $\varkappa^{02}(\mathbf{T})$ satisfying the K-hexahedron equations by choosing the positive solutions from equation (2.12) for $s \in \varkappa^2(\mathbf{T})$. $\square$

Remark VIII.5. The converse of Proposition VIII.3 (i.e., the counterpart of Theorems II.22(a) and IV.19(a)) does not hold for an arbitrary choice of $\mathbf{T}$. In other words, there exist piles $\mathbf{T}$ and arrays $\mathbf{x}$ indexed by $\varkappa^0(\mathbf{T})$ with nonzero components that are coherent solutions of the Kashaev equation, where $\mathbf{x}$ cannot be extended to an array indexed by $\varkappa^{02}(\mathbf{T})$ satisfying the K-hexahedron equations.

In order for a converse of Proposition VIII.3 (equivalently, a generalization of Theorems II.22(a) and IV.19(a)) to hold, one must impose an additional condition on the underlying cubical complexes; see Definition VIII.6 below.

Definition VIII.6. Let $\varkappa$ be a three-dimensional cubical complex that can be embedded into $\mathbb{R}^3$, cf. Definition III.2. (While this embeddability condition can be relaxed, it is satisfied in all subsequent applications. In fact, $\varkappa$ will always be the cubical complex associated to a pile of quadrangulations.) Let $\sim$ be the equivalence relation on $\varkappa^2$ generated by the equivalences $s_1 \sim s_2$ for all pairs (s_1, s_2) involving opposite faces of some 3-dimensional cube in $\varkappa^3$. Let $\varkappa_{\boxdot}$ denote the set of equivalence classes under this equivalence relation. Denote by $[s] \in \varkappa_{\boxdot}$ the equivalence class of $s \in \varkappa^2$. By analogy with Definition VII.13, denote by $\psi_\varkappa : \{-1,1\}^{\varkappa_{\boxdot}} \to \{-1,1\}^{\varkappa^3}$ the map sending an array $\mathbf{t} = (t_{[s]})_{[s]\in\varkappa_{\boxdot}}$ to the array $\psi_\varkappa(\mathbf{t}) = (u_C)_{C\in\varkappa^3}$ defined by $u_C = t_{[a]}t_{[b]}t_{[c]}$, where a, b, c are representatives of the three pairs of opposite 2-dimensional faces of C. We say that the cubical complex $\varkappa$ is *comfortable* if the following statements are equivalent for every $\mathbf{u} = (u_C) \in \{-1,1\}^{\varkappa^3}$:

(C1) $\mathbf{u}$ is in the image of $\psi_\varkappa$;

(C2) for every interior vertex $v \in \varkappa^0$ (cf. Definition III.2), we have

$$\prod_{C \ni v} u_C = 1, \tag{8.4}$$

the product over 3-dimensional cubes $C \in \varkappa^3$ containing v.

By Lemma VII.19, the standard tiling of $\mathbb{R}^3$ by unit cubes is comfortable.

Remark VIII.7. In Definition VIII.6, the statement (C1) always implies (C2). Indeed, if $\mathbf{u} = \psi_\varkappa(\mathbf{t})$ with $\mathbf{t} = \left(t_{[s]}\right)_{[s] \in \varkappa_\boxdot} \in \{-1, 1\}^{\varkappa_\boxdot}$, then for any interior vertex $v \in \varkappa^0$,

$$\prod_{v \in C \in \varkappa^3} u_C = \prod_{v \in s \in \varkappa^2} t_{[s]}^2 = 1. \tag{8.5}$$

Thus, in checking comfortableness, we simply must check that (C2) implies (C1).

We next state four results (Propositions VIII.8–VIII.9 and Theorems VIII.10–VIII.11) which the rest of this chapter is dedicated to proving. The reader may want to review Definitions III.6–III.7 before proceeding with the following proposition.

Proposition VIII.8. *Let* $\mathbf{T}$ *be a pile of quadrangulations of a polygon. Suppose that the divide associated to each quadrangulation in* $\mathbf{T}$ *is a pseudoline arrangement. Then* $\varkappa = \varkappa(\mathbf{T})$ *is comfortable. In particular, if* $\mathbf{T}$ *is a pile of* $\Diamond$*-tilings of the polygon* $\mathbf{P}_n$*, then* $\varkappa = \varkappa(\mathbf{T})$ *is comfortable.*

Proposition VIII.9. *There exists a pile* $\mathbf{T}$ *of quadrangulations of some polygon such that the cubical complex* $\varkappa = \varkappa(\mathbf{T})$ *is not comfortable.*

We next state a generalization of Theorems II.22 and IV.19.

Theorem VIII.10. *Let* $\mathbf{T}$ *be a pile of quadrangulations of a polygon such that* $\varkappa = \varkappa(\mathbf{T})$ *is comfortable. Any coherent solution of the Kashaev equation* $\mathbf{x} = (x_s)_{s \in \varkappa^0}$ *with nonzero components satisfying condition* (4.7) *can be extended to an array* $\tilde{\mathbf{x}} = (x_s)_{s \in \varkappa^{02}}$ *satisfying the K-hexahedron equations.*

However, Theorems II.22 and IV.19 don't generalize to cubical complexes that are not comfortable.

Theorem VIII.11. *Let* $\mathbf{T}$ *be a pile of quadrangulations of a polygon such that* $\varkappa = \varkappa(\mathbf{T})$ *is not comfortable. Then there exists a coherent solution of the Kashaev equation* $\mathbf{x}$ *indexed by* $\varkappa^0$ *which cannot be extended to an array indexed by* $\varkappa^{02}$ *satisfying the K-hexahedron equations.*

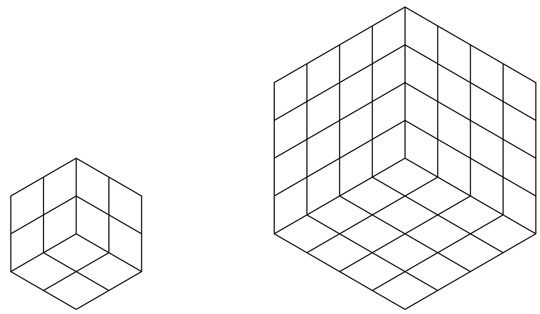
Note that Theorem IV.19(a) follows directly from Proposition VIII.8 and Theorem VIII.10. In Remark VIII.12 below, we explain that Theorem II.22(a) follows from Proposition VIII.8 and Theorem VIII.10 as well.

Remark VIII.12. Together, Theorem VIII.10 and Proposition VIII.8 imply Theorem II.22(a). For each cube $[-j,j]^3 \in \mathbb{R}^3$, project the "bottom" faces (i.e., $\{-j\} \times [-j,j] \times [-j,j]$, $[-j,j] \times \{-j\} \times [-j,j]$, $[-j,j] \times [-j,j] \times \{-j\}$) onto $\mathbb{R}^2$ to obtain a quadrangulation T_j of a region R_j, as shown in Figure 8.1. The divide associated to each quadrangulation T_i is a pseudoline arrangement. Hence, by Proposition VIII.8, for any pile $\mathbf{T}_i$ including T_i, $\varkappa(\mathbf{T}_i)$ is comfortable. Choose $\mathbf{T}_i$, so that we can associate the vertices of $\varkappa(\mathbf{T}_i)$ with $\{-j,\dots,j\}^3$, so that $\bigcup_{j=1}^{\infty} \varkappa^0(\mathbf{T}_i) = \mathbb{Z}^3$. Repeating the König's infinity lemma argument from the end of the proof of Theorem II.22(a), Theorem VIII.10 implies Theorem II.22.

The rest of this chapter is dedicated to proving Propositions VIII.8–VIII.9 and Theorems VIII.10–VIII.11.

We begin by proving Proposition VIII.8.

Lemma VIII.13. *Let* $\mathbf{T} = (T_0, \dots, T_\ell)$ *be a pile of quadrangulations of a polygon such that* $\varkappa(\mathbf{T})$ *is comfortable. Given* $0 \le i \le j \le \ell$, *let* $\mathbf{T}' = (T_i, \dots, T_j)$. *Then* $\varkappa(\mathbf{T}')$ *is comfortable.*



quadrangulation T_1 of R_1 quadrangulation T_2 of R_2

Figure 8.1: The quadrangulations T_j of regions R_j described in Remark VIII.12.

Proof. It suffices to check that (C2) implies (C1) for $\varkappa(\mathbf{T}')$ (see Remark VIII.7). Note that any $\mathbf{u} = (u_C)_{C \in \varkappa^3(\mathbf{T}')}$ satisfying (C2) can be extended to $\tilde{\mathbf{u}} = (u_C)_{C \in \varkappa^3(\mathbf{T})}$ satisfying (C2). Identifying $\varkappa_{\boxdot}(\mathbf{T})$ and $\varkappa_{\boxdot}(\mathbf{T}')$, the fact that there exists $\mathbf{t}$ such that $\psi_{\varkappa(\mathbf{T})}(\mathbf{t}) = \tilde{\mathbf{u}}$ implies that $\psi_{\varkappa(\mathbf{T}')}(\mathbf{t}) = \mathbf{u}$, as desired. $\square$

We can now prove Proposition VIII.8 in the special case where $\mathbf{T}$ be a pile of $\lozenge$-tilings of $\mathbf{P}_n$.

Lemma VIII.14. *Let* $\mathbf{T}$ *be a pile of* $\lozenge$*-tilings of* $\mathbf{P}_n$*. Then* $\varkappa = \varkappa(\mathbf{T})$ *is comfortable.*

Proof. Labeling the vertices of $\varkappa$ by subsets of $[n]$ (as in Chapter IV), we can label the cubes in $\varkappa^3$ by 3-element subsets of $[n]$ by taking the symmetric difference of the labels of any opposite vertices in the cube. Note that we can extend $\mathbf{T}$ to a longer pile $\mathbf{T}'$ so that for every $A \in \binom{[n]}{3}$, at least one cube of $\varkappa(\mathbf{T}')$ is labeled by A. Hence, by Proposition VIII.13, it suffices to prove the theorem under the additional assumption that each set in $\binom{[n]}{3}$ labels at least one cube in $\varkappa^3$.

Let A_1 be the set of $\mathbf{u} \in \{-1, 1\}^{\varkappa^3}$ satisfying (C1), and A_2 be the set of $\mathbf{u}$ satisfying (C2). Because $A_1 \subseteq A_2$, it suffices to show that $|A_1| \geq |A_2|$ in order to prove that $A_1 = A_2$. We claim that both A and B have size $2^{\binom{n-1}{2}}$.

First, we claim that $|A_1| \geq 2^{\binom{n-1}{2}}$. Identify each element $S \in \varkappa_{\boxdot}$ with a 2-

element subset of $[n]$ by taking the symmetric difference of the labels of any pair of opposite vertices of any tile in S. Note that if $\mathbf{u} = \psi_\varkappa(\mathbf{t})$, and a cube C labeled by $\{i,j,k\}$, then $u_C = t_{\{i,j\}}t_{\{i,k\}}t_{\{j,k\}}$. Define a map of vector spaces $f : \{-1,1\}^{\binom{[n]}{2}} \to \{-1,1\}^{\binom{[n]}{3}}$ where $f\left((t_S)_{S\in\binom{[n]}{2}}\right) = (u_C)_{C\in\binom{[n]}{3}}$ with

$$u_{\{i,j,k\}} = t_{\{i,j\}}t_{\{i,k\}}t_{\{j,k\}}. \tag{8.6}$$

If we fix $t_{\{1,2\}} = \cdots = t_{\{1,n\}} = 1$, then $u_{\{1,j,k\}} = t_{\{j,k\}}$, so the rank of f is at least the number of 2-element subsets of $\{2,\ldots,n\}$, i.e., $\binom{n-1}{2}$. Hence, it follows that $|A_1| \geq 2^{\binom{n-1}{2}}$.

Thus, in order to prove the proposition, we must show that $|A_2| \leq 2^{\binom{n-1}{2}}$. Note that there are $\binom{n-1}{2}$ vertices in the interior of any $\Diamond$-tiling of $\mathbf{P}_n$. In choosing $\mathbf{u}$ satisfying (C2), we can make an arbitrary choice of sign for any cube that shares its bottom vertex with T_0, but the signs of the remaining cubes is determined by condition (C2). Hence, because at most $\binom{n-1}{2}$ cubes can share their bottom vertices with T_0 (the bottom of a cube cannot be on the boundary of T_0), there are at most $2^{\binom{n-1}{2}}$ such $\mathbf{u}$ satisfying condition (C2), proving our claim. $\square$

We can now prove Proposition VIII.8 in its full generality.

Proof of Proposition VIII.8. Let $\mathbf{T} = (T_0,\ldots,T_\ell)$. We claim that we can "embed" the quadrangulations $T_0,\ldots,T_\ell$ in $\Diamond$-tilings of $\mathbf{P}_n$. Let $D_0,\ldots,D_\ell$ be the divides associated to $T_0,\ldots,T_\ell$. Because $D_0,\ldots,D_\ell$ are pseudoline arrangements connected by braid moves, we can extend $D_0,\ldots,D_\ell$ to pseudoline arrangements $\tilde{D}_0,\ldots,\tilde{D}_\ell$, still connected by braid moves, in which every pair of branches intersects exactly once. By Proposition III.8, there exists a pile $\tilde{\mathbf{T}} = (\tilde{T}_0,\ldots,\tilde{T}_\ell)$ of $\Diamond$-tilings of $\mathbf{P}_n$, for which the divides associated to $\tilde{T}_0,\ldots,\tilde{T}_\ell$ are $\tilde{D}_0,\ldots,\tilde{D}_\ell$.

By Lemma VIII.14, $\varkappa(\tilde{\mathbf{T}})$ is comfortable. The cubical complex $\varkappa(\tilde{\mathbf{T}})$ consists of $\varkappa = \varkappa(\mathbf{T})$, unioned with 2-dimensional faces that are not part of any 3-dimensional cube. Hence, it follows that $\varkappa$ is comfortable as well. $\square$

Proof of Proposition VIII.9. We describe a pile $\mathbf{T} = (T_0, \ldots, T_8)$ of quadrangulations of a square such that $\varkappa = \varkappa(\mathbf{T})$ is not comfortable. Let T_0 be as in Figure 8.2. It is easier to understand this example by looking at the divides associated to $T_0, \ldots, T_8$, displayed in Figure 8.3. Note that the divides associated to these quadranguations are not pseudoline arrangements. Note that $\varkappa$ has no interior vertices. Hence, every $\mathbf{u} \in \{-1, 1\}^{\varkappa^3}$ satisfies (C2). However, it is not difficult to check that if $\mathbf{u}$ satisfies (C1), then the sign on a given cube is determined by the sign on the other 7. Hence, $\varkappa$ is not comfortable. $\square$

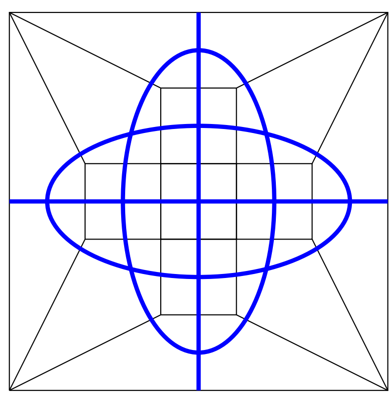


Figure 8.2: The quadrangulation T_0 from the proof of Proposition VIII.9, with the associated divide drawn on top in blue.

The rest of this chapter is dedicated to the proofs of Theorems VIII.10–VIII.11.

Definition VIII.15. Let $\mathbf{T} = (T_0, \ldots, T_\ell)$ be a pile of quadrangulations of a polygon, with $\varkappa = \varkappa(\mathbf{T})$, and $\mathbf{x} = (x_s)_{s \in \varkappa^0}$. We say that $\mathbf{x}$ is *generic* if for all extensions of $\mathbf{x}_{\text{init}}$ (the restriction of $\mathbf{x}$ to $\varkappa^0(T_0)$) to an array $\tilde{\mathbf{x}}$ indexed by $\varkappa^{02}(\mathbf{T})$ satisfying the K-hexahedron equations, the entries of $\tilde{\mathbf{x}}$ are all nonzero.

