On the Motion of Angled Crested Type Water Waves

by

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Abstract

We consider the two-dimensional water wave equation which is a model of ocean waves. The water wave equation is a free boundary problem for the Euler equation, where we assume that the fluid is inviscid, incompressible and irrotational and the air density is zero. In the case of zero surface tension, we show that the singular solutions constructed recently by Wu are rigid. In the case of non-zero surface tension, we construct an energy functional and prove an a priori estimate without assuming the Taylor sign condition. This energy reduces to the energy obtained by Kinsey and Wu in the zero surface tension case for angled crest water waves. We show that in an appropriate regime, the zero surface tension limit of our solutions is the one for the gravity water wave equation which includes waves with angled crests.

CHAPTER 1

Introduction

1.1 The Problem

Figure 1.1: Manhattan beach wave © Eino Mustonen (https://en.wikipedia.org/wiki/File:Manhattan_beach_wave.JPG)
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The motion of water waves in the ocean has been an eternal source of joy, wonder and a test bed of scientific activity. We have all seen fascinating phenomenon such as ripples in water, traveling waves, splashing, droplet formation, vortices etc. In the picture above we see another interesting phenomenon which is the formation of a wave with a sharp crest. A very natural question is to understand these kinds of waves in a scientific manner. There are many approaches to this problem such as by doing actual physical experiments, doing numerical simulations of these waves etc. We will take a mathematical approach to this problem and use rigorous mathematics in our analysis.

We will need a mathematical model of the fluid to analyze these waves. We assume a continuous model of the fluid (as compared to a model which takes into account the molecular nature of the fluid). To make the model mathematically tractable we will make rather severe assumptions (which may or may not be reasonable to understand the phenomenon in question). Some of the assumptions we make are:

1. The fluid is incompressible
2. The fluid has constant density 1
3. The temperature is a constant and doesn't play a role in the dynamics
4. There is no wind. We assume that air has a constant density of 0
5. There is no bottom, i.e. it is infinitely deep and infinitely wide
6. The fluid is irrotational, i.e. there are no vortices
7. There is no viscosity, i.e. no internal friction
8. We ignore the rotation of the earth and assume that gravity is a constant pointing in the downward direction

We assume that the fluid region at time t is $\Omega(t) \subset \mathbb{R}^d$ (where $d = 2, 3$) and that the fluid and air is separated by an interface $\Sigma(t)$. The physically relevant dimension is $d = 3$, however if the wave has a symmetry in a one direction (such as the one in the picture earlier), then we can reduce the dimension by one and assume that $d = 2$. In this case the boundary $\Sigma(t)$ being a one dimensional curve. Then by using Newton's laws we can derive the Euler equation which governs the motion of the fluid.

$$\begin{aligned} &\mathbf{v_t} + (\mathbf{v}.\nabla)\mathbf{v} = -\mathbf{i} - \nabla P \qquad \text{on } \Omega(t) \\ &\text{div } \mathbf{v} = 0, \quad \text{curl } \mathbf{v} = 0 \qquad \text{on } \Omega(t) \end{aligned} \tag{1.1}$$

Along with the boundary conditions

$$\begin{aligned} &P = -\sigma\partial_s\theta \qquad \text{on } \Sigma(t) \\ &(1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)) \\ &\mathbf{v} \to 0, \mathbf{v_t} \to 0 \qquad \text{as } |(x, y)| \to \infty \end{aligned} \tag{1.2}$$

Here $\mathbf{v}$ is the velocity vector, P is the pressure, $\theta =$ angle the interface makes with the $x - axis$, $\partial_s =$ arc length derivative, $\sigma =$ coefficient of surface tension ≥ 0. We also make the assumption that the interface tends to flat at infinity and that the velocity and acceleration of the fluid vanishes at infinity.

Note that the variables here are the shape of the fluid region Ω and the fluid velocity $\mathbf{v}$. The pressure can be recovered from the velocity by solving the following elliptic equation

$$\Delta P = -|\nabla \mathbf{v}|^2 \quad \text{on } \Omega(t) \qquad \text{and } P = -\sigma \partial_s \theta \qquad \text{on } \Sigma(t)$$

This equation is obtained by taking divergence in the Euler equation. Also observe that as the fluid is incompressible and irrotational, hence the fluid velocity is harmonic in Ω and hence is determined by its boundary values (assuming appropriate decay at infinity).

A typical wave with angled crest has the following shape

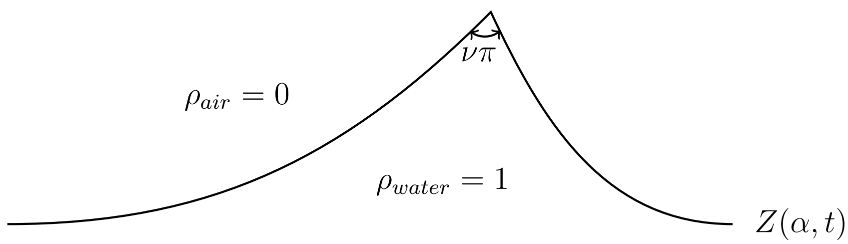

Figure 1.2: An angled crested wave

When the surface tension is non-zero, intuitively we would not expect sharp angles and hence typical waves will look like smoothened out versions of the above wave. If ϵ is a small parameter which captures how much the wave was smoothed, a typical wave will look like the following

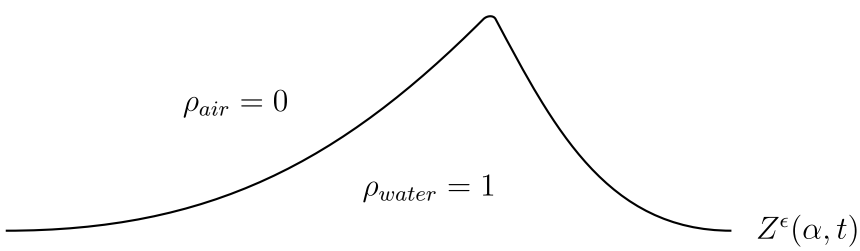

Figure 1.3: An approximate angled crested wave

In the last two decades, there has been a lot of work on the Cauchy problem for water waves. The local well-posedness of the water wave equation for smooth enough initial data is by now well established (see the next section for the literature). Recently, building upon the work of Kinsey and Wu [KW14], Wu [Wu15] proved the existence of solutions in the zero surface tension case, for initial interfaces with angled crests. In Wu [Wu18], the uniqueness of these solutions was shown.

In this thesis we explore two natural questions regarding such waves. The first question we address is how the singularities of the solution $Z(\alpha, t)$ (constructed in Wu [Wu15]) evolve in time, i.e. does the angle of the crest change? what is the acceleration at the tip? does the particle at the tip stay at the tip? We prove that the singularities are "rigid", in the sense as given by the statement of Main result 1 in §1.3. The other question we address is how these singular solutions can be approximated by solutions with non-zero surface tension. This is explained in Main result 2 in §1.3.

1.2 Previous Research

The study of special solutions of water waves has a long history and we will only concentrate on the literature regarding angled crests. Interestingly one of the earliest works on water waves was by Stokes [Sto80] who constructed traveling wave solutions with sharp crests of angle $120°$. This work was very influential however the work was not mathematically rigorous. After nearly a century Amick, Fraenkel, and Toland [AFT82] proved the existence of Stokes wave of greatest height which have a sharp crest of $120°$. Recently Wu [Wu12] proved the existence of self-similar solutions which have sharp crests of angle $< 90°$.

The earliest results on local well-posedness for the Cauchy problem are for small data in 2D and were obtained by Nalimov [Nal74], Yoshihara [Yos82, Yos83] and Craig [Cra85]. In the case of zero surface tension, Wu [Wu97, Wu99] obtained the proof of well-posedness for arbitrary data in Sobolev spaces. Later this result was extended to the case of bottom with finite depth, non-zero vorticity and in lowering the Sobolev regularity in [CL00, Lin05, Lan05, ZZ08, ABZ14a, HIT16, ABZ14b, dP16].

In the case of non-zero surface tension, the local well-posedness of the equation in Sobolev spaces was established by Beyer and Gunther in [BG98]. See also the works in [Igu01, Amb03, Sch05, CS07, ABZ11, Ngu17]. The zero surface tension limit of the water wave equations in Sobolev spaces was proved by Ambrose and Masmoudi [AM05, AM09]. See also the works in [OT02, SZ08, MZ09].

An important quantity related to the well-posedness of the problem in the zero surface tension case is the Taylor sign condition. This says that there should exist a constant $c > 0$ such that

$$-\frac{\partial P}{\partial n} \geq c > 0 \quad \text{ on } \Sigma(t)$$

In [Wu97] Wu proved that this condition is satisfied for the infinite bottom case if the interface is $C^{1,\alpha}$ for $\alpha > 0$. This was later shown to be true for flat bottoms and with perturbations to flat bottom by Lannes [Lan05] and was reproved again in [HIT16]. In all other results mentioned above for zero surface tension or for zero surface tension limit, the initial data is chosen in such a way that this condition is satisfied. In the zero surface tension case, for non $C^{1,\alpha}$ curves this condition is only satisfied in a weak sense with $-\frac{\partial P}{\partial n} \geq 0$ [Wu97, KW14]. This makes the quasilinear equation degenerate and hence standard energy estimates in Sobolev spaces do not work. Kinsey and Wu

[KW14] used a weighted Sobolev energy with the weight depending nonlinearly on the interface to deal with this case and proved a priori estimates for interfaces which can have angled crests. Wu [Wu15] obtained existence of solutions for the Cauchy problem that allows for such waves. Recently Wu [Wu18] proved the uniqueness of these solutions.

1.3 Results and outline of the Dissertation

In this thesis we will work with the model described earlier in the case of zero surface tension and positive surface tension. We now describe the results briefly.

We first work on the zero surface tension case. As mentioned earlier, Kinsey and Wu [KW14] (Theorem 2.4.1) proved a priori estimates for the water wave equation for the Cauchy problem. The fascinating aspect of this result was that the energy not only allows the initial interface to be smooth, but it also allows angled crests of angles $\nu\pi$ with $0 < \nu < \frac{1}{2}$. Building upon this work, Wu [Wu15] (Theorem 2.5.1) proved the existence of singular solutions.

In Chapter 2, we first lower the regularity of the energy in Kinsey Wu [KW14] by half spacial derivate in Theorem 2.4.2. We then consider the solutions constructed by Wu [Wu15] with initial data as interfaces with angled crests. The main result of this chapter is that the singularities of these solutions are "rigid", in the sense as given by Main result 1 below. This result in proved in Lemma 2.5.2, Theorem 2.5.3 and Corollary 2.5.4. In §2.6 we prove that the energy also allows cusps (which can be thought of as corresponding to $\nu = 0$).

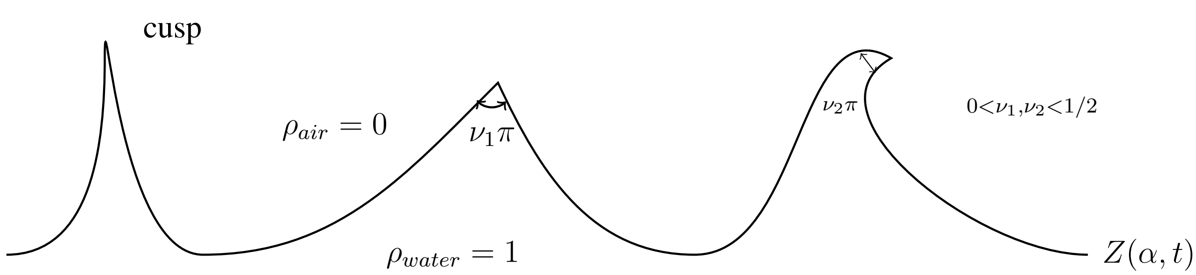

Figure 1.4: A wave with angled crests and cusps

Main Result 1 (Rigidity of Singularities): In its time of existence, the solutions of Wu from Theorem 2.5.1 satisfy

- Interface with angled crests remain angled crested
- Particles at the tip stay at the tip
- Acceleration at the tip $= -i$

- Angles do not change nor tilt

In Chapter 3 we focus on the question of whether these singular solutions in [Wu15] can be approximated by solutions with surface tension. We show that a natural extension of the work of Kinsey and Wu [KW14] to the case of non-zero surface tension does not allow interfaces with sharp crests. Hence in this case, we do not take the initial data as interfaces with angled crests, but a smoothened out version. We first parametrize the singular interface by the boundary value of a Riemann map Z from the lower half plane. To smoothen out the interface, we convolve with the Poisson kernel $Z^{\epsilon} = Z * P_{\epsilon}$ where $\epsilon > 0$ is the smoothing parameter. We also smoothen out the boundary values of the velocity $Z_t^{\epsilon} = Z_t * P_{\epsilon}$. Now with this smooth initial data, we solve the water wave equation with surface tension parameter σ and we denote these solutions as $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$. Hence we have two parameters $\epsilon, \sigma > 0$ with both of them going to zero and we want to say that these solutions should converge to the singular solution (Z, Z_t). We prove the following.

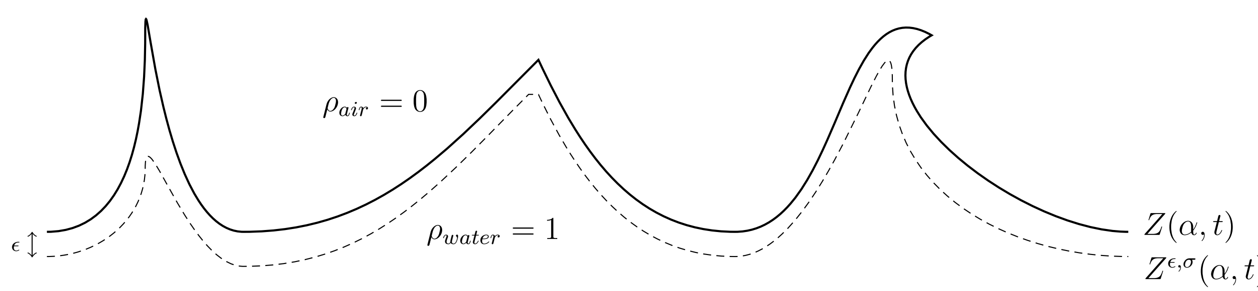

Figure 1.5: Waves with and without surface tension

Main result 2 (existence and convergence) We have

- Let $0 < \epsilon \leq 1$ and $\dfrac{\sigma}{\epsilon^{3/2}} \leq 1$, then there exists $T > 0$ independent of ϵ, σ so that the solutions $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ exist in $[0, T]$
- If in addition $\epsilon, \sigma \to 0$ with $\dfrac{\sigma}{\epsilon^{3/2}} \to 0$, then $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma}) \to (Z, Z_t)$ in $[0, T]$ in an appropriate norm.

In all previous local existence results for non-zero surface tension, the time of existence $T \lesssim \|\kappa\|_{\infty}^{-1}$ where κ is the curvature. Note that $\|\kappa\|_{\infty} \to \infty$ as $\epsilon \to 0$ and hence the time of existence of the solutions $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ goes to zero by previous existence results. Our result says that as long there is a balance between surface tension and smoothness $\sigma \lesssim \epsilon^{\frac{3}{2}}$, the solutions exist on a uniform time interval and in fact approximate the zero surface tension solution.

This result, which is proved in Proposition 3.4.9, is a consequence of our main results in Chapter 3. In Theorem 3.4.1 we prove the main apriori estimate which extends the apriori estimate of Kinsey-Wu [KW14] to the case of non-zero surface tension. The uniform existence result in the first part of the above result is a consequence of this theorem. The main convergence result is Theorem 3.4.8 which says that solutions with surface tension approximate the zero surface tension solutions as $\sigma \to 0$. The convergence result in the second part of the above result is a direct consequence of this theorem.

Chapter 4 is devoted to the proof of Theorem 3.4.1 and Chapter 5 to the proof of Theorem 3.4.8. Finally the appendix contains some of the most commonly used identities and estimates used throughout the thesis.

CHAPTER 2

Gravity Water Waves

In this chapter we will prove the a-priori estimate for angled crested water waves first proved in [KW14] and prove the rigidity of singularities of the singular solutions constructed in [Wu15]. The a-priori estimate is proved here primarily because it is a good warm up to our main result on surface tension in Chapter 3. We also reduce the regularity of the energy by half spacial weighted derivative and simplify the proof. We will not take the shortest route to prove the a-priori estimate as we will try to use arguments which generalize to the case of surface tension.

In §2.1 we establish the notation and prove some basic formulae. In §2.2 we derive the quasi-linear equation from which we obtain our energy. In §2.3 we give a heuristic explanation of the energy estimate and explain where the angle less than $90°$ restriction comes from. It is also explained why the energy we have is quite natural. In §2.4 we state and prove the energy estimate Theorem 2.4.2 for the case of zero surface tension. The energy for this result is lower order as compared to the one in [KW14]. In §2.5 we state and prove our main result Theorem 2.5.3 on the rigidity of the singular solutions constructed in [Wu15]. Finally in §2.6 we show that the energy from [KW14] allows angled crests and cusps.

2.1 Notation and Preliminaries

We will try to be as consistent as possible with the notation used in [KW14]. Most of this section is essentially taken directly from [KW14] except for some new definitions. The Fourier transform is defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} f(x)\, dx$$

In particular $\widehat{\partial_x f}(\xi) = -i\xi \hat{f}(\xi)$. We use this definition for the Fourier transform as we will be concerned with holomorphic functions on the lower half plane instead of the upper half plane. We will denote by $\mathcal{D}(\mathbb{R})$ the space of smooth functions with compact support in $\mathbb{R}$ and $\mathcal{D}'(\mathbb{R})$ will be the space of distributions. $\mathcal{S}(\mathbb{R})$ will denote the Schwartz space of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. A Fourier multiplier with symbol $a(\xi)$ is the operator T_a defined formally by the relation $\widehat{T_a f} = a(\xi)\hat{f}(\xi)$. The operators $|\partial_{\alpha'}|^s$ for $s \in \mathbb{R}$ are defined as the Fourier multipliers with symbol $|\xi|^s$. Note that the 1D Laplacian $\Delta = \partial_x^2$ is a Fourier multiplier with symbol $-|\xi|^2$ and hence $-\Delta = |\partial_{\alpha'}|^2$. The Sobolev spaces $H^s(\mathbb{R})$ for $s \geq 0$ is the space of functions with $\|f\|_{H^s} = \|(1+|\xi|)^{\frac{s}{2}}\hat{f}\|_2 < \infty$. The homogenous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R})$ is the space of functions modulo constants with $\|f\|_{\dot{H}^{\frac{1}{2}}} = \||\xi|^{\frac{1}{2}}\hat{f}(\xi)\|_2 < \infty$.

Let the interface $\Sigma(t) : z = z(\alpha, t) \in \mathbb{C}$ be given by a Lagrangian parametrization α satisfying $z_\alpha(\alpha, t) \neq 0$ for all $\alpha \in \mathbb{R}$. Hence $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the velocity of the fluid on the interface and $z_{tt}(\alpha, t) = (\mathbf{v_t} + (\mathbf{v}.\nabla)\mathbf{v})(z(\alpha, t), t)$ is the acceleration. Hence the Euler equation becomes

$$z_{tt}(\alpha, t) + i = -\hat{n}\frac{\partial P}{\partial \hat{n}}(z(\alpha, t))$$

where

$$\hat{n} = i\frac{z_\alpha}{|z_\alpha|} = ie^{i\theta} = \text{unit outward normal vector}$$

Define

$$a(\alpha, t) = -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial \hat{n}}(z(\alpha, t)) \ \in \mathbb{R}$$

So we get

$$\begin{aligned} &z_{tt} + i = iaz_\alpha \\ \text{Therefore} \quad &\bar{z}_{tt} - i = -ia\bar{z}_\alpha \end{aligned} \tag{2.1}$$

Let $\Phi(., t) : \Omega(t) \to P_-$ be a Riemann map satisfying $\lim_{z\to\infty} \Phi_z(z, t) = 1$ and define

$$h(\alpha, t) = \Phi(z(\alpha, t), t) \tag{2.2}$$

hence $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Let $h^{-1}(\alpha', t)$ be its inverse i.e.

$$h(h^{-1}(\alpha', t), t) = \alpha'$$

Note that for each fixed time t, a Riemann map $\Phi(\cdot, t) : \Omega(t) \to P_-$ with the condition $\lim_{z\to\infty} \Phi_z(z, t) = 1$ is not determined uniquely. We have one degree of freedom namely that of translation. To fix this, we impose the condition that $h_t(\alpha, t) \to 0$ as $|\alpha| \to \infty$ for all $t \geq 0$. With this, the only freedom left is that of the choice of translation of the Riemann map at $t = 0$, which does not play any role in the analysis. From now on, we will fix our Lagrangian parametrization at

$t = 0$ by imposing

$$h(\alpha, 0) = \alpha \quad \text{for all } \alpha \in \mathbb{R}$$

Hence the Lagrangian parametrization is the same as conformal parametrization at $t = 0$.

From now on compositions of functions will always be in the spatial variables. We write $f = f(\cdot, t), g = g(\cdot, t), f \circ g(\cdot, t) := f(g(\cdot, t), t)$. Define the operator U_g as given by $U_g f = f \circ g$ and when we say "precompose f with g", we mean $f \circ g$. Observe that $U_f U_g = U_{g \circ f}$. Let $[A, B] := AB - BA$ be the commutator of the operators A and B. If A is an operator and f is a function, then $(A + f)$ will represent the addition of the operators A and the multiplication operator T_f where $T_f(g) = fg$. For functions f, g, h we define the function

$$[f, g; h](\alpha') = \frac{1}{i\pi} \int \left(\frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) \left(\frac{g(\alpha') - g(\beta')}{\alpha' - \beta'} \right) h(\beta') \, d\beta'$$

Define the variables

$$\begin{array}{llll}
Z(\alpha', t) = z \circ h^{-1}(\alpha', t) & Z_{,\alpha'}(\alpha', t) = \partial_{\alpha'} Z(\alpha', t) & \text{Hence} & (\frac{z_\alpha}{h_\alpha}) \circ h^{-1} = Z_{,\alpha'} \\
Z_t(\alpha', t) = z_t \circ h^{-1}(\alpha', t) & Z_{t,\alpha'}(\alpha', t) = \partial_{\alpha'} Z_t(\alpha', t) & \text{Hence} & (\frac{z_{t\alpha}}{h_\alpha}) \circ h^{-1} = Z_{t,\alpha'} \\
Z_{tt}(\alpha', t) = z_{tt} \circ h^{-1}(\alpha', t) & Z_{tt,\alpha'}(\alpha', t) = \partial_{\alpha'} Z_{tt}(\alpha', t) & \text{Hence} & (\frac{z_{tt\alpha}}{h_\alpha}) \circ h^{-1} = Z_{tt,\alpha'}
\end{array}$$

Note that as $Z(\alpha', t) = z(h^{-1}(\alpha', t), t)$ we see that $\partial_t Z \neq Z_t$. Similarly $\partial_t Z_t \neq Z_{tt}$. The substitute for the time derivative is the material derivative. Define the operators

$$D_t = \text{material derivative} = \partial_t + b\partial_{\alpha'} \qquad \text{where } b = h_t \circ h^{-1}$$

$$D_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \qquad \overline{D}_{\alpha'} = \frac{1}{\overline{Z}_{,\alpha'}} \partial_{\alpha'} \qquad |D_{\alpha'}| = \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}$$

$\mathbb{H}$ = Hilbert transform = Fourier multiplier with symbol $sgn(\xi)$

$$\mathbb{H} f(\alpha') = \frac{1}{i\pi} p.v. \int \frac{1}{\alpha' - \beta'} f(\beta') \, d\beta'$$

$$\mathbb{P}_H = \text{Holomorphic projection} = \frac{\mathbb{I} + \mathbb{H}}{2}$$

$$\mathbb{P}_A = \text{Antiholomorphic projection} = \frac{\mathbb{I} - \mathbb{H}}{2}$$

$|\partial_{\alpha'}| = \sqrt{-\Delta} =$ Fourier multiplier with symbol $|\xi| \; = \; i\mathbb{H}\partial_{\alpha'}$

$|\partial_{\alpha'}|^{1/2} =$ Fourier multiplier with symbol $|\xi|^{1/2}$

Now we have $D_t Z = Z_t$ and $D_t Z_t = Z_{tt}$ and more generally $D_t(f(\cdot, t) \circ h^{-1}) = (\partial_t f(\cdot, t)) \circ h^{-1}$

or equivalently $\partial_t(F(\cdot,t)\circ h) = (D_t F(\cdot,t))\circ h$. This means that $D_t = U_h^{-1}\partial_t U_h$ i.e. D_t is the material derivative in Riemmanian coordinates. In this work, the material derivative D_t is more heavily used as compared to the time derivative ∂_t.

The Hilbert transform defined above satisfies the following property (see [Tit86])

Lemma 2.1.1. *Let $1 < p < \infty$ and let $F(z)$ be a holomorphic function in the lower half plane with $F(z) \to 0$ as $z \to \infty$. Then the following are equivalent*

1. $\sup_{y<0} \|F(\cdot + iy)\|_p < \infty$
2. *$F(z)$ has a boundary value f, non-tangentially almost everywhere with $f \in L^p$ and $\mathbb{H}(f) = f$.*

In particular this says if $\overline{\mathbf{v}}$ decays appropriately at infinity, then its boundary value $\overline{Z}_t$ will satisfy $\mathbb{H}\overline{Z}_t = \overline{Z}_t$. We now define a few more variables

$$\mathcal{A} = (ah_\alpha)\circ h^{-1}$$
$$A_1 = \mathcal{A}|Z_{,\alpha'}|^2 \qquad \text{Hence} \qquad \frac{A_1}{|Z_{,\alpha'}|} = -\frac{\partial P}{\partial \hat{n}}\circ h^{-1}$$
$$g = \theta\circ h^{-1}$$
$$\omega = e^{ig} = \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

With this notation, by precomposing (2.1) with h^{-1} we get the fundamental equation

$$\overline{Z}_{tt} - i = -i\frac{A_1}{Z_{,\alpha'}} \tag{2.3}$$

Let us now derive the formulae for A_1 and b.

2.1.1 Formula for A_1

Let $F = \overline{\mathbf{v}}$ and hence F is holomorphic in $\Omega(t)$ and $\bar{z}_t = F(z(\alpha,t),t)$. Hence

$$\bar{z}_{tt} = F_t(z(\alpha,t),t) + F_z(z(\alpha,t),t)z_t(\alpha,t) \qquad \bar{z}_{t\alpha} = F_z(z(\alpha,t),t)z_\alpha(\alpha,t)$$

$$\text{Hence} \qquad \bar{z}_{tt} = F_t\circ z + z_t\frac{\bar{z}_{t\alpha}}{z_\alpha}$$

Precomposing with h^{-1} we obtain $\overline{Z}_{tt} = F_t \circ Z + Z_t \dfrac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}}$. Now Multiply by $Z_{,\alpha'}$ and use (2.3) to get

$$A_1 = iZ_{,\alpha'} F_t \circ Z + Z_{,\alpha'} + iZ_t \overline{Z}_{t,\alpha'} \tag{2.4}$$

Note that the only non-holomorphic quantity in the above formula is $iZ_t\overline{Z}_{t,\alpha'}$. Hence apply $(\mathbb{I}-\mathbb{H})$ and use the fact that $\mathbb{H}(Z_{,\alpha'} - 1) = Z_{,\alpha'} - 1$ and $\mathbb{H}1 = 0$ to obtain

$$(\mathbb{I} - \mathbb{H})A_1 = 1 + i[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$$

Now take the real part to obtain the formula

$$A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'} \tag{2.5}$$

Note that the Taylor sign condition can be written as

$$-\frac{\partial P}{\partial \hat{n}} \circ h^{-1} = \frac{A_1}{|Z_{,\alpha'}|} \geq 0 \tag{2.6}$$

This formula was first derived by Wu [Wu97] to prove the strong Taylor sign condition for $C^{1,\alpha}$ interfaces with $\alpha > 0$ and was crucially used in Kinsey-Wu [KW14] to prove a-priori estimates for angled crest interfaces.

2.1.2 Formula for b

Recall that $h(\alpha,t) = \Phi(z(\alpha,t),t)$ and so by taking derivatives we get

$$h_t = \Phi_t \circ z + (\Phi_z \circ z)z_t \qquad h_\alpha = (\Phi_z \circ z)z_\alpha$$

$$\text{Hence} \quad h_t = \Phi_t \circ z + \frac{h_\alpha}{z_\alpha} z_t$$

Precomposing with h^{-1} we obtain $h_t \circ h^{-1} = \Phi_t \circ Z + \dfrac{Z_t}{Z_{,\alpha'}}$. Apply $(\mathbb{I} - \mathbb{H})$ and take real part, to get

$$b = \text{Re}(\mathbb{I} - \mathbb{H})\left(\frac{Z_t}{Z_{,\alpha'}}\right) \tag{2.7}$$

By taking a derivative and rearranging we obtain the relation

$$b_{\alpha'} = (\Phi_t \circ Z)_{\alpha'} + D_{\alpha'} Z_t + Z_t\left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) \tag{2.8}$$

This formula was first derived by Wu [Wu97] to prove the local well-posedness result in 2D.

2.2 The Quasilinear Equation

In [KW14] a quasilinear equation is derived for the variable $\overline{Z}_t$. We instead derive a quasilinear equation for the variable $\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}$. We do this as it is easier to reduce the regularity of the energy by using the quasilinear equation for this variable. This also foreshadows the quasilinear equation for the variable Θ in Chapter 3 which is critical to prove the energy estimate in the non-zero surface tension case. Let us first derive some simple but useful formulae:

a) We have

$$\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = \omega\partial_{\alpha'}\left(\frac{\overline{\omega}}{|Z_{,\alpha'}|}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} + \omega|D_{\alpha'}|\overline{\omega}$$

Observe that $\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}$ is real valued and $\omega|D_{\alpha'}|\overline{\omega}$ is purely imaginary. From this we obtain the relations

$$\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega) \tag{2.9}$$

b) As $\frac{z_\alpha}{|z_\alpha|} = e^{i\theta}$, we get that $\theta = \mathrm{Im}(\log z_\alpha)$. Hence $\partial_t\theta = \mathrm{Im}\frac{z_{t\alpha}}{z_\alpha} = -\mathrm{Im}\frac{\bar{z}_{t\alpha}}{\bar{z}_\alpha}$. Precomposing with h^{-1} we get

$$D_t g = -\mathrm{Im}(\overline{D}_{\alpha'}\overline{Z}_t) \tag{2.10}$$

c) We now record some frequently used commutator identities. They are easily seen by differentiating

$$[\partial_{\alpha'}, D_t] = b_{\alpha'}\partial_{\alpha'} \qquad [|D_{\alpha'}|, D_t] = \mathrm{Re}(D_{\alpha'}Z_t)|D_{\alpha'}| = \mathrm{Re}(\overline{D}_{\alpha'}\overline{Z}_t)|D_{\alpha'}|$$
$$[D_{\alpha'}, D_t] = (D_{\alpha'}Z_t)D_{\alpha'} \qquad [\overline{D}_{\alpha'}, D_t] = \left(\overline{D}_{\alpha'}\overline{Z}_t\right)\overline{D}_{\alpha'}$$

Using the commutator relation $[\partial_{\alpha'}, D_t] = b_{\alpha'}\partial_{\alpha'}$ we obtain the following formulae

$$D_t|Z_{,\alpha'}| = D_t e^{\mathrm{Re}\log Z_{,\alpha'}} = |Z_{,\alpha'}|\{\mathrm{Re}(D_{\alpha'}Z_t) - b_{\alpha'}\} \tag{2.11}$$

$$D_t\frac{1}{Z_{,\alpha'}} = \frac{-1}{Z_{,\alpha'}}(D_{\alpha'}Z_t - b_{\alpha'}) = \frac{1}{Z_{,\alpha'}}\left\{(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) + \overline{D}_{\alpha'}\overline{Z}_t\right\} \tag{2.12}$$

We will now derive the quasilinear equation. Define

$$J_0 = D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \tag{2.13}$$

Observe that J_0 is real valued. Apply D_t to the above formula for $D_t\frac{1}{Z_{,\alpha'}}$ to get

$$D_t^2\frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}\left\{(b_{\alpha'} - D_{\alpha'}Z_t)^2 + J_0 + D_t\overline{D}_{\alpha'}\overline{Z}_t\right\}$$

Now from (2.3) we see that

$$\begin{aligned} D_t\overline{D}_{\alpha'}\overline{Z}_t &= -(\overline{D}_{\alpha'}Z_t)^2 + \overline{D}_{\alpha'}\overline{Z}_{tt} \\ &= -(\overline{D}_{\alpha'}Z_t)^2 - \frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 - iA_1\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}} \end{aligned}$$

Define

$$Q_0 = (b_{\alpha'} - D_{\alpha'}Z_t)^2 - (\overline{D}_{\alpha'}\overline{Z}_t)^2 - \frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \tag{2.14}$$

Hence we see that

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}(J_0 + Q_0)$$

Now we apply $\partial_{\alpha'}$ and commute. First using $[\partial_{\alpha'}, D_t] = b_{\alpha'}\partial_{\alpha'}$ we see that

$$\begin{aligned} \left[\partial_{\alpha'}, D_t^2\right]\frac{1}{Z_{,\alpha'}} &= [\partial_{\alpha'}, D_t]D_t\frac{1}{Z_{,\alpha'}} + D_t[\partial_{\alpha'}, D_t]\frac{1}{Z_{,\alpha'}} \\ &= b_{\alpha'}\left(\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right) + D_t\left(b_{\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\ &= b_{\alpha'}\left(\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}} + D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + D_tb_{\alpha'}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \end{aligned}$$

Hence we get our main quasilinear equation as

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = D_{\alpha'}J_0 + R_0 \tag{2.15}$$

where

$$\begin{aligned} R_0 = {} & \left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)(J_0 + Q_0) + D_{\alpha'}Q_0 - b_{\alpha'}\left(\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}} + D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\ & - (D_t b_{\alpha'})\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - 2iA_1\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right)\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\ & - i\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \end{aligned} \tag{2.16}$$

2.3 Heuristics

We now give a brief heuristic explanation into the nature of the results. One of the main results in this chapter is Theorem 2.4.2 which is an apriori estimate for the energy $E(t)$. This energy allows smooth enough interfaces and interfaces with angled crests as initial data. This energy is lower order as compared to the energy of Kinsey-Wu [KW14] by half spacial derivative.

Local well-posedness results in water waves are generally proved in the following way:

1. Reduce equation to the boundary
2. Choose appropriate variables, coordinate system and derive a quasilinear equation/system
3. Take derivatives and write down the energy $E(t)$
4. Prove apriori estimate of the form $\frac{dE(t)}{dt} \leq P(E(t))$ where P is a polynomial with non-negative coefficients
5. Local existence using an approximation argument

A typical result using the above approach says that if the initial data satisfies $E(0) < \infty$, then there exists a unique solution to the water wave equation for a time $T = T(E(0)) > 0$ depending only on $E(0)$ and we have $\sup_{t\in[0,T]} E(t) < \infty$. Typically this energy $E(t)$ is equivalent to the Sobolev norm of the initial data. One way of proving local existence for rough initial data would be to lower the regularity of this Sobolev space. For gravity water waves, this was done in the work of [ABZ14b] where the Sobolev norm of the initial data corresponds to an interface being $C^{1,\alpha}$ where $\alpha > \frac{11}{24}$. Note that this does not allow interface with angled crests.

In Kinsey and Wu [KW14] and also in this chapter, the energy $E(t)$ is not equivalent to the standard Sobolev norm of the initial data. Instead the energy is equivalent to a weighted Sobolev norm with the weight depending nonlinearly on the interface. More precisely the weight can be thought of as the coefficient of the Taylor sign condition $w \approx -\frac{\partial P}{\partial \hat{n}}$. In conformal coordinates, this weight $w \approx 1$ when the interface is $C^{1,\alpha}$ for $\alpha > 0$ but behaves like $\omega(\alpha) \approx |\alpha|^{1-\nu}$ near an angled crest of angle $\nu\pi$. Hence the energy $E(t)$ in Theorem 2.4.2 behaves like the Sobolev norm for smooth enough interfaces and behaves completely different for interfaces with angled crests. The energy used in [KW14] and in Theorem 2.4.2 allows interfaces with angled crests with angles $\nu\pi$ for $0 < \nu < \frac{1}{2}$ and smooth enough interfaces which are $C^{2,\alpha}$ where $\alpha > \frac{1}{2}$.

The main goal here is to find an energy $E(t)$ which allows angled crests interfaces and then be able to prove an apriori estimate for this energy. To do this, we need to choose appropriate variables, coordinate systems and derive quasilinear equations from which we can construct the energy. It should be noted that there is no universal choice of variables or universal quasilinear equation from which we can start our analysis. We choose our variables and then derive quasilinear equations in such a way which helps us to suit our purposes. In [KW14] a quasilinear equation in the variable $\bar{Z}_t$ was derived and then the energy was constructed from this quasilinear equation. In this chapter we instead derived a quasilinear equation in the variable $\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$. This is done to reduce the regularity of the energy as compared to the energy in [KW14].

Observe that if we let $f = \partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}$, then we have $\mathbb{H}(f) = f$ and hence by using the relation $i\mathbb{H}\partial_{\alpha'} = |\partial_{\alpha'}|$, the quasilinear equation (2.15) can be written as

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}|\partial_{\alpha'}|\right)f = -2iA_1\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right)f + \text{ other l.o.t} \tag{2.17}$$

To obtain the energy, multiply the equation by $D_t\bar{f}$ and integrate to get the energy

$$\left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

The other two terms in the energy $E(t)$ in Theorem 2.4.2 namely $\|\bar{Z}_{t,\alpha'}\|_2^2$ and $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2$ are added as lower order terms and they are compatible with the energy above.

Let us now do a heuristic L^2 based energy estimate to understand the nature of the equation. If the interface is $C^{1,\alpha}$ then we have $0 < c_1 \leq \frac{1}{|Z_{,\alpha'}|} \leq c_2 < \infty$ and hence for smooth enough interfaces the main operator in (2.17) behaves like $\partial_t^2 + |\partial_{\alpha'}|$ for which standard energy estimates in Sobolev spaces work. As we are interested in interfaces with angled crests, let us assume that the interface has an angle of $\nu\pi$ at $\alpha' = 0$. Hence around $\alpha' = 0$ we have, $Z(\alpha') \sim (\alpha')^\nu$ and hence $\frac{1}{|Z_{,\alpha'}|}(\alpha') \sim |\alpha'|^{1-\nu}$. We can also approximate $A_1 \approx 1$ and so the equation behaves like

$$(\partial_t^2 + |\alpha'|^{2-2\nu}|\partial_{\alpha'}|)f = |\alpha'|^{1-2\nu} f + \text{ other l.o.t} \tag{2.18}$$

Note that the operator is no longer strictly hyperbolic but is only weakly hyperbolic. If we multiply by $\partial_t f$ and integrate, then we obtain control of $\partial_t f \in L^2$ and $|\alpha'|^{1-\nu} f \in \dot{H}^{\frac{1}{2}}$. For simplicity also assume that $f \in L^2$ which is compatible with $\partial_t f \in L^2$. To close the energy estimate, we need to control the L^2 norm of the right hand side. Hence to control the first term $|\alpha|^{1-2\nu} f \in L^2$, we would need $|\alpha'|^{1-2\nu} \in L^\infty_{loc}$ and hence we obtain the restriction $\nu \leq \frac{1}{2}$. This is the fundamental reason of the restrictions on the angles in [KW14]. Note that this restriction does not depend on the choice of f, but is purely a consequence of the structure of the quasilinear equation (2.18) and attempting to prove an L^2 based energy estimate.

Let us now understand the energy $\mathcal{E}$ defined right after Theorem 2.4.2 which is equivalent to the energy E. We now make an argument showing that the energy we have obtained here is very natural and not an artificial construct. Observe that the quasilinear equation is of the form

$$\left\{ D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|}\left(\frac{1}{|Z_{,\alpha'}|}\partial_\alpha\right)\mathbb{H} \right\} f = l.o.t$$

We show in the energy estimate that $1 \leq A_1 \leq 1 + \left\|\overline{Z}_{t,\alpha'}\right\|_2^2$ and hence we can consider $A_1 \approx 1$. Also in the special case of zero velocity $Z_t \equiv 0$, we actually have $A_1(\alpha') = 1$ for all $\alpha' \in \mathbb{R}$. Hence the main operators are D_t and $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}$. Recall that the operator $\mathbb{P}_H = \dfrac{\mathbb{I} + \mathbb{H}}{2}$ and has the property that for any smooth real valued function $f : \mathbb{R} \to \mathbb{R}$ vanishing at infinity, $\mathbb{P}_H(f) : \mathbb{R} \to \mathbb{C}$ is the boundary value of a holomorphic function in the lower half plane vanishing at infinity with $\text{Re}\{\mathbb{P}_H(f)\} = f/2$.

A very natural restriction is to have $\theta \in L^\infty$. The spaces L^∞ and $\dot{H}^{\frac{1}{2}}$ have the same scaling in dimension one and hence heuristically we will switch between the two. The energy $\mathcal{E}$ is obtained by applying the above operators to the relation $\theta \in L^\infty$ (or equivalently $g = \theta \circ h^{-1} \in L^\infty$), with suitable replacements of L^∞ with $\dot{H}^{\frac{1}{2}}$. We have the formula $D_t g = -\text{Im}(\overline{D}_{\alpha'}\overline{Z}_t)$ from (2.10) and hence heuristically we can replace $\overline{D}_{\alpha'}\overline{Z}_t$ with $D_t\mathbb{P}_H(g)$. The energy $\mathcal{E}$ has the heuristic representation

1) $\left\|\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_2 \sim \left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)^{\frac{1}{2}}\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

2) $\left\|\dfrac{1}{Z_{,\alpha'}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \sim \left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

3) $\left\|\overline{Z}_{t,\alpha'}\right\|_2 \sim D_t\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)^{-\frac{1}{2}}\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

4) $\left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \sim D_t\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)^{\frac{1}{2}}\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

To understand how the energy looks like in the arc length coordinate system, we first define $\underline{\mathfrak{a}} = -\left.\frac{\partial P}{\partial \hat{n}}\right|_{\mathbf{v}\equiv 0}$. Now in arc length coordinate system the main operators are then D_t and $\underline{\mathfrak{a}}\partial_s$ where we again write the material derivate as D_t, $\underline{\mathfrak{a}}$ corresponds to the weight $\frac{1}{|Z_{,\alpha'}|}$ and ∂_s is the arc length derivative which corresponds to the operator $\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}$. We let $\mathbb{P}_{hol}$ denote the linear operator defined by the property that for any smooth real valued function $f : \Sigma \to \mathbb{R}$ vanishing at infinity, $\mathbb{P}_{hol}(f) : \Sigma \to \mathbb{C}$ is the boundary value of a holomorphic function on Ω vanishing at infinity with $\mathrm{Re}\{\mathbb{P}_{hol}(f)\} = f/2$. We can do the same kind of heuristic representation as done above. Here we would have

1) $\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \sim (\underline{\mathfrak{a}}\partial_s)^{\frac{1}{2}}\mathbb{P}_{hol}(\theta) \in \dot{H}^{\frac{1}{2}}$

2) $\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \sim (\underline{\mathfrak{a}}\partial_s)\mathbb{P}_{hol}(\theta) \in \dot{H}^{\frac{1}{2}}$

3) $\left\|\overline{Z}_{t,\alpha'}\right\|_2 \sim D_t(\underline{\mathfrak{a}}\partial_s)^{-\frac{1}{2}}\mathbb{P}_{hol}(\theta) \in \dot{H}^{\frac{1}{2}}$

4) $\left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \sim D_t(\underline{\mathfrak{a}}\partial_s)^{\frac{1}{2}}\mathbb{P}_{hol}(\theta) \in \dot{H}^{\frac{1}{2}}$

Note that this is a very natural approach to construct the energy and it has the advantage that the energy automatically allows angled crests interfaces. This approach can be extended to the case of non-zero surface tension and this is exactly what we do in Chapter 3.

2.4 Apriori estimate

We first state the apriori estimate of [KW14]. The result there was proved in the periodic setting but the same proof works for the real line case also and we state the result for the real line case. Define the energy

$$\begin{aligned}\mathfrak{E}(t) = &\left\|\frac{1}{Z_{,\alpha'}}(t)\right\|_\infty^2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\right\|_2^2 + \left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}(t)\right\|_2^2 + \left\|\overline{Z}_{t,\alpha'}(t)\right\|_2^2 + \left\|D_{\alpha'}^2\overline{Z}_t(t)\right\|_2^2 \\ &+ \left\|\frac{1}{Z_{,\alpha'}}D_{\alpha'}^2\overline{Z}_t(t)\right\|_{\dot{H}^{\frac{1}{2}}}^2\end{aligned}$$

Theorem 2.4.1 ([KW14]). *Let $T' > 0$ and let (Z, Z_t) be a smooth solution*[1] *to the gravity water wave equation in the time interval $[0, T']$ with $\mathfrak{E}(t) < \infty$ for all $t \in [0, T']$. Then there exists $T = T(\mathfrak{E}(0)) > 0$ and $C = C(\mathfrak{E}(0)) > 0$ depending only on $\mathfrak{E}(0)$*[2]*, such that*

$$\sup_{[0,\min\{T,T'\}]} \mathfrak{E}(t) \leq C(\mathfrak{E}(0)) < \infty$$

We do not prove the theorem as stated above but prove a similar version. Let us now state the apriori estimate we prove here. Define the energy

$$E = \left\|\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|D_t\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2^2 + \left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

Theorem 2.4.2. *Let $T > 0$ and let (Z, Z_t) be a smooth solution*[3] *to the gravity water wave equation in the time interval $[0, T)$ with $E(t) < \infty$ for all $t \in [0, T)$. Then there exists a polynomial P with universal non-negative coefficients such that for all $t \in [0, T)$ we have*

$$\frac{dE(t)}{dt} \leq P(E(t))$$

We can also write the energy in a different way so that all the terms are holomorphic as is written for the energy $\mathfrak{E}$. Define

$$\mathcal{E} = \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2$$

Proposition 2.4.3. *There exists universal polynomials P_1, P_2 with non-negative coefficients so that for smooth solutions to the water wave equation we have*

$$E \leq P_1(\mathcal{E}) \quad \text{and} \quad \mathcal{E} \leq P_2(E)$$

Observe that the energy $\mathcal{E}$ is of lower regularity as compared to the energy $\mathfrak{E}$ by half weighted spacial derivative. If the interface doesn't has any singularities i.e. $|Z_{,\alpha'}| \in L^\infty$ and $\frac{1}{|Z_{,\alpha'}|} \in L^\infty$, then the energy is equivalent to the Sobolev norm. We have

Proposition 2.4.4. *There exists universal polynomials P_1, P_2 with non-negative coefficients so that*

[1] It is enough to assume that the solution satisfies $(Z_{,\alpha'} - 1, Z_t) \in C([0, T], H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}))$ for $s \geq 6$

[2] T(e) is decreasing with respect to e and $C(e)$ is increasing with respect to e

[3] It is enough to assume that the solution satisfies $(Z_{,\alpha'} - 1, Z_t) \in C([0, T], H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}))$ for $s \geq 6$

for smooth solutions to the water wave equation we have

$$\left\|\overline{Z}_{t,\alpha'}\right\|_{H^1} + \left\|\partial_{\alpha'} Z_{,\alpha'}\right\|_{H^{\frac{1}{2}}} \leq P_1\big(\mathcal{E} + \left\|Z_{,\alpha'}\right\|_\infty\big) \quad \textit{and}$$
$$\mathcal{E} \leq P_2\left(\left\|\overline{Z}_{t,\alpha'}\right\|_{H^1} + \left\|\partial_{\alpha'} Z_{,\alpha'}\right\|_{H^{\frac{1}{2}}} + \left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty\right)$$

The rest of the section is devoted to the proof of these statements. The approach of the proof is the same as that of [KW14].

2.4.1 Quantities controlled by the energy

In this section we control all the important terms controlled by the energy E. We will frequently use the estimates proved in the appendix to control the terms. In particular Proposition A.0.6, Proposition A.0.7, Corollary A.0.8 and Proposition A.0.9 are very frequently used.

The energy E is lower order as compared to the energy in Kinsey-Wu [KW14] by half weighted spacial derivative. In particular we do not have control of $D_{\alpha'}\overline{Z}_{tt} \in L^\infty$ which was heavily used in Kinsey-Wu [KW14] but we only have $D_{\alpha'}\overline{Z}_{tt} \in \dot{H}^{\frac{1}{2}}$ (which implies that $D_{\alpha'}\overline{Z}_{tt} \in BMO$ but not in L^∞). Because of this, the energy estimate becomes a little more subtle and we need to prove stronger control of existing terms. For e.g. in [KW14] it is shown that the terms $A_1, \overline{D}_{\alpha'}\overline{Z}_t, b_{\alpha'}, \frac{1}{\left|Z_{,\alpha'}\right|^2}\partial_{\alpha'} A_1 \in L^\infty$ and we show that in fact $A_1, \overline{D}_{\alpha'}\overline{Z}_t, b_{\alpha'}, \frac{1}{\left|Z_{,\alpha'}\right|^2}\partial_{\alpha'} A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$. Most of the terms controlled here in $\dot{H}^{\frac{1}{2}}$ are new. Also as we do not have control of $D_{\alpha'}\overline{Z}_{tt} \in L^\infty$ some of the proofs have to be changed, for e.g. the proof of $\frac{1}{\left|Z_{,\alpha'}\right|^3}\partial_{\alpha'}^2 A_1 \in L^2$ is different from [KW14].

In this section whenever we write $f \in L^2$, what we mean is that there exists a universal polynomial P with nonnegative coefficients such that $\|f\|_2 \leq P(E)$. Similar definitions for $f \in \dot{H}^{\frac{1}{2}}$ or $f \in L^\infty$. We define the norm $\|f\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} = \|f\|_\infty + \|f\|_{\dot{H}^{\frac{1}{2}}}$. We also define two new spaces $\mathcal{C}$ and $\mathcal{W}$:

1. If $w \in L^\infty$ and $|D_{\alpha'}|w \in L^2$, then we say $w \in \mathcal{W}$. Define

$$\|w\|_{\mathcal{W}} = \|w\|_\infty + \||D_{\alpha'}|w\|_2$$

2. If $f \in \dot{H}^{\frac{1}{2}}$ and $f|Z_{,\alpha'}| \in L^2$, then we say $f \in \mathcal{C}$. Define

$$\|f\|_{\mathcal{C}} = \|f\|_{\dot{H}^{\frac{1}{2}}} + \left(1 + \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\right)\|f|Z_{,\alpha'}|\|_2$$

We also define the norm $\|f\|_{\mathcal{W}\cap\mathcal{C}} = \|f\|_{\mathcal{W}} + \|f\|_{\mathcal{C}}$. The reason[4] for the introduction of these spaces is that we will frequently have situations where $f \in \dot{H}^{\frac{1}{2}}, w \in L^\infty$ and we want $fw \in \dot{H}^{\frac{1}{2}}$. We will also have situations where $f \in \dot{H}^{\frac{1}{2}}, g|Z_{,\alpha'}| \in L^2$ and we want $fg|Z_{,\alpha'}| \in L^2$. Clearly these are not true in general but in special cases this can be proved and the following lemma addresses this issue for a majority of the situations we encounter.

Lemma 2.4.5. *The following properties hold for the spaces $\mathcal{W}$ and $\mathcal{C}$*

1. *If $w_1, w_2 \in \mathcal{W}$, then $w_1 w_2 \in \mathcal{W}$. Moreover $\|w_1 w_2\|_{\mathcal{W}} \leq \|w_1\|_{\mathcal{W}}\|w_2\|_{\mathcal{W}}$*
2. *If $f \in \mathcal{C}$ and $w \in \mathcal{W}$, then $fw \in \mathcal{C}$. Moreover $\|fw\|_{\mathcal{C}} \lesssim \|f\|_{\mathcal{C}}\|w\|_{\mathcal{W}}$*
3. *If $f, g \in \mathcal{C}$, then $fg|Z_{,\alpha'}| \in L^2$. Moreover $\|fg|Z_{,\alpha'}|\|_2 \lesssim \|f\|_{\mathcal{C}}\|g\|_{\mathcal{C}}$*

In the lemma and in the definitions of $\mathcal{C}$ and $\mathcal{W}$, the function $\dfrac{1}{|Z_{,\alpha'}|}$ is used as a weight but there is nothing special about this function. We can define similar spaces and prove the lemma for any weight. The only property used of the weight is that $\left\|\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|}\right\|_2 < \infty$. See Proposition A.0.11 in the appendix for more details and for the proof of the lemma. In our case we are able to use the weight $\dfrac{1}{|Z_{,\alpha'}|}$ as $\left\|\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|}\right\|_2$ is controlled by the energy E_σ. We will now start controlling the terms:

1) $A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: Recall that $A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$ and hence

$$\|A_1\|_\infty \leq 1 + \|[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}\|_\infty \lesssim 1 + \|\overline{Z}_{t,\alpha'}\|_2^2$$

by Corollary A.0.8. Similarly by Proposition A.0.7 and Sobolev embedding we have

$$\|A_1\|_{\dot{H}^{\frac{1}{2}}} \leq \left\||\partial_{\alpha'}|^{\frac{1}{2}}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}\right\|_2 \lesssim \left\||\partial_{\alpha'}|^{\frac{1}{2}}Z_t\right\|_{BMO}\|\overline{Z}_{t,\alpha'}\|_2 \lesssim \|\overline{Z}_{t,\alpha'}\|_2^2$$

2) $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2, \partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^2, |D_{\alpha'}|\omega \in L^2$ and hence $\omega \in \mathcal{W}$

[4] Actually these spaces are not really necessary to prove the apriori estimate in the zero surface tension case but become critical in the non-zero surface case and we heavily need control of the terms in $\dot{H}^{\frac{1}{2}}$ which is the main use of these spaces. For e.g. we do not really need to prove $\overline{D}_{\alpha'}\overline{Z}_t \in \dot{H}^{\frac{1}{2}}$ to prove the apriori estimate in the zero surface tension case. However we will need this in the proof of the apriori estimate in the non-zero surface tension case.

Proof: Observe that $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2$ as it is part of the energy $E_{\sigma,0}$. Recall from (2.9) that

$$\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad\qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)$$

Hence $\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|}$ and $|D_{\alpha'}|\omega$ are in L^2. Also as $\|\omega\|_\infty = 1$ and $|D_{\alpha'}|w \in L^2$ we get that $w \in \mathcal{W}$. Now that we have shown that $\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^2$, we can use Lemma 2.4.5 from now on.

3) $D_{\alpha'}Z_t \in L^\infty$

Proof: This is a long proof as it is essentially a cyclic argument. We will assume that $\|D_{\alpha'}Z_t\|_\infty < \infty$ and show that we have $\|D_{\alpha'}Z_t\|_\infty^2 \leq C(E) + C(E)\|D_{\alpha'}Z_t\|_\infty$ and hence by using the inequality $ab \leq \dfrac{a^2}{2} + \dfrac{b^2}{2}$ on the second term we obtain a bound $\|D_{\alpha'}Z_t\|_\infty \leq C(E)$.
Step 1: We see that

$$\begin{aligned}
2\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right) &= (\mathbb{I}-\mathbb{H})(D_{\alpha'}Z_t) + (\mathbb{I}-\mathbb{H})\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\
&= 2D_{\alpha'}Z_t - (\mathbb{I}+\mathbb{H})(D_{\alpha'}Z_t) + (\mathbb{I}-\mathbb{H})\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\
&= 2D_{\alpha'}Z_t + \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} + [Z_t, \mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}
\end{aligned}$$

Hence $\quad \left\|\partial_{\alpha'}\mathbb{P}_A\left(\dfrac{Z_t}{Z_{,\alpha'}}\right)\right\|_\infty \lesssim \|D_{\alpha'}Z_t\|_\infty + \left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_2$

Step 2: Recall from (2.8) that $b_{\alpha'} = (\Phi_t \circ Z)_{\alpha'} + D_{\alpha'}Z_t + Z_t\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right)$. Applying $\mathrm{Re}(\mathbb{I}-\mathbb{H})$ we get

$$\begin{aligned}
b_{\alpha'} &= \mathrm{Re}\left\{(\mathbb{I}-\mathbb{H})\left(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}\right) + [Z_t, \mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} \\
&= \mathrm{Re}\left\{\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} + 2D_{\alpha'}Z_t + [Z_t, \mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}
\end{aligned}$$

Hence $\quad \|b_{\alpha'}\|_{L^\infty} \lesssim \left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_2 + \|D_{\alpha'}Z_t\|_{L^\infty}$

Step 3: Observe that as $(b_{\alpha'} - D_{\alpha'} - \overline{D}_{\alpha'}\overline{Z}_t)$ is real valued we have

$$\begin{aligned}|D_{\alpha'}|(b_{\alpha'} - D_{\alpha'} - \overline{D}_{\alpha'}\overline{Z}_t) &= \text{Re}\left\{\frac{\omega}{Z_{,\alpha'}}(\mathbb{I} - \mathbb{H})\partial_{\alpha'}(b_{\alpha'} - D_{\alpha'} - \overline{D}_{\alpha'}\overline{Z}_t)\right\} \\ &= \text{Re}\big\{\omega(\mathbb{I} - \mathbb{H})D_{\alpha'}(b_{\alpha'} - D_{\alpha'} - \overline{D}_{\alpha'}\overline{Z}_t)\big\} \\ &\quad - \text{Re}\left\{\omega\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}(b_{\alpha'} - D_{\alpha'} - \overline{D}_{\alpha'}\overline{Z}_t)\right\}\end{aligned}$$

From (2.8) we have $b_{\alpha'} = (\Phi_t \circ Z)_{\alpha'} + D_{\alpha'}Z_t + Z_t\Big(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\Big)$. Hence we get

$$\begin{aligned}&(\mathbb{I} - \mathbb{H})D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \\ &= (\mathbb{I} - \mathbb{H})\left\{-D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t + (D_{\alpha'}Z_t)\Big(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\Big) + \frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'}\Big(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\Big)\right\} \\ &= (\mathbb{I} - \mathbb{H})\left\{-D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t + (D_{\alpha'}Z_t)\Big(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\Big)\right\} + \left[\mathbb{P}_A\Big(\frac{Z_t}{Z_{,\alpha'}}\Big), \mathbb{H}\right]\partial_{\alpha'}\Big(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\Big)\end{aligned}$$

Now observe that as $D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t = D_{\alpha'}(\omega^2 D_{\alpha'}\overline{Z}_t)$ we have

$$\begin{aligned}(\mathbb{I} - \mathbb{H})D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t &= (\mathbb{I} - \mathbb{H})\big\{2w(D_{\alpha'}\omega)D_{\alpha'}\overline{Z}_t\big\} + (\mathbb{I} - \mathbb{H})\big\{\omega^2 D_{\alpha'}^2\overline{Z}_t\big\} \\ &= (\mathbb{I} - \mathbb{H})\big\{2w(D_{\alpha'}\omega)D_{\alpha'}\overline{Z}_t\big\} + \left[\frac{\omega^2}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\end{aligned}$$

Hence as $\||D_{\alpha'}|\omega\|_2 \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$ we have the estimate

$$\begin{aligned}&\big\||D_{\alpha'}|(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\big\|_2 \\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\{\|D_{\alpha'}Z_t\|_\infty + \left\|\partial_{\alpha'}\mathbb{P}_A\Big(\frac{Z_t}{Z_{,\alpha'}}\Big)\right\|_\infty + \|b_{\alpha'}\|_\infty\right\} \\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2\big\|\overline{Z}_{t,\alpha'}\big\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|D_{\alpha'}Z_t\|_\infty\end{aligned}$$

Step 4: Recall from (2.12) that $D_t\frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}(b_{\alpha'} - D_{\alpha'}Z_t)$ and hence

$$\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}} = D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) + D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t + (b_{\alpha'} - D_{\alpha'}Z_t)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$$

Now using $[\partial_{\alpha'}, D_t] = b_{\alpha'}\partial_{\alpha'}$ we see that

$$D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) + D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t + -D_{\alpha'}Z_t\Big(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\Big)$$

Hence we see that

$$\begin{aligned}\left\|D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_2 &\lesssim \left\|D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|D_{\alpha'}Z_t\|_\infty + \left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2\|\overline{Z}_{t,\alpha'}\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|D_{\alpha'}Z_t\|_\infty + \left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\end{aligned}$$

Step 5: Observe that $\partial_{\alpha'}(\overline{D}_{\alpha'}\overline{Z}_t)^2 = 2(\overline{Z}_{t,\alpha'})(\overline{D}_{\alpha'}^2\overline{Z}_t)$ and hence as $\overline{D}_{\alpha'}\overline{Z}_t$ decays at infinity we have by integrating

$$\begin{aligned}\|D_{\alpha'}Z_t\|_\infty^2 &\lesssim \left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_2 \\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2\|\overline{Z}_{t,\alpha'}\|_2^2 + \|\overline{Z}_{t,\alpha'}\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|D_{\alpha'}Z_t\|_\infty + \left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|\overline{Z}_{t,\alpha'}\|_2\end{aligned}$$

Hence using $ab \leq \dfrac{a^2}{2\epsilon} + \dfrac{\epsilon b^2}{2}$ for ϵ small on the second term we see that

$$\|D_{\alpha'}Z_t\|_\infty^2 \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2\|\overline{Z}_{t,\alpha'}\|_2^2 + \left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|\overline{Z}_{t,\alpha'}\|_2$$

proving the estimate

4) $\overline{D}_{\alpha'}^2\overline{Z}_t \in L^2, |D_{\alpha'}|^2\overline{Z}_t \in L^2, D_{\alpha'}^2\overline{Z}_t \in L^2$

Proof: As $D_{\alpha'}Z_t \in L^\infty$ by the proof above we already know that $D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$ and hence $\overline{D}_{\alpha'}^2\overline{Z}_t \in L^2$. Now

$$\overline{D}_{\alpha'}^2\overline{Z}_t = \overline{D}_{\alpha'}\big(\omega|D_{\alpha'}|\overline{Z}_t\big) = (\overline{D}_{\alpha'}\omega)|D_{\alpha'}|\overline{Z}_t + \omega^2|D_{\alpha'}|^2\overline{Z}_t$$

Now observe that $|D_{\alpha'}|\omega \in L^2$ and $|D_{\alpha'}|\overline{Z}_t \in L^\infty$ and hence the first term is in L^2. Hence we have $|D_{\alpha'}|^2\overline{Z}_t \in L^2$. A similar argument works for the rest.

5) $\overline{D}_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}, |D_{\alpha'}|\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}, D_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$

Proof: We will first prove $D_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$. Observe that $D_{\alpha'}\overline{Z}_t \in L^\infty$, $|D_{\alpha'}|D_{\alpha'}\overline{Z}_t \in L^2$ and hence we have $D_{\alpha'}\overline{Z}_t \in \mathcal{W}$. Now as $D_{\alpha'}\overline{Z}_t$ is holomorphic i.e. $\mathbb{H}D_{\alpha'}\overline{Z}_t = D_{\alpha'}\overline{Z}_t$ we see that $|\partial_{\alpha'}|D_{\alpha'}\overline{Z}_t = i\partial_{\alpha'}D_{\alpha'}\overline{Z}_t$. Hence we have

$$\left\|D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}^2 = \int\big(\overline{D}_{\alpha'}Z_t\big)\big(|\partial_{\alpha'}|D_{\alpha'}\overline{Z}_t\big)\,d\alpha' = i\int(Z_{t,\alpha'})\big(\overline{D}_{\alpha'}D_{\alpha'}\overline{Z}_t\big)\,d\alpha'$$

Hence $\left\|D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \sqrt{\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}\left\||D_{\alpha'}|D_{\alpha'}\overline{Z}_t\right\|_{L^2}}$. Also as $\left(D_{\alpha'}\overline{Z}_t\right)|Z_{,\alpha'}| = \overline{\omega}\overline{Z}_{t,\alpha'} \in L^2$, we have $D_{\alpha'}\overline{Z}_t \in \mathcal{C}$. Now as $|D_{\alpha'}|\overline{Z}_t = (D_{\alpha'}\overline{Z}_t)\omega$ we obtain

$$\left\||D_{\alpha'}|\overline{Z}_t\right\|_{\mathcal{W}\cap\mathcal{C}} \lesssim \left\|D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}\cap\mathcal{C}}\|\omega\|_{\mathcal{W}}$$

The rest are proved similarly.

6) $\partial_{\alpha'}\mathbb{P}_A\left(\dfrac{Z_t}{Z_{,\alpha'}}\right) \in L^\infty$

Proof: This was already proved when we proved that $D_{\alpha'}Z_t \in L^\infty$ and we have the estimate

$$\left\|\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_\infty \lesssim \|D_{\alpha'}Z_t\|_\infty + \left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

7) $b_{\alpha'} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and $\mathbb{H}(b_{\alpha'}) \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: Recall from (2.8) that $b_{\alpha'} = (\Phi_t \circ Z)_{\alpha'} + D_{\alpha'}Z_t + Z_t\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right)$. Applying $(\mathbb{I} - \mathbb{H})$ we get

$$\begin{aligned}(\mathbb{I} - \mathbb{H})b_{\alpha'} &= (\mathbb{I} - \mathbb{H})\left(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}}\right) + [Z_t, \mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\\ &= \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} + 2D_{\alpha'}Z_t + [Z_t, \mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\end{aligned}$$

Hence $\|(\mathbb{I} - \mathbb{H})b_{\alpha'}\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \lesssim \left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_2 + \|D_{\alpha'}Z_t\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}$

As $b_{\alpha'}$ is real valued, this implies $b_{\alpha'} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and $\mathbb{H}(b_{\alpha'}) \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

8) $|D_{\alpha'}|b_{\alpha'} \in L^2$ and hence $b_{\alpha'} \in \mathcal{W}$

Proof: In the proof of $D_{\alpha'}Z_t \in L^\infty$ we proved the estimate

$$\left\||D_{\alpha'}|(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_2 \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2\left\|\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|D_{\alpha'}Z_t\|_\infty$$

As $|D_{\alpha'}|D_{\alpha'}Z_t \in L^2$ we see that $|D_{\alpha'}|b_{\alpha'} \in L^2$.

9) $|D_{\alpha'}|A_1 \in L^2$ and hence $A_1 \in \mathcal{W}$, $\sqrt{A_1} \in \mathcal{W}$, $\dfrac{1}{A_1} \in \mathcal{W}$, $\dfrac{1}{\sqrt{A_1}} \in \mathcal{W}$

Proof: Observe that
$$\begin{aligned} |D_{\alpha'}|A_1 &= \text{Re}\left\{\frac{\omega}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}A_1\right\} \\ &= \text{Re}\{\omega(\mathbb{I}-\mathbb{H})D_{\alpha'}A_1\} - \text{Re}\left\{\omega\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}A_1\right\} \end{aligned}$$

Recall from (2.4) that $A_1 = iZ_{,\alpha'}F_t\circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'}$. Hence

$$\begin{aligned} (\mathbb{I}-\mathbb{H})D_{\alpha'}A_1 &= i(\mathbb{I}-\mathbb{H})\big((D_{\alpha'}Z_t)\overline{Z}_{t,\alpha'}\big) + i(\mathbb{I}-\mathbb{H})\left(\frac{Z_t}{Z_{,\alpha'}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right) \\ &= i(\mathbb{I}-\mathbb{H})\big((D_{\alpha'}Z_t)\overline{Z}_{t,\alpha'}\big) + i\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\mathbb{H}\right]\partial_{\alpha'}\overline{Z}_{t,\alpha'} \end{aligned}$$

Hence we have

$$\begin{aligned} \||D_{\alpha'}|A_1\|_2 &\lesssim \|(\mathbb{I}-\mathbb{H})D_{\alpha'}A_1\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|A_1\|_\infty \\ &\lesssim \|D_{\alpha'}Z_t\|_\infty\left\|\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_\infty\left\|\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|A_1\|_\infty \end{aligned}$$

Now as $A_1 \in L^\infty$ and $|D_{\alpha'}|A_1 \in L^2$, we have that $A_1 \in \mathcal{W}$. Similarly using the fact that $A_1 \geq 1$, we easily get that $\sqrt{A_1} \in \mathcal{W}$, $\dfrac{1}{A_1} \in \mathcal{W}$, $\dfrac{1}{\sqrt{A_1}} \in \mathcal{W}$

10) $D_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $|D_{\alpha'}|\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $|D_{\alpha'}|\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{C}$, $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{C}$

Proof: As $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \dot{H}^{\frac{1}{2}}$ and as $\sqrt{A_1}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \in L^2$ as $A_1 \in L^\infty$, by the definition of $\mathcal{C}$ we see that $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$. Hence we have

$$\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}} \lesssim \left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\frac{1}{\sqrt{A_1}}\right\|_{\mathcal{W}}$$

and we also have

$$\left\|\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}} \lesssim \left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\|\omega\|_{\mathcal{W}}$$

Similarly $D_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$. Now observe from (2.9)

$$\text{Re}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = |D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \qquad\qquad \text{Im}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i\left(\frac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right)$$

Hence $|D_{\alpha'}|\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{C}$ and $\dfrac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{C}$. Now again using $\omega \in \mathcal{W}$ and Lemma 2.4.5 we easily obtain $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{C}$

11) $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and hence $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in \mathcal{C}$

Proof: Observe that $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 = \text{Re}\left\{\dfrac{\omega^2\overline{\omega}^2}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|A_1\right\}$ and hence we first show that $\dfrac{\overline{\omega}^2}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|A_1 \in L^\infty \cap \mathcal{C}$. Now

$$\frac{\overline{\omega}^2}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|A_1 = (\mathbb{I}-\mathbb{H})\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}A_1\right) - \left[\frac{\overline{\omega}^2}{|Z_{,\alpha'}|}, \mathbb{H}\right]|D_{\alpha'}|A_1$$

Recall from (2.4) that $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'}$. Hence

$$\begin{aligned}(\mathbb{I}-\mathbb{H})\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}A_1\right) &= i(\mathbb{I}-\mathbb{H})\left\{\left(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}^2}\right)\overline{Z}_{t,\alpha'}\right\} + i(\mathbb{I}-\mathbb{H})\left\{Z_t\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\} \\ &= i\left[\frac{Z_{t,\alpha'}}{Z_{,\alpha'}^2}, \mathbb{H}\right]\overline{Z}_{t,\alpha'} + i[Z_t, \mathbb{H}]\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\end{aligned}$$

Hence we have

$$\begin{aligned}\left\|\frac{\overline{\omega}^2}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|A_1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} &\lesssim \|\overline{Z}_{t,\alpha'}\|_2\left(\left\|\partial_{\alpha'}\frac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}^2}\right\|_2 + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\right) \\ &\quad + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\||D_{\alpha'}|A_1\|_2\end{aligned}$$

and as $|D_{\alpha'}|A_1 \in L^2$, we have $\dfrac{\overline{\omega}^2}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|A_1 \in L^\infty \cap \mathcal{C}$. Now using the fact that $\omega \in \mathcal{W}$ and Lemma 2.4.5, we can conclude that $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in L^\infty \cap \mathcal{C}$.

12) $\dfrac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2 A_1 \in L^2$, $|D_{\alpha'}|\left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2$ and hence $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in \mathcal{W}$

Proof: Observe that $|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) = \mathrm{Re}\left\{\frac{\omega^3\overline{\omega}^3}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\}$ and hence it is enough to show that $\frac{\overline{\omega}^3}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2$. Now

$$\begin{aligned}
&\frac{\overline{\omega}^3}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\\
&= (\mathbb{I}-\mathbb{H})\left\{\overline{\omega}^2 D_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\} - \left[\frac{\overline{\omega}^3}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\\
&= (\mathbb{I}-\mathbb{H})\left\{D_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}A_1\right) - 2\overline{\omega}(D_{\alpha'}\overline{\omega})\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\}\\
&\quad - \left[\frac{\overline{\omega}^3}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)
\end{aligned}$$

Recall from (2.4) that $A_1 = iZ_{,\alpha'}F_t\circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'}$. Hence

$$\begin{aligned}
&(\mathbb{I}-\mathbb{H})\left\{D_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}A_1\right)\right\}\\
&= i(\mathbb{I}-\mathbb{H})\left\{D_{\alpha'}\left\{\left(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}^2}\right)\overline{Z}_{t,\alpha'} + Z_t\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\}\right\}\\
&= i(\mathbb{I}-\mathbb{H})\left\{\left(\partial_{\alpha'}\frac{Z_{t,\alpha'}}{Z_{,\alpha'}^2}\right)(D_{\alpha'}\overline{Z}_t) + 2(D_{\alpha'}Z_t)\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\}\\
&\quad + i\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\left\||D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\|_2 &\lesssim \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\left(\|D_{\alpha'}Z_t\|_\infty + \left\|\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_\infty\right)\\
&\quad + \left\|\partial_{\alpha'}\frac{Z_{t,\alpha'}}{Z_{,\alpha'}^2}\right\|_2\left\|D_{\alpha'}\overline{Z}_t\right\|_\infty + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty
\end{aligned}$$

Now the other term is easily controlled

$$\left\|\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2A_1\right\|_2 \lesssim \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty + \left\||D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\|_2$$

As $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in L^\infty$ and $|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2$ we get that $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in \mathcal{W}$.

13) $D_t A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: Recall that $A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$. This implies

$$D_t A_1 = -\text{Im}\big\{[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} - \big[b, Z_t; \overline{Z}_{t,\alpha'}\big]\big\}$$

$$\text{Hence} \quad \|D_t A_1\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \lesssim \big\|\overline{Z}_{t,\alpha'}\big\|_2 \big\|\overline{Z}_{tt,\alpha'}\big\|_2 + \|b_{\alpha'}\|_\infty \big\|\overline{Z}_{t,\alpha'}\big\|_2^2$$

14) $D_{\alpha'}\overline{Z}_{tt} \in \mathcal{C}$, $|D_{\alpha'}|\overline{Z}_{tt} \in \mathcal{C}$, $\overline{D}_{\alpha'}\overline{Z}_{tt} \in \mathcal{C}$ and $D_t D_{\alpha'} Z_t \in \mathcal{C}$. In particular $\overline{Z}_{tt,\alpha'} \in L^2$ and $D_t \overline{Z}_{t,\alpha'} \in L^2$

Proof: From (2.3) we see that

$$\overline{Z}_{tt} - i = -i\frac{A_1}{Z_{,\alpha'}}$$

and hence we have

$$\big\|D_{\alpha'}\overline{Z}_{tt}\big\|_{\mathcal{C}} \lesssim \|\overline{\omega}\|_{\mathcal{W}}^2 \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} A_1\right\|_{\mathcal{C}} + \|A_1\|_{\mathcal{W}} \left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}$$

Similarly we can prove $|D_{\alpha'}|\overline{Z}_{tt} \in \mathcal{C}$ and $\overline{D}_{\alpha'}\overline{Z}_{tt} \in \mathcal{C}$. Now see that

$$D_t D_{\alpha'} Z_t = -(D_{\alpha'} Z_t)^2 + D_{\alpha'} Z_{tt}$$

Hence we have

$$\|D_t D_{\alpha'} Z_t\|_{\mathcal{C}} \lesssim \|D_{\alpha'} Z_t\|_{\mathcal{W}} \|D_{\alpha'} Z_t\|_{\mathcal{C}} + \|D_{\alpha'} Z_{tt}\|_{\mathcal{C}}$$

As $D_{\alpha'}\overline{Z}_{tt} \in \mathcal{C}$ we see that $\overline{Z}_{tt,\alpha'} \in L^2$. Now $D_t \overline{Z}_{t,\alpha'} = -b_{\alpha'}\overline{Z}_{t,\alpha'} + \overline{Z}_{tt,\alpha'}$ and as $b_{\alpha'} \in L^\infty$ we that $D_t \overline{Z}_{t,\alpha'} \in L^2$.

15) $J_0 = D_t(b_{\alpha'} - D_{\alpha'} Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and hence $D_t b_{\alpha'} \in \dot{H}^{\frac{1}{2}}, \partial_{\alpha'} D_t b \in \dot{H}^{\frac{1}{2}}$

Proof: Recall from (2.8) that $b_{\alpha'} = (\Phi_t \circ Z)_{\alpha'} + D_{\alpha'} Z_t + Z_t\Big(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\Big)$. Hence

$$b_{\alpha'} - D_{\alpha'} Z_t - \overline{D}_{\alpha'}\overline{Z}_t = (\Phi_t \circ Z)_{\alpha'} + Z_t\Big(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\Big) - \overline{D}_{\alpha'}\overline{Z}_t$$

Observe that $(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)$ is real valued and hence by applying $\text{Re}(\mathbb{I} - \mathbb{H})$ we get

$$b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t = \text{Re}\left\{[Z_t, \mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{t,\alpha'}\right\}$$

Applying D_t we obtain

$$\begin{aligned}
&D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \\
&= \text{Re}\left\{[Z_{tt}, \mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + [Z_t, \mathbb{H}]\left(\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right) - \left[b, Z_t; \partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right] - \left[D_t\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{t,\alpha'} \right. \\
&\qquad \left. - \left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{tt,\alpha'} + \left[b, \frac{1}{\overline{Z}_{,\alpha'}}; \overline{Z}_{t,\alpha'}\right]\right\}
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} &\lesssim \left\|\overline{Z}_{tt,\alpha'}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\|_2 \\
&\quad + \|b_{\alpha'}\|_\infty\left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2
\end{aligned}$$

As $D_tD_{\alpha'}Z_t \in \mathcal{C}$ and $D_t\overline{D}_{\alpha'}\overline{Z}_t \in \mathcal{C}$, we get that $D_tb_{\alpha'} \in \dot{H}^{\frac{1}{2}}$. Now as $\partial_{\alpha'}D_tb = (b_{\alpha'})^2 + D_tb_{\alpha'}$ we get $\|\partial_{\alpha'}D_tb\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|b_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\|b_{\alpha'}\|_\infty + \|D_tb_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}$

16) $Q_0 \in \mathcal{W}$

Proof: Recall from (2.14) that

$$Q_0 = (b_{\alpha'} - D_{\alpha'}Z_t)^2 - (\overline{D}_{\alpha'}\overline{Z}_t)^2 - \frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1$$

Hence we see that

$$\|Q_0\|_{\mathcal{W}} \lesssim (\|b_{\alpha'}\|_{\mathcal{W}} + \|D_{\alpha'}Z_t\|_{\mathcal{W}})^2 + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\mathcal{W}}$$

17) $R_0 \in L^2$

Proof: Recall from (2.16) that

$$
\begin{aligned}
R_0 = &\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)(J_0 + Q_0) + D_{\alpha'}Q_0 - b_{\alpha'}\left(\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}} + D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\\
&- (D_t b_{\alpha'})\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - 2iA_1\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right)\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\\
&- i\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)
\end{aligned}
$$

Hence by writing $D_t b_{\alpha'} = J_0 + D_t D_{\alpha'}Z_t + D_t\overline{D}_{\alpha'}\overline{Z}_t$ we see that

$$
\begin{aligned}
&\|R_0\|_2\\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2(\|J_0\|_\infty + \|Q_0\|_\infty) + \|D_{\alpha'}Q_0\|_2 + \|b_{\alpha'}\|_\infty\left(\left\|\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\right)\\
&\quad + \|J_0\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left(\|D_t D_{\alpha'}Z_t\|_{\mathcal{C}} + \left\|D_t\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}}\right)\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\\
&\quad + \|A_1\|_\infty\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}} + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2
\end{aligned}
$$

18) $(\mathbb{I} - \mathbb{H})D_t^2\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \in L^2$

Proof: For a function f satisfying $\mathbb{P}_A f = 0$ we have

$$
\begin{aligned}
(\mathbb{I} - \mathbb{H})D_t^2 f &= [D_t, \mathbb{H}]D_t f + D_t[D_t, \mathbb{H}]f\\
&= [b, \mathbb{H}]\partial_{\alpha'}D_t f + D_t[b, \mathbb{H}]\partial_{\alpha'}f\\
&= 2[b, \mathbb{H}]\partial_{\alpha'}D_t f + [D_t b, \mathbb{H}]\partial_{\alpha'}f - [b, b; \partial_{\alpha'}f]
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\left\|(\mathbb{I} - \mathbb{H})D_t^2\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2\\
&\lesssim \|b_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \|\partial_{\alpha'}D_t b\|_{\dot{H}^{\frac{1}{2}}}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \|b_{\alpha'}\|_\infty^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2
\end{aligned}
$$

19) $(\mathbb{I} - \mathbb{H})\left\{i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} \in L^2$

Proof: Observe that

$$(\mathbb{I}-\mathbb{H})\left\{i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} = i\left[\frac{A_1}{|Z_{,\alpha'}|^2},\mathbb{H}\right]\partial_{\alpha'}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

Hence we have

$$\begin{aligned}
&\left\|(\mathbb{I}-\mathbb{H})\left\{i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\right\|_2 \\
&\lesssim \left(\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\mathcal{C}} + \|A_1\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\right)\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2
\end{aligned}$$

20) $|D_{\alpha'}|J_0 \in L^2$

Proof: As J_0 is real valued, we see that

$$\begin{aligned}
|D_{\alpha'}|J_0 &= \mathrm{Re}\left\{\frac{\omega}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}J_0\right\} \\
&= \mathrm{Re}\{\omega(\mathbb{I}-\mathbb{H})D_{\alpha'}J_0\} - \mathrm{Re}\left\{\omega\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}J_0\right\}
\end{aligned}$$

From equation (2.15) we have

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = D_{\alpha'}J_0 + R_0$$

Applying $(\mathbb{I}-\mathbb{H})$ to the above equation we obtain the estimate

$$\begin{aligned}
\||D_{\alpha'}|J_0\|_2 \lesssim{}& \left\|D_t^2\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 + \left\|(\mathbb{I}-\mathbb{H})\left\{i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\right\|_2 \\
&+ \|R_0\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|J_0\|_\infty
\end{aligned}$$

2.4.2 Closing the energy

We are now ready to close the energy E. To simplify the calculations we will use the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a universal non-negative polynomial P with $|a(t)-b(t)| \leq P(E(t))$. Observe that $\approx$ is an equivalence relation. With this notation proving Theorem 2.4.2 is equivalent to showing $\frac{dE(t)}{dt} \approx 0$.

Lemma 2.4.6. *Let $T > 0$ and let $f, g : \mathbb{R} \times [0, T) \to \mathbb{C}$ be C^1 functions decaying sufficiently fast at infinity. Then we have*

1. $\displaystyle \frac{d}{dt}\int g\, d\alpha' = \int D_t g\, d\alpha' + \int b_{\alpha'} g\, d\alpha'$ *and hence*

$$\left|\frac{d}{dt}\int |f|^2\, d\alpha'\right| \lesssim \|f\|_2 \|D_t f\|_2 + \|b_{\alpha'}\|_\infty \|f\|_2^2$$

2. $\displaystyle \left|\frac{d}{dt}\int (|\partial_{\alpha'}|\bar{f}) f\, d\alpha' - 2\mathrm{Re}\left\{\int (|\partial_{\alpha'}|\bar{f}) D_t f\, d\alpha'\right\}\right| \lesssim \left(\|b_{\alpha'}\|_\infty + \|\mathbb{H} b_{\alpha'}\|_\infty\right) \|f\|_{\dot{H}^{\frac{1}{2}}}^2$

Proof. The first identity follows directly from the fact that $D_t = \partial_t + b\partial_{\alpha'}$. Now

$$\begin{aligned}
\frac{d}{dt}\int (|\partial_{\alpha'}|\bar{f}) f\, d\alpha' &= 2\mathrm{Re}\left\{\int (|\partial_{\alpha'}|\bar{f}) \partial_t f\, d\alpha'\right\} \\
&= 2\mathrm{Re}\left\{\int (|\partial_{\alpha'}|\bar{f}) D_t f\, d\alpha'\right\} - 2\mathrm{Re}\left\{\int (|\partial_{\alpha'}|\bar{f})(b\partial_{\alpha'} f)\, d\alpha'\right\}
\end{aligned}$$

Hence we need to control the second term. Let $f^h, f^h_{\alpha'}, b^h, b^h_{\alpha'} : \bar{P}_- \to \mathbb{C}$ be the harmonic extension of $f, f_{\alpha'}, b, b_{\alpha'}$ respectively to the lower half plane and let us denote the Lebesgue measure on P_- by $d\mu$. Observe that $|\partial_{\alpha'}| f = \hat{n} \cdot \nabla f^h$ where $\hat{n}$ is the outward pointing unit normal. Hence by using the divergence theorem we have

$$\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}} (|\partial_{\alpha'}|\bar{f})(b\partial_{\alpha'} f)\, d\alpha' &= 2\mathrm{Re}\int_{P_-} \nabla \cdot \left((\nabla \bar{f}^h) b^h f^h_{\alpha'}\right) d\mu \\
&= 2\mathrm{Re}\int_{P_-} (\nabla \bar{f}^h \cdot \nabla b^h) f^h_{\alpha'}\, d\mu + 2\mathrm{Re}\int_{P_-} (\nabla \bar{f}^h \cdot \nabla f^h_{\alpha'}) b^h\, d\mu \\
&= 2\mathrm{Re}\int_{P_-} (\nabla \bar{f}^h \cdot \nabla b^h) f^h_{\alpha'}\, d\mu - \int_{P_-} \left|\nabla f^h\right|^2 b^h_{\alpha'}\, d\mu
\end{aligned}$$

where in the last step we used the fact that b is real valued. Hence we have

$$\begin{aligned}
\left|2\mathrm{Re}\int_{\mathbb{R}} (|\partial_{\alpha'}|\bar{f})(b\partial_{\alpha'} f)\, d\alpha'\right| &\lesssim \left\|\nabla b^h\right\|_{L^\infty(P_-)} \int_{P_-} \left|\nabla f^h\right|^2 d\mu \\
&\lesssim \left(\|b_{\alpha'}\|_\infty + \|\mathbb{H} b_{\alpha'}\|_\infty\right) \|f\|_{\dot{H}^{\frac{1}{2}}}^2
\end{aligned}$$

□

This lemma helps us move the time derivative inside the integral as a material derivative. We will now control the time derivative of the energy.

We will now close the energy. The first two terms are controlled by directly taking the time derivative and the last two terms are controlled by using the quasilinear equation (2.15). Using

Lemma 2.4.6 we see that

$$\frac{d}{dt}\left\|\overline{Z}_{t,\alpha'}\right\|_2^2 \lesssim \|b_{\alpha'}\|_\infty \left\|\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\overline{Z}_{t,\alpha'}\right\|_2 \left\|D_t\overline{Z}_{t,\alpha'}\right\|_2 \lesssim P(E)$$

and similarly

$$\frac{d}{dt}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 \lesssim \|b_{\alpha'}\|_\infty \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim P(E_\sigma)$$

Now we control the last two terms of E. Observe that the terms are of the form

$$\|D_t f\|_2^2 + \left\|\sqrt{A_1}\frac{f}{|Z_{,\alpha'}|}\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

Where $f = \partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}$ and note that $\mathbb{P}_H f = f$. We will simplify the time derivative of each of the terms individually before combining them.

1) As $b_{\alpha'} \in L^\infty$ we have from Lemma 2.4.6

$$\frac{d}{dt}\int |D_t f|^2 \, d\alpha' \approx 2\text{Re}\int (D_t^2 f)(D_t\bar{f}) \, d\alpha'$$

2) By using Lemma 2.4.6 we have

$$\frac{d}{dt}\int \left| |\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right|^2 d\alpha' \approx 2\text{Re}\int \left\{|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right\} D_t\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f}\right) d\alpha'$$

Observe that

$$D_t\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f}\right) = \left\{\frac{D_t A_1}{2A_1} + b_{\alpha'} - \text{Re}(D_{\alpha'}Z_t)\right\}\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f} + \frac{\sqrt{A_1}}{|Z_{,\alpha'}|}D_t\bar{f}$$

We note that for $f = \partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|}$ we have $\dfrac{f}{|Z_{,\alpha'}|} \in \mathcal{C}$. Hence

$$\begin{aligned}&\left\|\left\{\frac{D_t A_1}{2A_1} + b_{\alpha'} - \text{Re}(D_{\alpha'}Z_t)\right\}\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f}\right\|_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim \left\|\frac{f}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\sqrt{A_1}\right\|_{\mathcal{W}}\left\{\|D_t A_1\|_{\mathcal{W}}\left\|\frac{1}{A_1}\right\|_{\mathcal{W}} + \|b_{\alpha'}\|_{\mathcal{W}} + \|D_{\alpha'}Z_t\|_{\mathcal{W}}\right\}\end{aligned}$$

Hence we have

$$\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right|^2 d\alpha' \approx 2\text{Re}\int\left\{\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right\}(D_t\bar{f})\,d\alpha'$$

We simplify further using $|\partial_{\alpha'}| = i\mathbb{H}\partial_{\alpha'}$ and $\mathbb{H}f = f$

$$\begin{aligned}
&\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\\
&= i\left[\frac{\sqrt{A_1}}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right) + i\mathbb{H}\left\{\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)f + \frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right\}\\
&= i\left[\frac{\sqrt{A_1}}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right) + i\mathbb{H}\left\{\frac{1}{2}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)f + A_1\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right)f\right\}\\
&\quad - i\left[\frac{A_1}{|Z_{,\alpha'}|^2},\mathbb{H}\right]\partial_{\alpha'}f + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f
\end{aligned}$$

Hence we have the estimate

$$\begin{aligned}
&\left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right) - i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right\|_2\\
&\lesssim \left(\||D_{\alpha'}|A_1\|_2 + \left\|\sqrt{A_1}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\right)\left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right\|_{\dot{H}^{\frac{1}{2}}}\\
&\quad + \|A_1\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\frac{f}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}} + \|f\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}
\end{aligned}$$

As $D_t f \in L^2$ this shows that

$$\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right|^2 d\alpha' \approx 2\text{Re}\int\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right)(D_t\bar{f})\,d\alpha'$$

3) Combining the terms we obtain

$$\frac{d}{dt}\left\{\|D_t f\|_2^2 + \left\|\sqrt{A_1}\frac{f}{|Z_{,\alpha'}|}\right\|_{\dot{H}^{\frac{1}{2}}}^2\right\} \approx 2\text{Re}\int\left(D_t^2 f + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right)(D_t\bar{f})\,d\alpha'$$

For $f = \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}$ we obtain from (2.15)

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = D_{\alpha'}J_0 + R_0$$

We have already shown that $D_{\alpha'}J_0 \in L^2$ and $R_0 \in L^2$ and hence the integral is controlled. This completes the proof of Theorem 2.4.2

2.4.3 Equivalence of the energy and relation to Sobolev norm

Proof of Proposition 2.4.3. From the energy estimate we already have $\mathcal{E} \leq P_2(E)$ and hence we only need to show $E \leq P_1(\mathcal{E})$. We will now say that $f \in L^2$ if there exists a polynomial P such that $\|f\|_2 \leq P(\mathcal{E})$ in analogy to the notation in §2.4. Similar notation for the other spaces defined there. We now control terms

1. From following the proof in §2.4.1 we have $A_1 \in L^\infty$, $\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \in L^2$ and $\omega \in \mathcal{W}$.

2. We now show that $D_{\alpha'}\overline{Z}_t \in L^\infty$. Observe that

$$\partial_{\alpha'}(D_{\alpha'}\overline{Z}_t)^2 = 2(\overline{Z}_{t,\alpha'})(D_{\alpha'}^2\overline{Z}_t) = 2(\overline{Z}_{t,\alpha'})\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)D_{\alpha'}\overline{Z}_t + 2(\overline{Z}_{t,\alpha'})\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)$$

   Hence we have

$$\left\|D_{\alpha'}\overline{Z}_t\right\|_\infty^2 \leq 2\left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|D_{\alpha'}\overline{Z}_t\right\|_\infty + 2\left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2$$

   Now using the inequality $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ on the first term, we obtain $D_{\alpha'}\overline{Z}_t \in L^\infty$.

3. Following the apriori estimate we now have $|D_{\alpha'}|A_1 \in L^2$ and hence $\sqrt{A_1} \in \mathcal{W}$ and we also have $D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$. Hence we have

$$\left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\sqrt{A_1}\right\|_{\mathcal{W}}\|w\|_{\mathcal{W}}\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}$$

   and hence $\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \in \dot{H}^{\frac{1}{2}}$

4. Now following the proof of $D_{\alpha'}Z_t \in L^\infty$ in the apriori estimate we see that $D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \in L^2$. We also see that

$$D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) + D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t + -D_{\alpha'}Z_t\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

Hence we have

$$\begin{aligned}&\left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \\ &\lesssim \left\|D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \in L^2\right\|_2 + \left\|D_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_2 + \|D_{\alpha'}Z_t\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\end{aligned}$$

This completes the proof

□

Proof of Proposition 2.4.4. We prove each part separately:

1. Assume that $\mathcal{E} + \|Z_{,\alpha'}\|_\infty < \infty$. It is easy to see that $\overline{Z}_{t,\alpha'} \in H^1$. Observe that

$$\||D_{\alpha'}|Z_{,\alpha'}\|_2 \lesssim \|Z_{,\alpha'}\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

Hence $Z_{,\alpha'} \in \mathcal{W}$. Hence we see that

$$\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|Z_{,\alpha'}\|_{\mathcal{W}}\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}$$

Now we see that

$$\|\partial_{\alpha'}Z_{,\alpha'}\|_{\dot{H}^{\frac{1}{2}}} = \left\|Z_{,\alpha'}^2\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|Z_{,\alpha'}\|_\infty^2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} + \|Z_{,\alpha'}\|_\infty\|\partial_{\alpha'}Z_{,\alpha'}\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

2. Now assume that $\overline{Z}_{t,\alpha'} \in H^1, \partial_{\alpha'}Z_{,\alpha'} \in H^{\frac{1}{2}}$ and $\dfrac{1}{Z_{,\alpha'}} \in L^\infty$. We easily see that $\overline{Z}_{t,\alpha'} \in L^2, \dfrac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$ and $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2$. We see that

$$\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty^2\|\partial_{\alpha'}Z_{,\alpha'}\|_{\dot{H}^{\frac{1}{2}}} + \left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|\partial_{\alpha'}Z_{,\alpha'}\|_2$$

and from this we obtain

$$\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}$$

□

2.5 Main result on rigidity of singularities

We do not prove the existence of singular solutions but use the result of Wu namely Theorem 3.4 of [Wu15]. We will use the same notation used there and we now recall the notation and the existence result proved there. We will first describe the notion of generalized solution introduced in [Wu15].

Generalized Solution:

Now let $z' = x' + iy'$ where $x', y' \in \mathbb{R}$. Let (Z, Z_t) be a solution of (2.3) and let Ψ, F be holomorphic functions on P_- continuous on $\overline{P}_-$ such that

$$Z(\alpha', t) = \Psi(\alpha', t) \quad \overline{Z}_t(\alpha', t) = F(\alpha', t)$$

Hence Ψ and F are the shape of the domain and conjugate of the velocity in Riemann mapping coordinates respectively. We also need the pressure and so define $\mathfrak{B} : \overline{P}_- \to \mathbb{R}$ such that

$$\Delta\mathfrak{B} = -2|F_{z'}|^2 \quad \text{on } P_- \qquad \text{and } \mathfrak{B} = 0 \quad \text{on } \partial P_-$$

If these quantities satisfy the Euler equation in Riemann mapping coordinates, namely

$$\Psi_{z'}F_t - \Psi_t F_{z'} + \overline{F}F_{z'} - i\Psi_{z'} = -(\partial_{x'} - i\partial_{y'})\mathfrak{B} \quad \text{on } P_-$$

then the triple $(\Psi, F, \mathfrak{B})$ is said to be generalized solution to the gravity water wave equation. The main reason for the introduction of such a definition of the solution is that it allows self-intersecting solutions. This is because if $\Psi : P_- \to \mathbb{C}$ is one-one, then it is invertible and gives rise to a physical solution and we can obtain a solution to the Euler equation (1.1). If not then, it still makes sense to talk about this solution in a mathematical sense but it loses its physical meaning. Note that the quasilinear equation is solved in Riemann mapping coordinates and hence mathematically we are solving for the variables $(\Psi, F, \mathfrak{B})$ and the issue of the invertibility of Ψ is a separate one which has nothing to do with proving energy estimates or proving existence in terms of the variables $(\Psi, F, \mathfrak{B})$.

Existence of Singular solutions

We now describe the existence result of Wu namely Theorem 3.4 in [Wu15] which proves the existence of singular solutions to the water wave equation. Define the energy

$$\begin{aligned}\mathcal{E}_1(t) &= \sup_{y'<0}\left\|\frac{1}{\Psi_{z'}}(t)\right\|^2_{L^\infty(\mathbb{R},dx')} + \sup_{y'<0}\left\|\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)(t)\right\|^2_{L^2(\mathbb{R},dx')}\\ &\quad + \sup_{y'<0}\left\|\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)\right)(t)\right\|^2_{L^2(\mathbb{R},dx')} + \sup_{y'<0}\|F_{z'}(t)\|^2_{L^2(\mathbb{R},dx')}\\ &\quad + \sup_{y'<0}\left\|\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}F_{z'}\right)(t)\right\|^2_{L^2(\mathbb{R},dx')} + \sup_{y'<0}\left\|\frac{1}{\Psi_{z'}^2}\partial_{z'}\left(\frac{1}{\Psi_{z'}}F_{z'}\right)(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R},dx')}\end{aligned}$$

Observe that if the interface is smooth enough then this energy is equal to the energy

$$\begin{aligned}\mathfrak{E}(t) &= \left\|\frac{1}{Z_{,\alpha'}}(t)\right\|^2_\infty + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\right\|^2_2 + \left\|D^2_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\right\|^2_2 + \left\|\bar{Z}_{t,\alpha'}(t)\right\|^2_2 + \left\|D^2_{\alpha'}\bar{Z}_t(t)\right\|^2_2\\ &\quad + \left\|\frac{1}{Z_{,\alpha'}}D^2_{\alpha'}\bar{Z}_t(t)\right\|^2_{\dot{H}^{\frac{1}{2}}}\end{aligned}$$

where all these quantities are defined on the boundary $\Sigma(t)$. In [KW14] the apriori energy estimate is proved for the energy $\mathfrak{E}(t)$ for smooth enough interfaces. The reason for the introduction of the energy $\mathcal{E}_1$ is that this equivalence between $\mathfrak{E}$ and $\mathcal{E}_1$ is not at all clear for singular domains.

The initial data $(\Psi, F, \mathfrak{B})$ is chosen so that $\Psi(\cdot,0), F(\cdot,0) : P_- \to \mathbb{C}$ are holomorphic with the range of Ψ namely $\Omega(0) = \Psi(P_-)$ is a domain with the boundary $\partial\Omega(0)$ being a Jordan curve and $\lim_{z'\to\infty}\Psi_{z'} = 1$ (i.e. the interface tends to flat at infinity). $\mathfrak{B}(\cdot,0) : \bar{P}_- \to \mathbb{R}$ is chosen so that it is the unique solution to

$$\Delta\mathfrak{B} = -2|F_{z'}|^2 \quad \text{on } P_- \qquad \text{and } \mathfrak{B} = 0 \quad \text{on } \partial P_-$$

Also assume that

$$c_0 = \sup_{y'<0}\|F(x'+iy',0)\|_{L^2(\mathbb{R},dx')} + \sup_{y'<0}\left\|\frac{1}{\Psi_{z'}(x'+iy',0)} - 1\right\|_{L^2(\mathbb{R},dx')} < \infty$$

For such initial data Wu proved the following existence result.

Theorem 2.5.1 ([Wu15]). *Let the initial data $(\Psi, F, \mathfrak{B})(0)$ be as described above with $\mathcal{E}_1(0) < \infty$. Then there exists a time $T_0 > 0$ depending only on $\mathcal{E}_1(0)$ such that on $[0, T_0]$ the initial value problem of the gravity water wave equation* (2.3) *has a generalized solution $(\Psi, F, \mathfrak{B})(t)$ with the following properties:*

1. *$\Psi(\cdot,t)$ is holomorphic on P_- for each fixed $t \in [0, T_0]$, Ψ and $\frac{1}{\Psi_{z'}}$ are continuous on $\bar{P}_- \times$*

$[0, T_0]$ and Ψ is continuous differentiable on $P_- \times [0, T_0]$.

2. *$F(\cdot, t)$ is holomorphic on P_- for each fixed $t \in [0, T_0]$, F is continuous on $\overline{P}_- \times [0, T_0]$ and F is continuous differentiable on $P_- \times [0, T_0]$*

3. *$\mathfrak{B}$ is continuous on $\overline{P}_- \times [0, T_0]$ and $\mathfrak{B}$ is continuous differentiable with respect to the spacial variables on $P_- \times [0, T_0]$*

4. *For all $t \in [0, T_0]$ we have $\mathcal{E}_1(t) < \infty$ and*

$$\sup_{y'<0} \|F(x' + iy', t)\|_{L^2(\mathbb{R}, dx')} + \sup_{y'<0} \left\| \frac{1}{\Psi_{z'}(x' + iy', t)} - 1 \right\|_{L^2(\mathbb{R}, dx')} < \infty$$

The generalized solution gives rise to a solution $(\overline{\mathbf{v}}, P) = (F \circ \Psi^{-1}, \mathfrak{B} \circ \Psi^{-1})$ of the water wave equation (1.1) so long as $\Sigma(t) = \{Z = \Psi(\alpha', t) \,|\, \alpha' \in \mathbb{R}\}$ is a Jordan curve. Moreover if the initial interface is chort-arc with chord-arc constant δ, then there exists $T_1 > 0$ depending only on $\mathcal{E}_1(0)$ such that the interface remains chord-arc on the time interval $\left[0, \min\left\{T_0, \frac{\delta}{T_1}\right\}\right]$.

This result is proved by first mollifying the initial data, proving that the equation has a solution to the mollified initial data in the time interval $[0, T_0]$ where T_0 is independent of the mollification parameter ϵ and then letting $\epsilon \to 0$. This existence in the time interval $[0, T_0]$ to the smooth initial data uses the apriori estimate [KW14]. As we will need some elements of the proof we describe some of the notation and facts proved in the proof of this result.

Let the initial data be $(\Psi, F, \mathfrak{B})(0)$ and let $0 < \epsilon \leq 1$. Define

$$Z^\epsilon(\alpha', 0) = \Psi(\alpha' - \epsilon i, 0), \quad \overline{Z}_t^\epsilon(\alpha', 0) = F(\alpha' - \epsilon i, 0), \quad h^\epsilon(\alpha, 0) = \alpha$$
$$F^\epsilon(z', 0) = F(z' - \epsilon i, 0), \quad \Psi^\epsilon(z', 0) = \Psi(z' - \epsilon i, 0)$$

Similarly define $b^\epsilon = h_t^\epsilon \circ (h^\epsilon)^{-1}$. Then as part of the existence result of the smooth solution $(Z^\epsilon, \overline{Z}_t^\epsilon)(t)$ in $[0, T_0]$ it is shown that there exists a constant $C = C(\mathcal{E}_1(0))$ such that for all $t \in [0, T_0]$ and for all $0 < \epsilon \leq 1$

$$\|b_{\alpha'}^\epsilon(t)\|_\infty + \left\| \frac{h_{t\alpha}^\epsilon}{h_\alpha^\epsilon}(t) \right\|_\infty + \left\| \frac{z_{t\alpha}^\epsilon}{z_\alpha^\epsilon}(t) \right\|_\infty + \left\| \frac{1}{(Z_{,\alpha'}^\epsilon)^2} \partial_{\alpha'} Z_{t,\alpha'}^\epsilon(t) \right\|_{L^2(\mathbb{R}, d\alpha')} \leq C$$

Note that these terms are controlled as part of the energy estimate as was shown in §2.4. Using these it can be easily seen that there exists constants $0 < c_1, c_2 < \infty$ depending only on $\mathcal{E}_1(0)$ such that for all $(\alpha, t) \in \mathbb{R} \times [0, T_0]$ and for all $0 < \epsilon \leq 1$ we have

$$\begin{aligned} c_1 \leq |h_\alpha^\epsilon(\alpha, t)| \leq c_2 \\ \text{and } c_1|z_\alpha^\epsilon|(\alpha, 0) \leq |z_\alpha^\epsilon|(\alpha, t) \leq c_2|z_\alpha^\epsilon|(\alpha, 0) \\ \text{and } |z_{t\alpha}^\epsilon|(\alpha, t) \leq c_2|z_\alpha^\epsilon|(\alpha, 0) \end{aligned}$$

If $U \subset \mathbb{R}^n$ then we will use the notation $f_n \Rightarrow f$ on U to mean uniform convergence on compact subsets of U. As part of the proof of Theorem 2.5.1 it is shown that there exists a function $h : \mathbb{R} \times [0, T_0] \to \mathbb{R}$ so that $h(\cdot, t) : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and

$$h^\epsilon \Rightarrow h \quad \text{and} \quad (h^\epsilon)^{-1} \Rightarrow h^{-1} \qquad \text{on } \mathbb{R} \times [0, T_0]$$

as $\epsilon = \epsilon_j \to 0$. In the proof of Theorem 2.5.1 a subsequence is taken $\epsilon_j \to 0$ and for convenience it is replaced by ϵ. In the proof of our result Theorem 2.5.3 we will also use this notation and we will freely take a subsubsequence of the subsequence used in [Wu15] as it does not affect the result. As part of the proof it is also shown that there exists functions $u, w, q : \mathbb{R} \times [0, T_0] \to \mathbb{C}$ continuous and bounded on $\mathbb{R} \times [0, T_0]$ such that

$$z^\epsilon \Rightarrow z, \quad \frac{h_\alpha^\epsilon}{z_\alpha^\epsilon} \Rightarrow u, \quad z_t^\epsilon \Rightarrow w, \quad z_{tt}^\epsilon \Rightarrow q \qquad \text{on } \mathbb{R} \times [0, T_0]$$

as $\epsilon = \epsilon_j \to 0$. This gives us

$$Z^\epsilon \Rightarrow Z, \quad \frac{1}{Z_{,\alpha'}^\epsilon} \Rightarrow u \circ h^{-1}, \quad Z_t^\epsilon \Rightarrow w \circ h^{-1}, \quad Z_{tt}^\epsilon \Rightarrow q \circ h^{-1} \qquad \text{on } \mathbb{R} \times [0, T_0]$$

and we have $Z(\alpha', t) = \Psi(\alpha', t)$ for $\alpha' \in \mathbb{R}$ (Recall that $\Psi : \overline{P}_- \times [0, T_0] \to \mathbb{C}$) and also that $z(\alpha, t) = Z(h(\alpha, t), t)$. Hence $z(\alpha, t)$ is the Lagrangian parametrization of the interface and $Z(\alpha', t)$ is the parametrization in Riemann mapping coordinates. Now observe that

$$z^\epsilon(\alpha, t_2) - z^\epsilon(\alpha, t_1) = \int_{t_1}^{t_2} z_t^\epsilon(\alpha, s)\, ds$$

So by passing to the limit $\epsilon \to 0$, we obtain

$$z(\alpha, t_2) - z(\alpha, t_1) = \int_{t_1}^{t_2} w(\alpha, s)\, ds$$

As z is the position, hence w is indeed the velocity of the particles on the interface. In the same way q is the acceleration of the particles on the interface in Lagrangian coordinates.

Main result:

In all that follows let $T_0 > 0$ and let (F, Ψ, β) be a generalized solution of the water wave equation on the time interval $[0, T_0]$ given by Theorem 2.5.1. Define:

$$\text{Singular set } = S(t) = \left\{ \alpha' \in \mathbb{R} \,\middle|\, \frac{1}{\Psi_{z'}}(\alpha', 0, t) = 0 \right\}$$
$$\text{Non-Singular set} = NS(t) = \mathbb{R} \backslash S(t)$$

Note that the definition makes sense as $\frac{1}{\Psi_{z'}}$ is continuous on $\overline{P}_{-} \times [0, T_0]$. We will also identify $S(t)$ and $NS(t)$ as subsets of $\overline{P}_{-}$ so that it is meaningful to talk of sets such as $\overline{P}_{-} \backslash S(t)$. An important observation is that $S(t)$ is a set of measure zero. This is because it is the boundary value of a bounded holomorphic function $\frac{1}{\Psi_{z'}}$ and hence by the uniqueness theorem of F. and M. Riesz (see Theorem 17.13 in [Rud87]) its zero set on the boundary is of measure zero. Hence given any $\alpha \in S(t)$, there always exists a sequence $\alpha_n \in NS(t)$ such that $\alpha_n \to \alpha$. We have a description of dynamics of this set

Lemma 2.5.2. $S(t) = \{h(\alpha, t) \in \mathbb{R} \,|\, \alpha \in S(0)\}$ *and* $NS(t) = \{h(\alpha, t) \in \mathbb{R} \,|\, \alpha \in NS(0)\}$

This lemma says that the singularities propagate via the Lagrangian flow i.e. particles at singularities stay at the singularities. In particular the singularities are preserved in the sense that the interface doesn't smooth out or any new singularities form during the time in which $\mathcal{E}_1(t) < \infty$. This important fact is a simple consequence of the nature of the energy $\mathcal{E}_1$. We are now ready to state our main result.

Theorem 2.5.3. *Let* (F, Ψ, β) *be a solution as given by Theorem 2.5.1 and let* $\alpha \in S(0)$. *Then*

1. *The acceleration at the singularities* $q(\alpha, t) = -i$ *for all* $t \in [0, T_0]$

2. *Let* $\alpha_n \in NS(0)$ *be any sequence such that* $\alpha_n \to \alpha$. *Then*

$$\frac{\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(h(\alpha_n, t), t))}{\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(\alpha_n, 0)} \to 1 \quad \textit{as } n \to \infty$$

Corollary 2.5.4. *Let* (F, Ψ, β) *be a solution as given by Theorem 2.5.1 and assume that the there exists* $N \geq 1$ *isolated singularities of initial interface at locations* $\alpha_n \in S(0)$ *for* $1 \leq n \leq N$ *in conformal coordinates. Also assume that there exists unit vectors* $\beta_n, \gamma_n \in S^1$ *such that*

$$\lim_{\alpha \to \alpha_n^-} \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(\alpha, 0) = \beta_n \quad \textit{and} \lim_{\alpha \to \alpha_n^+} \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(\alpha, 0) = \gamma_n \quad \textit{for } 1 \leq i \leq N$$

Then for all $t \in [0, T_0]$ there are N isolated singularities of the interface at locations $h(\alpha_n, t) \in S(t)$ in conformal coordinates. We also have

$$\lim_{\alpha \to \alpha_n^-} \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(h(\alpha,t),t) = \beta_n \quad \textit{and} \lim_{\alpha \to \alpha_n^+} \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(h(\alpha,t),t) = \gamma_n \quad \textit{for } 1 \leq i \leq N$$

The proof is immediate from the theorem above. Hence these results in essence say that an initial interface with angled crests stays angled crested, the particle at the tip stays at the tip, the angle doesn't change, the angle doesn't tilt and the acceleration at the tip is the one due to gravity. In particular we can now say that the singularity is rigid. This also gives a complete description of the dynamics near the singularities as long as the energy $\mathcal{E}_1(t)$ remains finite.

2.5.1 Proof

The proof of the propagation of singularities and the fact that the acceleration at the singularities is the one due to gravity, follows easily from the nature of the energy. The proof of the constancy of the angle and that it doesn't tilt is a little more involved. To prove this, one important observation is the fact that if $\mathcal{E}_1(t) < \infty$, then the gradient of the velocity extends continuously to the boundary and that it vanishes at the singularities. This is proved in Proposition 2.5.8. The other important fact is that we can describe the dynamics of the angle at the non-singular points on the boundary in terms of the gradient of the velocity by a simple formula. This is proved in Proposition 2.5.9. The theorem follows easily by combining these two facts.

Proof of Lemma 2.5.2. We know that

$$\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon} \Rightarrow u$$

Now observe that

$$\partial_t\left(\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}\right) = \frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}\left(\frac{h_{t\alpha}^\epsilon}{h_\alpha^\epsilon} - \frac{z_{t\alpha}^\epsilon}{z_\alpha^\epsilon}\right)$$

Hence we have

$$\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(\alpha,t) = \frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(\alpha,0)\exp\left\{\int_0^t \left(\frac{h_{t\alpha}^\epsilon}{h_\alpha^\epsilon} - \frac{z_{t\alpha}^\epsilon}{z_\alpha^\epsilon}\right)(\alpha,s)\,ds\right\} \tag{2.19}$$

Hence there exists $c_1, c_2 > 0$ depending only on $\mathcal{E}_1(0)$ and T_0 such that

$$c_1\left|\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(\alpha,0)\right| \leq \left|\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(\alpha,t)\right| \leq c_2\left|\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(\alpha,0)\right| \qquad \text{for all } \alpha \in \mathbb{R}, t \in [0, T_0]$$

Therefore letting $\epsilon \to 0$ we obtain

$$c_1|u(\alpha,0)| \leq |u(\alpha,t)| \leq c_2|u(\alpha,0)| \qquad \text{for all } \alpha \in \mathbb{R}, t \in [0,T_0]$$

Now by using the relation

$$\frac{1}{\Psi_{z'}}(\alpha',0,t) = (u \circ h^{-1})(\alpha',t)$$

and the fact that $h(\alpha,0) = \alpha$ for all $\alpha \in \mathbb{R}$, the lemma is proved. $\square$

Lemma 2.5.5. *Let $g : \mathbb{R} \to \mathbb{C}$ be a continuous function and let $C > 0$ be a given constant. Let $A \subset \mathbb{R}$ be a set of full measure let $f : A \to \mathbb{C}$ be such that*

$$|f(x) - f(y)| \leq C|g(x) - g(y)| \qquad \textit{for all } x, y \in A$$

Then there exists a unique continuous function $\widetilde{f} : \mathbb{R} \to \mathbb{C}$ such that $\widetilde{f}|_A = f$

Proof. Clearly the constant C can be absorbed into the function g and so without loss of generality we assume $C = 1$. As A is a set of full measure, A is dense in $\mathbb{R}$. Fix $x \in \mathbb{R}$ and choose a sequence (x_n) with $x_n \in A$ and $x_n \to x$. Now

$$|f(x_n) - f(x_m)| \leq |g(x_n) - g(x_m)|$$

and hence $\{f(x_n)\}$ is a Cauchy sequence. Define $\widetilde{f}(x) = \lim_{n\to\infty} f(x_n)$.

We easily see that $\widetilde{f}$ is well defined. If (x'_n) is another sequence with $x'_n \in A$ and $x'_n \to x$, then

$$|f(x_n) - f(x'_n)| \leq |g(x_n) - g(x'_n)| \to 0 \quad \text{as } n \to \infty$$

From this we also see that $\widetilde{f}|_A = f$ and

$$|\widetilde{f}(x) - \widetilde{f}(y)| \leq |g(x) - g(y)| \qquad \text{for all } x, y \in \mathbb{R}$$

Hence $\widetilde{f}$ is a continuous function and $\widetilde{f}$ is unique, as a continuous function is determined by its values on a dense set. $\square$

Lemma 2.5.6. *Let $U \subset \mathbb{R}$ be an open set and let $f_n : U \times [0,T] \to \mathbb{C}$ be a sequence of smooth functions. Let $1 < p \leq \infty$ and suppose that for every closed interval $I \subset U$ there exists $C_I > 0$ depending only on the interval I, such that*

$$\sup_{[0,T]}\|f_n(t)\|_{L^\infty(I)} + \sup_{[0,T]}\|\partial_x f_n(t)\|_{L^p(I)} + \sup_{[0,T]}\|\partial_t f_n(t)\|_{L^\infty(I)} \leq C_I$$

Then there exists a continuous and bounded function $f : U \times [0,T] \to \mathbb{C}$ *and a subsequence* $\{f_{n_j}\}$ *such that* $f_{n_j} \to f$ *uniformly on compact subsets of* $U \times [0,T]$.

Proof. This is an easy consequence of Arzela-Ascoli theorem and is very similar to Lemma 5.1 in Wu [Wu15]. □

The following lemma is a more general version of the statement that the energy does not allows angled crests of angles greater than $90°$.

Lemma 2.5.7. *Let* $\alpha' \in S(t)$. *Then for all* $\delta > 0$ *we have*

$$\int_{\alpha'-\delta}^{\alpha'+\delta} |\Psi_{z'}|^2(s,y',t)\,ds \to \infty \quad \textit{as} \quad y' \to 0$$

Proof. We suppress the time dependence t and assume without loss of generality $\alpha' = 0$ that is $0 \in S(t)$. We have for $y' < 0$

$$\frac{1}{\Psi_{z'}^2}(x',y') = \frac{1}{\Psi_{z'}^2}(0,y') + \int_0^{x'} \partial_{z'}\left(\frac{1}{\Psi_{z'}^2}\right)(s,y')\,ds$$

Using the fact that the energy $\mathcal{E}_1 < \infty$ we see that $C = \sup_{y'<0}\left\|\partial_{z'}\left(\frac{1}{\Psi_{z'}^2}\right)(s,y')\right\|_{L^\infty(\mathbb{R},ds)} < \infty$. Hence we have

$$\frac{1}{|\Psi_{z'}|^2}(x',y') \leq \frac{1}{|\Psi_{z'}|^2}(0,y') + C|x'| \quad \text{for all } x' \in \mathbb{R}, y' < 0$$

From this we see that for $y' < 0$

$$\int_{-\delta}^{\delta} |\Psi_{z'}|^2(s,y')\,ds \geq \int_{-\delta}^{\delta} \frac{1}{\frac{1}{|\Psi_{z'}|^2}(0,y') + C|s|}\,ds$$

As $\frac{1}{\Psi_{z'}}$ is continuous on $\overline{P}_-$ and by assumption $\frac{1}{\Psi_{z'}}(0,0) = 0$, we have that $\frac{1}{\Psi_{z'}^2}(0,y') \to 0$ as $y' \to 0$ proving the lemma. □

Proposition 2.5.8. *For any fixed t,* $0 \leq t \leq T_0$ *the functions* $\left(\frac{1}{\Psi_{z'}}F_{z'}\right)(\cdot,\cdot,t)$ *and* $\left(\frac{1}{\Psi_{z'}}\overline{F}_{z'}\right)(\cdot,\cdot,t)$ *extend continuously to* $\overline{P}_-$ *with* $\left(\frac{1}{\Psi_{z'}}F_{z'}\right)(\alpha',0,t) = \left(\frac{1}{\Psi_{z'}}\overline{F}_{z'}\right)(\alpha',0,t) = 0$ *for all* $\alpha' \in S(t)$

Proof. We will suppress the dependence on t and first prove the result for $\frac{1}{\Psi_{z'}}F_{z'}$.

Step 1: Observe that $\sup_{y'<0}\left\|\frac{1}{\Psi_{z'}}F_{z'}(x',y')\right\|_{L^\infty(\mathbb{R},dx')} \leq C(\mathcal{E}_1)$. Hence by Fatou's theorem there exists $f \in L^\infty(\mathbb{R})$ and a set $A \subset \mathbb{R}$ of full measure such that for $y' < 0$ we have

$$\frac{1}{\Psi_{z'}} F_{z'}(\cdot, y') = f * K_{y'}$$
$$\text{and} \quad \frac{1}{\Psi_{z'}} F_{z'}(\alpha', y') \to f(\alpha') \quad \text{as } y' \to 0 \quad \text{for all } \alpha' \in A$$

Also as $\sup_{y'<0} \|F_{z'}(x', y')\|^2_{L^2(\mathbb{R}, dx')} < \infty$ and $F_{z'}$ is holomorphic, there exists $g_1 \in L^2(\mathbb{R})$ such that

$$F_{z'}(\cdot, y') \to g_1 \quad \text{in } L^2 \quad \text{as } y' \to 0$$

Similarly there exists $g_2 \in L^2$ such that

$$\frac{1}{\Psi_{z'}} \partial_{z'} \left(\frac{1}{\Psi_{z'}} F_{z'} \right)(\cdot, y') \to g_2 \quad \text{in } L^2 \quad \text{as } y' \to 0$$

Hence we see that

$$|F_{z'}| \left| \frac{1}{\Psi_{z'}} \partial_{z'} \left(\frac{1}{\Psi_{z'}} F_{z'} \right) \right| (\cdot, y') \to |g_1||g_2| \quad \text{in } L^1 \quad \text{as } y' \to 0$$

Define the function $h : \mathbb{R} \to \mathbb{R}$

$$h(\alpha) = \int_0^\alpha |g_1||g_2|(s)\, ds$$

Clearly h is a continuous function on $\mathbb{R}$. Now observe that for $y' < 0$ we have

$$\left| \left(\frac{1}{\Psi_{z'}} F_{z'} \right)^2 (\alpha_2, y') - \left(\frac{1}{\Psi_{z'}} F_{z'} \right)^2 (\alpha_1, y') \right| \leq 2 \int_{\alpha_1}^{\alpha_2} |F_{z'}| \left| \frac{1}{\Psi_{z'}} \partial_{z'} \left(\frac{1}{\Psi_{z'}} F_{z'} \right) \right| (s, y')\, ds$$

Now letting $y' \to 0$ we have

$$\left| f^2(\alpha_2) - f^2(\alpha_1) \right| \leq 2 |h(\alpha_2) - h(\alpha_1)| \quad \text{for all } \alpha_1, \alpha_2 \in A$$

Hence by Lemma 2.5.5 there exists $f_2 : \mathbb{R} \to \mathbb{C}$ a continuous function such that $f_2|_A = f^2$. Also observe that

$$\left| \left(\frac{1}{\Psi_{z'}} F_{z'} \right)^3 (\alpha_2, y') - \left(\frac{1}{\Psi_{z'}} F_{z'} \right)^3 (\alpha_1, y') \right| \leq C(\mathcal{E}_1) \int_{\alpha_1}^{\alpha_2} |F_{z'}| \left| \frac{1}{\Psi_{z'}} \partial_{z'} \left(\frac{1}{\Psi_{z'}} F_{z'} \right) \right| (s, y')\, ds$$

Hence via the same argument there exists $f_3 : \mathbb{R} \to \mathbb{C}$ a continuous function such that $f_3|_A = f^3$.

Step 2: Define the function $\widetilde{f} : \mathbb{R} \to \mathbb{C}$

$$\widetilde{f}(\alpha') = \begin{cases} (f_3/f_2)(\alpha') & \text{if } f_2(\alpha') \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We claim that $\widetilde{f}$ is a continuous function on $\mathbb{R}$ and $\widetilde{f}|_A = f$.

First note that both f_3 and f_2 are continuous. Fix $\alpha' \in \mathbb{R}$ and observe that if $f_2(\alpha') \neq 0$, then $\widetilde{f}$ is continuous at α'. Hence we need to prove the continuity of $\widetilde{f}$ at α' where $f_2(\alpha') = 0$. Define the function $f_{abs} : \mathbb{R} \to \mathbb{R}$ by $f_{abs} = \sqrt{|f_2|}$. Observe that f_{abs} is a continuous function on $\mathbb{R}$ and that $|f_3|(\alpha') = f_{abs}^3(\alpha')$ for all $\alpha' \in A$. As A is a set of full measure and both $|f_3|$ and f_{abs}^3 are continuous functions, we have $|f_3|(\alpha') = f_{abs}^3(\alpha')$ for all $\alpha' \in \mathbb{R}$. Hence we now see that $|\widetilde{f}(\alpha')| \leq f_{abs}(\alpha')$ for all $\alpha' \in \mathbb{R}$ and this inequality is enough to prove continuity of $\widetilde{f}$ at all α' where $f_2(\alpha') = 0$. Hence $\widetilde{f}$ is continuous on $\mathbb{R}$.

Recall that $f_2|_A = f^2$ and $f_2|_A = f^3$. If $\alpha' \in A$ and $f(\alpha') \neq 0$, then $\widetilde{f}(\alpha') = (f_3/f_2)(\alpha') = f(x)$. If $\alpha' \in A$ and $f(\alpha') = 0$, then we see that $f_2(\alpha') = 0$ and hence $\widetilde{f}(\alpha') = 0$. Hence $\widetilde{f}|_A = f$.

Step 3: As $\widetilde{f}|_A = f$ and A is a set of full measure we now have

$$\frac{1}{\Psi_{z'}} F_{z'}(\cdot, y') = \widetilde{f} * K_{y'} \quad \text{for all } y' < 0$$

As f is bounded, we see that $\widetilde{f}$ is a continuous and bounded function, and hence $\frac{1}{\Psi_{z'}} F_{z'}$ extends continuously to $\overline{P}_-$. Now let $\alpha' \in S(t)$ be a singular point. We proceed via contradiction and assume that $\frac{1}{\Psi_{z'}} F_{z'}(\alpha', 0) = c \neq 0$. Hence there exists $c_1, c_2, \delta > 0$ so that

$$0 < c_1 \leq \left| \frac{1}{\Psi_{z'}} F_{z'} \right| (s, y') \leq c_2 < \infty \quad \text{for all } s \in (\alpha' - \delta, \alpha' + \delta) \text{ and } -\delta \leq y' < 0$$

Observe that for $y' < 0$ we have

$$F_{z'}(\alpha', y') = \Psi_{z'}(\alpha', y') \left(\frac{1}{\Psi_{z'}} F_{z'}(\alpha', y') \right)$$

and hence we obtain

$$|F_{z'}(s, y')| \geq c_1 |\Psi_{z'}(s, y')| \quad \text{for all } s \in (\alpha' - \delta, \alpha' + \delta) \text{ and } -\delta \leq y' < 0$$

By integrating we get

$$\int_{\alpha'-\delta}^{\alpha'+\delta} |F_{z'}(s,y')|^2 \, ds \geq c_1^2 \int_{\alpha'-\delta}^{\alpha'+\delta} |\Psi_{z'}(s,y')|^2 \, ds \quad \text{for all } -\delta \leq y' < 0$$

Letting $y' \to 0$ and using Lemma 2.5.7 we see that

$$\lim_{y' \to 0} \int_{\alpha'-\delta}^{\alpha'+\delta} |F_{z'}(s,y')|^2 \, ds = \infty$$

which contradicts the finiteness of the energy, $\sup_{y'<0} \|F_{z'}(x',y')\|_{L^2(\mathbb{R},dx')} \leq \mathcal{E}_1 < \infty$

We have proven the result for $\frac{1}{\Psi_{z'}} F_{z'}$ and we now need to prove the result for $\frac{1}{\Psi_{z'}} \overline{F}_{z'}$. We observe that $\Psi_{z'}$ extends continuously to $\overline{P}_- \backslash S(t)$ and hence the functions $F_{z'}$ and $\frac{1}{\Psi_{z'}} \overline{F}_{z'}$ extend continuously to $\overline{P}_- \backslash S(t)$. As $\left| \frac{1}{\Psi_{z'}} \overline{F}_{z'} \right| = \left| \frac{1}{\Psi_{z'}} F_{z'} \right|$ on P_- and as $\frac{1}{\Psi_{z'}} F_{z'}$ extends continuously to $\overline{P}_-$ and vanishes on $S(t)$, this forces $\frac{1}{\Psi_{z'}} \overline{F}_{z'}$ to extend continuously to $\overline{P}_-$ and $\frac{1}{\Psi_{z'}} \overline{F}_{z'}(\alpha',0) = 0$ for all $\alpha' \in S(t)$.

□

Proposition 2.5.9. *Define the function $f : \mathbb{R} \times [0,T_0] \to \mathbb{C}$ as $f(\alpha',t) = \frac{1}{\Psi_{z'}} \overline{F}_{z'}(\alpha',0,t)$ where the definition makes sense by Proposition 2.5.8. Then*

$$\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(h(\alpha,t),t) = \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(\alpha,0) \exp\left\{ i\mathit{Im}\left(\int_0^t f(h(\alpha,s),s) \, ds \right) \right\} \quad \textit{for all } \alpha \in NS(0)$$

Proof. We will do this in steps.

Step 1: From (2.19) we see that

$$\frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(\alpha,t) = \frac{h_\alpha^\epsilon}{z_\alpha^\epsilon}(\alpha,0) \exp\left\{ \int_0^t \left(\frac{h_{t\alpha}^\epsilon}{h_\alpha^\epsilon} - \frac{z_{t\alpha}^\epsilon}{z_\alpha^\epsilon} \right)(\alpha,s) \, ds \right\}$$

and hence by inverting and by using $(\frac{z_\alpha^\epsilon}{h_\alpha^\epsilon}) = Z_{,\alpha'}^\epsilon \circ h^\epsilon$ we get

$$Z_{,\alpha'}^\epsilon(h^\epsilon(\alpha,t),t) = Z_{,\alpha'}^\epsilon(\alpha,0) \exp\left\{ \int_0^t \left(\frac{Z_{t,\alpha'}^\epsilon}{Z_{,\alpha'}^\epsilon} - b_{\alpha'}^\epsilon \right)(h^\epsilon(\alpha,s),s) \, ds \right\}$$

from this we easily obtain

$$\frac{Z_{,\alpha'}^\epsilon}{|Z_{,\alpha'}^\epsilon|}(h^\epsilon(\alpha,t),t) = \frac{Z_{,\alpha'}^\epsilon}{|Z_{,\alpha'}^\epsilon|}(\alpha,0) \exp\left\{ i\mathrm{Im}\left(\int_0^t \left(\frac{Z_{t,\alpha'}^\epsilon}{Z_{,\alpha'}^\epsilon} \right)(h^\epsilon(\alpha,s),s) \, ds \right) \right\} \quad \text{for all } \alpha \in \mathbb{R}$$

We now need to take the limit $\epsilon \to 0$. Recall that

$$\frac{1}{\Psi^{\epsilon}_{z'}} \Rightarrow \frac{1}{\Psi_{z'}} \quad \text{on } \overline{P}_{-} \times [0,T_0] \qquad \text{and } h^{\epsilon} \Rightarrow h \quad \text{on } \mathbb{R} \times [0,T_0]$$

Hence for all $t \in [0,T_0]$ and $\alpha \in NS(0)$ we have

$$\frac{Z^{\epsilon}_{,\alpha'}}{|Z^{\epsilon}_{,\alpha'}|}(h^{\epsilon}(\alpha,t),t) \to \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(h(\alpha,t),t) \quad \text{as } \epsilon \to 0$$

As for all $(\alpha',t) \in \mathbb{R} \times [0,T_0]$ we have $\left| \frac{Z^{\epsilon}_{t,\alpha'}}{Z^{\epsilon}_{,\alpha'}}(\alpha',t) \right| \leq C(\mathcal{E}_1(0))$, by dominated convergence it is enough to show that for all $t \in [0,T_0]$ and all $\alpha \in NS(0)$ we have

$$\frac{Z^{\epsilon}_{t,\alpha'}}{Z^{\epsilon}_{,\alpha'}}(h^{\epsilon}(\alpha,t),t) \to f(h(\alpha,t),t) \quad \text{as } \epsilon \to 0$$

Step 2: We first show that there exists a continuous function $g : NS(0) \times [0,T_0] \to \mathbb{C}$ such that

$$\frac{z^{\epsilon}_{t\alpha}}{h^{\epsilon}_{\alpha}} \Rightarrow g \quad \text{on} \quad NS(0) \times [0,T_0]$$

a) First observe that by the definition of $S(t)$, the function $\Psi_{z'}(\cdot,t)$ extends continuously to $\overline{P}_{-}\backslash S(t)$. Hence for any closed interval $I \subset NS(0)$ we have

$$\sup_{0<\epsilon<1} \|z^{\epsilon}_{\alpha}(\alpha,0)\|_{L^{\infty}(I)} \leq C_I$$

Now from the existence result Theorem 2.5.1 we see that

$$\sup_{[0,T_0]} |z^{\epsilon}_{t\alpha}(\alpha,t)| \leq |z^{\epsilon}_{\alpha}(\alpha,0)| C(\mathcal{E}_1(0))$$

Combining these estimates with the observation $0 < c_1 \leq |h^{\epsilon}_{\alpha}(t)| \leq c_2 < \infty$ for all $t \in [0,T_0]$ we have

$$\sup_{[0,T_0]} \left\| \frac{z^{\epsilon}_{t\alpha}}{h^{\epsilon}_{\alpha}}(\alpha,t) \right\|_{L^{\infty}(I)} \leq C_I$$

b) Observe that

$$\partial_{\alpha}\left(\frac{z^{\epsilon}_{t\alpha}}{h^{\epsilon}_{\alpha}}\right) = \left\{ \left(\frac{h_{\alpha}}{z_{\alpha}}\right) \frac{1}{z_{\alpha}} \partial_{\alpha}\left(\frac{z^{\epsilon}_{t\alpha}}{h^{\epsilon}_{\alpha}}\right) \right\} \left(\frac{z^2_{\alpha}}{h_{\alpha}}\right)$$

As $\left\{\left(\frac{h_\alpha}{z_\alpha}\right)\frac{1}{z_\alpha}\partial_\alpha\left(\frac{z^\epsilon_{t\alpha}}{h^\epsilon_\alpha}\right)\right\}\circ(h^\epsilon)^{-1} = \frac{1}{(Z^\epsilon_{,\alpha'})^2}\partial_{\alpha'}Z^\epsilon_{t,\alpha'}$ and as $0 < c_1 \leq |h^\epsilon_\alpha(t)| \leq c_2 < \infty$ we have

$$\left\|\left(\frac{h_\alpha}{z_\alpha}\right)\frac{1}{z_\alpha}\partial_\alpha\left(\frac{z^\epsilon_{t\alpha}}{h^\epsilon_\alpha}\right)(\cdot,t)\right\|_{L^2(\mathbb{R})} \leq C(\mathcal{E}_1(0)) \quad \text{for all } 0 \leq t \leq T_0$$

As $|z^\epsilon_\alpha(\alpha,t)| \leq |z^\epsilon_\alpha(\alpha,0)|C(\mathcal{E}_1(0))$ for all $0 \leq t \leq T_0$, we have for all closed intervals $I \subset NS(0)$

$$\sup_{[0,T_0]}\left\|\partial_\alpha\left(\frac{z^\epsilon_{t\alpha}}{h^\epsilon_\alpha}\right)\right\|_{L^2(I)} \leq C_I$$

c) We have

$$\partial_t\left(\frac{z^\epsilon_{t\alpha}}{h^\epsilon_\alpha}\right) = \frac{z^\epsilon_{tt\alpha}}{h^\epsilon_\alpha} - \left(\frac{h^\epsilon_{t,\alpha}}{h^\epsilon_\alpha}\right)\left(\frac{z^\epsilon_{t\alpha}}{h^\epsilon_\alpha}\right)$$

Now observe that

$$|z^\epsilon_{tt\alpha}(\alpha,t)| \leq |z^\epsilon_\alpha(\alpha,t)|\left|\frac{z^\epsilon_{tt\alpha}}{z^\epsilon_\alpha}\right|(\alpha,t)$$

Hence we have

$$\sup_{[0,T_0]}\left\|\frac{z^\epsilon_{tt\alpha}}{h^\epsilon_\alpha}(\alpha,t)\right\|_{L^\infty(I)} \leq C_I$$

which implies that

$$\sup_{[0,T_0]}\left\|\partial_t\left(\frac{z^\epsilon_{t\alpha}}{h^\epsilon_\alpha}\right)(\alpha,t)\right\|_{L^\infty(I)} \leq C_I$$

Hence by using Lemma 2.5.6 we can conclude the existence of such a function g.

Step 3: We now relate the function g to F and complete the proof. We have already shown in Proposition 2.5.8 that $\frac{1}{\Psi_{z'}}F_{z'}$ extends continuously to $\overline{P}_-$. As $\Psi_{z'}$ extends continuously to $\overline{P}_-\backslash S(t)$, we have that $F_{z'}$ extends continuously to $\overline{P}_-\backslash S(t)$ and so does $\overline{F}_{z'}$. Hence it makes sense to write the equality $\left(\frac{1}{\Psi_{z'}}\overline{F}_{z'}\right)(\alpha',0,t) = \left(\frac{1}{\Psi_{z'}}(\alpha',0,t)\right)\overline{F}_{z'}(\alpha',0,t)$ for all $\alpha' \in NS(t)$, where each function on the boundary is defined by its continuous extension. Now from step 2, we have by changing coordinates

$$Z^\epsilon_{t,\alpha'} \Rightarrow g\circ h^{-1} \quad \text{on the open set } \{(x,t) \in \mathbb{R}\times[0,T_0]\,|\,x \in NS(t)\}$$

Fix $t \in [0, T_0]$ and let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that $[\alpha_1, \alpha_2] \subset NS(t)$. Then

$$Z_t^\epsilon(\alpha_2, t) - Z_t^\epsilon(\alpha_1, t) = \int_{\alpha_1}^{\alpha_2} Z_{t,\alpha'}^\epsilon(s, t)\, ds$$

Letting $\epsilon \to 0$ and using the definition of F we have

$$\overline{F}(\alpha_2, 0, t) - \overline{F}(\alpha_2, 0, t) = \int_{\alpha_1}^{\alpha_2} (g \circ h^{-1})(s, t)\, ds$$

which implies that $\overline{F}_z(\alpha', 0, t) = (g \circ h^{-1})(\alpha', t)$ for all $\alpha' \in NS(t)$. Hence using the definitions of f and u we see that

$$f(\alpha', t) = (u \circ h^{-1})(g \circ h^{-1})(\alpha', t) \quad \text{for all } \alpha' \in NS(t)$$

We know that

$$\frac{1}{Z_{,\alpha'}^\epsilon} \Rightarrow u \circ h^{-1} \quad \text{on } \mathbb{R} \times [0, T_0]$$

Hence we have for all $t \in [0, T_0]$

$$\frac{Z_{t,\alpha'}^\epsilon}{Z_{,\alpha'}^\epsilon}(h^\epsilon(\alpha, t), t) \to (u \circ h^{-1})(g \circ h^{-1})(h(\alpha, t), t) = f(h(\alpha, t), t) \quad \text{for all } \alpha \in NS(0)$$

which proves the proposition. $\square$

Proof of Theorem 2.5.3. As $\alpha_n \in NS(0)$, using Proposition 2.5.9 we see that

$$\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(h(\alpha_n, t), t) = \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(\alpha_n, 0) \exp\left\{ i\mathrm{Im}\left(\int_0^t f(h(\alpha_n, s), s)\, ds \right) \right\}$$

Now from Proposition 2.5.8 we know that $f(\alpha', t)$ is continuous in α', and hence by dominated convergence theorem we have

$$\lim_{n \to \infty} \frac{\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(h(\alpha_n, t), t))}{\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(\alpha_n, 0)} = \exp\left\{ i\mathrm{Im}\left(\int_0^t f(h(\alpha, s), s)\, ds \right) \right\}$$

But by Proposition 2.5.8 we know that $f(h(\alpha, s), s) = 0$ for all $s \in [0, T_0]$ as $\alpha \in S(0)$. This proves the first part of the result. For the second part observe that

$$z_{tt}^{\epsilon} + i = i(A_1^{\epsilon} \circ h)\frac{h_{\alpha}^{\epsilon}}{z_{\alpha}^{\epsilon}}$$

Hence we have

$$|z_{tt}^{\epsilon} + i|(\alpha, t) \leq \frac{C}{|z_{\alpha}^{\epsilon}|(\alpha, t)} \qquad \text{for all } \alpha \in \mathbb{R}, 0 \leq t \leq T_0$$

where $C = C(\mathcal{E}_1(t))$. Now as $c_1|z_{\alpha}^{\epsilon}|(\alpha, 0) \leq |z_{\alpha}^{\epsilon}|(\alpha, t) \leq c_2|z_{\alpha}^{\epsilon}|(\alpha, 0)$ we obtain

$$|z_{tt}^{\epsilon} + i|(\alpha, t) \leq \frac{C}{|z_{\alpha}^{\epsilon}|(\alpha, 0)}$$

Now if $\alpha \in S(0)$, then $\dfrac{1}{\Psi_{z'}}(\alpha, 0) = 0$, and hence $\dfrac{1}{|z_{\alpha}^{\epsilon}|(\alpha, 0)} \to 0$ as $\epsilon \to 0$. Hence we have

$$q(\alpha, t) = -i$$

□

2.6 Examples: Angled crests and cusps

Note that if we assume that the interface and velocity is smooth near infinity and decays rapidly at infinity then to use Theorem 2.5.1 we only need to show the initial energy $\mathcal{E}_1(0) < \infty$ as

$$c_0 = \sup_{y'<0}\|F(x' + iy', 0)\|_{L^2(\mathbb{R}, dx')} + \sup_{y'<0}\left\|\frac{1}{\Psi_{z'}(x' + iy', 0)} - 1\right\|_{L^2(\mathbb{R}, dx')}$$

automatically satisfies $c_0 < \infty$ from $\mathcal{E}_1(0) < \infty$ and the fact that initial data decays at infinity. Also we see that it is easy to construct F such that

$$\sup_{y'<0}\|F(x' + iy', 0)\|_{H^2(\mathbb{R}, dx')} < \infty$$

which automatically controls all terms of $\mathcal{E}_1$ containing F if we have $\sup_{y'<0}\left\|\dfrac{1}{\Psi_{z'}}(0)\right\|_{L^\infty(\mathbb{R}, dx')} < \infty$. Hence we will now construct domains such that

1. $\sup_{y'<0}\left\|\dfrac{1}{\Psi_{z'}}(t)\right\|_{L^\infty(\mathbb{R}, dx')} < \infty$

2. $\sup_{y'<0}\left\|\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)(t)\right\|_{L^2(\mathbb{R},dx')} < \infty$ and

3. $\sup_{y'<0}\left\|\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)\right)(t)\right\|_{L^2(\mathbb{R},dx')} < \infty$

1. Smooth domains

Observe that is boundary is of class $C^{1,\alpha}$ with $0 < \alpha \leq 1$ then there exists constants $0 < c_1, c_2 < \infty$ such that $c_1 \leq |Z_{,\alpha'}|(\alpha', 0) \leq c_2$. Hence if $Z_{,\alpha'} - 1 \in H^2$ then we easily see that $\mathcal{E}_1(0) < \infty$. Hence if the domain is smooth then the theorem Theorem 2.5.1 applies.

2. Angled crests

A regular smooth curve in the plane is a smooth mapping $\gamma : I \to \mathbb{C}$ such that $\gamma'(t) \neq 0$ for all $t \in I$ where I is an interval. Consider a domain Ω with $0 \in \partial\Omega$ such that the boundary of Ω at 0 consists of two regular smooth arcs such that the opening angle of Ω at 0 is $\nu\pi$ i.e. there is a corner at $0 \in \partial\Omega$. Assume that $0 < \nu < 2$. Then we have a local description of Ψ near $z = 0$.