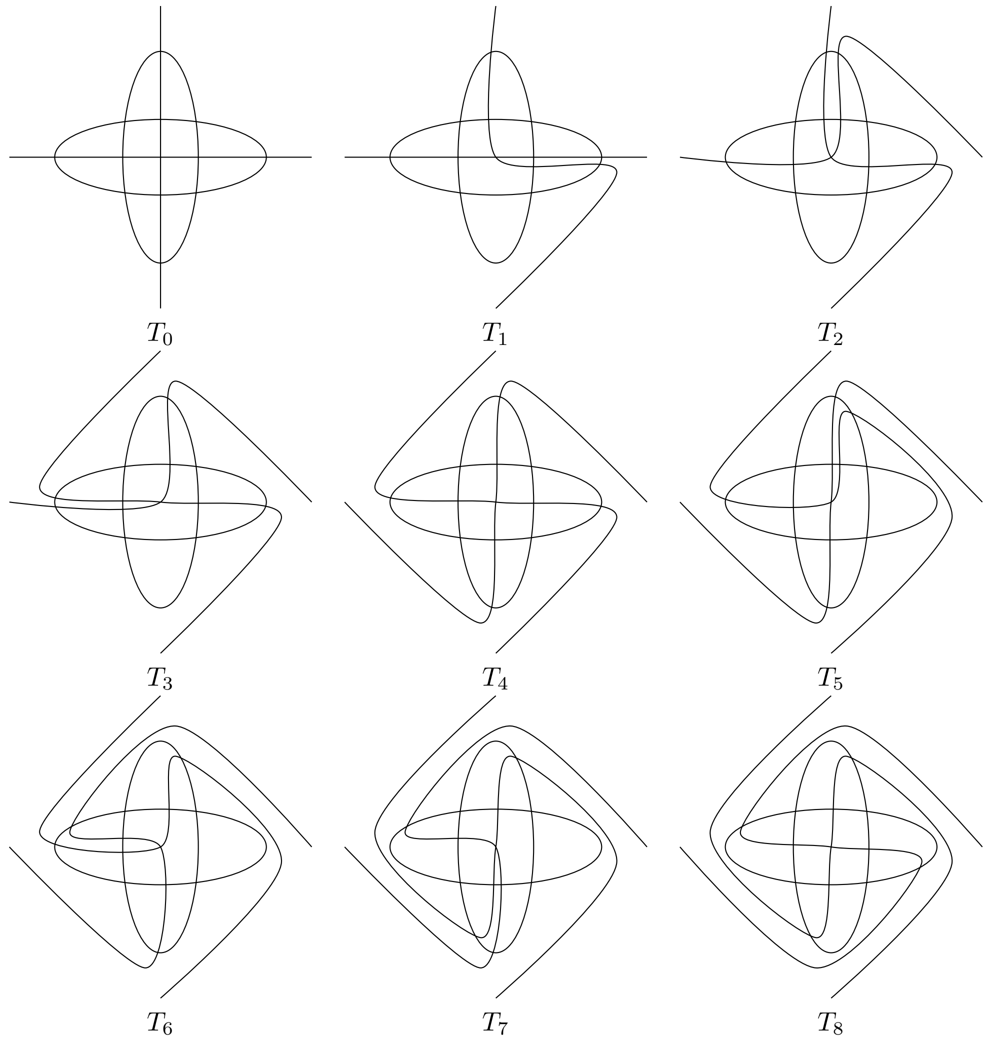


Figure 8.3: The divides associated to the quadrangulations $T_0, \dots, T_8$ from the proof of Proposition VIII.9.

Definition VIII.16. Let $\mathbf{T} = (T_0, \dots, T_\ell)$ be a pile of quadrangulations of a polygon, with $\varkappa = \varkappa(\mathbf{T})$. Given an array $\tilde{\mathbf{x}}_{\mathrm{init}} = (x_s)_{s \in \varkappa^{02}(T_0)}$ and $\mathbf{t} = \big(t_{[s]}\big)_{[s] \in \varkappa_{\boxdot}}$, set $\mathbf{t} \cdot \tilde{\mathbf{x}}_{\mathrm{init}} = (y_s)_{s \in \varkappa^{02}(T_0)}$, where

$$y_s = \begin{cases} x_s & \text{if } s \in \varkappa^0(T_0) \\ t_{[s]} x_s & \text{if } s \in \varkappa^2(T_0). \end{cases} \tag{8.7}$$

Given a generic array $\tilde{\mathbf{x}}_{\mathrm{init}}$ indexed by $\varkappa^{02}(T_0)$, define $(\tilde{\mathbf{x}}_{\mathrm{init}})^{\uparrow \varkappa^{02}}$ to be the unique extension of $\tilde{\mathbf{x}}_{\mathrm{init}}$ to an array indexed by $\varkappa^{02}$ satisfying the K-hexahedron equations.

Define $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow \varkappa^0}$ to be the restriction of $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow \varkappa^{02}}$ to $\varkappa^0$.

Lemma VIII.17. *Let $\mathbf{T} = (T_0, \ldots, T_\ell)$ be a pile of quadrangulations of a polygon, with $\varkappa = \varkappa(\mathbf{T})$. Fix a generic array $\tilde{\mathbf{x}}_{\text{init}}$ indexed by $\varkappa^{02}(T_0)$ satisfying equation 2.12 for $s \in \varkappa^2(T_0)$, and $\mathbf{t} \in \{-1, 1\}^{\varkappa_\boxdot}$. Then the following are equivalent:*

- *$\psi_\varkappa(\mathbf{t})$ has value 1 on $C_1, \ldots, C_{i-1}$, and value -1 on C_i;*
- *$(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow \varkappa^0}$ and $(\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}})^{\uparrow \varkappa^0}$ agree at $\varkappa^0(T_0), \ldots, \varkappa^0(T_{i-1})$ but not at $\varkappa^0(T_i)$.*

Proof. The proof follows directly from Lemma VII.11. □

Lemma VIII.18. *Let $\mathbf{T} = (T_0, \ldots, T_\ell)$ be a pile of quadrangulations of a polygon, with $\varkappa = \varkappa(\mathbf{T})$. Let $\mathbf{x}$ and $\mathbf{x}'$ be generic and distinct coherent solutions of the Kashaev equation, both indexed by $\varkappa^0$, such that $\mathbf{x}$ and $\mathbf{x}'$ agree at $\varkappa^0(T_0)$. Let i be the minimum value such that $\mathbf{x}$ and $\mathbf{x}'$ do not agree at $\varkappa^0(T_i)$. Then the cube C_i shares its bottom vertex with T_0.*

Proof. Assume (for contradiction) that C_i doesn't share its bottom vertex with T_0. Then the bottom vertex of C_i must be an interior vertex of $\varkappa$. Hence, by the coherence and genericity of $\mathbf{x}$ and $\mathbf{x}'$, the values of $\mathbf{x}$ and $\mathbf{x}'$ are uniquely determined by their values at $\varkappa^0(T_0), \ldots, \varkappa^0(T_{i-1})$, which are the same for $\mathbf{x}$ and $\mathbf{x}'$. Hence, $\mathbf{x}$ and $\mathbf{x}'$ agree at the top vertex of C_i, and thus agree at $\varkappa^0(T_i)$, a contradiction. □

We can now prove a weaker version of Theorem VIII.10, under the additional constraint of genericity.

Corollary VIII.19. *Let $\mathbf{T}$ be a pile of quadrangulations of a polygon such that $\varkappa = \varkappa(\mathbf{T})$ is comfortable. Any generic, coherent solution of the Kashaev equation $\mathbf{x} = (x_s)_{s \in \varkappa^0}$ can be extended to $\tilde{\mathbf{x}} = (x_s)_{s \in \varkappa^{02}}$ satisfying the K-hexahedron equations.*

Proof. Choose an arbitrary extension of $\mathbf{x}_{\text{init}}$, the restriction of $\mathbf{x}$ to $\varkappa^0(T_0)$ to an array $\tilde{\mathbf{x}}'_{\text{init}}$ indexed by $\varkappa^{02}(T_0)$ satisfying equation (2.12) for $s \in \varkappa^2(T_0)$. The result follows once we can show that there exists $\mathbf{t} \in \{-1,1\}^{\varkappa^{\boxdot}}$ such that $(\mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \varkappa^{02}}$ agrees with $\mathbf{x}$ on $\varkappa^0$.

We proceed by induction, and assume that there exists $\mathbf{t}$ such that $(\mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \varkappa^{02}}$ agrees with $\mathbf{x}$ on $\varkappa^0(T_j)$ for $j = 0, \dots, i-1$. If $(\mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \varkappa^{02}}$ agrees with $\mathbf{x}$ on $\varkappa^0(T_j)$ for $j = 0, \dots, i$, we are done. Suppose that $(\mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \varkappa^{02}}$ does not agree with $\mathbf{x}$ on $\varkappa^0(T_i)$. We need to find $\mathbf{t}_i$ such that $(\mathbf{t}_i \mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \varkappa^{02}}$ agrees with $\mathbf{x}$ on $\varkappa^0(T_j)$ for $j = 0, \dots, i$. By Lemma VIII.17, this is equivalent to finding $\mathbf{t}_i$ such that $\psi_\varkappa(\mathbf{t}_i)$ is 1 on $C_1, \dots, C_{i-1}$, and -1 on C_i. By Lemma VIII.18, the cube C_i shares its bottom vertex with T_0. Hence, there exists $\mathbf{u} = (u_s) \in \{-1,1\}^{\varkappa^3}$ satisfying (C2) such that $u_{C_1} = \cdots = u_{C_{i-1}} = 1$ and $u_{C_i} = -1$. (For example, choose $\mathbf{u}$ so that $u_{C_i} = -1$, and $u_C = 1$ for all other cubes C that share a bottom vertex with T_0. Then the remaining values are determined by condition (C2).) Because $\varkappa$ is comfortable, there exists $\mathbf{t}_i$ such that $\psi_\varkappa(\mathbf{t}_i) = \mathbf{u}$, as desired. $\square$

Proof of Theorem VIII.10. We need to loosen the condition that $\mathbf{x}$ is generic from Corollary VIII.19 to the conditions that $\mathbf{x}$ has nonzero components and satisfies condition (4.7).

Let $\mathbf{x} \in (\mathbb{C}^*)^{\varkappa^0}$ be a coherent solution of the Kashaev equation with nonzero components that satisfies condition (4.7). It is straightforward to show that there exists a sequence $\mathbf{x}_1, \mathbf{x}_2, \dots \in (\mathbb{C}^*)^{\varkappa^0}$ of generic, coherent solutions of the Kashaev equation that converge pointwise to $\mathbf{x}$. By Corollary VIII.19, there exist $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \dots \in (\mathbb{C}^*)^{\varkappa^{02}}$ satisfying the K-hexahedron equations such that $\tilde{\mathbf{x}}_i$ restricts to $\mathbf{x}_i$. There exists a subsequence of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \dots$ that converges to an array $\tilde{\mathbf{x}}$. (For each $s \in \varkappa^2(\mathbf{T})$, we can partition the sequence $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \dots$ into two sequences, each of which converges at s. Because $\varkappa^2(\mathbf{T})$ is finite, the claim follows.) The array $\tilde{\mathbf{x}}$ must satisfy the K-hexahedron equations and restrict to $\mathbf{x}$, so we are done. □

In order to complete the proof of Theorem VIII.11, we will need the following technical lemma.

Lemma VIII.20. *Let $\mathbf{T} = (T_0, \ldots, T_\ell)$ be a pile of quadrangulations of a polygon such that $\varkappa(\mathbf{T})$ is not comfortable, but $\varkappa(T_0, \ldots, T_{\ell-1})$ is comfortable. Let C_ℓ be the cube of $\varkappa$ corresponding to the flip from $T_{\ell-1}$ to T_ℓ.*

(a) Let v be the bottom vertex of the cube C_ℓ, i.e., let v be the vertex of $T_{\ell-1}$ not in T_ℓ. Then v is in T_0.

(b) Let $\mathbf{w} = (w_C)_{C\in\varkappa^3}$ where $w_{C_\ell} = -1$, and $w_C = 1$ for $C \neq C_\ell$. Then $\mathbf{w}$ is not in the image of $\psi_\varkappa$.

Proof. Let $\varkappa' = \varkappa(T_0, \ldots, T_{\ell-1})$. Let

- a_1 be the number of $\mathbf{u} \in \{-1, 1\}^{(\varkappa')^3}$ satisfying (C1),
- a_2 be the number of $\mathbf{u} \in \{-1, 1\}^{(\varkappa')^3}$ satisfying (C2),
- b_1 be the number of $\mathbf{u} \in \{-1, 1\}^{\varkappa^3}$ satisfying (C1), and
- b_2 be the number of $\mathbf{u} \in \{-1, 1\}^{\varkappa^3}$ satisfying (C2).

Because a_1, a_2, b_1, b_2 enumerate the elements of vector fields over $\mathbb{F}_2$, all four quantities must be powers of 2. Because $\varkappa'$ is comfortable, $a_1 = a_2$. Because $\varkappa$ is not comfortable, $b_1 < b_2$. It is clear that $b_1 \leq 2a_1$ and $b_2 \leq 2a_2$. Hence, it follows that $a_1 = a_2 = b_1 = b_2/2$.

Assume (for contradiction) that v is not in T_0, so v is in the interior of $\varkappa$. But then if $\mathbf{u} = (u_C)_{C\in\varkappa^3}$ satisfies (C2),

$$u_{C_\ell} = \prod_{C\in\varkappa^3 : w\in C\neq C_\ell} u_C, \tag{8.8}$$

so $a_2 = b_2$, a contradiction. Hence, we have proved (a).

Because $a_1 = b_1$, it follows that for each $\mathbf{u}' \in \{-1,1\}^{(\varkappa')^3}$ satisfying (C1), there exists exactly one $\mathbf{u} \in \{-1,1\}^{\varkappa^3}$ satisfying (C1) that restricts to $\mathbf{u}'$. Because $\mathbf{u} = (u_C)_{C\in\varkappa^3}$ and $\mathbf{u}' = (u_C)_{C\in(\varkappa')^3}$ where $u_C = 1$ for all C satisfy (C1), $\mathbf{w}$ cannot satisfy (C1), proving (b). $\square$

Proof of Theorem VIII.11. Without loss of generality, we assume that $\varkappa(T_0,\ldots,T_{\ell-1})$ is comfortable. (If not, let m be minimum so that $\varkappa(T_0,\ldots,T_m)$ is not comfortable, but $\varkappa(T_0,\ldots,T_{m-1})$ is comfortable. If we can prove the theorem for $\varkappa(T_0,\ldots,T_m)$, it follows that it holds for $\varkappa(\mathbf{T})$.)

We now construct an array $\mathbf{x}$ satisfying the desired conditions. Let C be the cube of $\varkappa$ corresponding to the flip from $T_{\ell-1}$ to T_ℓ, and let v be the top vertex of C (i.e., v is the new vertex in T_i). Choose arbitrary positive values for $\mathbf{x}_{\text{init}}$. Extend $\mathbf{x}_{\text{init}}$ to $\mathbf{x}$ by the positive Kashaev recurrence until we reach v, where we choose the other value such that $K^C(\mathbf{x}) = 0$.

By construction, $\mathbf{x}$ restricted to $\varkappa(T_0,\ldots,T_{\ell-1})$ satisfies the positive Kashaev recurrence, and hence is a coherent solution of the Kashaev equation. By Lemma VIII.20(a), no vertices of C are in the interior of $\varkappa$. Hence, $\mathbf{x}$ is a coherent solution of the Kashaev equation.

Next, we show that $\mathbf{x}$ cannot be extended to an array indexed by $\varkappa^{02}$ satisfying the K-hexahedron equations. Let $\mathbf{x}_{\text{pK}}$ be the array satisfying the positive Kashaev recurrence that restricts to $\mathbf{x}_{\text{init}}$ at T_0 (so $\mathbf{x}_{\text{pK}}$ agrees with $\mathbf{x}$ everywhere except v). Let $\tilde{\mathbf{x}}_{\text{pK}}$ be an extension of $\mathbf{x}_{\text{pK}}$ to $\varkappa^{02}$ satisfying the K-hexahedron equations. Assume (for contradiction) that there exists $\mathbf{t} \in \{-1,1\}^{\tilde{\varkappa}_2}$ such that $\tilde{\mathbf{x}}(\mathbf{t}\cdot(\tilde{\mathbf{x}}_{\text{pK}})_0)$ restricts to $\mathbf{x}$. Hence, by Lemma VIII.17, $\psi_\varkappa(\mathbf{t})$ has value -1 at C, and value 1 everywhere else. But Lemma VIII.20(b) says that array is not in the image of $\psi_\varkappa$, a contradiction. Hence, no such $\mathbf{t}$ exists, so $\mathbf{x}$ cannot be extended to an array indexed by $\varkappa^{02}$ satisfying the K-hexahedron equations. $\square$

CHAPTER IX

Proofs of Corollary IV.23 and Theorem IV.26

This chapter contains the proofs of Corollary IV.23 and Theorem IV.26.

We use the following lemma in proving Corollary IV.23.

Lemma IX.1. *Let* $\mathbf{T}$ *be a pile of* $\lozenge$*-tilings of* $\mathbf{P}_n$*, with* $\varkappa = \varkappa(\mathbf{T})$*. Let* $\mathbf{x} = (x_s) \in (\mathbb{C}^*)^{\varkappa^0}$ *be a coherent solution of the Kashaev equation satisfying condition* (4.5)*. Suppose* $s_1, s_2 \in \varkappa^0$ *are labeled by the same subset of* $[n]$*. Then* $x_{s_1} = x_{s_2}$*.*

Proof. Note that due to the homogeneity of the Kashaev equation and the coherence equations (equation (IV.17)), we can rescale the components of $\mathbf{x}$ to obtain a standard array. Hence, we can assume that $\mathbf{x}$ is standard.

By Theorem IV.19(a), we can extend $\mathbf{x}$ to an array $\tilde{\mathbf{x}}$ indexed by $\varkappa^{02}$ satisfying the K-hexahedron equations. Note that we can choose a sequence $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \dots$ of standard arrays indexed by $\varkappa^{02}$ satisfying the K-hexahedron equations converging to $\tilde{\mathbf{x}}$ such that the restriction of $\tilde{\mathbf{x}}_i$ to $\varkappa^{02}(T)$ for any tiling T in $\mathbf{T}$ is generic. By Theorems IV.9 and IV.12, there exist symmetric $n \times n$ matrices such that $\tilde{\mathbf{x}}_i = \tilde{\mathbf{x}}_{\varkappa(\mathbf{T})}(M_i)$. Hence, the components of $\tilde{\mathbf{x}}_i$ at s_1 and s_2 must agree, so the components of $\tilde{\mathbf{x}}$ at s_1 and s_2 must agree. $\square$

Proof of Corollary IV.23. The first bullet point implies the second two by Corollary IV.14, and it is obvious that the third implies the second. Thus, we need to show that the second bullet point implies the first.