Theorem 2.6.1 ([Wig65])**.** *Let Ω a domain as described above with $0 \in \partial\Omega$. Let $\Psi : \overline{P}_- \to \overline{\Omega}$ be a Riemann map with $\Psi(0) = 0$ and let $s \geq 0$ be an integer. Then there exists an $\epsilon = \epsilon(s) > 0$ and continuous functions $\phi_s : \overline{P}_- \cap B(0,\epsilon)$ with $\phi_s(0) \neq 0$ such that*

$$\partial_z^s \Psi(z) = z^{\nu - s}\phi_s(z) \quad \textit{for all} \quad z \in \bar{P}_- \cap B(0,\epsilon) \setminus \{0\}$$

Hence we now consider an interface with $N \geq 1$ angled crests with angles $\nu_i\pi$ where $0 < \nu_i < \frac{1}{2}$. Near an angled crest, if we change coordinates so that it is at the origin, then by the above lemma we see that $\Psi_{z'}(z) \sim z^{\nu-1}$ near 0. Hence we see that near $z = 0$ we have

1. $\frac{1}{\Psi_{z'}}(z) \sim z^{1-\nu} \in L^\infty_{loc}$

2. $\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)(z) \sim z^{-\nu} \in L^2_{loc}$

3. $\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\partial_{z'}\left(\frac{1}{\Psi_{z'}}\right)\right)(z) \sim z^{1-3\nu} \in L^2_{loc}$

Hence the energy $\mathcal{E}_1(0) < \infty$ and hence angles $\nu\pi$ with $0 < \nu < \frac{1}{2}$ are allowed.

3. Cusps

A regular analytic curve in the plane is an analytic mapping $\gamma : I \to \mathbb{C}$ such that $\gamma'(t) \neq 0$ for all $t \in I$ where I is an interval. Consider a domain Ω with $0 \in \partial\Omega$ such that the boundary of Ω at 0 consists of two regular analytic arcs such that the opening angle of Ω at 0 vanishes i.e. there is a cusp at $0 \in \partial\Omega$. By an analytic change of coordinates near 0 we can assume that the boundary of Ω near 0 consists of two arcs one of which is the positive real axis. Assume that there exists an $R > 0$ such that

$$\Omega \cap \overline{B}(0, R) = \{z \in \mathbb{C} \,|\, |z| \leq R, 0 < arg(z) < \theta(|z|)\}$$

where $\theta(t)$ is a real power series that converges on $(-2R, 2R)$ and is positive on $(0, R)$. Then we have the following description of the Riemann mapping near 0.

Theorem 2.6.2 ([KL16]). *Let Ω a domain as described above with $0 \in \partial\Omega$. Let $\theta(t) = \sum_{j=1}^{\infty} a_j t^j$ be the power series of $\theta(t)$ around 0 with $a_1 \neq 0$. Let $\Psi : \overline{P}_- \to \overline{\Omega}$ be a Riemann map with $\Psi(0) = 0$. Then for $s \geq 1$ there exists $\epsilon = \epsilon(s) > 0$ and continuous functions $\phi_s : \overline{P}_- \cap B(0, \epsilon)$ with $\phi_s(0) \neq 0$ such that*

$$\partial_z^s \Psi(z) = \frac{z^{-s}}{\log(z)^2} \phi_s(z) \quad \textit{for all} \quad z \in \bar{P}_- \cap B(0, \epsilon) \setminus \{0\}$$

Hence we now consider an interface an analytic cusp as described above. Using the result above we see that near $z = 0$ we have

1. $\dfrac{1}{\Psi_{z'}}(z) \sim z \log(z)^2 \in L^\infty_{loc}$
2. $\partial_{z'}\left(\dfrac{1}{\Psi_{z'}}\right)(z) \sim \log(z)^2 \in L^2_{loc}$
3. $\dfrac{1}{\Psi_{z'}}\partial_{z'}\left(\dfrac{1}{\Psi_{z'}}\partial_{z'}\left(\dfrac{1}{\Psi_{z'}}\right)\right)(z) \sim z \log(z)^6 \in L^2_{loc}$

Hence the energy $\mathcal{E}_1(0) < \infty$ and hence cusps are allowed. It is interesting to note that Theorem 2.5.3 and Corollary 2.5.4 applies to cusps as well.

CHAPTER 3

Gravity Capillary Water Waves

This chapter is dedicated to establishing the groundwork of our main results on surface tension. In §3.1 we establish the notation and prove some basic formulae including the formula for the Taylor sign condition. In §3.2 we derive the quasilinear equations from which we obtain our energy E_σ. In §3.3 we give a heuristic explanation of the energy estimate for the energy E_σ and explain why we do not allow singular interfaces as initial data for $\sigma > 0$. It is also explained why the energy we have is quite natural and how it can be seen coming from the structure of the quasilinear equation. We also explain the novelties and the main difficulties in proving the energy estimate.

In §3.4 we state our main results. The first main result is Theorem 3.4.1 which is an apriori estimate for the case of non-zero surface tension. This result extends the result of [KW14] to the case of positive surface tension. The second main result is the convergence result Theorem 3.4.8 which says that solutions with surface tension approximate the zero surface tension solutions as $\sigma \to 0$. Using both of these main results, we give an example Proposition 3.4.9 which demonstrates the effectiveness of these results. Finally we §3.5 we give an outline for the rest of the thesis.

3.1 Notation and Preliminaries

We will use all the notation used in Chapter 2 and add a few more definitions due to surface tension. As the equation changes due to surface tension we repeat the argument of the derivation of the Euler equation on the boundary.

Let the interface $\Sigma(t) : z = z(\alpha, t) \in \mathbb{C}$ be given by a Lagrangian parametrization α satisfying $z_\alpha(\alpha, t) \neq 0$ for all $\alpha \in \mathbb{R}$. Hence $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the velocity of the fluid on the interface and $z_{tt}(\alpha, t) = (\mathbf{v_t} + (\mathbf{v}.\nabla)\mathbf{v})(z(\alpha, t), t)$ is the acceleration. As $\frac{z_\alpha}{|z_\alpha|}(\alpha, t) = e^{i\theta(\alpha,t)}$ and $\frac{1}{|z_\alpha|}\partial_\alpha$ is the arc length derivative in Lagrangian coordinates, the pressure can be rewritten as

$$P(z(\alpha, t), t) = i\sigma\frac{1}{z_\alpha}\partial_\alpha\frac{z_\alpha}{|z_\alpha|}(\alpha, t)$$

Note that $\frac{1}{z_\alpha}\partial_\alpha \frac{z_\alpha}{|z_\alpha|}$ is purely imaginary. Hence the Euler equation becomes

$$z_{tt}(\alpha,t)+i=-\hat{n}\frac{\partial P}{\partial \hat{n}}(z(\alpha,t))-\hat{t}\frac{\partial P}{\partial \hat{t}}(z(\alpha,t))$$

where

$$\hat{t}=\frac{z_\alpha}{|z_\alpha|}=e^{i\theta}=\text{unit tangent vector}$$
$$\hat{n}=i\frac{z_\alpha}{|z_\alpha|}=ie^{i\theta}=\text{unit outward normal vector}$$

Define

$$a(\alpha,t)=-\frac{1}{|z_\alpha|}\frac{\partial P}{\partial \hat{n}}(z(\alpha,t))\ \in\mathbb{R}$$

Hence we get

$$z_{tt}+i=iaz_\alpha-\frac{z_\alpha}{|z_\alpha|}\frac{1}{|z_\alpha|}\partial_\alpha(P(z(\alpha,t),t))$$
$$\text{Therefore}\quad \bar{z}_{tt}-i=-ia\bar{z}_\alpha-i\sigma\frac{1}{z_\alpha}\partial_\alpha\frac{1}{z_\alpha}\partial_\alpha\frac{z_\alpha}{|z_\alpha|} \tag{3.1}$$

As in the case with zero surface tension, let $\Phi(.,t):\Omega(t)\to P_-$ be a Riemann map satisfying $\lim_{z\to\infty}\Phi_z(z,t)=1$ and define

$$h(\alpha,t)=\Phi(z(\alpha,t),t) \tag{3.2}$$

hence $h:\mathbb{R}\to\mathbb{R}$ is a homeomorphism. Let $h^{-1}(\alpha',t)$ be its inverse i.e.

$$h(h^{-1}(\alpha',t),t)=\alpha'$$

As was done in Chapter 2, from now on, we will fix our Lagrangian parametrization at $t=0$ by imposing

$$h(\alpha,0)=\alpha\quad \text{for all } \alpha\in\mathbb{R}$$

Hence the Lagrangian parametrization is the same as conformal parametrization at $t=0$. We define Z, Z_t, Z_{tt} and the operators $D_t, D_{\alpha'}, |D_{\alpha'}|$ etc. as in Chapter 2. We now define some new variables

$$\mathcal{A}=(ah_\alpha)\circ h^{-1}$$

$$A_{1,\sigma} = \mathcal{A}|Z_{,\alpha'}|^2 \qquad \text{Hence} \quad \frac{A_{1,\sigma}}{|Z_{,\alpha'}|} = -\frac{\partial P}{\partial \hat{n}} \circ h^{-1}$$

$$A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$$

$$g = \theta \circ h^{-1} \qquad \text{Hence} \quad \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} = e^{ig} \text{ and } |D_{\alpha'}|g = (\partial_s \theta) \circ h^{-1} = -iD_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

$$\Theta = (\mathbb{I} + \mathbb{H})|D_{\alpha'}|g = -i(\mathbb{I} + \mathbb{H})D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

$$\omega = e^{ig} = \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \qquad \text{Hence} \quad |D_{\alpha'}|\omega = i\omega\text{Re}\Theta$$

Observe that $\text{Re}\Theta = \kappa \circ h^{-1}$ where κ is the curvature. Note the definition of A_1 is consistent with the definition in Chapter 2 as we had proved that $A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$ in (2.5). With this notation, by precomposing (3.1) with h^{-1} we get

$$\overline{Z}_{tt} - i = -i\frac{A_{1,\sigma}}{Z_{,\alpha'}} - i\sigma D_{\alpha'}D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \tag{3.3}$$

Let us now derive the formulae of $A_{1,\sigma}$. The formula for b and $b_{\alpha'}$ is the same as in (2.7) and (2.8) and surface tension does not affect the formula.

3.1.1 Formula for $A_{1,\sigma}$

Let $F = \overline{\mathbf{v}}$ and hence F is holomorphic in $\Omega(t)$ and $\bar{z}_t = F(z(\alpha,t),t)$. Hence

$$\bar{z}_{tt} = F_t(z(\alpha,t),t) + F_z(z(\alpha,t),t)z_t(\alpha,t) \qquad \bar{z}_{t\alpha} = F_z(z(\alpha,t),t)z_\alpha(\alpha,t)$$

$$\text{Hence} \quad \bar{z}_{tt} = F_t \circ z + z_t\frac{\bar{z}_{t\alpha}}{z_\alpha}$$

Precomposing with h^{-1} we obtain $\overline{Z}_{tt} = F_t \circ Z + Z_t\dfrac{\overline{Z}_{t,\alpha'}}{Z_{,\alpha'}}$. Now Multiply by $Z_{,\alpha'}$ and use (3.3) to get

$$A_{1,\sigma} = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'} - \sigma\partial_{\alpha'}D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

Apply $(\mathbb{I} - \mathbb{H})$ and use the fact that $\mathbb{H}(Z_{,\alpha'} - 1) = Z_{,\alpha'} - 1$ and $\mathbb{H}1 = 0$ to obtain

$$(\mathbb{I} - \mathbb{H})A_{1,\sigma} = 1 + i[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'} - \sigma(\mathbb{I} - \mathbb{H})\partial_{\alpha'}D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

Now take the real part

$$A_{1,\sigma} = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'} + \sigma\partial_{\alpha'}\mathbb{H}D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

Hence

$$A_{1,\sigma} = A_1 + \sigma\partial_{\alpha'}\mathbb{H}D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \qquad \text{and in particular} \quad A_{1,\sigma}\big|_{\sigma=0} = A_1 \tag{3.4}$$

Using this relation above we get another formula for A_1

$$A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'} - \sigma\partial_{\alpha'}(\mathbb{I} + \mathbb{H})D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \tag{3.5}$$

Note that the only non-holomorphic quantity in the above formula is $iZ_t\overline{Z}_{t,\alpha'}$. Also note that as $A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$ by the calculation in [KW14, Wu16] we have that $A_1 \geq 1$. From (3.4) the Taylor sign condition term can be written as

$$-\frac{\partial P}{\partial \hat{n}} \circ h^{-1} = \frac{A_{1,\sigma}}{|Z_{,\alpha'}|} = \frac{1}{|Z_{,\alpha'}|}\big(A_1 + \sigma|\partial_{\alpha'}|(\kappa \circ h^{-1})\big) \tag{3.6}$$

where $\kappa \circ h^{-1} = -iD_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ is the curvature in conformal coordinates. Hence we see that for large values of σ, the Taylor sign condition may fail. For $\sigma = 0$, this formula was first derived by Wu [Wu97] to prove the Taylor sign condition for $C^{1,\alpha}$ interfaces with $\alpha > 0$ and was crucially used in Kinsey-Wu [KW14] to prove apriori estimates for angled crest interfaces. We will also use this formula in an essential way.

3.1.2 Fundamental Equation

Substituting the formula for $A_{1,\sigma}$ in equation (3.3), we get

$$\overline{Z}_{tt} - i = -i\frac{A_1}{Z_{,\alpha'}} - i\sigma D_{\alpha'}\mathbb{H}D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} - i\sigma D_{\alpha'}D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

Now combine the second and third term and use $\Theta = -i(\mathbb{I} + \mathbb{H})D_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ to get the fundamental equation

$$\overline{Z}_{tt} - i = -i\frac{A_1}{Z_{,\alpha'}} + \sigma D_{\alpha'}\Theta \tag{3.7}$$

Note that as A_1 does not depend on σ, the effect of surface tension is that it adds a holomorphic quantity to the conjugate of the acceleration. We also see that

$$\frac{A_1}{|Z_{,\alpha'}|} = -\frac{\partial P}{\partial \hat{n}} \circ h^{-1}\Big|_{\sigma=0} \geq 0$$

and hence it represents the Taylor sign condition in the absence of surface tension. As the equation is written in terms of A_1 and not $A_{1,\sigma}$, our energy E_σ will always be positive irrespective of the value of surface tension.

3.2 The Quasilinear Equations

We will now use the fundamental equation (3.7) as the starting point to derive our quasilinear equations. Our main equation is for the variable $\overline{D}_{\alpha'}\overline{Z}_t$ which is obtained by applying the operators $\overline{D}_{\alpha'}D_t$ to the equation (3.7). We also obtain equations for $\overline{Z}_t$, $\overline{Z}_{t,\alpha'}$ and Θ which should be thought of lower order and auxiliary equations. Let us first derive some simple but useful formulas:

a) Observe that for any complex valued function f, $\mathbb{H}(\text{Re}f) = i\text{Im}(\mathbb{H}f)$ and $\mathbb{H}(i\text{Im}f) = \text{Re}(\mathbb{H}f)$. Hence we get the following useful identities

$$(\mathbb{I}+\mathbb{H})(\text{Re}f) = f - i\text{Im}(\mathbb{I}-\mathbb{H})f \tag{3.8}$$

$$(\mathbb{I}+\mathbb{H})(i\text{Im}f) = f - \text{Re}(\mathbb{I}-\mathbb{H})f \tag{3.9}$$

b) As we will frequently work with the operator $|D_{\alpha'}|^3$ we record some commonly used expansions

$$\begin{aligned}
|D_{\alpha'}|^2 f &= \left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|f + \frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2 f \\
|D_{\alpha'}|^3 f &= \left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|f + \left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)^2|D_{\alpha'}|f + 3\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2 f \\
&\quad + \frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^3 f
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
|D_{\alpha'}|^3 f &= \partial_{\alpha'}\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right)\right\} - \frac{3}{2}\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2 f \\
&\quad - 2\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)^2|D_{\alpha'}|f - \frac{1}{2}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|f
\end{aligned} \tag{3.11}$$

We will now derive formulas for Θ, $D_t\Theta$ and $D_t^2\Theta$. All three of them are derived similarly.

3.2.1 Formula for Θ

We know that $\text{Re}\Theta = -\text{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$. Applying $(\mathbb{I}+\mathbb{H})$ to this formula and using the identities (3.8) and (3.9) we get

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\text{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \tag{3.12}$$

As $\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$ is holomorphic, this implies that the second term in the above formula is lower order. Hence $\Theta \approx i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$ and therefore Θ and $\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$ have the same regularity.

3.2.2 Formula for $D_t\Theta$

Apply $|D_{\alpha'}|$ on the formula for $D_t g$ in (2.10) to obtain

$$D_t|D_{\alpha'}|g = -\text{Im}(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t) - \text{Re}(\overline{D}_{\alpha'}\overline{Z}_t)|D_{\alpha'}|g$$

As $|D_{\alpha'}|g = \text{Re}\Theta$, hence $\text{Re}(\overline{D}_{\alpha'}\overline{Z}_t)|D_{\alpha'}|g = \text{Re}\big\{(\overline{D}_{\alpha'}\overline{Z}_t)\text{Re}\Theta\big\} = \text{Im}\big\{i(\overline{D}_{\alpha'}\overline{Z}_t)\text{Re}\Theta\big\}$. Also observe that $D_t|D_{\alpha'}|g = \text{Re}(D_t\Theta)$. Hence we have

$$\text{Re}(D_t\Theta) = -\text{Im}\big\{(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\} \tag{3.13}$$

Now apply $(\mathbb{I}+\mathbb{H})$ on both sides and use the identities (3.8) and (3.9) to get

$$D_t\Theta = i(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t - i\text{Re}(\mathbb{I}-\mathbb{H})\big\{(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\} + i\text{Im}(\mathbb{I}-\mathbb{H})D_t\Theta \tag{3.14}$$

Note that $D_{\alpha'}\overline{Z}_t$ and Θ are holomorphic and as will be shown in the energy estimate, this causes the second and third term in the above formula to be of lower order. Hence $D_t\Theta \approx i(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t$.

3.2.3 Formula for $D_t^2\Theta$

Apply D_t on the formula for $\text{Re}(D_t\Theta)$ in (3.13) to obtain

$$\begin{aligned}
&\text{Re}(D_t^2\Theta)\\
&= -\text{Im}\big\{D_t(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\}\\
&= -\text{Im}\big\{(|D_{\alpha'}| + i\text{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t\big\} + \text{Im}\big\{\text{Re}(\overline{D}_{\alpha'}\overline{Z}_t)|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t - i\text{Re}(D_t\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\}
\end{aligned}$$

Now apply $(\mathbb{I}+\mathbb{H})$ on both sides and use the identities (3.8) and (3.9) to get

$$\begin{aligned}
D_t^2\Theta = {}& i(|D_{\alpha'}| + i\text{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t - i\text{Re}(\mathbb{I}-\mathbb{H})\big\{(|D_{\alpha'}| + i\text{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t\big\}\\
&+ i\text{Im}(\mathbb{I}-\mathbb{H})D_t^2\Theta + (\mathbb{I}+\mathbb{H})\text{Im}\big\{\text{Re}(\overline{D}_{\alpha'}\overline{Z}_t)|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t - i\text{Re}(D_t\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\}
\end{aligned} \tag{3.15}$$

Again in this formula only the first term is the main term and all other terms are lower order. Hence $D_t^2\Theta \approx i(|D_{\alpha'}| + i\text{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t$.

3.2.4 Equation for $\overline{Z}_t$

Apply D_t to the fundamental equation (3.7)

$$\overline{Z}_{ttt} = -i\frac{D_tA_1}{Z_{,\alpha'}} - i\frac{A_1}{Z_{,\alpha'}}\left(Z_{,\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right) - \sigma(D_{\alpha'}Z_t)D_{\alpha'}\Theta + \sigma D_{\alpha'}D_t\Theta$$

Now use the formula for $D_t\dfrac{1}{Z_{,\alpha'}}$ from (2.12) and $D_t\Theta$ from (3.14) to obtain

$$\begin{aligned}
\overline{Z}_{ttt} = {}& -i\frac{1}{Z_{,\alpha'}}\big(D_tA_1 + A_1(b_{\alpha'} - D_{\alpha'}Z_t\overline{D}_{\alpha'}\overline{Z}_t)\big) - i\frac{A_1}{Z_{,\alpha'}}\overline{D}_{\alpha'}\overline{Z}_t - \sigma(D_{\alpha'}Z_t)D_{\alpha'}\Theta\\
&+ i\sigma D_{\alpha'}(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t - i\sigma D_{\alpha'}\text{Re}(\mathbb{I}-\mathbb{H})\big\{(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\}\\
&+ i\sigma D_{\alpha'}\text{Im}(\mathbb{I}-\mathbb{H})D_t\Theta
\end{aligned}$$

Let us define the real valued variable J_1 as

$$\begin{aligned}
J_1 = {}& D_tA_1 + A_1\big(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t\big) + \sigma\partial_{\alpha'}\text{Re}(\mathbb{I}-\mathbb{H})\big\{(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\}\\
&- \sigma\partial_{\alpha'}\text{Im}(\mathbb{I}-\mathbb{H})D_t\Theta
\end{aligned} \tag{3.16}$$

Using this we get

$$\overline{Z}_{ttt} + i\frac{A_1}{Z_{,\alpha'}}\overline{D}_{\alpha'}\overline{Z}_t - i\sigma D_{\alpha'}(|D_{\alpha'}| + i\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t = -\sigma(D_{\alpha'}Z_t)D_{\alpha'}\Theta - i\frac{J_1}{Z_{,\alpha'}} \tag{3.17}$$

We modify this equation slightly to get an equation appropriate for the computation of the lower order term in the energy. Rewrite the above equation as

$$\begin{aligned}&\overline{Z}_{ttt}+i\frac{A_1}{Z_{,\alpha'}}\overline{D}_{\alpha'}\overline{Z}_t-i\sigma D_{\alpha'}\left\{\left(|D_{\alpha'}|\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}+\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right\}\\&=-\sigma(D_{\alpha'}Z_t)D_{\alpha'}\Theta-\sigma D_{\alpha'}\left\{(\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\right\}-i\frac{J_1}{Z_{,\alpha'}}\end{aligned}$$

Multiply by $Z_{,\alpha'}$ and rearrange to get

$$\begin{aligned}&\overline{Z}_{ttt}Z_{,\alpha'}+iA_1\overline{D}_{\alpha'}\overline{Z}_t-i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\\&=i\sigma\partial_{\alpha'}\left\{\left(|D_{\alpha'}|\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}\right\}-\sigma(D_{\alpha'}Z_t)\partial_{\alpha'}\Theta-\sigma\partial_{\alpha'}\left\{(\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\right\}-iJ_1\end{aligned}\tag{3.18}$$

This equation gives rise to the energy $E_{\sigma,1}$ in the energy estimate.

3.2.5 Equation for $\overline{D}_{\alpha'}\overline{Z}_t$

Apply $\overline{D}_{\alpha'}$ to the equation (3.17) and use commutator identities to get

$$\begin{aligned}&D_t^2\overline{D}_{\alpha'}\overline{Z}_t+i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t-i\sigma\overline{D}_{\alpha'}D_{\alpha'}(|D_{\alpha'}|+i\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\\&=-(\overline{D}_{\alpha'}\overline{Z}_t)\overline{D}_{\alpha'}\overline{Z}_{tt}-2(\overline{D}_{\alpha'}\overline{Z}_t)(D_t\overline{D}_{\alpha'}\overline{Z}_t)-i\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right)(\overline{D}_{\alpha'}\overline{Z}_t)\\&\quad-\sigma\overline{D}_{\alpha'}\{(D_{\alpha'}Z_t)D_{\alpha'}\Theta\}-i\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1-\frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\end{aligned}$$

Observe that $-(\overline{D}_{\alpha'}\overline{Z}_t)\overline{D}_{\alpha'}\left(\overline{Z}_{tt}+i\frac{A_1}{Z_{,\alpha'}}\right)=-\sigma(\overline{D}_{\alpha'}\overline{Z}_t)\overline{D}_{\alpha'}D_{\alpha'}\Theta$. Hence we get

$$\begin{aligned}&D_t^2\overline{D}_{\alpha'}\overline{Z}_t+i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t-i\sigma\overline{D}_{\alpha'}D_{\alpha'}(|D_{\alpha'}|+i\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\\&=-2(\overline{D}_{\alpha'}\overline{Z}_t)(D_t\overline{D}_{\alpha'}\overline{Z}_t)-2\sigma\mathrm{Re}(D_{\alpha'}Z_t)\overline{D}_{\alpha'}D_{\alpha'}\Theta-\sigma(\overline{D}_{\alpha'}D_{\alpha'}Z_t)D_{\alpha'}\Theta\\&\quad-i\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1-\frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\end{aligned}$$

Now

$$\overline{D}_{\alpha'}D_{\alpha'} = \frac{1}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\left(\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}|D_{\alpha'}|\right) = \left(\frac{1}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{|Z_{,\alpha'}|}{Z_{,\alpha'}}\right)|D_{\alpha'}| + |D_{\alpha'}|^2 = (|D_{\alpha'}| - i\mathrm{Re}\Theta)|D_{\alpha'}|$$

Hence $\overline{D}_{\alpha'}D_{\alpha'}(|D_{\alpha'}| + i\mathrm{Re}\Theta)$

$$\begin{aligned}
&= (|D_{\alpha'}| - i\mathrm{Re}\Theta)|D_{\alpha'}|(|D_{\alpha'}| + i\mathrm{Re}\Theta) \\
&= (|D_{\alpha'}| - i\mathrm{Re}\Theta)(|D_{\alpha'}| + i\mathrm{Re}\Theta)|D_{\alpha'}| + (|D_{\alpha'}| - i\mathrm{Re}\Theta)(i\mathrm{Re}(|D_{\alpha'}|\Theta)) \\
&= \big(|D_{\alpha'}|^2 + i\mathrm{Re}(|D_{\alpha'}|\Theta) + (\mathrm{Re}\Theta)^2\big)|D_{\alpha'}| + i\mathrm{Re}\big(|D_{\alpha'}|^2\Theta\big) + i\mathrm{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}| \\
&\quad + (\mathrm{Re}\Theta)\mathrm{Re}(|D_{\alpha'}|\Theta) \\
&= |D_{\alpha'}|^3 + \big(2i\mathrm{Re}(|D_{\alpha'}|\Theta) + (\mathrm{Re}\Theta)^2\big)|D_{\alpha'}| + i\mathrm{Re}\big(|D_{\alpha'}|^2\Theta\big) + (\mathrm{Re}\Theta)\mathrm{Re}(|D_{\alpha'}|\Theta)
\end{aligned}$$

Therefore $\overline{D}_{\alpha'}D_{\alpha'}(|D_{\alpha'}| + i\mathrm{Re}\Theta) \approx |D_{\alpha'}|^3$. Hence we get the main equation for $\overline{D}_{\alpha'}\overline{Z}_t$

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\overline{D}_{\alpha'}\overline{Z}_t = R_1 - i\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1 - i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1 \tag{3.19}$$

where

$$\begin{aligned}
R_1 &= -2(\overline{D}_{\alpha'}\overline{Z}_t)(D_t\overline{D}_{\alpha'}\overline{Z}_t) - 2\sigma\mathrm{Re}(D_{\alpha'}Z_t)\overline{D}_{\alpha'}D_{\alpha'}\Theta - \sigma(\overline{D}_{\alpha'}D_{\alpha'}Z_t)D_{\alpha'}\Theta \\
&\quad + i\sigma\big(2i\mathrm{Re}(|D_{\alpha'}|\Theta) + (\mathrm{Re}\Theta)^2\big)|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t - \sigma\mathrm{Re}\big(|D_{\alpha'}|^2\Theta\big)\overline{D}_{\alpha'}\overline{Z}_t \\
&\quad + i\sigma(\mathrm{Re}\Theta)(\mathrm{Re}(|D_{\alpha'}|\Theta))\overline{D}_{\alpha'}\overline{Z}_t
\end{aligned} \tag{3.20}$$

and J_1 was defined in (3.16). This equation gives rise to the energy $E_{\sigma,4}$ in the energy estimate.

3.2.6 Equation for $\overline{Z}_{t,\alpha'}$

Multiply the equation for $\overline{D}_{\alpha'}\overline{Z}_t$ in (3.19) by $\overline{Z}_{,\alpha'}$ to get the equation

$$\begin{aligned}
&\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\overline{Z}_{t,\alpha'} \\
&= R_1\overline{Z}_{,\alpha'} - i\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1 - iD_{\alpha'}J_1 - \overline{Z}_{,\alpha'}\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}
\end{aligned} \tag{3.21}$$

This equation gives rise to the energy $E_{\sigma,2}$ in the energy estimate. This equation will also be useful to get estimates for the term $D_{\alpha'}J_1$.

3.2.7 Equation for Θ

Apply $\overline{D}_{\alpha'}$ to the fundamental equation (3.7) and use $\overline{D}_{\alpha'} D_{\alpha'} = (|D_{\alpha'}| - i\mathrm{Re}\Theta)|D_{\alpha'}|$ to get

$$D_t \overline{D}_{\alpha'} \overline{Z}_t + i\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) - \sigma(|D_{\alpha'}| - i\mathrm{Re}\Theta)|D_{\alpha'}|\Theta = -\big(\overline{D}_{\alpha'}\overline{Z}_t\big)^2$$

Now applying the operator $i(|D_{\alpha'}| + i\mathrm{Re}\Theta)$ we obtain

$$\begin{aligned}
&i(|D_{\alpha'}| + i\mathrm{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t - |D_{\alpha'}|\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) - i\sigma(|D_{\alpha'}| + i\mathrm{Re}\Theta)(|D_{\alpha'}| - i\mathrm{Re}\Theta)|D_{\alpha'}|\Theta \\
&= -2i\big(\overline{D}_{\alpha'}\overline{Z}_t\big)\big(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\big) + (\mathrm{Re}\Theta)\left\{\big(\overline{D}_{\alpha'}\overline{Z}_t\big)^2 + i\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right)\right\}
\end{aligned}$$

Observe that

$$(|D_{\alpha'}| + i\mathrm{Re}\Theta)(|D_{\alpha'}| - i\mathrm{Re}\Theta)|D_{\alpha'}|\Theta = |D_{\alpha'}|^3\Theta - i\mathrm{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}|\Theta + (\mathrm{Re}\Theta)^2|D_{\alpha'}|\Theta$$

Hence we get

$$\begin{aligned}
&i(|D_{\alpha'}| + i\mathrm{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t - |D_{\alpha'}|\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) - i\sigma|D_{\alpha'}|^3\Theta \\
&= -2i\big(\overline{D}_{\alpha'}\overline{Z}_t\big)\big(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\big) + (\mathrm{Re}\Theta)\left\{\big(\overline{D}_{\alpha'}\overline{Z}_t\big)^2 + i\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) + i\sigma(\mathrm{Re}\Theta)|D_{\alpha'}|\Theta\right\} \\
&\quad + \sigma\mathrm{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}|\Theta
\end{aligned}$$

Now recall from (3.12) that $\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}(\mathbb{I} - \mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$. Therefore

$$\begin{aligned}
-|D_{\alpha'}|\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) &= -|D_{\alpha'}|\left\{\frac{A_1}{|Z_{,\alpha'}|}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + \frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\} \\
&= i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta - \frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathrm{Re}(\mathbb{I} - \mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\
&\quad - \left(|D_{\alpha'}|\frac{A_1}{|Z_{,\alpha'}|}\right)\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - |D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&i(|D_{\alpha'}| + i\mathrm{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta - i\sigma|D_{\alpha'}|^3\Theta \\
&= -2i\big(\overline{D}_{\alpha'}\overline{Z}_t\big)\big(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\big) + (\mathrm{Re}\Theta)\left\{\big(\overline{D}_{\alpha'}\overline{Z}_t\big)^2 + i\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) + i\sigma(\mathrm{Re}\Theta)|D_{\alpha'}|\Theta\right\} \\
&\quad + \sigma\mathrm{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}|\Theta + \left(|D_{\alpha'}|\frac{A_1}{|Z_{,\alpha'}|}\right)\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + |D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \\
&\quad + \frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)
\end{aligned}
$$

Recall from (3.15) that

$$
\begin{aligned}
D_t^2\Theta &= i(|D_{\alpha'}| + i\mathrm{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\big\{(|D_{\alpha'}| + i\mathrm{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t\big\} \\
&\quad + i\mathrm{Im}(\mathbb{I}-\mathbb{H})D_t^2\Theta + (\mathbb{I}+\mathbb{H})\mathrm{Im}\big\{\mathrm{Re}(\overline{D}_{\alpha'}\overline{Z}_t)|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t - i\mathrm{Re}(D_t\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\}
\end{aligned}
$$

Hence replacing the term $i(|D_{\alpha'}| + i\mathrm{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t$ in the equation with $D_t^2\Theta$ we get our main equation as

$$
\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\Theta = R_2 + iJ_2 \tag{3.22}
$$

where

$$
\begin{aligned}
R_2 &= -2i\big(\overline{D}_{\alpha'}\overline{Z}_t\big)\big(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\big) + (\mathrm{Re}\Theta)\left\{\big(\overline{D}_{\alpha'}\overline{Z}_t\big)^2 + i\overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) + i\sigma(\mathrm{Re}\Theta)|D_{\alpha'}|\Theta\right\} \\
&\quad + \sigma\mathrm{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}|\Theta + \left(|D_{\alpha'}|\frac{A_1}{|Z_{,\alpha'}|}\right)\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + |D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \\
&\quad + (\mathbb{I}+\mathbb{H})\mathrm{Im}\big\{\mathrm{Re}(\overline{D}_{\alpha'}\overline{Z}_t)|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t - i\mathrm{Re}(D_t\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\} \\
&\quad + \frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)
\end{aligned} \tag{3.23}
$$

$$
J_2 = \mathrm{Im}(\mathbb{I}-\mathbb{H})\big(D_t^2\Theta\big) - \mathrm{Re}(\mathbb{I}-\mathbb{H})\big\{(|D_{\alpha'}| + i\mathrm{Re}\Theta)D_t\overline{D}_{\alpha'}\overline{Z}_t\big\} \tag{3.24}
$$

Note that the variable J_2 is real valued. This equation gives rise to the energy $E_{\sigma,3}$ in the energy estimate.

3.3 Heuristics

Similar to the heuristic explanation in §2.3 we now give a brief heuristic explanation into the nature of our results. One of our main results is Theorem 3.4.1 which is an apriori estimate for the energy E_σ. This result extends the apriori estimate of Kinsey-Wu [KW14] to include the effect of surface tension $\sigma \geq 0$. For the sake of convenience we will repeat a few of the arguments mentioned in §2.3.

Local well-posedness results in water waves are generally proved in the following way:

1. Reduce equation to the boundary
2. Choose appropriate variables, coordinate system and derive a quasilinear equation/system
3. Take derivatives and write down the energy $E(t)$
4. Prove apriori estimate of the form $\frac{dE(t)}{dt} \leq P(E(t))$ where P is a polynomial with non-negative coefficients
5. Local existence using an approximation argument

A typical result using the above approach says that if the initial data satisfies $E(0) < \infty$, then there exists a unique solution to the water wave equation for a time $T = T(E(0)) > 0$ depending only on $E(0)$ and we have $\sup_{t\in[0,T]} E(t) < \infty$. Typically this energy $E(t)$ is equivalent to the Sobolev norm of the initial data. One way of proving local existence for rough initial data would be to lower the regularity of this Sobolev space. For gravity water waves, this was done in the work of [ABZ14b] where the Sobolev norm of the initial data corresponds to an interface being $C^{1,\alpha}$ where $\alpha > \frac{11}{24}$. Note that this does not allow interface with angled crests.

In Kinsey and Wu [KW14], the energy $E(t)$ is not equivalent to the standard Sobolev norm of the initial data. Instead the energy is equivalent to a weighted Sobolev norm with the weight depending nonlinearly on the interface. More precisely the weight can be thought of as the coefficient of the Taylor sign condition $w \approx -\frac{\partial P}{\partial \hat{n}}$. This weight $w \approx 1$ when the interface is $C^{1,\alpha}$ for $\alpha > 0$ but behaves like $\omega(\alpha) \approx \alpha^{1-\nu}$ near for an interface with an angled crest of angle $\nu\pi$. Hence the energy $E(t)$ in Theorem 2.4.2 behaves like the Sobolev norm for smooth enough interfaces and behaves completely different for interfaces with angled crests. The energy used in [KW14] and in Theorem 2.4.2 allows interfaces with angled crests with angles $\nu\pi$ for $0 < \nu < \frac{1}{2}$ and smooth enough interfaces which are $C^{2,\alpha}$ where $\alpha > \frac{1}{2}$.

In the case of non-zero surface tension, observe that the fundamental equation from (3.7) is

$$\bar{Z}_{tt} - i = -i\frac{A_1}{Z_{,\alpha'}} + \sigma D_{\alpha'}\Theta$$

Multiply the above equation by $\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ and take real parts to get

$$\text{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}(\overline{Z}_{tt} - i)\right) = \sigma|D_{\alpha'}|^2 g$$

Observe that $|D_{\alpha'}|^2 g = (\partial_s^2\theta) \circ h^{-1}$ and hence we have the estimate $\|\sigma\partial_s^2\theta\|_\infty \leq \|Z_{tt}\|_\infty + 1$. This estimate holds for all smooth enough solutions.

If we assume that the above computation makes sense even for non-smooth solutions, then this argument shows that the acceleration has to be infinite for non-smooth solutions when $\sigma > 0$. This is in stark contrast to the case of $\sigma = 0$, where the acceleration remains finite even for angled crested interfaces, as was shown in [KW14]. Hence for the case of $\sigma > 0$, we will not allow the initial data to have any singularities, as we work in the regime of finite acceleration. However we will allow the L^∞ norm of the curvature of the initial data to be very large.

The main goal here is to find an energy $E_\sigma(t)$ which allows angled crested interfaces for $\sigma = 0$ and which allows large curvature for $\sigma > 0$, and then prove an apriori estimate for this energy. To do this, we need to choose appropriate variables, coordinate systems and derive quasilinear equations from which we can construct the energy. It should be noted that there is no universal choice of variables or universal quasilinear equation from which we can start our analysis. We choose our variables and then derive quasilinear equations in such a way which helps us to suit our purposes.

There are multiple issues involved in finding such an energy and then proving the energy estimate. We will mostly concentrate on the structural issues which were involved in finding such an energy and only briefly go over the analytical difficulties in proving the energy estimate. The key ideas and difficulties in proving Theorem 3.4.1 are as follows:

1) Structure of the quasilinear equations and the Taylor sign condition:

In §3.2 we derive quasilinear equations in conformal coordinates of the form

$$\left\{D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)\mathbb{H} - i\sigma\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^3\mathbb{H}\right\}f = l.o.t \tag{3.25}$$

where $\mathbb{H} =$ Hilbert transform $\quad i\mathbb{H}\partial_{\alpha'} = |\partial_{\alpha'}|$

$D_t =$ material derivative $\quad \frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'} =$ arc length derivative on the interface

Here $f = \overline{Z}_t Z_{,\alpha'}, \overline{Z}_{t,\alpha'}, \overline{D}_{\alpha'}\overline{Z}_t$ or Θ and the right hand side consists of lower order terms. Note that all there variables are either holomorphic or almost holomorphic i.e. $\mathbb{H}f = f$. Hence by using the relation $i\mathbb{H}\partial_{\alpha'} = |\partial_{\alpha'}|$ we see that the quasilinear equation looks like

$$\left\{D_t^2 + \left(\frac{A_1}{|Z_{,\alpha'}|}\right)\frac{1}{|Z_{,\alpha'}|}|\partial_{\alpha'}| - \sigma\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^2\frac{1}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\right\}f = l.o.t$$

For the operator to be hyperbolic, we need the coefficient $\frac{A_1}{|Z_{,\alpha'}|} \geq 0$. Observe that we have $(-\nabla P\cdot\hat{n}|_{\sigma=0})\circ h^{-1} = \frac{A_1}{|Z_{,\alpha'}|}$ which is always non-negative from the definition of A_1 and also does not depend on σ. Hence the operator is hyperbolic for all values of $\sigma \geq 0$. This is in stark contrast to previous results on zero surface tension limit. In previous works, the structure of the quasilinear equation had the coefficient as $-\nabla P\cdot\hat{n}$ which is the coefficient in the Taylor sign condtion instead of $\nabla P\cdot\hat{n}|_{\sigma=0}$. Recall from (3.6) that

$$-\frac{\partial P}{\partial\hat{n}}\circ h^{-1} = \frac{A_{1,\sigma}}{|Z_{,\alpha'}|} = \frac{1}{|Z_{,\alpha'}|}\big(A_1 + \sigma|\partial_{\alpha'}|(\kappa\circ h^{-1})\big)$$

where $\kappa\circ h^{-1} = -iD_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ is the curvature in conformal coordinates. Hence we see that for large values of σ, the Taylor sign condition may fail i.e. $-\nabla P\cdot\hat{n}$ may become negative. In all previous works on zero surface tension, there was a restriction that σ needs to be small so that the restriction $-\nabla P\cdot\hat{n} \geq 0$ is satisfied. By following this approach we have managed to remove this restriction. The better structure of this quasilinear equation also helps us significantly in understanding the behavior of singular solutions.

2) The energy estimate:

We will now explain the construction of the energy, the choice of variables and do a heuristic energy estimate to understand the difficulties. Note that the quasilinear equation is of the form

$$\left\{D_t^2 + \left(\frac{A_1}{|Z_{,\alpha'}|}\right)\frac{1}{|Z_{,\alpha'}|}|\partial_{\alpha'}| - \sigma\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^2\frac{1}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\right\}f = l.o.t$$

If we multiply by $D_t\bar{f}$ then by integrating we get the energy

$$\|D_tf\|_2^2 + \left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

This is done for $f = \overline{Z}_{t,\alpha'}$ and $f = \Theta$ to obtain the energies $E_{\sigma,2}$ and $E_{\sigma,3}$ respectively. If we instead we multiply the equation by $|\partial_{\alpha'}|D_t\bar{f}$ and integrate we get the energy

$$\|D_t f\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}f\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}|D_{\alpha'}|f\right\|_2^2$$

This is done for $f = \overline{Z}_{t,\alpha'}Z_{,\alpha'}$ and $f = \overline{D}_{\alpha'}\overline{Z}_t$ to obtain the energies $E_{\sigma,1}$ and $E_{\sigma,4}$ respectively. The energy $E_{\sigma,0}$ is a collection of lower order terms which are needed to close the energy estimate. We will explain its importance in the next item.

Let us now do a heuristic energy estimate. If the interface is $C^{1,\alpha}$ then we have $0 < c_1 \leq \frac{1}{|Z_{,\alpha'}|} \leq c_2 < \infty$ and hence for smooth enough interfaces the main operator in (3.25) behaves like $\partial_t^2 + |\partial_{\alpha'}| + \sigma|\partial_{\alpha'}|^3$ for which standard energy estimates in Sobolev spaces work. If the interface has an angled crest of angle $\nu\pi$ at $\alpha = 0$, then $Z(\alpha) \sim \alpha^\nu$ and hence $\frac{1}{|Z_\alpha|} \sim |\alpha|^{1-\nu}$ near $\alpha = 0$ and hence the quasilinear equation near $\alpha = 0$ behaves like

$$\begin{aligned}&\left\{\partial_t^2 + |\alpha|^{2-2\nu}|\partial_\alpha| + \sigma|\alpha|^{3-3\nu}|\partial_\alpha|^3\right\}f \\ &= |\alpha|^{1-2\nu}f + \sigma|\alpha|^{2-3\nu}|\partial_\alpha|^2 f + \sigma|\alpha|^{1-3\nu}|\partial_\alpha|f + \sigma|\alpha|^{-3\nu}f + \text{other l.o.t}\end{aligned}$$

We have included a few simplified versions of the lower order terms to demonstrate the issues in proving an energy estimate. We obtain our energies by multiplying the above equation with either $\partial_t f$ or $|\partial_\alpha|\partial_t f$ and then integrating. If we multiply the equation by $\partial_t f$ and integrate, we obtain control of $\partial_t f \in L^2, |\alpha|^{1-\nu}f \in \dot{H}^{\frac{1}{2}}$ and $\sigma^{\frac{1}{2}}|\alpha|^{\frac{3}{2}-\frac{3}{2}\nu}|\partial_\alpha|f \in \dot{H}^{\frac{1}{2}}$. For simplicity also assume that $f \in L^2$ which is compatible with $\partial_t f \in L^2$. To close the energy estimate, we need to control the L^2 norm of the right hand side. Hence to control the first term $|\alpha|^{1-2\nu}f \in L^2$, we obtain the restriction $\nu \leq \frac{1}{2}$ which is one of the main reasons of the restrictions on the angles in Kinsey and Wu [KW14]. Note that we cannot control the term $\sigma|\alpha|^{2-3\nu}|\partial_\alpha|^2 f \in L^2$ as we only have control of $3/2$ derivatives on f from the energy. For smooth enough interfaces, it was observed by Ambrose-Masmoudi [AM05] that by carefully choosing f (by using variables derived from θ), this term does not appear in the quasilinear equation and we follow the same approach. We do not use the modified tangential velocity as in [AM05] and instead use the more natural material derivative D_t along with the variable θ to obtain our highest order quasilinear equation.

Assuming we can manage to remove this term, we still need to control $\sigma|\alpha|^{1-3\nu}|\partial_\alpha|f \in L^2$ and $\sigma|\alpha|^{-3\nu}f \in L^2$. As we only have $f \in L^2$, there is no way we can control the term $\sigma|\alpha|^{-3\nu}f \in L^2$ and this is the reason why we do not allow angled crest data if $\sigma > 0$. Hence if we work with the smooth interface $Z^\epsilon = Z * P_\epsilon$ where P_ϵ is the Poisson kernel, then this has the effect of changing $|\alpha| \mapsto |-i\epsilon + \alpha|$ near $\alpha = 0$. Hence to close the energy estimate, we obtain the restriction $\sigma\epsilon^{-3\nu} \lesssim 1$. For the interface Z^ϵ, the curvature $\kappa \sim \epsilon^{-\nu}$ and hence this can be written as $\sigma\kappa^3 \lesssim 1$. A similar argument for $\sigma|\alpha|^{1-3\nu}|\partial_\alpha|f \in L^2$ also yields the same restriction. Note that these restrictions do not depend on the choice of f, but is purely a consequence of the structure of the quasilinear equation and attempting to prove an L^2 based energy estimate.

A key difficulty in implementing this heuristic energy estimate is to find a suitable f and obtain a corresponding quasilinear equation for this variable. As it so happens, for most choices of variables that we choose for which we can obtain a quasilinear equation, the structure of the quasilinear equation will still be the same as one given by (3.25). Hence we now focus just on the choice of the variable f and hope that we can derive a quasilinear equation for this variable. We need the following properties for f

(a) f needs to have appropriate weights so that the energy allows angled crests solutions when $\sigma = 0$

(b) The quasilinear equation for f should not have terms like $\sigma|\alpha|^{2-3\nu}|\partial_\alpha|^2 f$ in the errors, to be able to close the energy estimate.

For $\sigma = 0$, the second restriction does not show up and this was seen in the work of Kinsey-Wu [KW14] where the weighted derivative $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}$ was used to obtain higher order energies. If we work in the smooth case, then we don't need to worry about the first restriction and this was seen in the work of Ambrose-Masmoudi [AM05] where the arc length derivative $\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}$ was used to obtain the higher order energies. However we are working in the regime where both of the restrictions apply and these are rather severe restrictions on f. A simple computation shows that in particular if f has such properties, then no weighted derivate of the form $w\partial_\alpha f$ will also satisfy the same properties and hence obtaining higher order energies is non obvious.

Fortunately instead of working with spacial derivatives, if we work with the material derivative D_t, then both of the difficulties are resolved i.e. if f satisfies a quasilinear equation satisfying both the restrictions then the variable $D_t f$ will also satisfy both the restrictions. This is the fundamental reason why we use the material derivate quite heavily in this work. The highest order energy in E_σ namely $E_{\sigma,4}$ corresponds to an $f = \overline{D}_{\alpha'}\overline{Z}_t$ which is related to the material derivative of the angle by the relation $\mathrm{Im}(\overline{D}_{\alpha'}\overline{Z}_t) = -(\partial_t\theta)\circ h^{-1}$. This variable $\overline{D}_{\alpha'}\overline{Z}_t$ also has the useful property that it is almost holomorphic i.e. $\mathbb{H}(\overline{D}_{\alpha'}\overline{Z}_t) \approx \overline{D}_{\alpha'}\overline{Z}_t$, which is quite useful to prove the energy estimate. The energy $E_{\sigma,4}$ is the most important part of the energy E_σ.

The energy $E_{\sigma,3}$ corresponding to the variable Θ should be thought of as a complimentary energy to $E_{\sigma,4}$. For $\sigma = 0$ the energies $E_{\sigma,3}$ and $E_{\sigma,4}$ are equivalent, whereas for $\sigma > 0$ the energy $E_{\sigma,4}$ is higher order as compared to $E_{\sigma,3}$. The energies $E_{\sigma,1}$ and $E_{\sigma,2}$ can be thought of as lower order energies as compared to $E_{\sigma,4}$.

3) Lower order terms:

We now explain the introduction of the energy $E_{\sigma,0}$. Note that

$$E_{\sigma,0} = \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\infty}^{2} + \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{2}^{6} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{2}^{2} + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^{2}\frac{1}{Z_{,\alpha'}}\right\|_{2}^{2}$$

The last two terms can actually be obtained from the quasilinear equation for the variable $f = \mathbb{P}_H(g)$ and the by multiplying the equation by $|\partial_{\alpha'}|\mathbb{P}_H(g)$ and integrating. The reason we do not do this is that is simply more convenient to just add these terms directly. The introduction of the first two terms is more complicated. As it turns out, we can replace the first two terms with the term $\left\|\sigma^{\frac{1}{3}}Z_{,\alpha'}\right\|_{\infty}^{3}$ and we would be still be able to close the energy estimate. One way to think about this term is as by the following argument: It is very tempting to think that the quasilinear equation is of the form

$$\left\{D_t^2 + iA_1\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\mathbb{H} - i\left(\sigma^{\frac{1}{3}}|Z_{,\alpha'}|\right)^3\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)^3\mathbb{H}\right\}f = l.o.t$$

We can then think of $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}$ as the main weighted derivative (as was done in Kinsey-Wu [KW14]) and assume $\sigma^{\frac{1}{3}}|Z_{,\alpha'}| \in L^{\infty}$. Indeed adding the term $\left\|\sigma^{\frac{1}{3}}Z_{,\alpha'}\right\|_{\infty}$ in the energy E_{σ} considerably simplifies the proof of the apriori estimate Theorem 3.4.1 and the proof of the convergence result Theorem 5.0.1. However this has the drawback of yielding a worse result in terms of the scaling, as instead of the restriction $\frac{\sigma}{\epsilon^{\frac{3}{2}}} \leq 1$ in Proposition 3.4.9, we would need the more restrictive condition of $\frac{\sigma}{\epsilon^{3}} \leq 1$. The restriction $\frac{\sigma}{\epsilon^{\frac{3}{2}}} \leq 1$ is more natural as this corresponds to letting $\nu \uparrow \frac{1}{2}$ in $\sigma\kappa^3 \sim \sigma\epsilon^{-3\nu} \lesssim 1$ which we saw in the heuristic energy estimate. Our energy E_{σ} has no lower order terms with respect to the scaling which keeps gravity constant (See Theorem 3.4.1). These are the main reasons for the introduction of the first two terms in $E_{\sigma,0}$ instead of the simpler $\left\|\sigma^{\frac{1}{3}}Z_{,\alpha'}\right\|_{\infty}^{3}$.

4) Analytical difficulties:

In addition to the structural difficulties due to surface tension explained above, we also face numerous analytical difficulties. Even in the special case of $\sigma = 0$, the energy $E_{\sigma}|_{\sigma=0}$ is lower order as compared to the energy in Kinsey-Wu [KW14] by half weighted spacial derivative and we crucially do not have $D_{\alpha'}\overline{Z}_{tt} \in L^{\infty}$. This makes our estimates, even in the case of

$\sigma = 0$, much more subtle. In addition we now have a lot of nonlinear terms due to surface tension which we need to control. To overcome these issues, we define weighted function spaces adapted to our problem and prove estimates for these spaces (see Lemma 2.4.5). We use these function spaces along with estimates from harmonic analysis to control the nonlinear terms.

Let us now explain the energy $\mathcal{E}_\sigma$ heuristically (E_σ and $\mathcal{E}_\sigma$ are equivalent) defined in §3.4 and show that it is a natural energy. Observe that the quasilinear equation is of the form

$$\left\{D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|}\left(\frac{1}{|Z_{,\alpha'}|}\partial_\alpha\right)\mathbb{H} - i\sigma\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^3\mathbb{H}\right\}f = l.o.t$$

We show in the energy estimate that $1 \leq A_1 \leq 1 + \left\|\bar{Z}_{t,\alpha'}\right\|_2^2$ and hence we can consider $A_1 \approx 1$. Also in the special case of zero velocity $Z_t \equiv 0$, we actually have $A_1(\alpha') = 1$ for all $\alpha' \in \mathbb{R}$. Hence the main operators are D_t, $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}$ and $\dfrac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}$. Recall that the operator $\mathbb{P}_H = \dfrac{\mathbb{I}+\mathbb{H}}{2}$ and has the property that for any smooth real valued function $f : \mathbb{R} \to \mathbb{R}$ vanishing at infinity, $\mathbb{P}_H(f) : \mathbb{R} \to \mathbb{C}$ is the boundary value of a holomorphic function in the lower half plane vanishing at infinity with $\text{Re}\{\mathbb{P}_H(f)\} = f/2$.

A very natural restriction is to have $\theta \in L^\infty$. The spaces L^∞ and $\dot{H}^{\frac{1}{2}}$ have the same scaling in dimension one and hence heuristically we will switch between the two. The energy $\mathcal{E}_\sigma$ is obtained by applying the above operators to the relation $\theta \in L^\infty$ (or equivalently $g = \theta \circ h^{-1} \in L^\infty$), with suitable replacements of L^∞ with $\dot{H}^{\frac{1}{2}}$. The energy $\mathcal{E}_\sigma = \mathcal{E}_{\sigma,1} + \mathcal{E}_{\sigma,2}$ and using this $\mathcal{E}_{\sigma,1}$ has the heuristic representation

1) $\displaystyle \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \sim \left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)^{\frac{1}{2}}\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

2) $\displaystyle \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \sim \left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

3) $\displaystyle \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \sim \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^{\frac{1}{2}}\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

4) $\displaystyle \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \sim \left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)^{-\frac{1}{2}}\left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^{\frac{3}{2}}\mathbb{P}_H(g) \in L^\infty$

5) $\displaystyle \left\|\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 \sim \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^{\frac{3}{2}}\mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

6) $\displaystyle \left\| \frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_{\dot{H}^{\frac{1}{2}}} \sim \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right)^{\frac{1}{2}} \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|} \partial_{\alpha'} \right)^{\frac{3}{2}} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

7) $\displaystyle \|\sigma \partial_{\alpha'} \Theta\|_{\dot{H}^{\frac{1}{2}}} \sim \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right)^{-1} \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|} \partial_{\alpha'} \right)^{3} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

8) $\displaystyle \left\| \frac{\sigma}{Z_{,\alpha'}} \partial_{\alpha'}^3 \frac{1}{Z_{,\alpha'}} \right\|_{2} \sim \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right)^{-\frac{1}{2}} \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|} \partial_{\alpha'} \right)^{3} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

9) $\displaystyle \left\| \frac{\sigma}{Z_{,\alpha'}^2} \partial_{\alpha'}^3 \frac{1}{Z_{,\alpha'}} \right\|_{\dot{H}^{\frac{1}{2}}} \sim \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|} \partial_{\alpha'} \right)^{3} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

We have the formula $D_t g = -\mathrm{Im}(\overline{D}_{\alpha'} \overline{Z}_t)$ from (2.10) and hence heuristically we can replace $\overline{D}_{\alpha'} \overline{Z}_t$ with $D_t \mathbb{P}_H(g)$. Hence $\mathcal{E}_{\sigma,2}$ has the heuristic representation

1) $\displaystyle \|\overline{Z}_{t,\alpha'}\|_2 \sim D_t \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right)^{-\frac{1}{2}} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

2) $\displaystyle \left\| \frac{1}{Z_{,\alpha'}^2} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \sim D_t \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right)^{\frac{1}{2}} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

3) $\displaystyle \left\| \frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \sim D_t \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right)^{-1} \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|} \partial_{\alpha'} \right)^{\frac{3}{2}} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

4) $\displaystyle \left\| \frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}} \partial_{\alpha'}^2 \overline{Z}_{t,\alpha'} \right\|_2 \sim D_t \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|} \partial_{\alpha'} \right)^{\frac{3}{2}} \mathbb{P}_H(g) \in \dot{H}^{\frac{1}{2}}$

To understand how the energy looks like in the arc length coordinate system, we first define $\underline{\mathfrak{a}} = -\left.\dfrac{\partial P}{\partial \hat{n}}\right|_{\sigma=0,\mathbf{v}\equiv 0}$. Now in arc length coordinate system the main operators are then $D_t, \underline{\mathfrak{a}}\partial_s$ and $\sigma^{\frac{1}{3}}\partial_s$ where we again write the material derivate as D_t, $\underline{\mathfrak{a}}$ corresponds to the weight $\dfrac{1}{|Z_{,\alpha'}|}$ and ∂_s is the arc length derivative which corresponds to the operator $\dfrac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}$. We let $\mathbb{P}_{hol}$ denote the linear operator defined by the property that for any smooth real valued function $f : \Sigma \to \mathbb{R}$ vanishing at infinity, $\mathbb{P}_{hol}(f) : \Sigma \to \mathbb{C}$ is the boundary value of a holomorphic function on Ω vanishing at infinity with $\mathrm{Re}\{\mathbb{P}_{hol}(f)\} = f/2$. We can do the same kind of heuristic representation as done above. Here we would have

1) $\displaystyle \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2 \sim (\underline{\mathfrak{a}}\partial_s)^{\frac{1}{2}} \mathbb{P}_{hol}(\theta) \in \dot{H}^{\frac{1}{2}}$

2) $\left\| \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{\dot{H}^{\frac{1}{2}}} \sim (\underline{\mathfrak{a}}\partial_s)\mathbb{P}_{hol}(\theta) \in \dot{H}^{\frac{1}{2}}$

3) $\left\| \sigma^{\frac{1}{6}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2 \sim (\sigma^{\frac{1}{3}}\partial_s)^{\frac{1}{2}}\mathbb{P}_{hol}(\theta) \in \dot{H}^{\frac{1}{2}}$

and so on. This is exactly analogous to the heuristic representation we say for the energy $\mathcal{E}$ for the gravity water waves.

3.4 Main results

We now describe our results. Define

$$
\begin{aligned}
E_{\sigma,0} &= \left\| \sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_\infty^2 + \left\| \sigma^{\frac{1}{6}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2^6 + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2^2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_2^2 \\
E_{\sigma,1} &= \left\| (\overline{Z}_{tt} - i) Z_{,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\| \sqrt{A_1} \overline{Z}_{t,\alpha'} \right\|_2^2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2^2 \\
E_{\sigma,2} &= \left\| D_t \overline{Z}_{t,\alpha'} \right\|_2^2 + \left\| \sqrt{A_1} \frac{\overline{Z}_{t,\alpha'}}{|Z_{,\alpha'}|} \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}}^2 \\
E_{\sigma,3} &= \| D_t \Theta \|_2^2 + \left\| \sqrt{A_1} \frac{\Theta}{|Z_{,\alpha'}|} \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\|_{\dot{H}^{\frac{1}{2}}}^2 \\
E_{\sigma,4} &= \left\| D_t \overline{D}_{\alpha'} \overline{Z}_t \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\| \sqrt{A_1} |D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t \right\|_2^2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t \right\|_2^2 \\
E_\sigma &= E_{\sigma,0} + E_{\sigma,1} + E_{\sigma,2} + E_{\sigma,3} + E_{\sigma,4}
\end{aligned}
$$

Observe that the variables used above are all very natural. Z_t and Z_{tt} are the velocity and acceleration on the boundary respectively, Θ is twice the holomorphic projection of the curvature and $\overline{D}_{\alpha'}\overline{Z}_t$ is related to the material derivative of the angle by the relation $\mathrm{Im}\big(\overline{D}_{\alpha'}\overline{Z}_t\big) = -(\partial_t\theta) \circ h^{-1}$ from (2.10). The weight $\frac{1}{|Z_{,\alpha'}|}$ is related to the Taylor sign condition from (3.6).

The energies $E_{\sigma,i}$ for $1 \leq i \leq 4$ are obtained from the quasilinear equations derived in §3.2 whereas the energy $E_{\sigma,0}$ is added as a lower order term. For $\sigma = 0$, the energies $E_{\sigma,3}$ and $E_{\sigma,4}$ are equivalent, but for $\sigma > 0$ the energy $E_{\sigma,4}$ is higher order than $E_{\sigma,3}$. We now state our main result.

Theorem 3.4.1. *Let $T > 0$ and let (Z, Z_t) be a smooth solution [1] to the capillary gravity water wave equation with surface tension $\sigma \geq 0$ in the time interval $[0,T)$ with $E_\sigma(t) < \infty$ for all*

[1]It is enough to assume that the solution satisfies $(Z_{,\alpha'} - 1, Z_t) \in C([0,T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}))$ for $s \geq 10$

$t \in [0,T)$. *Then there exists a polynomial P with universal non-negative coefficients such that for all* $t \in [0,T)$ *we have*

$$\frac{dE_\sigma(t)}{dt} \leq P(E_\sigma(t))$$

Remark 3.4.2. We mention a minor technical point in the statement of the theorem. The energy $E_{\sigma,0}$ contains a term which is the L^∞ norm of a function and hence may not in general be differentiable in time even for smooth solutions. Hence for this term we replace the time derivative by the upper Dini derivative $\limsup_{s\to 0^+} \frac{\|f(t+s)\|_\infty - \|f(t)\|_\infty}{s}$

Properties:

1. Energy is positve for all σ : For any smooth data decaying sufficiently at infinity, the energy is positive and finite for all $\sigma \geq 0$. In particular the energy is positive even if the Taylor sign condition fails which may happen for large values of σ as may be observed from the formula of $A_{1,\sigma}$. This is in contrast to the work in [AM05], where the authors assume σ small to obtain a positive energy.

2. Energy estimate is completely independent of σ: In the above theorem the polynomial P has universal coefficients and hence is independent of σ

3. Energy allows angled crest solutions for $\sigma = 0$: If we put $\sigma = 0$ in the energy E_σ, then it allows solutions with angled crest with angle less than $90°$. These are exactly the solutions allowed by the energy obtained by Kinsey and Wu in [KW14] and our energy is equivalent to theirs. (More precisely the above energy for $\sigma = 0$ is lower order as compared to the energy in [KW14] by half spacial derivative)

4. Energy does not allow angled crest solution for $\sigma > 0$: In the proof of this theorem we show the estimate $\left\|\partial_s\theta\right\|_\infty \leq \sigma^{-\frac{1}{3}}P(E_\sigma)$ and hence for $\sigma > 0$ the curvature is finite, which automatically excludes angled crest solutions. Note however that for small values of σ, the energy allows data with quite large curvature of the order of $\sigma^{-\frac{1}{3}}$. (See Proposition 3.4.9 where for any given $\epsilon > 0$ arbitrarily small, we construct examples where $E_\sigma = O(1)$ and the curvature of the initial data grows like $\sigma^{-\frac{1}{3}+\epsilon}$ as $\sigma \to 0$)

5. Scaling: Let $Z_\lambda(\alpha',t) = \lambda^{-1}Z(\lambda\alpha', \sqrt{\lambda}t)$ and $Z_{t\lambda} = \lambda^{-\frac{1}{2}}Z_t(\lambda\alpha', \sqrt{\lambda}t)$. Then as can be seen directly from the equation, $(Z_\lambda, Z_{t\lambda})$ is a solution to the water wave equation with the same gravity $-\mathbf{k}$ but surface tension $\sigma\lambda^{-2}$. The energy E_σ has the following property: If $(E_\sigma)_\lambda$ represents the energy corresponding to the data $(Z_\lambda, Z_{t\lambda})$, then we have $(E_\sigma)_\lambda \leq E_\sigma$ for $\lambda \leq 1$ and $(E_\sigma)_\lambda \geq E_\sigma$ for $\lambda \geq 1$. This says that there are no lower order terms in the energy with respect to this scaling.

We now give a more convenient expression for the energy and its relation to the Sobolev norm of the data. Define

$$\begin{aligned}
\mathcal{E}_{\sigma,1} &= \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\sigma^{\frac{1}{6}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^6 + \left\|\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2 \\
&\quad + \left\|\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\sigma\partial_{\alpha'}\Theta\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2^2 \\
&\quad + \left\|\frac{\sigma}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}^2 \\
\mathcal{E}_{\sigma,2} &= \left\|\bar{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\bar{Z}_{t,\alpha'}\right\|_2^2 \\
\mathcal{E}_\sigma &= \mathcal{E}_{\sigma,1} + \mathcal{E}_{\sigma,2}
\end{aligned}$$

Proposition 3.4.3. *There exists universal polynomials P_1, P_2 with non-negative coefficients so that for smooth solutions to the water wave equation with surface tension $\sigma \geq 0$ we have*

$$E_\sigma \leq P_1(\mathcal{E}_\sigma) \quad \textit{and } \mathcal{E}_\sigma \leq P_2(E_\sigma)$$

For $\sigma = 0$, this energy is the same as the energy $\mathcal{E}$ from §2.4 and hence we can see that the energy allows angled crest interfaces as shown in §2.6. For $\sigma > 0$, in contrast we see that most of the terms with surface tension in $\mathcal{E}_{\sigma,1}$ do not allow angled crest interfaces. Indeed one can directly see that the natural extension of the energy of Kinsey-Wu [KW14] does not allow any singularities in the interface. Observe that in Kinsey-Wu [KW14], the quantity $\bar{Z}_{tt,\alpha'} \in L^2$ and $\partial_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) \in L^2$. Hence if we assume these when $\sigma > 0$, then by the equation (3.7) we see that $\sigma\partial_{\alpha'}D_{\alpha'}\Theta \in L^2$. Hence $|D_{\alpha'}|\Theta \in L^\infty_{loc}$ and as $\mathrm{Re}\Theta = \kappa \circ h^{-1}$, we see that $\partial_s\kappa \in L^\infty_{loc}$. Hence the interface has to be at least $C^{2,1}$ which rules out any type of singularity.

In fact when $\sigma > 0$, the condition $\mathcal{E}_\sigma < \infty$ forces the interface to be $C^{3,\alpha}$ for $0 < \alpha < 1$ which follows from the proposition below. However even though the interface is quite smooth, we still do not have a good bound on the curvature and in fact the curvature can be very large (See Proposition 3.4.9 to see that the curvature can be of order $\sigma^{-\frac{1}{3}}$). The following simple proposition says that if we ignore the weights in $\mathcal{E}_\sigma$, then we immediately obtain the Sobolev norm.

Proposition 3.4.4. *There exists universal polynomials P_1, P_2 with non-negative coefficients so that* *for smooth solutions to the water wave equation with surface tension $\sigma > 0$ we have*

$$\left\|\overline{Z}_{t,\alpha'}\right\|_{H^2} + \|\partial_{\alpha'} Z_{,\alpha'}\|_{H^{2.5}} \leq P_1\left(\mathcal{E}_\sigma + \|Z_{,\alpha'}\|_\infty + \frac{1}{\sigma}\right) \quad \textit{and}$$
$$\mathcal{E}_\sigma \leq P_2\left(\left\|\overline{Z}_{t,\alpha'}\right\|_{H^2} + \|\partial_{\alpha'} Z_{,\alpha'}\|_{H^{2.5}} + \left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty + \sigma\right)$$

Remark 3.4.5. If $\mathcal{E}_\sigma < \infty$ and $\sigma > 0$, then we in fact have $Z_{,\alpha'} \in L^\infty$ but the norm $\|Z_{,\alpha'}\|_\infty$ depends on $\sigma^{-\frac{1}{3}}$ and the rate at which $Z_{,\alpha'} \to 1$ as $|\alpha'| \to \infty$. We show in the proof of Theorem 3.4.1 that $\|\sigma^{\frac{1}{3}}\kappa\|_\infty \leq C(\mathcal{E}_\sigma)$ and hence the curvature $\kappa \in L^\infty$. Therefore by the Kellogg-Warschawski theorem (see chapter 3 of [Pom92]), the derivative of the Riemann mapping extends continuously to the boundary and hence $Z_{,\alpha'} \in L^\infty_{loc}$. As $Z_{,\alpha'} \to 1$ when $|\alpha'| \to \infty$, we have that $Z_{,\alpha'} \in L^\infty$.

We can now use local well-posedness results in Sobolev spaces along with the above apriori estimate Theorem 3.4.1 to obtain an existence result in terms of the energy $\mathcal{E}_\sigma$. We use the existence result of Alazard-Burq-Zuily [ABZ11] to obtain the following

Corollary 3.4.6. *Let $\sigma > 0$ and assume the initial data $(Z, Z_t)|_{t=0}$ satisfies the following conditions*

1. *The interface Z is a graph and Im$(Z) \in L^2$, $Z_{,\alpha'} - 1 \in L^2$, $Z_t \in L^2$*
2. *$\mathcal{E}_\sigma(0) < \infty$*

Then there exists a $T > 0$ depending only on $\mathcal{E}_\sigma(0)$ such that the initial value problem to the equation (3.7) *has a unique solution (Z, Z_t) in the time interval $[0, T]$ satisfying $\sup_{t\in[0,T]} \mathcal{E}_\sigma(t) < \infty$.*

Remark 3.4.7. In the above statement, the restrictions of Z being a graph and Im$(Z) \in L^2$ come from the existence result [ABZ11]. We will remove these restrictions in an upcoming article.

The novelty of the above result is that the time of existence depends only on $\mathcal{E}_\sigma(0)$. The usefulness of the energy $\mathcal{E}_\sigma$ comes from the fact that there are interfaces (such as smooth interfaces close to being angled crest) for which the $C^{1,\alpha}$ norm (for any $\alpha > 0$) of the interface of the initial data is very large but $\mathcal{E}_\sigma$ remains bounded. This translates into a longer time of existence if we use the energy $\mathcal{E}_\sigma$ instead of the Sobolev norm. See Proposition 3.4.9 below for an explicit example.