Next, suppose $\mathbf{T} = (T_0, \ldots, T_\ell)$ is a pile of $\lozenge$-tilings of $\mathbf{P}_n$ in which every $I \subseteq [n]$ labels at least one vertex of $\varkappa(\mathbf{T})$, and $\mathbf{x} = \mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation. Let $\mathbf{T}' = (T_0, \ldots, T_\ell, \ldots, T_{\ell'})$ be an extension of $\mathbf{T}$ where $\mathbf{T}'$ contains the tiling $T_{\min,n}$. By Lemma IX.1, $\mathbf{x}_{\varkappa(\mathbf{T}')}(\bar{\mathbf{x}})$ is the unique extension of $\mathbf{x}$ to $\varkappa^0(\mathbf{T}')$. By Theorem IV.19(a), there exists an array $\tilde{\mathbf{x}}$ indexed by $\varkappa^{02}(\mathbf{T}')$ extending $\mathbf{x}_{\varkappa(\mathbf{T}')}(\bar{\mathbf{x}})$ that satisfies the K-hexahedron equations. By Theorem IV.10, Proposition IV.11, and Corollary IV.15 (all of which are due to Kenyon and Pemantle [4]), there exists a unique symmetric matrix M such that $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_{\varkappa(\mathbf{T}')}(M)$, so M satisfies condition (4.8). $\square$

Next, we shall work towards a proof of Theorem IV.26.

Proposition IX.2. *Let M be an $n \times n$ symmetric matrix, and let $\bar{\mathbf{x}} = \bar{\mathbf{x}}(M)$. Then for all $I \subseteq [n]$ and $A \in \binom{[n]}{4}$, equation (4.14) holds.*

Proof. Note that it suffices to prove Proposition IX.2 for generic, symmetric M, because any symmetric matrix can be written as a limit of generic, symmetric matrices. Fix a generic, symmetric $n \times n$ matrix M for the rest of the proof.

For $I \subset [n]$ and distinct $i, j \in [n]$, let

$$x_{I,\{i,j\}} = (-1)^{\lfloor (|I'|+1)/2 \rfloor} M_{I' \cup \{i\}}^{I' \cup \{j\}}, \tag{9.1}$$

where $I' = I \setminus \{i, j\}$. Note that if $\mathbf{T}$ is a pile of $\lozenge$-tilings of $\mathbf{P}_n$, a cube of $\varkappa(\mathbf{T})$ containing vertices labeled by I and $I \cup \{i, j, k\}$ for $i, j, k \notin I$ has top/bottom vertices labeled by $I \cup \{j\}$ and $I \cup \{i, k\}$. Hence, by Lemma VII.2 and the fact that $\tilde{\mathbf{x}}_{\varkappa(\mathbf{T})}(M)$ satisfies the K-hexahedron equations, where $\mathbf{T}$ is any pile of $\lozenge$-tilings of $\mathbf{P}_n$, it follows that

$$K_{I,\{i,j,k\}}(\bar{\mathbf{x}}) = \pm x_{I,\{i,j\}} x_{I,\{\{i,k\}} x_{I,\{j,k\}}, \tag{9.2}$$

where the plus sign appears on the right-hand side of equation (9.2) if either

- $i, k \in I$ and $j \notin I$, or
- $j \in I$ and $i, k \notin I$,

and the minus sign appears otherwise.

Let $I \subseteq [n]$ and $A \in \binom{[n]}{4}$. It is straightforward to check that an even number of $\{i < j < k\} \in \binom{A}{3}$ satisfy neither of the bullet points above. Hence, by equation (9.2), it follows that

$$\prod_{J \in \binom{A}{3}} K_{I,J}(\bar{\mathbf{x}}) = \prod_{J \in \binom{A}{2}} x_{I,J}^2 = \prod_{J \in \binom{A}{2}} L_{I,J}(\bar{\mathbf{x}}), \tag{9.3}$$

as desired. $\square$

The reader may want to review Example III.17 before proceeding with the following lemma.

Lemma IX.3. *Fix an array $\bar{\mathbf{x}} = (x_I)_{I \subseteq [4]}$, satisfying the conditions that*

- *$x_I \neq 0$ for all $I \subseteq [4]$;*
- *for any $I \subseteq [4]$ and distinct $i, j \in [4]$, $L_{I,\{i,j\}} \neq 0$;*
- *for all $I \subseteq [4]$ and distinct $i, j, k \in [4]$, equation (4.12) holds;*
- *for all $I \subseteq [4]$, equation (4.13) holds.*

Let $\mathbf{T}_1 = (T_{1,0}, \ldots, T_{1,4}), \mathbf{T}_2 = (T_{2,0}, \ldots, T_{2,4}) \in \mathcal{C}(4)$ be the two distinct piles in $\mathcal{C}(4)$. Let $\tilde{\mathbf{x}}_1 \in (\mathbb{C}^*)^{\varkappa^{02}(\mathbf{T}_1)}$ be an extension of $\mathbf{x}_{\varkappa(\mathbf{T}_1)}(\bar{\mathbf{x}})$ satisfying the K-hexahedron equations. Then there exists an extension $\tilde{\mathbf{x}}_2 \in (\mathbb{C}^*)^{\varkappa^{02}(\mathbf{T}_2)}$ of $\mathbf{x}_{\varkappa(\mathbf{T}_2)}(\bar{\mathbf{x}})$ satisfying the K-hexahedron equations that agrees with $\tilde{\mathbf{x}}_1$ on $\varkappa^{02}(T_{1,0}) = \varkappa^{02}(T_{2,0})$.

Proof. By the homogeneity of equations (4.12)–(4.13), we can rescale the components of $\bar{\mathbf{x}}$ so that $x_\varnothing = 1$. Hence, it follows from Corollary IV.25 that there exists an extension $\tilde{\mathbf{x}}'_1 \in (\mathbb{C}^*)^{\varkappa^{02}(\mathbf{T}_1)}$ of $\mathbf{x}_{\varkappa(\mathbf{T}_1)}(\bar{\mathbf{x}})$ and an extension $\tilde{\mathbf{x}}'_2 \in (\mathbb{C}^*)^{\varkappa^{02}(\mathbf{T}_2)}$ of $\mathbf{x}_{\varkappa(\mathbf{T}_2)}(\bar{\mathbf{x}})$ that agree on $\varkappa^{02}(T_{1,0}) = \varkappa^{02}(T_{2,0})$.

Let $\tilde{\mathbf{x}}'_{\text{init}}$ be the restriction of $\tilde{\mathbf{x}}'_1$ to $\varkappa^{02}(T_{1,0})$. Let $\mathbf{t} \in \{-1,1\}^{\varkappa^2(T_{1,0})}$ (where we associate $\varkappa^2(T_{1,0})$ with $\tilde{\varkappa}_2(\mathbf{T}_1)$) so that $\tilde{\mathbf{x}}_1 = (\mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \varkappa^{02}(\mathbf{T}_1)}$. By Lemma VIII.17, because $\tilde{\mathbf{x}}'_1$ and $\tilde{\mathbf{x}}_1$ agree on $\varkappa^0(\mathbf{T}_1)$, $\psi_{\varkappa(\mathbf{T}_1)}(\mathbf{t})$ has value 1 at every cube of $\varkappa(\mathbf{T}_1)$. Because $\psi_{\varkappa(\mathbf{T}_1)}(\mathbf{t})$ has value 1 at every cube of $\varkappa(\mathbf{T}_1)$, $\psi_{\varkappa(\mathbf{T}_2)}(\mathbf{t})$ has value 1 at every cube of $\varkappa(\mathbf{T}_2)$. Hence, $(\mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \varkappa^2(\mathbf{T}_2)}$ agrees with $\tilde{\mathbf{x}}_1$ on $\varkappa^{02}(T_{1,0}) = \varkappa^{02}(T_{2,0})$ and restricts to $\mathbf{x}_{\varkappa(\mathbf{T}_2)}(\bar{\mathbf{x}})$. $\square$

The reader may want to review Definition III.21 before proceeding with the following definition.

Definition IX.4. Let $\mathbf{T}_1 = (T_{1,0}, \ldots, T_{1,\ell})$ and $\mathbf{T}_2 = (T_{2,0}, \ldots, T_{2,\ell})$ be two piles, such that the directed cubical complexes $\varkappa(\mathbf{T}_1)$ and $\varkappa(\mathbf{T}_2)$ are related by a flip. Label the vertices of $\varkappa(\mathbf{T}_1)$ and $\varkappa(\mathbf{T}_2)$ involved in the 3-flip by subsets of $[4]$, as in Figure 9.1. Let $\mathbf{x}_1 \in (\mathbb{C}^*)^{\varkappa^0(\mathbf{T}_1)}$ and $\mathbf{x}_2 \in (\mathbb{C}^*)^{\varkappa^0(\mathbf{T}_2)}$ be arrays satisfying condition (4.5). We say that the pair $(\mathbf{x}_1, \mathbf{x}_2)$ is *K-flipped* when

- $\mathbf{x}_1$ and $\mathbf{x}_2$ agree everywhere, except at the vertex at which $\varkappa(\mathbf{T}_1)$ and $\varkappa(\mathbf{T}_2)$ differ;
- writing $\bar{\mathbf{x}} = (x_I)_{I \subseteq [4]}$, where x_I is the component of $\mathbf{x}_1$ and/or $\mathbf{x}_2$ at the vertex labeled by I, $\bar{\mathbf{x}}$ satisfies equation (4.12) for all $I \subseteq [4]$ and distinct $i,j,k \in [4]$ and equation (4.13) for all $I \subseteq [4]$.

Lemma IX.5. *Let $\mathbf{T}_1 = (T_{1,0}, \ldots, T_{1,\ell})$ and $\mathbf{T}_2 = (T_{2,0}, \ldots, T_{2,\ell})$ be two piles, such that the directed cubical complexes $\varkappa(\mathbf{T}_1)$ and $\varkappa(\mathbf{T}_2)$ are related by a flip. Let $\mathbf{x}_1 \in$*

$(\mathbb{C}^*)^{\varkappa^0(\mathbf{T}_1)}$ *and* $\mathbf{x}_2 \in (\mathbb{C}^*)^{\varkappa^0(\mathbf{T}_2)}$ *be arrays satisfying condition* (4.5)*, such that the pair* $(\mathbf{x}_1, \mathbf{x}_2)$ *is K-flipped.*

(a) Then $\mathbf{x}_1$ *is a coherent solution of the Kashaev equation if and only if* $\mathbf{x}_2$ *is a coherent solution of the Kashaev equation.*

(b) Suppose $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are both coherent solutions of the Kashaev equation. Let* $\tilde{\mathbf{x}}_1 \in (\mathbb{C}^*)^{\varkappa^{02}(\mathbf{T}_1)}$ *be an extension of* $\mathbf{x}_1$ *to* $\varkappa^{02}(\mathbf{T}_1)$*, and let* $\tilde{\mathbf{x}}_{\mathrm{init}}$ *be the restriction of* $\tilde{\mathbf{x}}_1$ *to* $\varkappa^{02}(T_{1,0})$*. Then, identifying* $\varkappa^{02}(T_{1,0})$ *and* $\varkappa^{02}(T_{2,0})$*,* $(\tilde{\mathbf{x}}_{\mathrm{init}})^{\uparrow \varkappa^{02}(\mathbf{T}_2)}$ *restricts to* $\mathbf{x}_2$*.*

Proof. This result follows from Lemma IX.3. $\square$

Lemma IX.6. *There exists a sequence* $\mathbf{T}_0, \ldots, \mathbf{T}_{\binom{n}{3}-\binom{n-1}{2}} \in \mathcal{C}(n)$ *of piles, where we write* $\mathbf{T}_i = (T_{i,0}, \ldots, T_{i,\binom{n}{3}})$ *for* $i = 0, \ldots, \binom{n}{3} - \binom{n-1}{2}$*, such that*

- *the directed cubical complexes* $\varkappa(\mathbf{T}_{i-1})$ *and* $\varkappa(\mathbf{T}_i)$ *are related by a flip for* $i = 1, \ldots, \ell$*;*
- *the directed cubes of* $\varkappa(\mathbf{T}_0)$ *corresponding to the flips between* $T_{0,i-1}$ *and* $T_{0,i}$ *for* $i = 1, \ldots, \binom{n-1}{2}$ *share their bottom vertex with* $T_{0,0}$*;*
- *for* $i = 1, \ldots, \binom{n}{3} - \binom{n-1}{2}$*,* $T_{0,j} = \cdots = T_{i-1,j}$ *for* $j = i + \binom{n-1}{2}, \ldots, \binom{n}{3}$*;*
- *for* $i = 1, \ldots, \binom{n}{3} - \binom{n-1}{2}$*, the directed cube of* $\varkappa(\mathbf{T}_{i-1})$ *corresponding to the flip between* $T_{i-1,i-1+\binom{n}{3}-\binom{n-1}{2}}$ *and* $T_{i-1,i-1+\binom{n}{3}-\binom{n-1}{2}}$ *is the top of the four cubes of* $\varkappa(\mathbf{T}_{i-1})$ *involved in the flip between* $\varkappa(\mathbf{T}_{i-1})$ *and* $\varkappa(\mathbf{T}_i)$*.*

Remark IX.7. The idea behind Lemma IX.6 is as follows. Let C_i be the cube of $\varkappa(\mathbf{T}_0)$ corresponding to the flip between $T_{0,i-1}$ and $T_{0,i}$. The second bullet point states that the cubes $C_1, \ldots, C_{\binom{n-1}{2}}$ share their bottom vertex with $T_{0,0}$, and hence do not have their bottom vertices in the interior of $\varkappa(\mathbf{T}_0)$. As a consequence of the remaining bullet points, there is a sequence of $\binom{n}{3} - \binom{n-1}{2}$ flips on the directed cubical complex $\varkappa(\mathbf{T}_0)$ in which $C_{\binom{n-1}{2}+1}, \dots, C_{\binom{n}{3}}$ (in that order) are the top cubes involved in the flips.

Proof of Lemma IX.6. Define a total order $<_{\text{lex}}$ on $\binom{[n]}{k}$, where $\{i_1 < \cdots < i_k\} <_{\text{lex}} \{i'_1 < \cdots < i'_k\}$ when there exists j such that $i_\ell = i'_\ell$ for $\ell < j$, and $i_j < i'_j$. Set $\{\alpha_1 <_{\text{lex}} \cdots <_{\text{lex}} \alpha_{\binom{n}{3}}\} = \binom{[n]}{3}$. Note that the permutation $\sigma_0 = (\alpha_1, \dots, \alpha_{\binom{n}{3}})$ of $\binom{[n]}{3}$ is admissible (see Definition III.23). Let $\mathbf{T}_0$ be the pile corresponding to $(\alpha_1, \dots, \alpha_{\binom{n}{3}})$ (see Theorem III.25). Note that $1 \in \alpha_i$ for $i = 1, \dots, \binom{n-1}{2}$, so the second bullet point holds.

We now construct the piles $\mathbf{T}_1, \dots, \mathbf{T}_{\binom{n}{3}-\binom{n-1}{2}}$ inductively as follows. For $i = 1, \dots, \binom{n}{3} - \binom{n-1}{2}$, the admissible permutation σ_i corresponding to $\mathbf{T}_i$ should have the following properties:

- The inversion set of σ_i is $\{\{1\} \cup \alpha_{\binom{n-1}{2}+1}, \dots, \{1\} \cup \alpha_{\binom{n-1}{2}+i}\}$. Hence, the inversion sets of σ_{i-1} and σ_i differ by the element $\{1\} \cup \alpha_{\binom{n-1}{2}+i}$, so $\varkappa(\mathbf{T}_{i-1})$ and $\varkappa(\mathbf{T}_i)$ are related by a flip. Hence, the first bullet point holds.
- Writing $\sigma_{i-1} = (\beta_1, \dots, \beta_{\binom{n}{3}})$, $\beta_j = \alpha_j$ for $j = i + \binom{n-1}{2}, \dots, \binom{n}{3}$. Hence, the third bullet point holds. Because $\beta_{i+\binom{n-1}{2}} = \alpha_{i+\binom{n-1}{2}}$ and the flip between $\varkappa(\mathbf{T}_{i-1})$ and $\varkappa(\mathbf{T}_i)$ consists of the inclusion of $\{1\} \cup \alpha_{i+\binom{n-1}{2}}$ to the inversion set, the fourth bullet point follows.

For $i = 1, \dots, \binom{n}{3} - \binom{n-1}{2}$, write $\sigma_{i-1} = (\beta_1, \dots, \beta_{\binom{n}{3}})$. We want to obtain σ_i. Write $\alpha_{i+\binom{n-1}{2}} = \beta_{i+\binom{n-1}{2}} = \{i_1 < i_2 < i_3\}$. Let $(\gamma_1, \dots, \gamma_{i+\binom{n-1}{2}-4})$ be the subsequence of $(\beta_1, \dots, \beta_{i+\binom{n-1}{2}-4})$ excluding $\{1, i_1, i_2\}$, $\{1, i_1, i_3\}$, and $\{1, i_2, i_3\}$. Setting

(9.4)

$$\sigma_i = \left(\gamma_1, \dots, \gamma_{i+\binom{n-1}{2}-4}, \{i_1, i_2, i_3\}, \{1, i_2, i_3\}, \{1, i_1, i_3\}, \{1, i_1, i_2\}, \beta_{i+\binom{n-1}{2}}, \dots, \beta_{\binom{n}{3}}\right),$$

it is straightforward to check that σ_i is an admissible permutation with the desired properties. $\square$

Remark IX.8. The pile $\mathbf{T}_0$ constructed in the proof of Lemma IX.6 is a representative for the smallest element of the third higher Bruhat order. The sequence $\left(\varkappa\left(\mathbf{T}_0\right), \ldots, \varkappa\left(\mathbf{T}_{\binom{n}{3}-\binom{n-1}{2}}\right)\right)$, where $\mathbf{T}_0, \ldots, \mathbf{T}_{\binom{n}{3}-\binom{n-1}{2}}$ are the piles constructed in the proof of Lemma IX.6, are the first $\binom{n}{3}-\binom{n-1}{2}+1$ elements for a representative for the smallest element of the fourth higher Bruhat order. See [8] or [10] for further discussion of higher Bruhat orders.