Result on Convergence:

We now explain our results about convergence. Let A be a solution to the water wave equation with surface tension σ and B a solution to the water wave equation with no surface tension. Hence we want to show that

$$A \to B \quad \text{as} \quad \sigma \to 0$$

in an appropriate sense. We will denote by f_a the function f for solution A and f_b for solution B. For e.g. $(\overline{Z}_{t,\alpha'})_a$ and $(\overline{Z}_{t,\alpha'})_b$ denotes the spacial derivative of the velocity for the respective solutions. Note that with this notation equation (3.7) becomes

$$(\overline{Z}_{tt})_a - i = -i\left(\frac{A_1}{Z_{,\alpha'}}\right)_a + \sigma(D_{\alpha'}\Theta)_a \quad \text{where as} \quad (\overline{Z}_{tt})_b - i = -i\left(\frac{A_1}{Z_{,\alpha'}}\right)_b$$

We also have the operators

$$(|D_{\alpha'}|)_a = \frac{1}{|Z_{,\alpha'}|_a}\partial_{\alpha'} \quad (|D_{\alpha'}|)_b = \frac{1}{|Z_{,\alpha'}|_b}\partial_{\alpha'} \quad \text{etc.}$$

Now let h_a, h_b be the change of coordinate diffeomorphisms as defined in (2.2) for the solutions A and B and let the material derivatives by given by $(D_t)_a = U_{h_a}^{-1}\partial_t U_{h_a}$ and $(D_t)_b = U_{h_b}^{-1}\partial_t U_{h_b}$. We define

$$\widetilde{h} = h_b \circ h_a^{-1} \quad \text{and} \quad \widetilde{U} = U_{\widetilde{h}} = U_{h_a}^{-1}U_{h_b}$$

While taking the difference of the two solutions, we want to subtract in Lagrangian coordinates and then bring it to the Riemmanian coordinate system of A. The reason we want to subtract in the Lagrangian coordinate system is that, in our proof of the energy estimate we mainly used the material derivative, and in Lagrangian coordinate system the material derivative for both the solutions is given by the same operator ∂_t. The operator $\widetilde{U}$ first takes a function in the Riemmanian coordinate system of B to the Lagrangian coordinate system and then to the Riemmanian coordinate system of A.

We define $\Delta(f) = f_a - \widetilde{U}(f_b)$. For e.g.

$$\Delta(\overline{Z}_{t,\alpha'}) = (\overline{Z}_{t,\alpha'})_a - \widetilde{U}(\overline{Z}_{t,\alpha'})_b \qquad \Delta(\overline{Z}_{tt}) = -i\left\{\left(\frac{A_1}{Z_{,\alpha'}}\right)_a - \widetilde{U}\left(\frac{A_1}{Z_{,\alpha'}}\right)_b\right\} + \sigma(D_{\alpha'}\Theta)_a$$

where we have written $\widetilde{U}(f)_b$ instead of $\widetilde{U}(f_b)$ for easier readability. To state our convergence result, we first need to define a few more norms. We define the higher order energy for zero surface tension solutions as

$$\mathcal{E}_{high} = \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \|\overline{Z}_{t,\alpha'}\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

This energy $\mathcal{E}_{high}$ is equivalent to the energy in Kinsey-Wu [KW14] where an apriori estimate for the energy is proved (The energy in Kinsey-Wu also has the term $\left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty$ which is not present in $\mathcal{E}_{high}$). Note that this energy is half weighted spacial derivative i.e. $\left(\frac{1}{\left|Z_{,\alpha'}\right|^2}\partial_{\alpha'}\right)^{\frac{1}{2}}$ higher order than the energy $\mathcal{E}_\sigma|_{\sigma=0}$. We reprove this energy estimate in §5.1 as we need additional control of terms from the energy which is not proved as part of the proof in [KW14]. We also define an auxiliary energy

$$
\begin{aligned}
(\mathcal{E}_{\sigma,aux})_b = &\left\|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_b\right\|_\infty^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_b\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_b\right\|_2^2 \\
&+ \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right)_b\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\bar{Z}_{t,\alpha'}\right)_b\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^2\bar{Z}_{t,\alpha'}\right)_b\right\|_{\dot{H}^{\frac{1}{2}}}^2
\end{aligned}
$$

Note carefully that the index here is B which is the zero surface tension solution. Hence this term couples the zero surface tension solution B with the coefficient of surface tension σ from the capillary gravity water wave solution A. This energy is needed as part of the energy for the difference due to technical reasons which will be explained in §5.2. Define the difference of the solutions by taking the difference in the terms of the energy $\mathcal{E}_\sigma$ and adding $(\mathcal{E}_{\sigma,aux})_b$

$$
\begin{aligned}
\mathcal{E}_{\Delta,1} = &\left\|\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2^2 + \left\|\Delta\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\left(\sigma^{\frac{1}{6}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^6 \\
&+ \left\|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_\infty^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2 \\
&+ \left\|(\sigma\partial_{\alpha'}\Theta)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\left(\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^2 + \left\|\left(\frac{\sigma}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2 \\
\mathcal{E}_{\Delta,2} = &\left\|\Delta\left(\bar{Z}_{t,\alpha'}\right)\right\|_2^2 + \left\|\Delta\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right)\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right)_a\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\bar{Z}_{t,\alpha'}\right)_a\right\|_2^2 \\
\mathcal{E}_\Delta = &\left(\mathcal{E}_{\sigma,aux}\right)_b + \mathcal{E}_{\Delta,1} + \mathcal{E}_{\Delta,2}
\end{aligned}
$$

We now state our main result on convergence. We will state the theorem here only for the special case of the two solutions having the same initial data and a more general result is stated in Chapter 5. The existence part of this result follows from earlier results. For $\sigma = 0$, one can use the existence result Theorem 3.4 of Wu [Wu15], where it is shown that for data in the class $\mathcal{E}_{high}(0) < \infty$, the solution exists in a time interval $[0, T]$ with T depending only on $\mathcal{E}_{high}(0)$ and satisfies $\sup_{[0,T]}\mathcal{E}_{high}(t) < \infty$. For $\sigma > 0$ we can use Corollary 3.4.6 for an existence result in terms of $\mathcal{E}_\sigma$. The main result is the estimate for the difference of the solutions.

Theorem 3.4.8. *Fix $\sigma_0 > 0$ and let (Z^σ, Z_t^σ) and (Z, Z_t) be solutions to the equation* (3.7) *with the same smooth initial data $(Z^\sigma, Z_t^\sigma)|_{t=0} = (Z, Z_t)|_{t=0}$ for $0 < \sigma \leq \sigma_0$ and $\sigma = 0$ respectively. Assume that there is a constant $M > 0$ such that*

$$(\mathcal{E}_{high})(0), (\mathcal{E}_{\sigma_0})(0) \leq M$$

Then there exists a time $T > 0$ depending only on M, such that the solutions (Z^σ, Z_t^σ) and (Z, Z_t) exist in the time interval $[0, T]$ and as $\sigma \to 0$, we have $(Z^\sigma, Z_t^\sigma) \to (Z, Z_t)$ in $[0, T]$ in the sense that $\sup_{t\in[0,T]} \mathcal{E}_\Delta(Z^\sigma, Z)(t) \leq e^{C(M)T}\mathcal{E}_\Delta(Z^\sigma, Z)(0) \lesssim \sigma \to 0$, where $\mathcal{E}_\Delta(Z^\sigma, Z)$ is the energy $\mathcal{E}_\Delta$ with Z^σ being the solution A and Z being the solution B.

This result should be contrasted with the result of Ambrose-Masmoudi [AM05] where the convergence is proved in Sobolev spaces. The importance of the above result is the fact that the constant $C(M)$ appearing in the estimate for $\mathcal{E}_\Delta$

$$\sup_{t\in[0,T]} \mathcal{E}_\Delta(Z^\sigma, Z)(t) \leq e^{C(M)T}\mathcal{E}_\Delta(Z^\sigma, Z)(0)$$

depends only on M which in turn depends only on the energy $\mathcal{E}_{high}$ and $\mathcal{E}_\sigma$ of the initial data. If the initial interface is close to being angled crest, then $\mathcal{E}_{high}$ and $\mathcal{E}_\sigma$ remain bounded ($\mathcal{E}_\sigma$ remains finite provided the surface tension is small enough depending on how close it is an angled crest interface) where as the $C^{1,\alpha}$ norm (for any $0 < \alpha \leq 1$) of the interface Z blows up as the interface gets closer to being angled crested. Hence this result allows us to control the difference of the solutions independent of how close the initial interface is to an angled crest interface. It is also worthwhile to note that in the proof we show that the angle of the interface $\theta^\sigma \to \theta$ in L^∞ as $\sigma \to 0$. Hence the approximation between the solutions with surface tension and zero surface tension is quite strong.

We now give an example which demonstrates the usefulness of the above results. First observe that $\Psi_z \neq 0$ for $z \in P_-$ as Ψ is a Riemann map. Let $F : \bar{P}_- \to \overline{\Omega}$ be the holomorphic function vanishing at infinity with boundary value $\overline{Z}_t$ i.e. $F = \overline{\mathbf{v}} \circ \Psi$ and hence F is the conjugate of the velocity in Riemann mapping coordinates. Let $z = x + iy$ and define the quantity

$$\begin{aligned}
M = {} & \sup_{y<0}\left\|\Psi_z^{\frac{1}{2}}\partial_z\left(\frac{1}{\Psi_z}\right)\right\|_{L^{\frac{4}{3}}(\mathbb{R},dx)} + \sup_{y<0}\left\|\partial_z\left(\frac{1}{\Psi_z}\right)\right\|_{L^2(\mathbb{R},dx)} + \sup_{y<0}\left\|\frac{1}{\Psi_z}\partial_z\left(\frac{1}{\Psi_z}\right)\right\|_{L^\infty(\mathbb{R},dx)} \\
& + \sup_{y<0}\left\|\frac{1}{\Psi_z}\partial_z^2\left(\frac{1}{\Psi_z}\right)\right\|_{L^1(\mathbb{R},dx)} + \sup_{y<0}\left\|\frac{1}{\Psi_z^2}\partial_z^2\left(\frac{1}{\Psi_z}\right)\right\|_{L^2(\mathbb{R},dx)} + \sup_{y<0}\left\|\frac{1}{\Psi_z^3}\partial_z^3\left(\frac{1}{\Psi_z}\right)\right\|_{L^1(\mathbb{R},dx)} \\
& + \sup_{y<0}\left\|\frac{1}{\Psi_z}\right\|_{L^\infty(\mathbb{R},dx)} + \sup_{y<0}\|F_z\|_{H^{2.5}(\mathbb{R},dx)}
\end{aligned}$$

By an analogous argument as was done in §2.6, $M < \infty$ allows interfaces with angled crests of angles $\nu\pi$ with $0 < \nu < \frac{1}{2}$ and also allows cusps. We can now state the example.

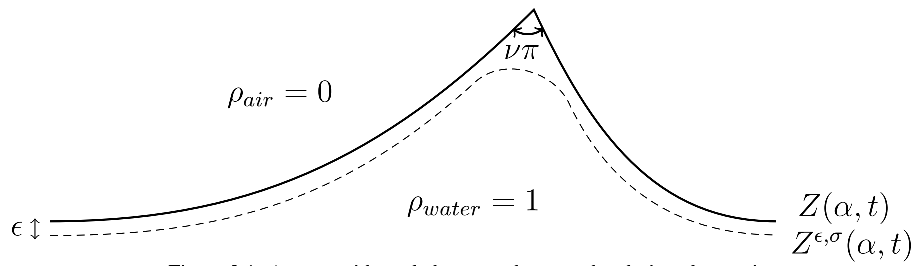

Figure 3.1: A wave with angled crest and a smooth solution close to it

Proposition 3.4.9. *Consider an initial data $(Z, Z_t)|_{t=0}$ with Z being a graph of a function, decaying rapidly at infinity and satisfying $M < \infty$. Let (Z, Z_t) be the solution of equation (3.7) for $\sigma = 0$ with initial data $(Z, Z_t)|_{t=0}$ as obtained in [Wu15]. For $0 < \epsilon \leq 1$ and $\sigma \geq 0$ denote by $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ the solution to the equation (3.7) with value of surface tension σ and with initial data $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})|_{t=0} = (Z * P_\epsilon, Z_t * P_\epsilon)|_{t=0}$ where P_ϵ is the Poisson kernel. For $\sigma = 0$ we denote $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ by $(Z^\epsilon, Z_t^\epsilon)$. These solutions satisfy the following properties:*

1) If $\dfrac{\sigma}{\epsilon^{\frac{3}{2}}} \leq 1$, then there exists a time $T > 0$ independent of ϵ and σ such that the solutions $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ exist in the time interval $[0, T]$.

2) In addition if we let $\dfrac{\sigma}{\epsilon^{\frac{3}{2}}} \to 0$, then the solutions $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma}) \to (Z^\epsilon, Z_t^\epsilon)$ in the sense that $\sup\limits_{t\in[0,T]} \mathcal{E}_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(t) \leq C(M)\dfrac{\sigma}{\epsilon^{\frac{3}{2}}} \to 0$ where $C(M)$ is a constant which depends only on the initial data $(Z, Z_t)|_{t=0}$ and is independent of σ and ϵ.

If the interface has only one angled crest of angle $\nu\pi$ with $0 < \nu < \frac{1}{2}$, then the curvature $\partial_s\theta$ of the interface $Z^{\epsilon,\sigma}$ at $t = 0$ behaves as $\partial_s\theta \sim \epsilon^{-\nu}$ as $\epsilon \to 0$, and hence if $\sigma = \epsilon^{\frac{3}{2}}$, then $\partial_s\theta \sim \sigma^{-\frac{2\nu}{3}}$. Hence as $\nu \uparrow \frac{1}{2}$, the rate of growth of the curvature of $Z^{\epsilon,\sigma}$ tends to $\sigma^{-\frac{1}{3}}$ as $\sigma \to 0$.

Remark 3.4.10. In previous results on zero surface tension limit for large data, even if σ is very small, the time of existence $T \lesssim \|\kappa\|_\infty^{-1}$. Hence as the interface $Z^{\epsilon,\sigma}$ with a single angled crest of angle $\nu\pi$ has curvature $\|\kappa\|_\infty \sim \epsilon^{-\nu}$, this yields $T \lesssim \epsilon^\nu \to 0$ as $\epsilon \to 0$. The above example says that these solutions in fact exist on a much larger time interval and that the time of existence is at least $O(1)$ even as $\epsilon \to 0$, provided there is a balance between surface tension and smoothness $\sigma \sim \epsilon^{\frac{3}{2}}$.

Remark 3.4.11. In the recent paper [Wu18], it is proved that $(Z^\epsilon, Z_t^\epsilon) \to (Z, Z_t)$ as $\epsilon \to 0$ in the sense that $\sup_{t\in[0,T]} \mathcal{F}(Z^\epsilon, Z)(t) \to 0$ where $\mathcal{F}(Z^\epsilon, Z)$ is a positive functional which compares the solutions $(Z^\epsilon, Z_t^\epsilon)$ and (Z, Z_t) and it also shown that the solution (Z, Z_t) constructed in [Wu15] is unique. Combining this with our result above we have that as $\epsilon, \sigma \to 0$ and $\dfrac{\sigma}{\epsilon^{\frac{3}{2}}} \to 0$, we have $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma}) \to (Z, Z_t)$ in the sense that $\sup\limits_{t\in[0,T]} \mathcal{E}_{\Delta}(Z^{\epsilon,\sigma}, Z^{\epsilon})(t) + \sup\limits_{t\in[0,T]} \mathcal{F}(Z^{\epsilon}, Z)(t) \to 0$

Remark 3.4.12. As mentioned earlier, the restriction of Z being a graph comes solely from the existence result [ABZ11] and can be removed.

The above proposition is direct consequence of Theorem 3.4.8. To use the theorem, we just show that if $\sigma/\epsilon^{\frac{3}{2}} \leq 1$ then $\mathcal{E}_{high}(Z^{\epsilon})(0), \mathcal{E}_{\sigma}(Z^{\epsilon})(0) \leq C(M)$ where $C(M)$ is a constant which depends only on the initial data $(Z, Z_t)|_{t=0}$. We also prove $E_{\Delta}(Z^{\epsilon}, Z^{\epsilon,\sigma})(0) \leq C(M)\sigma/\epsilon^{\frac{3}{2}}$ from which the result follows. To see the scaling $\sigma/\epsilon^{\frac{3}{2}}$, observe that one of the terms in $\mathcal{E}_{\sigma}$ is $\sigma\partial_{\alpha'}\Theta \in \dot{H}^{\frac{1}{2}}$. As $\Theta^{\epsilon} = \Theta * P_{\epsilon}$ we have

$$\|\sigma\partial_{\alpha'}\Theta^{\epsilon}\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\Theta\|_2 \frac{\sigma}{\epsilon^{\frac{3}{2}}} \leq C(M)\frac{\sigma}{\epsilon^{\frac{3}{2}}}$$

where we have used the fact that for an angled crest wave with angle $< 90°$, we have $\Theta \in L^2$. We show that this kind of estimate occurs for every term in $\mathcal{E}_{\sigma}$ and E_{Δ}, and hence the result follows. The scaling is inherent from the main operators in the quasilinear equation. Observe that

$$\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)^{-\frac{3}{2}} \left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}\right)^3 \sim \sigma\partial_{\alpha'}^{\frac{3}{2}}$$

which naturally gives us the factor $\sigma/\epsilon^{\frac{3}{2}}$.

3.5 Outline of the Proof

There are two main theorems, one on existence and one on convergence. The proof of Theorem 3.4.1 is proved in Chapter 4 where we also prove the equivalence of the energy E_{σ} and $\mathcal{E}_{\sigma}$ and its relation to the Sobolev norm and we also prove the existence result. The proof strategy is the same as we did for the zero surface tension case namely by first obtaining quasilinear equations, controlling the quantities controlled by the energy and then closing the energy.

The convergence result is proved in Chapter 5. We first prove the apriori estimate for the energy E_{high}. After this we prove the estimate for the energy $\mathcal{E}_{\sigma,aux}$ which was part of the energy $\mathcal{E}_{\Delta}$. After this we prove the main result Theorem 3.4.8. We actually prove a more general result which can handle initial data with different interfaces and Theorem 3.4.8 is a special case of this result. We then proceed to give the proof of the example stated earlier.

CHAPTER 4

The Energy $\mathcal{E}_\sigma$

In this chapter we will first prove Theorem 3.4.1. We first control the terms from E_σ in §4.1 and then close the energy estimate in §4.2 completing the proof. We then prove Proposition 3.4.3 in §4.3 and prove Proposition 3.4.4 in §4.4. Finally we prove Corollary 3.4.6 in §4.5.

4.1 Quantities controlled by E_σ

Now we come to main part of the section. Here we control all the important terms controlled by the energy E_σ. We will frequently use the estimates proved in the appendix to control the terms. In particular Proposition A.0.6, Proposition A.0.7, Corollary A.0.8 and Proposition A.0.9 are very frequently used.

In this section whenever we write $f \in L^2$, what we mean is that there exists a universal polynomial P with nonnegative coefficients such that $\|f\|_2 \leq P(E_\sigma)$. Similar definitions for $f \in \dot{H}^{\frac{1}{2}}$ or $f \in L^\infty$. We define the norm $\|f\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} = \|f\|_\infty + \|f\|_{\dot{H}^{\frac{1}{2}}}$. The spaces $\mathcal{C}$ and $\mathcal{W}$ are as defined in §2.4.1 and we will very frequently use Lemma 2.4.5. In this section we will sometimes use the function $Z_{,\alpha'}^{1/2}$. This is defined as

$$Z_{,\alpha'}^{1/2} = e^{\frac{1}{2}\log(Z_{,\alpha'})} \qquad \text{where } \log(Z_{,\alpha'}) \to 0 \text{ as } |\alpha'| \to \infty$$

Note that there is no ambiguity in the definition of $\log(Z_{,\alpha'})$ as it is continuous and we have fixed its value at infinity.

For $\sigma = 0$ the energy E_σ is lower order as compared to the energy in Kinsey-Wu [KW14] by half weighted spacial derivative and is equivalent to the energy E defined in §2.4. A few of the terms for $\sigma = 0$ have to be proved differently as compared to §2.4.1 due to the differing forms of the energy. Of course estimates for terms involving surface tension are all new.

1) $\overline{Z}_{t,\alpha'} \in L^2, |D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t \in L^2$

Proof: This is true as $A_1 \geq 1$ and as $E_{\sigma,1}$ and $E_{\sigma,4}$ are part of the energy

2) $A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: The proof is the same as in §2.4.1

3) $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2, \partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^2, |D_{\alpha'}|\omega \in L^2$ and hence $\omega \in \mathcal{W}$

Proof: The proof is the same as in §2.4.1

4) $\overline{D}_{\alpha'}\overline{Z}_t \in L^\infty, |D_{\alpha'}|\overline{Z}_t \in L^\infty, D_{\alpha'}\overline{Z}_t \in L^\infty$

Proof: We only need to prove that $\overline{D}_{\alpha'}\overline{Z}_t \in L^\infty$ and the rest follows. Observe that

$$\partial_{\alpha'}\big(\overline{D}_{\alpha'}\overline{Z}_t\big)^2 = 2(\overline{Z}_{t,\alpha'})\big(\overline{D}_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\big)$$

As $\overline{D}_{\alpha'}\overline{Z}_t$ decays at infinity, by integrating we get

$$\left\|(\overline{D}_{\alpha'}\overline{Z}_t)^2\right\|_\infty \lesssim \int \left|\overline{Z}_{t,\alpha'}\right|\left|\overline{D}_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right| d\alpha' \lesssim \left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}\left\||D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_{L^2}$$

Hence $\left\|\overline{D}_{\alpha'}\overline{Z}_t\right\|_\infty \lesssim \sqrt{\left\|\overline{Z}_{t,\alpha'}\right\|_{L^2}\left\||D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_{L^2}}$

5) $\overline{D}_{\alpha'}^2\overline{Z}_t \in L^2, |D_{\alpha'}|^2\overline{Z}_t \in L^2, D_{\alpha'}^2\overline{Z}_t \in L^2$

Proof: We already know that $|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t \in L^2$ and hence $\overline{D}_{\alpha'}^2\overline{Z}_t \in L^2$. Now

$$\overline{D}_{\alpha'}^2\overline{Z}_t = \overline{D}_{\alpha'}\big(\omega|D_{\alpha'}|\overline{Z}_t\big) = (\overline{D}_{\alpha'}\omega)|D_{\alpha'}|\overline{Z}_t + \omega^2|D_{\alpha'}|^2\overline{Z}_t$$

Now observe that $|D_{\alpha'}|\omega \in L^2$ and $|D_{\alpha'}|\overline{Z}_t \in L^\infty$ and hence the first term is in L^2. Hence we have $|D_{\alpha'}|^2\overline{Z}_t \in L^2$. A similar argument works for the rest.

6) $\overline{D}_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}, |D_{\alpha'}|\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}, D_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$

Proof: The proof is the same as in §2.4.1

7) $\partial_{\alpha'}\mathbb{P}_A\left(\dfrac{Z_t}{Z_{,\alpha'}}\right) \in L^\infty$

Proof: The proof is the same as in §2.4.1

8) $|D_{\alpha'}|A_1 \in L^2$ and hence $A_1 \in \mathcal{W}$, $\sqrt{A_1} \in \mathcal{W}$, $\dfrac{1}{A_1} \in \mathcal{W}$, $\dfrac{1}{\sqrt{A_1}} \in \mathcal{W}$

Proof: The proof is the same as in §2.4.1 and the only change is that instead of the using the formula from (2.4) i.e. $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'}$, we use the formula from (3.5) which is $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'} - \sigma\partial_{\alpha'}(\mathbb{I}+\mathbb{H})D_{\alpha'}\dfrac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ which does not affect the estimate as the only difference is the addition of a holomorphic quantity which vanishes under the application $(\mathbb{I}-\mathbb{H})$.

9) $\Theta \in L^2$, $D_t\Theta \in L^2$

Proof: Recall from (3.12) that

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

As the Hilbert transform is bounded on L^2 we easily see that $\|\Theta\|_2 \lesssim \left\|\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_2$. We have $D_t\Theta \in L^2$ as it part of the energy $E_{\sigma,3}$

10) $\dfrac{\Theta}{|Z_{,\alpha'}|} \in \mathcal{C}$

Proof: We know from $E_{\sigma,3}$ that $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\Theta \in \dot{H}^{\frac{1}{2}}$. Now as $\|\sqrt{A_1}\Theta\|_2 \lesssim \|A_1\|_\infty^{\frac{1}{2}}\|\Theta\|_2$ we now have $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\Theta \in \mathcal{C}$. Hence we get $\left\|\dfrac{\Theta}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}} \lesssim \left\|\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\Theta\right\|_{\mathcal{C}}\left\|\dfrac{1}{\sqrt{A_1}}\right\|_{\mathcal{W}}$

11) $D_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $|D_{\alpha'}|\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $|D_{\alpha'}|\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{C}$, $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{C}$

Proof: Observe that

$$\begin{aligned}\frac{\Theta}{|Z_{,\alpha'}|} &= i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}\left\{\frac{1}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} \\ &= i\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}} + i\mathrm{Re}\left\{\left[\frac{1}{|Z_{,\alpha'}|},\mathbb{H}\right]\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} - i\mathrm{Re}\left\{\left[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\end{aligned}$$

Hence $\left\|\overline{D}_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\dfrac{\Theta}{|Z_{,\alpha'}|}\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_2^2$ which implies that $\overline{D}_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$. As

$\overline{\omega} \in \mathcal{W}$, by Lemma 2.4.5 we get $D_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$ and $|D_{\alpha'}|\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$. Observe that

$$\text{Re}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = |D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \qquad \text{Im}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i\left(\frac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right)$$

Hence $|D_{\alpha'}|\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{C}$ and $\dfrac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{C}$. Now again using $\omega \in \mathcal{W}$ and Lemma 2.4.5 we easily obtain $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{C}$

12) $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and hence $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in \mathcal{C}$

Proof: The proof is the same as in §2.4.1 and the only change is that instead of the using the formula from (2.4) i.e. $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'}$, we use the formula from (3.5) which is $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'} - \sigma\partial_{\alpha'}(\mathbb{I}+\mathbb{H})D_{\alpha'}\dfrac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ which does not affect the estimate as the only difference is the addition of a holomorphic quantity which vanishes under the application $(\mathbb{I}-\mathbb{H})$.

13) $\dfrac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2A_1 \in L^2$, $|D_{\alpha'}|\left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2$ and hence $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in \mathcal{W}$

Proof: The proof is the same as in §2.4.1 and the only change is that instead of the using the formula from (2.4) i.e. $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'}$, we use the formula from (3.5) which is $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'} - \sigma\partial_{\alpha'}(\mathbb{I}+\mathbb{H})D_{\alpha'}\dfrac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ which does not affect the estimate as the only difference is the addition of a holomorphic quantity which vanishes under the application $(\mathbb{I}-\mathbb{H})$.

14) $b_{\alpha'} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and $\mathbb{H}(b_{\alpha'}) \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: The proof is the same as in §2.4.1

15) $|D_{\alpha'}|b_{\alpha'} \in L^2$ and hence $b_{\alpha'} \in \mathcal{W}$

Proof: The proof is the same as in §2.4.1

16) $\partial_{\alpha'} D_t \dfrac{1}{Z_{,\alpha'}} \in L^2$, $D_t \partial_{\alpha'} \dfrac{1}{Z_{,\alpha'}} \in L^2$

Proof: Recall from (2.12) that $D_t \dfrac{1}{Z_{,\alpha'}} = \dfrac{1}{Z_{,\alpha'}}(b_{\alpha'} - D_{\alpha'} Z_t)$ and hence

$$\partial_{\alpha'} D_t \frac{1}{Z_{,\alpha'}} = \Big(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\Big)(b_{\alpha'} - D_{\alpha'} Z_t) + D_{\alpha'} b_{\alpha'} - D_{\alpha'}^2 Z_t$$

Hence

$$\left\|\partial_{\alpha'} D_t \frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_2 (\|b_{\alpha'}\|_\infty + \|D_{\alpha'} Z_t\|_\infty) + \||D_{\alpha'}| b_{\alpha'}\|_2 + \left\|D_{\alpha'}^2 Z_t\right\|_2$$

Similarly we have

$$\left\|D_t \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim \left\|\partial_{\alpha'} D_t \frac{1}{Z_{,\alpha'}}\right\|_2 + \|b_{\alpha'}\|_\infty \left\|\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right\|_2$$

17) $\overline{Z}_{tt,\alpha'} \in L^2$

Proof: From $E_{\sigma,2}$ we have $D_t \overline{Z}_{t,\alpha'} \in L^2$. Hence $\left\|\overline{Z}_{tt,\alpha'}\right\|_2 \lesssim \left\|D_t \overline{Z}_{t,\alpha'}\right\|_2 + \|b_{\alpha'}\|_\infty \left\|\overline{Z}_{t,\alpha'}\right\|_2$

18) $\overline{D}_{\alpha'} \overline{Z}_{tt} \in \mathcal{C}, |D_{\alpha'}| \overline{Z}_{tt} \in \mathcal{C}, D_t \overline{D}_{\alpha'} \overline{Z}_t \in \mathcal{C}$ and $D_t |D_{\alpha'}| \overline{Z}_t \in \mathcal{C}$

Proof: From $E_{\sigma,4}$ we have that $D_t \overline{D}_{\alpha'} \overline{Z}_t \in \dot{H}^{\frac{1}{2}}$. Observe that

$$D_t \overline{D}_{\alpha'} \overline{Z}_t = \overline{D}_{\alpha'} \overline{Z}_{tt} - (\overline{D}_{\alpha'} \overline{Z}_t)^2$$

and as $\overline{D}_{\alpha'} \overline{Z}_t \in \mathcal{C} \cap \mathcal{W}$, by using Lemma 2.4.5 we get that $(\overline{D}_{\alpha'} \overline{Z}_t)^2 \in \mathcal{C}$. Hence $\overline{D}_{\alpha'} \overline{Z}_{tt} \in \dot{H}^{\frac{1}{2}}$. As $\overline{Z}_{tt,\alpha'} \in L^2$ we get that $\overline{D}_{\alpha'} \overline{Z}_{tt} \in \mathcal{C}$. By again using the equation above, we get that $D_t \overline{D}_{\alpha'} \overline{Z}_t \in \mathcal{C}$. By using $\overline{\omega} \in \mathcal{W}, \overline{D}_{\alpha'} \overline{Z}_{tt} \in \mathcal{C}$ in Lemma 2.4.5, we obtain $|D_{\alpha'}| \overline{Z}_{tt} \in \mathcal{C}$. Now observe that

$$D_t |D_{\alpha'}| \overline{Z}_t = |D_{\alpha'}| \overline{Z}_{tt} - \mathrm{Re}(\overline{D}_{\alpha'} \overline{Z}_t) |D_{\alpha'}| \overline{Z}_t$$

As $\overline{D}_{\alpha'} \overline{Z}_t \in \mathcal{C}$ we get that $\mathrm{Re}(\overline{D}_{\alpha'} \overline{Z}_t) \in \mathcal{C}$. Also as $|D_{\alpha'}| \overline{Z}_t \in \mathcal{W}$, using Lemma 2.4.5 we obtain $\mathrm{Re}(\overline{D}_{\alpha'} \overline{Z}_t) |D_{\alpha'}| \overline{Z}_t \in \mathcal{C}$. Hence $D_t |D_{\alpha'}| \overline{Z}_t \in \mathcal{C}$.

19) $D_t A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: The proof is the same as in §2.4.1

20) $D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and hence $D_t b_{\alpha'} \in \dot{H}^{\frac{1}{2}}, \partial_{\alpha'}D_t b \in \dot{H}^{\frac{1}{2}}$

Proof: The proof is the same as in §2.4.1

21) $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^\infty$, $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty$, $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega \in L^\infty$, $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\mathrm{Re}\Theta \in L^\infty$

Proof: $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^\infty$ as it part of the energy $E_{\sigma,0}$. Now recall from (2.9) that

$$\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)$$

Hence we easily obtain $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty$ and $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega \in L^\infty$. Now from (2.9) we have $\mathrm{Re}\Theta = -iD_{\alpha'}\omega$ and this implies that $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\mathrm{Re}\Theta \in L^\infty$

22) $\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2$, $\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^2$, $\dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega \in L^2$

Proof: $\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2$ as it part of the energy $E_{\sigma,0}$. Again using (2.9) we can control the other terms.

23) $\sigma\partial_{\alpha'}\Theta \in \dot{H}^{\frac{1}{2}}$

Proof: We first note that $(\overline{Z}_{tt} - i)Z_{,\alpha'} \in \dot{H}^{\frac{1}{2}}$ as it part of the energy $E_{\sigma,1}$. But from the fundamental equation (3.7) we get

$$(\overline{Z}_{tt} - i)Z_{,\alpha'} = -iA_1 + \sigma\partial_{\alpha'}\Theta$$

We have already proven that $A_1 \in \dot{H}^{\frac{1}{2}}$ and hence $\sigma\partial_{\alpha'}\Theta \in \dot{H}^{\frac{1}{2}}$

24) $\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta \in L^2$

Proof: As $\Theta \in L^2, \sigma\partial_{\alpha'}\Theta \in \dot{H}^{\frac{1}{2}}$ we have $\widehat{\Theta} \in L^2, \sigma|\xi|^{\frac{3}{2}}\widehat{\Theta} \in L^2$. Hence we simply interpolate between them

$$\left\|\sigma^{\frac{2}{3}}|\xi||\widehat{\Theta}|\right\|_2 = \left\|\left(\sigma|\xi|^{\frac{3}{2}}|\widehat{\Theta}|\right)^{\frac{2}{3}}(|\widehat{\Theta}|)^{\frac{1}{3}}\right\|_2 \lesssim \left\|\left(\sigma|\xi|^{\frac{3}{2}}|\widehat{\Theta}|\right)^{\frac{2}{3}}\right\|_3\left\||\widehat{\Theta}|^{\frac{1}{3}}\right\|_6 \lesssim \left\|\sigma|\xi|^{\frac{3}{2}}\widehat{\Theta}\right\|_2^{\frac{2}{3}}\left\|\widehat{\Theta}\right\|_2^{\frac{1}{3}}$$

25) $\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} \in L^2, \sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|} \in L^2, \frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega \in L^2, \sigma^{\frac{2}{3}}\partial_{\alpha'}|D_{\alpha'}|\omega \in L^2$

Proof: Recall the formula of Θ

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

by differentiating we get

$$\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta = i\sigma^{\frac{2}{3}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - i\sigma^{\frac{2}{3}}\mathrm{Re}\left\{\partial_{\alpha'}\left[\frac{\omega}{Z_{,\alpha'}^{1/2}},\mathbb{H}\right]\left(Z_{,\alpha'}^{1/2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}$$

Hence

$$\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \lesssim \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2 + \left\|\sigma^{\frac{1}{2}}\partial_{\alpha'}\frac{\omega}{Z_{,\alpha'}^{1/2}}\right\|_\infty\left\|\sigma^{\frac{1}{6}}Z_{,\alpha'}^{1/2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

From this and (2.9) we get

$$\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

$$\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2 \lesssim \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2$$

$$\left\|\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right\|_2 \lesssim \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

and we easily obtain $\sigma^{\frac{2}{3}}\partial_{\alpha'}|D_{\alpha'}|\omega \in L^2$ from $\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega \in L^2$ and we have

$$\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}|D_{\alpha'}|\omega\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2$$

26) $\sigma^{\frac{1}{3}}\Theta \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: As $\Theta \in L^2, \sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta \in L^2$ we have $\widehat{\Theta} \in L^2, \sigma^{\frac{2}{3}}|\xi|\widehat{\Theta} \in L^2$. Hence we simply interpolate between them

$$\left\|\sigma^{\frac{1}{3}}|\xi|^{\frac{1}{2}}|\widehat{\Theta}|\right\|_2 = \left\|\left(\sigma^{\frac{2}{3}}|\xi||\widehat{\Theta}|\right)^{\frac{1}{2}}(|\widehat{\Theta}|)^{\frac{1}{2}}\right\|_2 \lesssim \left\|\left(\sigma^{\frac{2}{3}}|\xi||\widehat{\Theta}|\right)^{\frac{1}{2}}\right\|_4\left\||\widehat{\Theta}|^{\frac{1}{2}}\right\|_4 \lesssim \left\|\sigma^{\frac{2}{3}}|\xi|\widehat{\Theta}\right\|_2^{\frac{1}{2}}\left\|\widehat{\Theta}\right\|_2^{\frac{1}{2}}$$

Hence $\sigma^{\frac{1}{3}}\Theta \in \dot{H}^{\frac{1}{2}}$. Now as Θ decays at infinity we have

$$\left\|\sigma^{\frac{1}{3}}\Theta\right\|_\infty^2 = \left\|\sigma^{\frac{2}{3}}\Theta^2\right\|_\infty \lesssim \sigma^{\frac{2}{3}}\int\left|\partial_{\alpha'}(\Theta^2)\right| d\alpha' \lesssim \|\Theta\|_2\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\|_2$$

27) $\sigma^{\frac{1}{3}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$, $\sigma^{\frac{1}{3}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$, $\sigma^{\frac{1}{3}}|D_{\alpha'}|\omega \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: This is proved by exactly the same argument used above to show $\sigma^{\frac{1}{3}}\Theta \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

28) $\sigma\partial_{\alpha'}D_{\alpha'}\Theta \in L^2$, $\sigma|D_{\alpha'}|\partial_{\alpha'}\Theta \in L^2$, $\sigma\partial_{\alpha'}|D_{\alpha'}|\Theta \in L^2$

Proof: Taking a derivative in the fundamental equation (3.7) we get

$$\bar{Z}_{tt,\alpha'} = -iD_{\alpha'}A_1 - iA_1\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} + \sigma\partial_{\alpha'}D_{\alpha'}\Theta$$

Hence $\|\sigma\partial_{\alpha'}D_{\alpha'}\Theta\|_2 \lesssim \left\|\bar{Z}_{tt,\alpha'}\right\|_2 + \||D_{\alpha'}|A_1\|_2 + \|A_1\|_\infty\left\|\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_2$. From this we get that $\|\sigma|D_{\alpha'}|\partial_{\alpha'}\Theta\|_2 \lesssim \|\sigma\partial_{\alpha'}D_{\alpha'}\Theta\|_2 + \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2$. We can prove $\sigma\partial_{\alpha'}|D_{\alpha'}|\Theta \in L^2$ similarly.

29) $\dfrac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in L^2$, $\dfrac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\dfrac{1}{|Z_{,\alpha'}|} \in L^2$, $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\omega \in L^2$

Proof: We first observe that

$$\begin{aligned}\left\|\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - \frac{\sigma}{\bar{Z}_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2 &\lesssim \left\|\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \\ &\quad + \left\|\sigma^{\frac{1}{3}}|D_{\alpha'}|\omega\right\|_\infty\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\end{aligned}$$

Hence the difference between them is controlled. This implies that we replace them with each other whenever we want. Now

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

by differentiating we get

$$\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\Theta = i\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + i\mathrm{Re}\left\{\left[\frac{\sigma}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}$$
$$- i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}$$

Now we can replace $(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}$ with $(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\}$ and rewrite it as $\left[\frac{\sigma}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}$. Hence we have the estimate

$$\begin{aligned}
&\left\|\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \\
&\lesssim \left\|\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\Theta\right\|_2 + \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\{\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2 + \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2\right\} \\
&+ \left\|\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty + \left\|\sigma^{\frac{1}{3}}|D_{\alpha'}|\omega\right\|_\infty\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2
\end{aligned}$$

Hence $\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}} \in L^2$. By using (2.9) we get that $\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|} \in L^2$ and similarly we also have $\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2 D_{\alpha'}\omega \in L^2$. From this we see that

$$\left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\omega\right\|_2 \lesssim \left\|\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2 D_{\alpha'}\omega\right\|_2 + \left\|\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty$$
$$+ \left\|\sigma^{\frac{1}{3}}|D_{\alpha'}|\omega\right\|_\infty\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2$$

30) $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} \in L^2$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|} \in L^2$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega \in L^2$ and $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta \in L^2$

Proof: $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} \in L^2$ as it part of the energy $E_{\sigma,0}$. Now as

$$\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)$$

We have the estimates

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

Now recall the formula for Θ

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

Hence we have

$$\begin{aligned}&\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\|_2\\ &\lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left(\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\right)\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\end{aligned}$$

31) $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \in \mathcal{W}$, $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \in \mathcal{W}$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega \in \mathcal{W}$, $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\mathrm{Re}\Theta \in \mathcal{W}$

Proof: We will only show that $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \in \mathcal{W}$ and the rest are proved similarly. As $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \in L^\infty$ we only need to show $|D_{\alpha'}|\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \in L^2$. Now

$$\left\||D_{\alpha'}|\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

32) $\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: As $\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta$ decays at infinity, we use Proposition A.0.10 with $w = \frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}$ to get

$$\left\|\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 \lesssim \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2\left\|\sigma\partial_{\alpha'}|D_{\alpha'}|\Theta\right\|_2 + \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2^2\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2^2$$

33) $\dfrac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$, $\dfrac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$, $\dfrac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: This is proved by exactly the same argument used above to show $\dfrac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

34) $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \in \mathcal{C}$

Proof: It was proved earlier that $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta \in L^2$. Also $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \in \dot{H}^{\frac{1}{2}}$ as it part of the energy $E_{\sigma,3}$

35) $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{C}$, $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\omega \in \mathcal{C}$

Proof: As $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2$, $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{|Z_{,\alpha'}|} \in L^2$, $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega \in L^2$ we only have to prove the $\dot{H}^{\frac{1}{2}}$ estimates. Now observe that

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - \frac{\omega\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}} \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}$$

Hence the difference between them is controlled. This implies that we replace them with each other whenever we want. Now

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

by differentiating we get

$$\begin{aligned}\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta = i\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + i\mathrm{Re}\left\{\left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\\ - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\end{aligned}$$

Now we can replace $(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}$ with $(\mathbb{I}-\mathbb{H})\left\{\frac{\omega\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\}$ and rewrite it as $\left[\frac{\omega\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}},\mathbb{H}\right]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}$. Hence we have the estimate

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\sigma^{\frac{1}{2}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\sigma^{\frac{1}{2}}\partial_{\alpha'}^2\frac{\omega}{|Z_{,\alpha'}|^{\frac{3}{2}}}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

Note that we can easily show $\sigma^{\frac{1}{2}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\in L^2$, $\sigma^{\frac{1}{2}}\partial_{\alpha'}^2\frac{w}{|Z_{,\alpha'}|^{\frac{3}{2}}}\in L^2$ by using Leibniz rule and controlling each individual term. Hence $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\in\mathcal{C}$, $\frac{\omega\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\in\mathcal{C}$. As $\overline{\omega}\in\mathcal{W}$ by using Lemma 2.4.5 we get $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\in\mathcal{C}$. Now using (2.9) we easily get $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\in\mathcal{C}$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_{\alpha'}\omega\in\mathcal{C}$. Hence

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\omega\right\|_{\mathcal{C}} \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_{\alpha'}\omega\right\|_{\mathcal{C}}\|\omega\|_{\mathcal{W}} + \left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_{\mathcal{W}}\|\omega\|_{\mathcal{W}}$$

36) $\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta\in\mathcal{C}$, $\sigma D_{\alpha'}^2\Theta\in\mathcal{C}$, $\sigma|D_{\alpha'}|^2\Theta\in\mathcal{C}$, $\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta\in\mathcal{C}$

Proof: Applying the derivative $\overline{D}_{\alpha'}$ to the fundamental equation (3.7) we get

$$\overline{D}_{\alpha'}\overline{Z}_{tt} = -iA_1\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}} - \frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 + \sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta$$

Hence we get

$$\left\|\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta\right\|_{\mathcal{C}} \lesssim \left\|\overline{D}_{\alpha'}\overline{Z}_{tt}\right\|_{\mathcal{C}} + \left\|\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\|A_1\|_{\mathcal{W}} + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\mathcal{C}}$$

Now as $\overline{\omega}\in\mathcal{W}$, by Lemma 2.4.5 we get $\sigma D_{\alpha'}^2\Theta\in\mathcal{C}$. Now we see that

$$\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta = \sigma\left(\omega|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right) + \frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta$$

Hence we have

$$\left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta\right\|_{\mathcal{C}} \lesssim \left\|\sigma\bar{D}_{\alpha'}D_{\alpha'}\Theta\right\|_{\mathcal{C}} + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}}\|\omega\|_{\mathcal{W}}$$

By a similar argument we get $\sigma|D_{\alpha'}|^2\Theta \in \mathcal{C}$

37) $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{C}$, $\sigma\partial_{\alpha'}^3\dfrac{1}{|Z_{,\alpha'}|^3} \in \mathcal{C}$, $\dfrac{\sigma}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^3\omega \in \mathcal{C}$

Proof: As $\dfrac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in L^2$, $\dfrac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\dfrac{1}{|Z_{,\alpha'}|} \in L^2$, $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\omega \in L^2$ we only need to show the $\dot{H}^{\frac{1}{2}}$ estimates. Now observe that

$$\begin{aligned}\left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - \frac{\omega\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}} &\lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\omega\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}} \\ &\quad + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\end{aligned}$$

Hence the difference between them is controlled. This implies that we replace them with each other whenever we want. Now

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

by differentiating we get

$$\begin{aligned}\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta = i\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + i\mathrm{Re}\left\{\left[\frac{\sigma}{|Z_{,\alpha'}|^2},\mathbb{H}\right]\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} \\ - i\mathrm{Re}(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\end{aligned}$$

Now we can replace $(\mathbb{I}-\mathbb{H})\left\{\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\left(\dfrac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right)\right\}$ with $(\mathbb{I}-\mathbb{H})\left\{\dfrac{\omega\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}}\right\}$

and rewrite it as $\left[\dfrac{\omega\sigma}{|Z_{,\alpha'}|^2}, \mathbb{H}\right]\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}}$. Hence we have the estimate

$$\begin{aligned}
&\left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\sigma\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|^2}\right\|_2\left\|\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\sigma\partial_{\alpha'}^3\frac{\omega}{|Z_{,\alpha'}|^2}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \\
&\quad + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\omega\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}} + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}
\end{aligned}$$

Note that we can easily show $\sigma\partial_{\alpha'}^3\dfrac{1}{|Z_{,\alpha'}|^2} \in L^2$, $\sigma\partial_{\alpha'}^3\dfrac{\omega}{|Z_{,\alpha'}|^2} \in L^2$ by using Leibniz rule and controlling each individual term. Hence $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\left(\dfrac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right) \in \mathcal{C}$, $\dfrac{\omega\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$. As $\overline{\omega} \in \mathcal{W}$ by using Lemma 2.4.5 we get $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$. Now using (2.9) we easily get $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{C}$, $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2 D_{\alpha'}\omega \in \mathcal{C}$. Hence

$$\begin{aligned}
\left\|\frac{\sigma}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^3\omega\right\|_{\mathcal{C}} &\lesssim \left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2 D_{\alpha'}\omega\right\|_{\mathcal{C}}\|\omega\|_{\mathcal{W}} + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\omega\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}}\|\omega\|_{\mathcal{W}} \\
&\quad + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_{\mathcal{W}}\|\omega\|_{\mathcal{W}} \\
\left\|\sigma\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|^3}\right\|_{\mathcal{C}} &\lesssim \left\|\sigma\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}} + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}} \\
&\quad + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}^2\left\|\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}
\end{aligned}$$

38) $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$, $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t \in L^2$ and $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$

Proof: We have $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$ as it part of the energy $E_{\sigma,2}$. Now observe that

$$\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right\|_2 \lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\infty}\left\|\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2$$

We prove $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$ similarly.

39) $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in L^2$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\overline{D}_{\alpha'}\overline{Z}_t \in L^2$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t \in L^2$ and in the same way $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}|D_{\alpha'}|^2\overline{Z}_{t,\alpha'} \in L^2$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}^2\overline{Z}_t \in L^2$

Proof: Note that $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t \in L^2$ as it part of the energy $E_{\sigma,4}$. Now we have

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\overline{D}_{\alpha'}\overline{Z}_t\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_2 + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\||D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_2$$

Similarly we see that

$$\begin{aligned}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2 &\lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\overline{D}_{\alpha'}\overline{Z}_t\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\||D_{\alpha'}|\overline{Z}_t\right\|_\infty \\ &\quad + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\end{aligned}$$

We also have

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}|D_{\alpha'}|^2\overline{Z}_{t,\alpha'}\right\|_2 \lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2$$

The estimate for $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}^2\overline{Z}_t \in L^2$ is shown in a similar way.

40) $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{W}\cap\mathcal{C}$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$, $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$ and also $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$

Proof: Note that $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$ as it part of the energy $E_{\sigma,1}$ and $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}}$

as it part of the energy $E_{\sigma,2}$. Hence $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{C}$. Now observe that

$$\begin{aligned}
&\left\| |D_{\alpha'}|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2 \\
&\lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2
\end{aligned}$$

Now as $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}$ decays at infinity, we use Proposition A.0.10 with $w = \dfrac{1}{|Z_{,\alpha'}|}$ to get

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_\infty^2 \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \left\||D_{\alpha'}|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2$$

Hence we have proved that $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{W}\cap\mathcal{C}$. Now we see that

$$\begin{aligned}
&\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}\cap\mathcal{C}} \\
&\lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}\cap\mathcal{C}} \|\omega\|_{\mathcal{W}} + \left\||D_{\alpha'}|\overline{Z}_t\right\|_{\mathcal{W}\cap\mathcal{C}} \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}}
\end{aligned}$$

We prove $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}, \dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$ similarly.

41) $\dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$, $\dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t \in L^2$, $\dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$

Proof: We interpolate between $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$ and $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$. We simply decompose $\left|\dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right| = \left|\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right|^{\frac{1}{3}}\left|\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right|^{\frac{2}{3}}$ and use Holder inequality to obtain

$$\left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^{\frac{1}{3}}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^{\frac{2}{3}}$$

We also see that

$$\left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2 \left\||D_{\alpha'}|\overline{Z}_t\right\|_\infty$$

We prove $\dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$ similarly.

42) $\dfrac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^2$, $\sigma^{\frac{1}{3}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t \in L^2$, $\sigma^{\frac{1}{3}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$

Proof: We observe that

$$\left\|\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2 \lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2$$

Similarly we have

$$\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right\|_2^2 \lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right\|_2 \left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right\|_2$$

We prove $\sigma^{\frac{1}{3}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in L^2$ in the same way as above.

43) $\sigma^{\frac{1}{6}}\dfrac{\overline{Z}_{t,\alpha'}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \in \mathcal{W}$

Proof: We use Proposition A.0.10 with $w = \dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}$ to get

$$\left\|\sigma^{\frac{1}{6}}\frac{\overline{Z}_{t,\alpha'}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\right\|_\infty^2 \lesssim \left\|\overline{Z}_{t,\alpha'}\right\|_2 \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right\|_2 + \left\|\overline{Z}_{t,\alpha'}\right\|_2^2 \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2^2$$

We also have

$$\left\||D_{\alpha'}|\left(\sigma^{\frac{1}{6}}\frac{\overline{Z}_{t,\alpha'}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\right)\right\|_2 \lesssim \left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2 \left\||D_{\alpha'}|\overline{Z}_t\right\|_\infty$$

44) $\sigma^{\frac{1}{6}}\partial_{\alpha'}\mathbb{P}_A\left(\dfrac{Z_t}{Z_{,\alpha'}^{1/2}}\right) \in L^\infty$

Proof: We see that

$$\begin{aligned}2\sigma^{\frac{1}{6}}\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}^{1/2}}\right) &= \sigma^{\frac{1}{6}}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{t,\alpha'}}{Z_{,\alpha'}^{1/2}}\right) + \sigma^{\frac{1}{6}}(\mathbb{I}-\mathbb{H})\left(Z_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}^{1/2}}\right) \\ &= 2\sigma^{\frac{1}{6}}\frac{Z_{t,\alpha'}}{Z_{,\alpha'}^{1/2}} + \sigma^{\frac{1}{6}}\left[\frac{1}{Z_{,\alpha'}^{1/2}},\mathbb{H}\right]Z_{t,\alpha'} + \sigma^{\frac{1}{6}}[Z_t,\mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}^{1/2}}\right)\end{aligned}$$

Hence we have the estimate

$$\left\|\sigma^{\frac{1}{6}}\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}^{1/2}}\right)\right\|_{\infty} \lesssim \left\|\sigma^{\frac{1}{6}}\frac{\overline{Z}_{t,\alpha'}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\right\|_{\infty} + \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|Z_{t,\alpha'}\|_2$$

45) $\dfrac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: We first observe that

$$\left\|\partial_{\alpha'}\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2 \lesssim \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\infty} + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2$$

We now use Proposition A.0.10 with $w = \dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}$ to get

$$\begin{aligned}&\left\|\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 \\ &\lesssim \left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2 + \left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2^2\end{aligned}$$

46) $\sigma^{\frac{1}{3}}\partial_{\alpha'}b_{\alpha'} \in L^2$

Proof: Recall from (2.8) that $b_{\alpha'} = (\Phi_t \circ Z)_{\alpha'} + D_{\alpha'}Z_t + Z_t\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right)$. Applying $(\mathbb{I}-\mathbb{H})\partial_{\alpha'}$ we get

$$(\mathbb{I}-\mathbb{H})\partial_{\alpha'}b_{\alpha'} = (\mathbb{I}-\mathbb{H})\big(\partial_{\alpha'}D_{\alpha'}Z_t\big) + (\mathbb{I}-\mathbb{H})\left\{Z_{t,\alpha'}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\} + (\mathbb{I}-\mathbb{H})\left\{Z_t\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\}$$

Now we see that

$$Z_t\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} = \frac{Z_t}{2}\left(Z_{,\alpha'}^{1/2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)^2 + \frac{Z_t}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\left(Z_{,\alpha'}^{1/2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

hence

$$(\mathbb{I}-\mathbb{H})\left\{Z_t\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\} = \frac{1}{2}[Z_t,\mathbb{H}]\left(Z_{,\alpha'}^{1/2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)^2 + \left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}^{1/2}}\right)\right]\partial_{\alpha'}\left(Z_{,\alpha'}^{1/2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

Now as $b_{\alpha'}$ is real valued, by taking real part of $(\mathbb{I}-\mathbb{H})\partial_{\alpha'}b_{\alpha'}$ we obtain the estimate

$$\begin{aligned}
&\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}b_{\alpha'}\right\|_2 \\
&\lesssim \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}D_{\alpha'}Z_t\right\|_2 + \left\{\left\|\sigma^{\frac{1}{6}}\frac{Z_{t,\alpha'}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\right\|_\infty + \left\|\sigma^{\frac{1}{6}}\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}^{1/2}}\right)\right\|_\infty\right\}\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \\
&\quad + \left\|Z_{t,\alpha'}\right\|_2\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2
\end{aligned}$$

47) $\dfrac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'} \in L^2$

Proof: This is obtained by interpolating between $\sigma^{\frac{1}{3}}\partial_{\alpha'}b_{\alpha'} \in L^2$ and $|D_{\alpha'}|b_{\alpha'} \in L^2$. We have

$$\left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right\|_2^2 \lesssim \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}b_{\alpha'}\right\|_2\left\||D_{\alpha'}|b_{\alpha'}\right\|_2$$

48) $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'} \in L^\infty$

Proof: In the proof of $b_{\alpha'} \in L^\infty$ we showed that

$$b_{\alpha'} = \mathrm{Re}\left\{\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]Z_{t,\alpha'} + 2D_{\alpha'}Z_t + [Z_t,\mathbb{H}]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}$$

Now taking the derivative $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}$ we obtain

$$
\begin{aligned}
&\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'} \\
&= \mathrm{Re}\Bigg\{\left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]Z_{t,\alpha'} + \left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}Z_{t,\alpha'}\right) \\
&\qquad - \left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}},\frac{1}{Z_{,\alpha'}};Z_{t,\alpha'}\right] + \frac{2\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t + \left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}Z_{t,\alpha'},\mathbb{H}\right]\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \\
&\qquad + [Z_t,\mathbb{H}]\partial_{\alpha'}\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + \left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}},Z_t;\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right]\Bigg\}
\end{aligned}
$$

Hence we have the estimate

$$
\begin{aligned}
&\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right\|_\infty \\
&\lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\bar{Z}_{t,\alpha'}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\bar{Z}_{t,\alpha'}\right\|_2 \\
&\quad + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t\right\|_\infty
\end{aligned}
$$

49) $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'} \in L^2$, $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|b_{\alpha'} \in L^2$ and $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}b_{\alpha'} \in L^2$

Proof: We will first show that $(\mathbb{I}-\mathbb{H})\left\{\dfrac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\} \in L^2$. We recall the formula of $b_{\alpha'}$ from (2.8) as $b_{\alpha'} = (\Phi_t\circ Z)_{\alpha'} + D_{\alpha'}Z_t + Z_t\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right)$. Hence

$$
\begin{aligned}
&(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\} \\
&= (\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}Z_t + \left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}Z_{t,\alpha'}\right)\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) + 2(D_{\alpha'}Z_t)\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\} \\
&\quad + (\mathbb{I}-\mathbb{H})\left\{\left(\frac{Z_t}{Z_{,\alpha'}}\right)\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\}
\end{aligned}
$$

Now

$$(\mathbb{I}-\mathbb{H})\left\{\left(\frac{Z_t}{Z_{,\alpha'}}\right)\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\} = -\frac{1}{2}[Z_t,\mathbb{H}]\left\{\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\} + \left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)$$

Hence we have

$$\begin{aligned}&\left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\}\right\|_2\\
&\lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}Z_t\right\|_2 + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}Z_{t,\alpha'}\right\|_2\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\\
&\quad+\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\{\|D_{\alpha'}Z_t\|_\infty + \left\|\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_\infty + \|Z_{t,\alpha'}\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\right\}\end{aligned}$$

Now lets come back to prove $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'} \in L^2$. We see that

$$\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'} = \mathrm{Re}\left\{\frac{\sigma^{\frac{1}{2}}\omega^{\frac{3}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2 b_{\alpha'}\right\}$$

Hence it is enough to show that $\dfrac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2 b_{\alpha'} \in L^2$. Now we have

$$\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2 b_{\alpha'} = -\left[\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}},\mathbb{H}\right]\partial_{\alpha'}^2 b_{\alpha'} + (\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\}$$

From this we finally have the estimate

$$\begin{aligned}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\|_2 &\lesssim \|b_{\alpha'}\|_\infty\left\{\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\right\}\\
&\quad+\left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\}\right\|_2\end{aligned}$$

We also see that

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|b_{\alpha'}\right\|_2 \lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty \||D_{\alpha'}|b_{\alpha'}\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\|_2$$

The other term $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}b_{\alpha'} \in L^2$ is obtained similarly.

50) $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1 \in L^\infty$

Proof: We know that $A_1 = 1 - \mathrm{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$ and hence

$$\begin{aligned}&\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1 \\ &= -\mathrm{Im}\left\{\left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}Z_t, \mathbb{H}\right]\overline{Z}_{t,\alpha'} + [Z_t, \mathbb{H}]\partial_{\alpha'}\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\overline{Z}_{t,\alpha'}\right) - \left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}, Z_t; \overline{Z}_{t,\alpha'}\right]\right\}\end{aligned}$$

Hence we have

$$\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1\right\|_\infty \lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty \|\overline{Z}_{t,\alpha'}\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \|\overline{Z}_{t,\alpha'}\|_2$$

51) $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1 \in L^2$

Proof: Observe that $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1 = \mathrm{Re}\left\{\dfrac{\sigma^{\frac{1}{2}}\omega^{\frac{3}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2 A_1\right\}$ and hence it is enough to show that $\dfrac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2 A_1 \in L^2$. Now

$$\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2 A_1 = (\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right\} - \left[\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}, \mathbb{H}\right]\partial_{\alpha'}^2 A_1$$

Recall from (3.5) that $A_1 = iZ_{,\alpha'}F_t \circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'} - \sigma\partial_{\alpha'}(\mathbb{I}+\mathbb{H})D_{\alpha'}\dfrac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$. Hence

$$
\begin{aligned}
(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right\} &= i(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}\left\{(\partial_{\alpha'}Z_{t,\alpha'})(\overline{Z}_{t,\alpha'}) + 2(Z_{t,\alpha'})(\partial_{\alpha'}\overline{Z}_{t,\alpha'})\right\}\right\} \\
&\quad + i[Z_t, \mathbb{H}]\left\{\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\}
\end{aligned}
$$

With

$$
\begin{aligned}
&[Z_t, \mathbb{H}]\left\{\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\} \\
&\quad = [Z_t, \mathbb{H}]\partial_{\alpha'}\left\{\frac{\sigma^{\frac{1}{2}}}{(Z_{,\alpha'})^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\} - \frac{3}{2}[Z_t, \mathbb{H}]\left\{\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right\|_2 &\lesssim \left\{\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\right\}\|A_1\|_\infty \\
&\quad + \|Z_{t,\alpha'}\|_2\left\{\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_\infty + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\right\}
\end{aligned}
$$

52) $(\mathbb{I}-\mathbb{H})D_t^2\Theta \in L^2$, $(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'} \in L^2$, $(\mathbb{I}-\mathbb{H})D_t^2 D_{\alpha'}\overline{Z}_t \in \dot{H}^{\frac{1}{2}}$

Proof: For a function f satisfying $\mathbb{P}_A f = 0$ we have

$$
\begin{aligned}
(\mathbb{I}-\mathbb{H})D_t^2 f &= [D_t, \mathbb{H}]D_t f + D_t[D_t, \mathbb{H}]f \\
&= [b, \mathbb{H}]\partial_{\alpha'}D_t f + D_t[b, \mathbb{H}]\partial_{\alpha'}f \\
&= 2[b, \mathbb{H}]\partial_{\alpha'}D_t f + [D_t b, \mathbb{H}]\partial_{\alpha'}f - [b, b; \partial_{\alpha'}f]
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\left\|(\mathbb{I}-\mathbb{H})D_t^2\Theta\right\|_2 &\lesssim \|b_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\|D_t\Theta\|_2 + \|\partial_{\alpha'}D_t b\|_{\dot{H}^{\frac{1}{2}}}\|\Theta\|_2 + \|b_{\alpha'}\|_\infty^2\|\Theta\|_2 \\
\left\|(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\right\|_2 &\lesssim \|b_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\left\|D_t\overline{Z}_{t,\alpha'}\right\|_2 + \|\partial_{\alpha'}D_t b\|_{\dot{H}^{\frac{1}{2}}}\left\|\overline{Z}_{t,\alpha'}\right\|_2 + \|b_{\alpha'}\|_\infty^2\left\|\overline{Z}_{t,\alpha'}\right\|_2 \\
\left\|(\mathbb{I}-\mathbb{H})D_t^2 D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \|b_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\left\|D_t D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_{\alpha'}D_t b\|_{\dot{H}^{\frac{1}{2}}}\left\|D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\quad + \|b_{\alpha'}\|_\infty^2\left\|D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}
\end{aligned}
$$

53) $\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\Theta \in L^2$, $\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in L^2$, $\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3 D_{\alpha'}\overline{Z}_t \in \dot{H}^{\frac{1}{2}}$

Proof: We use (3.10) for a function f satisfying $\mathbb{P}_A f = 0$ to get

$$\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3 f = \sigma(\mathbb{I}-\mathbb{H})\left\{\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|f + \left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)^2|D_{\alpha'}|f\right\} + \sigma\left[\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|^3}, \mathbb{H}\right]\partial_{\alpha'}^2 f + \sigma\left[\frac{1}{|Z_{,\alpha'}|^3}, \mathbb{H}\right]\partial_{\alpha'}^3 f$$

Hence we have

$$\begin{aligned}
&\left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_2 \\
&\lesssim \left\|\sigma\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|^3}\right\|_{\dot{H}^{\frac{1}{2}}}\left\|\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} \\
&\quad + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty^2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\Theta\right\|_2 \\
&\lesssim \left\|\sigma\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|^3}\right\|_{\dot{H}^{\frac{1}{2}}}\left\|\Theta\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}} \\
&\quad + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}\left\|\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3 D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\sigma\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|^3}\right\|_{\dot{H}^{\frac{1}{2}}}\left\|D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}} \\
&\quad + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}\left\|\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}}
\end{aligned}$$

54) $\left[D_t^2, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$, $\left[D_t^2, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$

Proof: We will only show $\left[D_t^2, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$ and $\left[D_t^2, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$ is proved similarly. We recall from (2.12) that $D_t\frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}\left\{(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) + \overline{D}_{\alpha'}\overline{Z}_t\right\}$ and hence

$$
\begin{aligned}
\left[D_t^2, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} &= \overline{Z}_{t,\alpha'}D_t^2\frac{1}{Z_{,\alpha'}} + 2(D_t\overline{Z}_{t,\alpha'})D_t\frac{1}{Z_{,\alpha'}} \\
&= \overline{Z}_{t,\alpha'}D_t\left\{\frac{1}{Z_{,\alpha'}}\left\{(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) + \overline{D}_{\alpha'}\overline{Z}_t\right\}\right\} \\
&\quad + 2(D_{\alpha'}\overline{Z}_{tt} - b_{\alpha'}D_{\alpha'}\overline{Z}_t)(b_{\alpha'} - D_{\alpha'}Z_t) \\
&= (D_{\alpha'}\overline{Z}_t)(b_{\alpha'} - D_{\alpha'}Z_t)^2 + (D_{\alpha'}\overline{Z}_t)D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \\
&\quad + (D_{\alpha'}\overline{Z}_t)D_t\overline{D}_{\alpha'}\overline{Z}_t + 2(D_{\alpha'}\overline{Z}_{tt} - b_{\alpha'}D_{\alpha'}\overline{Z}_t)(b_{\alpha'} - D_{\alpha'}Z_t)
\end{aligned}
$$

Now we have the estimates

$$
\begin{aligned}
\left\| |Z_{,\alpha'}|(D_{\alpha'}\overline{Z}_t)D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_2 &\lesssim \left\|\overline{Z}_{t,\alpha'}\right\|_2\left\|D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_\infty \\
\left\|(D_{\alpha'}\overline{Z}_t)D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \left\|D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}\left\|D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_\infty \\
&\quad + \left\|D_{\alpha'}\overline{Z}_t\right\|_\infty\left\|D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_{\dot{H}^{\frac{1}{2}}}
\end{aligned}
$$

This implies that $(D_{\alpha'}\overline{Z}_t)D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \in \mathcal{C}$. Hence we have

$$
\begin{aligned}
\left\|\left[D_t^2, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} &\lesssim \left\|D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}}\|b_{\alpha'} - D_{\alpha'}Z_t\|_{\mathcal{W}}^2 + \left\|(D_{\alpha'}\overline{Z}_t)D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)\right\|_{\mathcal{C}} \\
&\quad + \left(\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_{\mathcal{C}} + \|b_{\alpha'}\|_{\mathcal{W}}\left\|D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}}\right)(\|b_{\alpha'}\|_{\mathcal{W}} + \|D_{\alpha'}Z_t\|_{\mathcal{W}}) \\
&\quad + \left\|D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}}\left\|D_t\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}}
\end{aligned}
$$

55) $\left[i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$, $\left[i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$

Proof: Observe that $\left[i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} = iA_1(|D_{\alpha'}|\overline{Z}_t)|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}$ and so

$$
\left\|\left[i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} \lesssim \|A_1\|_{\mathcal{W}}\left\||D_{\alpha'}|\overline{Z}_t\right\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}
$$

The other term is proved similarly.

56) $(\mathbb{I}-\mathbb{H})\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}}$, $(\mathbb{I}-\mathbb{H})\left[i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}}$ and we also have $|Z_{,\alpha'}|\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in L^2$, $|Z_{,\alpha'}|\left[i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in L^2$

Proof: We will only show that the first term $(\mathbb{I}-\mathbb{H})\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$ and the third term $|Z_{,\alpha'}|\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in L^2$ and the other terms are proved similarly. Note that we are not making the stronger claim that $\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in \mathcal{C}$ which was proved for other commutators above. This is not true and the use of $(\mathbb{I}-\mathbb{H})$ in the $\dot{H}^{\frac{1}{2}}$ estimate is essential. We have

$$\begin{aligned}\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} &= i\sigma\left(|D_{\alpha'}|^3\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} + 3i\sigma\left(|D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'} \\ &\quad + 3i\sigma\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)|D_{\alpha'}|^2\overline{Z}_{t,\alpha'}\end{aligned}$$

We control each term seperately:

(a) We use the expansion in (3.10) to get

$$\begin{aligned}&\sigma\left(|D_{\alpha'}|^3\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} \\ &= \sigma\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right)\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} + \sigma\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)^2\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} \\ &\quad + 3\sigma\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} + \sigma\left(\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}\end{aligned}$$

Hence we have the estimate

$$\begin{aligned}&\left\|\sigma\left(|D_{\alpha'}|^3\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} \\ &\lesssim \left\{\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}} + \left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}^2\right. \\ &\quad \left. + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}} + \left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\right\}\left\||D_{\alpha'}|\overline{Z}_t\right\|_{\mathcal{W}}\end{aligned}$$

(b) We observe that

$$\sigma\left(|D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'} = \sigma\left\{\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|\frac{1}{Z_{,\alpha'}} + \frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\}|D_{\alpha'}|\overline{Z}_{t,\alpha'}$$

Hence we have the estimate

$$\begin{aligned}\left\|\sigma\left(|D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} \lesssim{}& \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}} \\ &+ \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}}\end{aligned}$$

(c) We observe that

$$\sigma\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)|D_{\alpha'}|^2\overline{Z}_{t,\alpha'} = \sigma\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)\left\{\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'} + \frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\}$$

The first term is easily controlled

$$\begin{aligned}&\left\|\sigma\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} \\ &\lesssim \left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}}\end{aligned}$$

Hence we are only left with $\sigma\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}$. We see that

$$\left\|\sigma|Z_{,\alpha'}|\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2 \lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2$$

This conclude the proof of $|Z_{,\alpha'}|\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'} \in L^2$. To finish the $\dot{H}^{\frac{1}{2}}$ estimate we rewrite the term $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}$ as $\frac{\omega^2}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}$ and commute one derivative outside to obtain

$$\begin{aligned}(\mathbb{I}-\mathbb{H})\left\{\sigma\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right)\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\} ={}& -2(\mathbb{I}-\mathbb{H})\left\{\sigma\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)^2\frac{\omega}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\} \\ &+ \sigma\left[\frac{\omega}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\end{aligned}$$

We can bound each of the terms

$$\begin{aligned}
&\left\|\sigma\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)^2\frac{\omega}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} \\
&\lesssim \left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}}\|\omega\|_{\mathcal{W}}
\end{aligned}$$

and also

$$\begin{aligned}
&\left\|\sigma\left[\frac{\omega}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\left\{\left\|\frac{\sigma^{\frac{2}{3}}}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}^2\omega\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\infty}+\||D_{\alpha'}|\omega\|_2\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\infty}^2\right. \\
&\qquad +\left\|\sigma^{\frac{1}{3}}\overline{D}_{\alpha'}\omega\right\|_{\infty}\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2+\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\infty}\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 \\
&\qquad \left.+\left\|\frac{\sigma}{\overline{Z}_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2\right\}
\end{aligned}$$

57) $R_1 \in \mathcal{C}$

Proof: We recall from (3.20) the formula of R_1

$$\begin{aligned}
R_1 &= -2(\overline{D}_{\alpha'}\overline{Z}_t)(D_t\overline{D}_{\alpha'}\overline{Z}_t) - 2\sigma\mathrm{Re}(D_{\alpha'}Z_t)\overline{D}_{\alpha'}D_{\alpha'}\Theta - \sigma(\overline{D}_{\alpha'}D_{\alpha'}Z_t)D_{\alpha'}\Theta \\
&\quad + i\sigma\big(2i\mathrm{Re}(|D_{\alpha'}|\Theta)+(\mathrm{Re}\Theta)^2\big)|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t - \sigma\mathrm{Re}\big(|D_{\alpha'}|^2\Theta\big)\overline{D}_{\alpha'}\overline{Z}_t \\
&\quad + i\sigma(\mathrm{Re}\Theta)(\mathrm{Re}(|D_{\alpha'}|\Theta))\overline{D}_{\alpha'}\overline{Z}_t
\end{aligned}$$

All the terms are easily controlled

$$\begin{aligned}
&\|R_1\|_{\mathcal{C}} \\
&\lesssim \left\|\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}}\left\|D_t\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}}+\|D_{\alpha'}Z_t\|_{\mathcal{W}}\left\|\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta\right\|_{\mathcal{C}}+\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}D_{\alpha'}Z_t\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}} \\
&\quad +\left\{\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}}+\left\|\frac{\Theta}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\mathrm{Re}\Theta\right\|_{\mathcal{W}}\right\}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}} \\
&\quad +\left\|\sigma|D_{\alpha'}|^2\Theta\right\|_{\mathcal{W}}\left\|\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}}+\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_{\mathcal{W}}\|\overline{\omega}\|_{\mathcal{W}}\left\|\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}}
\end{aligned}$$

58) $J_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: We recall from (3.16) the formula for J_1

$$\begin{aligned} J_1 = D_tA_1 + A_1\big(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t\big) + \sigma\partial_{\alpha'}\mathrm{Re}(\mathbb{I}-\mathbb{H})\big\{(|D_{\alpha'}| + i\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\} \\ - \sigma\partial_{\alpha'}\mathrm{Im}(\mathbb{I}-\mathbb{H})D_t\Theta \end{aligned}$$

We have already shown that $D_tA_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and we have

$$\begin{aligned} \big\|A_1\big(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t\big)\big\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \lesssim & \|A_1\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}\big\|b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t\big\|_\infty \\ & + \|A_1\|_\infty\big\|b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t\big\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \end{aligned}$$

Let us now control the other terms.

(a) Observe that

$$\begin{aligned} \sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})\big\{|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\big\} &= \sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})\left\{\left(|D_{\alpha'}|\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} + \frac{1}{|Z_{,\alpha'}|\overline{Z}_{,\alpha'}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\} \\ &= \sigma\left[|D_{\alpha'}|\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\overline{Z}_{t,\alpha'} + \sigma\left[\frac{1}{|Z_{,\alpha'}|\overline{Z}_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}\overline{Z}_{t,\alpha'} \end{aligned}$$

Hence we have

$$\begin{aligned} &\big\|\sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})\big\{|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\big\}\big\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \\ &\lesssim \big\|\overline{Z}_{t,\alpha'}\big\|_2\left\{\left\|\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2 + \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_\infty \right. \\ &\qquad \left. + \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\right\} \end{aligned}$$

(b) We note that $i\mathrm{Re}\Theta = D_{\alpha'}\omega$ and hence we have

$$\begin{aligned} \sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})\big\{(i\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\} &= \sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})\big\{(D_{\alpha'}\omega)\overline{D}_{\alpha'}\overline{Z}_t\big\} \\ &= \sigma\partial_{\alpha'}\left[\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega, \mathbb{H}\right]\overline{Z}_{t,\alpha'} \end{aligned}$$

From this we obtain

$$
\begin{aligned}
&\left\|\sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})\left\{(i\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\right\}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\{\left\|\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\omega\right\|_2 + \left\|\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty + \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\sigma^{\frac{1}{3}}|D_{\alpha'}|\omega\right\|_\infty \right.\\
&\qquad \left. + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty^2 \||D_{\alpha'}|\omega\|_2\right\}\left\|\overline{Z}_{t,\alpha'}\right\|_2
\end{aligned}
$$

(c) We see that as $\mathbb{P}_A\Theta = 0$ we have

$$
\sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})D_t\Theta = \sigma\partial_{\alpha'}[D_t,\mathbb{H}]\Theta = \sigma\partial_{\alpha'}[b,\mathbb{H}]\partial_{\alpha'}\Theta
$$

Hence we have

$$
\|\sigma\partial_{\alpha'}(\mathbb{I}-\mathbb{H})D_t\Theta\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \lesssim \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}b_{\alpha'}\right\|_2\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2
$$

59) $|D_{\alpha'}|J_1 \in L^2$ and hence $J_1 \in \mathcal{W}$

Proof: As J_1 is real valued we have

$$
|D_{\alpha'}|J_1 = \mathrm{Re}\left\{\frac{\omega}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}J_1\right\} = \mathrm{Re}\left\{\omega(\mathbb{I}-\mathbb{H})D_{\alpha'}J_1 - \omega\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}J_1\right\}
$$

We recall the equation of $\overline{Z}_{t,\alpha'}$ from (3.21)

$$
\begin{aligned}
&\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\overline{Z}_{t,\alpha'} \\
&= R_1\overline{Z}_{,\alpha'} - i\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1 - iD_{\alpha'}J_1 - \overline{Z}_{,\alpha'}\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}
\end{aligned}
$$

By applying $(\mathbb{I}-\mathbb{H})$ to the above equation we get

$$
\begin{aligned}
&\|(\mathbb{I}-\mathbb{H})D_{\alpha'}J_1\|_2 \\
&\lesssim \left\|(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\right\|_2 + \|A_1\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_2 \\
&\quad + \|R_1|Z_{,\alpha'}|\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|J_1\|_\infty + \left\||Z_{,\alpha'}|\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\|_2
\end{aligned}
$$

Hence we easily have

$$\| |D_{\alpha'}| J_1 \|_2 \lesssim \| (\mathbb{I} - \mathbb{H}) D_{\alpha'} J_1 \|_2 + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2 \| J_1 \|_\infty$$

60) $R_2 \in L^2$

Proof: We recall from (3.23) the formula for R_2

$$\begin{aligned} R_2 = & -2i\left(\overline{D}_{\alpha'} \overline{Z}_t\right)\left(|D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t\right) + (\mathrm{Re}\Theta)\left\{ \left(\overline{D}_{\alpha'} \overline{Z}_t\right)^2 + i \overline{D}_{\alpha'}\left(\frac{A_1}{Z_{,\alpha'}}\right) + i\sigma(\mathrm{Re}\Theta)|D_{\alpha'}|\Theta \right\} \\ & + \sigma \mathrm{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}|\Theta + \left(|D_{\alpha'}| \frac{A_1}{|Z_{,\alpha'}|}\right)\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) + |D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1\right) \\ & + (\mathbb{I} + \mathbb{H}) \mathrm{Im}\left\{ \mathrm{Re}(\overline{D}_{\alpha'} \overline{Z}_t)|D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t - i\mathrm{Re}(D_t\Theta) \overline{D}_{\alpha'} \overline{Z}_t \right\} \\ & + \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \mathrm{Re}(\mathbb{I} - \mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) \end{aligned}$$

Most of the terms are easily controlled and we have

$$\begin{aligned} & \left\| R_2 - \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \mathrm{Re}(\mathbb{I} - \mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) \right\|_2 \\ & \lesssim \left\| \overline{D}_{\alpha'} \overline{Z}_t \right\|_\infty \left(\left\| |D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t \right\|_2 + \| D_t \Theta \|_2 \right) + \|\Theta\|_2 \left\| \overline{D}_{\alpha'} \overline{Z}_t \right\|_\infty^2 + \|\Theta\|_2 \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_\infty \\ & \quad + \left\| \frac{\Theta}{|Z_{,\alpha'}|} \right\|_{\mathcal{C}} \left\| |D_{\alpha'}| \frac{1}{Z_{,\alpha'}} \right\|_{\mathcal{C}} \|A_1\|_\infty + \left\| \frac{\Theta}{|Z_{,\alpha'}|} \right\|_{\mathcal{C}} \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\|_{\mathcal{C}} \left\| \sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \mathrm{Re}\Theta \right\|_\infty \\ & \quad + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\|_{\mathcal{C}}^2 + \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_\infty \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2 + \left\| |D_{\alpha'}| \frac{1}{|Z_{,\alpha'}|} \right\|_{\mathcal{C}} \left\| |D_{\alpha'}| \frac{1}{Z_{,\alpha'}} \right\|_{\mathcal{C}} \|A_1\|_\infty \\ & \quad + \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \right\|_{\mathcal{W}} \end{aligned}$$

We now control the last term. We have

$$\begin{aligned} \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \mathrm{Re}(\mathbb{I} - \mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) = & \ \mathrm{Re}(\mathbb{I} - \mathbb{H})\left\{ \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) \right\} \\ & - \mathrm{Re}\left\{ \left[\frac{A_1}{|Z_{,\alpha'}|^2}, \mathbb{H} \right] \partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) \right\} \end{aligned}$$

The first term can be written as

$$(\mathbb{I}-\mathbb{H})\left\{\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} = \left[\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega, \mathbb{H}\right]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} + \left[\frac{A_1\omega}{|Z_{,\alpha'}|^2}, \mathbb{H}\right]\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}$$

Hence we have

$$\begin{aligned}
&\left\|\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\mathrm{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\{\|A_1\|_{\mathcal{W}}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right\|_{\mathcal{C}} + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\mathcal{C}}\|\omega\|_{\mathcal{W}}\right. \\
&\qquad\left. + \left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\|A_1\|_{\mathcal{W}}\|\omega\|_{\mathcal{W}}\right\}
\end{aligned}$$

61) $J_2 \in L^2$

Proof: Let us recall the equation of Θ from (3.22)

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\Theta = R_2 + iJ_2$$

Observe that J_2 is real valued. Hence by applying $\mathrm{Im}(\mathbb{I}-\mathbb{H})$ to the above equation we get

$$\begin{aligned}
\|J_2\|_2 &\lesssim \left\|(\mathbb{I}-\mathbb{H})D_t^2\Theta\right\|_2 + \left(\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\mathcal{C}} + \|A_1\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\right)\|\Theta\|_2 \\
&\quad + \left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\Theta\right\|_2 + \|R_2\|_2
\end{aligned}$$

62) $\sigma\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}}$, $\sigma\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}}$

Proof: We will only prove $\sigma\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}}$ and the other one is proved exactly in the same way. We first observe that

$$\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right) = \frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}\left\{\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'} + \omega|D_{\alpha'}|^2\overline{Z}_{t,\alpha'}\right\}$$
$$= \sigma\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{\omega}|D_{\alpha'}|^2\overline{Z}_{t,\alpha'} + \sigma\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'}$$
$$+ \sigma(D_{\alpha'}\omega)|D_{\alpha'}|^2\overline{Z}_{t,\alpha'} + \sigma|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}$$

Hence

$$\left\|\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right) - \sigma|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_2$$
$$\lesssim \left(\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_\infty + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\right)\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}|D_{\alpha'}|^2\overline{Z}_{t,\alpha'}\right\|_2$$
$$+ \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2\left\|\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_\infty$$

Now we have

$$\sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} = \left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\left\{\sigma|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} - \frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\}$$
$$+ \sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\frac{1}{Z_{,\alpha'}}(\mathbb{P}_H + \mathbb{P}_A)\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)$$

We can control each of the terms

(a) The first term is easily controlled

$$\left\|\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\left\{\sigma|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} - \frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\}\right\|_{\dot{H}^{\frac{1}{2}}}$$
$$\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right) - \sigma|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_2$$

(b) We have

$$\sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\frac{1}{Z_{,\alpha'}}\mathbb{P}_H\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right) = \sigma\left[\frac{1}{Z_{,\alpha'}^2},\mathbb{H}\right]\partial_{\alpha'}^2\mathbb{P}_H\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)$$

and hence we obtain

$$\begin{aligned}
&\left\|\sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\frac{1}{Z_{,\alpha'}}\mathbb{P}_H\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\|_{\dot{H}^{\frac12}} \\
&\lesssim \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\left\{\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2+\left\|\sigma^{\frac13}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\sigma^{\frac23}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\right\}
\end{aligned}$$

(c) We see that

$$\begin{aligned}
&\frac{2\sigma}{Z_{,\alpha'}}\mathbb{P}_A\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right) \\
&= -\sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)+\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \\
&\quad +(\mathbb{I}-\mathbb{H})\left\{\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)-\sigma|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\}
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\left\|\sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\frac{1}{Z_{,\alpha'}}\mathbb{P}_A\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\|_{\dot{H}^{\frac12}} \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\{\left\|\sigma^{\frac23}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\sigma^{\frac13}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_\infty+\left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_2\right. \\
&\qquad\left.+\left\|\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)-\sigma|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_2\right\}
\end{aligned}$$

63) $(\mathbb{I}-\mathbb{H})D_t^2\overline{D}_{\alpha'}\overline{Z}_t\in\dot{H}^{\frac12}$

Proof: We have already shown that $(\mathbb{I}-\mathbb{H})D_t^2D_{\alpha'}\overline{Z}_t\in\dot{H}^{\frac12}$. Hence

$$\begin{aligned}
&(\mathbb{I}-\mathbb{H})\left\{D_t^2D_{\alpha'}\overline{Z}_t\right\} \\
&= (\mathbb{I}-\mathbb{H})\left\{\left[D_t^2,\frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}+\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]D_t^2\overline{Z}_{t,\alpha'}+\frac{1}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}
\end{aligned}$$

Let us recall the equation of $\overline{Z}_{t,\alpha'}$ from (3.21)

$$\begin{aligned}
&\left(D_t^2+i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}-i\sigma|D_{\alpha'}|^3\right)\overline{Z}_{t,\alpha'} \\
&= R_1\overline{Z}_{,\alpha'}-i\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1-iD_{\alpha'}J_1-\overline{Z}_{,\alpha'}\left[D_t^2+i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}-i\sigma|D_{\alpha'}|^3,\frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}
\end{aligned}$$

From this we obtain

$$\begin{aligned}\left\|\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]D_t^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \left\|\sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}\\ &\quad+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\{\|R_1|Z_{,\alpha'}|\|_2+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|J_1\|_\infty+\||D_{\alpha'}|J_1\|_2\right\}\\ &\quad+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\{\left\||Z_{,\alpha'}|\left[D_t^2+i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}-i\sigma|D_{\alpha'}|^3,\frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\|_2\right.\\ &\qquad\left.+\|A_1\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\right\}\end{aligned}$$

We can similarly prove that $\left[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]D_t^2\overline{Z}_{t,\alpha'}\in\dot{H}^{\frac{1}{2}}$. Using this we have

$$\begin{aligned}&\left\|\frac{1}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}\\ &\lesssim\left\|\left[D_t^2,\frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}}+\left\|\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]D_t^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}+\left\|(\mathbb{I}-\mathbb{H})D_t^2D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}\end{aligned}$$

As $(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\in L^2$, this implies that $\frac{1}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\in\mathcal{C}$. Now as $\omega\in\mathcal{W}$, by using Lemma 2.4.5 we get that $\frac{1}{\overline{Z}_{,\alpha'}}(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\in\mathcal{C}$. Now

$$\begin{aligned}&(\mathbb{I}-\mathbb{H})\left\{D_t^2\overline{D}_{\alpha'}\overline{Z}_t\right\}\\ &=(\mathbb{I}-\mathbb{H})\left\{\left[D_t^2,\frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}+\left[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]D_t^2\overline{Z}_{t,\alpha'}+\frac{1}{\overline{Z}_{,\alpha'}}(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\end{aligned}$$

Hence we obtain

$$\begin{aligned}&\left\|(\mathbb{I}-\mathbb{H})D_t^2\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}\\ &\lesssim\left\|\left[D_t^2,\frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}}+\left\|\left[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]D_t^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}+\left\|\frac{1}{\overline{Z}_{,\alpha'}}(\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}\end{aligned}$$

64) $\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\in\dot{H}^{\frac{1}{2}}$

Proof: We have already shown that $\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3 D_{\alpha'}\overline{Z}_t \in \dot{H}^{\frac{1}{2}}$. Hence

$$\begin{aligned}
&\left\|\frac{\sigma}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\sigma\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\sigma(\mathbb{I}-\mathbb{H})\left\{\left[|D_{\alpha'}|^3,\frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\quad + \left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3 D_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}
\end{aligned}$$

As $\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in L^2$, this implies that $\dfrac{\sigma}{Z_{,\alpha'}}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \mathcal{C}$. Now as $\omega \in \mathcal{W}$, by using Lemma 2.4.5 we get that $\dfrac{\sigma}{\overline{Z}_{,\alpha'}}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \mathcal{C}$. Hence

$$\begin{aligned}
&\left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\sigma(\mathbb{I}-\mathbb{H})\left\{\left[|D_{\alpha'}|^3,\frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\sigma\left[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\quad + \left\|\frac{\sigma}{\overline{Z}_{,\alpha'}}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}
\end{aligned}$$

65) $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1 \in \dot{H}^{\frac{1}{2}}$ and hence $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1 \in \mathcal{C}$

Proof: Let us recall the equation of $\overline{D}_{\alpha'}\overline{Z}_t$ from (3.19)

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\overline{D}_{\alpha'}\overline{Z}_t = R_1 - i\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1 - i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1$$

We see that

$$i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t = \left(\frac{2i\omega A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right)D_{\alpha'}\overline{Z}_t + \frac{i\omega^2 A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t$$

Now observe that $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1$ is real valued. Hence by applying $\mathrm{Im}(\mathbb{I}-\mathbb{H})$ to the equation of

$\overline{D}_{\alpha'}\overline{Z}_t$ we get

$$
\begin{aligned}
&\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}\|J_1\|_{\mathcal{W}} + \left\|(\mathbb{I}-\mathbb{H})D_t^2\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\quad + \left\|D_{\alpha'}^2\overline{Z}_t\right\|_2\left(\|A_1\|_\infty\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2 + \|A_1\|_\infty\||D_{\alpha'}|\omega\|_2 + \||D_{\alpha'}|A_1\|_2\right) \\
&\quad + \left\|D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}}\|\omega\|_{\mathcal{W}}\|A_1\|_{\mathcal{W}}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right\|_{\mathcal{C}} + \|R_1\|_{\mathcal{C}}
\end{aligned}
$$

4.2 Closing the energy estimate for E_σ

We are now ready to close the energy E_σ. To simplify the calculations we will use the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a universal non-negative polynomial P with $|a(t) - b(t)| \leq P(E_\sigma(t))$. Observe that $\approx$ is an equivalence relation. With this notation proving Theorem 3.4.1 is equivalent to showing $\frac{dE_\sigma(t)}{dt} \approx 0$.

4.2.1 Controlling $E_{\sigma,0}$

Recall that

$$
E_{\sigma,0} = \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2 + \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^6 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2
$$

We control the terms individually

1) As mentioned in Remark 3.4.2 we will substitute the time derivative with the upper Dini derivative for the L^∞ term. Define $f(\alpha', t) = \left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)(\alpha', t)$. Hence we have

$$
\limsup_{s\to 0^+}\frac{\|f(\cdot, t+s)\|_\infty^2 - \|f(\cdot,t)\|_\infty^2}{s} = 2\|f(\cdot,t)\|_\infty \limsup_{s\to 0^+}\frac{\|f(\cdot,t+s)\|_\infty - \|f(\cdot,t)\|_\infty}{s}
$$

Now as $\left\| \sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{\infty}(t)$ is part of the energy we only need to concentrate on the second term. As $\partial_t(f(\cdot,t) \circ h) = (D_t f(\cdot,t)) \circ h$ we use Proposition A.0.13 to get

$$
\begin{aligned}
&\limsup_{s \to 0^+} \frac{\|f(\cdot, t+s)\|_{\infty} - \|f(\cdot,t)\|_{\infty}}{s} \\
&= \limsup_{s \to 0^+} \frac{\|(f \circ h)(\cdot, t+s)\|_{\infty} - \|(f \circ h)(\cdot,t)\|_{\infty}}{s} \\
&\leq \|(D_t f)(\cdot, t)\|_{\infty}
\end{aligned}
$$

Recall from (2.11) that $D_t|Z_{,\alpha'}| = |Z_{,\alpha'}|\big(\mathrm{Re}(D_{\alpha'} Z_t) - b_{\alpha'}\big)$ and hence we have

$$
\begin{aligned}
D_t\left(|Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) &= \frac{1}{2}\big(\mathrm{Re}(D_{\alpha'} Z_t) - b_{\alpha'}\big)\left(|Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - b_{\alpha'} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \\
&\quad + |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} D_t \frac{1}{Z_{,\alpha'}}
\end{aligned}
$$

Now as $D_t \frac{1}{Z_{,\alpha'}} = \frac{1}{Z_{,\alpha'}}(b_{\alpha'} - D_{\alpha'} Z_t)$ we obtain

$$
\begin{aligned}
D_t\left(|Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) &= \frac{1}{2}\big(\mathrm{Re}(D_{\alpha'} Z_t) - b_{\alpha'} - 2D_{\alpha'} Z_t\big)\left(|Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \\
&\quad + |Z_{,\alpha'}|^{\frac{1}{2}} D_{\alpha'}(b_{\alpha'} - D_{\alpha'} Z_t)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left\| D_t\left(\sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|_{\infty} &\lesssim (\|D_{\alpha'} Z_t\|_{\infty} + \|b_{\alpha'}\|_{\infty}) \left\| \sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{\infty} \\
&\quad + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} b_{\alpha'} \right\|_{\infty} + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_{\infty} \\
&\lesssim P(E_\sigma)
\end{aligned}
$$

2) By using the calculation above we first obtain

$$
\begin{aligned}
\left\| D_t\left(\sigma^{\frac{1}{6}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\|_{2} &\lesssim (\|D_{\alpha'} Z_t\|_{\infty} + \|b_{\alpha'}\|_{\infty}) \left\| \sigma^{\frac{1}{6}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{2} \\
&\quad + \left\| \frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} b_{\alpha'} \right\|_{2} + \left\| \frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_{2}
\end{aligned}
$$

Hence by using Lemma 2.4.6 we get

$$\begin{aligned}
&\frac{d}{dt}\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^6 \\
&\lesssim \|b_{\alpha'}\|_\infty\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^6 + \left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^5\left\|D_t\left(\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \\
&\lesssim P(E_\sigma)
\end{aligned}$$

3) By using Lemma 2.4.6 we obtain

$$\frac{d}{dt}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 \lesssim \|b_{\alpha'}\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim P(E_\sigma)$$

4) We first note that

$$D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} = D_{\alpha'}b_{\alpha'} - D_{\alpha'}^2 Z_t - \left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)D_{\alpha'}Z_t$$

From this we see that

$$\begin{aligned}
\left\|D_t\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\|_2 &\lesssim (\|b_{\alpha'}\|_\infty + \|D_{\alpha'}Z_t\|_\infty)\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}b_{\alpha'}\right\|_2 \\
&\quad + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2 Z_t\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t\right\|_\infty
\end{aligned}$$

Hence by using Lemma 2.4.6 we get

$$\begin{aligned}
&\frac{d}{dt}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 \\
&\lesssim \|b_{\alpha'}\|_\infty\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\|D_t\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \\
&\lesssim P(E_\sigma)
\end{aligned}$$

4.2.2 Controlling $E_{\sigma,1}$

Recall that

$$E_{\sigma,1} = \left\|(\overline{Z}_{tt} - i)Z_{,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\sqrt{A_1}\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2$$

We will first simplify the time derivative of each of the individual terms before combining them.

1) As $b_{\alpha'}, \mathbb{H}b_{\alpha'} \in L^\infty$, by using Lemma 2.4.6 we get

$$\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\right\}\right|^2 d\alpha' \approx 2\text{Re}\int\left\{|\partial_{\alpha'}|\big((Z_{tt} + i)\overline{Z}_{,\alpha'}\big)\right\}D_t\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big)d\alpha'$$

Now

$$\begin{aligned} D_t\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big) &= \overline{Z}_{ttt}Z_{,\alpha'} + (D_{\alpha'}Z_t - b_{\alpha'})(\overline{Z}_{tt} - i)Z_{,\alpha'} \\ &= \overline{Z}_{ttt}Z_{,\alpha'} + (D_{\alpha'}Z_t - b_{\alpha'})(-iA_1 + \sigma\partial_{\alpha'}\Theta) \end{aligned}$$

and we observe that

$$\begin{aligned} \|(D_{\alpha'}Z_t - b_{\alpha'})(-iA_1 + \sigma\partial_{\alpha'}\Theta)\|_{\dot{H}^{\frac{1}{2}}} \lesssim{}& \Big(\|D_{\alpha'}Z_t\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} + \|b_{\alpha'}\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}\Big)\|A_1\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \\ &+ \Big(\big\|\sigma^{\frac{1}{3}}\partial_{\alpha'}D_{\alpha'}Z_t\big\|_2 + \big\|\sigma^{\frac{1}{3}}\partial_{\alpha'}b_{\alpha'}\big\|_2\Big)\big\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\big\|_2 \end{aligned}$$

Hence we have

$$\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\right\}\right|^2 d\alpha' \approx 2\text{Re}\int(\overline{Z}_{ttt}Z_{,\alpha'})|\partial_{\alpha'}|\big((Z_{tt} + i)\overline{Z}_{,\alpha'}\big)d\alpha'$$

2) As $b_{\alpha'}, A_1, D_tA_1 \in L^\infty$ and $\overline{Z}_{t,\alpha'} \in L^2$ we get

$$\begin{aligned} \frac{d}{dt}\int A_1\left|\overline{Z}_{t,\alpha'}\right|^2 d\alpha' &= \int(b_{\alpha'}A_1 + D_tA_1)\left|\overline{Z}_{t,\alpha'}\right|^2 d\alpha' + 2\text{Re}\int A_1\overline{Z}_{t,\alpha'}(-b_{\alpha'}Z_{t,\alpha'} + Z_{tt,\alpha'})\,d\alpha' \\ &\approx 2\text{Re}\int A_1\overline{Z}_{t,\alpha'}Z_{tt,\alpha'}\,d\alpha' \end{aligned}$$

Now we have

$$\begin{aligned} Z_{tt,\alpha'} = \partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}(Z_{tt} + i)\overline{Z}_{,\alpha'}\right) &= \overline{D}_{\alpha'}\big((Z_{tt} + i)\overline{Z}_{,\alpha'}\big) + \left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)(Z_{tt} + i)\overline{Z}_{,\alpha'} \\ &= \overline{D}_{\alpha'}\big((Z_{tt} + i)\overline{Z}_{,\alpha'}\big) + \left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)\big(iA_1 + \sigma\partial_{\alpha'}\overline{\Theta}\big) \end{aligned}$$

and we see that

$$\left\|\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)\left(iA_1+\sigma\partial_{\alpha'}\overline{\Theta}\right)\right\|_2 \lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\|A_1\|_\infty + \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2$$

Hence

$$\frac{d}{dt}\int A_1\left|\overline{Z}_{t,\alpha'}\right|^2 d\alpha' \approx 2\mathrm{Re}\int A_1\overline{Z}_{t,\alpha'}\overline{D}_{\alpha'}\left((Z_{tt}+i)\overline{Z}_{,\alpha'}\right)d\alpha'$$

Now as $\partial_{\alpha'} = \partial_{\alpha'}(\mathbb{I}+\mathbb{H}-\mathbb{H}) = i|\partial_{\alpha'}| + (\mathbb{I}+\mathbb{H})\partial_{\alpha'}$ and $(Z_{tt}+i)\overline{Z}_{,\alpha'} = iA_1 + \sigma\partial_{\alpha'}\overline{\Theta}$ we obtain

$$\overline{D}_{\alpha'}\left((Z_{tt}+i)\overline{Z}_{,\alpha'}\right) = \frac{i}{\overline{Z}_{,\alpha'}}|\partial_{\alpha'}|\left((Z_{tt}+i)\overline{Z}_{,\alpha'}\right) + \frac{i}{\overline{Z}_{,\alpha'}}(\mathbb{I}+\mathbb{H})\partial_{\alpha'}A_1$$

We see that

$$\begin{aligned}\left\|\frac{1}{\overline{Z}_{,\alpha'}}(\mathbb{I}+\mathbb{H})\partial_{\alpha'}A_1\right\|_2 &= \left\|\left[\frac{1}{\overline{Z}_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}A_1 + (\mathbb{I}+\mathbb{H})\overline{D}_{\alpha'}A_1\right\|_2 \\ &\lesssim \left\|\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2\|A_1\|_\infty + \left\|\overline{D}_{\alpha'}A_1\right\|_2\end{aligned}$$

Hence we finally have

$$\frac{d}{dt}\int A_1\left|\overline{Z}_{t,\alpha'}\right|^2 d\alpha' \approx 2\mathrm{Re}\int\left(iA_1\overline{D}_{\alpha'}\overline{Z}_t\right)|\partial_{\alpha'}|\left((Z_{tt}+i)\overline{Z}_{,\alpha'}\right)d\alpha'$$

3) By Lemma 2.4.6 we get

$$\sigma\frac{d}{dt}\int\left|\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right|^2 d\alpha' \approx 2\sigma\mathrm{Re}\int\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)D_t\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}Z_{t,\alpha'}\right)d\alpha'$$

Using $D_t|Z_{,\alpha'}| = |Z_{,\alpha'}|\{\mathrm{Re}(D_{\alpha'}Z_t) - b_{\alpha'}\}$ and $[\partial_{\alpha'}, D_t] = b_{\alpha'}\partial_{\alpha'}$ we obtain

$$\begin{aligned}\sigma^{\frac{1}{2}}D_t\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}Z_{t,\alpha'}\right) &= \left(-\frac{3}{2}b_{\alpha'} - \frac{1}{2}\mathrm{Re}(D_{\alpha'}Z_t)\right)\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}Z_{t,\alpha'}\right) \\ &\quad - \left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right)Z_{t,\alpha'} + \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}Z_{tt,\alpha'}\end{aligned}$$

As $b_{\alpha'}, \text{Re}(D_{\alpha'}Z_t), \dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'} \in L^{\infty}$ and $Z_{t,\alpha'} \in L^2$ we have

$$\sigma\frac{d}{dt}\int\left|\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right|^2 d\alpha' \approx 2\sigma\text{Re}\int\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}Z_{tt,\alpha'}\right)d\alpha'$$

Now

$$\begin{aligned}\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}Z_{tt,\alpha'} &= \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\left(\frac{1}{\overline{Z}_{,\alpha'}}(Z_{tt}+i)\overline{Z}_{,\alpha'}\right)\\ &= \left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right)(Z_{tt}+i)\overline{Z}_{,\alpha'} + 2\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)\partial_{\alpha'}\left\{(Z_{tt}+i)\overline{Z}_{,\alpha'}\right\}\\ &\quad + \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}\overline{Z}_{,\alpha'}}\partial_{\alpha'}^2\left\{(Z_{tt}+i)\overline{Z}_{,\alpha'}\right\}\end{aligned}$$

Using $(Z_{tt}+i)\overline{Z}_{,\alpha'} = iA_1 + \sigma\partial_{\alpha'}\overline{\Theta}$ we obtain the estimate

$$\begin{aligned}&\left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right)(Z_{tt}+i)\overline{Z}_{,\alpha'}\right\|_2\\ &\lesssim \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2\|A_1\|_\infty + \left\|\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_\infty\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2\end{aligned}$$

and also

$$\begin{aligned}&\left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)\partial_{\alpha'}\left\{(Z_{tt}+i)\overline{Z}_{,\alpha'}\right\}\right\|_2\\ &\lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left(\left\||D_{\alpha'}|A_1\right\|_2 + \left\|\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\Theta\right\|_2\right)\end{aligned}$$

Hence

$$\begin{aligned}&\sigma\frac{d}{dt}\int\left|\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right|^2 d\alpha'\\ &\approx 2\sigma\text{Re}\int\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}\overline{Z}_{,\alpha'}}\partial_{\alpha'}^2\left\{(Z_{tt}+i)\overline{Z}_{,\alpha'}\right\}\right)d\alpha'\end{aligned}$$

Now using $\partial_{\alpha'} = i|\partial_{\alpha'}| + (\mathbb{I}+\mathbb{H})\partial_{\alpha'}$ and $(Z_{tt}+i)\overline{Z}_{,\alpha'} = iA_1 + \sigma\partial_{\alpha'}\overline{\Theta}$ we obtain

$$\begin{aligned}\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}\overline{Z}_{,\alpha'}}\partial_{\alpha'}^2\big\{(Z_{tt}+i)\overline{Z}_{,\alpha'}\big\} &= \frac{i\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}\overline{Z}_{,\alpha'}}\partial_{\alpha'}|\partial_{\alpha'}|\big\{(Z_{tt}+i)\overline{Z}_{,\alpha'}\big\} \\ &\quad + \frac{i\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}\overline{Z}_{,\alpha'}}(\mathbb{I}+\mathbb{H})\partial_{\alpha'}^2 A_1\end{aligned}$$

By commuting the Hilbert transform outside we obtain the estimate

$$\begin{aligned}&\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}(\mathbb{I}+\mathbb{H})\partial_{\alpha'}^2 A_1\right\|_2 \\ &\lesssim \|A_1\|_\infty\left(\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\right) + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right\|_2\end{aligned}$$

Hence

$$\sigma\frac{d}{dt}\int\left|\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right|^2 d\alpha' \approx 2\text{Re}\int\left\{-i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\}|\partial_{\alpha'}|\big((Z_{tt}+i)\overline{Z}_{,\alpha'}\big)d\alpha'$$

4) Now by combining all the three terms we obtain

$$\frac{d}{dt}E_{\sigma,1} \approx 2\text{Re}\int\left\{\overline{Z}_{ttt}Z_{,\alpha'} + iA_1\overline{D}_{\alpha'}\overline{Z}_t - i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\}|\partial_{\alpha'}|\big((Z_{tt}+i)\overline{Z}_{,\alpha'}\big)d\alpha'$$

Recall from (3.18) that

$$\begin{aligned}&\overline{Z}_{ttt}Z_{,\alpha'} + iA_1\overline{D}_{\alpha'}\overline{Z}_t - i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right) \\ &= i\sigma\partial_{\alpha'}\left\{\left(|D_{\alpha'}|\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}\right\} - \sigma(D_{\alpha'}Z_t)\partial_{\alpha'}\Theta - \sigma\partial_{\alpha'}\big\{(\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\} - iJ_1\end{aligned}$$

Hence it is sufficient to show that each of the terms on the right hand side is in $\dot{H}^{\frac{1}{2}}$. We have already shown that $J_1 \in \dot{H}^{\frac{1}{2}}$. We also have

$$\begin{aligned}\|\sigma(D_{\alpha'}Z_t)\partial_{\alpha'}\Theta\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}D_{\alpha'}Z_t\right\|_2\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2 + \|D_{\alpha'}Z_t\|_\infty\|\sigma\partial_{\alpha'}\Theta\|_{\dot{H}^{\frac{1}{2}}} \\ \left\|\sigma\partial_{\alpha'}\big\{(\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\}\right\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_2\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right\|_2 + \left\|\overline{D}_{\alpha'}\overline{Z}_t\right\|_\infty\|\sigma\partial_{\alpha'}\Theta\|_{\dot{H}^{\frac{1}{2}}} \\ &\quad + \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\text{Re}\Theta\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}}\end{aligned}$$

Now observe that

$$i\sigma\partial_{\alpha'}\left\{\left(|D_{\alpha'}|\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}\right\} = i\sigma\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} + i\sigma\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'} + i\sigma\left(\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_t$$

We have the estimates

$$\begin{aligned}&\left\|\sigma\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}\\ &\lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{\mathcal{W}}\left\||D_{\alpha'}|\overline{Z}_t\right\|_{\mathcal{C}}\end{aligned}$$

and

$$\left\|\sigma\left(\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{\mathcal{W}}\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}}$$

To control the last term we can use Proposition A.0.11 with $f = \sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}, w = \frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}$ and $h = \frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\overline{Z}_{t,\alpha'}$. Hence

$$\begin{aligned}&\left\|\sigma\left(\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}\\ &\lesssim \left\|\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}\left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\overline{Z}_{t,\alpha'}\right\|_{\infty} + \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2\left\|\sigma^{\frac{1}{3}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right\|_2\\ &\quad + \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{\overline{Z}_{,\alpha'}}\right\|_2\left\|\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\overline{Z}_{t,\alpha'}\right\|_{\infty}\end{aligned}$$

This completes the proof of $\frac{d}{dt}E_{\sigma,1}(t) \lesssim P(E_\sigma(t))$

4.2.3 Controlling $E_{\sigma,2}$ and $E_{\sigma,3}$

Recall that both $E_{\sigma,2}$ and $E_{\sigma,3}$ are of the form

$$E_{\sigma,i} = \|D_t f\|_2^2 + \left\|\sqrt{A_1}\frac{f}{|Z_{,\alpha'}|}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

Where $f = \overline{Z}_{t,\alpha'}$ for $i = 2$ and $f = \Theta$ for $i = 3$. Also note that $\mathbb{P}_H f = f$ for these choices of f. We will simplify the time derivative of each of the terms individually before combining them.

1) As $b_{\alpha'} \in L^\infty$ we have from Lemma 2.4.6

$$\frac{d}{dt}\int |D_t f|^2 \, d\alpha' \approx 2\text{Re}\int (D_t^2 f)(D_t \bar{f}) \, d\alpha'$$

2) By using Lemma 2.4.6 we have

$$\frac{d}{dt}\int \left| |\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right|^2 d\alpha' \approx 2\text{Re}\int \left\{|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right\} D_t\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f}\right) d\alpha'$$

Observe that

$$D_t\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f}\right) = \left\{\frac{D_t A_1}{2A_1} + b_{\alpha'} - \text{Re}(D_{\alpha'} Z_t)\right\}\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f} + \frac{\sqrt{A_1}}{|Z_{,\alpha'}|}D_t \bar{f}$$

We note that for $f = \overline{Z}_{t,\alpha'}$ or $f = \Theta$ we have $\dfrac{f}{|Z_{,\alpha'}|} \in \mathcal{C}$. Hence

$$\begin{aligned}
&\left\|\left(\frac{D_t A_1}{2A_1} + b_{\alpha'} - \text{Re}(D_{\alpha'} Z_t)\right)\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\bar{f}\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\frac{f}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\sqrt{A_1}\right\|_{\mathcal{W}}\left\{\|D_t A_1\|_{\mathcal{W}}\left\|\frac{1}{A_1}\right\|_{\mathcal{W}} + \|b_{\alpha'}\|_{\mathcal{W}} + \|D_{\alpha'} Z_t\|_{\mathcal{W}}\right\}
\end{aligned}$$

Hence we have

$$\frac{d}{dt}\int \left| |\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right|^2 d\alpha' \approx 2\text{Re}\int \left\{\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right\}(D_t \bar{f}) \, d\alpha'$$

We simplify further using $|\partial_{\alpha'}| = i\mathbb{H}\partial_{\alpha'}$ and $\mathbb{H}f = f$

$$
\begin{aligned}
&\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\\
&= i\left[\frac{\sqrt{A_1}}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right) + i\mathbb{H}\left\{\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)f + \frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right\}\\
&= i\left[\frac{\sqrt{A_1}}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right) + i\mathbb{H}\left\{\frac{1}{2}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)f + A_1\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right)f\right\}\\
&\quad - i\left[\frac{A_1}{|Z_{,\alpha'}|^2},\mathbb{H}\right]\partial_{\alpha'}f + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f
\end{aligned}
$$

Hence we have the estimate

$$
\begin{aligned}
&\left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}|\partial_{\alpha'}|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right) - i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right\|_2\\
&\lesssim \left(\||D_{\alpha'}|A_1\|_2 + \left\|\sqrt{A_1}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\right)\left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right\|_{\dot{H}^{\frac{1}{2}}} + \|A_1\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\left\|\frac{f}{|Z_{,\alpha'}|}\right\|_{\mathcal{C}}\\
&\quad + \|f\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}
\end{aligned}
$$

As $D_t f \in L^2$ this shows that

$$
\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right|^2 d\alpha' \approx 2\text{Re}\int\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right)(D_t\bar{f})\,d\alpha'
$$

3) By Lemma 2.4.6 we have

$$
\sigma\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right)\right|^2 d\alpha' \approx 2\sigma\text{Re}\int\left\{|\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right)\right\}D_t\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\bar{f}\right)d\alpha'
$$

We note that

$$
\sigma^{\frac{1}{2}}D_t\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\bar{f}\right) = \sigma^{\frac{1}{2}}\left(\frac{1}{2}b_{\alpha'} - \frac{3}{2}\text{Re}(D_{\alpha'}Z_t)\right)\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\bar{f}\right) + \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\bar{f}
$$

As $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f \in \mathcal{C}$ for $f = \overline{Z}_{t,\alpha'}$ or $f = \Theta$ we obtain

$$\left\| \sigma^{\frac{1}{2}} \left(\frac{1}{2} b_{\alpha'} - \frac{3}{2}\text{Re}(D_{\alpha'} Z_t) \right) \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \bar{f} \right) \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim (\|b_{\alpha'}\|_{\mathcal{W}} + \|D_{\alpha'} Z_t\|_{\mathcal{W}}) \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right\|_{\mathcal{C}}$$

Hence

$$\sigma \frac{d}{dt} \int \left| |\partial_{\alpha'}|^{\frac{1}{2}} \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \right|^2 d\alpha' \approx -2\sigma \text{Re} \int \partial_{\alpha'} \left\{ \frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} |\partial_{\alpha'}| \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \right\} D_t \bar{f} \, d\alpha'$$

Now we see that

$$\begin{aligned} \sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} |\partial_{\alpha'}| \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \right\} &= i\sigma \partial_{\alpha'} \left[\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \\ &\quad + i\sigma \mathbb{H} \partial_{\alpha'} \left\{ \frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \right\} \end{aligned}$$

We have the estimate

$$\begin{aligned} &\left\| \sigma \partial_{\alpha'} \left[\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}, \mathbb{H} \right] \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \right\|_2 \\ &\lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right\|_{\dot{H}^{\frac{1}{2}}} \left\{ \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 \frac{1}{|Z_{,\alpha'}|} \right\|_2 + \left\| \sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right\|_\infty \left\| \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right\|_2 \right\} \end{aligned}$$

By using the expansion in (3.11) for $f = \overline{Z}_{t,\alpha'}$ we get

$$\begin{aligned} &\left\| \sigma |D_{\alpha'}|^3 \overline{Z}_{t,\alpha'} - \sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right) \right\} \right\|_2 \\ &\lesssim \left\| \sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right\|_\infty \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}} \partial_{\alpha'}^2 \overline{Z}_{t,\alpha'} \right\|_2 + \left\| \sigma^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right\|_\infty^2 \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \\ &\quad + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 \frac{1}{|Z_{,\alpha'}|} \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_\infty \end{aligned}$$

Similarly using the expansion in (3.11) for $f = \Theta$ we obtain

$$\begin{aligned}
&\left\| \sigma|D_{\alpha'}|^3\Theta - \sigma\partial_{\alpha'}\left\{ \frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \right) \right\} \right\|_2 \\
&\lesssim \left\| |D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \right\|_{\mathcal{C}} \left\| \frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta \right\|_{\mathcal{C}} + \left\| \sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \right\|_\infty \left\| |D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \right\|_{\mathcal{C}} \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \right\|_{\mathcal{C}} \\
&\quad + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|} \right\|_{\mathcal{C}} \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \right\|_{\mathcal{C}}
\end{aligned}$$

Using these we now have

$$\sigma\frac{d}{dt}\int\left| |\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f \right) \right|^2 d\alpha' \approx -2\text{Re}\int \big(i\sigma\mathbb{H}|D_{\alpha'}|^3 f\big)D_t\bar{f}\,d\alpha'$$

But we have already shown that $(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3 f \in L^2$ for both $f = \overline{Z}_{t,\alpha'}$ and $f = \Theta$. Hence we finally have

$$\sigma\frac{d}{dt}\int\left| |\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f \right) \right|^2 d\alpha' \approx 2\text{Re}\int \big(-i\sigma|D_{\alpha'}|^3 f\big)D_t\bar{f}\,d\alpha'$$

4) Combining all three terms we obtain

$$\frac{d}{dt}E_{\sigma,i} \approx 2\text{Re}\int\left(D_t^2 f + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f - i\sigma|D_{\alpha'}|^3 f \right)(D_t\bar{f})\,d\alpha'$$

For $f = \overline{Z}_{t,\alpha'}$ we obtain from (3.21)

$$\begin{aligned}
&\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3 \right)\overline{Z}_{t,\alpha'} \\
&= R_1\overline{Z}_{,\alpha'} - i\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} \right)J_1 - iD_{\alpha'}J_1 - \overline{Z}_{,\alpha'}\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}} \right]\overline{Z}_{t,\alpha'}
\end{aligned}$$

We have already shown that $R_1 \in \mathcal{C}$, $J_1 \in \mathcal{W}$, $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2$ and the last term in L^2. Now for $f = \Theta$ we have from (3.22)

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3 \right)\Theta = R_2 + iJ_2$$

In this case too we have shown that $R_2, J_2 \in L^2$. Hence this shows that

$$\frac{d}{dt}E_{\sigma,i}(t) \lesssim P(E_\sigma(t)) \qquad \text{for } i = 2,3$$

4.2.4 Controlling $E_{\sigma,4}$

Recall that

$$E_{\sigma,4} = \left\|D_t\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\sqrt{A_1}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_2^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_2^2$$

As before we first simplify the terms individually before combining them.

1) By Lemma 2.4.6 and the fact that $|\partial_{\alpha'}| = i\mathbb{H}\partial_{\alpha'}$ we have

$$\begin{aligned}\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(D_t\overline{D}_{\alpha'}\overline{Z}_t\right)\right|^2 d\alpha' &\approx 2\text{Re}\int (D_t^2\overline{D}_{\alpha'}\overline{Z}_t)|\partial_{\alpha'}|(D_tD_{\alpha'}Z_t)\,d\alpha' \\ &\approx 2\text{Re}\int(\mathbb{H}D_t^2\overline{D}_{\alpha'}\overline{Z}_t)\{-i\partial_{\alpha'}(D_tD_{\alpha'}Z_t)\}\,d\alpha'\end{aligned}$$

But we have shown that $(\mathbb{I}-\mathbb{H})D_t^2\overline{D}_{\alpha'}\overline{Z}_t \in \dot{H}^{\frac{1}{2}}$. Hence we have

$$\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(D_t\overline{D}_{\alpha'}\overline{Z}_t\right)\right|^2 d\alpha' \approx 2\text{Re}\int(D_t^2\overline{D}_{\alpha'}\overline{Z}_t)\{-i\partial_{\alpha'}(D_tD_{\alpha'}Z_t)\}\,d\alpha'$$

2) By Lemma 2.4.6 and as $b_{\alpha'} \in L^\infty$ we have

$$\begin{aligned}\frac{d}{dt}\int A_1\left||D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right|^2 d\alpha' &\approx \int\left(\frac{D_tA_1}{A_1}\right)A_1\left||D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right|^2 d\alpha' \\ &\quad + 2\text{Re}\int A_1\left(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)D_t(|D_{\alpha'}|D_{\alpha'}Z_t)\,d\alpha'\end{aligned}$$

As $\dfrac{D_tA_1}{A_1} \in L^\infty$, the first term is controlled. We now see that

$$D_t|D_{\alpha'}|D_{\alpha'}Z_t = -\text{Re}(D_{\alpha'}Z_t)|D_{\alpha'}|D_{\alpha'}Z_t + |D_{\alpha'}|D_tD_{\alpha'}Z_t$$

Now as $\text{Re}(D_{\alpha'}Z_t) \in L^\infty$ we obtain

$$\frac{d}{dt}\int A_1\left||D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right|^2 d\alpha' \approx 2\text{Re}\int\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)\{-i\partial_{\alpha'}(D_tD_{\alpha'}Z_t)\}\,d\alpha'$$

3) By Lemma 2.4.6 and as $b_{\alpha'} \in L^\infty$ we have

$$\begin{aligned}
&\sigma\frac{d}{dt}\int\left|\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right|^2 d\alpha' \\
&\approx 2\sigma\text{Re}\int\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\}D_t\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_t\right\}d\alpha'
\end{aligned}$$

We see that

$$\begin{aligned}
&\sigma^{\frac{1}{2}}D_t\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_t\right\} \\
&= \left(-\frac{3}{2}\text{Re}(D_{\alpha'}Z_t) - \frac{b_{\alpha'}}{2}\right)\left\{\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_t\right\} - \text{Re}\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t\right)(|D_{\alpha'}|D_{\alpha'}Z_t) \\
&\quad + \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_tD_{\alpha'}Z_t
\end{aligned}$$

As $D_{\alpha'}Z_t, b_{\alpha'} \in L^\infty$, the first term is controlled in L^2. The second term is also in L^2 as we have $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t \in L^\infty$ and $|D_{\alpha'}|D_{\alpha'}Z_t \in L^2$. Hence we have

$$\sigma\frac{d}{dt}\int\left|\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right|^2 d\alpha' \approx 2\text{Re}\int\left(-i\sigma|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\right)\{-i\partial_{\alpha'}(D_tD_{\alpha'}Z_t)\}\,d\alpha'$$

4) Combining the three terms we get

$$\frac{d}{dt}E_{\sigma,4} \approx 2\text{Re}\int\left(D_t^2\overline{D}_{\alpha'}\overline{Z}_t + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t - i\sigma|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\right)\{-i\partial_{\alpha'}(D_tD_{\alpha'}Z_t)\}\,d\alpha'$$

From equation (3.19) we see that

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\overline{D}_{\alpha'}\overline{Z}_t = R_1 - i\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1 - i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1$$

But we have already shown that $R_1, \dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1 \in \dot{H}^{\frac{1}{2}}$ and the second term is controlled in

$\dot{H}^{\frac{1}{2}}$ by the estimate

$$\left\| \left(\overline{D}_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) J_1 \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\| \overline{D}_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{\mathcal{C}} \|J_1\|_{\mathcal{W}}$$

Hence we have

$$\frac{d}{dt} E_{\sigma,4}(t) \lesssim P(E_\sigma(t))$$

This concludes the proof of Theorem 3.4.1

4.3 Equivalence of E_σ and $\mathcal{E}_\sigma$

Proof of Proposition 3.4.3. Let us first prove that $\mathcal{E}_\sigma \leq P_2(E_\sigma)$. Note that in §4.1 we have pretty much controlled all the terms in $\mathcal{E}_\sigma$. The terms which are not controlled $\dfrac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}} \partial_{\alpha'}^2 \dfrac{1}{Z_{,\alpha'}}$ and $\dfrac{\sigma}{Z_{,\alpha'}^2} \partial_{\alpha'}^3 \dfrac{1}{Z_{,\alpha'}}$ can be easily controlled in $\dot{H}^{\frac{1}{2}}$ as we have that $\omega \in \mathcal{W}$ and we have already shown that $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'}^2 \dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $\dfrac{\sigma}{|Z_{,\alpha'}|^2} \partial_{\alpha'}^3 \dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$.

Let us now show that $E_\sigma \leq P_1(\mathcal{E}_\sigma)$. We will now say that $f \in L^2$ if there exists a polynomial P such that $\|f\|_2 \leq P(\mathcal{E}_\sigma)$ in analogy to the notation in §4.1. Similar notation for the other spaces defined there. We now control the terms.

1. First observe that $E_{\sigma,0}$ is already controlled by $\mathcal{E}_\sigma$.

2. As we have $\overline{Z}_{t,\alpha'} \in L^2$, we see that $A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$. As $\sigma \partial_{\alpha'} \Theta \in \dot{H}^{\frac{1}{2}}$ we have that $(\overline{Z}_{tt} - i) Z_{,\alpha'} \in \dot{H}^{\frac{1}{2}}$ by using equation (3.7). Hence we see that $E_{\sigma,1}$ is controlled.

3. We now show that $D_{\alpha'} \overline{Z}_t \in L^\infty$. Observe that

$$\partial_{\alpha'} (D_{\alpha'} \overline{Z}_t)^2 = 2(\overline{Z}_{t,\alpha'})(D_{\alpha'}^2 \overline{Z}_t) = 2(\overline{Z}_{t,\alpha'}) \left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) D_{\alpha'} \overline{Z}_t + 2(\overline{Z}_{t,\alpha'}) \left(\frac{1}{Z_{,\alpha'}^2} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right)$$

Hence we have

$$\left\| D_{\alpha'} \overline{Z}_t \right\|_\infty^2 \leq 2 \left\| \overline{Z}_{t,\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2 \left\| D_{\alpha'} \overline{Z}_t \right\|_\infty + 2 \left\| \overline{Z}_{t,\alpha'} \right\|_2 \left\| \frac{1}{Z_{,\alpha'}^2} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2$$

Now using the inequality $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ on the first term, we obtain $D_{\alpha'} \overline{Z}_t \in L^\infty$.

4. Following the apriori estimate for E_σ we now have the terms $|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t \in L^2$, $\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^2$, $\omega \in \mathcal{W}$, $\overline{D}_{\alpha'}\overline{Z}_t \in \mathcal{W}\cap\mathcal{C}$, $\partial_{\alpha'}\mathbb{P}_A\left(\dfrac{Z_t}{Z_{,\alpha'}}\right) \in L^\infty$, $A_1 \in \mathcal{W}$, $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in \mathcal{W}\cap\mathcal{C}$, $\Theta \in L^2$, $D_t\Theta \in L^2$ by formula (3.14). By using the proof of $D_{\alpha'}\dfrac{1}{Z_{,\alpha'}}$ in §4.1 we see that $|D_{\alpha'}|\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$ and $\dfrac{\Theta}{|Z_{,\alpha'}|} \in \mathcal{C}$. Hence we have $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\Theta \in \mathcal{C}$. Also $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\overline{Z}_{t,\alpha'} \in \mathcal{C}$ is easily shown.

5. Again by following the proof of E_σ we see that $\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{W}$, $\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta \in L^2$, $\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2$ etc. and $\sigma^{\frac{1}{3}}\Theta \in L^\infty\cap\dot{H}^{\frac{1}{2}}$, $\sigma^{\frac{1}{3}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^\infty\cap\dot{H}^{\frac{1}{2}}$ etc. Hence we now have $\dfrac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\Theta \in L^2$, $\sigma\partial_{\alpha'}D_{\alpha'}\Theta \in L^2$ by following the proof of $\dfrac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in L^2$ in §4.1. In particular we now have $D_t\overline{Z}_{t,\alpha'} \in L^2$ by using equation (3.7).

6. By following the proof of $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in L^2$ in §4.1 we obtain $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t \in L^2$. We use Proposition A.0.10 with $f = \dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}$ and $w = \dfrac{1}{|Z_{,\alpha'}|}$ to similarly obtain $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{C}$. Hence $E_{\sigma,2}$ is controlled.

7. By following the proof of $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$ in §4.1 we see that $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \in \mathcal{C}$. Hence we see that $E_{\sigma,3}$ is controlled.

8. As $\omega \in \mathcal{W}$ we have $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$. Hence by following the proof of $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$ in §4.1 we obtain $\dfrac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta \in \mathcal{C}$, $\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta \in \mathcal{C}$ etc. Hence by using equation (3.7) we now have $\overline{D}_{\alpha'}\overline{Z}_{tt} \in \mathcal{C}$ and hence $D_t\overline{D}_{\alpha'}\overline{Z}_t \in \mathcal{C}$. Hence $E_{\sigma,4}$ is controlled and this finishes the proof of Proposition 3.4.3

□

4.4 Relation with Sobolev norm

Proof of Proposition 3.4.4. We prove each part seperately:

1. Let $\sigma > 0$ and assume that $\mathcal{E}_\sigma + \|Z_{,\alpha'}\|_\infty < \infty$. Hence we have that $\overline{Z}_{t,\alpha'} \in L^2$ and we have

$$\left\|\partial_{\alpha'}^2 \overline{Z}_{t,\alpha'}\right\|_2 \lesssim \frac{1}{\sigma^{\frac{1}{2}}}\|Z_{,\alpha'}\|_\infty^{\frac{5}{2}}\left\|\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2 \quad \text{and} \quad \||D_{\alpha'}|Z_{,\alpha'}\|_2 \lesssim \|Z_{,\alpha'}\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

Hence $\overline{Z}_{t,\alpha'} \in H^2$ and as $Z_{,\alpha'} \in L^\infty$ we obtain $Z_{,\alpha'} \in \mathcal{W}$. From this we see that

$$\left\|\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \frac{1}{\sigma}\|Z_{,\alpha'}\|_{\mathcal{W}}^2\left\|\frac{\sigma}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}}$$

Hence $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in H^{2.5}$. As $Z_{,\alpha'} \in L^\infty$, we clearly have $\partial_{\alpha'}Z_{,\alpha'} \in L^2$ as. Now for $s \geq 1$ we see that

$$\begin{aligned}\||\partial_{\alpha'}|^s\partial_{\alpha'}Z_{,\alpha'}\|_2 &= \left\||\partial_{\alpha'}|^s\left(Z_{,\alpha'}^2\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \\ &\lesssim \||\partial_{\alpha'}|^s Z_{,\alpha'}\|_2\|Z_{,\alpha'}\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty + \|Z_{,\alpha'}\|_\infty^2\left\||\partial_{\alpha'}|^s\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\end{aligned}$$

Using this for $s = 1, 2, 2.5$ sequentially we obtain $\partial_{\alpha'}Z_{,\alpha'} \in H^{2.5}$.

2. Now assume that $\sigma > 0$ and $\left\|\overline{Z}_{t,\alpha'}\right\|_{H^2} + \|\partial_{\alpha'}Z_{,\alpha'}\|_{H^{2.5}} + \left\|\dfrac{1}{Z_{,\alpha'}}\right\|_\infty < \infty$. We first observe that $\mathcal{E}_{\sigma,2}$ is easily controlled and that $\sigma^{\frac{1}{6}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^2$, $\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^\infty$. Now we have

$$\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim \left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty^2\|\partial_{\alpha'}Z_{,\alpha'}\|_2$$

and hence for $s \geq 1$ we have

$$\begin{aligned}\left\||\partial_{\alpha'}|^s\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 &= \left\||\partial_{\alpha'}|^s\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}Z_{,\alpha'}\right)\right\|_2 \\ &\lesssim \left\||\partial_{\alpha'}|^s\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty\|\partial_{\alpha'}Z_{,\alpha'}\|_\infty + \left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty^2\||\partial_{\alpha'}|^s\partial_{\alpha'}Z_{,\alpha'}\|_2\end{aligned}$$

Using this for $s = 1, 2, 2.5$ sequentially we obtain $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in H^{2.5}$. Hence we easily see that

$\dfrac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2$ and $\dfrac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in L^2$. We also have the estimates

$$\begin{aligned}
\left\|\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 \\
\left\|\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \sigma^{\frac{1}{2}}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}^{\frac{3}{2}}\left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} + \sigma^{\frac{1}{2}}\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}^{\frac{1}{2}}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2
\end{aligned}$$

and similarly

$$\left\|\frac{\sigma}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \sigma\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}^2\left\|\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} + \sigma\left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2$$

We are only left with $\sigma\partial_{\alpha'}\Theta$. We first observe that as $Z_{,\alpha'} = e^{f+ig}$ we have

$$\partial_{\alpha'}Z_{,\alpha'} = Z_{,\alpha'}\partial_{\alpha'}(f+ig) \quad \text{and} \quad \partial_{\alpha'}^2 Z_{,\alpha'} = Z_{,\alpha'}\{\partial_{\alpha'}(f+ig)\}^2 + Z_{,\alpha'}\partial_{\alpha'}^2(f+ig)$$

and hence we have

$$\left\|\partial_{\alpha'}\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\right\|_{L^2\cap L^\infty} = \|\partial_{\alpha'}g\|_{L^2\cap L^\infty} \lesssim \left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}\|\partial_{\alpha'}Z_{,\alpha'}\|_{L^2\cap L^\infty}$$

and also

$$\begin{aligned}
\left\|\partial_{\alpha'}^2\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\right\|_2 &= \left\|\partial_{\alpha'}(e^{ig}\partial_{\alpha'}g)\right\|_2 \\
&\lesssim \|\partial_{\alpha'}g\|_2\|\partial_{\alpha'}g\|_\infty + \left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}\left\|\partial_{\alpha'}^2 Z_{,\alpha'}\right\|_2 + \left\|\frac{1}{Z_{,\alpha'}}\right\|_{\infty}^2\|\partial_{\alpha'}Z_{,\alpha'}\|_\infty\|\partial_{\alpha'}Z_{,\alpha'}\|_2
\end{aligned}$$

From this we see that

$$\left\||\partial_{\alpha'}|^{\frac{3}{2}}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \lesssim \left\|\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\partial_{\alpha'}^2\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\right\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2$$

Hence $\sigma\partial_{\alpha'}\Theta \in \dot{H}^{\frac{1}{2}}$ by using the formula (3.12).

□

4.5 Existence

Proof of Corollary 3.4.6. We fix $\sigma > 0$ and let $\mathcal{E}_\sigma(0) < \infty$. Now by Remark 3.4.5 we see that $Z_{,\alpha'} \in L^\infty$. Hence by Proposition 3.4.4, we see that $\partial_{\alpha'} Z_{,\alpha'} \in H^{2.5}$ and $\overline{Z}_{t,\alpha'} \in H^2$ which shows that the interface is $C^{3,\alpha}$ for any $0 < \alpha < 1$. Hence we can change the coordinate system to the Eulerian coordinate system and we have that $(\eta, \mathbf{v})|_{t=0} \in H^{3.5+s}(\mathbb{R}) \times H^{2+s}(\mathbb{R})$ for $0 \leq s < \frac{1}{2}$ where η is the height of the interface and $\mathbf{v}$ is the velocity on the boundary. Now by using the existence result of Alazard-Burq-Zuily [ABZ11] we now know that there is a unique solution in a time period $[0, T']$ where T' depends on the norm $\|\eta\|_{H^{3.5+s}} + \|\mathbf{v}\|_{H^{2+s}}$ of the initial data which in turn depends on norm $\|(Z - \alpha', Z_t)|_{t=0}\|_{H^{4.5} \times H^3}$. Now we wish to use Theorem 3.4.1 but as the solution is not smooth, we cannot directly use it.

This is easily remedied by mollifying the data by the Poisson kernel, $(Z^\epsilon, Z_t^\epsilon)|_{t=0} = (Z * P_\epsilon, Z_t * P_\epsilon)|_{t=0}$ and then solving the water wave equation with surface tension $\sigma > 0$ with the initial data $(Z^\epsilon, Z_t^\epsilon)|_{t=0}$ in the same way as above. Now we can apply Theorem 3.4.1 on these solutions $(Z^\epsilon, Z_t^\epsilon)$ as they are smooth, and hence these solutions exist in a time interval $[0, T]$ where T depends only on $\mathcal{E}_\sigma(Z^\epsilon, Z_t^\epsilon)(0) \leq \mathcal{E}_\sigma(Z, Z_t)(0)$ due to the holomorphicity of the terms in $\mathcal{E}_\sigma$. Now letting $\epsilon \to 0$, we obtain a solution to the water wave equation with initial data $(Z, Z_t)|_{t=0}$ in the time interval $[0, T]$ with $\partial_{\alpha'} Z_{,\alpha'} \in H^{2.5}$ and $\overline{Z}_{t,\alpha'} \in H^2$, and by uniqueness the time of existence of the solution is $max\{T, T'\}$.

□

CHAPTER 5

Convergence

In this chapter we prove Theorem 3.4.8. Note that this theorem was stated for the initial data of the two solutions being the same. We will prove a more general result here which can handle different initial data. Let us now state the result precisely.

First recall the higher order energy for the zero surface tension solutions

$$\mathcal{E}_{high} = \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

We define an auxiliary energy for the zero surface tension solutions needed in the statement of the convergence result. Let $\lambda > 0$ be any real number then we define

$$\begin{aligned} E_{\lambda,aux} = &\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_2^2 \\ &+ \left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\|_2^2 + \left\|\left\{\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\}\right\|_{\dot{H}^{\frac{1}{2}}}^2 \end{aligned}$$

We recall the notation used for convergence. Let A be a solution to the water wave equation with surface tension σ and B a solution to the water wave equation with no surface tension. We will denote by f_a the function f for solution A and f_b for solution B. For e.g. $(\overline{Z}_{t,\alpha'})_a$ and $(\overline{Z}_{t,\alpha'})_b$ denotes the spacial derivative of the velocity for the respective solutions. Note that with this notation equation (3.7) becomes

$$(\overline{Z}_{tt})_a - i = -i\left(\frac{A_1}{Z_{,\alpha'}}\right)_a + \sigma(D_{\alpha'}\Theta)_a \quad \text{where as} \quad (\overline{Z}_{tt})_b - i = -i\left(\frac{A_1}{Z_{,\alpha'}}\right)_b$$

We also have the operators

$$(|D_{\alpha'}|)_a = \frac{1}{|Z_{,\alpha'}|_a}\partial_{\alpha'} \quad (|D_{\alpha'}|)_b = \frac{1}{|Z_{,\alpha'}|_b}\partial_{\alpha'} \qquad \text{etc.}$$

Now let h_a, h_b be the change of coordinate diffeomorphisms as defined in (2.2) for the solutions A and B and let the material derivatives by given by $(D_t)_a = U_{h_a}^{-1}\partial_t U_{h_a}$ and $(D_t)_b = U_{h_b}^{-1}\partial_t U_{h_b}$. We define

$$\widetilde{h} = h_b \circ h_a^{-1} \qquad \text{and} \qquad \widetilde{U} = U_{\widetilde{h}} = U_{h_a}^{-1}U_{h_b}$$

While taking the difference of the two solutions, we want to subtract in Lagrangian coordinates and then bring it to the Riemmanian coordinate system of A. The reason we want to subtract in the Lagrangian coordinate system is that, in our proof of the energy estimate we mainly used the material derivative, and in Lagrangian coordinate system the material derivative for both the solutions is given by the same operator ∂_t. The operator $\widetilde{U}$ first takes a function in the Riemmanian coordinate system of B to the Lagrangian coordinate system and then to the Riemmanian coordinate system of A.

We define $\Delta(f) = f_a - \widetilde{U}(f_b)$. For e.g.

$$\Delta(\overline{Z}_{t,\alpha'}) = (\overline{Z}_{t,\alpha'})_a - \widetilde{U}(\overline{Z}_{t,\alpha'})_b \qquad \Delta(\overline{Z}_{tt}) = -i\left\{\left(\frac{A_1}{Z_{,\alpha'}}\right)_a - \widetilde{U}\left(\frac{A_1}{Z_{,\alpha'}}\right)_b\right\} + \sigma(D_{\alpha'}\Theta)_a$$

We are now ready to define the energy for the difference of the solutions. Define

$$\begin{aligned}
E_{\Delta,0} = {} & \left\|\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_\infty^2 + \left\|\left(\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^6 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^2 \\
& + \|\Delta(\omega)\|_\infty^2 + \left\|\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2^2 + \left\|\widetilde{h}_{\alpha'} - 1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 + \left\||D_{\alpha'}|_a(\widetilde{h}_{\alpha'} - 1)\right\|_2^2 \\
& + \left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|_b}\right) - 1\right\|_\infty^2
\end{aligned}$$

$$E_{\Delta,1} = \left\|\Delta\left\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\right\}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|(\sqrt{A_1})_a\Delta(\overline{Z}_{t,\alpha'})\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a\right\|_2^2$$

$$E_{\Delta,2} = \left\|\Delta(D_t\overline{Z}_{t,\alpha'})\right\|_2^2 + \left\|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a\Delta(\overline{Z}_{t,\alpha'})\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

$$E_{\Delta,3} = \|\Delta(D_t\Theta)\|_2^2 + \left\|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a\Delta(\Theta)\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

$$E_{\Delta,4} = \left\|\Delta(D_t\overline{D}_{\alpha'}\overline{Z}_t)\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|(\sqrt{A_1})_a|D_{\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t)\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

$$E_\Delta = (E_{\sigma,aux})_b + E_{\Delta,0} + E_{\Delta,1} + E_{\Delta,2} + E_{\Delta,3} + E_{\Delta,4}$$

Note that here $(E_{\sigma,aux})_b$ is the energy $E_{\lambda,aux}$ for solution B with the value $\lambda = \sigma$. Hence $(E_{\sigma,aux})_b$ couples the zero surface tension solution B with the value of surface tension σ of the solution A. This term is added due to technical reasons and is explained in more detail in §5.2. The other terms in the energy come from taking a difference in the energy E_σ. We now state our result about convergence.

Theorem 5.0.1. *Consider two smooth solutions $(Z, Z_t)_a(t)$, $(Z, Z_t)_b(t)$ in $[0, T]$ to the equation* (3.7) *with surface tension σ and with surface tension $\sigma = 0$ respectively. Assume that there exists an $M > 0$ such that*

$$\sup_{t\in[0,T]}(\mathcal{E}_{high})_b(t),\ \sup_{t\in[0,T]}(\mathcal{E}_\sigma)_a(t),\ \left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|_b}\right)\right\|_\infty(0),\ \left\|\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}\big(|Z_{,\alpha'}|_b\big)\right\|_\infty(0) \leq M$$

Then there exists a constant $C(M)$ depending only on M so that

$$\frac{d}{dt}E_\Delta(t) \leq C(M)E_\Delta(t) \quad \textit{for all } t \in [0, T)$$

In the above theorem, the restriction

$$\left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|_b}\right)\right\|_\infty(0),\ \left\|\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}\big(|Z_{,\alpha'}|_b\big)\right\|_\infty(0) \leq M$$

forces the initial interfaces of the two solutions to be close to each other. The theorem simplifies considerably if we work with the same initial data for the two solutions A and B which was how Theorem 3.4.8 was stated. For $\sigma = 0$, one can use the existence result Theorem 3.4 of Wu [Wu15], where it is shown that for data in the class $\mathcal{E}_{high}(0) < \infty$, the solution exists in a time interval $[0, T]$ with T depending only on $\mathcal{E}_{high}(0)$ and satisfies $\sup_{[0,T]}\mathcal{E}_{high}(t) < \infty$. For $\sigma > 0$ we can use Corollary 3.4.6 for an existence result in terms of $\mathcal{E}_\sigma$. Also see that the energy E_Δ controls $\|\Delta(\omega)\|_\infty$ and hence this says that $\theta_a - \theta_b \in L^\infty$ and the difference goes to zero as $A \to B$ as stated in §3.4.

The rest of this chapter is devoted to the proof of Theorem 5.0.1 and also the proof of the example stated in Proposition 3.4.9. We will first prove the apriori estimate for $\mathcal{E}_{high}$ in §5.1. We then use this estimate and prove an apriori estimate for $E_{\lambda,aux}$ in terms of $\mathcal{E}_{high}$ in §5.2. We then prove Theorem 5.0.1 in §5.3. Finally we prove the example for convergence Proposition 3.4.9 in §5.4.

5.1 Higher order energy $\mathcal{E}_{high}$

We will now prove the higher order energy estimate for $\sigma = 0$. Note that we already have an energy estimate for $\sigma = 0$ by simply taking the special case of $\sigma = 0$ in the energy E_σ in Theorem 3.4.1. This higher order energy is equivalent to the energy used in Kinsey-Wu [KW14] (more precisely we do not have the term $\left\|\frac{1}{Z_{,\alpha'}}\right\|_\infty$ in the energy). The higher order energy corresponds to the energy from the equation for $D_{\alpha'}^2\overline{Z}_t$. We will control a lot of quantities which are not controlled in [KW14] such as $|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \in \mathcal{W}\cap\mathcal{C}$, $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{W}\cap\mathcal{C}$, $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t \in \mathcal{C}$ etc. The control of all these terms are needed in §5.2 which in turn is used to prove Theorem 5.0.1.