Lemma IX.9. *Let $\bar{\mathbf{x}} = (x_I)_{I\subseteq[n]}$ be an array satisfying the conditions that $L_{I,\{i,j\}} \neq 0$ for any $I \subseteq [n]$ and distinct $i, j \in [n]$, and $x_\varnothing = 1$. Suppose that for all $I \subseteq [n]$ and distinct $i, j, k \in [n]$, equation* (4.12) *holds, and for all $I \subseteq [n]$ and $A \in \binom{[n]}{4}$, equation* (4.14) *holds. Then there exists $\mathbf{T} \in \mathcal{C}(n)$ such that $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation.*

Proof. Let $\mathbf{T}_0, \ldots, \mathbf{T}_{\binom{n}{3}-\binom{n-1}{2}} \in \mathcal{C}(n)$ be a sequence of piles satisfying the conditions of Lemma IX.6, where we write $\mathbf{T}_i = (T_{i,0}, \ldots, T_{i,\binom{n}{3}})$ for $i = 0, \ldots, \binom{n}{3}-\binom{n-1}{2}$. We will show that $\mathbf{x}_{\varkappa(\mathbf{T}_0)}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation.

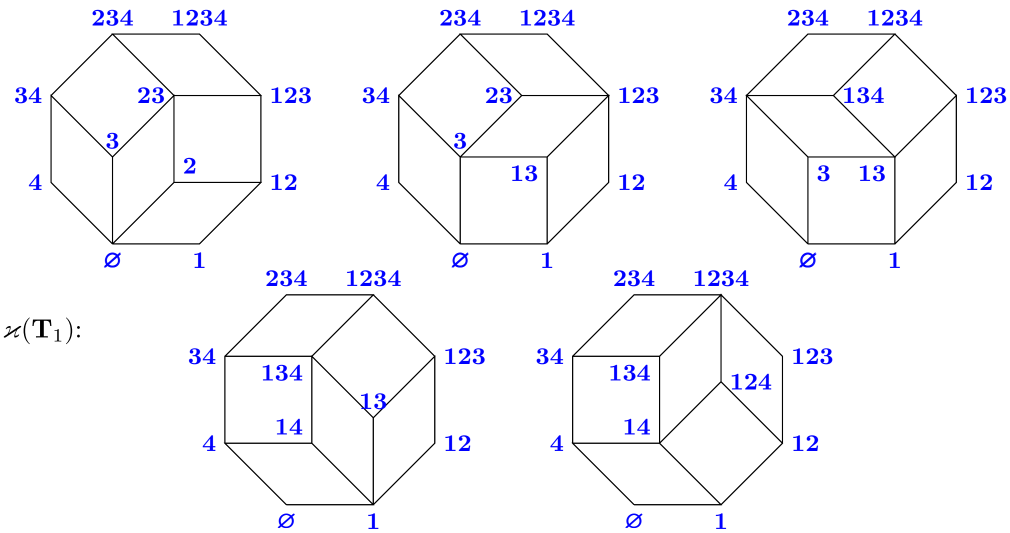
We claim that $\mathbf{x}_{\varkappa(T_{0,0},\ldots,T_{0,j})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation for $j = 1, \ldots, \binom{n}{3}$ and proceed by induction. Because equation (4.12) holds for all $I \subseteq [n]$ and distinct $i, j, k \in [n]$, $\mathbf{x}_{\varkappa(T_{0,0},\ldots,T_{0,j})}(\bar{\mathbf{x}})$ satisfies the Kashaev equation. Hence, we only have to check coherence, i.e., we need to check that equation (4.6) holds for every interior vertex of $\varkappa(T_{0,0}, \ldots, T_{0,j})$. For $j = 1, \ldots, \binom{n-1}{2}$, none of the vertices $\varkappa^0(T_{0,0}, \ldots, T_{0,j})$ are interior vertices of $\varkappa(T_{0,0}, \ldots, T_{0,j})$, so $\mathbf{x}_{\varkappa(T_{0,0},\ldots,T_{0,j})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation. By our inductive hypothesis, $\mathbf{x}_{\varkappa(T_{0,0},\ldots,T_{0,j-1})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation. By construction, $\varkappa(T_{i-1,0},\ldots,T_{i-1,j-1})$ and $\varkappa(T_{i,0},\ldots,T_{i,j-1})$ are related by a flip for $i=1,\ldots,j-\binom{n-1}{2}-1$. Hence, the pairs $\left(\mathbf{x}_{\varkappa(T_{i-1,0},\ldots,T_{i-1,j-1})}(\bar{\mathbf{x}}),\mathbf{x}_{\varkappa(T_{i,0},\ldots,T_{i,j-1})}(\bar{\mathbf{x}})\right)$ are K-flipped for $i=1,\ldots,j-\binom{n-1}{2}-1$ by the conditions of the lemma. By repeated applications of Lemma IX.5(a), it follows that $\mathbf{x}_{\varkappa\left(T_{j-\binom{n-1}{2}-1,0},\ldots,T_{j-\binom{n-1}{2}-1,j-1}\right)}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation. By construction, the cube of $\varkappa\left(T_{j-\binom{n-1}{2}-1,0},\ldots,T_{j-\binom{n-1}{2}-1,j}\right)$ corresponding to the flip between $T_{j-\binom{n-1}{2}-1,j-1}$ and $T_{j-\binom{n-1}{2}-1,j}$ is the top of four cubes where a flip can take place. Hence, because equation (4.14) holds for all $I\subseteq[n]$ and $A\in\binom{[n]}{4}$, $\mathbf{x}_{\varkappa\left(T_{j-\binom{n-1}{2}-1,0},\ldots,T_{j-\binom{n-1}{2}-1,j}\right)}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation. By construction, $\varkappa(T_{i-1,0},\ldots,T_{i-1,j})$ and $\varkappa(T_{i-1,0},\ldots,T_{i-1,j})$ are related by a flip for $i=1,\ldots,j-\binom{n-1}{2}-1$, so the pairs $\left(\mathbf{x}_{\varkappa(T_{i-1,0},\ldots,T_{i-1,j})}(\bar{\mathbf{x}}),\mathbf{x}_{\varkappa(T_{i,0},\ldots,T_{i,j})}(\bar{\mathbf{x}})\right)$ are K-flipped for $i=1,\ldots,j-\binom{n-1}{2}-1$. Thus, by repeated applications of Lemma IX.5(a), it follows that $\mathbf{x}_{\varkappa(T_{0,0},\ldots,T_{0,j})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation. $\square$

Proof of Theorem IV.26. By Corollary IV.23, the first two bullet points are equivalent, and by Corollary IV.23 and Proposition IX.2, the first bullet point implies the third bullet point. Hence, we just need to show that the third bullet point implies the first.

Suppose that the third bullet point holds. By Lemma IX.9, there exists $\mathbf{T}\in\mathcal{C}(n)$ such that $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ is a coherent solution of the Kashaev equation. Hence, by Theorem IV.19, there exists an extension $\tilde{\mathbf{x}}\in(\mathbb{C}^*)^{\varkappa^{02}(\mathbf{T})}$ of $\mathbf{x}_{\varkappa(\mathbf{T})}(\bar{\mathbf{x}})$ to $\varkappa^{02}(\mathbf{T})$ that satisfies the K-hexahedron equations. Hence, there exists a symmetric matrix M such that $\tilde{\mathbf{x}}=\tilde{\mathbf{x}}_{\varkappa(\mathbf{T})}(M)$. Let $\tilde{\mathbf{x}}_{\text{init}}$ be the restriction of $\tilde{\mathbf{x}}$ to $T_{\min,n}$.

Given any $\mathbf{T}'\in\mathcal{C}(n)$, by Proposition III.22, there exists a sequence of piles $\mathbf{T}_0,\ldots,\mathbf{T}_\ell$ with $\mathbf{T}=\mathbf{T}_0$ and $\mathbf{T}'=\mathbf{T}_\ell$ such that $\varkappa(\mathbf{T}_{i-1})$ and $\varkappa(\mathbf{T}_i)$ are related by a flip for $i = 1, \ldots, \ell$. Hence, $(\mathbf{x}_{\varkappa(\mathbf{T}_{i-1})}(\bar{\mathbf{x}}), \mathbf{x}_{\varkappa(\mathbf{T}_i)}(\bar{\mathbf{x}}))$ is K-flipped, so by repeated applications of Lemma IX.5(a) and (b), $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow \varkappa^{02}(\mathbf{T}')}$ restricts to $\mathbf{x}_{\varkappa(\mathbf{T}')}(\bar{\mathbf{x}})$. Because every $I \subseteq [n]$ labels a vertex in $\varkappa(\mathbf{T}')$ for some $\mathbf{T}' \in \mathcal{C}(n)$, $\bar{\mathbf{x}} = \bar{\mathbf{x}}(M)$, as desired. $\square$

In $\varkappa(\mathbf{T}_1)$:



In $\varkappa(\mathbf{T}_2)$:

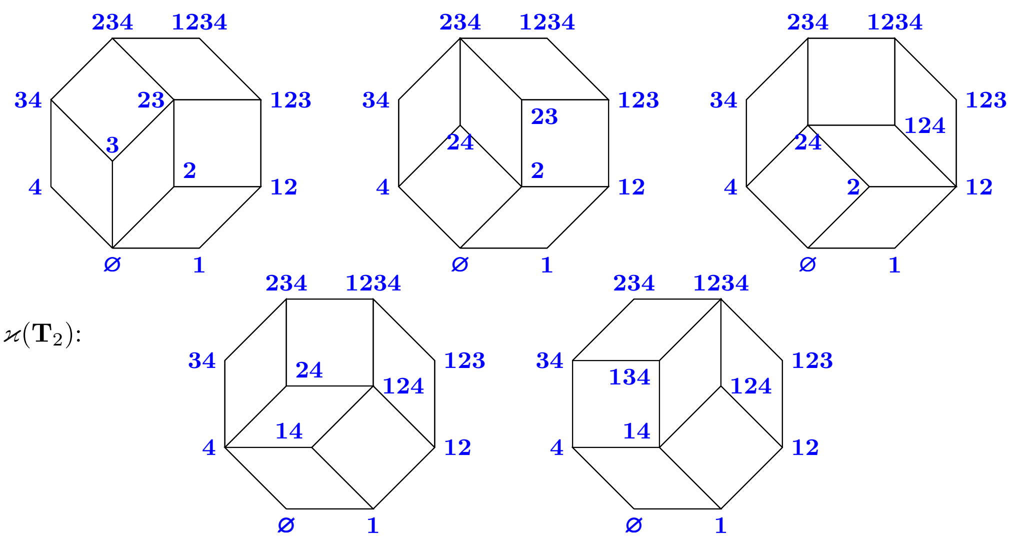


Figure 9.1: Labeling the vertices involved in a flip between $\varkappa(\mathbf{T}_1)$ and $\varkappa(\mathbf{T}_2)$ in Lemma IX.5 with subsets of [4] (in blue).

CHAPTER X

Generalizations of the Kashaev Equation

In this chapter, we describe an axiomatic setup for equations similar to the Kashaev equation and the examples from Chapters V–VI. This allows us to prove all of the results from Chapters V–VI. This chapter is organized as follows:

- Proposition X.2 and Lemma X.8 generalize Propositions II.8, V.4, VI.1, and VI.7.
- In Definition X.11, given a polynomial equation resembling the Kashaev equation, we describe how to obtain a set of equations with the same properties as the K-hexahedron equations.
- In Definition X.18, we define certain signs that appear in our generalized "coherence" equation (10.56).
- Theorem X.24, the main result in this chapter, generalizes Theorems II.22, V.9, VI.4, and VI.10. The proof of Theorem X.24 is nearly identical to the proof of Theorem II.22 from Chapter VII.

Definition X.1. For $d \geq 1$ and $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$, we denote by

$$[\mathbf{a}] = \left\{(b_1, \ldots, b_d) \in \mathbb{Z}^d : 0 \leq |b_i| \leq |a_i| \text{ and } a_i b_i \geq 0 \text{ for all } i\right\} \tag{10.1}$$

the set of integer points in the $|a_1| \times \cdots \times |a_d|$ box with opposite vertices $(0, \ldots, 0)$ and $(a_1, \ldots, a_d)$. For $\mathbf{b} = (b_1, \ldots, b_d), \mathbf{c} = (c_1, \ldots, c_d) \in \mathbb{Z}^d$, we write $\mathbf{b} \odot \mathbf{c} =$

$(b_1c_1, \ldots, b_dc_d) \in \mathbb{Z}^d$. Denote by $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^d$ the all 1's vector, and set $\mathbf{1}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$, with 1 in the ith place, for $i = 1, \ldots, d$. Let

$$\mathbf{z}_{[\mathbf{a}]} = \{z_{\mathbf{i}} : \mathbf{i} \in [\mathbf{a}]\} \tag{10.2}$$

be a set of indeterminates. For $i = 1, \ldots, d$, let $\pi_{\mathbf{a},i} : \mathbf{z}_{[\mathbf{a}]} \to \mathbf{z}_{[\mathbf{a}]}$ be the involution defined by

$$z_{(j_1,\ldots,j_d)} \mapsto z_{(j_1,\ldots,a_i-j_i,\ldots,j_d)}, \tag{10.3}$$

i.e., we "flip" the index of each variable in its ith coordinate. The action of $\pi_{\mathbf{a},i}$ extends from $\mathbf{z}_{[\mathbf{a}]}$ to the polynomial ring $\mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$. Given an array $\mathbf{x} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d}$ and integer vectors $v \in \mathbb{Z}^d$, $\mathbf{a} \in \mathbb{Z}^d_{\geq 0}$, and $\boldsymbol{\alpha} \in \{-1, 1\}^d$, we denote by $\mathbf{x}_{v+[\mathbf{a}\odot\boldsymbol{\alpha}]} \in \mathbb{C}^{[\mathbf{a}]}$ the array whose entries are

$$(\mathbf{x}_{v+[\mathbf{a}\odot\boldsymbol{\alpha}]})_{\mathbf{i}} = x_{v+\mathbf{i}\odot\boldsymbol{\alpha}}, \text{ for } \mathbf{i} \in [\mathbf{a}]. \tag{10.4}$$

In particular,

$$(\mathbf{x}_{v+[\mathbf{a}]})_{\mathbf{i}} = x_{v+\mathbf{i}}, \text{ for } \mathbf{i} \in [\mathbf{a}]. \tag{10.5}$$

Thus, given a polynomial $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$, the number $f(\mathbf{x}_{v+[\boldsymbol{\alpha}\odot\mathbf{a}]}) \in \mathbb{C}$ is obtained by setting $z_{\mathbf{i}} = x_{v+\boldsymbol{\alpha}\odot\mathbf{i}}$ for each variable $z_{\mathbf{i}}$ for $\mathbf{i} \in [\mathbf{a}]$. We say that $\mathbf{x} \in \mathbb{C}^{\mathbb{Z}^d}$ *satisfies* f if $f(\mathbf{x}_{v+[\mathbf{a}]}) = 0$ for all $v \in \mathbb{Z}^d$.

Proposition X.2. *Let $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d_{\geq 1}$, and a polynomial $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ satisfy the following conditions:*

- *(X.2.1) f is invariant under the action of $\pi_{\mathbf{a},i}$ for $i = 1, \ldots, d$;*
- *(X.2.2) f has degree 2 with respect to the variable $z_{\mathbf{a}}$; as a quadratic polynomial in $z_{\mathbf{a}}$, f has discriminant D which factors as a product $D = f_1 \cdots f_d$, where each polynomial $f_i \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}-\mathbf{1}_i]}]$ is invariant under the action of $\pi_{\mathbf{a}-\mathbf{1}_i,j}$ for $j = 1, \ldots, d$.*

Then for any $\mathbf{x} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d}$ *satisfying* f*, we have, for all* $v \in \mathbb{Z}^d$*:*

(10.6)
$$\left(\prod_{\boldsymbol{\alpha} \in \{-1,0\}^d} \frac{\partial f}{\partial z_{\mathbf{a}}} (\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) \right)^2 = \left(\prod_{i=1}^{d} \prod_{\substack{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_d)\in\{-1,0\}^d \\ \beta_i=0}} f_i(\mathbf{x}_{v+\boldsymbol{\beta}+[\mathbf{a}]}) \right)^2 .$$

Moreover, for all $v \in Z^d$*, we have:*

(10.7)
$$\begin{aligned} &\left(\prod_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d) \\ \alpha_1,\ldots,\alpha_d\in\{-1,0\} \\ \alpha_1+\cdots+\alpha_d \text{ even}}} \frac{\partial f}{\partial z_{\mathbf{a}}} (\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) \right)^2 \\ =& \left(\prod_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d) \\ \alpha_1,\ldots,\alpha_d\in\{-1,0\} \\ \alpha_1+\cdots+\alpha_d \text{ odd}}} \frac{\partial f}{\partial z_{\mathbf{a}}} (\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) \right)^2 \\ =& \prod_{i=1}^{d} \prod_{\substack{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_d)\in\{-1,0\}^d \\ \beta_i=0}} f_i(\mathbf{x}_{v+\boldsymbol{\beta}+[\mathbf{a}]}). \end{aligned}$$

Remark X.3. The subscripts $v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]$ appearing on the left-hand side of (10.6) run over all boxes of size $a_1 \times \cdots \times a_d$ containing $v + [\mathbf{a}-\mathbf{1}]$. The subscripts $v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]$ appearing on the right-hand side of (10.6) run over $i = 1, \ldots, d$ and boxes of size $a_1 \times \cdots \times a_{i-1} \times (a_i - 1) \times a_{i+1} \times \cdots \times a_d$ containing $v + [\mathbf{a}-\mathbf{1}]$. In particular, when $a_1 = \cdots = a_d = 1$, all of these products are over boxes of a certain size containing the vertex v. For example, in the case $\mathbf{a} = (1, 2)$ (like in Proposition VI.7), the boxes we are considering on the left-hand side of (10.6) are given in the top row of Figure 6.2, while the boxes we are considering on the right-hand side of (10.6) are given in the bottom row of Figure 6.2.