5.1.1 Equation for $D_{\alpha'}^2\overline{Z}_t$

Plugging in $\sigma = 0$ in the equation for $\overline{Z}_t$ from (3.17) we obtain

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\overline{Z}_t = -i\frac{J_1}{Z_{,\alpha'}}$$

with $J_1 = D_tA_1 + A_1\left(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t\right)$ and $\overline{Z}_{tt} - i = -i\dfrac{A_1}{Z_{,\alpha'}}$. Applying $D_{\alpha'}^2$ to the above equation we obtain

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)D_{\alpha'}^2\overline{Z}_t = -iD_{\alpha'}^2\left(\frac{J_1}{Z_{,\alpha'}}\right) + \left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, D_{\alpha'}^2\right]\overline{Z}_t$$

Let us try to simplify the terms above

a) Recall that $[D_{\alpha'}, D_t] = (D_{\alpha'}Z_t)D_{\alpha'}$ and $[\partial_{\alpha'}, D_t] = b_{\alpha'}\partial_{\alpha'}$

b) We see that

$$\begin{aligned}\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, D_{\alpha'}\right] &= D_t[D_t, D_{\alpha'}] + [D_t, D_{\alpha'}]D_t + [(Z_{tt}+i)D_{\alpha'}, D_{\alpha'}] \\ &= \left\{-2(D_{\alpha'}Z_{tt}) + 2(D_{\alpha'}Z_t)^2\right\}D_{\alpha'} - 2(D_{\alpha'}Z_t)D_{\alpha'}D_t\end{aligned}$$

c) We have the relation

$$
\begin{aligned}
&\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, D_{\alpha'}\right]D_{\alpha'} \\
&= \left\{-2(D_{\alpha'}Z_{tt}) + 2(D_{\alpha'}Z_t)^2\right\}D_{\alpha'}^2 - 2(D_{\alpha'}Z_t)D_{\alpha'}D_tD_{\alpha'} \\
&= \left\{-2(D_{\alpha'}Z_{tt}) + 2(D_{\alpha'}Z_t)^2\right\}D_{\alpha'}^2 - 2(D_{\alpha'}Z_t)D_{\alpha'}\{-(D_{\alpha'}Z_t)D_{\alpha'} + D_{\alpha'}D_t\} \\
&= 2(D_{\alpha'}Z_t)(D_{\alpha'}^2Z_t)D_{\alpha'} + \left\{-2(D_{\alpha'}Z_{tt}) + 4(D_{\alpha'}Z_t)^2\right\}D_{\alpha'}^2 - 2(D_{\alpha'}Z_t)D_{\alpha'}^2D_t
\end{aligned}
$$

d) We similarly have

$$
\begin{aligned}
&D_{\alpha'}\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, D_{\alpha'}\right] \\
&= \left\{-2(D_{\alpha'}^2Z_{tt}) + 4(D_{\alpha'}Z_t)(D_{\alpha'}^2Z_t)\right\}D_{\alpha'} + \left\{-2(D_{\alpha'}Z_{tt}) + 2(D_{\alpha'}Z_t)^2\right\}D_{\alpha'}^2 \\
&\quad - 2(D_{\alpha'}^2Z_t)D_{\alpha'}D_t - 2(D_{\alpha'}Z_t)D_{\alpha'}^2D_t
\end{aligned}
$$

e) Hence we have

$$
\begin{aligned}
&\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, D_{\alpha'}^2\right] \\
&= \left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, D_{\alpha'}\right]D_{\alpha'} + D_{\alpha'}\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, D_{\alpha'}\right] \\
&= \left\{-2(D_{\alpha'}^2Z_{tt}) + 6(D_{\alpha'}Z_t)(D_{\alpha'}^2Z_t)\right\}D_{\alpha'} + \left\{-4(D_{\alpha'}Z_{tt}) + 6(D_{\alpha'}Z_t)^2\right\}D_{\alpha'}^2 \\
&\quad - 2(D_{\alpha'}^2Z_t)D_{\alpha'}D_t - 4(D_{\alpha'}Z_t)D_{\alpha'}^2D_t
\end{aligned}
$$

f) We see that

$$
\begin{aligned}
-iD_{\alpha'}^2\left(\frac{J_1}{Z_{,\alpha'}}\right) &= -iD_{\alpha'}\left\{\frac{\overline{\omega}^2}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1 + J_1\left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\} \\
&= -i\overline{\omega}^3|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) - 2i\overline{\omega}(D_{\alpha'}\overline{\omega})\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \\
&\quad - i(D_{\alpha'}J_1)\left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - iJ_1\left(D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)
\end{aligned}
$$

Combining the above identities we get the equation for $D_{\alpha'}^2 \overline{Z}_t$

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)D_{\alpha'}^2 \overline{Z}_t = -i\overline{\omega}^3 |D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} J_1\right) + R_3 \tag{5.1}$$

where

$$\begin{aligned} R_3 = &\left\{-2(D_{\alpha'}^2 Z_{tt}) + 6(D_{\alpha'}Z_t)(D_{\alpha'}^2 Z_t)\right\}(D_{\alpha'}\overline{Z}_t) + \left\{-4(D_{\alpha'}Z_{tt}) + 6(D_{\alpha'}Z_t)^2\right\}(D_{\alpha'}^2\overline{Z}_t) \\ & - 2(D_{\alpha'}^2 Z_t)(D_{\alpha'}\overline{Z}_{tt}) - 4(D_{\alpha'}Z_t)(D_{\alpha'}^2\overline{Z}_{tt}) - 2i\overline{\omega}(D_{\alpha'}\overline{\omega})\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} J_1\right) \\ & - i(D_{\alpha'}J_1)\left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - iJ_1\left(D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right) \end{aligned} \tag{5.2}$$

5.1.2 Statement of the energy estimate for E_{high}

We will now write down the energy and prove the energy estimate. We define the higher order energy for zero surface tension solutions as

$$E_{high} = E_\sigma|_{\sigma=0} + \int \left|D_t D_{\alpha'}^2 \overline{Z}_t\right|^2 d\alpha' + \int \left| |\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}D_{\alpha'}^2\overline{Z}_t\right)\right|^2 d\alpha'$$

Theorem 5.1.1. *Let $T > 0$ and let (Z, Z_t) be a smooth solution to the gravity water wave equation with zero surface tension in the time interval $[0, T)$ with $E_{high}(t) < \infty$ for all $t \in [0, T)$. Then there exists a polynomial P with universal non-negative coefficients such that for all $t \in [0, T)$ we have*

$$\frac{dE_{high}(t)}{dt} \leq P(E_{high}(t))$$

5.1.3 Quantities controlled by the energy E_{high}

In this section whenever we write $f \in L^2$, what we mean is that there exists a universal polynomial P with nonnegative coefficients such that $\|f\|_2 \leq P(E_{high})$. Similar definitions for $f \in \dot{H}^{\frac{1}{2}}$, $f \in L^\infty$, $f \in \mathcal{C}$ or $f \in \mathcal{W}$ where the definitions for the spaces $\mathcal{C}$ and $\mathcal{W}$ are as in §4.1. Note that E_{high} controls the energy $E_\sigma|_{\sigma=0}$ and hence we already have control of a lot of quantities as proved in §4.1. We will freely use the quantities controlled by $E_\sigma|_{\sigma=0}$ to prove the above theorem. In particular we will also be making use of Lemma 2.4.5. Let us now establish the quantities controlled by E_{high} which are not controlled by $E_\sigma|_{\sigma=0}$.

1) $D_tD_{\alpha'}^2\overline{Z}_t \in L^2$, $D_{\alpha'}^2\overline{Z}_{tt} \in L^2$, $|D_{\alpha'}|^2\overline{Z}_t \in L^2$ and $D_{\alpha'}^2Z_{tt} \in L^2$, $|D_{\alpha'}|D_tD_{\alpha'}Z_t \in L^2$

Proof: We see that
$$\begin{aligned} D_tD_{\alpha'}^2\overline{Z}_t &= \left[D_t, D_{\alpha'}^2\right]\overline{Z}_t + D_{\alpha'}^2\overline{Z}_{tt} \\ &= D_{\alpha'}\left\{-(D_{\alpha'}Z_t)D_{\alpha'}\overline{Z}_t\right\} - (D_{\alpha'}Z_t)D_{\alpha'}^2\overline{Z}_t + D_{\alpha'}^2\overline{Z}_{tt} \\ &= -(D_{\alpha'}^2Z_t)D_{\alpha'}\overline{Z}_t - 2(D_{\alpha'}Z_t)D_{\alpha'}^2\overline{Z}_t + D_{\alpha'}^2\overline{Z}_{tt} \end{aligned}$$

Now $D_tD_{\alpha'}^2\overline{Z}_t \in L^2$ as it part of the energy and hence we have

$$\left\|D_{\alpha'}^2\overline{Z}_{tt}\right\|_2 \lesssim \left\|D_tD_{\alpha'}^2\overline{Z}_t\right\|_2 + \left\|D_{\alpha'}^2Z_t\right\|_2\left\|D_{\alpha'}\overline{Z}_t\right\|_\infty + \left\|D_{\alpha'}^2\overline{Z}_t\right\|_2\|D_{\alpha'}Z_t\|_\infty$$

Now we observe that

$$D_{\alpha'}^2\overline{Z}_{tt} = D_{\alpha'}\left(\overline{\omega}|D_{\alpha'}|\overline{Z}_{tt}\right) = \overline{\omega}(|D_{\alpha'}|\overline{\omega})|D_{\alpha'}|\overline{Z}_{tt} + \overline{\omega}^2|D_{\alpha'}|^2\overline{Z}_{tt}$$

Hence we have

$$\left\||D_{\alpha'}|^2\overline{Z}_{tt}\right\|_2 \lesssim \left\|D_{\alpha'}^2\overline{Z}_{tt}\right\|_2 + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{\omega}\right\|_{\mathcal{C}}\left\||D_{\alpha'}|\overline{Z}_{tt}\right\|_{\mathcal{C}}$$

The terms $D_{\alpha'}^2Z_{tt} \in L^2$, $|D_{\alpha'}|D_tD_{\alpha'}Z_t \in L^2$ are proven similarly.

2) $D_{\alpha'}\overline{Z}_{tt} \in \mathcal{W}\cap\mathcal{C}$, $|D_{\alpha'}|\overline{Z}_{tt} \in \mathcal{W}\cap\mathcal{C}$ and $\overline{D}_{\alpha'}\overline{Z}_{tt} \in \mathcal{W}\cap\mathcal{C}$

Proof: We already know that $D_{\alpha'}\overline{Z}_{tt} \in \mathcal{C}$ and $D_{\alpha'}^2\overline{Z}_{tt} \in L^2$. Hence using $f = D_{\alpha'}\overline{Z}_{tt}$ and $w = \frac{1}{Z_{,\alpha'}}$ in Proposition A.0.10 we obtain

$$\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_\infty^2 \lesssim \left\|\overline{Z}_{tt,\alpha'}\right\|_2\left\|D_{\alpha'}^2\overline{Z}_{tt}\right\|_2$$

We also have $\||D_{\alpha'}|Z_{tt}\|_{\mathcal{W}\cap\mathcal{C}} \lesssim \|\overline{\omega}\|_{\mathcal{W}}\|D_{\alpha'}Z_{tt}\|_{\mathcal{W}\cap\mathcal{C}}$ and $\overline{D}_{\alpha'}\overline{Z}_{tt} \in \mathcal{W}\cap\mathcal{C}$ is shown similarly.

3) $|D_{\alpha'}|\dfrac{1}{Z_{,\alpha'}} \in L^\infty$, $|D_{\alpha'}|\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty$ and $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in L^\infty$

Proof: We observe that as $\overline{Z}_{tt} - i = -i\dfrac{A_1}{Z_{,\alpha'}}$ we have

$$|D_{\alpha'}|\overline{Z}_{tt} = -i\frac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 - iA_1|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}$$

As $A_1 \geq 1$ we have

$$\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty \lesssim \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty + \left\||D_{\alpha'}|\overline{Z}_{tt}\right\|_\infty$$

Hence $|D_{\alpha'}|\dfrac{1}{Z_{,\alpha'}} \in L^\infty$. Now recall from (2.9) that

$$\mathrm{Re}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = |D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i\left(\frac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right)$$

Hence we easily see that $|D_{\alpha'}|\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty$ and $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in L^\infty$

4) $|D_{\alpha'}|^2\dfrac{1}{Z_{,\alpha'}} \in L^2$, $D_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2$ and similarly $|D_{\alpha'}|^2\dfrac{1}{|Z_{,\alpha'}|} \in L^2$, $\dfrac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^3\omega \in L^2$. We also have $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2$, $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta \in L^2$

Proof: As $\overline{Z}_{tt} - i = -i\dfrac{A_1}{Z_{,\alpha'}}$ we have

$$\begin{aligned}
&|D_{\alpha'}|^2\overline{Z}_{tt} \\
&= |D_{\alpha'}|\left(-i\frac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 - iA_1|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right) \\
&= -i\overline{\omega}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) - i(|D_{\alpha'}|\overline{\omega})\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) - i(|D_{\alpha'}|A_1)\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right) \\
&\quad - iA_1|D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}
\end{aligned}$$

As $A_1 \geq 1$ we see that

$$\begin{aligned}
\left\||D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim \left\||D_{\alpha'}|^2\overline{Z}_{tt}\right\|_2 + \left\||D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\|_2 + \left\||D_{\alpha'}|\omega\right\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty \\
+ \left\||D_{\alpha'}|A_1\right\|_2\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty
\end{aligned}$$

Now we see that

$$D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} = D_{\alpha'}\left(\overline{\omega}|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right) = \overline{\omega}(|D_{\alpha'}|\overline{\omega})|D_{\alpha'}|\frac{1}{Z_{,\alpha'}} + \overline{\omega}^2|D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}$$

Hence we have

$$\left\|D_{\alpha'}^2 \frac{1}{Z_{,\alpha'}}\right\|_2 \lesssim \|D_{\alpha'}\overline{\omega}\|_2\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty + \left\||D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}\right\|_2$$

Now using the formula (2.9) we have

$$\text{Re}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = |D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \qquad \text{Im}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i\left(\frac{\overline{\omega}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right)$$

Hence we have

$$\begin{aligned}
\left\||D_{\alpha'}|^2\frac{1}{|Z_{,\alpha'}|}\right\|_2 &\lesssim \||D_{\alpha'}|\omega\|_2\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty + \left\||D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}\right\|_2 \\
\left\|\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2\omega\right\|_2 &\lesssim \||D_{\alpha'}|\omega\|_2\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty + \left\||D_{\alpha'}|^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \||D_{\alpha'}|\overline{\omega}\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right\|_\infty \\
&\quad + \left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\||D_{\alpha'}|\omega\|_2
\end{aligned}$$

The estimate $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} \in L^2$ is shown similarly. Now recall the formula of Θ from (3.12)

$$\Theta = i\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}} - i\text{Re}(\mathbb{I}-\mathbb{H})\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)$$

We have

$$\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \lesssim \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2$$

and hence we obtain

$$\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\Theta\right\|_2 \lesssim \left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2$$

5) $|D_{\alpha'}|\frac{1}{Z_{,\alpha'}} \in \mathcal{W}\cap\mathcal{C}$, $|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|} \in \mathcal{W}\cap\mathcal{C}$ and $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega \in \mathcal{W}\cap\mathcal{C}$

Proof: The inclusion in $\mathcal{C}$ is known as it is part of energy estimate for E_σ for $\sigma = 0$. Now we have already shown all the quantities in L^∞ and using the above estimates like $|D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \in L^2$ we are done.

6) $D_t b_{\alpha'} \in L^\infty$, $\partial_{\alpha'} D_t b \in L^\infty$

Proof: We already know that $E_\sigma|_{\sigma=0}$ controls $D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty$. Now

$$D_t D_{\alpha'} Z_t = -(D_{\alpha'}Z_t)^2 + D_{\alpha'}Z_{tt}$$

and hence as $D_{\alpha'}Z_t \in L^\infty$, $D_{\alpha'}Z_{tt} \in L^\infty$ we have $D_t D_{\alpha'} Z_t \in L^\infty$. Hence we have $D_t b_{\alpha'} \in L^\infty$. We now have $\partial_{\alpha'} D_t b = b_{\alpha'}^2 + D_t b_{\alpha'}$ and as $b_{\alpha'} \in L^\infty$ we see that $\partial_{\alpha'} D_t b \in \infty$.

7) $\frac{1}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t \in \mathcal{C}, \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \in \mathcal{C}$, $\frac{1}{|Z_{,\alpha'}|^3} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \in \mathcal{C}$ and $\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \in \mathcal{C}$ and similarly $D_t^2 \overline{D}_{\alpha'} \overline{Z}_t \in \mathcal{C}$

Proof: From the energy we know that $\frac{\sqrt{A_1}}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t \in \dot{H}^{\frac{1}{2}}$. Hence as $\sqrt{A_1} D_{\alpha'}^2 \overline{Z}_t \in L^2$ we see that $\frac{\sqrt{A_1}}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t \in \mathcal{C}$. Hence we see that

$$\left\| \frac{1}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t \right\|_{\mathcal{C}} \lesssim \left\| \frac{\sqrt{A_1}}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t \right\|_{\mathcal{C}} \left\| \frac{1}{\sqrt{A_1}} \right\|_{\mathcal{W}}$$

As $\omega \in \mathcal{W}$ we see that $\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \in \mathcal{C}$. Now we have

$$\frac{1}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t = \left(D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) |D_{\alpha'}| \overline{Z}_t + \frac{\omega^2}{|Z_{,\alpha'}|^3} \partial_{\alpha'} \overline{Z}_{t,\alpha'}$$

Hence we have

$$\left\| \frac{1}{|Z_{,\alpha'}|^3} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_{\mathcal{C}} \lesssim \|\overline{\omega}\|_{\mathcal{W}}^2 \left\| \frac{1}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t \right\|_{\mathcal{C}} + \|\overline{\omega}\|_{\mathcal{W}}^2 \left\| D_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{\mathcal{C}} \left\| |D_{\alpha'}| \overline{Z}_t \right\|_{\mathcal{W}}$$

Now we see that

$$\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t = \left(|D_{\alpha'}| \frac{1}{\overline{Z}_{,\alpha'}} \right) |D_{\alpha'}| \overline{Z}_t + \frac{\omega}{|Z_{,\alpha'}|^3} \partial_{\alpha'} \overline{Z}_{t,\alpha'}$$

which implies

$$\left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \right\|_{\mathcal{C}} \lesssim \left\| |D_{\alpha'}| \frac{1}{\overline{Z}_{,\alpha'}} \right\|_{\mathcal{C}} \left\| |D_{\alpha'}| \overline{Z}_t \right\|_{\mathcal{W}} + \|w\|_{\mathcal{W}} \left\| \frac{1}{|Z_{,\alpha'}|^3} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_{\mathcal{C}}$$

Now we recall the equation for $\overline{D}_{\alpha'} \overline{Z}_t$ for $\sigma = 0$ from (3.19) and (3.20)

$$\left(D_t^2 + i \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \right) \overline{D}_{\alpha'} \overline{Z}_t = -2(\overline{D}_{\alpha'} \overline{Z}_t)(D_t \overline{D}_{\alpha'} \overline{Z}_t) - i \left(\overline{D}_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) J_1 - i \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} J_1$$

Hence we have

$$\begin{aligned} \left\| D_t^2 \overline{D}_{\alpha'} \overline{Z}_t \right\|_{\mathcal{C}} \lesssim & \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \right\|_{\mathcal{C}} + \left\| \overline{D}_{\alpha'} \overline{Z}_t \right\|_{\mathcal{W}} \left\| D_t \overline{D}_{\alpha'} \overline{Z}_t \right\|_{\mathcal{C}} + \left\| \overline{D}_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{\mathcal{C}} \|J_1\|_{\mathcal{W}} \\ & + \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_{\mathcal{C}} \end{aligned}$$

8) $(\mathbb{I} - \mathbb{H}) D_t^2 D_{\alpha'}^2 \overline{Z}_t \in L^2$

Proof: For a function f satisfying $\mathbb{P}_A f = 0$ we have

$$\begin{aligned} (\mathbb{I} - \mathbb{H}) D_t^2 f &= [D_t, \mathbb{H}] D_t f + D_t [D_t, \mathbb{H}] f \\ &= [b, \mathbb{H}] \partial_{\alpha'} D_t f + D_t [b, \mathbb{H}] \partial_{\alpha'} f \\ &= 2[b, \mathbb{H}] \partial_{\alpha'} D_t f + [D_t b, \mathbb{H}] \partial_{\alpha'} f - [b, b; \partial_{\alpha'} f] \end{aligned}$$

As $\mathbb{P}_A D_{\alpha'}^2 \overline{Z}_t = 0$ we obtain

$$\left\| (\mathbb{I} - \mathbb{H}) D_t^2 D_{\alpha'}^2 \overline{Z}_t \right\|_2 \lesssim \|b_{\alpha'}\|_\infty \left\| D_t D_{\alpha'}^2 \overline{Z}_t \right\|_2 + \|\partial_{\alpha'} D_t b\|_{\dot{H}^{\frac{1}{2}}} \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_2 + \|b_{\alpha'}\|_\infty^2 \left\| D_{\alpha'}^2 \overline{Z}_t \right\|_2$$

9) $(\mathbb{I} - \mathbb{H}) \left(i \dfrac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t \right) \in L^2$

Proof: We see that

$$(\mathbb{I} - \mathbb{H}) \left(i \frac{A_1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t \right) = i \left[\frac{A_1}{|Z_{,\alpha'}|^2}, \mathbb{H} \right] \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t$$

and hence we have

$$\left\|(\mathbb{I}-\mathbb{H})\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\|_2 \lesssim \left\|D_{\alpha'}^2\overline{Z}_t\right\|_2\left\{\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty + \|A_1\|_\infty\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\right\}$$

10) $R_3 \in L^2$

Proof: We recall from (5.2) the formula of R_3

$$\begin{aligned} R_3 &= \left\{-2(D_{\alpha'}^2 Z_{tt}) + 6(D_{\alpha'}Z_t)(D_{\alpha'}^2 Z_t)\right\}(D_{\alpha'}\overline{Z}_t) + \left\{-4(D_{\alpha'}Z_{tt}) + 6(D_{\alpha'}Z_t)^2\right\}(D_{\alpha'}^2\overline{Z}_t) \\ &\quad - 2(D_{\alpha'}^2 Z_t)(D_{\alpha'}\overline{Z}_{tt}) - 4(D_{\alpha'}Z_t)(D_{\alpha'}^2\overline{Z}_{tt}) - 2i\overline{\omega}(D_{\alpha'}\overline{\omega})\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \\ &\quad - i(D_{\alpha'}J_1)\left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - iJ_1\left(D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right) \end{aligned}$$

Hence we easily have the estimate

$$\begin{aligned} &\|R_3\|_2 \\ &\lesssim \left\{\left\|D_{\alpha'}^2 Z_{tt}\right\|_2 + \|D_{\alpha'}Z_t\|_\infty\left\|D_{\alpha'}^2 Z_t\right\|_2\right\}\left\|D_{\alpha'}\overline{Z}_t\right\|_\infty + \left\{\|D_{\alpha'}Z_{tt}\|_\infty + \|D_{\alpha'}Z_t\|_\infty^2\right\}\left\|D_{\alpha'}^2\overline{Z}_t\right\|_2 \\ &\quad + \left\|D_{\alpha'}^2 Z_t\right\|_2\left\|D_{\alpha'}\overline{Z}_{tt}\right\|_\infty + \|D_{\alpha'}Z_t\|_\infty\left\|D_{\alpha'}^2\overline{Z}_{tt}\right\|_2 + \|D_{\alpha'}\overline{\omega}\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right\|_\infty \\ &\quad + \|D_{\alpha'}J_1\|_2\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty + \|J_1\|_\infty\left\|D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 \end{aligned}$$

11) $|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \in L^2$ and hence $\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1 \in L^\infty$

Proof: As J_1 is real valued we have

$$\begin{aligned}
&|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\\
&= \text{Re}(\mathbb{I}-\mathbb{H})\left\{\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\}\\
&= \text{Re}\left\{\left[\frac{1}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) - \omega^3\left[\frac{\overline{\omega}^3}{|Z_{,\alpha'}|},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\}\\
&\quad + \text{Re}\left\{\omega^3(\mathbb{I}-\mathbb{H})\left\{\frac{\overline{\omega}^3}{|Z_{,\alpha'}|}\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\}\right\}
\end{aligned}$$

Now applying $(\mathbb{I}-\mathbb{H})$ on the equation for $D_{\alpha'}^2\overline{Z}_t$ from (5.1) we obtain

$$\begin{aligned}
\left\|(\mathbb{I}-\mathbb{H})\left\{\overline{\omega}^3|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\}\right\|_2 &\lesssim \left\|(\mathbb{I}-\mathbb{H})D_t^2D_{\alpha'}^2\overline{Z}_t\right\|_2 + \|R_3\|_2\\
&\quad + \left\|(\mathbb{I}-\mathbb{H})\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\|_2
\end{aligned}$$

Hence we have

$$\begin{aligned}
\left\||D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\|_2 &\lesssim \left\{\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2 + \||D_{\alpha'}|\overline{\omega}\|_2\right\}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right\|_\infty\\
&\quad + \left\|(\mathbb{I}-\mathbb{H})\left\{\overline{\omega}^3|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\}\right\|_2
\end{aligned}$$

Now we just use Proposition A.0.10 with the functions $f = \dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1$ and $w = \dfrac{1}{|Z_{,\alpha'}|}$ and we easily get that $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1 \in L^\infty$

5.1.4 Closing the energy estimate for E_{high}

We now complete the proof of Theorem 5.1.1. Recall that

$$E_{high} = E_\sigma|_{\sigma=0} + \int\left|D_tD_{\alpha'}^2\overline{Z}_t\right|^2 d\alpha' + \int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}D_{\alpha'}^2\overline{Z}_t\right)\right|^2 d\alpha'$$

Analogous to what we did in the energy estimate of E_σ, we simplify the calculations by the use of the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a universal non-negative polynomial P with $|a(t) - b(t)| \leq P(E_{high}(t))$. Observe that $\approx$ is an equivalence relation. With this notation, proving Theorem 5.1.1 is equivalent to showing $\frac{dE_{high}(t)}{dt} \approx 0$.

Now we know from Theorem 3.4.1 that

$$\frac{dE_\sigma(t)}{dt} \leq P(E_\sigma(t))$$

and hence this is true for $\sigma = 0$ with the same polynomial P. Hence we have

$$\frac{d(E_\sigma|_{\sigma=0})(t)}{dt} \leq P((E_\sigma|_{\sigma=0})(t)) \leq P(E_{high}(t))$$

Hence we only need to control the time derivative of $E_{high} - E_\sigma|_{\sigma=0}$. Hence

$$\frac{dE_{high}(t)}{dt} \approx \frac{d}{dt}\left\{\int \left|D_t D_{\alpha'}^2 \overline{Z}_t\right|^2 d\alpha' + \int \left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t\right)\right|^2 d\alpha'\right\}$$

The right hand side is the time derivative of

$$\int |D_t f|^2 d\alpha' + \int \left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|} f\right)\right|^2 d\alpha'$$

where $f = D_{\alpha'}^2 \overline{Z}_t$ and we have $\mathbb{P}_H f = f$. We have already computed the time derivative of such functionals in the §4.2.3. Hence using that computation we get

$$\frac{dE_{high}(t)}{dt} \approx 2\mathrm{Re} \int \left(D_t^2 f + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} f\right)(D_t \bar{f})\, d\alpha'$$

As $D_t D_{\alpha'}^2 \overline{Z}_t \in L^2$ we only need to show that the other term in in L^2. Now the equation for $D_{\alpha'}^2 \overline{Z}_t$ from (5.1) implies

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right) D_{\alpha'}^2 \overline{Z}_t = -i\overline{\omega}^3 |D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} J_1\right) + R_3$$

As we have shown $|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} J_1\right) \in L^2$ and $R_3 \in L^2$ we have shown that this term is in L^2 and the proof of Theorem 5.1.1 is complete.

5.1.5 Equivalence of E_{high} and $\mathcal{E}_{high}$

We now give a simpler description of the energy E_{high}. Define

$$\mathcal{E}_{high} = \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{1}{Z_{,\alpha'}^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

Proposition 5.1.2. *There exists universal polynomials P_1, P_2 with non-negative coefficients so that for smooth solutions to the water wave equation with no surface tension we have*

$$E_{high} \leq P_1(\mathcal{E}_{high}) \quad \textit{and } \mathcal{E}_{high} \leq P_2(E_{high})$$

Proof. Let $E_{high} < \infty$. We have already pretty much controlled all the terms of $\mathcal{E}_{high}$. The term which is not controlled is $\dfrac{1}{Z_{,\alpha'}^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'}$ which can be easily controlled in $\dot{H}^{\frac{1}{2}}$ as we have that $\omega \in \mathcal{W}$ and we have already shown that $\dfrac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{C}$.

Now we assume that $\mathcal{E}_{high} < \infty$. We use Proposition A.0.10 with $f = D_{\alpha'}\dfrac{1}{Z_{,\alpha'}}$ and $w = \dfrac{1}{Z_{,\alpha'}}$ to obtain

$$\begin{aligned}
&\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^4 \\
&\lesssim \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^2\left\|D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2^4
\end{aligned}$$

Now use the inequality $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ on the first term to obtain $D_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^\infty \cap \dot{H}^{\frac{1}{2}}$. Hence by Proposition 3.4.3 we know that $E_\sigma|_{\sigma=0}$ is controlled. Hence we only need to show $D_t D_{\alpha'}^2\overline{Z}_t \in L^2$ and $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}D_{\alpha'}^2\overline{Z}_t \in \dot{H}^{\frac{1}{2}}$. Now following the proof of $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}}$ in §5.1.3 we see that $|D_{\alpha'}|^2\overline{Z}_{tt} \in L^2$. Following the proof of $|D_{\alpha'}|^2\overline{Z}_{tt} \in L^2$, we see that $D_{\alpha'}^2\overline{Z}_{tt} \in L^2$ and $D_t D_{\alpha'}^2\overline{Z}_t \in L^2$.

Now as $\omega \in \mathcal{W}$ and $\dfrac{1}{Z_{,\alpha'}^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{C}$, we see that $\dfrac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{C}$. Hence following the proof of $\dfrac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{C}$, we see that $\dfrac{1}{|Z_{,\alpha'}|}D_{\alpha'}^2\overline{Z}_t \in \mathcal{C}$. Now as $\sqrt{A_1} \in \mathcal{W}$, we easily obtain

$\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|} D_{\alpha'}^2 \overline{Z}_t \in \mathcal{C}$ and hence E_{high} is controlled. □

5.2 Auxiliary energy $\mathcal{E}_{\lambda,aux}$

In this section we again consider a solution to the water wave equation with zero surface tension and show that the energy $E_{\lambda,aux}$ defined at the start of §5, is controlled as long as $\mathcal{E}_{high}$ is finite. This energy depends on the chosen constant $\lambda > 0$. When we put $\lambda = \sigma$, we will denote this energy by $E_{\sigma,aux}$ and this energy will be used in the next section where we prove the estimate for convergence. The reason for the necessity of this energy is as follows:

Suppose $(Z, Z_t)_a$ is a solution to the water wave equation with zero surface tension and we have another solution $(Z, Z_t)_b$ is a solution to the water wave equation with surface tension σ. The subscript denotes which solution we are talking about. Then while proving the estimate for the difference of the solutions in the proof of Theorem 5.0.1, we will need to control terms which essentially looks like (see §5.3.2)

$$2\text{Re}\int \left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_{tt}\right)_a \left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_b d\alpha'$$

Now $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_b \in L^2$ as it will follow directly from the energy (The energy E_σ controls terms like this). Note however that the energy E_σ does not give us control of anything higher order than this. Hence we will therefore need to control $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_{tt}\right)_a \in L^2$ from a different source. These types of terms are controlled by $E_{\sigma,aux}$.

5.2.1 Equation for $\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t$

Let us recall the equation of $D_{\alpha'}^2\overline{Z}_t$ from (5.1)

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)D_{\alpha'}^2\overline{Z}_t = -i\overline{\omega}^3|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) + R_3$$

with R_3 as given in (5.2) along with the identities $J_1 = D_tA_1 + A_1\big(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t\big)$ and

$\overline{Z}_{tt} - i = -i\dfrac{A_1}{Z_{,\alpha'}}$. Applying $\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}$ to the above equation we obtain

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t = -i\overline{\omega}^3\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) + R_4$$

where

$$\begin{aligned} R_4 = &-3i\overline{\omega}^2\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{\omega}\right)|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}R_3 \\ &+ \left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\right]D_{\alpha'}^2\overline{Z}_t \end{aligned}$$

Let us try to simplify the terms above

a) $D_t\dfrac{1}{Z_{,\alpha'}^{1/2}} = \dfrac{-1}{2Z_{,\alpha'}^{3/2}}D_tZ_{,\alpha'} = \dfrac{-1}{2Z_{,\alpha'}^{1/2}}(D_{\alpha'}Z_t - b_{\alpha'})$

b) $\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}^{1/2}} = \dfrac{1}{2}Z_{,\alpha'}^{1/2}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} = \left(\dfrac{1}{2}Z_{,\alpha'}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right)\dfrac{1}{Z_{,\alpha'}^{1/2}}$

c) $\left[\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}, D_t\right] = \dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}[\partial_{\alpha'}, D_t] + \left[\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}, D_t\right]\partial_{\alpha'} = \left(\dfrac{b_{\alpha'}}{2} + \dfrac{D_{\alpha'}Z_t}{2}\right)\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}$

d) $\left[\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}, D_t^2\right]$

$$\begin{aligned} &= D_t\left[\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}, D_t\right] + \left[\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}, D_t\right]D_t \\ &= \left\{\frac{D_tb_{\alpha'}}{2} + \frac{D_tD_{\alpha'}Z_t}{2} - \left(\frac{b_{\alpha'}}{2} + \frac{D_{\alpha'}Z_t}{2}\right)^2\right\}\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} + \left(\frac{b_{\alpha'}}{2} + \frac{D_{\alpha'}Z_t}{2}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_t \end{aligned}$$

e) $\left[\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}, i\dfrac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right]$

$$= \left\{2iA_1\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right) + \frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 - \frac{iA_1}{2}\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}$$

Combining the above formulae we have

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t = -i\overline{\omega}^3\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) + R_4 \qquad (5.3)$$

where

$$
\begin{aligned}
R_4 = &-\left\{\frac{D_t b_{\alpha'}}{2} + \frac{D_t D_{\alpha'} Z_t}{2} - \left(\frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2}\right)^2\right\}\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} R_3 \\
&-\left\{2iA_1\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right) + \frac{i}{|Z_{,\alpha'}|^2}\partial_{\alpha'} A_1 - \frac{iA_1}{2}\left(\bar{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \bar{Z}_t \\
&- 3i\bar{\omega}^2\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\bar{\omega}\right)|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} J_1\right) - \left(\frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_t D_{\alpha'}^2 \bar{Z}_t
\end{aligned} \tag{5.4}
$$

and R_3 is as defined in (5.2).

5.2.2 Statement of the energy estimate for $E_{\lambda,aux}$

The energy $E_{\lambda,aux}$ defined at the start of §5 depends on the chosen constant $\lambda > 0$. Recall that it is given by

$$
\begin{aligned}
E_{\lambda,aux} = &\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2\bar{Z}_t\right\|_2^2 \\
&+ \left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2\bar{Z}_t\right)\right\|_2^2 + \left\|\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2\bar{Z}_t\right)\right\|_{\dot{H}^{\frac{1}{2}}}^2
\end{aligned}
$$

Theorem 5.2.1. *Let $T > 0$ and let (Z, Z_t) be a smooth solution to the gravity water wave equation with zero surface tension in the time interval $[0, T)$ with $\mathcal{E}_{high}(t), E_{\lambda,aux}(t) < \infty$ for all $t \in [0, T)$. Then there exists a polynomial P with universal non-negative coefficients such that for all $t \in [0, T)$ we have*

$$\frac{dE_{\lambda,aux}(t)}{dt} \leq P(\mathcal{E}_{high}(t))E_{\lambda,aux}(t)$$

Remark 5.2.2. As in the case of energy E_σ, the energy $E_{\lambda,aux}$ contains a term which is the L^∞ norm of a function and hence we replace the time derivative of this term by the upper Dini derivative.

5.2.3 Quantities controlled by the energy $E_{\lambda,aux}$

In this section whenever we write $f \in L^2_{\lambda^\alpha}$, what we mean is that there exists a universal polynomial P with nonnegative coefficients such that $\|f\|_2 \leq (E_{\lambda,aux})^\alpha P(\mathcal{E}_{high})$. Similar definitions for $f \in \dot{H}^{\frac{1}{2}}_{\lambda^\alpha}$ and $f \in L^\infty_{\lambda^\alpha}$. We define the spaces $\mathcal{C}_{\lambda^\alpha}$ and $\mathcal{W}_{\lambda^\alpha}$ as follows

1. If $w \in L^\infty_{\lambda^\alpha}$ and $|D_{\alpha'}|w \in L^2_{\lambda^\alpha}$, then we say $f \in \mathcal{W}_{\lambda^\alpha}$. Define

$$\|w\|_{\mathcal{W}_{\lambda^\alpha}} = \|w\|_{\mathcal{W}} = \|w\|_\infty + \||D_{\alpha'}|w\|_2$$

2. If $f \in \dot{H}^{\frac{1}{2}}_{\lambda^\alpha}$ and $f|Z_{,\alpha'}| \in L^2_{\lambda^\alpha}$, then we say $f \in \mathcal{C}_{\lambda^\alpha}$. Define

$$\|f\|_{\mathcal{C}_{\lambda^\alpha}} = \|f\|_{\mathcal{C}} = \|f\|_{\dot{H}^{\frac{1}{2}}} + \left(1 + \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\right)\|f|Z_{,\alpha'}|\|_2$$

Analogous to Lemma 2.4.5 we have the following lemma

Lemma 5.2.3. *Let $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 = \alpha_3$. Then the following properties hold for the spaces $\mathcal{W}_{\lambda^\alpha}$ and $\mathcal{C}_{\lambda^\alpha}$*

1. *If $w_1 \in \mathcal{W}_{\lambda^{\alpha_1}}$, $w_2 \in \mathcal{W}_{\lambda^{\alpha_2}}$, then $w_1 w_2 \in \mathcal{W}_{\lambda^{\alpha_3}}$. Moreover we have the estimate $\|w_1 w_2\|_{\mathcal{W}_{\lambda^{\alpha_3}}} \leq \|w_1\|_{\mathcal{W}_{\lambda^{\alpha_1}}}\|w_2\|_{\mathcal{W}_{\lambda^{\alpha_2}}}$*
2. *If $f \in \mathcal{C}_{\lambda^{\alpha_1}}$ and $w \in \mathcal{W}_{\lambda^{\alpha_2}}$, then $fw \in \mathcal{C}_{\lambda^{\alpha_3}}$. Moreover $\|fw\|_{\mathcal{C}_{\lambda^{\alpha_3}}} \lesssim \|f\|_{\mathcal{C}_{\lambda^{\alpha_1}}}\|w\|_{\mathcal{W}_{\lambda^{\alpha_2}}}$*
3. *If $f \in \mathcal{C}_{\lambda^{\alpha_1}}$, $g \in \mathcal{C}_{\lambda^{\alpha_2}}$, then $fg|Z_{,\alpha'}| \in L^2_{\lambda^{\alpha_3}}$. Moreover $\|fg|Z_{,\alpha'}|\|_2 \lesssim \|f\|_{\mathcal{C}_{\lambda^{\alpha_1}}}\|g\|_{\mathcal{C}_{\lambda^{\alpha_2}}}$*

When we write $f \in L^2$ we mean $f \in L^2_{\lambda^\alpha}$ with $\alpha = 0$. Similar notation for $\dot{H}^{\frac{1}{2}}, L^\infty, \mathcal{C}$ and $\mathcal{W}$. This notation is now consistent with the notation used in §5.1.3. Let us now control the important terms controlled by the energy $E_{\lambda,aux}$.

1) $\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty_{\sqrt{\lambda}}$, $\lambda^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z^{1/2}_{,\alpha'}} \in L^\infty_{\sqrt{\lambda}}$ and $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega \in L^\infty_{\sqrt{\lambda}}$

Proof: From the energy we already know that $\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in L^\infty_{\sqrt{\lambda}}$. Hence we easily have $\lambda^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z^{1/2}_{,\alpha'}} \in L^\infty_{\sqrt{\lambda}}$. Recall from (2.9) that

$$\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)$$

Hence we obtain $\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in L^\infty_{\sqrt{\lambda}}$ and $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega \in L^\infty_{\sqrt{\lambda}}$

2) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{|Z_{,\alpha'}|} \in L^2_{\sqrt{\lambda}}$ and $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega \in L^2_{\sqrt{\lambda}}$ and hence we have that $\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{W}_{\sqrt{\lambda}}$, $\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{W}_{\sqrt{\lambda}}$ and $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega \in \mathcal{W}_{\sqrt{\lambda}}$

Proof: $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2_{\sqrt{\lambda}}$ as it is part of the energy. Recall from (2.9) that

$$\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)$$

Hence taking derivatives we obtain

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2$$

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\left\{\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \||D_{\alpha'}|\omega\|_2\right\} + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2$$

We also see that

$$\left\||D_{\alpha'}|\left(\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2$$

The rest are proven similarly.

3) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2 Z_t \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in L^2_{\sqrt{\lambda}}$ and similarly we have $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}Z_t \in L^2_{\sqrt{\lambda}}$ and $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_t \in L^2_{\sqrt{\lambda}}$

Proof: We already have $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in L^2_{\sqrt{\lambda}}$ as it is part of the energy. We now have

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t = \left(\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)(|D_{\alpha'}|D_{\alpha'}\overline{Z}_t) + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t$$

and hence

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t\right\|_2 \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\||D_{\alpha'}|D_{\alpha'}\overline{Z}_t\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_2$$

Now we have

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t = \left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_t + 2\left(\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right) + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}$$

From this we obtain

$$\begin{aligned}&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2 \\ &\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\||D_{\alpha'}|\overline{Z}_t\right\|_\infty + \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \\ &\quad + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t\right\|_2\end{aligned}$$

Hence we have $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in L^2_{\sqrt{\lambda}}$. By taking conjugation and retracing the steps backwards we easily obtain the other estimates $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}^2\overline{Z}_t \in L^2_{\sqrt{\lambda}}$, $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\overline{D}_{\alpha'}\overline{Z}_t \in L^2_{\sqrt{\lambda}}$ and $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_t \in L^2_{\sqrt{\lambda}}$.

4) $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$, $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$, $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$ and similarly $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|Z_t \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$

Proof: We first see that

$$\left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2 \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2$$

Now using Proposition A.0.10 with $f = \dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}$ and $w = \dfrac{1}{|Z_{,\alpha'}|}$ we see that

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2^2$$

As $|Z_{,\alpha'}|f \in L^2_{\sqrt{\lambda}}$ and $w' \in L^2$ we have $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$. Now we have

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t = \left(\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)|D_{\alpha'}|\overline{Z}_t + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}$$

Hence

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}} \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}_{\sqrt{\lambda}}}\left\||D_{\alpha'}|\overline{Z}_t\right\|_{\mathcal{W}\cap\mathcal{C}} + \|\overline{\omega}\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}}$$

The estimates $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|Z_t \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$ are proven similarly.

5) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1 \in \mathcal{W}_{\sqrt{\lambda}}$

Proof: The proofs for $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1 \in L^\infty$ and $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2A_1 \in L^2$ are exactly the same as for the energy E_σ in §4.1. Hence we have the estimates

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1\right\|_\infty \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty \|\overline{Z}_{t,\alpha'}\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \|\overline{Z}_{t,\alpha'}\|_2$$

$$\begin{aligned}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2A_1\right\|_2 &\lesssim \left\{\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\right\}\|A_1\|_\infty \\ &\quad + \|Z_{t,\alpha'}\|_2\left\{\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_\infty + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\right\}\end{aligned}$$

From this we get

$$\left\||D_{\alpha'}|\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1\right)\right\|_2 \lesssim \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1\right\|_\infty + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2A_1\right\|_2$$

6) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^3A_1 \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2_{\sqrt{\lambda}}$ and hence $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2A_1 \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$

Proof: We will first show that $(\mathbb{I}-\mathbb{H})\left\{\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^3A_1\right\} \in L^2_{\sqrt{\lambda}}$. For the water wave equation with no surface tension we have from (3.5) that $A_1 = iZ_{,\alpha'}F_t\circ Z + Z_{,\alpha'} + iZ_t\overline{Z}_{t,\alpha'}$. Hence

$$\begin{aligned}&(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^3A_1\right\} \\ &= i(\mathbb{I}-\mathbb{H})\left\{\left(\frac{Z_t}{Z_{,\alpha'}}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^3\overline{Z}_{t,\alpha'}\right\} \\ &\quad + i(\mathbb{I}-\mathbb{H})\left\{\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2Z_{t,\alpha'}\right)(D_{\alpha'}\overline{Z}_t) + 3\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}Z_{t,\alpha'}\right)\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right. \\ &\qquad\qquad \left. + 3(D_{\alpha'}Z_t)\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right)\right\}\end{aligned}$$

Now we see that

$$i(\mathbb{I}-\mathbb{H})\left\{\left(\frac{Z_t}{Z_{,\alpha'}}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^3\overline{Z}_{t,\alpha'}\right\} = -\frac{5i}{2}[Z_t,\mathbb{H}]\left\{\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right)\right\} + i\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right)$$

hence we have the estimate

$$\begin{aligned}&\left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^3 A_1\right\}\right\|_2 \\ &\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2\left\||D_{\alpha'}|\overline{Z}_t\right\|_\infty + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2 \\ &\quad + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2\left\{\|Z_{t,\alpha'}\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_\infty\right\}\end{aligned}$$

Now lets come back to proving the main estimate $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2_{\sqrt{\lambda}}$. Now as A_1 is real valued we see that

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) = \lambda^{\frac{1}{2}}\mathrm{Re}\left\{\frac{\omega^{\frac{7}{2}}\overline{\omega}^{\frac{7}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\}$$

Hence it is enough to show that $\dfrac{\lambda^{\frac{1}{2}}\overline{\omega}^{\frac{7}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2\left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2_{\sqrt{\lambda}}$. We have

$$
\begin{aligned}
&\frac{\lambda^{\frac{1}{2}}\overline{\omega}^{\frac{7}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}(\mathbb{I}-\mathbb{H})\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \\
&= -\left[\frac{\lambda^{\frac{1}{2}}\overline{\omega}^{\frac{7}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}},\mathbb{H}\right]\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) + (\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}\overline{\omega}^{\frac{7}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\} \\
&= (\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^3 A_1\right\} - \left[\frac{\lambda^{\frac{1}{2}}\overline{\omega}^{\frac{7}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}},\mathbb{H}\right]\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \\
&\quad + (\mathbb{I}-\mathbb{H})\left\{\frac{2\lambda^{\frac{1}{2}}\overline{\omega}^2}{Z_{,\alpha'}^{\frac{3}{2}}}\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)^2\partial_{\alpha'}A_1 + \frac{2\lambda^{\frac{1}{2}}\overline{\omega}^2}{Z_{,\alpha'}^{\frac{3}{2}}}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right)\partial_{\alpha'}A_1\right. \\
&\qquad\qquad \left. + \frac{4\lambda^{\frac{1}{2}}\overline{\omega}^2}{Z_{,\alpha'}^{\frac{3}{2}}}\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right)\partial_{\alpha'}^2 A_1\right\}
\end{aligned}
$$

Hence we have the estimate

$$
\begin{aligned}
&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\|_2 \\
&\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty + \left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^3 A_1\right\}\right\|_2 \\
&\quad + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty\left\{\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty \||D_{\alpha'}|\omega\|_2\right\} \\
&\quad + \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\{\left(\||D_{\alpha'}|\omega\|_2 + \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\right)\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty\right. \\
&\qquad\qquad \left. + \left\|\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2 A_1\right\|_2\right\}
\end{aligned}
$$

From this we easily have the estimate

$$\begin{aligned}&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^3 A_1\right\|_2\\ &\lesssim \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty\left\{\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right\|_2\right\}\\ &\quad+\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2 A_1\right\|_2+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right)\right\|_2\end{aligned}$$

and similarly

$$\left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^2 A_1\right)\right\|_2\lesssim\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2 A_1\right\|_2+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^3 A_1\right\|_2$$

Hence now using Proposition A.0.10 with $f=\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 A_1$ and $w=\dfrac{1}{|Z_{,\alpha'}|}$ we obtain

$$\begin{aligned}&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 A_1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2\\ &\lesssim\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right\|_2\left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^2 A_1\right)\right\|_2+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right\|_2^2\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2^2\end{aligned}$$

7) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\in L^\infty_{\sqrt{\lambda}}$

Proof: The proof of this estimate the same as the proof of $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\in L^\infty$ for the energy E_σ in §4.1. Hence we have the estimate

$$\begin{aligned}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right\|_\infty&\lesssim\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\bar{Z}_{t,\alpha'}\right\|_2+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\bar{Z}_{t,\alpha'}\right\|_2\\ &\quad+\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\bar{Z}_{t,\alpha'}\right\|_2+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t\right\|_\infty\end{aligned}$$

8) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'} \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|b_{\alpha'} \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}b_{\alpha'} \in L^2_{\sqrt{\lambda}}$ and hence we also have $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'} \in \mathcal{W}_{\sqrt{\lambda}}$

Proof: The proof of this estimate is the same as the proof of $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'} \in L^2$ for the energy E_σ in §4.1. Hence we have the estimate

$$\begin{aligned}\left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\}\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 D_{\alpha'}Z_t\right\|_2 + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}Z_{t,\alpha'}\right\|_2\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \\ + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\{\|D_{\alpha'}Z_t\|_\infty + \left\|\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right\|_\infty \right. \\ \left. + \|Z_{t,\alpha'}\|_2\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\right\}\end{aligned}$$

and using this we have

$$\begin{aligned}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\|_2 \lesssim \|b_{\alpha'}\|_\infty\left\{\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\right\} \\ + \left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2 b_{\alpha'}\right\}\right\|_2\end{aligned}$$

The estimates for the other terms are also shown similarly.

9) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\bar{Z}_{tt,\alpha'} \in L^2_{\sqrt{\lambda}}$

Proof: We know that $\bar{Z}_{tt} - i = -i\dfrac{A_1}{Z_{,\alpha'}}$. Hence we have

$$\begin{aligned}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\bar{Z}_{tt,\alpha'}\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right\|_2 + \||D_{\alpha'}|A_1\|_2\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \\ + \|A_1\|_\infty\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\end{aligned}$$

10) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\bar{Z}_{tt,\alpha'} \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2\bar{Z}_{tt} \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2 Z_{tt} \in L^2_{\sqrt{\lambda}}$ and similarly we

also have $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_t D_{\alpha'}^2 \overline{Z}_t \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} |D_{\alpha'}| D_t D_{\alpha'} Z_t \in L^2_{\sqrt{\lambda}}$

Proof: From the energy we have $D_t\left(\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right) \in L^2_{\sqrt{\lambda}}$. Hence we have

$$\begin{aligned}
&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_t D_{\alpha'}^2 \overline{Z}_t\right\|_2 \\
&\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_2 \left\{\|b_{\alpha'}\|_\infty + \|D_{\alpha'} Z_t\|_\infty\right\} + \left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right)\right\|_2
\end{aligned}$$

Using the commutator $[D_{\alpha'}, D_t] = (D_{\alpha'} Z_t) D_{\alpha'}$ we see that

$$D_t D_{\alpha'}^2 \overline{Z}_t = -2(D_{\alpha'} Z_t) D_{\alpha'}^2 \overline{Z}_t - (D_{\alpha'} \overline{Z}_t) D_{\alpha'}^2 Z_t + D_{\alpha'}^2 \overline{Z}_{tt}$$

Hence we have

$$\begin{aligned}
&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_{tt}\right\|_2 \\
&\lesssim \left\{\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'} Z_t\right\|_\infty + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'} \overline{Z}_t\right\|_\infty\right\}\left(\left\|D_{\alpha'}^2 Z_t\right\|_2 + \left\|D_{\alpha'}^2 \overline{Z}_t\right\|_2\right) \\
&\quad + \left\||D_{\alpha'}| \overline{Z}_t\right\|_\infty \left\{\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'}^2 Z_t\right\|_2\right\} + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_t D_{\alpha'}^2 \overline{Z}_t\right\|_2
\end{aligned}$$

Now we have the identity

$$D_{\alpha'}^2 \overline{Z}_{tt} = \left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{tt,\alpha'} + \frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'} \overline{Z}_{tt,\alpha'}$$

Hence we have

$$\begin{aligned}
\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 \overline{Z}_{tt,\alpha'}\right\|_2 &\lesssim \left\||D_{\alpha'}| \overline{Z}_{tt}\right\|_\infty \left\{\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\right\} \\
&\quad + \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} \overline{Z}_{tt}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_{tt}\right\|_2
\end{aligned}$$

We can prove $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2 Z_{tt} \in L^2_{\sqrt{\lambda}}$ similarly. Now observe that

$$|D_{\alpha'}|D_t D_{\alpha'} Z_t = |D_{\alpha'}|\big(D_{\alpha'}Z_{tt} - (D_{\alpha'}Z_t)^2\big) = \omega D_{\alpha'}^2 Z_{tt} - 2(|D_{\alpha'}|Z_t)D_{\alpha'}^2 Z_t$$

Hence we have

$$\begin{aligned}
&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_t D_{\alpha'}Z_t\right\|_2 \\
&\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty \left\|D_{\alpha'}^2 Z_{tt}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2 Z_{tt}\right\|_2 \\
&\quad + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|Z_t\right\|_\infty \left\|D_{\alpha'}^2 Z_t\right\|_2 + \||D_{\alpha'}|Z_t\|_\infty \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2 Z_t\right\|_2
\end{aligned}$$

11) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'} \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\overline{Z}_{t,\alpha'} \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_{tt} \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$ and $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_{tt} \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$

Proof: Let $f = \dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}$ and $w = \dfrac{1}{|Z_{,\alpha'}|}$. Then we see that $\dfrac{f}{w} \in L^2_{\sqrt{\lambda}}$. Now we have

$$\left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right)\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{tt,\alpha'}\right\|_2 + \left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_\infty \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right\|_2$$

As $w' \in L^2$, using Proposition A.0.10 we obtain

$$\begin{aligned}
\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} &\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right\|_2 \left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right)\right\|_2 \\
&\quad + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right\|_2^2 \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2^2
\end{aligned}$$

Hence we have $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'} \in \mathcal{W}_{\sqrt{\lambda}} \cap \mathcal{C}_{\sqrt{\lambda}}$. Now

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\overline{Z}_{t,\alpha'} = -\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\big(b_{\alpha'}\overline{Z}_{t,\alpha'}\big) + \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}$$

From this we obtain

$$
\begin{aligned}
&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}} \\
&\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right\|_{\mathcal{W}_{\sqrt{\lambda}}}\left\||D_{\alpha'}|\overline{Z}_t\right\|_{\mathcal{W}\cap\mathcal{C}} + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right\|_{\mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}} \\
&\quad + \|b_{\alpha'}\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}}
\end{aligned}
$$

We also have

$$
\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_{tt}\right\|_{\infty} \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}_{\sqrt{\lambda}}}\left\||D_{\alpha'}|\overline{Z}_{tt}\right\|_{\mathcal{W}\cap\mathcal{C}} + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right\|_{\infty}
$$

We prove $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_{tt} \in L^{\infty}_{\sqrt{\lambda}}$ similarly.

12) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\dfrac{1}{Z_{,\alpha'}} \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\dfrac{1}{|Z_{,\alpha'}|} \in L^2_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^3\omega \in L^2_{\sqrt{\lambda}}$

Proof: We know that $\overline{Z}_{tt} - i = -i\dfrac{A_1}{Z_{,\alpha'}}$ and as $A_1 \geq 1$ we have

$$
\begin{aligned}
&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2 \\
&\lesssim \left\|\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2A_1\right\|_2\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\infty} + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\infty}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 \\
&\quad + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\overline{Z}_{tt}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^3A_1\right\|_2
\end{aligned}
$$

Recall from (2.9) that

$$
\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)
$$

Hence by taking derivatives we obtain

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|}\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega\right\|_2\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right\|_\infty\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2$$

and similarly

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^3\omega\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega\right\|_2\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right\|_\infty\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2$$

13) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{|Z_{,\alpha'}|} \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$ and $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\omega \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$

Proof: We first note that

$$\left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \lesssim \left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2$$

Now taking $f = \dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}}$ and $w = \dfrac{1}{|Z_{,\alpha'}|}$ in Proposition A.0.10 we obtain

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2^2$$

As $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in L^2_{\sqrt{\lambda}}$ this shows that $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in \mathcal{W}_{\sqrt{\lambda}}\cap\mathcal{C}_{\sqrt{\lambda}}$. We can prove the other estimates similarly by taking derivatives in

$$\mathrm{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \mathrm{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)$$

14) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t \in \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in \mathcal{C}_{\sqrt{\lambda}}$ and similarly $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 D_{\alpha'}Z_t \in \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_t \in \mathcal{C}_{\sqrt{\lambda}}$

Proof: We first see that $\dfrac{\lambda^{\frac{1}{2}}\sqrt{A_1}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in L^2_{\sqrt{\lambda}}$ as $A_1 \in L^\infty$ and $\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in L^2_{\sqrt{\lambda}}$ as it is part of the energy. Hence $\dfrac{\lambda^{\frac{1}{2}}\sqrt{A_1}}{|Z_{,\alpha'}|Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in \mathcal{C}_{\sqrt{\lambda}}$. From this we obtain

$$\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_{\mathcal{C}_{\sqrt{\lambda}}} \lesssim \left\|\frac{\lambda^{\frac{1}{2}}\sqrt{A_1}}{|Z_{,\alpha'}|Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_{\mathcal{C}_{\sqrt{\lambda}}}\left\|\frac{1}{\sqrt{A_1}}\right\|_{\mathcal{W}}$$

As $\omega^{\frac{1}{2}} \in \mathcal{W}$ we see that $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in \mathcal{C}_{\sqrt{\lambda}}$. Now we have

$$\begin{aligned}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}_{\sqrt{\lambda}}} &\lesssim \|\omega\|_{\mathcal{W}}\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}_{\sqrt{\lambda}}}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}} \\ &\quad + \|\omega\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_{\mathcal{C}_{\sqrt{\lambda}}}\end{aligned}$$

Now observe that

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t = \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}\left\{\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\overline{Z}_{t,\alpha'} + \frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\}$$

Hence we have

$$\begin{aligned}&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}_{\sqrt{\lambda}}} \\ &\lesssim \|\omega\|_{\mathcal{W}}\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}_{\sqrt{\lambda}}}\left\|\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}} + \|\omega\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 D_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}_{\sqrt{\lambda}}} \\ &\quad + \|\omega\|_{\mathcal{W}}\||D_{\alpha'}|\overline{Z}_t\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{C}_{\sqrt{\lambda}}} + \|\omega\|_{\mathcal{W}}\left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_{\mathcal{C}_{\sqrt{\lambda}}}\end{aligned}$$

Now the estimates $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2 D_{\alpha'}Z_t \in \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_{\alpha'}Z_t \in \mathcal{C}_{\sqrt{\lambda}}$ are proven similarly.

15) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta \in L^2_{\sqrt{\lambda}}$

Proof: Using formula (3.8) we see that

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta = (\mathbb{I}+\mathbb{H})\text{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\} + i\text{Im}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\}$$

Now using the formula (2.9) we see that

$$\text{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\} = -i\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}(D_{\alpha'}\omega)$$

Hence we have

$$\left\|\text{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\}\right\|_2 \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty \||D_{\alpha'}|\omega\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega\right\|_2$$

We also have

$$(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\} = \left[\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}},\mathbb{H}\right]\partial_{\alpha'}\Theta$$

and hence we have

$$\left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\}\right\|_2 \lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty \|\Theta\|_2$$

16) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \in \mathcal{C}_{\sqrt{\lambda}}$, $\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}\Theta \in \mathcal{C}_{\sqrt{\lambda}}$

Proof: Using formula (3.8) we see that

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta = (\mathbb{I}+\mathbb{H})\text{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\} + i\text{Im}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\}$$

Using the formula (2.9) we see that

$$\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\} = -i\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}(D_{\alpha'}\omega)$$

Hence we have

$$\begin{aligned}&\left\|\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\}\right\|_{\mathcal{C}_{\sqrt{\lambda}}}\\ &\lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_{\mathcal{W}_{\sqrt{\lambda}}}\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right\|_{\mathcal{C}} + \|\overline{\omega}\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\omega\right\|_{\mathcal{C}_{\sqrt{\lambda}}}\end{aligned}$$

Now observe that

$$(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\} = \lambda^{\frac{1}{2}}\left[\frac{\omega^{\frac{1}{2}}}{|Z_{,\alpha'}|},\mathbb{H}\right]\frac{1}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\Theta$$

Hence we have

$$\left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\}\right\|_{\mathcal{C}_{\sqrt{\lambda}}} \lesssim \left(\||D_{\alpha'}|\omega\|_2 + \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\right)\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\|_2$$

Hence $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta \in \mathcal{C}_{\sqrt{\lambda}}$. Now as $\overline{\omega}^{\frac{3}{2}} \in \mathcal{W}$ we obtain the other estimate easily by multiplying.

17) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta \in L^2_{\sqrt{\lambda}}$

Proof: Using formula (3.8) we see that

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta = (\mathbb{I}+\mathbb{H})\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\} + i\mathrm{Im}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\}$$

We control the term individually.

(a) Using the formula (3.13) we see that

$$\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\} = -\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\mathrm{Im}\left\{|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t + i(\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\right\}$$

Hence we have

$$\left\|\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\}\right\|_2 \lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\|_2 \left\|\overline{D}_{\alpha'}\overline{Z}_t\right\|_\infty + \|\Theta\|_2\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_\infty$$

(b) We note that

$$(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\} = \left[\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}},\mathbb{H}\right]\partial_{\alpha'}D_t\Theta - \omega^{\frac{1}{2}}\left[\frac{\lambda^{\frac{1}{2}}\overline{\omega}^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}},\mathbb{H}\right]\partial_{\alpha'}D_t\Theta + \omega^{\frac{1}{2}}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_t\Theta\right\}$$

and as $\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\Theta$ is holomorphic we have

$$\begin{aligned}(\mathbb{I}-\mathbb{H})&\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_t\Theta\right\}\\ &= (\mathbb{I}-\mathbb{H})\left\{\left(\frac{b_{\alpha'}}{2}+\frac{D_{\alpha'}Z_t}{2}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\Theta\right\} + [b,\mathbb{H}]\partial_{\alpha'}\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\Theta\right\}\end{aligned}$$

From this we obtain

$$\begin{aligned}&\left\|(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\}\right\|_2\\ &\lesssim (\|b_{\alpha'}\|_\infty + \|D_{\alpha'}Z_t\|_\infty)\left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\Theta\right\|_2\\ &\quad + \left(\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_\infty + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_2\right)\|D_t\Theta\|_2\end{aligned}$$

18) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta \in \mathcal{C}_{\sqrt{\lambda}}$

Proof: Note that we only need to prove the $\dot{H}^{\frac{1}{2}}$ estimate. Using formula (3.8) we see that

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta = (\mathbb{I}+\mathbb{H})\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta\right\} + i\mathrm{Im}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta\right\}$$

We control the term individually.

(a) Using the formula (3.13) we see that

$$\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta\right\} = -\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\mathrm{Im}\left\{|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t + i(\mathrm{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\right\}$$

Hence we have

$$\begin{aligned}
&\left\|\mathrm{Re}\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta\right\}\right\|_{\mathcal{C}_{\sqrt{\lambda}}}\\
&\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}_{\sqrt{\lambda}}} + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}_{\sqrt{\lambda}}}\left\|\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{W}}\\
&\quad + \left\|\frac{\Theta}{|Z_{,\alpha'}|}\right\|_{\mathcal{W}}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\mathcal{C}_{\sqrt{\lambda}}}
\end{aligned}$$

(b) We note that

$$\begin{aligned}
&(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta\right\}\\
&= \left[\frac{1}{|Z_{,\alpha'}|},\mathbb{H}\right]\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta - \omega^{\frac{3}{2}}\frac{\overline{\omega}^{\frac{3}{2}}}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\}
\end{aligned}$$

As we have

$$\left\|\left[\frac{1}{|Z_{,\alpha'}|},\mathbb{H}\right]\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\|_2$$

We only need to show the second term is in $\dot{H}^{\frac{1}{2}}$. Now as $\omega^{\frac{3}{2}} \in \mathcal{W}$, it is enough to show that $\frac{\overline{\omega}^{\frac{3}{2}}}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\} \in \mathcal{C}_{\sqrt{\lambda}}$. As $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta \in L^2_{\sqrt{\lambda}}$ we only need to show the $\dot{H}^{\frac{1}{2}}$ estimate. Now

$$\begin{aligned} &\frac{\overline{\omega}^{\frac{3}{2}}}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\} \\ &= -\left[\frac{\overline{\omega}^{\frac{3}{2}}}{|Z_{,\alpha'}|},\mathbb{H}\right]\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta + (\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta\right\} \end{aligned}$$

Now as $\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}\Theta$ is holomorphic we have

$$\begin{aligned} &(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}D_t\Theta\right\} \\ &= (\mathbb{I}-\mathbb{H})\left\{\left(-\frac{b_{\alpha'}}{2}+\frac{3D_{\alpha'}Z_t}{2}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\} + [b,\mathbb{H}]\partial_{\alpha'}\left\{\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\} \end{aligned}$$

From this we obtain

$$\begin{aligned} &\left\|\frac{\overline{\omega}^{\frac{3}{2}}}{|Z_{,\alpha'}|}(\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\}\right\|_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim \left\{\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right\|_2 + \||D_{\alpha'}|\omega\|_2\right\}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_t\Theta\right\|_2 \\ &\quad + (\|b_{\alpha'}\|_{\mathcal{W}} + \|D_{\alpha'}Z_t\|_{\mathcal{W}})\left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\mathcal{C}_{\sqrt{\lambda}}} + \|b_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right\|_{\dot{H}^{\frac{1}{2}}} \end{aligned}$$

19) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}R_3 \in L^2_{\sqrt{\lambda}}$

Proof: Recall from (5.2) the formula of R_3

$$\begin{aligned} R_3 &= \left\{-2(D_{\alpha'}^2 Z_{tt}) + 6(D_{\alpha'}Z_t)(D_{\alpha'}^2 Z_t)\right\}(D_{\alpha'}\overline{Z}_t) + \left\{-4(D_{\alpha'}Z_{tt}) + 6(D_{\alpha'}Z_t)^2\right\}(D_{\alpha'}^2\overline{Z}_t) \\ &\quad - 2(D_{\alpha'}^2 Z_t)(D_{\alpha'}\overline{Z}_{tt}) - 4(D_{\alpha'}Z_t)(D_{\alpha'}^2\overline{Z}_{tt}) - 2i\overline{\omega}(D_{\alpha'}\overline{\omega})\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \\ &\quad - i(D_{\alpha'}J_1)\left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - iJ_1\left(D_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right) \end{aligned}$$

Define

$$A = \left\| \lambda^{\frac{1}{2}} |Z_{,\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_\infty + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}\omega \right\|_\infty + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_2 + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}} \partial_{\alpha'}^3 \frac{1}{Z_{,\alpha'}} \right\|_2$$
$$+ \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_\infty + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \overline{Z}_{tt,\alpha'} \right\|_\infty + \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}} \partial_{\alpha'}^2 \overline{Z}_{t,\alpha'} \right\|_2$$
$$+ \left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}} \partial_{\alpha'}^2 \overline{Z}_{tt,\alpha'} \right\|_2$$

and

$$B = 1 + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_2 + \| |D_{\alpha'}|\omega \|_2 + \left\| |D_{\alpha'}| \frac{1}{Z_{,\alpha'}} \right\|_\infty + \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\|_2 + \left\| \overline{Z}_{t,\alpha'} \right\|_2$$
$$+ \left\| \overline{Z}_{tt,\alpha'} \right\|_2 + \left\| |D_{\alpha'}| \overline{Z}_t \right\|_\infty + \left\| |D_{\alpha'}| \overline{Z}_{tt} \right\|_\infty + \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 + \| J_1 \|_\infty$$
$$+ \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \overline{Z}_{tt,\alpha'} \right\|_2 + \left\| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_\infty + \left\| |D_{\alpha'}| \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} J_1 \right) \right\|_2$$

Now by expanding and taking derivatives we have the following estimate

$$\left\| \frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} R_3 \right\|_2 \lesssim AB^4$$

20) $R_4 \in L^2_{\sqrt{\lambda}}$

Proof: Recall from (5.4) the formula of R_4

$$R_4 = -\left\{ \frac{D_t b_{\alpha'}}{2} + \frac{D_t D_{\alpha'} Z_t}{2} - \left(\frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right)^2 \right\} \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}} \partial_{\alpha'} R_3$$
$$- \left\{ 2iA_1 \left(|D_{\alpha'}| \frac{1}{|Z_{,\alpha'}|} \right) + \frac{i}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 - \frac{iA_1}{2} \left(\overline{D}_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}} \partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t$$
$$- 3i\overline{\omega}^2 \left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}} \partial_{\alpha'} \overline{\omega} \right) |D_{\alpha'}| \left(\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} J_1 \right) - \left(\frac{b_{\alpha'}}{2} + \frac{D_{\alpha'} Z_t}{2} \right) \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}} \partial_{\alpha'} D_t D_{\alpha'}^2 \overline{Z}_t$$

Hence we have the estimate

$$
\begin{aligned}
&\|R_4\|_2 \\
&\lesssim \left\{\|D_t b_{\alpha'}\|_\infty + \|D_t D_{\alpha'} Z_t\|_\infty + (\|b_{\alpha'}\|_\infty + \|D_{\alpha'} Z_t\|_\infty)^2\right\}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_2 \\
&\quad + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} R_3\right\|_2 + \left\{\|A_1\|_\infty \left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_\infty + \left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} A_1\right\|_\infty\right\}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_2 \\
&\quad + \|A_1\|_\infty \left\||D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right\|_\infty \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty \left\||D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} J_1\right)\right\|_2 \\
&\quad + \left\{\|b_{\alpha'}\|_\infty + \|D_{\alpha'} Z_t\|_\infty\right\}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'} D_t D_{\alpha'}^2 \overline{Z}_t\right\|_2
\end{aligned}
$$

21) $(\mathbb{I} - \mathbb{H}) D_t^2 \left(\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right) \in L^2$

Proof: For a function f satisfying $\mathbb{P}_A f = 0$ we have

$$
\begin{aligned}
(\mathbb{I} - \mathbb{H}) D_t^2 f &= [D_t, \mathbb{H}] D_t f + D_t [D_t, \mathbb{H}] f \\
&= [b, \mathbb{H}] \partial_{\alpha'} D_t f + D_t [b, \mathbb{H}] \partial_{\alpha'} f \\
&= 2[b, \mathbb{H}] \partial_{\alpha'} D_t f + [D_t b, \mathbb{H}] \partial_{\alpha'} f - [b, b; \partial_{\alpha'} f]
\end{aligned}
$$

As $\mathbb{P}_A\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right) = 0$ we obtain

$$
\begin{aligned}
&\left\|(\mathbb{I} - \mathbb{H}) D_t^2 \left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right)\right\|_2 \\
&\lesssim \|b_{\alpha'}\|_\infty \left\|D_t \left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right)\right\|_2 + \|\partial_{\alpha'} D_t b\|_{\dot{H}^{\frac{1}{2}}} \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_2 \\
&\quad + \|b_{\alpha'}\|_\infty^2 \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right\|_2
\end{aligned}
$$

22) $(\mathbb{I} - \mathbb{H})\left\{ i\dfrac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'} D_{\alpha'}^2 \overline{Z}_t\right)\right\} \in L^2$

Proof: We see that

$$(\mathbb{I}-\mathbb{H})\left\{i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\} = i\left[\frac{A_1}{|Z_{,\alpha'}|^2},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)$$

and hence we have

$$\begin{aligned}&\left\|(\mathbb{I}-\mathbb{H})\left\{i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\}\right\|_2 \\ &\lesssim \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_2\left\{\left\|\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right\|_\infty + \|A_1\|_\infty\left\||D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right\|_\infty\right\}\end{aligned}$$

23) $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \in L^2_{\sqrt{\lambda}}$

Proof: As J_1 is real valued we see that

$$\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) = \text{Re}\left\{\frac{\lambda^{\frac{1}{2}}\omega^{\frac{7}{2}}\overline{\omega}^3}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\}$$

and we see that

$$\begin{aligned}\frac{\lambda^{\frac{1}{2}}\overline{\omega}^3}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}(\mathbb{I}-\mathbb{H})|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) &= -\left[\frac{\lambda^{\frac{1}{2}}\overline{\omega}^3}{Z_{,\alpha'}^{1/2}},\mathbb{H}\right]\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \\ &\quad + (\mathbb{I}-\mathbb{H})\left\{\frac{\lambda^{\frac{1}{2}}\overline{\omega}^3}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\}\end{aligned}$$

Now recall the of equation of $\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t$ from (5.3)

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t = -i\overline{\omega}^3\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) + R_4$$

Applying $(\mathbb{I}-\mathbb{H})$ to the above equation we obtain the estimate

$$\begin{aligned}&\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\|_2\\&\lesssim\left\{\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right\|_\infty\right\}\left\||D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right)\right\|_2+\|R_4\|_2\\&\quad+\left\|(\mathbb{I}-\mathbb{H})D_t^2\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\|_2+\left\|(\mathbb{I}-\mathbb{H})\left\{i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\}\right\|_2\end{aligned}$$

5.2.4 Closing the energy estimate for $E_{\lambda,aux}$

We now complete the proof of Theorem 5.2.1. Recall that

$$\begin{aligned}E_{\lambda,aux}&=\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2+\left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2+\left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2+\left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_2^2\\&\quad+\left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\|_2^2+\left\|\left\{\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\}\right\|_{\dot{H}^{\frac{1}{2}}}^2\end{aligned}$$

Analogous to what we did in the energy estimate of E_σ, we simplify the calculations by the use of the following notation: If $a(t),b(t)$ are functions of time we write $a\approx b$ if there exists a universal non-negative polynomial P with $|a(t)-b(t)|\leq P(E_{high}(t))E_{\lambda,aux}(t)$. Observe that $\approx$ is an equivalence relation. With this notation, proving Theorem 5.2.1 is equivalent to showing $\frac{dE_{\lambda,aux}(t)}{dt}\approx 0$. We control the first four directly and for the last two terms we use the equation (5.3).