Before proving Proposition X.2, we give several examples of polynomials discussed in previous chapters that satisfy conditions (X.2.1)–(X.2.2). In the examples below, we write $z_{i_1\cdots i_d} = z_{(i_1,\ldots,i_d)}$ for $(i_1, \ldots, i_d) \in \mathbb{Z}^d$.

Example X.4. Our first example is the Kashaev equation. Let $\mathbf{a} = (1,1,1) \in \mathbb{Z}^3$, and let

$$f = 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 - 4(s + t) \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}], \tag{10.8}$$

where

$$\begin{gathered} a = z_{000}z_{111}, \quad b = z_{100}z_{011}, \quad c = z_{010}z_{101}, \quad d = z_{001}z_{110}, \\ s = z_{000}z_{011}z_{101}z_{110}, \quad t = z_{100}z_{010}z_{001}z_{111}. \end{gathered} \tag{10.9}$$

The polynomial f is invariant not only under the action of the $\pi_{\mathbf{a},i}$, but under all symmetries of the cube. Its discriminant (as a polynomial in z_{111}) D factors as a product $D = f_1 f_2 f_3$, with

$$\begin{aligned} f_1 &= 16(z_{000}z_{011} + z_{010}z_{001}); \\ f_2 &= z_{000}z_{101} + z_{100}z_{001}; \\ f_3 &= z_{000}z_{110} + z_{100}z_{010}. \end{aligned} \tag{10.10}$$

Hence, f satisfies conditions (X.2.1)–(X.2.2). Therefore, Proposition II.8 is a special case of Proposition X.2.

Example X.5. Let $\mathbf{a} = (1,1) \in \mathbb{Z}^2$, and let

$$\begin{aligned} f = {} & z_{00}^2 + z_{10}^2 + z_{01}^2 + z_{11}^2 - 2(z_{00}z_{10} + z_{10}z_{11} + z_{11}z_{01} + z_{01}z_{00}) - 6(z_{00}z_{11} + z_{10}z_{01}) \\ & \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]. \end{aligned} \tag{10.11}$$

The polynomial f is invariant not only under the action of the $\pi_{\mathbf{a},i}$, but under all symmetries of the square. Its discriminant (as a polynomial in z_{11}) D factors as a product $D = f_1 f_2$, with

$$\begin{aligned} f_1 &= 32(z_{00} + z_{01}); \\ f_2 &= z_{00} + z_{10}. \end{aligned} \tag{10.12}$$

Hence, f satisfies conditions (X.2.1)–(X.2.2). Therefore, Proposition V.4 is a special case of Proposition X.2.

Example X.6. Fix $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Let $\mathbf{a} = (3) \in \mathbb{Z}^1$, and let

$$f = z_0^2 z_3^2 + \alpha_1 z_1^2 z_2^2 + \alpha_2 z_0 z_1 z_2 z_3 + \alpha_3 (z_0 z_2^3 + z_1^3 z_3) \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]. \tag{10.13}$$

The polynomial f is invariant under the action of $\pi_{\mathbf{a},1}$, i.e., replacing z_i by z_{3-i}. Its discriminant (as a polynomial in z_{11}) $D = f_1$, with

$$f_1 = \alpha_3^2 z_1^6 + 2\alpha_2\alpha_3 z_0 z_1^4 z_2 + (\alpha_2^2 - 4\alpha_1) z_0^2 z_1^2 z_2^2 - 4\alpha_3 z_0^3 z_2^3. \tag{10.14}$$

Hence, f satisfies conditions (X.2.1)–(X.2.2). Therefore, Proposition VI.1 is a special case of Proposition X.2.

Example X.7. Fix $\alpha_1, \alpha_2 \in \mathbb{C}$. Let $\mathbf{a} = (1, 2) \in \mathbb{Z}^2$, and let

$$\begin{aligned} f = {} & z_{00}^2 z_{12}^2 + z_{10}^2 z_{02}^2 + \frac{\alpha_2^2 - \alpha_1^2}{4} z_{01}^2 z_{11}^2 - \alpha_1 (z_{00} z_{02} z_{11}^2 + z_{10} z_{12} z_{01}^2) \\ & - 2 z_{00} z_{10} z_{02} z_{12} - \alpha_2 (z_{00} z_{12} z_{01} z_{11} + z_{10} z_{02} z_{01} z_{11}) \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]. \end{aligned} \tag{10.15}$$

The polynomial f is invariant under the action of $\pi_{\mathbf{a},1}$ and $\pi_{\mathbf{a},2}$. Its discriminant (as a polynomial in z_{12}) factors as a product $D = f_1 f_2$, with

$$\begin{aligned} f_1 &= \alpha_1 z_{01}^2 + 4 z_{00} z_{02}; \\ f_2 &= \alpha_1 (z_{00}^2 z_{11}^2 + z_{01}^2 z_{10}^2) + 2\alpha_2 z_{00} z_{01} z_{10} z_{11}. \end{aligned} \tag{10.16}$$

Hence, f satisfies conditions (X.2.1)–(X.2.2). Therefore, Proposition VI.7 is a special case of Proposition X.2.

Proof of Proposition X.2. Let g denote the coefficient of $z_{\mathbf{a}}^2$ in f (viewed as a polynomial in $z_{\mathbf{a}}$). It is easy to check that

$$D = f_1 \cdots f_d = \left(\frac{\partial f}{\partial z_{\mathbf{a}}} \right)^2 - 4fg. \tag{10.17}$$

Because $\mathbf{x}$ satisfies f, we have

$$\left(\frac{\partial f}{\partial z_{\mathbf{a}}} \right)^2 (\mathbf{x}_{v - (\mathbf{a}-\mathbf{1}) \odot \boldsymbol{\alpha} + [(1+2\boldsymbol{\alpha}) \odot \mathbf{a}]}) = (f_1 \cdots f_d)(\mathbf{x}_{v - (\mathbf{a}-\mathbf{1}) \odot \boldsymbol{\alpha} + [(1+2\boldsymbol{\alpha}) \odot \mathbf{a}]}). \tag{10.18}$$

Because each f_i is invariant under the action of $\pi_{\mathbf{a}-\mathbf{1}_i,j}$ for $j = 1,\dots,d$, we have

$$\left(\frac{\partial f}{\partial z_{\mathbf{a}}}\right)^2 (\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(1+2\boldsymbol{\alpha})\odot\mathbf{a}]}) = \prod_{i=1}^{d} f_i(\mathbf{x}_{v+\boldsymbol{\alpha}\odot(\mathbf{1}-\mathbf{1}_i)+[\mathbf{a}]}). \tag{10.19}$$

Given $i \in \{1,\dots,d\}$ and $\boldsymbol{\beta} = (\beta_1,\dots,\beta_d) \in \{-1,0\}^d$ with $\beta_i = 0$, there exist exactly two $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d) \in \{-1,0\}^d$ with $(\mathbf{1}-\mathbf{1}_i)\odot\boldsymbol{\alpha} = \boldsymbol{\beta}$: one with $\alpha_i = 0$ and the other with $\alpha_i = 1$. Hence, $\alpha_1 + \cdots + \alpha_d$ is even for one such choice of $\boldsymbol{\alpha}$, and odd for the other. Taking the product over $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d) \in \{-1,0\}^d$ with $\alpha_1 + \cdots + \alpha_d$ odd (or even) in (10.19), we obtain (10.7). Equation (10.6) follows. $\square$

Lemma X.8. *Let* $\mathbf{a} = (a_1,\dots,a_d) \in \mathbb{Z}^d_{\geq 1}$, *and let* $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ *be a polynomial satisfying conditions (X.2.1)–(X.2.2). Let* $\mathbf{x} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d}$ *be an array satisfying* f. *Fix* $v \in \mathbb{Z}^d$ *and* $\gamma \in \{-1,1\}$. *Then*

$$\prod_{\boldsymbol{\alpha}\in\{-1,0\}^d} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) = \gamma \prod_{i=1}^{d} \prod_{\substack{\boldsymbol{\beta}=(\beta_1,\dots,\beta_d)\in\{-1,0\}^d \\ \beta_i=0}} f_i(\mathbf{x}_{v+\boldsymbol{\beta}+[\mathbf{a}]}) \tag{10.20}$$

if and only if

$$\prod_{\substack{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d) \\ \alpha_1,\dots,\alpha_d\in\{-1,0\} \\ \alpha_1+\cdots+\alpha_d \text{ even}}} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) = \gamma \prod_{\substack{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d) \\ \alpha_1,\dots,\alpha_d\in\{-1,0\} \\ \alpha_1+\cdots+\alpha_d \text{ odd}}} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}). \tag{10.21}$$

Proof. This follows immediately from Proposition X.2. $\square$

Definition X.9. Given $\mathbf{a} = (a_1,\dots,a_d) \in \mathbb{Z}^d_{\geq 1}$ and $1 \leq i \leq d$, let

$$F_i^{\mathbf{a}} = \{v + [\mathbf{a}-\mathbf{1}_i] : v \in \mathbb{Z}^d\} \tag{10.22}$$

denote the set of boxes of size $a_1 \times \cdots \times a_{i-1} \times (a_i - 1) \times a_{i+1} \times \cdots \times a_d$ in $\mathbb{Z}^d$. Set

$$F^{\mathbf{a}} = \bigcup_{i=1}^{d} F_i^{\mathbf{a}}. \tag{10.23}$$

Given an array $\tilde{\mathbf{x}} = (x_s)_{s\in\mathbb{Z}^d \cup F^{\mathbf{a}}}$ with $\mathbf{x} = (x_s)_{s\in\mathbb{Z}^d}$, $i \in \{1,\ldots,d\}$, and $v \in \mathbb{Z}^d$, we remark that $\mathbf{x}_{v+[\mathbf{a}-\mathbf{1}_i]}$ (with $\mathbf{x}$ bold) refers to the array defined in Definition X.1, whereas $x_{v+[\mathbf{a}-\mathbf{1}_i]}$ (with x not bold) refers to the component of $\tilde{\mathbf{x}}$ indexed by $v+[\mathbf{a}-\mathbf{1}_i] \in F^{\mathbf{a}}$.

Definition X.10. For $\mathbf{a} \in \mathbb{Z}^d_{\geq 1}$, define $[\mathbf{a}^*]$ by

$$[\mathbf{a}^*] = ([\mathbf{a}] \setminus \{\mathbf{a}\}) \cup \bigcup_{i=1}^{d} \{[\mathbf{a}-\mathbf{1}_i]\}. \tag{10.24}$$

In other words, the set $[\mathbf{a}^*]$ consists of $[\mathbf{a}]\setminus\{\mathbf{a}\} \subset \mathbb{Z}^d$, along with the sets $v+[\mathbf{a}-\mathbf{1}_i] \in F^{\mathbf{a}}$ for $i = 1,\ldots,d$.

We want to develop a generalization of the K-hexahedron equations for arrays indexed by $\mathbb{Z}^d \cup F^{\mathbf{a}}$. Suppose that $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ is a polynomial satisfying conditions (X.2.1)–(X.2.2), with the polynomials $f_1,\ldots,f_d$ from condition (X.2.2) fixed. Let g and h be the coefficients of $z_{\mathbf{a}}^2$ and $z_{\mathbf{a}}$ in f, viewed as a polynomial in $z_{\mathbf{a}}$. We consider arrays $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ such that

$$x_{v+\mathbf{a}} = \frac{-h(\mathbf{x}_{v+[\mathbf{a}]}) + \prod_{i=1}^{d} x_{v+[\mathbf{a}-\mathbf{1}_i]}}{2g(\mathbf{x}_{v+[\mathbf{a}]})} \text{ for all } v \in \mathbb{Z}^d, \tag{10.25}$$

$$x_{v+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} = r_i(x_{v+s} : s \in [\mathbf{a}^*]) \text{ for } i = 1,\ldots,d \text{ and for all } v \in \mathbb{Z}^d, \tag{10.26}$$

$$x^2_{v+[\mathbf{a}-\mathbf{1}_i]} = f_i(\mathbf{x}_{v+[\mathbf{a}-\mathbf{1}_i]}) \text{ for } i = 1,\ldots,d \text{ and for all } v \in \mathbb{Z}^d, \tag{10.27}$$

where $r_1,\ldots,r_d$ are some rational functions in the variables z_s for $s \in [\mathbf{a}^*]$. Note that if $\tilde{\mathbf{x}}$ satisfies conditions (10.25) and (10.27), then by the quadratic formula, its restriction $\mathbf{x} = (x_s)_{s\in\mathbb{Z}^d}$ satisfies f. In the following definition, we formulate the properties that our tuple of rational functions $(r_1,\ldots,r_d)$ should have in order for the subsequent developments to follow.

Definition X.11. Let $\mathbf{a} = (a_1,\ldots,a_d) \in \mathbb{Z}^d_{\geq 1}$, and let $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ be a polynomial satisfying conditions (X.2.1)–(X.2.2). Fix the polynomials $f_1,\ldots,f_d$ from condition (X.2.2). Let g be the coefficient of $z_{\mathbf{a}}^2$ in f, viewed as a polynomial in $z_{\mathbf{a}}$. For $i = 1, \dots, d$, let $r_i = \frac{p_i}{q_i}$ be rational functions in the variables z_s for $s \in [\mathbf{a}^*]$, with p_i, q_i polynomials in these variables. We say that $(r_1, \dots, r_d)$ is *adapted to* $(f; f_1, \dots, f_d)$ if there exist signs $\beta_1, \dots, \beta_d \in \{-1, 1\}$ such that the following properties hold for $i = 1, \dots, d$:

- the denominator q_i of r_i is of the form

$$q_i = g^{b_i} \prod_{j \in \{1,\dots,d\} \setminus \{i\}} z_{[\mathbf{a}-\mathbf{1}_j]}^{b_{ij}}, \tag{10.28}$$

  where $b_i \in \mathbb{Z}_{\geq 0}$ and $b_{ij} \in \{0, 1\}$;

- for all arrays $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ satisfying (10.25), (10.27), and

$$q_i(x_{v+s} : s \in [\mathbf{a}^*]) \neq 0 \text{ for all } v \in \mathbb{Z}^d; \tag{10.29}$$

$$g(\mathbf{x}_{v+[\mathbf{a}]}) \neq 0 \text{ for all } v \in \mathbb{Z}^d; \tag{10.30}$$

  the following condition holds:

$$r_i(x_{v+s} : s \in [\mathbf{a}^*]) = \beta_i \frac{\left(\frac{\partial f}{\partial z_{\mathbf{a}}}\right)\left(\mathbf{x}_{v+a_i\mathbf{1}_i+[\mathbf{a}\odot(\mathbf{1}-2\mathbf{1}_i)]}\right)}{\prod_{j\in\{1,\dots,d\}-\{i\}} x_{v+[\mathbf{a}-\mathbf{1}_j]}}. \tag{10.31}$$

Note that one can obtain a tuple $(r_1, \dots, r_d)$ adapted to $(f; f_1, \dots, f_d)$ by choosing the signs $\beta_1, \dots, \beta_d \in \{-1, 1\}$ and using condition (10.25) to replace all instances of $x_{v+\mathbf{a}}$ in (10.31).

In the following proposition, we show that with $(r_1, \dots, r_d)$ adapted to $(f; f_1, \dots, f_d)$, the recurrence (10.25)–(10.26) "propagates" the condition (10.27).

Proposition X.12. *Let $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}_{\geq 1}^d$, and let $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ be a polynomial satisfying conditions (X.2.1)–(X.2.2). Fix the polynomials $f_1, \dots, f_d$ from condition (X.2.2). Let g be the coefficient of $z_{\mathbf{a}}^2$ in f, viewed as a polynomial in $z_{\mathbf{a}}$. Let $(r_1, \dots, r_d)$ be a d-tuple of rational functions in the variables z_s for $s \in [\mathbf{a}^*]$*

adapted to $(f; f_1, \dots, f_d)$*. Fix* $v \in \mathbb{Z}^d$*. Let* $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ *be an array satisfying conditions* (10.25)–(10.26), (10.29)–(10.30)*, and*

$$x^2_{v+[\mathbf{a}-\mathbf{1}_i]} = f_i(\mathbf{x}_{v+[\mathbf{a}-\mathbf{1}_i]}) \text{ for } i = 1, \dots, d. \tag{10.32}$$

Then

$$x^2_{v+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} = f_i(\mathbf{x}_{v+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]}) \text{ for } i = 1, \dots, d. \tag{10.33}$$

Proof. By identity (10.17),

$$\begin{aligned} \left(\frac{\partial f}{\partial z_{\mathbf{a}}}\right)^2 (\mathbf{x}_{v+a_i\mathbf{1}_i+[\mathbf{a}\odot(\mathbf{1}-2\mathbf{1}_i)]}) &= (f_1 \cdots f_d)(\mathbf{x}_{v+a_i\mathbf{1}_i+[\mathbf{a}\odot(\mathbf{1}-2\mathbf{1}_i)]}) \\ &= f_i(\mathbf{x}_{v+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]}) \prod_{j\in\{1,\dots,d\}-\{i\}} f_j(\mathbf{x}_{v+[\mathbf{a}-\mathbf{1}_j]}). \end{aligned} \tag{10.34}$$

Hence,

$$\begin{aligned} x^2_{v+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} &= \beta_i^2 \frac{\left(\frac{\partial f}{\partial z_{\mathbf{a}}}\right)^2 (\mathbf{x}_{v+a_i\mathbf{1}_i+[\mathbf{a}\odot(\mathbf{1}-2\mathbf{1}_i)]})}{\prod_{j\in\{1,\dots,d\}-\{i\}} x^2_{v+[\mathbf{a}-\mathbf{1}_j]}} \\ &= \frac{f_i(\mathbf{x}_{v+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]}) \prod_{j\in\{1,\dots,d\}-\{i\}} f_j(\mathbf{x}_{v+[\mathbf{a}-\mathbf{1}_j]})}{\prod_{j\in\{1,\dots,d\}-\{i\}} f_j(\mathbf{x}_{v+[\mathbf{a}-\mathbf{1}_j]})} \\ &= f_i(\mathbf{x}_{v+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]}), \end{aligned} \tag{10.35}$$

as desired. ∎

We now describe d-tuples of rational functions adapted to $(f; f_1, \dots, f_d)$ for the four polynomials f in the Examples X.4–X.7.

Example X.13. Continuing with Example X.4, let us write

$$\begin{aligned} z_{i_1\left(i_2+\frac{1}{2}\right)\left(i_3+\frac{1}{2}\right)} &= z_{(i_1,i_2,i_3)+[\mathbf{a}-\mathbf{1}_1]}; \\ z_{\left(i_1+\frac{1}{2}\right)i_2\left(i_3+\frac{1}{2}\right)} &= z_{(i_1,i_2,i_3)+[\mathbf{a}-\mathbf{1}_2]}; \\ z_{\left(i_1+\frac{1}{2}\right)\left(i_2+\frac{1}{2}\right)i_3} &= z_{(i_1,i_2,i_3)+[\mathbf{a}-\mathbf{1}_3]}. \end{aligned} \tag{10.36}$$

Set

$$r_1(z_s : s \in [\mathbf{a}^*]) = \frac{4z_{\frac{1}{2}0\frac{1}{2}}z_{\frac{1}{2}\frac{1}{2}0} + z_{0\frac{1}{2}\frac{1}{2}}z_{100}}{z_{000}}; \tag{10.37}$$

$$r_2(z_s : s \in [\mathbf{a}^*]) = \frac{z_{0\frac{1}{2}\frac{1}{2}}z_{\frac{1}{2}\frac{1}{2}0} + 4z_{\frac{1}{2}0\frac{1}{2}}z_{010}}{4z_{000}}; \tag{10.38}$$

$$r_3(z_s : s \in [\mathbf{a}^*]) = \frac{z_{0\frac{1}{2}\frac{1}{2}}z_{\frac{1}{2}0\frac{1}{2}} + 4z_{\frac{1}{2}\frac{1}{2}0}z_{001}}{4z_{000}}. \tag{10.39}$$

It can be checked that (r_1, r_2, r_3) is adapted to $(f; f_1, f_2, f_3)$ by following the construction at the end of Definition X.11 with $\beta_1 = \beta_2 = \beta_3 = -1$ and using condition (10.27). Note that r_1, r_2, r_3 matches the right-hand-sides of (2.21)–(2.23) and the K-hexahedron equations (2.21)–(2.24) and (2.12) are the same as conditions (10.25)–(10.27) if each $z_{v+[\mathbf{a}-\mathbf{1}_1]}$ for $v \in \mathbb{Z}^3$ is rescaled by a factor of 4.

Example X.14. Continuing with Example X.5, let us write

$$\begin{aligned} z_{i_1\left(i_2+\frac{1}{2}\right)} &= z_{(i_1,i_2)+[\mathbf{a}-\mathbf{1}_1]}; \\ z_{\left(i_1+\frac{1}{2}\right)i_2} &= z_{(i_1,i_2)+[\mathbf{a}-\mathbf{1}_2]}. \end{aligned} \tag{10.40}$$

Set

$$r_1(z_s : s \in [\mathbf{a}^*]) = z_{0\frac{1}{2}} + 8z_{\frac{1}{2}0}; \tag{10.41}$$

$$r_2(z_s : s \in [\mathbf{a}^*]) = z_{\frac{1}{2}0} + \frac{1}{4}z_{0\frac{1}{2}}. \tag{10.42}$$

It can be checked that (r_1, r_2) is adapted to $(f; f_1, f_2)$ by following the construction at the end of Definition X.11 with $\beta_1 = \beta_2 = -1$ and using condition (10.27). Note that r_1, r_2 matches the right-hand-sides of (5.13)–(5.14) and the conditions (5.12)–(5.16) are the same as conditions (10.25)–(10.27) if each $z_{v+[\mathbf{a}-\mathbf{1}_1]}$ for $v \in \mathbb{Z}^2$ is rescaled by a factor of $4\sqrt{2}$.

Example X.15. Continuing with Example X.6, let us write

$$w_{i+1} = z_{i+[\mathbf{a}-\mathbf{1}_1]}. \tag{10.43}$$

Set

$$r_1(z_s : s \in [\mathbf{a}^*]) = \frac{\alpha_3^2 z_1^6 + \alpha_2\alpha_3 z_0 z_1^4 z_2 + 2\alpha_3 z_0^3 z_2^3 + w_1^2 + (-2\alpha_3 z_1^3 - \alpha_2 z_0 z_1 z_2) w_1}{2z_0^3}. \tag{10.44}$$

It can be checked that (r_1) is adapted to $(f; f_1)$ by following the construction at the end of Definition X.11 with $\beta_1 = 1$ and using condition (10.27). Note that r_1 matches the right-hand side of equation (6.8), and conditions (6.7)–(6.9) are the same as conditions (10.25)–(10.27).

Example X.16. Continuing with Example X.7, let us write

$$\begin{aligned} w_{i_1(i_2+1)} &= z_{(i_1,i_2)+[\mathbf{a}-\mathbf{1}_1]}; \\ w_{\left(i_1+\frac{1}{2}\right)\left(i_2+\frac{1}{2}\right)} &= z_{(i_1,i_2)+[\mathbf{a}-\mathbf{1}_2]}. \end{aligned} \tag{10.45}$$

Set

$$r_1(z_s : s \in [\mathbf{a}^*]) = \frac{z_{10} w_{01} + w_{\frac{1}{2}\frac{1}{2}}}{z_{00}}; \tag{10.46}$$

$$r_2(z_s : s \in [\mathbf{a}^*]) = \frac{z_{01}(\alpha_1 z_{01} z_{10} + \alpha_2 z_{00} z_{11}) w_{01} + (\alpha_1 z_{01}^2 + 2 z_{00} z_{02}) w_{\frac{1}{2}\frac{1}{2}}}{2z_{00}^2}. \tag{10.47}$$

It can be checked that (r_1, r_2) is adapted to $(f; f_1, f_2)$ by following the construction at the end of Definition X.11 with $\beta_1 = \beta_2 = -1$ and using condition (10.27). Note that r_1, r_2 matches the right-hand side of equation (6.24)–(6.25), and conditions (6.23)–(6.27) are the same as conditions (10.25)–(10.27).

Lemma X.17. *Let $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}_{\geq 1}^d$. Let $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ be a polynomial that is irreducible over $\mathbb{C}$ and satisfies conditions (X.2.1)–(X.2.2). Fix the polynomials $f_1, \ldots, f_d$ from condition (X.2.2). Let $(r_1, \ldots, r_d)$ be a tuple of rational functions in the variables z_s for $s \in [\mathbf{a}^*]$ that is adapted to $(f; f_1, \ldots, f_d)$. Then for all $\boldsymbol{\alpha} \in \{-1,1\}^d$, there exists a unique sign $\gamma_{\boldsymbol{\alpha}} \in \{-1,1\}$ such the following condition holds* *for all arrays* $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ *satisfying* (10.25)–(10.27) *and* (10.29)–(10.30)*:*

$$\frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v+[\boldsymbol{\alpha}\odot\mathbf{a}]}) = \gamma_{\boldsymbol{\alpha}} \prod_{i=1}^{d} x_{v+[\boldsymbol{\alpha}\odot(\mathbf{a}-\mathbf{1}_i)]} \text{ for all } v \in \mathbb{Z}^d. \tag{10.48}$$

Definition X.18. For f and $(r_1, \ldots, r_d)$ as in Lemma X.17, we call the signs $(\gamma_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\{-1,1\}^d}$ given in Lemma X.17 the *propagation signs* corresponding to $(f; f_1, \cdots, f_d; r_1, \ldots, r_d)$.

The proof of Lemma X.17 relies on the following lemma.

Lemma X.19. *Let* $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d_{\geq 1}$. *Let* $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ *be a polynomial that is irreducible over* $\mathbb{C}$ *and satisfies conditions (X.2.1)–(X.2.2). Let* $j \neq k \in \{1, \ldots, d\}$. *Then there exists* $\alpha_{jk} \in \{-1, 1\}$ *such that for all* $\boldsymbol{\alpha} \in \{0,1\}^d$,

$$\frac{\partial f}{\partial z_{\mathbf{a}\odot\boldsymbol{\alpha}}} \frac{\partial f}{\partial z_{\mathbf{a}\odot\boldsymbol{\alpha}+(\mathbf{a}-2\mathbf{a}\odot\boldsymbol{\alpha})\odot(\mathbf{1}_j+\mathbf{1}_k)}} - \alpha_{jk} \frac{\partial f}{\partial z_{\mathbf{a}\odot\boldsymbol{\alpha}+(\mathbf{a}-2\mathbf{a}\odot\boldsymbol{\alpha})\odot\mathbf{1}_j}} \frac{\partial f}{\partial z_{\mathbf{a}\odot\boldsymbol{\alpha}+(\mathbf{a}-2\mathbf{a}\odot\boldsymbol{\alpha})\odot\mathbf{1}_k}} \tag{10.49}$$

is a multiple of f.

Proof. Because f satisfies conditions (X.2.1)–(X.2.2),

$$\begin{aligned}
&\left(\frac{\partial f}{\partial z_{\mathbf{a}}}\right)^2 \left(\frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_j-\mathbf{1}_k)}}\right)^2 - \left(\frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_j)}}\right)^2 \left(\frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_k)}}\right)^2 \\
&\qquad \equiv (f_1 \cdots f_d)(\pi_{\mathbf{a},j}\pi_{\mathbf{a},k}(f_1 \cdots f_d)) - (\pi_{\mathbf{a},j}(f_1 \cdots f_d))(\pi_{\mathbf{a},k}(f_1 \cdots f_d)) \\
&\qquad = (1-1) f_j f_k (\pi_{\mathbf{a},j}(f_j))(\pi_{\mathbf{a},k}(f_k)) \prod_{i\in\{1,\ldots,d\}\setminus\{j,k\}} f_j^2 \\
&\qquad = 0
\end{aligned} \tag{10.50}$$

mod f. Hence, by the irreducibility of f, there exists $\alpha_{jk} \in \{-1, 1\}$ such that

$$\frac{\partial f}{\partial z_{\mathbf{a}}} \frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_j-\mathbf{1}_k)}} - \alpha_{jk} \frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_j)}} \frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_k)}} \tag{10.51}$$

is a multiple of f. The full lemma follows from condition (X.2.1). $\square$

Proof of Lemma X.17. We proceed by induction on the number of -1s in $\boldsymbol{\alpha}$. When $\boldsymbol{\alpha} = \mathbf{1}$, then $\gamma_{\boldsymbol{\alpha}} = 1$ by condition (10.25). If $\boldsymbol{\alpha}$ contains one -1, say $\boldsymbol{\alpha} = \mathbf{1} - \mathbf{1}_i$, then $\gamma_{\boldsymbol{\alpha}} = \beta_i$, where β_i is the sign from Definition X.11.

Suppose $\ell \geq 2$, and $\boldsymbol{\alpha}$ has ℓ -1s, including -1s at positions j and k. Then by Lemma X.19 and our inductive hypothesis,

$$\begin{aligned} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v+[\boldsymbol{\alpha}\odot\mathbf{a}]}) &= \frac{\alpha_{jk} \frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_j)}}(\mathbf{x}_{v+[\boldsymbol{\alpha}\odot\mathbf{a}]}) \frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_k)}}(\mathbf{x}_{v+[\boldsymbol{\alpha}\odot\mathbf{a}]})}{\frac{\partial f}{\partial z_{\mathbf{a}\odot(\mathbf{1}-\mathbf{1}_j-\mathbf{1}_k)}}(\mathbf{x}_{v+[\boldsymbol{\alpha}\odot\mathbf{a}]})} \\ &= \alpha_{jk}\gamma_{\boldsymbol{\alpha}+2(\mathbf{1}_j+\mathbf{1}_k)}\gamma_{\boldsymbol{\alpha}+2\mathbf{1}_j}\gamma_{\boldsymbol{\alpha}+2\mathbf{1}_k} \prod_{i=1}^{d} x_{v+[\boldsymbol{\alpha}\odot(\mathbf{a}-\mathbf{1}_i)]}, \end{aligned} \tag{10.52}$$

so setting $\gamma_{\boldsymbol{\alpha}} = \alpha_{jk}\gamma_{\boldsymbol{\alpha}+2(\mathbf{1}_j+\mathbf{1}_k)}\gamma_{\boldsymbol{\alpha}+2\mathbf{1}_j}\gamma_{\boldsymbol{\alpha}+2\mathbf{1}_k} \in \{-1, 1\}$, we obtain the desired result.

$\square$

Example X.20. Let us continue with Examples X.4 and X.13. Following the argument in the proof of Lemma X.17, it can be shown that $\gamma_{\boldsymbol{\alpha}} = 1$ if $\boldsymbol{\alpha} = \pm\mathbf{1}$, and $\gamma_{\boldsymbol{\alpha}} = -1$ otherwise. Note that this fact is equivalent to Lemma VII.2.

Example X.21. Let us continue with Examples X.5 and X.14. Following the argument in the proof of Lemma X.17, it can be shown that $\gamma_{\boldsymbol{\alpha}} = 1$ if $\boldsymbol{\alpha} = \mathbf{1}$, and $\gamma_{\boldsymbol{\alpha}} = -1$ otherwise. In particular, $\gamma_{-\mathbf{1}} = 1$. Hence, if $\tilde{\mathbf{x}} = (x_s) \in (\mathbb{C}^*)^{\mathbb{Z}^2 \cup F_{\mathbf{a}}}$ satisfies conditions (5.12)–(5.16), it follows that $(x_{-s})_{s \in \mathbb{Z}^2 \cup F_{\mathbf{a}}}$ cannot satisfy conditions (5.12)–(5.16).

Example X.22. Let us continue with Examples X.6 and X.15. It is straightforward to show that $\gamma_{(1)} = \gamma_{(-1)} = 1$.

Example X.23. Let us continue with Examples X.7 and X.16. Following the argument in the proof of Lemma X.17, it can be shown that $\gamma_{\boldsymbol{\alpha}} = 1$ if $\boldsymbol{\alpha} = \pm\mathbf{1}$, and $\gamma_{\boldsymbol{\alpha}} = -1$ otherwise.

We now state the main theorem of this chapter.

Theorem X.24. *Let* $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d_{\geq 1}$. *Let* $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ *be a polynomial that is irreducible over* $\mathbb{C}$ *and satisfies conditions (X.2.1)–(X.2.2). Fix the polynomials* $f_1, \dots, f_d$ *from condition (X.2.2). Let* g *and* h *be the coefficients of* $z_{\mathbf{a}}^2$ *and* $z_{\mathbf{a}}$ *in* f, *viewed as a polynomial in* $z_{\mathbf{a}}$. *Let* $(r_1, \dots, r_d)$ *be a tuple of rational functions in the variables* z_s *for* $s \in [\mathbf{a}^*]$ *that is adapted to* $(f; f_1, \dots, f_d)$. *Let* $(\gamma_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \{-1,1\}^d}$ *be the propagation signs corresponding to* $(f; f_1, \dots, f_d; r_1, \dots, r_d)$.

(a) Let $\mathbf{x} = (x_s)_{s \in \mathbb{Z}^d}$ *be an array such that*

$$\mathbf{x} \textit{ satisfies } f; \tag{10.53}$$

$$\frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v+[\mathbf{a}]}) \neq 0 \textit{ for all } v \in \mathbb{Z}^d \textit{ if } d > 1; \tag{10.54}$$

$$g(\mathbf{x}_{v+[\mathbf{a}]}) \neq 0 \textit{ for all } v \in \mathbb{Z}^d; \tag{10.55}$$

$$\begin{aligned} &\prod_{\boldsymbol{\alpha} \in \{-1,0\}^d} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) \\ &= \left(\prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \gamma_{\boldsymbol{\alpha}}\right) \prod_{i=1}^{d} \prod_{\boldsymbol{\beta}=(\beta_1,\dots,\beta_d) \in \{-1,0\}^d : \beta_i = 0} f_i(\mathbf{x}_{v+\boldsymbol{\beta}+[\mathbf{a}]}) \textit{ for all } v \in \mathbb{Z}^d. \end{aligned} \tag{10.56}$$

Then $\mathbf{x}$ *can be extended to an array* $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ *satisfying* (10.25)–(10.27).

(b) Conversely, if $\tilde{\mathbf{x}} = (x_s) \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ *satisfies conditions* (10.25)–(10.27) *and* (10.29)–(10.30), *then the restriction of* $\tilde{\mathbf{x}}$ *to* $\mathbb{Z}^d$ *satisfies* f *and the condition* (10.56).

The following proposition states that the sign $\prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \gamma_{\boldsymbol{\alpha}}$ in (10.56) is independent of the choice of $(r_1, \dots, r_d)$ if $d \geq 2$.

Proposition X.25. *Let* $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d_{\geq 1}$ *with* $d \geq 2$. *Let* $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ *be a polynomial that is irreducible over* $\mathbb{C}$ *and satisfies conditions (X.2.1)–(X.2.2). Then there exists a sign* $\gamma \in \{-1, 1\}$ *such that for any tuple of rational functions* $(r_1, \dots, r_d)$

in the variables z_s for $s \in [\mathbf{a}^*]$ that is adapted to $(f; f_1, \ldots, f_d)$, we have

$$\gamma = \prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \gamma_{\boldsymbol{\alpha}}, \tag{10.57}$$

where $(\gamma_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \{-1,1\}^d}$ are the propagation signs corresponding to $(f;f_1,\ldots,f_d;r_1,\ldots,r_d)$.