1. Controlling the time derivative of $\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2$ proceeds exactly the same as that of controlling $\left\|\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2$ which we did in proving the estimate for E_σ. Using that computation we obtain

$$\begin{aligned}\left\|D_t\left(\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_\infty&\lesssim(\|D_{\alpha'}Z_t\|_\infty+\|b_{\alpha'}\|_\infty)\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty\\&\quad+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right\|_\infty+\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t\right\|_\infty\end{aligned}$$

and we have the estimate

$$\begin{aligned}
&\limsup_{s\to 0^+} \frac{\left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2(t+s) - \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2(t)}{s}\\
&\lesssim \left\|\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty(t)\left\|D_t\left(\lambda^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_\infty(t)\\
&\lesssim P(\mathcal{E}_{high}(t))E_{\lambda,aux}(t)
\end{aligned}$$

2. We first observe that

$$D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right) = -\left(\frac{b_{\alpha'}}{2}+\frac{D_{\alpha'}Z_t}{2}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'} + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\left(-b_{\alpha'}\overline{Z}_{t,\alpha'}+\overline{Z}_{tt,\alpha'}\right)$$

Hence

$$\begin{aligned}
\left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2 &\lesssim \{\|b_{\alpha'}\|_\infty + \|D_{\alpha'}Z_t\|_\infty\}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\\
&\quad + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right\|_\infty\left\|\overline{Z}_{t,\alpha'}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{tt,\alpha'}\right\|_2
\end{aligned}$$

Now by using Lemma 2.4.6 we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\left|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right|^2 d\alpha'\\
&\lesssim \|b_{\alpha'}\|_\infty\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2\left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2\\
&\lesssim P(\mathcal{E}_{high})E_{\lambda,aux}
\end{aligned}$$

3. We observe that

$$\begin{aligned}
&D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\\
&= -\left(\frac{b_{\alpha'}}{2}+\frac{D_{\alpha'}Z_t}{2}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\left(D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\\
&= -\left(\frac{b_{\alpha'}}{2}+\frac{D_{\alpha'}Z_t}{2}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}} + \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}\left(D_{\alpha'}b_{\alpha'} - D_{\alpha'}^2 Z_t - \left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)D_{\alpha'}Z_t\right)
\end{aligned}$$

Hence we have

$$\left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \lesssim \{\|b_{\alpha'}\|_\infty + \|D_{\alpha'}Z_t\|_\infty\}\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2Z_t\right\|_2 + \left\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_2\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t\right\|_\infty + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}b_{\alpha'}\right\|_2$$

Now by using Lemma 2.4.6 we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\left|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right|^2 d\alpha' \\
&\lesssim \|b_{\alpha'}\|_\infty\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2\left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \\
&\lesssim P(\mathcal{E}_{high})E_{\lambda,aux}
\end{aligned}$$

4. Using Lemma 2.4.6 we see that

$$\begin{aligned}
&\frac{d}{dt}\int\left|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right|^2 d\alpha' \\
&\lesssim \|b_{\alpha'}\|_\infty\left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right\|_2\left\|D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right)\right\|_2 \\
&\lesssim P(\mathcal{E}_{high})E_{\lambda,aux}
\end{aligned}$$

5. The quantity left to control is the time derivative of

$$\int|D_tf|^2d\alpha' + \int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}f\right)\right|^2 d\alpha'$$

where $f = \frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t$ and we have $\mathbb{P}_H f = f$. We have already computed the time derivative of such functionals in the §4.2.3. Hence using that computation we have

$$\frac{dE_{\lambda,aux}(t)}{dt} \approx 2\text{Re}\int\left(D_t^2 f + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right)(D_t\bar{f})\,d\alpha'$$

As $D_t\left(\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right) \in L^2_{\sqrt{\lambda}}$ we only need to show that the other term in in $L^2_{\sqrt{\lambda}}$. Now the equation for $\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t$ from (5.3) implies

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t = -i\overline{\omega}^3\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) + R_4$$

As we have shown $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \in L^2_{\sqrt{\lambda}}$ and $R_3 \in L^2_{\sqrt{\lambda}}$ we have shown that this term is in $L^2_{\sqrt{\lambda}}$ and the proof of Theorem 5.2.1 is complete.

5.2.5 Equivalence of $E_{\lambda,aux}$ and $\mathcal{E}_{\lambda,aux}$

We now give a simpler description of the energy $E_{\lambda,aux}$. Define

$$\begin{aligned}\mathcal{E}_{\lambda,aux} = &\left\|\lambda^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\|_\infty^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right\|_2^2\\ &+ \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_2^2 + \left\|\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^{\frac{1}{2}}}^2\end{aligned}$$

Proposition 5.2.4. *There exists universal polynomials P_1, P_2 with non-negative coefficients so that for smooth solutions to the water wave equation with no surface tension we have*

$$E_{\lambda,aux} \leq P_1(\mathcal{E}_{high})\mathcal{E}_{\lambda,aux} \quad \textit{and } \mathcal{E}_{\lambda,aux} \leq P_2(\mathcal{E}_{high})E_{\lambda,aux}$$

Proof. Let $E_{\lambda,aux} < \infty$. We have already pretty much controlled all the terms of $\mathcal{E}_{\lambda,aux}$ and the only term not controlled is $\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}}$. This term can be easily controlled by using the fact that $\omega \in \mathcal{W}$ and $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in \mathcal{C}_{\sqrt{\lambda}}$.

Now assume that $\mathcal{E}_{\lambda,aux} < \infty$. We see that the first three terms of $E_{\lambda,aux}$ are controlled. Now following the proof of $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in L^2_{\sqrt{\lambda}}$ in §5.2.3, we see that $\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t \in L^2_{\sqrt{\lambda}}$, $\frac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2 Z_t \in L^2_{\sqrt{\lambda}}$. Hence we now have $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}Z_t \in L^\infty_{\sqrt{\lambda}}$, $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t \in L^\infty_{\sqrt{\lambda}}$. Now following the proof of $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}} \in L^2_{\sqrt{\lambda}}$, we obtain that $\frac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\overline{Z}_{tt} \in L^2_{\sqrt{\lambda}}$. Now we follow the proof of $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^3\overline{Z}_{tt} \in L^2_{\sqrt{\lambda}}$ and obtain $D_t\left(\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right) \in L^2_{\sqrt{\lambda}}$.

We now use $\omega \in \mathcal{W}$ to see that $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in \mathcal{C}_{\sqrt{\lambda}}$. Hence again by following the proof of $\dfrac{\lambda^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in \mathcal{C}_{\sqrt{\lambda}}$ we see that $\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\left(\dfrac{\lambda^{\frac{1}{2}}}{Z_{,\alpha'}^{1/2}}\partial_{\alpha'}D_{\alpha'}^2\overline{Z}_t\right) \in \mathcal{C}_{\sqrt{\lambda}}$ and therefore the proposition is proved. □

5.3 Apriori estimate for E_Δ

As mentioned in §3.4, let A be a solution to the water wave equation with surface tension σ and B a solution to the water wave equation with no surface tension and denote by f_a the function f for solution A and f_b for solution B. Recall that the material derivatives are by given by $(D_t)_a = U_{h_a}^{-1}\partial_t U_{h_a}$, $(D_t)_b = U_{h_b}^{-1}\partial_t U_{h_b}$ and we have $\widetilde{h} = h_b \circ h_a^{-1}$ and $\widetilde{U} = U_{\widetilde{h}} = U_{h_a}^{-1}U_{h_b}$. Define the operators

$$(\mathcal{H}f)(x) = \frac{1}{i\pi}p.v.\int \frac{\widetilde{h}_{\beta'}(\beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}f(\beta')\,d\beta'$$
$$(\widetilde{\mathcal{H}}f)(x) = \frac{1}{i\pi}p.v.\int \frac{1}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}f(\beta')\,d\beta'$$

Recall from §3.4 that $\Delta(f) = f_a - \widetilde{U}(f_b)$. For convenience we will often write it as $\Delta(f) = f_a - \widetilde{U}(f)_b$. We also define the following notation

$$[f_1, f_2; \partial_{\alpha'}f_3]_{\widetilde{h}} = \frac{1}{i\pi}\int\left(\frac{f_1(\alpha') - f_1(\beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}\right)\left(\frac{f_2(\alpha') - f_2(\beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}\right)\partial_{\beta'}f_3(\beta')\,d\beta'$$

The following two lemmas are proved by a straightforward calculation.

Lemma 5.3.1. *Let $\widetilde{U}, \mathcal{H}, \widetilde{\mathcal{H}}$ be defined as above. Then*

1. $(D_t)_a\widetilde{U} = \widetilde{U}(D_t)_b$
2. $\partial_{\alpha'}\widetilde{U} = \widetilde{h}_{\alpha'}\widetilde{U}\partial_{\alpha'} \quad \partial_{\alpha'}\widetilde{U}^{-1} = \dfrac{1}{\widetilde{h}_{\alpha'} \circ \widetilde{h}^{-1}}\widetilde{U}^{-1}\partial_{\alpha'}$ *and* $\widetilde{h}_{\alpha'} = U_{h_a}^{-1}\left(\dfrac{(h_\alpha)_b}{(h_\alpha)_a}\right)$
3. $\mathcal{H}\widetilde{U} = \widetilde{U}\mathbb{H}$

4. $\widetilde{U}[f,\mathbb{H}]\partial_{\alpha'}g = [(\widetilde{U}f),\widetilde{\mathcal{H}}]\partial_{\alpha'}(\widetilde{U}g)$

5. $\widetilde{U}[f_1,f_2;\partial_{\alpha'}f_3] = [(\widetilde{U}f_1),(\widetilde{U}f_2);\partial_{\alpha'}(\widetilde{U}f_3)]_{\widetilde{h}}$

Lemma 5.3.2. *Let Δ be defined as above. Then*

1. $\Delta(f_1f_2\cdots f_n) = \displaystyle\sum_{i=0}^{n-2}\big\{\widetilde{U}(f_1)_b\cdots\widetilde{U}(f_i)_b\big\}\Delta(f_{i+1})\big\{(f_{i+2})_a\cdots(f_n)_a\big\}$

2. $\Delta[f,\mathbb{H}]\partial_{\alpha'}g = [\Delta f,\mathbb{H}]\partial_{\alpha'}(g_a) + \big[\widetilde{U}(f)_b,\mathbb{H}-\widetilde{\mathcal{H}}\big]\partial_{\alpha'}(g_a) + \widetilde{U}\big\{[f_b,\mathbb{H}]\partial_{\alpha'}\big(\widetilde{U}^{-1}(\Delta g)\big)\big\}$

3. $$\begin{aligned}\Delta[f_1,f_2;\partial_{\alpha'}f_3] &= [\Delta f_1,(f_2)_a;\partial_{\alpha'}(f_3)_a] + \big[\widetilde{U}(f_1)_b,\Delta f_2;\partial_{\alpha'}(f_3)_a\big]\\ &\quad + \big[\widetilde{U}(f_1)_b,\widetilde{U}(f_2)_b;\partial_{\alpha'}(\Delta f_3)\big]\\ &\quad + \Big\{\big[\widetilde{U}(f_1)_b,\widetilde{U}(f_2)_b;\partial_{\alpha'}\widetilde{U}(f_3)_b\big] - \big[\widetilde{U}(f_1)_b,\widetilde{U}(f_2)_b;\partial_{\alpha'}\widetilde{U}(f_3)_b\big]_{\widetilde{h}}\Big\}\end{aligned}$$

The following lemma gives us control of some basic quantities required for the proof of Theorem 5.0.1.

Lemma 5.3.3. *Assume the hypothesis of Theorem 5.0.1. Then we have the following*

1. $\|\widetilde{h}_{\alpha'}\|_{L^\infty}(t), \left\|\dfrac{1}{\widetilde{h}_{\alpha'}}\right\|_{L^\infty}(t) \leq C(M)$ *for all* $t\in[0,T)$

2. $\left|\dfrac{\widetilde{h}(\alpha',t)-\widetilde{h}(\beta',t)}{\alpha'-\beta'}\right|, \left|\dfrac{\alpha'-\beta'}{\widetilde{h}(\alpha',t)-\widetilde{h}(\beta',t)}\right| \leq C(M)$ *for all* $\alpha'\neq\beta'$ *and* $t\in[0,T)$

3. $\|\widetilde{U}f\|_2 \leq C(M)\|f\|_2$ *and* $\|\widetilde{U}f\|_{\dot{H}^{\frac{1}{2}}} \leq C(M)\|f\|_{\dot{H}^{\frac{1}{2}}}$*. These estimates are also true for the operator* $\widetilde{U}^{-1}$ *instead of* $\widetilde{U}$*. Hence we have* $\|\mathcal{H}(f)\|_2 \leq C(M)\|f\|_2$, $\|\mathcal{H}(f)\|_{\dot{H}^{\frac{1}{2}}} \leq C(M)\|f\|_{\dot{H}^{\frac{1}{2}}}$ *and* $\|\widetilde{\mathcal{H}}(f)\|_2 \leq C(M)\|f\|_2$

4. $\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}(t), \left\|\dfrac{1}{\widetilde{h}_{\alpha'}}\right\|_{\dot{H}^{\frac{1}{2}}}(t) \leq C(M)$ *for all* $t\in[0,T)$

5. $\left\||Z_{,\alpha'}|_a\widetilde{U}\left(\dfrac{1}{|Z_{,\alpha'}|}\right)_b\right\|_\infty(t), \left\|\dfrac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\right\|_\infty(t) \leq C(M)$ *for all* $t\in[0,T)$

6. $\|(|D_{\alpha'}|)_a\widetilde{h}_{\alpha'}\|_2(t) \leq C(M)$ *for all* $t\in[0,T)$

7. $\left\|(|D_{\alpha'}|)_a\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\dfrac{1}{|Z_{,\alpha'}|}\right)_b\right\}\right\|_2(t), \left\|(|D_{\alpha'}|)_a\left\{\dfrac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\right\}\right\|_\infty(t) \leq C(M)$ *for all* $t\in[0,T)$

Proof. We will prove each of the estimates individually.

1. We know that $h_a(\alpha', 0) = h_b(\alpha', 0) = \alpha'$ at time $t = 0$. Now observe that

$$U_h^{-1}\left(\frac{h_{t\alpha}}{h_\alpha}\right) = b_{\alpha'} \quad \text{and } \partial_t h_\alpha = \left(\frac{h_{t\alpha}}{h_\alpha}\right) h_\alpha \quad \text{and } \partial_t \frac{1}{h_\alpha} = -\left(\frac{h_{t\alpha}}{h_\alpha}\right)\frac{1}{h_\alpha}$$

and as $\|(b_{\alpha'})_b\|_\infty$ is controlled by E_{high} and $\|(b_{\alpha'})_a\|_\infty$ is controlled by E_σ, we see that h_α and $\frac{1}{h_\alpha}$ remain bounded by $C(M)$ for both solutions A and B. Now

$$(D_t)_a \widetilde{h}_{\alpha'} = U_{h_a}^{-1}\left(\partial_t \frac{(h_\alpha)_b}{(h_\alpha)_a}\right) = \widetilde{h}_{\alpha'} U_{h_a}^{-1}\left\{\left(\frac{h_{t\alpha}}{h_\alpha}\right)_b - \left(\frac{h_{t\alpha}}{h_\alpha}\right)_a\right\} = \widetilde{h}_{\alpha'}(\widetilde{U}(b_{\alpha'})_b - (b_{\alpha'})_a)$$

Hence as $\widetilde{h}_{\alpha'} = 1$ at time $t = 0$, we see that $\|\widetilde{h}_{\alpha'}\|_\infty(t) \leq C(M)$. Similarly for $\dfrac{1}{\widetilde{h}_{\alpha'}}$.

2. This is an easy consequence of $\|\widetilde{h}_{\alpha'}\|_{L^\infty}(t), \left\|\dfrac{1}{\widetilde{h}_{\alpha'}}\right\|_{L^\infty}(t) \leq C(M)$ and the fact that $\widetilde{h}$ is a homeomorphism.

3. We see that

$$\|\widetilde{U}f\|_2^2 = \int \left|f(\widetilde{h}(\alpha'))\right|^2 d\alpha' = \int \frac{|f(s)|^2}{(\widetilde{h}_{\alpha'} \circ \widetilde{h}^{-1})(s)}\, ds \leq C(M)\|f\|_2^2$$

Similarly we have that

$$\begin{aligned}
\|\widetilde{U}f\|_{\dot{H}^{\frac{1}{2}}}^2 &= \frac{1}{2\pi}\int\int \frac{|f(\widetilde{h}(\alpha')) - f(\widetilde{h}(\beta'))|^2}{(\alpha' - \beta')^2}\, d\alpha'\, d\beta' \\
&= \frac{1}{2\pi}\int\int \left(\frac{|f(x) - f(y)|^2}{(\widetilde{h}^{-1}(x) - \widetilde{h}^{-1}(y))^2}\right)\frac{1}{(\widetilde{h}_{\alpha'} \circ \widetilde{h}^{-1})(x)(\widetilde{h}_{\alpha'} \circ \widetilde{h}^{-1})(y)}\, dx\, dy \\
&\leq C(M)\frac{1}{2\pi}\int\int \frac{|f(x) - f(y)|^2}{(\alpha' - \beta')^2}\, dx\, dy \\
&\leq C(M)\|f\|_{\dot{H}^{\frac{1}{2}}}^2
\end{aligned}$$

In the same way we can prove the estimates for $\widetilde{U}^{-1}$ instead of $\widetilde{U}$.

4. Using Lemma 2.4.6 we see that

$$\begin{aligned}\frac{d}{dt}\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}^2 &\lesssim C(M)\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\|(D_t)_a\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim C(M)\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}\|\widetilde{h}_{\alpha'}(\widetilde{U}(b_{\alpha'})_b - (b_{\alpha'})_a)\|_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim C(M)\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}(\|\widetilde{h}_{\alpha'}\|_\infty + \|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}) \\ &\lesssim C(M)(\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}^2 + 1)\end{aligned}$$

Now as $\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}(0) = 0$, we obtain that $\|\widetilde{h}_{\alpha'}\|_{\dot{H}^{\frac{1}{2}}}(t) \leq C(M)$ for $t \in [0, T)$.

5. We observe that

$$\begin{aligned}&(D_t)_a\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\} \\ &= \left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\mathrm{Re}\Big\{(D_{\alpha'}Z_t)_a - \widetilde{U}(D_{\alpha'}Z_t)_b - (b_{\alpha'})_a + \widetilde{U}(b_{\alpha'})_b\Big\}\end{aligned}$$

Hence we have that $\left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\|_\infty(t) \leq C(M)$. The other estimate is proven similarly.

6. We first observe that

$$\begin{aligned}(D_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'} &= -\mathrm{Re}(D_{\alpha'}Z_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'} + |D_{\alpha'}|_a(D_t)_a\widetilde{h}_{\alpha'} \\ &= -\mathrm{Re}(D_{\alpha'}Z_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'} + |D_{\alpha'}|_a\Big\{\widetilde{h}_{\alpha'}(\widetilde{U}(b_{\alpha'})_b) - (b_{\alpha'})_a\Big\} \\ &= -\mathrm{Re}(D_{\alpha'}Z_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'} + (|D_{\alpha'}|_a\widetilde{h}_{\alpha'})(\widetilde{U}(b_{\alpha'})_b - (b_{\alpha'})_a) \\ &\quad + \widetilde{h}_{\alpha'}\left\{\widetilde{h}_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\right)\widetilde{U}(|D_{\alpha'}|b_{\alpha'})_b - (|D_{\alpha'}|b_{\alpha'})_a\right\}\end{aligned}$$

Hence using Lemma 2.4.6 we have

$$\begin{aligned}\frac{d}{dt}\left\||D_{\alpha'}|_a\widetilde{h}_{\alpha'}\right\|_2^2 &\leq C(M)\left\||D_{\alpha'}|_a\widetilde{h}_{\alpha'}\right\|_2^2 + \left\||D_{\alpha'}|_a\widetilde{h}_{\alpha'}\right\|_2\left\|(D_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'}\right\|_2 \\ &\leq C(M)\left\{\left\||D_{\alpha'}|_a\widetilde{h}_{\alpha'}\right\|_2^2 + 1\right\}\end{aligned}$$

As $|D_{\alpha'}|_a\widetilde{h}_{\alpha'} = 0$ at time $t = 0$, we are done

7. Observe that

$$
\begin{aligned}
&(|D_{\alpha'}|)_a\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\\
&=\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\left\{-\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a+\widetilde{h}_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\right)\widetilde{U}\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}
\end{aligned}
$$

Hence we see that $\left\|(|D_{\alpha'}|)_a\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\right\|_2(t)\leq C(M)$ for all $t\in[0,T)$.

□

5.3.1 Quantities controlled by E_Δ

In this section whenever we write $f\in L^2_{\Delta^\alpha}$, what we mean is that there exists a constant $C(M)$ depending only on M such that $\|f\|_2\leq C(M)(E_\Delta)^\alpha$. Similar definitions for $f\in L^1_{\Delta^\alpha}$, $f\in\dot{H}^{\frac{1}{2}}_{\Delta^\alpha}$ and $f\in L^\infty_{\Delta^\alpha}$. When we write $f\in L^2$ we mean $f\in L^2_{\Delta^\alpha}$ with $\alpha=0$. Similar notation for $\dot{H}^{\frac{1}{2}},L^\infty,\mathcal{C}$ and $\mathcal{W}$. This notation is now consistent with the notation used in §4.1 and §5.1.3. It is important to note that if $f\in L^2$ and $f\in L^2_{\Delta^\alpha}$, then we have $f\in L^2_{\Delta^\beta}$ for all $0<\beta<\alpha$. We say that $a\approx_{L^2_{\Delta^\alpha}} b$ if $a-b\in L^2_{\Delta^\alpha}$. It should be noted that $\approx_{L^2_{\Delta^\alpha}}$ is an equivalence relation. Similar definitions for $\approx_{L^1_{\Delta^\alpha}},\approx_{L^\infty_{\Delta^\alpha}}$ and $\approx_{\dot{H}^{\frac{1}{2}}_{\Delta^\alpha}}$.

We define the spaces $\mathcal{C}_{\Delta^\alpha}$ and $\mathcal{W}_{\Delta^\alpha}$ as follows

1. If $w\in L^\infty_{\Delta^\alpha}$ and $|D_{\alpha'}|_a w\in L^2_{\Delta^\alpha}$, then we say $f\in\mathcal{W}_{\Delta^\alpha}$. Define

$$\|w\|_{\mathcal{W}_{\Delta^\alpha}}=\|w\|_\infty+\||D_{\alpha'}|_a w\|_2$$

2. If $f\in\dot{H}^{\frac{1}{2}}_{\Delta^\alpha}$ and $f|Z_{,\alpha'}|_a\in L^2_{\Delta^\alpha}$, then we say $f\in\mathcal{C}_{\Delta^\alpha}$. Define

$$\|f\|_{\mathcal{C}_{\Delta^\alpha}}=\|f\|_{\dot{H}^{\frac{1}{2}}}+\left(1+\left\|\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a\right\|_2\right)\|f|Z_{,\alpha'}|_a\|_2$$

Similar to $\approx_{L^2_{\Delta^\alpha}}$ we also define the relations $\approx_{\mathcal{W}_{\Delta^\alpha}}$ and $\approx_{\mathcal{C}_{\Delta^\alpha}}$. Now analogous to Lemma 2.4.5 we have the following lemma

Lemma 5.3.4. *Let $\alpha_1,\alpha_2,\alpha_3\geq 0$ with $\alpha_1+\alpha_2=\alpha_3$. Then the following properties hold for the spaces $\mathcal{W}_{\Delta^\alpha}$ and $\mathcal{C}_{\Delta^\alpha}$*

1. *If* $w_1 \in \mathcal{W}_{\Delta^{\alpha_1}}$, $w_2 \in \mathcal{W}_{\Delta^{\alpha_2}}$, *then* $w_1 w_2 \in \mathcal{W}_{\Delta^{\alpha_3}}$. *Moreover we have* $\|w_1 w_2\|_{\mathcal{W}_{\Delta^{\alpha_3}}} \leq \|w_1\|_{\mathcal{W}_{\Delta^{\alpha_1}}} \|w_2\|_{\mathcal{W}_{\Delta^{\alpha_2}}}$

2. *If* $f \in \mathcal{C}_{\Delta^{\alpha_1}}$ *and* $w \in \mathcal{W}_{\Delta^{\alpha_2}}$, *then* $fw \in \mathcal{C}_{\Delta^{\alpha_3}}$. *Moreover* $\|fw\|_{\mathcal{C}_{\Delta^{\alpha_3}}} \lesssim \|f\|_{\mathcal{C}_{\Delta^{\alpha_1}}} \|w\|_{\mathcal{W}_{\Delta^{\alpha_2}}}$

3. *If* $f \in \mathcal{C}_{\Delta^{\alpha_1}}$, $g \in \mathcal{C}_{\Delta^{\alpha_2}}$, *then* $fg|Z_{,\alpha'}| \in L^2_{\Delta^{\alpha_3}}$. *Moreover* $\|fg|Z_{,\alpha'}|\|_2 \lesssim \|f\|_{\mathcal{C}_{\Delta^{\alpha_1}}} \|g\|_{\mathcal{C}_{\Delta^{\alpha_2}}}$

In this section, we will need to commute weights and derivatives with the operators $\widetilde{U}, \Delta$ quite frequently and hence the following lemma will be very frequently used.

Lemma 5.3.5. *Let* f, g *be smooth functions and let* $\alpha \in \mathbb{R}$. *Then*

1. *If* $g_a \widetilde{U}(\partial_{\alpha'} f)_b \in L^2$ *then*
   (a) $g_a \partial_{\alpha'} \widetilde{U}(f)_b \in L^2$
   (b) $g_a \widetilde{U}(\partial_{\alpha'} f)_b \approx_{L^2_{\sqrt{\Delta}}} g_a \partial_{\alpha'} \widetilde{U}(f)_b$
   (c) $g_a \Delta(\partial_{\alpha'} f) \approx_{L^2_{\sqrt{\Delta}}} g_a \partial_{\alpha'} \Delta(f)$

   These estimates are also true if we replace $(L^2, L^2_{\sqrt{\Delta}})$ *with* $(L^\infty, L^\infty_{\sqrt{\Delta}})$, $(L^\infty \cap \dot{H}^{\frac{1}{2}}, L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}})$, $(\mathcal{W}, \mathcal{W}_{\sqrt{\Delta}})$ *or* $(\mathcal{C}, \mathcal{C}_{\sqrt{\Delta}})$.

2. *If* $g_a \widetilde{U}(f)_b \in L^2$ *then*
   (a) $(g|Z_{,\alpha'}|^\alpha)_a \widetilde{U}\big(|Z_{,\alpha'}|^{-\alpha} f\big)_b \in L^2$
   (b) $g_a \widetilde{U}(f)_b \approx_{L^2_{\sqrt{\Delta}}} (g|Z_{,\alpha'}|^\alpha)_a \widetilde{U}\big(|Z_{,\alpha'}|^{-\alpha} f\big)_b$
   (c) $g_a \Delta(f) \approx_{L^2_{\sqrt{\Delta}}} (g|Z_{,\alpha'}|^\alpha)_a \Delta(|Z_{,\alpha'}|^{-\alpha} f)$

   These estimates are also true if we replace $(L^2, L^2_{\sqrt{\Delta}})$ *with* $(L^\infty, L^\infty_{\sqrt{\Delta}})$, $(\mathcal{W}, \mathcal{W}_{\sqrt{\Delta}})$ *or* $(\mathcal{C}, \mathcal{C}_{\sqrt{\Delta}})$.

3. *If* $g_a \widetilde{U}(f)_b \in L^2$ *then*
   (a) $(g\omega^\alpha)_a \widetilde{U}(\omega^{-\alpha} f)_b \in L^2$
   (b) $g_a \widetilde{U}(f)_b \approx_{L^2_{\sqrt{\Delta}}} (g\omega^\alpha)_a \widetilde{U}(\omega^{-\alpha} f)_b$
   (c) $g_a \Delta(f) \approx_{L^2_{\sqrt{\Delta}}} (g\omega^\alpha)_a \Delta(\omega^{-\alpha} f)$

   These estimates are also true if we replace $(L^2, L^2_{\sqrt{\Delta}})$ *with* $(L^\infty, L^\infty_{\sqrt{\Delta}})$, $(\mathcal{W}, \mathcal{W}_{\sqrt{\Delta}})$ *or* $(\mathcal{C}, \mathcal{C}_{\sqrt{\Delta}})$.

Proof. We prove each of the statements individually.

1. We first observe that the energy E_Δ controls $(\widetilde{h}_{\alpha'}-1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}} \cap \mathcal{W}_{\sqrt{\Delta}}$. Now notice that $g_a \partial_{\alpha'} \widetilde{U}(f)_b = \widetilde{h}_{\alpha'} g_a \widetilde{U}(\partial_{\alpha'} f)_b$. As $\widetilde{h}_{\alpha'} \in L^\infty$ we see that $g_a \partial_{\alpha'} \widetilde{U}(f)_b \in L^2$. Now we have

$$g_a \partial_{\alpha'} \widetilde{U}(f)_b - g_a \widetilde{U}(\partial_{\alpha'} f)_b = (\widetilde{h}_{\alpha'}-1) g_a \widetilde{U}(\partial_{\alpha'} f)_b$$

and hence $\|g_a \partial_{\alpha'} \widetilde{U}(f)_b - g_a \widetilde{U}(\partial_{\alpha'} f)_b\|_2 \leq \|\widetilde{h}_{\alpha'}-1\|_\infty \|g_a \widetilde{U}(\partial_{\alpha'} f)_b\|_2 \leq C(M)(E_\Delta)^{\frac{1}{2}}$. The other estimates are shown similarly using the fact that $\widetilde{h}_{\alpha'} \in L^\infty \cap \dot{H}^{\frac{1}{2}} \cap \mathcal{W}$ and $(\widetilde{h}_{\alpha'}-1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}} \cap \mathcal{W}_{\sqrt{\Delta}}$.

2. The energy E_Δ controls $\Delta w \in L^\infty_{\sqrt{\Delta}}$ and $\Delta\left(\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}\right) \in L^2_{\sqrt{\Delta}}$. Hence by using (2.9) we see that $\Delta\left(\partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|}\right) \in L^2_{\sqrt{\Delta}}$ and $\Delta(|D_{\alpha'}|\omega) \in L^2_{\sqrt{\Delta}}$.

   Now as $|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b - 1 \in L^\infty_{\sqrt{\Delta}}$, we use $|x^\alpha - 1| \leq C(\alpha)|x-1|\max(x^\alpha, 1)$ for $x>0$, $\alpha \in \mathbb{R}$ to see that $\left(|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right)^\alpha - 1 \in L^\infty_{\sqrt{\Delta}}$. In particular we have $\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b - 1 \in L^\infty_{\sqrt{\Delta}}$. Now we have

$$\begin{aligned}&(|D_{\alpha'}|)_a\left\{|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\} \\ &= \left\{|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\left\{-\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a + \widetilde{h}_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\right)\widetilde{U}\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\end{aligned}$$

   Hence we see that $|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b - 1 \in \mathcal{W}_{\sqrt{\Delta}}$ or more generally $\left(|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right)^\alpha - 1 \in \mathcal{W}_{\sqrt{\Delta}}$. Now coming back we see that

$$(g|Z_{,\alpha'}|^\alpha)_a \widetilde{U}\big(|Z_{,\alpha'}|^{-\alpha} f\big)_b = \left(|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right)^\alpha g_a \widetilde{U}(f)_b$$

   Now as $|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b \in L^\infty$, we see that $(g|Z_{,\alpha'}|^\alpha)_a \widetilde{U}\big(|Z_{,\alpha'}|^{-\alpha} f\big)_b \in L^2$. Now

$$(g|Z_{,\alpha'}|^\alpha)_a \widetilde{U}\big(|Z_{,\alpha'}|^{-\alpha} f\big)_b - g_a \widetilde{U}(f)_b = \left\{\left(|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right)^\alpha - 1\right\} g_a \widetilde{U}(f)_b$$

   Hence we have $\left\|(g|Z_{,\alpha'}|^\alpha)_a \widetilde{U}\big(|Z_{,\alpha'}|^{-\alpha} f\big)_b - g_a \widetilde{U}(f)_b\right\|_2 \leq C(M)(E_\Delta)^{\frac{1}{2}}$. The other estimates are proven similarly using $|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b \in L^\infty \cap \mathcal{W}$ and $\left\{\left(|Z_{,\alpha'}|_a \widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right)^\alpha - 1\right\} \in L^\infty_{\sqrt{\Delta}} \cap \mathcal{W}_{\sqrt{\Delta}}$.

3. This is proved exactly the same as above. Here we use the estimate $\Delta(w) \in L^{\infty}_{\sqrt{\Delta}}$ and $|w| = 1$ to see that $\omega_a \widetilde{U}(\omega^{-1})_b - 1 \in L^{\infty}_{\sqrt{\Delta}}$ or more generally $\omega_a^{\alpha} \widetilde{U}(\omega^{-\alpha})_b - 1 \in L^{\infty}_{\sqrt{\Delta}}$.

   Now we observe that

$$
\begin{aligned}
&|D_{\alpha'}|_a \Big(\omega_a \widetilde{U}(\omega^{-1})_b\Big) \\
&= \Big(\omega_a \widetilde{U}(\omega^{-1})_b\Big)\bigg\{\bigg(\frac{\overline{\omega}}{|Z_{,\alpha'}|}\bigg)\partial_{\alpha'}\omega - \widetilde{h}_{\alpha'}\bigg(\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\bigg)\widetilde{U}\bigg(\frac{\overline{\omega}}{|Z_{,\alpha'}|}\partial_{\alpha'}\omega\bigg)_b\bigg\}
\end{aligned}
$$

   From this we see that $\omega_a \widetilde{U}(\omega^{-1})_b - 1 \in \mathcal{W}_{\sqrt{\Delta}}$ or more generally $\omega_a^{\alpha} \widetilde{U}(\omega^{-\alpha})_b - 1 \in \mathcal{W}_{\sqrt{\Delta}}$. Now using

$$
(g\omega^{\alpha})_a \widetilde{U}\big(\omega^{-\alpha} f\big)_b - g_a \widetilde{U}(f)_b = g_a \widetilde{U}(f)_b \Big\{\omega_a^{\alpha}\widetilde{U}(\omega^{-\alpha})_b - 1\Big\}
$$

   We easily reach the desired conclusion.

□

We now state some important estimates which we will use in this section to prove convergence. See the appendix for the proof.

Lemma 5.3.6. *Let $\mathbb{H}$ be the Hilbert transform and let $\mathcal{H}, \widetilde{\mathcal{H}}$ be defined as in the start of this section and let $f, f_1, f_2, f_3, g \in \mathcal{S}(\mathbb{R})$. Let M be defined as in Lemma 5.3.3 and we will suppress the dependence of M i.e. we write $a \lesssim b$ instead of $a \leq C(M)b$. With this notation we have the following estimates*

1. $\|(\mathbb{H} - \mathcal{H})f\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f\|_2$
2. $\|(\mathbb{H} - \mathcal{H})f\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f\|_{\dot{H}^{\frac{1}{2}}}$
3. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]g\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f'\|_2\|g\|_1$
4. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]g\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f\|_{\dot{H}^{\frac{1}{2}}}\|g\|_2$
5. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f'\|_{\infty}\|g\|_2$
6. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f'\|_2\|g\|_{\infty}$
7. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f'\|_2\|g\|_{\dot{H}^{\frac{1}{2}}}$
8. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\{\|f'\|_{\dot{H}^{\frac{1}{2}}}\|g\|_2 + \|f'g\|_2\}$
9. $\|\partial_{\alpha'}[f_1, [f_2, \mathbb{H} - \widetilde{\mathcal{H}}]]\partial_{\alpha'} f_3\|_2 \lesssim \|\widetilde{h}_{\alpha'} - 1\|_{\infty}\|f_1'\|_2\|f_2'\|_2\|f_3'\|_2$

10. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]g\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|f'\|_2 \|g\|_2$

11. $\|[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|f'\|_\infty \|g\|_{\dot{H}^{\frac{1}{2}}}$

12. $\left\|[f_1, f_2; f_3] - [f_1, f_2; f_3]_{\widetilde{h}}\right\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|f_1'\|_\infty \|f_2'\|_2 \|f_3\|_2$

13. $\left\|[f_1, f_2; \partial_{\alpha'} f_3] - [f_1, f_2; \partial_{\alpha'} f_3]_{\widetilde{h}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|f_1'\|_\infty \|f_2'\|_\infty \|f_3\|_{\dot{H}^{\frac{1}{2}}}$

We note that we have already shown that $(\widetilde{h}_{\alpha'} - 1) \in \mathcal{W}_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and for any $\alpha \in \mathbb{R}$ we have $\left\{ |Z_{,\alpha'}|_a \widetilde{U}\left(\dfrac{1}{|Z_{,\alpha'}|}\right)_b \right\}^\alpha - 1 \in \mathcal{W}_{\sqrt{\Delta}}$ and $\omega_a^\alpha \widetilde{U}(\omega^{-\alpha})_b - 1 \in \mathcal{W}_{\sqrt{\Delta}}$.

Let us now control the main terms controlled by E_Δ. The proof of these estimates follows exactly analogous to the proof of terms in E_σ and by using Lemma 5.3.2, 5.3.5 and 5.3.6. As the proofs are just simple modifications of the proofs in §4.1 we will just control a few terms and show how it is done and the rest are proved analogously. A few terms require a bit more work and we give more details for those terms.

1. $\Delta(\overline{Z}_{t,\alpha'}) \in L^2_{\sqrt{\Delta}}, \partial_{\alpha'}\Delta(\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$ and $|D_{\alpha'}|_a \Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}, \Delta(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$

   Proof: As $(A_1)_a \geq 1$, we see that E_Δ controls $\Delta(\overline{Z}_{t,\alpha'}) \in L^2_{\sqrt{\Delta}}$ and $|D_{\alpha'}|_a \Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$. We obtain the other two estimates by using Lemma 5.3.5.

2. $\Delta(A_1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

   Proof: Recall that $A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}$ and hence

$$
\begin{aligned}
&\Delta(A_1) \\
&= -\text{Im}\left\{\Delta[Z_t, \mathbb{H}]\partial_{\alpha'}\overline{Z}_t\right\} \\
&= -\text{Im}\left\{[\Delta Z_t, \mathbb{H}]\partial_{\alpha'}(\overline{Z}_t)_a + \left[\widetilde{U}(Z_t)_b, \mathbb{H} - \widetilde{\mathcal{H}}\right]\partial_{\alpha'}(\overline{Z}_t)_a + \widetilde{U}\left\{[(Z_t)_b, \mathbb{H}]\partial_{\alpha'}\left(\widetilde{U}^{-1}(\Delta\overline{Z}_t)\right)\right\}\right\}
\end{aligned}
$$

   Hence $\|\Delta(A_1)\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \leq C(M)(E_\Delta)^{\frac{1}{2}}$

3. $\Delta\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right) \in L^2_{\sqrt{\Delta}}, \Delta\left(\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|}\right) \in L^2_{\sqrt{\Delta}}, \Delta(|D_{\alpha'}|\omega) \in L^2_{\sqrt{\Delta}}$ and hence $\Delta(\omega) \in \mathcal{W}_{\sqrt{\Delta}}$

   Proof: Observe that $\Delta\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right) \in L^2_{\sqrt{\Delta}}$ as it is part of the energy $E_{\Delta,0}$. Recall from (2.9) that

$$
\text{Re}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = \partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|} \qquad \text{Im}\left(\frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) = i(\overline{\omega}|D_{\alpha'}|\omega)
$$

Using $\Delta(w) \in L^\infty_{\sqrt{\Delta}}$ we obtain $\Delta\left(\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right) \in L^2_{\sqrt{\Delta}}$ and $\Delta(|D_{\alpha'}|\omega) \in L^2_{\sqrt{\Delta}}$. Now using Lemma 5.3.5 we obtain $|D_{\alpha'}|_a\Delta(w) \in L^2_{\sqrt{\Delta}}$ and hence using $\Delta(\omega) \in L^\infty_{\sqrt{\Delta}}$ from $E_{\Delta,0}$ we obtain $\Delta(\omega) \in \mathcal{W}_{\sqrt{\Delta}}$.

4. $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}, \Delta(|D_{\alpha'}|\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}$ and $\Delta(D_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}$

   Proof: First observe that $|Z_{,\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \approx_{L^2_{\sqrt{\Delta}}} \Delta(\omega\overline{Z}_{t,\alpha'}) \in L^2_{\sqrt{\Delta}}$. Hence we have

$$\partial_{\alpha'}(\Delta(\overline{D}_{\alpha'}\overline{Z}_t))^2 = 2\big\{|Z_{,\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t)\big\}|D_{\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^1_\Delta$$

   Hence $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}$. The other ones are obtained easily by using $\Delta(\omega) \in L^\infty_{\sqrt{\Delta}}$

5. $\Delta(\overline{D}^2_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}, \Delta(|D_{\alpha'}|^2\overline{Z}_t) \in L^2_{\sqrt{\Delta}}, \Delta(D^2_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$

   Proof: We already know that $\Delta(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$ and hence $\Delta(\overline{D}^2_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$. Now

$$\overline{D}^2_{\alpha'}\overline{Z}_t = \overline{D}_{\alpha'}\big(\omega|D_{\alpha'}|\overline{Z}_t\big) = (\overline{D}_{\alpha'}\omega)|D_{\alpha'}|\overline{Z}_t + \omega^2|D_{\alpha'}|^2\overline{Z}_t$$

   Applying Δ to the above equation we easily get $\Delta(|D_{\alpha'}|^2\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$. The estimate for $\Delta(D^2_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$ is proven similarly.

6. $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}, \Delta(|D_{\alpha'}|\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}, \Delta(D_{\alpha'}\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}$

   Proof: As $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}$ and $|D_{\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$ we see that $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}}$. Now $|Z_{,\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$ and using Proposition A.0.10 with $f = \Delta(\overline{D}_{\alpha'}\overline{Z}_t)$ and $w = \dfrac{1}{|Z_{,\alpha'}|_a}$ we see that

$$\begin{aligned}\big\|\Delta(\overline{D}_{\alpha'}\overline{Z}_t)\big\|^2_{\dot{H}^{\frac{1}{2}}} \lesssim \big\||Z_{,\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t)\big\|_2\left\|\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|_a}\Delta(\overline{D}_{\alpha'}\overline{Z}_t)\right)\right\|_2 \\ + \big\||Z_{,\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t)\big\|^2_2\left\|\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|_a}\right\|^2_2\end{aligned}$$

   From this we obtain $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}$. As $\overline{\omega} \in \mathcal{W}$, using Lemma 5.3.5 we see that $\Delta(|D_{\alpha'}|\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}$. The other one is proven similarly.

7. $\Delta\left\{\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right)\right\} \in L^\infty_{\sqrt{\Delta}}$

Proof: We have shown the formula

$$2\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right) = 2D_{\alpha'}Z_t + \left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]Z_{t,\alpha'} + [Z_t, \mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}$$

Now applying Δ to the formula above, the estimate follows easily.

8. $\Delta(|D_{\alpha'}|A_1) \in L^2_{\sqrt{\Delta}}$ and hence $\Delta(A_1) \in \mathcal{W}_{\sqrt{\Delta}}, \Delta(\sqrt{A_1}) \in \mathcal{W}_{\sqrt{\Delta}}, \Delta\left(\frac{1}{A_1}\right) \in \mathcal{W}_{\sqrt{\Delta}}$ and also $\Delta\left(\frac{1}{\sqrt{A_1}}\right) \in \mathcal{W}_{\sqrt{\Delta}}$

   Proof: Following the proof of $|D_{\alpha'}|A_1 \in L^2$ in §4.1 and using Lemma 5.3.2, 5.3.5 and 5.3.6, we can easily show that $\Delta(|D_{\alpha'}|A_1) \in L^2_{\sqrt{\Delta}}$ and this shows that $\Delta(A_1) \in \mathcal{W}_{\sqrt{\Delta}}$. Now this implies that $\frac{1}{(A_1)_a}\widetilde{U}(A_1)_b - 1 \in \mathcal{W}_{\sqrt{\Delta}}$. The inequality $|x^\alpha - 1| \leq C(\alpha)|x-1|\max(x^\alpha, 1)$ for $x > 0$, $\alpha \in \mathbb{R}$ implies that we have $\left(\frac{1}{(A_1)_a}\widetilde{U}(A_1)_b\right)^\alpha - 1 \in \mathcal{W}_{\sqrt{\Delta}}$. Choosing suitable values of α imply all the other estimates.

9. $\Delta(\Theta) \in L^2_{\sqrt{\Delta}}$ and $\Delta(D_t\Theta) \in L^2_{\sqrt{\Delta}}$

   Proof: The proof of $\Delta(\Theta) \in L^2_{\sqrt{\Delta}}$ follows the same was as it was shown in §4.1 and $\Delta(D_t\Theta) \in L^2_{\sqrt{\Delta}}$ as it part of E_Δ.

10. $\frac{1}{|Z_{,\alpha'}|_a}\Delta(\Theta) \in \mathcal{C}_{\sqrt{\Delta}}, \Delta\left(\frac{\Theta}{|Z_{,\alpha'}|}\right) \in \mathcal{C}_{\sqrt{\Delta}}$

    Proof: The energy E_Δ controls $\frac{\sqrt{(A_1)_a}}{|Z_{,\alpha'}|_a}\Delta(\Theta) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and hence by using $\frac{1}{\sqrt{(A_1)_a}} \in \mathcal{W}_{\sqrt{\Delta}}$ we have $\frac{1}{|Z_{,\alpha'}|_a}\Delta(\Theta) \in \mathcal{C}_{\sqrt{\Delta}}$ The other one is obtained by using Lemma 5.3.5.

From now on we will just state the estimates for which the proof follows exactly as in §4.1 and can be easily obtained by using Lemma 5.3.2, 5.3.5 and 5.3.6 as shown in the above examples. For estimates which do not follow this pattern we give more details.

11. $\Delta\left(D_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \in \mathcal{C}_{\sqrt{\Delta}}, \Delta\left(|D_{\alpha'}|\frac{1}{Z_{,\alpha'}}\right) \in \mathcal{C}_{\sqrt{\Delta}}, \Delta\left(|D_{\alpha'}|\frac{1}{|Z_{,\alpha'}|}\right) \in \mathcal{C}_{\sqrt{\Delta}}$ and similarly we have $\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\omega\right) \in \mathcal{C}_{\sqrt{\Delta}}$

12. $\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and hence $\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in \mathcal{C}_{\sqrt{\Delta}}$

13. $\Delta\left(\frac{1}{|Z_{,\alpha'}|^3}\partial_{\alpha'}^2 A_1\right) \in L^2_{\sqrt{\Delta}}$, $|D_{\alpha'}|\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in L^2_{\sqrt{\Delta}}$ and $\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in \mathcal{W}_{\sqrt{\Delta}}$

14. $\Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and $\Delta(\mathbb{H}(b_{\alpha'})) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

15. $\Delta(|D_{\alpha'}|b_{\alpha'}) \in L^2_{\sqrt{\Delta}}$ and hence $\Delta(b_{\alpha'}) \in \mathcal{W}_{\sqrt{\Delta}}$

16. $\Delta\left\{\partial_{\alpha'}D_t\frac{1}{Z_{,\alpha'}}\right\} \in L^2_{\sqrt{\Delta}}$, $\Delta\left\{D_t\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right\} \in L^2_{\sqrt{\Delta}}$

17. $\Delta(\overline{Z}_{tt,\alpha'}) \in L^2_{\sqrt{\Delta}}$

18. $\Delta(\overline{D}_{\alpha'}\overline{Z}_{tt}) \in \mathcal{C}_{\sqrt{\Delta}}$, $\Delta(|D_{\alpha'}|\overline{Z}_{tt}) \in \mathcal{C}_{\sqrt{\Delta}}$, $\Delta(D_t\overline{D}_{\alpha'}\overline{Z}_t) \in \mathcal{C}_{\sqrt{\Delta}}$ and $\Delta(D_t|D_{\alpha'}|\overline{Z}_t) \in \mathcal{C}_{\sqrt{\Delta}}$

19. $\Delta(D_tA_1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

20. $\Delta(D_t(b_{\alpha'} - D_{\alpha'}Z_t - \overline{D}_{\alpha'}\overline{Z}_t)) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and hence $\Delta(D_tb_{\alpha'}) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$, $\Delta(\partial_{\alpha'}D_tb) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

Now we start controlling terms with surface tension. Note that these estimates are only for the solution A and hence the estimates have already been shown in §4.1. For most of the estimates we will have that the power of σ will be the same as that of the power of Δ. For e.g. we have $\left(\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$ and both σ and Δ are raised to the same power $1/6$. However the estimates derived from $\sigma\partial_{\alpha'}\Theta, \sigma\partial_{\alpha'}D_{\alpha'}\Theta$ and $\sigma D^2_{\alpha'}\Theta$ will not follow this pattern. For e.g. we have $(\sigma^{\frac{1}{3}}\Theta) \in L^\infty_{\Delta^{\frac{1}{6}}}$ and not $(\sigma^{\frac{1}{3}}\Theta) \in L^\infty_{\Delta^{\frac{1}{3}}}$. The reason is that E_Δ controls $\Delta\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ not $\dot{H}^{\frac{1}{2}}_{\Delta}$ and hence we have $(\sigma\partial_{\alpha'}\Theta) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. Similarly for $\sigma\partial_{\alpha'}D_{\alpha'}\Theta$ and $\sigma D^2_{\alpha'}\Theta$.

21. $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^\infty_{\sqrt{\Delta}}$, $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^\infty_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right)_a \in L^\infty_{\sqrt{\Delta}}$ and $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\mathrm{Re}\Theta\right)_a \in L^\infty_{\sqrt{\Delta}}$

22. $\left(\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$, $\left(\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$, $\left(\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$

23. $(\sigma\partial_{\alpha'}\Theta)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

Proof: Note carefully that we claim the estimate $(\sigma\partial_{\alpha'}\Theta)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and not $(\sigma\partial_{\alpha'}\Theta)_a \in \dot{H}^{\frac{1}{2}}_{\Delta}$. From the fundamental equation (3.7) we have

$$\Delta\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big) = -i\Delta(A_1) + (\sigma\partial_{\alpha'}\Theta)_a$$

and we know that E_Δ controls $\Delta\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. As $\Delta(A_1) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$, we obtain the above estimate.

24. $\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$

25. $\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$, $\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$, $\left(\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$ and similarly we have $\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}|D_{\alpha'}|\omega\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$

Proof: Note carefully that we have $L^2_{\Delta^{\frac{1}{3}}}$ and not $L^2_{\Delta^{\frac{2}{3}}}$ in the above estimate. First see that $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^2$ as $(E_\sigma)_a(t) \leq C(M)$ and $(E_\sigma)_a$ controls it. But we have already shown above that $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\sqrt{\Delta}}$. Hence we have $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\Delta^\beta}$ for all $0 \leq \beta \leq 1/2$. In a similar way we can show that $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^2_{\Delta^\beta}$ etc. for all $0 \leq \beta \leq 1/2$. Due to this argument, the proof of the above estimates follow in the same way as is shown in §4.1.

26. $(\sigma^{\frac{1}{3}}\Theta)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$

27. $\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$, $\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$, $\left(\sigma^{\frac{1}{3}}|D_{\alpha'}|\omega\right)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$

28. $(\sigma\partial_{\alpha'}D_{\alpha'}\Theta)_a \in L^2_{\sqrt{\Delta}}$, $(\sigma|D_{\alpha'}|\partial_{\alpha'}\Theta)_a \in L^2_{\sqrt{\Delta}}$, $(\sigma\partial_{\alpha'}|D_{\alpha'}|\Theta)_a \in L^2_{\sqrt{\Delta}}$

29. $\left(\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^3\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\omega\right)_a \in L^2_{\sqrt{\Delta}}$

30. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\omega\right)_a \in L^2_{\sqrt{\Delta}}$ and also $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right)_a \in L^2_{\sqrt{\Delta}}$

31. $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in \mathcal{W}_{\sqrt{\Delta}}$, $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in \mathcal{W}_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\omega\right)_a \in \mathcal{W}_{\sqrt{\Delta}}$ and $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\mathrm{Re}\Theta\right)_a \in \mathcal{W}_{\sqrt{\Delta}}$

32. $\left(\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right)_a \in L^\infty_{\Delta^{\frac{5}{12}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{5}{12}}}$

33. $\left(\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in L^\infty_{\Delta^{\frac{5}{12}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{5}{12}}}$, $\left(\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial^2_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in L^\infty_{\Delta^{\frac{5}{12}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{5}{12}}}$ and similarly $\left(\frac{\sigma^{\frac{5}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial^2_{\alpha'}\omega\right)_a \in L^\infty_{\Delta^{\frac{5}{12}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{5}{12}}}$

34. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$

35. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial^2_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial^2_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial^2_{\alpha'}\omega\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$

36. $\left(\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, $\left(\sigma D^2_{\alpha'}\Theta\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, $\left(\sigma|D_{\alpha'}|^2\Theta\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, $\left(\frac{\sigma}{|Z_{,\alpha'}|^2}\partial^2_{\alpha'}\Theta\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$

37. $\left(\frac{\sigma}{|Z_{,\alpha'}|^2}\partial^3_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, $\left(\frac{\sigma}{|Z_{,\alpha'}|^2}\partial^3_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, $\left(\sigma\partial^3_{\alpha'}\frac{1}{|Z_{,\alpha'}|^3}\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$ and similarly $\left(\frac{\sigma}{|Z_{,\alpha'}|^3}\partial^3_{\alpha'}\omega\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$

38. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right)_a \in L^2_{\sqrt{\Delta}}$ and $\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\sqrt{\Delta}}$

39. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial^2_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial^2_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\sqrt{\Delta}}$ and also $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}|D_{\alpha'}|^2\overline{Z}_{t,\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}^2_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\sqrt{\Delta}}$

40. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right)_a \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}$ and similarly we have $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}\overline{Z}_t\right)_a \in \mathcal{W}_{\sqrt{\Delta}} \cap \mathcal{C}_{\sqrt{\Delta}}$

41. $\left(\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$, $\left(\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$, $\left(\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$

42. $\left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$, $\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}|D_{\alpha'}|\overline{Z}_t\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$, $\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$

43. $\left(\sigma^{\frac{1}{6}}\frac{\overline{Z}_{t,\alpha'}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\right)_a \in \mathcal{W}_{\Delta^{\frac{1}{6}}}$

44. $\left\{\sigma^{\frac{1}{6}}\partial_{\alpha'}\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}^{1/2}}\right)\right\}_a \in L^\infty_{\Delta^{\frac{1}{6}}}$

45. $\left(\frac{\sigma^{\frac{1}{3}}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in L^\infty_{\Delta^{\frac{1}{3}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{3}}}$

46. $\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}b_{\alpha'}\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$

47. $\left(\frac{\sigma^{\frac{1}{6}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right)_a \in L^2_{\Delta^{\frac{1}{6}}}$

48. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}b_{\alpha'}\right)_a \in L^\infty_{\sqrt{\Delta}}$

49. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial^2_{\alpha'}b_{\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$, $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|b_{\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$ and $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_{\alpha'}b_{\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$

50. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1\right)_a \in L^\infty_{\sqrt{\Delta}}$

51. $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial^2_{\alpha'}A_1\right)_a \in L^2_{\sqrt{\Delta}}$

52. $\Delta\big((\mathbb{I}-\mathbb{H})D_t^2\Theta\big) \in L^2_{\sqrt{\Delta}}, \Delta\big((\mathbb{I}-\mathbb{H})D_t^2\overline{Z}_{t,\alpha'}\big) \in L^2_{\sqrt{\Delta}}, \Delta\big((\mathbb{I}-\mathbb{H})D_t^2 D_{\alpha'}\overline{Z}_t\big) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

Proof: The proofs of the first two estimates follow in the same way as in §4.1. However the proof of $\Delta\big((\mathbb{I}-\mathbb{H})D_t^2 D_{\alpha'}\overline{Z}_t\big) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ requires more work. We use the identity from Proposition A.0.2 for the holomorphic function $D_{\alpha'}\overline{Z}_t$

$$
\begin{aligned}
&(\mathbb{I}-\mathbb{H})D_t^2 D_{\alpha'}\overline{Z}_t \\
&= 2\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right]\partial_{\alpha'}(D_t D_{\alpha'}\overline{Z}_t) - \left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right); \partial_{\alpha'} D_{\alpha'}\overline{Z}_t\right] \\
&\quad + \frac{1}{4}(\mathbb{I}-\mathbb{H})\left\{\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'}\right)[Z_t, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t\right\} - \frac{1}{4}(\mathbb{I}-\mathbb{H})\left\{\left([Z_t, \mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)^2 D_{\alpha'}\overline{Z}_t\right\} \\
&\quad + \frac{1}{2}\left[[Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\overline{Z}_{t,\alpha'}\right) + [Z_{tt}, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t
\end{aligned}
$$

Now we apply Δ to the above equation and handle each term individually. It is easy to see that

(a) $\Delta\left\{\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{H}\right]\partial_{\alpha'}(D_t D_{\alpha'}\overline{Z}_t)\right\} \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

(b) $\Delta\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right), \mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right); \partial_{\alpha'} D_{\alpha'}\overline{Z}_t\right] \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

(c) $\Delta\left(\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right] Z_{t,\alpha'}\right) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

(d) $\Delta\big([Z_t, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t\big) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

(e) $\Delta\left([Z_t, \mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

(f) $\Delta(D_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

(g) $\partial_{\alpha'}\Delta\left([Z_t, [Z_t, \mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) \in L^2_{\sqrt{\Delta}}$

(h) $\partial_{\alpha'}\Delta\left(\frac{1}{Z_{,\alpha'}^2}\overline{Z}_{t,\alpha'}\right) \in L^2_{\sqrt{\Delta}}$.

(i) $\Delta\big([Z_{tt}, \mathbb{H}]D_{\alpha'}^2\overline{Z}_t\big) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

Hence we have the required estimate.

53. $\big(\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\Theta\big)_a \in L^2_{\sqrt{\Delta}}, \big(\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\big)_a \in L^2_{\sqrt{\Delta}}, \big(\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3 D_{\alpha'}\overline{Z}_t\big)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

54. $\Delta\left\{\left[D_t^2, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\} \in \mathcal{C}_{\sqrt{\Delta}}, \Delta\left\{\left[D_t^2, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\} \in \mathcal{C}_{\sqrt{\Delta}}$

55. $\Delta\left\{\left[i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\} \in \mathcal{C}_{\sqrt{\Delta}}, \Delta\left\{\left[i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\} \in \mathcal{C}_{\sqrt{\Delta}}$

56. $\left\{(\mathbb{I}-\mathbb{H})\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}, \left\{(\mathbb{I}-\mathbb{H})\left[i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and we also have $\left\{|Z_{,\alpha'}|\left[i\sigma|D_{\alpha'}|^3, \frac{1}{Z_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}_a \in L^2_{\sqrt{\Delta}}, \left\{|Z_{,\alpha'}|\left[i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\right\}_a \in L^2_{\sqrt{\Delta}}$

57. $\Delta(R_1) \in \mathcal{C}_{\sqrt{\Delta}}$

58. $\Delta(J_1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

59. $\Delta(|D_{\alpha'}|J_1) \in L^2_{\sqrt{\Delta}}$ and hence $\Delta(J_1) \in \mathcal{W}_{\sqrt{\Delta}}$

60. $\Delta(R_2) \in L^2_{\sqrt{\Delta}}$

61. $\Delta(J_2) \in L^2_{\sqrt{\Delta}}$

62. $\left(\sigma\left[\frac{1}{Z_{,\alpha'}}, \mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}, \left(\sigma\left[\frac{1}{\overline{Z}_{,\alpha'}}, \mathbb{H}\right]|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

63. $\Delta\big\{(\mathbb{I}-\mathbb{H})D_t^2\overline{D}_{\alpha'}\overline{Z}_t\big\} \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

64. $\big(\sigma(\mathbb{I}-\mathbb{H})|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\big)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$

65. $\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and hence $\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1\right) \in \mathcal{C}_{\sqrt{\Delta}}$

5.3.2 Closing the energy estimate for E_Δ

Analogous to the energy estimate of E_σ, we simplify the calculations by the use of the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a constant $C(M)$ depending only on M (where M was defined in Theorem 5.0.1) with $|a(t) - b(t)| \leq C(M)E_\Delta(t)$.

Observe that $\approx$ is an equivalence relation. With this notation, proving Theorem 5.0.1 is equivalent to showing $\frac{dE_\Delta(t)}{dt} \approx 0$. Note that we have already shown in that

$$\frac{d}{dt}E_{\lambda,aux} \leq P(\mathcal{E}_{high})E_{\lambda,aux}$$

Hence by plugging in $\lambda = \sigma$ and noting that $(\mathcal{E}_{high})_b$ is controlled by $C(M)$ we have

$$\frac{d}{dt}(E_{\sigma,aux})_b \leq P((\mathcal{E}_{high})_b)(E_{\sigma,aux})_b \leq C(M)E_\Delta$$

Now we control the other components of E_Δ.

Controlling $E_{\Delta,0}$

We recall that

$$\begin{aligned} E_{\Delta,0} = &\left\|\left(\sigma^{\frac{1}{2}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_\infty^2 + \left\|\left(\sigma^{\frac{1}{6}}|Z_{,\alpha'}|^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^6 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^2 \\ &+ \|\Delta(\omega)\|_\infty^2 + \left\|\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2^2 + \left\|\widetilde{h}_{\alpha'} - 1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 + \left\||D_{\alpha'}|_a(\widetilde{h}_{\alpha'} - 1)\right\|_2^2 \\ &+ \left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|_b}\right) - 1\right\|_\infty^2 \end{aligned}$$

The time derivatives of the first three quantities follow the same way as was done in controlling the time derivative of $E_{\sigma,0}$. Now we control the other quantities

1. We observe that

$$(D_t)_a\Delta(\omega) = \Delta(D_t\omega) = -\Delta\big(\mathrm{Im}(\overline{D}_{\alpha'}\overline{Z}_t)\big) = -\mathrm{Im}\big(\Delta(\overline{D}_{\alpha'}\overline{Z}_t)\big) \in L^\infty_{\sqrt{\Delta}}$$

   Hence by Proposition A.0.13 we have

$$\frac{d}{dt}\|\Delta(\omega)\|_\infty^2 \lesssim \|\Delta\omega\|_\infty\|(D_t)_a\Delta(\omega)\|_\infty \leq C(M)E_\Delta$$

2. By using Lemma 2.4.6 we obtain

$$\begin{aligned} \frac{d}{dt}\left\|\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2^2 &\lesssim \|(b_{\alpha'})_a\|_\infty\left\|\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2^2 + \left\|\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2\left\|(D_t)_a\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2 \\ &\leq C(M)E_\Delta \end{aligned}$$

3. By the calculation of Lemma 5.3.3 we have

$$(D_t)_a \widetilde{h}_{\alpha'} = \widetilde{h}_{\alpha'}(\widetilde{U}(b_{\alpha'})_b - (b_{\alpha'})_a) = -\widetilde{h}_{\alpha'}\Delta(b_{\alpha'})$$

As $\Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ we have that $(D_t)_a\widetilde{h}_{\alpha'} \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. Hence by Proposition A.0.13 and Lemma 2.4.6 we have

$$\frac{d}{dt}\left\|\widetilde{h}_{\alpha'} - 1\right\|^2_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \leq C(M)\left\|\widetilde{h}_{\alpha'} - 1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}\left\|(D_t)_a\widetilde{h}_{\alpha'}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \leq C(M)E_\Delta$$

4. By the calculation of Lemma 5.3.3 we have

$$\begin{aligned}(D_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'} &= -\mathrm{Re}(D_{\alpha'}Z_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'} + (|D_{\alpha'}|_a\widetilde{h}_{\alpha'})(\widetilde{U}(b_{\alpha'})_b - (b_{\alpha'})_a) \\ &\quad + \widetilde{h}_{\alpha'}\left\{\widetilde{h}_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\right)\widetilde{U}(|D_{\alpha'}|b_{\alpha'})_b - (|D_{\alpha'}|b_{\alpha'})_a\right\}\end{aligned}$$

Now as $|D_{\alpha'}|_a\widetilde{h}_{\alpha'} \in L^2_{\sqrt{\Delta}}$, $\widetilde{h}_{\alpha'} - 1 \in L^\infty_{\sqrt{\Delta}}$, $\dfrac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b - 1 \in L^\infty_{\sqrt{\Delta}}$ and $\Delta(|D_{\alpha'}|b_{\alpha'}) \in L^2_{\sqrt{\Delta}}$ we see that $(D_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'} \in L^2_{\sqrt{\Delta}}$. Hence by Lemma 2.4.6

$$\frac{d}{dt}\left\||D_{\alpha'}|_a(\widetilde{h}_{\alpha'} - 1)\right\|_2^2 \leq C(M)\left\||D_{\alpha'}|_a(\widetilde{h}_{\alpha'} - 1)\right\|_2\left\|(D_t)_a|D_{\alpha'}|_a\widetilde{h}_{\alpha'}\right\|_2 \leq C(M)E_\Delta$$

5. By the calculation of Lemma 5.3.3 we have

$$\begin{aligned}&(D_t)_a\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\} \\ &= \left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\mathrm{Re}\left\{(D_{\alpha'}Z_t)_a - \widetilde{U}(D_{\alpha'}Z_t)_b - (b_{\alpha'})_a + \widetilde{U}(b_{\alpha'})_b\right\}\end{aligned}$$

Now as $\Delta(D_{\alpha'}Z_t) \in L^\infty_{\sqrt{\Delta}}$ and $\Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}}$ we see that $(D_t)_a\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\} \in L^\infty_{\sqrt{\Delta}}$. Hence by Proposition A.0.13 we have

$$\begin{aligned}\frac{d}{dt}\left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|_b}\right) - 1\right\|_\infty^2 &\leq \left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|_b}\right) - 1\right\|_\infty\left\|(D_t)_a\left\{|Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|}\right)_b\right\}\right\|_\infty \\ &\leq C(M)E_\Delta\end{aligned}$$

Controlling $E_{\Delta,1}$

Recall that

$$E_{\Delta,1} = \left\|\Delta\left\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\right\}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|(\sqrt{A_1})_a\Delta(\overline{Z}_{t,\alpha'})\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a\right\|_2^2$$

We will first simplify the time derivative of each of the individual terms before combining them.