Remark X.26. By Lemma X.8, given $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ satisfying conditions (X.2.1)–(X.2.2) and an array $\mathbf{x} = (x_s)_{s \in \mathbb{Z}^d}$ satisfying f, the following are equivalent:

- $\mathbf{x}$ satisfies condition (10.56);
- $\mathbf{x}$ satisfies

$$\begin{aligned} &\prod_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d) \\ \alpha_1,\ldots,\alpha_d \in \{-1,0\} \\ \alpha_1+\cdots+\alpha_d \text{ even}}} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) \\ &\qquad = \gamma \prod_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d) \\ \alpha_1,\ldots,\alpha_d \in \{-1,0\} \\ \alpha_1+\cdots+\alpha_d \text{ odd}}} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) \text{ for all } v \in \mathbb{Z}^d. \end{aligned} \tag{10.58}$$

Hence, one can replace condition (10.56) in Theorem X.24 by condition (10.58).

Example X.27. Continuing with Examples X.4, X.13, and X.20, Theorem II.22 is a special case of Theorem X.24. Theorem II.9 is a special case of Theorem X.24(b), where we require all values of $\tilde{\mathbf{x}}$, including the values indexed by $F^{\mathbf{a}}$, to be positive.

Example X.28. Continuing with Examples X.5, X.14, and X.21, Theorem V.9 is a special case of Theorem X.24. Theorem V.7 is a special case of Theorem X.24(b), where we require $x_s > 0$ for $s \in \mathbb{Z}^2_{\{0,1,2,\ldots\}}$, $s \in \mathbb{Z}^2_{\{0,1,2,\ldots\}} + [\mathbf{a} - \mathbf{1}_1]$, and $s \in \mathbb{Z}^2_{\{0,1,2,\ldots\}} + [\mathbf{a} - \mathbf{1}_2]$.

Example X.29. Continuing with Examples X.6, X.15, and X.22, Theorem VI.4 is a special case of Theorem X.24. Theorem VI.3 is a special case of Theorem X.24(b), where we require all values of $\tilde{\mathbf{x}}$, including the values indexed by $F^{\mathbf{a}}$, to be positive.

Example X.30. Continuing with Examples X.7, X.16, and X.23, Theorem VI.10 is a special case of Theorem X.24. Theorem VI.9 is a special case of Theorem X.24(b), where we require all values of $\tilde{\mathbf{x}}$, including the values indexed by $F^{\mathbf{a}}$, to be positive.

Before we prove Theorem X.24, we first prove Proposition X.25. Proposition X.25 follows from the lemma below.

Lemma X.31. *Let $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d_{\geq 1}$. Let $f \in \mathbb{C}[\mathbf{z}_{[\mathbf{a}]}]$ be a polynomial that is irreducible over $\mathbb{C}$ and satisfies conditions (X.2.1)–(X.2.2). Fix $j \in \{1, \dots, d\}$. Let $(r_1, \dots, r_d)$ and $(\tilde{r}_1, \dots, \tilde{r}_d)$ be tuples of rational functions in the variables z_s for $s \in [\mathbf{a}^*]$ that are adapted to $(f; f_1, \dots, f_d)$, such that $\tilde{r}_j = -r_j$ and $\tilde{r}_i = r_i$ for $i \neq j$. Let $(\gamma_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \{-1,1\}^d}$ and $(\tilde{\gamma}_{\boldsymbol{\alpha}} \in \{-1, 1\})_{\boldsymbol{\alpha} \in \{-1,1\}^d}$ be the propagation signs corresponding to $(f; f_1, \dots, f_d; r_1, \dots, r_d)$ and $(f; f_1, \dots, f_d; \tilde{r}_1, \dots, \tilde{r}_d)$, respectively. Given $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \{-1, 1\}^d$, the following are equivalent:*

- $\gamma_{\boldsymbol{\alpha}} = \tilde{\gamma}_{\boldsymbol{\alpha}}$;
- $\alpha_j = 1$.

Proof. We proceed by induction on the number of -1s in $\boldsymbol{\alpha}$. Note that $\tilde{\gamma}_{\mathbf{1}} = \gamma_{\mathbf{1}} = 1$, $\tilde{\gamma}_{\mathbf{1}-2\mathbf{1}_j} = -\gamma_{\mathbf{1}-2\mathbf{1}_j}$, and $\tilde{\gamma}_{\mathbf{1}-2\mathbf{1}_i} = \gamma_{\mathbf{1}-2\mathbf{1}_i}$ for $i \neq j$. Suppose $\boldsymbol{\alpha}$ has at least two -1s. As we showed in the proof to Lemma X.17, if k_1 and k_2 are distinct values such that $\alpha_{k_1} = \alpha_{k_2} = 1$, then $\gamma_{\boldsymbol{\alpha}} = \alpha_{k_1 k_2} \gamma_{\boldsymbol{\alpha}+2(\mathbf{1}_{k_1}+\mathbf{1}_{k_2})} \gamma_{\boldsymbol{\alpha}+2\mathbf{1}_{k_1}} \gamma_{\boldsymbol{\alpha}+2\mathbf{1}_{k_2}}$ and $\tilde{\gamma}_{\boldsymbol{\alpha}} = \alpha_{k_1 k_2} \tilde{\gamma}_{\boldsymbol{\alpha}+2(\mathbf{1}_{k_1}+\mathbf{1}_{k_2})} \tilde{\gamma}_{\boldsymbol{\alpha}+2\mathbf{1}_{k_1}} \tilde{\gamma}_{\boldsymbol{\alpha}+2\mathbf{1}_{k_2}}$. If $\alpha_j = 1$, let k_1, k_2 be distinct values such that $\alpha_{k_1} = \alpha_{k_2} = -1$, so

(10.59)

$$\tilde{\gamma}_{\boldsymbol{\alpha}} = \alpha_{k_1 k_2} \tilde{\gamma}_{\boldsymbol{\alpha}+2(\mathbf{1}_{k_1}+\mathbf{1}_{k_2})} \tilde{\gamma}_{\boldsymbol{\alpha}+2\mathbf{1}_{k_1}} \tilde{\gamma}_{\boldsymbol{\alpha}+2\mathbf{1}_{k_2}} = \alpha_{k_1 k_2} \gamma_{\boldsymbol{\alpha}+2(\mathbf{1}_{k_1}+\mathbf{1}_{k_2})} \gamma_{\boldsymbol{\alpha}+2\mathbf{1}_{k_1}} \gamma_{\boldsymbol{\alpha}+2\mathbf{1}_{k_2}} = \gamma_{\boldsymbol{\alpha}}.$$

If $\alpha_j = -1$, let $k \neq j$ be a value such that $\alpha_k = -1$, so

$$\tilde{\gamma}_{\boldsymbol{\alpha}} = \alpha_{jk} \tilde{\gamma}_{\boldsymbol{\alpha}+2(\mathbf{1}_j+\mathbf{1}_k)} \tilde{\gamma}_{\boldsymbol{\alpha}+2\mathbf{1}_j} \tilde{\gamma}_{\boldsymbol{\alpha}+2\mathbf{1}_k} = -\alpha_{jk} \gamma_{\boldsymbol{\alpha}+2(\mathbf{1}_j+\mathbf{1}_k)} \gamma_{\boldsymbol{\alpha}+2\mathbf{1}_j} \gamma_{\boldsymbol{\alpha}+2\mathbf{1}_k} = -\gamma_{\boldsymbol{\alpha}}. \tag{10.60}$$

$\square$

Proof of Proposition X.25. It suffices to prove the proposition for tuples of rational functions $(r_1, \ldots, r_d)$ and $(\tilde{r}_1, \ldots, \tilde{r}_d)$ satisfying the conditions of Lemma X.31. Let $(\gamma_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \{-1,1\}^d}$ and $(\tilde{\gamma}_{\boldsymbol{\alpha}} \in \{-1,1\})_{\boldsymbol{\alpha} \in \{-1,1\}^d}$ be the propagation signs corresponding to $(f; f_1, \ldots, f_d; r_1, \ldots, r_d)$ and $(f; f_1, \ldots, f_d; \tilde{r}_1, \ldots, \tilde{r}_d)$, respectively. Then by Lemma X.31,

$$\prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \tilde{\gamma}_{\boldsymbol{\alpha}} = (-1)^{2^{d-1}} \prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \gamma_{\boldsymbol{\alpha}} = \prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \gamma_{\boldsymbol{\alpha}}, \tag{10.61}$$

because $d \geq 2$. $\square$

The remainder of this chapter is dedicated to the proof of Theorem X.24. For the rest of this chapter, we fix all quantities given in Theorem X.24, and set

$$a = a_1 + \cdots + a_d. \tag{10.62}$$

Proof of Theorem X.24(b). Let $\mathbf{x}$ be the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^d$. It is clear that $\mathbf{x}$ must satisfy f. By Lemma X.17,

$$\begin{aligned} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) &= \gamma_{\mathbf{1}+2\boldsymbol{\alpha}} \prod_{i=1}^{d} x_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot(\mathbf{a}-\mathbf{1}_i)]} \\ &= \gamma_{\mathbf{1}+2\boldsymbol{\alpha}} \prod_{i=1}^{d} x_{v+\boldsymbol{\alpha}\odot(\mathbf{1}-\mathbf{1}_i)+[\mathbf{a}-\mathbf{1}_i]}. \end{aligned} \tag{10.63}$$

Taking the product over $\boldsymbol{\alpha} \in \{-1,0\}^d$, we get

$$\begin{aligned} &\prod_{\boldsymbol{\alpha} \in \{-1,0\}^d} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{v-(\mathbf{a}-\mathbf{1})\odot\boldsymbol{\alpha}+[(\mathbf{1}+2\boldsymbol{\alpha})\odot\mathbf{a}]}) \\ &= \left(\prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \gamma_{\boldsymbol{\alpha}}\right) \prod_{i=1}^{d} \prod_{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_d) \in \{-1,0\}^d : \beta_i = 0} x^2_{v+\boldsymbol{\beta}+[\mathbf{a}-\mathbf{1}_i]} \\ &= \left(\prod_{\boldsymbol{\alpha} \in \{-1,1\}^d} \gamma_{\boldsymbol{\alpha}}\right) \prod_{i=1}^{d} \prod_{\boldsymbol{\beta}=(\beta_1,\ldots,\beta_d) \in \{-1,0\}^d : \beta_i = 0} f_i(\mathbf{x}_{v+\boldsymbol{\beta}+[\mathbf{a}]}). \end{aligned} \tag{10.64}$$

$\square$

The proof of Theorem X.24(a) below is nearly identical to the proof of Theorem II.22(a).

Definition X.32. For $U \subseteq \mathbb{Z}$, let $\mathbb{Z}^d_U$ denote the set

$$\mathbb{Z}^d_U = \{(i_1, \ldots, i_d) \in \mathbb{Z}^d : i_1 + \cdots i_d \in U\}. \tag{10.65}$$

For $U \subseteq \mathbb{Z}$, we will also use the notation

$$F_{\mathbf{a},U} = \{v + [\mathbf{a} - \mathbf{1}_i] : v \in \mathbb{Z}_U, i \in \{1, \ldots, d\}\}. \tag{10.66}$$

In particular, we will be interested in $\mathbb{Z}^d_{a,\mathrm{init}} = \mathbb{Z}^d_{\{0,\ldots,a-1\}}$ and $F^{\mathbf{a}}_{\mathrm{init}} = F_{\mathbf{a},\{0\}}$.

Definition X.33. We say that an array $\tilde{\mathbf{x}}_{\mathrm{init}}$ indexed by $\mathbb{Z}^d_{a,\mathrm{init}} \cup F^{\mathbf{a}}_{\mathrm{init}}$ satisfying condition (10.27) is *generic* if there exists an extension of $\tilde{\mathbf{x}}_{\mathrm{init}}$ to an array $\tilde{\mathbf{x}}$ indexed by $\mathbb{Z}^d \cup F^{\mathbf{a}}$ satisfying equations (10.25)–(10.27) where the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^d$ satisfies conditions (10.54)–(10.55). Similarly, we say that an array $\mathbf{x}_{\mathrm{init}}$ indexed by $\mathbb{Z}^d_{a,\mathrm{init}}$ is generic if every extension of $\mathbf{x}_{\mathrm{init}}$ to an array $\tilde{\mathbf{x}}_{\mathrm{init}}$ indexed by $\mathbb{Z}^d_{a,\mathrm{init}} \cup F^{\mathbf{a}}_{\mathrm{init}}$ satisfying condition (10.27) is generic.

Definition X.34. Let $\tilde{\mathbf{x}}_{\mathrm{init}}$ be a generic array indexed by $\mathbb{Z}^d_{a,\mathrm{init}} \cup F^{\mathbf{a}}_{\mathrm{init}}$ satisfying condition (10.27). We denote by $(\tilde{\mathbf{x}}_{\mathrm{init}})^{\uparrow \mathbb{Z}^d \cup F^{\mathbf{a}}}$ the unique extension of $\tilde{\mathbf{x}}_{\mathrm{init}}$ to $\mathbb{Z}^d \cup F^{\mathbf{a}}$ where $(\tilde{\mathbf{x}}_{\mathrm{init}})^{\uparrow \mathbb{Z}^d \cup F^{\mathbf{a}}}$ satisfies equations (10.25)–(10.27).

The next lemma generalizes Lemma VII.11.

Lemma X.35. *Let $S = [\mathbf{a}] \cup \{b\mathbf{1}_i + [\mathbf{a} - \mathbf{1}_i] : b \in \{0,1\}, i \in \{1, \ldots, d\}\}$, i.e., S is the set of vertices of $[\mathbf{a}] \subset \mathbb{Z}^d$ and boxes of $F^{\mathbf{a}}$ completely contained in $[\mathbf{a}]$. Fix values $t_i \in \{-1, 1\}$ for $i = 1, \ldots, d$. Suppose $\tilde{\mathbf{x}} = (x_s)_{s \in S}$ and $\tilde{\mathbf{y}} = (y_s)_{s \in S}$ are arrays of complex numbers such that*

- *$\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ both satisfy equations* (10.25)–(10.27)*, with the denominators in equations* (10.25)–(10.26) *non-vanishing,*

- *$y_s = x_s$ for $s \in [\mathbf{a}] - \{\mathbf{a}\}$,*
- *$y_{[\mathbf{a}-\mathbf{1}_i]} = t_i x_{[\mathbf{a}-\mathbf{1}_i]}$ for $i = 1, \ldots, d$, and*
- *$\prod_{i=1}^d t_i = 1$.*

Then the following equations hold:

$$y_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} = t_i x_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} \text{ for } i = 1, \ldots, d; \tag{10.67}$$

$$y_{\mathbf{a}} = x_{\mathbf{a}}. \tag{10.68}$$

Proof. Note that

$$y_{\mathbf{a}} - x_{\mathbf{a}} = \left(\prod_{i=1}^d t_i - 1\right) \frac{\prod_{i=1}^d x_{[\mathbf{a}-\mathbf{1}_i]}}{2g(\mathbf{x}_{[\mathbf{a}-\mathbf{1}_i]})} = 0, \tag{10.69}$$

so $y_{\mathbf{a}} = x_{\mathbf{a}}$. By (10.48),

$$\begin{aligned} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{a_i\mathbf{1}_i+[(\mathbf{1}-2\mathbf{1}_i)\odot\mathbf{a}]}) &= \gamma_{(\mathbf{1}-2\mathbf{1}_i)} \prod_{j=1}^d x_{a_i\mathbf{1}_i+[((\mathbf{1}-2\mathbf{1}_i))\odot(\mathbf{a}-\mathbf{1}_j)]} \\ &= \gamma_{(\mathbf{1}-2\mathbf{1}_i)} x_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} \prod_{j\in\{1,\ldots,d\}\backslash\{i\}} x_{[\mathbf{a}-\mathbf{1}_j]} \end{aligned} \tag{10.70}$$

and

$$\begin{aligned} \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{y}_{a_i\mathbf{1}_i+[(\mathbf{1}-2\mathbf{1}_i)\odot\mathbf{a}]}) &= \gamma_{(\mathbf{1}-2\mathbf{1}_i)} \prod_{j=1}^d y_{a_i\mathbf{1}_i+[((\mathbf{1}-2\mathbf{1}_i))\odot(\mathbf{a}-\mathbf{1}_j)]} \\ &= \gamma_{(\mathbf{1}-2\mathbf{1}_i)} y_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} \prod_{j\in\{1,\ldots,d\}\backslash\{i\}} y_{[\mathbf{a}-\mathbf{1}_j]} \\ &= \gamma_{(\mathbf{1}-2\mathbf{1}_i)} y_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} \prod_{j\in\{1,\ldots,d\}\backslash\{i\}} t_j x_{[\mathbf{a}-\mathbf{1}_j]} \\ &= \gamma_{(\mathbf{1}-2\mathbf{1}_i)} t_i y_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} \prod_{j\in\{1,\ldots,d\}\backslash\{i\}} x_{[\mathbf{a}-\mathbf{1}_j]}. \end{aligned} \tag{10.71}$$

Because

$$\frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{x}_{a_i\mathbf{1}_i+[(\mathbf{1}-2\mathbf{1}_i)\odot\mathbf{a}]}) = \frac{\partial f}{\partial z_{\mathbf{a}}}(\mathbf{y}_{a_i\mathbf{1}_i+[(\mathbf{1}-2\mathbf{1}_i)\odot\mathbf{a}]}), \tag{10.72}$$

it follows that $x_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} = t_i y_{\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]}$. □

Definition X.36. Define an equivalence relation on $F^{\mathbf{a}}$ by setting $s_1 \sim s_2$ if and only if $s_1 = v + [\mathbf{a} - \mathbf{1}_i]$ and $s_2 = v + \beta \mathbf{1}_i + [\mathbf{a} - \mathbf{1}_i]$ for some $1 \leq i \leq d$, $v \in \mathbb{Z}^d$, and $\beta \in \mathbb{Z}$. Let $F^{\mathbf{a}}_{\boxdot}$ denote the set of equivalence classes under this equivalence relation. Denote by $[s] \in F^{\mathbf{a}}_{\boxdot}$ the equivalence class of $s \in F^{\mathbf{a}}$.

Definition X.37. Define an action of $\{-1,1\}^{F^{\mathbf{a}}_{\boxdot}}$ on arrays indexed by $\mathbb{Z}^d_{a,\text{init}} \cup F^{\mathbf{a}}_{\text{init}}$ as follows: given $\mathbf{t} = (t_s)_{s \in F^{\mathbf{a}}_{\boxdot}} \in \{-1,1\}^{F^{\mathbf{a}}_{\boxdot}}$ and $\tilde{\mathbf{x}}_{\text{init}} = (x_s)_{s \in \mathbb{Z}^d_{a,\text{init}} \cup F^{\mathbf{a}}_{\text{init}}}$, define $\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}} = (\tilde{x}_s)_{\mathbb{Z}^d_{a,\text{init}} \cup F^{\mathbf{a}}_{\text{init}}}$, where

$$\tilde{x}_s = \begin{cases} x_s & \text{if } s \in \mathbb{Z}^d_{a,\text{init}} \\ t_{[s]} x_s & \text{if } s \in F^{\mathbf{a}}_{\text{init}}. \end{cases} \tag{10.73}$$

Definition X.38. For $\mathbf{t} = (t_s) \in \{-1,1\}^{F^{\mathbf{a}}_{\boxdot}}$, define $\psi(\mathbf{t}) = (u_s) \in \{-1,1\}^{\mathbb{Z}^d + \mathbf{a}/2}$ by

$$u_{s+\mathbf{a}/2} = \prod_{i=1}^{d} t_{[s+[\mathbf{a}-\mathbf{1}_i]]} \tag{10.74}$$

for $s \in \mathbb{Z}^d$.

The next lemma generalizes Lemma VII.18.

Lemma X.39. *Let $\tilde{\mathbf{x}}_{\text{init}}$ be a generic array indexed by $\mathbb{Z}^d_{a,\text{init}} \cup F^{\mathbf{a}}_{\text{init}}$ satisfying condition (10.27). Let $\mathbf{t} \in \{-1,1\}^{F^{\mathbf{a}}_{\boxdot}}$, and $\mathbf{u} = (u_s)_{s \in \mathbb{Z}^d + \mathbf{a}/2} = \psi(\mathbf{t})$. Let $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow \mathbb{Z}^d \cup F^{\mathbf{a}}} = (x_s)_{s \in \mathbb{Z}^d \cup F^{\mathbf{a}}}$, and $(\mathbf{t} \cdot \tilde{\mathbf{x}}_{\text{init}})^{\uparrow \mathbb{Z}^d \cup F^{\mathbf{a}}} = (y_s)_{s \in \mathbb{Z}^d \cup F^{\mathbf{a}}}$. Suppose $v \in \mathbb{Z}^d_{\{a,a+1,\dots\}}$ satisfies the condition that $u_{w-\mathbf{a}/2} = 1$ for all $w \in \mathbb{Z}^d_{\{a,a+1,\dots\}}$ with $w \leq v$. Then:*

(a) $y_v = x_v$;

(b) $y_{v-\mathbf{a}+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]} = t_{[v-\mathbf{a}+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]]} x_{v-\mathbf{a}+\mathbf{1}_i+[\mathbf{a}-\mathbf{1}_i]}$ for $i = 1, \dots, d$.

Proof. We prove parts (a) and (b) together by induction. Assume that we have proved parts (a) and (b) for all $w \in \mathbb{Z}^d_{\{a,a+1,\dots\}}$ with $w < v$. By construction, $x_w = y_w$ for all $w \in \mathbb{Z}^d_{a,\text{init}}$ and statement (b) holds for all $w \in \mathbb{Z}^d_{\{a-1\}}$. Hence, $y_{v-s'} =$

$x_{v-s'}$ for $s \in [\mathbf{a}] - \{\mathbf{0}\}$, and $y_{v-\mathbf{a}+[\mathbf{a}-\mathbf{1}_i]} = t_{[v-\mathbf{a}+[\mathbf{a}-\mathbf{1}_i]]} x_{v-\mathbf{a}+[\mathbf{a}-\mathbf{1}_i]}$ for $i = 1, \ldots, d$. Because $u_{v-\mathbf{a}/2} = 1$, statements (a) and (b) follow from Lemma X.35. $\square$

The next lemma generalizes Lemma VII.19.

Lemma X.40. *An array $\mathbf{u} = (u_s) \in \{-1,1\}^{\mathbb{Z}^d + \mathbf{a}/2}$ is in the image of ψ (see Definition X.38) if and only if for every $v \in \mathbb{Z}^3$,*

$$\prod_{\boldsymbol{\alpha} \in \{0,1\}^d} u_{v+\mathbf{a}/2+\boldsymbol{\alpha}} = 1. \tag{10.75}$$

Proof. First, suppose $\mathbf{u} = \psi(\mathbf{t})$, where $\mathbf{t} = (t_s) \in \{-1,1\}^{F^{\mathbf{a}}_{\boxdot}}$. Then for any $v \in \mathbb{Z}^d$,

$$\prod_{\boldsymbol{\alpha} \in \{0,1\}^d} u_{v+\mathbf{a}/2+\boldsymbol{\alpha}} = \prod_{\boldsymbol{\alpha} \in \{0,1\}^d} \prod_{i=1}^{d} t_{[v+[\mathbf{a}-\mathbf{1}_i]]} = \prod_{i=1}^{d} \prod_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d) \in \{0,1\}^d \\ \alpha_i = 0}} t^2_{[v+\boldsymbol{\alpha}+[\mathbf{a}-\mathbf{1}_i]]} = 1. \tag{10.76}$$

Next, suppose that condition (10.75) holds. It is clear that $\mathbf{u}$ is uniquely determined by its components at $S = \{(v_1, \ldots, v_d) + \mathbf{a}/2 : v_1 \cdots v_d = 0\}$ and condition (10.75). For $v = (v_1, \ldots, v_d) \in \mathbb{Z}^d_{\{0\}}$ and $i \in \{1, \ldots, d\}$, set

$$t_{[v+[\mathbf{a}-\mathbf{1}_i]]} = \begin{cases} \prod_{j=1}^{i} u_{(v_1,\ldots,v_{j-1},0,v_{j+1},\ldots,v_d)+\mathbf{a}/2} & \text{if } v_1, \ldots, v_{i-1} \neq 0; \\ 1 & \text{otherwise.} \end{cases} \tag{10.77}$$

Set $\mathbf{t} = (t_s) \in \{-1,1\}^{F^{\mathbf{a}}_{\boxdot}}$. It is straightforward to check that $\psi(\mathbf{t})$ agrees with $\mathbf{u}$ at S. Hence, because $\psi(\mathbf{t})$ and $\mathbf{u}$ both satisfy condition (10.75), it follows that $\mathbf{u} = \psi(\mathbf{t})$. $\square$

The next lemma generalizes Lemma VII.21.

Lemma X.41. *Let $\hat{\mathbf{x}}$ be an array indexed by $\mathbb{Z}^d_{\{0,1,\ldots,a+d-1\}}$. Assume that $\hat{\mathbf{x}}$ satisfying f, and, moreover, its restriction to $\mathbb{Z}^d_{a,\mathrm{init}}$ is generic. Then there exists an array $\tilde{\mathbf{x}}$ indexed by $\mathbb{Z}^d \cup F^{\mathbf{a}}$ satisfying equations* (10.25)–(10.27) *and extending $\hat{\mathbf{x}}$.*

Proof. For $i = a, \ldots, a+d-1$, we will show by induction on i that there exists an array $\tilde{\mathbf{x}}_{\text{init}}$ indexed by $\mathbb{Z}^d_{a,\text{init}} \cup F^{\mathbf{a}}_{\text{init}}$ satisfying (10.27) such that $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow \mathbb{Z}^d \cup F^{\mathbf{a}}}$ agrees with $\hat{\mathbf{x}} = (x_s)_{s \in \mathbb{Z}^d_{\{0,\ldots,a+d-1\}}}$ on $\mathbb{Z}^d_{\{0,\ldots,i\}}$. Let $\tilde{\mathbf{x}}'_{\text{init}}$ be an array indexed by $\mathbb{Z}^d_{a,\text{init}} \cup F^{\mathbf{a}}_{\text{init}}$ satisfying (10.27) such that $(\tilde{\mathbf{x}}'_{\text{init}})^{\uparrow \mathbb{Z}^d \cup F^{\mathbf{a}}} = (y_s)_{s \in \mathbb{Z}^d \cup F^{\mathbf{a}}}$ agrees with $\hat{\mathbf{x}}$ on $\mathbb{Z}^d_{\{0,\ldots,i-1\}}$. (For $i = a$, we can obtain $\tilde{\mathbf{x}}'_{\text{init}}$ be taking an arbitrary extension of $\mathbf{x}_{\text{init}}$ to $\mathbb{Z}^d_{a,\text{init}} \cup F^{\mathbf{a}}_{\text{init}}$ satisfying condition (10.27). For $i > a$, we have shown that $\tilde{\mathbf{x}}'_{\text{init}}$ exists by induction.) Choose $\tilde{\mathbf{u}} = (u_s) \in \{-1,1\}^{\mathbb{Z}^d_{\{a,\ldots,a+d-1\}} - \mathbf{a}/2}$ so that

- $u_{s-\mathbf{a}/2} = 1$ if $s \in \mathbb{Z}^d_{\{i\}}$ and $x_s = y_s$;
- $u_{s-\mathbf{a}/2} = -1$ if $s \in \mathbb{Z}^d_{\{i\}}$ and $x_s = y_s$;
- $u_{s-\mathbf{a}/2} = 1$ if $s \in \mathbb{Z}^d_{\{j\}}$ for $0 \leq j < i$.

Extend $\tilde{\mathbf{u}}$ to $\mathbf{u} = (u_s)_{s \in \mathbb{Z}^d + \mathbf{a}/2}$ by condition (10.27). By Lemma X.40, there exists $\mathbf{t} \in \{-1,1\}^{F^{\mathbf{a}}_{\boxdot}}$ such that $\mathbf{u} = \psi(\mathbf{t})$. Set $\tilde{\mathbf{x}}_{\text{init}} = \mathbf{t} \cdot \tilde{\mathbf{x}}'_{\text{init}}$. Then by Lemma X.39, $(\tilde{\mathbf{x}}_{\text{init}})^{\uparrow \mathbb{Z}^d \cup F^{\mathbf{a}}}$ agrees with $\hat{\mathbf{x}}$ on $\mathbb{Z}^d_{\{0,\ldots,i\}}$, as desired. □

We can now prove a weaker version of Theorem X.24(a), under the additional constraint of genericity.

Corollary X.42. *Let $\mathbf{x} = (x_s)_{s \in \mathbb{Z}^d}$ be an array that satisfies f and condition* (10.56), *and whose restriction to $\mathbb{Z}^d_{a,\text{init}}$ is generic. Then $\mathbf{x}$ can be extended to an array $\tilde{\mathbf{x}}$ indexed by $\mathbb{Z}^d \cup F^{\mathbf{a}}$ satisfying equations* (10.25)–(10.27).

Proof. Let $\mathbf{x} = (x_s)_{s \in \mathbb{Z}^d}$ be an array that satisfies f and condition (10.56), and whose restriction to $\mathbb{Z}^d_{a,\text{init}}$ is generic. By Lemma X.41, there exists an array $\tilde{\mathbf{x}}$ indexed by $\mathbb{Z}^d \cup F^{\mathbf{a}}$ satisfying equations (10.25)–(10.27) that agrees with $\mathbf{x}$ on $\mathbb{Z}^d_{\{0,\ldots,a+d-1\}}$. Let $\mathbf{x}'$ be the restriction of $\tilde{\mathbf{x}}$ to $\mathbb{Z}^d$. By Theorem X.24(b), $\mathbf{x}'$ is a coherent solution of the Kashaev equation. There is a unique solution of f satisfying condition (10.56) agreeing with $\mathbf{x}$ at $\mathbb{Z}^d_{\{0,\ldots,a+d-1\}}$, as condition (10.56) gives the remaining values as rational expressions in the values at $\mathbb{Z}^d_{\{0,\ldots,a+d-1\}}$, where the denominators do not vanish because conditions (10.54)–(10.55) hold for $\mathbf{x}$ (as $\mathbf{x}$ is generic). Hence, $\mathbf{x}' = \mathbf{x}$, as desired. □

Proof of Theorem X.24(a). We need to loosen the genericity condition in Corollary X.42 to the condition that $\mathbf{x}$ satisfies (10.54)–(10.55).

Let $\mathbf{x}$ satisfy f along with conditions (10.54)–(10.55) and condition (10.56). Let $A_j = [-j, j]^d \cap \mathbb{Z}^d$, and let $B_j = \left\{s \in F^{\mathbf{a}} : s \subseteq [-j, j]^d\right\}$. We claim that if there exist $\tilde{\mathbf{x}}_j \in \mathbb{C}^{A_j \cup B_j}$ satisfying equations (10.25)–(10.27) that agree with $\mathbf{x}$ on A_j for all j, then there exists $\tilde{\mathbf{x}} \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ satisfying equations (10.25)–(10.27) that agrees with $\mathbf{x}$ on $\mathbb{Z}^d$. Construct an infinite tree T as follows:

- The vertices of T are arrays indexed by $A_j \cup B_j$ satisfying equations (10.25)–(10.27) that agree with $\mathbf{x}$ on A_j (over $j \in \mathbb{Z}_{\geq 0}$).
- Add an edge between $\tilde{\mathbf{x}}_j \in \mathbb{C}^{A_j \cup B_j}$ and $\tilde{\mathbf{x}}_{j+1} \in \mathbb{C}^{A_{j+1} \cup B_{j+1}}$ if $\tilde{\mathbf{x}}_{j+1}$ restricts to $\tilde{\mathbf{x}}_j$.

Thus, T is an infinite tree in which every vertex has finite degree. By König's infinity lemma, there exists an infinite path $\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \ldots$ in T with $\tilde{\mathbf{x}}_j \in \mathbb{C}^{A_j \cup B_j}$. Thus, there exists $\tilde{\mathbf{x}} \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ restricting to $\tilde{\mathbf{x}}_j$ for all $j \in \mathbb{Z}_{\geq 0}$, so $\tilde{\mathbf{x}}$ satisfies equations (10.25)–(10.27) and agrees with $\mathbf{x}$ on $\mathbb{Z}^d$.

Given $j \in \mathbb{Z}_{\geq 0}$, we claim that there exists $\tilde{\mathbf{x}} \in \mathbb{C}^{A_j \cup B_j}$ satisfying equations (10.25)–(10.27) that agrees with $\mathbf{x}$ on A_j. Because $\mathbf{x}$ satisfies conditions (10.54)–(10.55), there exists a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ of arrays satisfying f along with conditions (10.54)–(10.55) and condition (10.56), whose restrictions to $\mathbb{Z}^d_{a,\mathrm{init}}$ are generic. By Corollary X.42, there exist $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \cdots \in \mathbb{C}^{\mathbb{Z}^d \cup F^{\mathbf{a}}}$ satisfying equations (10.25)–(10.27) such that $\tilde{\mathbf{x}}_i$ restricts to $\mathbf{x}_i$. However, the sequence $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots$ does not necessarily converge. Let $\tilde{\mathbf{x}}'_1, \tilde{\mathbf{x}}'_2, \cdots \in \mathbb{C}^{A_j \cup B_j}$ be the restrictions of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots$ to $A_j \cup B_j$. There exists a subsequence of $\tilde{\mathbf{x}}'_1, \tilde{\mathbf{x}}'_2, \ldots$ that converges to some $\tilde{\mathbf{x}} \in \mathbb{C}^{A_j \cup B_j}$. (For each $s \in B_j$, we can partition the sequence $\tilde{\mathbf{x}}'_1, \tilde{\mathbf{x}}'_2, \ldots$ into two sequences, each of which converges at s. Because B_j is finite, the claim follows.) The array $\tilde{\mathbf{x}}$ must satisfy equations (10.25)–(10.27) and agree with $\mathbf{x}$ on A_j, so we are done. $\square$

BIBLIOGRAPHY

[1] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[2] Sergey Fomin, Pavlo Pylyavskyy, and Eugenii Shustin. Morsifications and mutations. arXiv:1711.10598, 2017.

[3] R. M. Kashaev. On discrete three-dimensional equations associated with the local Yang-Baxter relation. *Lett. Math. Phys.*, 38(4):389–397, 1996.

[4] Richard Kenyon and Robin Pemantle. Principal minors and rhombus tilings. *J. Phys. A*, 47(47):474010, 17, 2014.

[5] Richard Kenyon and Robin Pemantle. Double-dimers, the Ising model and the hexahedron recurrence. *J. Combin. Theory Ser. A*, 137:27–63, 2016.

[6] Dmitry Kozlov. *Combinatorial algebraic topology*. Springer, 2008.

[7] Alexander Leaf. The cluster algebra nature of the Kashaev equation and hexahedron recurrence. In preparation.

[8] Yu. I. Manin and V. V. Schechtman. Arrangements of hyperplanes, higher braid groups and higher Bruhat orders. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 289–308. Academic Press, Boston, MA, 1989.

[9] Luke Oeding. Set-theoretic defining equations of the variety of principal minors of symmetric matrices. *Algebra Number Theory*, 5(1):75–109, 2011.

[10] Günter M. Ziegler. Higher Bruhat orders and cyclic hyperplane arrangements. *Topology*, 32(2):259–279, 1993.