1. By using Lemma 2.4.6 we get

$$\begin{aligned}&\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\Delta\left\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\right\}\right|^2 d\alpha' \\ &\approx 2\text{Re}\int\left\{|\partial_{\alpha'}|\Delta\big((Z_{tt} + i)\overline{Z}_{,\alpha'}\big)\right\}(D_t)_a\Delta\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big)d\alpha'\end{aligned}$$

Now $(D_t)_a\Delta\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big) = \Delta\big(D_t(\overline{Z}_{tt} - i)Z_{,\alpha'}\big)$ and we have

$$\begin{aligned}D_t\big((\overline{Z}_{tt} - i)Z_{,\alpha'}\big) &= \overline{Z}_{ttt}Z_{,\alpha'} + (D_{\alpha'}Z_t - b_{\alpha'})(\overline{Z}_{tt} - i)Z_{,\alpha'} \\ &= \overline{Z}_{ttt}Z_{,\alpha'} + (D_{\alpha'}Z_t - b_{\alpha'})(-iA_1 + \sigma\partial_{\alpha'}\Theta)\end{aligned}$$

Now applying Δ above and working as in the proof of $E_{\sigma,1}$ we obtain

$$\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\Delta\left\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\right\}\right|^2 d\alpha' \approx 2\text{Re}\int\left\{|\partial_{\alpha'}|\Delta\big((Z_{tt} + i)\overline{Z}_{,\alpha'}\big)\right\}\Delta\big(\overline{Z}_{ttt}Z_{,\alpha'}\big)d\alpha'$$

2. We see that

$$\begin{aligned}&\frac{d}{dt}\int(A_1)_a\left|\Delta(\overline{Z}_{t,\alpha'})\right|^2 d\alpha' \\ &= \int(b_{\alpha'}A_1 + D_tA_1)_a\left|\Delta(\overline{Z}_{t,\alpha'})\right|^2 d\alpha' + 2\text{Re}\int(A_1)_a\Delta(\overline{Z}_{t,\alpha'})\Delta(-b_{\alpha'}Z_{t,\alpha'} + Z_{tt,\alpha'})\, d\alpha' \\ &\leq C(M)E_\Delta\end{aligned}$$

3. By following the proof of time derivative of $E_{\sigma,1}$ we have

$$
\begin{aligned}
&\sigma\frac{d}{dt}\int\left|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a\right|^2 d\alpha' \\
&\approx 2\mathrm{Re}\int\left\{-i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\}_a |\partial_{\alpha'}|\big((Z_{tt}+i)\overline{Z}_{,\alpha'}\big)_a d\alpha' \\
&= 2\mathrm{Re}\int\left\{-i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\}_a |\partial_{\alpha'}|\Delta\big((Z_{tt}+i)\overline{Z}_{,\alpha'}\big) d\alpha' \\
&\quad + 2\mathrm{Re}\int\left\{-i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)\right\}_a |\partial_{\alpha'}|\widetilde{U}\big((Z_{tt}+i)\overline{Z}_{,\alpha'}\big)_b d\alpha'
\end{aligned}
$$

We now show that the second term is controlled. Observe that $((Z_{tt}+i)\overline{Z}_{,\alpha'})_b = i(A_1)_b$ and that $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$. Hence we only need to show that $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}|\partial_{\alpha'}|\widetilde{U}(A_1)_b \in L^2_{\sqrt{\Delta}}$. Now

$$
\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}|\partial_{\alpha'}|\widetilde{U}(A_1)_b = i\left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}},\mathbb{H}\right]\partial_{\alpha'}^2\widetilde{U}(A_1)_b + i\mathbb{H}\left\{\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}^2\widetilde{U}(A_1)_b\right\}
$$

The first term is easily shown to be in $L^2_{\sqrt{\Delta}}$ as

$$
\left\|\left[\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}},\mathbb{H}\right]\partial_{\alpha'}^2\widetilde{U}(A_1)_b\right\|_2 \lesssim \left\|\sigma^{\frac{1}{2}}\partial_{\alpha'}^2\frac{1}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\right\|_2\left\|\widetilde{U}(A_1)_b\right\|_\infty
$$

Hence it is enough to show that $\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}^2\widetilde{U}(A_1)_b \in L^2_{\sqrt{\Delta}}$. We see that

$$
\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}^2\widetilde{U}(A_1)_b = \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}\left(\widetilde{h}_{\alpha'}\widetilde{U}(\partial_{\alpha'}A_1)_b\right)
$$

and hence we have the estimate

$$
\begin{aligned}
&\left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}^2\widetilde{U}(A_1)_b\right\|_2 \\
&\leq C(M)\big\||D_{\alpha'}|_a\widetilde{h}_{\alpha'}\big\|_2\left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1\right)_b\right\|_\infty + C(M)\left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2A_1\right)_b\right\|_2
\end{aligned}
$$

Now $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}A_1\right)_b \in L^2_{\sqrt{\Delta}}$ and $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2 A_1\right)_b \in L^2_{\sqrt{\Delta}}$ as they are part of $(E_{\sigma,aux})_b$. Hence we have shown

$$\begin{aligned}
&\sigma\frac{d}{dt}\int\left|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a\right|^2 d\alpha' \\
&\approx 2\text{Re}\int\left\{-i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)_a\right\}|\partial_{\alpha'}|\Delta\big((Z_{tt}+i)\overline{Z}_{,\alpha'}\big)d\alpha'
\end{aligned}$$

4. Now combining the terms we have

$$\frac{d}{dt}E_{\Delta,1} \approx 2\text{Re}\int\left\{\Delta\big(\overline{Z}_{ttt}Z_{,\alpha'}\big) - i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)_a\right\}|\partial_{\alpha'}|\Delta\big((Z_{tt}+i)\overline{Z}_{,\alpha'}\big)d\alpha'$$

Recall from (3.18) that

$$\begin{aligned}
&\overline{Z}_{ttt}Z_{,\alpha'} + iA_1\overline{D}_{\alpha'}\overline{Z}_t - i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right) \\
&= i\sigma\partial_{\alpha'}\left\{\left(|D_{\alpha'}|\frac{1}{\overline{Z}_{,\alpha'}}\right)\overline{Z}_{t,\alpha'}\right\} - \sigma(D_{\alpha'}Z_t)\partial_{\alpha'}\Theta - \sigma\partial_{\alpha'}\big\{(\text{Re}\Theta)\overline{D}_{\alpha'}\overline{Z}_t\big\} - iJ_1
\end{aligned}$$

Now we just apply Δ to the above equation and control the quantities. We see that $\Delta(J_1) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$, $\Delta(A_1\overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and the other terms with σ are controlled as in the proof of $E_{\sigma,1}$. Hence

$$\Delta\big(\overline{Z}_{ttt}Z_{,\alpha'}\big) - i\sigma\partial_{\alpha'}\left(\frac{1}{\overline{Z}_{,\alpha'}}|D_{\alpha'}|\overline{Z}_{t,\alpha'}\right)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$$

and hence we have shown that $\frac{d}{dt}E_{\Delta,1} \leq C(M)E_\Delta$.

Controlling $E_{\Delta,2}$ and $E_{\Delta,3}$

Note that both $E_{\Delta,2}$ and $E_{\Delta,3}$ are of the form

$$E_{\Delta,3} = \|\Delta(D_t f)\|_2^2 + \left\|\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a\Delta(f)\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2$$

Where $f = \overline{Z}_{t,\alpha'}$ for $i = 2$ and $f = \Theta$ for $i = 3$. Also note that $\mathbb{P}_H f = f$ for these choices of f. We will simplify the time derivative of each of the terms individually before combining them.

1. From Lemma 2.4.6 we have

$$\frac{d}{dt}\int |\Delta(D_t f)|^2 \, d\alpha' \approx 2\text{Re}\int (\Delta(D_t^2 f))\Delta(D_t\bar{f})\, d\alpha'$$

2. By following the proof of time derivative of $E_{\sigma,2}, E_{\sigma,3}$ we have

$$\begin{aligned}&\frac{d}{dt}\int \left| |\partial_{\alpha'}|^{\frac{1}{2}}\left\{\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \Delta(f)\right\}\right|^2 d\alpha' \\ &\approx 2\text{Re}\int \left\{\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a |\partial_{\alpha'}|\left(\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \Delta(f)\right)\right\}\Delta(D_t\bar{f})\, d\alpha'\end{aligned}$$

Now we see that

$$\begin{aligned}\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a |\partial_{\alpha'}|\left(\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \Delta(f)\right) &\approx_{L^2_{\sqrt{\Delta}}} i\mathbb{H}\left\{\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \partial_{\alpha'}\left(\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \Delta(f)\right)\right\} \\ &\approx_{L^2_{\sqrt{\Delta}}} i\mathbb{H}\left\{\left(\frac{A_1}{|Z_{,\alpha'}|^2}\right)_a \partial_{\alpha'}\Delta(f)\right\}\end{aligned}$$

Now using Lemma 5.3.5 we have

$$\begin{aligned}i\mathbb{H}\left\{\left(\frac{A_1}{|Z_{,\alpha'}|^2}\right)_a \partial_{\alpha'}\Delta(f)\right\} &\approx_{L^2_{\sqrt{\Delta}}} i\mathbb{H}\left\{\Delta\left(\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right)\right\} \\ &\approx_{L^2_{\sqrt{\Delta}}} i\mathbb{H}\left\{\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right\}_a - i\mathbb{H}\left\{\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right\}_b\end{aligned}$$

Now as $\left\{\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right\}_b \in L^2$ we can replace $\mathbb{H}$ in the second term with $\mathcal{H}$. Hence we have

$$\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a |\partial_{\alpha'}|\left(\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \Delta(f)\right) \approx_{L^2_{\sqrt{\Delta}}} \Delta\left\{i\mathbb{H}\left(\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right)\right\}$$

We can simplify the above term by using $\mathbb{H}f = f$. We see that

$$i\mathbb{H}\left(\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right) = -i\left[\frac{A_1}{|Z_{,\alpha'}|^2}, \mathbb{H}\right]\partial_{\alpha'}f + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f$$

Now apply Δ to the above equation. We can easily control the first term and hence we have

$$\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a |\partial_{\alpha'}|\left(\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \Delta(f)\right) \approx_{L^2_{\sqrt{\Delta}}} \Delta\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} f\right)$$

Finally using this we obtain

$$\frac{d}{dt}\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left\{\left(\frac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a \Delta(f)\right\}\right|^2 d\alpha' \approx 2\mathrm{Re}\int \Delta\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} f\right)\Delta(D_t\bar{f})\, d\alpha'$$

3. By using the argument in controlling $E_{\sigma,2}, E_{\sigma,3}$ we have

$$\begin{aligned}
&\frac{d}{dt}\sigma\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right)_a\right|^2 d\alpha' \\
&\approx -2\sigma\mathrm{Re}\int \partial_{\alpha'}\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}|\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right)\right\}_a (D_t\bar{f})_a\, d\alpha' \\
&\approx -2\sigma\mathrm{Re}\int \partial_{\alpha'}\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}|\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right)\right\}_a \Delta(D_t\bar{f})\, d\alpha' \\
&\quad - 2\sigma\mathrm{Re}\int \partial_{\alpha'}\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}|\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right)\right\}_a \widetilde{U}(D_t\bar{f})_b\, d\alpha'
\end{aligned}$$

We now show that the second term is controlled by using $(E_{\sigma,aux})_b$. We see that

$$\begin{aligned}
&- 2\sigma\mathrm{Re}\int \partial_{\alpha'}\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}|\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right)\right\}_a \widetilde{U}(D_t\bar{f})_b\, d\alpha' \\
&= 2\sigma\mathrm{Re}\int |\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} f\right)_a\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}_a}\partial_{\alpha'}\widetilde{U}(D_t\bar{f})_b\right) d\alpha'
\end{aligned}$$

Now we know that $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} D_t\Theta\right)_b \in \mathcal{C}_{\sqrt{\Delta}}$ and $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'} D_t\overline{Z}_{t,\alpha'}\right)_b \in \mathcal{C}_{\sqrt{\Delta}}$ as they are both controlled by $(E_{\sigma,aux})_b$. Hence we also have that $\dfrac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}_a}\partial_{\alpha'}\widetilde{U}(D_t\overline{Z}_{t,\alpha'})_b \in \mathcal{C}_{\sqrt{\Delta}}$ and

$\frac{1}{|Z_{,\alpha'}|_a^{\frac{3}{2}}}\partial_{\alpha'}\widetilde{U}(D_t\Theta)_b \in \mathcal{C}_{\sqrt{\Delta}}$ by using Lemma 5.3.5. Therefore we now have

$$\begin{aligned}&\frac{d}{dt}\sigma\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right)_a\right|^2 d\alpha'\\ &\approx -2\sigma\mathrm{Re}\int \partial_{\alpha'}\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}|\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right)\right\}_a \Delta(D_t\bar{f})\,d\alpha'\end{aligned}$$

Now using the proof in $E_{\sigma,2}, E_{\sigma,3}$ we obtain

$$\sigma\partial_{\alpha'}\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}|\partial_{\alpha'}|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right)\right\}_a \approx_{L^2_{\sqrt{\Delta}}} \left(i\sigma|D_{\alpha'}|^3 f\right)_a$$

So we finally have

$$\frac{d}{dt}\sigma\int\left||\partial_{\alpha'}|^{\frac{1}{2}}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}f\right)_a\right|^2 d\alpha' \approx -2\int\left(i\sigma|D_{\alpha'}|^3 f\right)_a\Delta(D_t\bar{f})\,d\alpha'$$

4. Now combining all three terms we have

$$\frac{d}{dt}E_{\Delta,i} \approx 2\mathrm{Re}\int\left\{\Delta(D_t^2 f) + \Delta\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}f\right) - i\sigma(|D_{\alpha'}|^3 f)_a\right\}\Delta(D_t\bar{f})\,d\alpha'$$

For $f = \overline{Z}_{t,\alpha'}$ we obtain from (3.21)

$$\begin{aligned}&\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\overline{Z}_{t,\alpha'}\\ &= R_1\overline{Z}_{,\alpha'} - i\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1 - iD_{\alpha'}J_1 - \overline{Z}_{,\alpha'}\left[D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3, \frac{1}{\overline{Z}_{,\alpha'}}\right]\overline{Z}_{t,\alpha'}\end{aligned}$$

Hence applying Δ on both sides, we easily see that the terms on the right hand side are in $L^2_{\sqrt{\Delta}}$. Similarly for $f = \Theta$ we have from (3.22)

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\Theta = R_2 + iJ_2$$

In this case also we apply Δ on both sides and see that the terms on the right are controlled.

Hence we have shown that for $i = 2, 3$ we have

$$\frac{d}{dt} E_{\Delta,i} \leq C(M) E_{\Delta}$$

Controlling $E_{\Delta,4}$

Recall that

$$E_{\Delta,4} = \left\| \Delta(D_t \overline{D}_{\alpha'} \overline{Z}_t) \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\| (\sqrt{A_1})_a |D_{\alpha'}|_a \Delta(\overline{D}_{\alpha'} \overline{Z}_t) \right\|_2^2 + \left\| \left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t \right)_a \right\|_{\dot{H}^{\frac{1}{2}}}^2$$

We again simply the terms individually before combining them

1. By Lemma 2.4.6 we have

$$\frac{d}{dt} \int \left| |\partial_{\alpha'}|^{\frac{1}{2}} \Delta\big(D_t \overline{D}_{\alpha'} \overline{Z}_t\big) \right|^2 d\alpha' \approx 2\mathrm{Re} \int \Delta(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t) |\partial_{\alpha'}| \Delta(D_t D_{\alpha'} Z_t) \, d\alpha'$$

   Now as $\Delta((\mathbb{I} - \mathbb{H}) D_t^2 \overline{D}_{\alpha'} \overline{Z}_t) \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ we see that

$$\Delta(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t) \approx_{\dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}} \Delta(\mathbb{H} D_t^2 \overline{D}_{\alpha'} \overline{Z}_t) \approx_{\dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}} \mathbb{H}(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t)_a - \mathcal{H} \widetilde{U}(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t)_b$$

   But we know that $(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t)_b \in \dot{H}^{\frac{1}{2}}$ as it is controlled by $(E_{high})_b$. Hence we now have $(\mathbb{H} - \mathcal{H}) \widetilde{U}(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t)_b \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. From this we get

$$\Delta(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t) \approx_{\dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}} \mathbb{H} \Delta(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t)$$

   Now we use the fact that $|\partial_{\alpha'}| = i \mathbb{H} \partial_{\alpha'}$ to obtain

$$\frac{d}{dt} \int \left| |\partial_{\alpha'}|^{\frac{1}{2}} \Delta\big(D_t \overline{D}_{\alpha'} \overline{Z}_t\big) \right|^2 d\alpha' \approx 2\mathrm{Re} \int \Delta(D_t^2 \overline{D}_{\alpha'} \overline{Z}_t) \{ -i \partial_{\alpha'} \Delta(D_t D_{\alpha'} Z_t) \} \, d\alpha'$$

2. By following the proof of control of $E_{\sigma,4}$ we see that

$$\begin{aligned} &\frac{d}{dt} \int (A_1)_a \left| |D_{\alpha'}|_a \Delta(\overline{D}_{\alpha'} \overline{Z}_t) \right|^2 d\alpha' \\ &\approx 2\mathrm{Re} \int \left(i \left(\frac{A_1}{|Z_{,\alpha'}|^2} \right)_a \partial_{\alpha'} \Delta(\overline{D}_{\alpha'} \overline{Z}_t) \right) \{ -i \partial_{\alpha'} \Delta(D_t D_{\alpha'} Z_t) \} \, d\alpha' \end{aligned}$$

Now we know that $\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)_b \in \mathcal{C}$ as it is controlled by $(E_{high})_b$. Hence as $(A_1)_b \in \mathcal{W}$, we have $\left(\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)_b \in \mathcal{C}$. Hence we see that

$$i\left(\frac{A_1}{|Z_{,\alpha'}|^2}\right)_a \partial_{\alpha'}\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \approx_{\dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}} \Delta\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)$$

From this we get

$$\frac{d}{dt}\int (A_1)_a \left||D_{\alpha'}|_a \Delta(\overline{D}_{\alpha'}\overline{Z}_t)\right|^2 d\alpha' \approx 2\mathrm{Re}\int \Delta\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right)\{-i\partial_{\alpha'}\Delta(D_tD_{\alpha'}Z_t)\}\, d\alpha'$$

3. By following the proof of control of $E_{\sigma,4}$ we see that

$$\begin{aligned}
&\sigma\frac{d}{dt}\int\left|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_a\right|^2 d\alpha' \\
&\approx 2\sigma\mathrm{Re}\int\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\}_a\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_tD_{\alpha'}Z_t\right\}_a \\
&\approx 2\sigma\mathrm{Re}\int\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\}_a\left\{\frac{1}{|Z_{,\alpha'}|_a^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|_a\Delta(D_tD_{\alpha'}Z_t)\right\} \\
&\quad + 2\sigma\mathrm{Re}\int\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\}_a\left\{\frac{1}{|Z_{,\alpha'}|_a^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|_a\widetilde{U}(D_tD_{\alpha'}Z_t)_b\right\}
\end{aligned}$$

We now show that the second term is controlled. We first observe that

$$\begin{aligned}
&\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|_a\widetilde{U}(D_tD_{\alpha'}Z_t)_b \\
&= \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{1}{2}}}\partial_{\alpha'}\left(\frac{\widetilde{h}_{\alpha'}}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\widetilde{U}(|D_{\alpha'}|D_tD_{\alpha'}Z_t)_b\right) \\
&= |Z_{,\alpha'}|_a^{\frac{1}{2}}\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\right)_b|D_{\alpha'}|_a\left(\frac{\widetilde{h}_{\alpha'}}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\right)\widetilde{U}\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_tD_{\alpha'}Z_t\right)_b \\
&\quad + \frac{\widetilde{h}_{\alpha'}}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|_a^{\frac{1}{2}}}\partial_{\alpha'}\widetilde{U}(|D_{\alpha'}|D_tD_{\alpha'}Z_t)_b\right)
\end{aligned}$$

Now we know that $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}D_tD_{\alpha'}Z_t\right)_b \in L^\infty_{\sqrt{\Delta}}$ and $\left(\frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|D_tD_{\alpha'}Z_t\right)_b \in L^2_{\sqrt{\Delta}}$ as they are controlled by $(E_{\sigma,aux})_b$. We also know that $\widetilde{h}_{\alpha'}\mathcal{W}$, $\dfrac{1}{|Z_{,\alpha'}|_a}\widetilde{U}(|Z_{,\alpha'}|)_b \in \mathcal{W}$ and hence the above terms are controlled. Hence we have

$$
\begin{aligned}
&\sigma\frac{d}{dt}\int\left|\left(\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_a\right|^2 d\alpha' \\
&\approx 2\sigma\mathrm{Re}\int\left\{\frac{1}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right\}_a\left\{\frac{1}{|Z_{,\alpha'}|_a^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|_a\Delta(D_tD_{\alpha'}Z_t)\right\} \\
&\approx 2\mathrm{Re}\int\left\{-i\sigma|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\right\}_a\{-i\partial_{\alpha'}\Delta(D_tD_{\alpha'}Z_t)\}\, d\alpha'
\end{aligned}
$$

4. Combining the three terms we obtain

$$
\begin{aligned}
\frac{d}{dt}E_{\Delta,4} \approx 2\mathrm{Re}\int\{-i\partial_{\alpha'}\Delta(D_tD_{\alpha'}Z_t)\}\Bigg\{\Delta(D_t^2\overline{D}_{\alpha'}\overline{Z}_t) + \Delta\left(i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\overline{D}_{\alpha'}\overline{Z}_t\right) \\
- i\sigma(|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t)_a\Bigg\}\, d\alpha'
\end{aligned}
$$

From equation (3.19) we see that

$$
\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'} - i\sigma|D_{\alpha'}|^3\right)\overline{D}_{\alpha'}\overline{Z}_t = R_1 - i\left(\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)J_1 - i\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}J_1
$$

Now we apply Δ to the above equation and see that the terms on the right hand side terms are controlled in $\dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. Hence we have

$$
\frac{d}{dt}E_{\Delta,4} \leq C(M)E_{\Delta}
$$

This concludes the proof of Theorem 5.0.1.

5.3.3 Equivalence of E_Δ and $\mathcal{E}_\Delta$

We now give a simpler description of the energy E_Δ. Define

$$
\mathcal{E}_{\Delta,0} = \|\Delta(\omega)\|_\infty^2 + \left\|\widetilde{h}_{\alpha'} - 1\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 + \left\||D_{\alpha'}|_a(\widetilde{h}_{\alpha'} - 1)\right\|_2^2 + \left\||Z_{,\alpha'}|_a\widetilde{U}\left(\frac{1}{|Z_{,\alpha'}|_b}\right) - 1\right\|_\infty^2
$$

$$\mathcal{E}_{\Delta,1} = \left\|\Delta\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_2^2 + \left\|\Delta\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\left(\sigma^{\frac{1}{6}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^6$$
$$+ \left\|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\right\|_\infty^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2$$
$$+ \left\|(\sigma\partial_{\alpha'}\Theta)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\left(\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_a\right\|_2^2 + \left\|\left(\frac{\sigma}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_a\right\|_{\dot{H}^{\frac{1}{2}}}^2$$
$$\mathcal{E}_{\Delta,2} = \left\|\Delta(\overline{Z}_{t,\alpha'})\right\|_2^2 + \left\|\Delta\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right)_a\right\|_2^2$$
$$\mathcal{E}_{\Delta} = (\mathcal{E}_{\sigma,aux})_b + \mathcal{E}_{\Delta,0} + \mathcal{E}_{\Delta,1} + \mathcal{E}_{\Delta,2}$$

where $(\mathcal{E}_{\sigma,aux})_b$ is the energy $\mathcal{E}_{\lambda,aux}$ defined in §5.2.5 with $\lambda = \sigma$ for the zero surface tension solution B. Hence this term couples the zero surface tension solution B with the coefficient of surface tension σ from the capillary gravity water wave solution A. Note that if the two solutions have the same initial data, then $\mathcal{E}_{\Delta,0}(0) = 0$ and hence we obtain the representation of the energy as stated in §3.4.

Proposition 5.3.7. *There exists universal polynomials P_1, P_2 with non-negative coefficients so that for smooth solutions to the water wave equation with no surface tension we have*

$$E_\Delta \leq P_1(\mathcal{E}_{high})\mathcal{E}_\Delta \quad \textit{and } \mathcal{E}_\Delta \leq P_2(\mathcal{E}_{high})E_\Delta$$

Proof. We will continue to use the same notation as in §5.3.1 except for a few minor modifications. In the definitions, instead of using the energy E_Δ we will use the energy $\mathcal{E}_\Delta$. So now whenever we write $f \in L^2_{\Delta^\alpha}$, what we mean is that there exists a constant $C(M)$ depending only on M such that $\|f\|_2 \leq C(M)(\mathcal{E}_\Delta)^\alpha$. Similar modifications for $f \in L^1_{\Delta^\alpha}$, $f \in \dot{H}^{\frac{1}{2}}_{\Delta^\alpha}$ and $f \in L^\infty_{\Delta^\alpha}$. The definitions of the spaces $\mathcal{C}_{\Delta^\alpha}$ and $\mathcal{W}_{\Delta^\alpha}$ remain the same except for the fact that we have now changed the underlying definition of the spaces $L^2_{\Delta^\alpha}, \dot{H}^{\frac{1}{2}}_{\Delta^\alpha}$ and $L^\infty_{\Delta^\alpha}$. Similarly the definitions of $\approx_{L^2_{\Delta^\alpha}}, \approx_{L^1_{\Delta^\alpha}}, \approx_{L^\infty_{\Delta^\alpha}}, \approx_{\dot{H}^{\frac{1}{2}}_{\Delta^\alpha}}, \approx_{\mathcal{W}_{\Delta^\alpha}}$ and $\approx_{\mathcal{C}_{\Delta^\alpha}}$ remain the same except the changes to the underlying spaces. Observe that there is no change to Lemma 5.3.4.

We now make the important observation that Lemma 5.3.5 still remains true with the new definitons. This is because in the proof of Lemma 5.3.5, the only properties of E_Δ used were the control of $(\widetilde{h}_{\alpha'} - 1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}} \cap \mathcal{W}_{\sqrt{\Delta}}$, $\Delta w \in L^\infty_{\sqrt{\Delta}}$, $\Delta\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right) \in L^2_{\sqrt{\Delta}}$ and the term $|Z_{,\alpha'}|_a\widetilde{U}\left(\dfrac{1}{|Z_{,\alpha'}|}\right)_b - 1 \in L^\infty_{\sqrt{\Delta}}$. All of the these quantities are also controlled by $\mathcal{E}_\Delta$ and hence the lemma still holds.

Now we already know that $(E_{\sigma,aux})_b \leq P_1(\mathcal{E}_{high})(\mathcal{E}_{\sigma,aux})_b$ by Proposition 5.2.4. Also $E_{\Delta,0}$ is clearly controlled by $\mathcal{E}_\Delta$. Hence we only need to show that $E_{\Delta,i}$ for $1 \leq i \leq 4$ is controlled by $\mathcal{E}_\Delta$ which we now prove.

1. Controlling $E_{\Delta,1}$: From $\mathcal{E}_{\Delta,2}$ we have $\Delta(\overline{Z}_{t,\alpha'}) \in L^2_{\sqrt{\Delta}}$. Hence we have $(\sqrt{A_1})_a\Delta(\overline{Z}_{t,\alpha'}) \in L^2_{\sqrt{\Delta}}$. Hence now via §5.3.1 we have $\Delta(A_1) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. Now we know from (3.7) that $(\overline{Z}_{tt} - i)Z_{,\alpha'} = -iA_1 + \sigma\partial_{\alpha'}\Theta$ and hence

$$\Delta\big\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\big\} = -i\Delta(A_1) + (\sigma\partial_{\alpha'}\Theta)_a$$

   As $(\sigma\partial_{\alpha'}\Theta)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ we see that $\Delta\big\{(\overline{Z}_{tt} - i)Z_{,\alpha'}\big\} \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. From $\mathcal{E}_\Delta$ we also clearly see that $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in L^2_{\sqrt{\Delta}}$ and hence $E_{\Delta,1}$ is controlled.

2. Controlling $E_{\Delta,2}$: We prove this step by step.

   (a) As $\mathcal{E}_\Delta$ controls $\Delta\left(\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right) \in L^2_{\sqrt{\Delta}}$, from §5.3.1 we easily obtain $\Delta\left(\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|}\right) \in L^2_{\sqrt{\Delta}}$, $\Delta(|D_{\alpha'}|\omega) \in L^2_{\sqrt{\Delta}}$ and $\Delta(\omega) \in \mathcal{W}_{\sqrt{\Delta}}$.

   (b) As $\mathcal{E}_\Delta$ controls $\Delta\left(\dfrac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right) \in L^2_{\sqrt{\Delta}}$, by using Lemma 5.3.5 repeatedly we also have $\dfrac{1}{(Z_{,\alpha'})_a^2}\partial_{\alpha'}\Delta(\overline{Z}_{t,\alpha'}) \in L^2_{\sqrt{\Delta}}$. Hence we see that

$$\begin{aligned}\partial_{\alpha'}\left(\frac{1}{(Z_{,\alpha'})_a}\Delta(\overline{Z}_{t,\alpha'})\right)^2 &= 2\left(\frac{1}{(Z_{,\alpha'})_a}\Delta(\overline{Z}_{t,\alpha'})\right)\partial_{\alpha'}\left(\frac{1}{(Z_{,\alpha'})_a}\Delta(\overline{Z}_{t,\alpha'})\right) \\ &= 2\left(\frac{1}{(Z_{,\alpha'})_a}\Delta(\overline{Z}_{t,\alpha'})\right)\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_a\Delta(\overline{Z}_{t,\alpha'}) \\ &\quad + 2\Delta(\overline{Z}_{t,\alpha'})\left(\frac{1}{(Z_{,\alpha'})_a^2}\partial_{\alpha'}\Delta(\overline{Z}_{t,\alpha'})\right)\end{aligned}$$

   From this we obtain

$$\begin{aligned}\left\|\frac{1}{(Z_{,\alpha'})_a}\Delta(\overline{Z}_{t,\alpha'})\right\|_\infty^2 &\lesssim \left\|\frac{1}{(Z_{,\alpha'})_a}\Delta(\overline{Z}_{t,\alpha'})\right\|_\infty\left\|\Delta(\overline{Z}_{t,\alpha'})\right\|_2 \\ &\quad + \left\|\Delta(\overline{Z}_{t,\alpha'})\right\|_2\left\|\frac{1}{(Z_{,\alpha'})_a^2}\partial_{\alpha'}\Delta(\overline{Z}_{t,\alpha'})\right\|_2\end{aligned}$$

   Now using the inequality $ab \leq \dfrac{\epsilon a^2}{2} + \dfrac{b^2}{2\epsilon}$, we see that $\dfrac{1}{(Z_{,\alpha'})_a}\Delta(\overline{Z}_{t,\alpha'}) \in L^\infty_{\sqrt{\Delta}}$. Now by using Lemma 5.3.5 and the fact that $\Delta(\omega) \in L^\infty_{\sqrt{\Delta}}$ we see that $\Delta(D_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}$,

$\Delta(|D_{\alpha'}|\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}$ and $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^\infty_{\sqrt{\Delta}}$

(c) Observe that

$$\Delta\big(D_{\alpha'}^2\overline{Z}_t\big) = \Delta\left\{\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)D_{\alpha'}\overline{Z}_t\right\} + \Delta\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)$$

Hence we have $\Delta\big(D_{\alpha'}^2\overline{Z}_t\big) \in L^2_{\sqrt{\Delta}}$. Similarly we can also show $\Delta\big(|D_{\alpha'}|^2\overline{Z}_t\big) \in L^2_{\sqrt{\Delta}}$ and $\Delta\big(\overline{D}_{\alpha'}^2\overline{Z}_t\big) \in L^2_{\sqrt{\Delta}}$. Now using Lemma 5.3.5 we see that $\Delta\big(|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\big) \in L^2_{\sqrt{\Delta}}$ and $|D_{\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$. This in particular implies $(\sqrt{A_1})_a|D_{\alpha'}|_a\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in L^2_{\sqrt{\Delta}}$ which is part of $E_{\Delta,4}$.

(d) Following the proof in §5.3.1 we see that $\Delta(D_{\alpha'}\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}}\cap\mathcal{C}_{\sqrt{\Delta}}$, $\Delta(|D_{\alpha'}|\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}}\cap\mathcal{C}_{\sqrt{\Delta}}$ and $\Delta(\overline{D}_{\alpha'}\overline{Z}_t) \in \mathcal{W}_{\sqrt{\Delta}}\cap\mathcal{C}_{\sqrt{\Delta}}$. Hence using Lemma 5.3.5 we have $\dfrac{1}{|Z_{,\alpha'}|_a}\Delta(\overline{Z}_{t,\alpha'}) \in \mathcal{C}_{\sqrt{\Delta}}$. As $(\sqrt{A_1})_a \in \mathcal{W}$ we now obtain $\left(\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a\Delta(\overline{Z}_{t,\alpha'}) \in \mathcal{C}_{\sqrt{\Delta}}$ and hence we have controlled the second term of $E_{\Delta,2}$.

(e) Following the proof in §5.3.1 we see that $\Delta(|D_{\alpha'}|A_1) \in L^2_{\sqrt{\Delta}}$ and hence we have $\Delta(A_1) \in \mathcal{W}_{\sqrt{\Delta}}$ and $\Delta(\sqrt{A_1}) \in \mathcal{W}_{\sqrt{\Delta}}$

(f) From §5.3.1 we see that $\Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}} \cap \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$, $\Delta(|D_{\alpha'}|b_{\alpha'}) \in L^2_{\sqrt{\Delta}}$ and $\Delta(b_{\alpha'}) \in \mathcal{W}_{\sqrt{\Delta}}$

(g) As $(\sigma\partial_{\alpha'}\Theta)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$ and $\Theta_a \in L^2$, by interpolation we see that $(\sigma^{\frac{2}{3}}\partial_{\alpha'}\Theta)_a \in L^2_{\Delta^{\frac{1}{3}}}$ and $(\sigma^{\frac{1}{3}}\Theta)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$

(h) From §5.3.1 we see that $\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}}\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$, $\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\dfrac{1}{|Z_{,\alpha'}|}\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$ and similarly $\left(\dfrac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\omega\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$, $\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}|D_{\alpha'}|\omega\right)_a \in L^2_{\Delta^{\frac{1}{3}}}$. In the same way we have $\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$, $\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}\dfrac{1}{|Z_{,\alpha'}|}\right)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$ and also $\left(\sigma^{\frac{1}{3}}|D_{\alpha'}|\omega\right)_a \in L^\infty_{\Delta^{\frac{1}{6}}} \cap \dot{H}^{\frac{1}{2}}_{\Delta^{\frac{1}{6}}}$.

(i) Following the proof in §4.1 we see that $\left(\dfrac{\sigma}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\Theta\right)_a \in L^2_{\sqrt{\Delta}}$ and from this we easily get $(\sigma\partial_{\alpha'}D_{\alpha'}\Theta)_a \in L^2_{\sqrt{\Delta}}$.

(j) We now recall from (3.7)

$$\overline{Z}_{tt} - i = -i\frac{A_1}{Z_{,\alpha'}} + \sigma D_{\alpha'}\Theta$$

Taking derivatives on both sides and applying Δ we get

$$\Delta(\overline{Z}_{tt,\alpha'}) = -i\Delta\left(A_1\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - i\Delta(D_{\alpha'}A_1) + (\sigma\partial_{\alpha'}D_{\alpha'}\Theta)_a$$

Hence we see that $\Delta(\overline{Z}_{tt,\alpha'}) \in L^2_{\sqrt{\Delta}}$. As $D_t\overline{Z}_{t,\alpha'} = -b_{\alpha'}\overline{Z}_{t,\alpha'}$ and $\Delta(b_{\alpha'}) \in L^\infty_{\sqrt{\Delta}}$ we obtain $\Delta(D_t\overline{Z}_{t,\alpha'})$ which is the first term of $E_{\Delta,2}$.

(k) By exactly following the proof of $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in L^2$ in §4.1, we can easily prove that $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\sqrt{\Delta}}$. Now we use can Proposition A.0.10 with $\omega = \dfrac{1}{|Z_{,\alpha'}|_a}$ and $f = \left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a$ and we obtain $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_a \in \dot{H}^{\frac{1}{2}}_{\sqrt{\Delta}}$. Hence $E_{\Delta,2}$ is controlled.

3. Controlling $E_{\Delta,2}$: We prove this step by step.

   (a) By (3.12) we see that $\Delta(\Theta) \in L^2_{\sqrt{\Delta}}$. Similarly from (3.14) we obtain $\Delta(D_t\Theta) \in L^2_{\sqrt{\Delta}}$.

   (b) As we have $\Delta\left(D_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right) \in \mathcal{C}_{\sqrt{\Delta}}$, using Lemma 5.3.5 we see that $\Delta\left(\overline{D}_{\alpha'}\dfrac{1}{Z_{,\alpha'}}\right) \in \mathcal{C}_{\sqrt{\Delta}}$. Now following the proof of $\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1 \in \mathcal{C}$ in §4.1, we can easily show that $\Delta\left(\dfrac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}A_1\right) \in \mathcal{C}_{\sqrt{\Delta}}$.

   (c) Following the proof of $\overline{D}_{\alpha'}\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$ in §4.1, we see that $\Delta\left(\dfrac{\Theta}{|Z_{,\alpha'}|}\right) \in \mathcal{C}_{\sqrt{\Delta}}$. Hence by Lemma 5.3.5 we see that $\dfrac{1}{|Z_{,\alpha'}|_a}\Delta(\Theta) \in \mathcal{C}_{\sqrt{\Delta}}$ and $\left(\dfrac{\sqrt{A_1}}{|Z_{,\alpha'}|}\right)_a\Delta(\Theta) \in \mathcal{C}_{\sqrt{\Delta}}$.

   (d) As $\left(\dfrac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}}\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$, by using Lemma 5.3.5 we see that $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}}\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$. Now by following the proof of $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}^2\dfrac{1}{Z_{,\alpha'}} \in \mathcal{C}$ in §4.1, we easily obtain $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{3}{2}}}\partial_{\alpha'}\Theta\right)_a \in \mathcal{C}_{\sqrt{\Delta}}$. Hence $E_{\Delta,3}$ is controlled.

4. Controlling $E_{\Delta,4}$: Observe that we have already controlled the second term. Now by following the proof of $\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'} \in L^2$ in §4.1, we see that $\left(\dfrac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}|D_{\alpha'}|\overline{D}_{\alpha'}\overline{Z}_t\right)_a \in L^2_{\sqrt{\Delta}}$ and so

the third term is controlled. Now by following the proof of $\frac{\sigma}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}} \in \mathcal{C}$ and $\frac{\sigma}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\Theta \in \mathcal{C}$ in §4.1, we see that $(\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta)_a \in \mathcal{C}_{\sqrt{\Delta}}$. Hence by applying $\overline{D}_{\alpha'}$ in the formula (3.7) we obtain

$$\Delta(D_{\alpha'}\overline{Z}_{tt}) = -i\Delta\left(A_1\overline{D}_{\alpha'}\frac{1}{Z_{,\alpha'}}\right) - i\Delta\left(\frac{1}{|Z_{,\alpha'}|^2}A_1\right) + (\sigma\overline{D}_{\alpha'}D_{\alpha'}\Theta)_a$$

Hence $\Delta(\overline{D}_{\alpha'}\overline{Z}_{tt}) \in \mathcal{C}_{\sqrt{\Delta}}$ which shows that $E_{\Delta,4}$ is controlled, completing the proof.

□

5.4 Example

In this section we prove Proposition 3.4.9. We will first need a few basic facts about Riemann mapping and convolution. Let $P_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + x^2}$ be the Poisson kernel. The following property is easily proved using basic properties of convolution.

Lemma 5.4.1. *If $f \in L^q(\mathbb{R})$, then for $s \geq 0$ an integer we have*

$$\|(\partial_{\alpha'}^s f) * P_\epsilon\|_p \lesssim \|f\|_q \epsilon^{-s-\left(\frac{1}{q}-\frac{1}{p}\right)} \quad \textit{for } 1 \leq q \leq p \leq \infty$$

Similarly for $s \in \mathbb{R}, s \geq 0$ we have

$$\|(|\partial_{\alpha'}|^s f) * P_\epsilon\|_p \lesssim \|f\|_q \epsilon^{-s-\left(\frac{1}{q}-\frac{1}{p}\right)} \quad \textit{for } 1 \leq q \leq p \leq \infty$$

Here the constants in the inequality depend only on the values of s, p and q.

Let us define $\tau = \frac{\sigma}{\epsilon^{3/2}}$. Now to prove that the time of existence of $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ is uniform, we use Theorem 3.4.8 and hence we only need to show that if $\tau \leq 1$, then

$$\mathcal{E}_{high}(Z^{\epsilon,\sigma})(0), \mathcal{E}_\sigma(Z^{\epsilon,\sigma})(0) \leq C(M)$$

Similarly to prove the convergence statement, we only need to show that

$$\mathcal{E}_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(0) \leq C(M)\tau$$

We now prove both of these statements. To simplify the proof we will suppress the dependence of M in the inequalities i.e. when we write $a \lesssim b$, we mean that there exists a constant $C(M)$

depending only on M such that $a \leq C(M)b$. As we only need to prove the estimates for $t = 0$, we will suppress the time dependence of the solutions e.g. we will write $(Z * P_\epsilon, Z_t * P_\epsilon)|_{t=0}$ by $(Z, Z_t)_\epsilon$ for simplicity. We show uniform time of existence and convergence separately.

Part 1: We easily see that $\mathcal{E}_{high} \lesssim 1$ for the initial data of $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ as all the quantities in $\mathcal{E}_{high}$ are boundary values of holomorphic functions and $\mathcal{E}_{high}$ for the initial data of (Z, Z_t) is bounded by M. We now control the terms in $\mathcal{E}_\sigma$.

1. Observe that $(Z, Z_t)_\epsilon = (Z, Z_t)_{\epsilon'} * P_{\epsilon - \epsilon'}$ for $0 < \epsilon' < \epsilon$ and hence by Lemma 5.4.1

   $$\left\| \left(Z_{,\alpha'}^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 \lesssim \left\| \left(Z_{,\alpha'}^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)_{\epsilon'} \right\|_{\frac{4}{3}} (\epsilon - \epsilon')^{-\frac{1}{4}} \leq \sup_{y<0} \left\| \Psi_z^{\frac{1}{2}} \partial_z \left(\frac{1}{\Psi_z} \right) \right\|_{L^{\frac{4}{3}}(\mathbb{R},dx)} (\epsilon - \epsilon')^{-\frac{1}{4}}$$

   Hence letting $\epsilon' \to 0$, we obtain $\left\| \left(\sigma^{\frac{1}{6}} Z_{,\alpha'}^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 \lesssim \tau^{\frac{1}{6}}$.

2. Following the same argument as above, we use $\sup_{y<0} \left\| \Psi_z^{\frac{1}{2}} \partial_z \left(\frac{1}{\Psi_z} \right) \right\|_{L^{\frac{4}{3}}(\mathbb{R},dx)} \leq M$ and also Lemma 5.4.1 to obtain $\left\| \left(\sigma^{\frac{1}{2}} Z_{,\alpha'}^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_\infty \lesssim \tau^{\frac{1}{2}}$.

3. We use $\sup_{y<0} \left\| \partial_z \left(\frac{1}{\Psi_z} \right) \right\|_{L^2(\mathbb{R},dx)} \leq M$ and Lemma 5.4.1 to get $\left\| \left(\sigma^{\frac{1}{3}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_\infty \lesssim \tau^{\frac{1}{3}}$. This also implies that $\left\| \left(\sigma^{\frac{1}{3}} \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right)_\epsilon \right\|_\infty \lesssim \tau^{\frac{1}{3}}$

4. We use $\sup_{y<0} \left\| \partial_z \left(\frac{1}{\Psi_z} \right) \right\|_{L^2(\mathbb{R},dx)} \leq M$ and Lemma 5.4.1 to get $\left\| \left(\sigma^{\frac{2}{3}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 \lesssim \tau^{\frac{2}{3}}$.

5. We use $\sup_{y<0} \left\| \frac{1}{\Psi_z} \partial_z^2 \left(\frac{1}{\Psi_z} \right) \right\|_{L^1(\mathbb{R},dx)} \leq M$ and Lemma 5.4.1 to get $\left\| \sigma \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 \lesssim \tau$. Hence we have

   $$\left\| \left(\frac{\sigma}{Z_{,\alpha'}} \partial_{\alpha'}^3 \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 \lesssim \left\| \sigma \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 + \left\| \left(\sigma^{\frac{1}{3}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_\infty \left\| \left(\sigma^{\frac{2}{3}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 \lesssim \tau$$

   We similarly show that

   $$\left\| \left\{ \sigma Z_{,\alpha'}^{\frac{1}{2}} \partial_{\alpha'} \left(\frac{1}{Z_{,\alpha'}^{\frac{3}{2}}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right) \right\}_\epsilon \right\|_2 \lesssim \tau \quad \text{and} \quad \left\| \sigma \partial_{\alpha'} \left(\frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right)_\epsilon \right\|_2 \lesssim \tau$$

6. We observe that

$$\left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2^2 \lesssim \left\|\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2\left\|\sigma\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 \lesssim \tau$$

Hence we also have

$$\left\|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_{\mathcal{W}} = \left\|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_\infty + \left\|\left\{|D_{\alpha'}|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)\right\}_\epsilon\right\|_2 \lesssim \tau^{\frac{1}{2}}$$

7. Observe that $\|\Theta_\epsilon\|_2 \lesssim \left\|\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2$. Hence by using $\sup_{y<0}\left\|\partial_z\left(\frac{1}{\Psi_z}\right)\right\|_{L^2(\mathbb{R},dx)} \leq M$ and Lemma 5.4.1 to get $\|(\sigma\partial_{\alpha'}\Theta)_\epsilon\|_{\dot{H}^{\frac{1}{2}}} \lesssim \tau$.

8. We observe that

$$\left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \left\|\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2\left\|\left\{\sigma Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)\right\}_\epsilon\right\|_2 \lesssim \tau$$

9. We use $\sup_{y<0}\left\|\frac{1}{\Psi_z^2}\partial_z^2\left(\frac{1}{\Psi_z}\right)\right\|_{L^2(\mathbb{R},dx)} \leq M$ and Lemma 5.4.1 to get

$$\left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 \lesssim \tau^{\frac{2}{3}}$$

From this we also obtain

$$\begin{aligned}\left\|\left(\frac{\sigma^{\frac{2}{3}}}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 &\lesssim \left\|\sigma^{\frac{2}{3}}\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 + \left\|\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_\infty\left\|\left(\sigma^{\frac{2}{3}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 \\ &\lesssim \tau^{\frac{2}{3}}\end{aligned}$$

10. We now as usual use $\sup_{y<0}\left\|\frac{1}{\Psi_z^2}\partial_z^2\left(\frac{1}{\Psi_z}\right)\right\|_{L^2(\mathbb{R},dx)} \leq M$ and Lemma 5.4.1 to obtain the estimate

$\left\|\sigma\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \tau$. From this we also obtain

$$\begin{aligned}
&\left\|\left(\frac{\sigma}{Z_{,\alpha'}^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \left\|\sigma\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_{\mathcal{W}}\left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{3}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_{\mathcal{C}} \\
&\lesssim \tau
\end{aligned}$$

11. We see that as $\epsilon \leq 1$ we have $\sigma^{\frac{1}{2}} = \tau^{\frac{1}{2}}\epsilon^{\frac{3}{4}} \leq \tau^{\frac{1}{2}}$. Hence using $\sup_{y<0}\left\|\frac{1}{\Psi_z}\right\|_{L^\infty(\mathbb{R},dx)} \leq M$ and $\sup_{y<0}\|F_z\|_{H^{2.5}(\mathbb{R},dx)} \leq M$ we obtain

$$\left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_\epsilon\right\|_2 \lesssim \sigma^{\frac{1}{2}}\left\|\frac{1}{(Z_{,\alpha'})_\epsilon}\right\|_\infty^{\frac{1}{2}}\left\|(\partial_{\alpha'}\overline{Z}_{t,\alpha'})_\epsilon\right\|_2 \lesssim \tau^{\frac{1}{2}}$$

12. We again use $\sup_{y<0}\left\|\frac{1}{\Psi_z}\right\|_{L^\infty(\mathbb{R},dx)} \leq M$ and $\sup_{y<0}\|F_z\|_{H^{2.5}(\mathbb{R},dx)} \leq M$ to obtain

$$\left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right)_\epsilon\right\|_2 \lesssim \sigma^{\frac{1}{2}}\left\|\frac{1}{(Z_{,\alpha'})_\epsilon}\right\|_\infty^{\frac{5}{2}}\left\|(\partial_{\alpha'}^2\overline{Z}_{t,\alpha'})_\epsilon\right\|_2 \lesssim \tau^{\frac{1}{2}}$$

This completes the proof of uniform time of existence in Proposition 3.4.9. We now prove the convergence aspect of the proposition.

Part 2: We now show that as $\tau \to 0$, the solutions $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma}) \to (Z^\epsilon, Z_t^\epsilon)$. As mentioned before we only need to show that $\mathcal{E}_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(0) \leq C(M)\tau$. Recall that $\mathcal{E}_\Delta = (\mathcal{E}_{\sigma,aux})_\epsilon + \mathcal{E}_{\Delta,1} + \mathcal{E}_{\Delta,2}$ where $(\mathcal{E}_{\sigma,aux})_\epsilon$ is given by

$$\begin{aligned}
(\mathcal{E}_{\sigma,aux})_\epsilon = &\left\|\left(\sigma^{\frac{1}{2}}Z_{,\alpha'}^{\frac{1}{2}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_\infty^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2^2 \\
&+ \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{1}{2}}}\partial_{\alpha'}\overline{Z}_{t,\alpha'}\right)_\epsilon\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{5}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right)_\epsilon\right\|_2^2 + \left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^2\overline{Z}_{t,\alpha'}\right)_\epsilon\right\|_{\dot{H}^{\frac{1}{2}}}^2
\end{aligned}$$

Now in part 1 above we showed that $\mathcal{E}_\sigma \lesssim 1$ and all the terms with σ are bounded by τ. Hence $\mathcal{E}_{\Delta,1} + \mathcal{E}_{\Delta,2} \lesssim \tau$. We now only have to control $(\mathcal{E}_{\sigma,aux})_\epsilon$. Observe that only two terms of $(\mathcal{E}_{\sigma,aux})_\epsilon$

have been controlled which we now control.

1. We use $\sup_{y<0}\left\|\frac{1}{\Psi_z^3}\partial_z^3\left(\frac{1}{\Psi_z}\right)\right\|_{L^1(\mathbb{R},dx)} \leq M$ and Lemma 5.4.1 to get

$$\left\|\sigma\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^3}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 \lesssim \tau$$

From this we also obtain

$$\begin{aligned}\left\|\left(\frac{\sigma}{Z_{,\alpha'}^3}\partial_{\alpha'}^4\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 &\lesssim \left\|\left(\frac{1}{Z_{,\alpha'}}\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_\infty\left\|\left(\frac{\sigma}{Z_{,\alpha'}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 + \left\|\sigma\partial_{\alpha'}\left(\frac{1}{Z_{,\alpha'}^3}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 \\ &\lesssim \tau\end{aligned}$$

Now we have

$$\begin{aligned}&\left\|\left(\sigma|Z_{,\alpha'}|^2\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^5}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)\right)_\epsilon\right\|_2 \\ &\lesssim \left\|\left(\frac{\sigma}{Z_{,\alpha'}^3}\partial_{\alpha'}^4\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 + \left\|\left(\sigma^{\frac{1}{3}}\partial_{\alpha'}\frac{1}{|Z_{,\alpha'}|}\right)_\epsilon\right\|_\infty\left\|\left(\frac{\sigma^{\frac{2}{3}}}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 \\ &\lesssim \tau\end{aligned}$$

and from this we see that

$$\left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{2}{3}}}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2 \lesssim \left\|\left(\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}^2\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2\left\|\left(\sigma|Z_{,\alpha'}|^2\partial_{\alpha'}\left(\frac{1}{|Z_{,\alpha'}|^5}\partial_{\alpha'}^3\frac{1}{Z_{,\alpha'}}\right)\right)_\epsilon\right\|_2 \lesssim \tau$$

2. We easily see that

$$\begin{aligned}&\left\|\left(\frac{\sigma^{\frac{1}{2}}}{Z_{,\alpha'}^{\frac{7}{2}}}\partial_{\alpha'}^2\bar{Z}_{t,\alpha'}\right)_\epsilon\right\|_{\dot{H}^{\frac{1}{2}}} \\ &\lesssim \sigma^{\frac{1}{2}}\left\|\frac{1}{(Z_{,\alpha'})_\epsilon}\right\|_\infty^{\frac{7}{2}}\left\|(\partial_{\alpha'}^2\bar{Z}_{t,\alpha'})_\epsilon\right\|_{\dot{H}^{\frac{1}{2}}} + \sigma^{\frac{1}{2}}\left\|\frac{1}{(Z_{,\alpha'})_\epsilon}\right\|_\infty^{\frac{5}{2}}\left\|\left(\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)_\epsilon\right\|_2\left\|\left(\partial_{\alpha'}^2\bar{Z}_{t,\alpha'}\right)_\epsilon\right\|_2 \\ &\lesssim \tau^{\frac{1}{2}}\end{aligned}$$

This finishes the proof of $\mathcal{E}_\Delta(Z^{\epsilon,\sigma}, Z^\epsilon)(0) \leq C(M)\tau$ and hence we have proved the convergence result.

To see the rate of growth of curvature, observe that curvature of the interface in Riemannian coordinates = $(\partial_s \theta) \circ h^{-1} = \dfrac{1}{|Z_{,\alpha'}|}\partial_{\alpha'} g$ where $Z_{,\alpha'} = e^{f+ig}$. Hence the curvature of the interface Z^ϵ is given by $\dfrac{1}{(|Z_{,\alpha'}|)_\epsilon}\partial_{\alpha'}(g_\epsilon)$. Now if the interface Z has an angled crest at $\alpha' = 0$, then we see that $\partial_{\alpha'}(g_\epsilon)(0) \sim \epsilon^{-1}$ as g has a jump for $\epsilon = 0$. But we know from Theorem 2.6.1 that $(Z_{,\alpha'})_\epsilon(0) \sim \epsilon^{\nu-1}$. Hence $\epsilon^{-\nu}$ is a lower bound on the curvature. To see that the upper bound is the same, we observe from formula (2.9) that

$$\frac{1}{(|Z_{,\alpha'}|)_\epsilon}\partial_{\alpha'}(g_\epsilon) \leq \left|\partial_{\alpha'}\frac{1}{Z^\epsilon_{,\alpha'}}\right| \lesssim \epsilon^{-\nu}$$

Hence we have that the curvature of $Z^\epsilon \sim \epsilon^{-\nu}$.

APPENDIX A

Here we will prove all the identities and commutator estimates used in closing the energy. We will state most of the statements only for functions in the Schwartz class and it can be extended to more general functions by an approximation argument. Let us first recall some of the notation used. Let $D_t = \partial_t + b\partial_{\alpha'}$ where b is as defined in (2.7) and recall that $[f, g; h]$ is defined as

$$[f_1, f_2; f_3](\alpha') = \frac{1}{i\pi}\int\left(\frac{f_1(\alpha') - f_1(\beta')}{\alpha' - \beta'}\right)\left(\frac{f_2(\alpha') - f_2(\beta')}{\alpha' - \beta'}\right)f_3(\beta')\,d\beta'$$

We also have the operators

$$(\mathcal{H}f)(x) = \frac{1}{i\pi}p.v.\int \frac{\widetilde{h}_{\beta'}(\beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}f(\beta')\,d\beta'$$
$$(\widetilde{\mathcal{H}}f)(x) = \frac{1}{i\pi}p.v.\int \frac{1}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}f(\beta')\,d\beta'$$

and the notation

$$[f_1, f_2; \partial_{\alpha'} f_3]_{\widetilde{h}} = \frac{1}{i\pi}\int\left(\frac{f_1(\alpha') - f_1(\beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}\right)\left(\frac{f_2(\alpha') - f_2(\beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}\right)\partial_{\beta'} f_3(\beta')\,d\beta'$$

Proposition A.0.1. *Let $f, g, h \in \mathcal{S}(\mathbb{R})$. Then we have the following identities*

1. $h\partial_{\alpha'}[f, \mathbb{H}]\partial_{\alpha'} g = [h\partial_{\alpha'} f, \mathbb{H}]\partial_{\alpha'} g + [f, \mathbb{H}]\partial_{\alpha'}(h\partial_{\alpha'} g) - [h, f; \partial_{\alpha'} g]$
2. $D_t[f, \mathbb{H}]\partial_{\alpha'} g = [D_t f, \mathbb{H}]\partial_{\alpha'} g + [f, \mathbb{H}]\partial_{\alpha'}(D_t g) - [b, f; \partial_{\alpha'} g]$

Proof. The second identity is a direct consequence of the first. Now we see that

$$\begin{aligned}
&h(\alpha')\partial_{\alpha'}[f, \mathbb{H}]\partial_{\alpha'} g \\
&= h(\alpha')\partial_{\alpha'}\left(\frac{1}{i\pi}\int \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}\partial_{\beta'} g(\beta')\,d\beta'\right)
\end{aligned}$$

$$
\begin{aligned}
&= h(\alpha')f'(\alpha')\left(\frac{1}{i\pi}\int\frac{1}{\alpha'-\beta'}\partial_{\beta'}g(\beta')\,d\beta'\right) - \frac{1}{i\pi}\int\frac{f(\alpha')-f(\beta')}{(\alpha'-\beta')^2}h(\beta')\partial_{\beta'}g(\beta')\,d\beta' \\
&\quad - \frac{1}{i\pi}\int\left(\frac{h(\alpha')-h(\beta')}{\alpha'-\beta'}\right)\left(\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right)\partial_{\beta'}g(\beta')\,d\beta' \\
&= \frac{1}{i\pi}\int\frac{h(\alpha')f'(\alpha')-h(\beta')f'(\beta')}{\alpha'-\beta'}\partial_{\beta'}g(\beta')\,d\beta' + \frac{1}{i\pi}\int\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\partial_{\beta'}(h(\beta')\partial_{\beta'}g(\beta'))\,d\beta' \\
&\quad - \frac{1}{i\pi}\int\left(\frac{h(\alpha')-h(\beta')}{\alpha'-\beta'}\right)\left(\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right)\partial_{\beta'}g(\beta')\,d\beta'
\end{aligned}
$$

We now observe that the quantity above is exactly the same as the right hand side of the above proposition. Hence the proposition is proved. □

The following identities say that the material derivative of a holomorphic function remain essentially holomorphic.

Proposition A.0.2. *Let $f \in \mathcal{S}(\mathbb{R})$ with $\mathbb{P}_A f = 0$. Then we have the following identities*

1. $(\mathbb{I}-\mathbb{H})D_t f = (\mathbb{I}-\mathbb{H})(Z_t D_{\alpha'} f)$
2. $(\mathbb{I}-\mathbb{H})D_t^2 f$

$$
\begin{aligned}
&= 2\left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\mathbb{H}\right]\partial_{\alpha'}(D_t f) - \left[\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right),\mathbb{P}_A\left(\frac{Z_t}{Z_{,\alpha'}}\right);\partial_{\alpha'}f\right] \\
&\quad + \frac{1}{4}(\mathbb{I}-\mathbb{H})\left\{\left(\left[\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]Z_{t,\alpha'}\right)[Z_t,\mathbb{H}]D_{\alpha'}f\right\} - \frac{1}{4}(\mathbb{I}-\mathbb{H})\left\{\left([Z_t,\mathbb{H}]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}\right)^2 f\right\} \\
&\quad + \frac{1}{2}\left[[Z_t,[Z_t,\mathbb{H}]]\partial_{\alpha'}\frac{1}{Z_{,\alpha'}},\mathbb{H}\right]\partial_{\alpha'}\left(\frac{f}{Z_{,\alpha'}}\right) + [Z_{tt},\mathbb{H}]D_{\alpha'}f
\end{aligned}
$$

Proof. See [Wu15] Appendix B and section 4 for the proof of the above identities. □

Proposition A.0.3. *Let $H \in C^1(\mathbb{R}), A_i \in C^1(\mathbb{R})$ for $i = 1,\cdots m$ and $F \in C^\infty(\mathbb{R})$. Define*

$$
\begin{aligned}
C_1(H,A,f)(x) &= p.v.\int F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\Pi_{i=1}^m(A_i(x)-A_i(y))}{(x-y)^{m+1}}f(y)\,dy \\
C_2(H,A,f)(x) &= p.v.\int F\left(\frac{H(x)-H(y)}{x-y}\right)\frac{\Pi_{i=1}^m(A_i(x)-A_i(y))}{(x-y)^{m}}\partial_y f(y)\,dy
\end{aligned}
$$

then there exists constants c_1, c_2, c_3, c_4 depending only on F and $\|H'\|_\infty$ so that

1. $\|C_1(H,A,f)\|_2 \le c_1\|A_1'\|_\infty\cdots\|A_m'\|_\infty\|f\|_2$
2. $\|C_1(H,A,f)\|_2 \le c_2\|A_1'\|_2\|A_2'\|_\infty\cdots\|A_m'\|_\infty\|f\|_\infty$
3. $\|C_2(H,A,f)\|_2 \le c_3\|A_1'\|_\infty\cdots\|A_m'\|_\infty\|f\|_2$

4. $\|C_2(H,A,f)\|_2 \leq c_4\|A_1'\|_2\|A_2'\|_\infty \cdots \|A_m'\|_\infty\|f\|_\infty$

Proof. The first estimate is a theorem by Coifman, McIntosh and Meyer [CMM82]. See also chapter 9 of [MC97]. Estimate 2 is a consequence of the Tb theorem and a proof can be found in [Wu09]. The third and fourth estimates can be obtained from the first two by integration by parts. □

Corollary A.0.4. *Let $H \in C^1(\mathbb{R})$, $A_i \in C^1(\mathbb{R})$ for $i = 1, \cdots m$ and let $\delta > 0$ be such that*

$$\delta \leq \left|\frac{H(x)-H(y)}{x-y}\right| \leq \frac{1}{\delta} \quad \textit{for all } x \neq y$$

Let $0 \leq k \leq m+1$ and define

$$T(A,f)(x) = p.v. \int \frac{\Pi_{i=1}^m (A_i(x)-A_i(y))}{(x-y)^{m+1-k}(H(x)-H(y))^k} f(y)\,dy$$

then we have the estimates

1. $\|T(A,f)\|_2 \leq C(\|H'\|_\infty, \delta)\|A_1'\|_\infty \cdots \|A_m'\|_\infty\|f\|_2$
2. $\|T(A,f)\|_2 \leq C(\|H'\|_\infty, \delta)\|A_1'\|_2\|A_2'\|_\infty \cdots \|A_m'\|_\infty\|f\|_\infty$

Proof. If $k = 0$, then the result follows directly from Proposition A.0.3. If $k \geq 1$, we choose a smooth function F with compact support such that $F(x) = 0$ if $|x| \leq \frac{\delta}{2}$ or $|x| \geq \frac{2}{\delta}$ and $F(x) = x^{-k}$ if $\delta \leq |x| \leq \frac{1}{\delta}$. The result now follows from Proposition A.0.3. □

Proposition A.0.5. *Let $T : \mathcal{D}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ be a linear operator with kernel $K(x,y)$ such that on the open set $\{(x,y) : x \neq y\} \subset \mathbb{R} \times \mathbb{R}$, $K(x,y)$ is a function satisfying*

$$|K(x,y)| \leq \frac{C_0}{|x-y|} \quad \textit{and} \quad |\nabla_x K(x,y)| \leq \frac{C_0}{|x-y|^2}$$

where C_0 is a constant. If T is continuous on $L^2(\mathbb{R})$ with $\|T\|_{L^2\to L^2} \leq C_0$ and if $T(1) = 0$, then T is bounded on $\dot{H}^s$ for $0 < s < 1$ with $\|T\|_{\dot{H}^s \to \dot{H}^s} \lesssim C_0$

Proof. This proposition is a direct consequence of the result of Lemarie [Lem85] where only weak boundedness of T on L^2 (in the sense of David and Journe) is assumed. As boundedness on L^2 implies weak boundedness, the proposition follows. See also chapter 10 of [MC97] for another proof of the result of Lemarie. □

Proposition A.0.6. *Let $f \in \mathcal{S}(\mathbb{R})$. Then we have*

1. $\|f\|_\infty \lesssim \|f\|_{H^s}$ *if $s > \frac{1}{2}$ and for $s = \frac{1}{2}$ we have* $\|f\|_{BMO} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}$

2. $\displaystyle\int\left|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right|^2 d\beta' \lesssim \|f'\|_2^2$

3. $\displaystyle\left\|\sup_{\beta'}\left|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right|\right\|_{L^2(\mathbb{R},d\alpha')} \lesssim \|f'\|_2$

4. $\displaystyle\|f\|_{\dot{H}^{\frac{1}{2}}}^2 = \frac{1}{2\pi}\iint\left|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right|^2 d\beta'\, d\alpha'$

5. $\displaystyle\left\|\partial_{\beta'}\left(\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right)\right\|_{L^2(\mathbb{R}^2,d\alpha'\,d\beta')} \lesssim \|f'\|_{\dot{H}^{\frac{1}{2}}}$

Proof. 1) is the standard Sobolev embedding and 2) is a consequence of Hardy's inequality.
3) We see that

$$\sup_{\beta'}\left|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right| = \sup_{\beta'}\left|\int_0^1 f'(\beta'+s(\alpha'-\beta'))\,ds\right| \leq M(f')(\alpha')$$

where M is the uncentered Hardy Littlewood maximal operator. As the maximal operator is bounded on L^2, the estimate follows.
4) Observe that as $|\partial_{\alpha'}| = i\mathbb{H}\partial_{\alpha'}$ and $\mathbb{H}(1)=0$ we have

$$\begin{aligned}
\|f\|_{\dot{H}^{\frac{1}{2}}}^2 &= -\frac{1}{\pi}\int \bar{f}(\alpha')\partial_{\alpha'}\left(\int \frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\,d\beta'\right)d\alpha' \\
&= \frac{1}{\pi}\int \bar{f}(\alpha')\left(\int \frac{f(\alpha')-f(\beta')}{(\alpha'-\beta')^2}\,d\beta'\right)d\alpha' \\
&= \frac{1}{\pi}\iint\left|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right|^2 d\beta'\,d\alpha' + \frac{1}{\pi}\iint \frac{f(\alpha')-f(\beta')}{(\alpha'-\beta')^2}\bar{f}(\beta')\,d\beta'\,d\alpha'
\end{aligned}$$

Now we switch α' and β' in the second term and hence we get

$$\begin{aligned}
\frac{1}{\pi}\int \bar{f}(\alpha')\left(\int \frac{f(\alpha')-f(\beta')}{(\alpha'-\beta')^2}\,d\beta'\right)d\alpha' &= \frac{1}{\pi}\iint\left|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right|^2 d\beta'\,d\alpha' \\
&\quad - \frac{1}{\pi}\int \bar{f}(\alpha')\left(\int \frac{f(\alpha')-f(\beta')}{(\alpha'-\beta')^2}\,d\beta'\right)d\alpha'
\end{aligned}$$

The identity now follows.

5) We see that

$$\begin{aligned}\partial_{\beta'}\left(\frac{f(\alpha')-f(\beta')}{(\alpha'-\beta')}\right) &= \frac{f(\alpha')-f(\beta')}{(\alpha'-\beta')^2} - \frac{f'(\beta')}{\alpha'-\beta'} \\ &= \int_0^1 \frac{f'(\beta'+s(\alpha'-\beta')) - f'(\beta')}{(\alpha'-\beta')}\, ds \\ &= \int_0^1 s\left[\frac{f'(\beta'+sl)-f'(\beta')}{sl}\right] ds \quad \text{using } \alpha' = \beta' + l\end{aligned}$$

Hence we have

$$\begin{aligned}\left\|\partial_{\beta'}\left(\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right)\right\|_{L^2(\mathbb{R}^2, d\alpha'\, d\beta')} &\lesssim \int_0^1 s\left\|\frac{f'(\beta'+sl)-f'(\beta')}{sl}\right\|_{L^2(\mathbb{R}^2, d\beta'\, dl)} ds \\ &\lesssim \int_0^1 \sqrt{s}\|f'\|_{\dot{H}^{\frac{1}{2}}}\, ds \\ &\lesssim \|f'\|_{\dot{H}^{\frac{1}{2}}}\end{aligned}$$

□

Proposition A.0.7. *Let $f, g, h \in \mathcal{S}(\mathbb{R})$ and $s, a \in \mathbb{R}$. Then we have the following estimates*

1. $\||\partial_{\alpha'}|^s(fg)\|_2 \lesssim \||\partial_{\alpha'}|^s f\|_2\|g\|_\infty + \|f\|_\infty\||\partial_{\alpha'}|^s g\|_2$ *for* $s > 0$
2. $\left\||\partial_{\alpha'}|^s[f, \mathbb{H}](|\partial_{\alpha'}|^a g)\right\|_2 \lesssim \left\||\partial_{\alpha'}|^{s+a} f\right\|_{BMO}\|g\|_2$ *for* $s, a \geq 0$
3. $\left\||\partial_{\alpha'}|^s[f, \mathbb{H}](|\partial_{\alpha'}|^a g)\right\|_2 \lesssim \left\||\partial_{\alpha'}|^{s+a} f\right\|_2\|g\|_{BMO}$ *for* $s \geq 0$ *and* $a > 0$
4. $\left\|\left[f, |\partial_{\alpha'}|^{\frac{1}{2}}\right] g\right\|_2 \lesssim \left\||\partial_{\alpha'}|^{\frac{1}{2}} f\right\|_{BMO}\|g\|_2$

Proof. These estimates are all variants of the Kato Ponce commutator estimate and are proved using the paraproduct decomposition. See [KP88] for the first estimate, [HIT16] for the second and third and [Li16] for the last one. □

Corollary A.0.8. *Let $f, g, h \in \mathcal{S}(\mathbb{R})$ and $m, n \in \mathbb{Z}$. Then we have the following estimates*

1. $\|fg\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}\|g\|_\infty + \|f\|_\infty\|g\|_{\dot{H}^{\frac{1}{2}}}$
2. $\|fg\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|f'\|_2\|g\|_2 + \|f\|_\infty\|g\|_{\dot{H}^{\frac{1}{2}}}$
3. $\|\partial_{\alpha'}^m[f, \mathbb{H}]\partial_{\alpha'}^n g\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \lesssim \left\|\partial_{\alpha'}^{(m+n+1)} f\right\|_2\|g\|_2$ *for* $m, n \geq 0$
4. $\|\partial_{\alpha'}^m[f, \mathbb{H}]\partial_{\alpha'}^n g\|_2 \lesssim \|\partial_{\alpha'}^{(m+n)} f\|_\infty\|g\|_2$ *for* $m, n \geq 0$
5. $\|\partial_{\alpha'}^m[f, \mathbb{H}]\partial_{\alpha'}^n g\|_2 \lesssim \|\partial_{\alpha'}^{(m+n)} f\|_2\|g\|_\infty$ *for* $m \geq 0$ *and* $n \geq 1$

Proof. The first, fourth and fifth estimate follow easily. For 2) observe that

$$|\partial_{\alpha'}|^{\frac{1}{2}}(fg) = [|\partial_{\alpha'}|^{\frac{1}{2}}, f]g + f|\partial_{\alpha'}|^{\frac{1}{2}}g$$

and hence

$$\|fg\|_{\dot{H}^{\frac{1}{2}}} \lesssim \||\partial_{\alpha'}|^{\frac{1}{2}}f\|_{BMO}\|g\|_2 + \|f\|_\infty\|g\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|f'\|_2\|g\|_2 + \|f\|_\infty\|g\|_{\dot{H}^{\frac{1}{2}}}$$

3) The $\dot{H}^{\frac{1}{2}}$ estimate follows from the previous proposition and hence we only need to show the L^∞ estimate. We note that

$$\begin{aligned}
\partial_{\alpha'}^m[f, \mathbb{H}]\partial_{\alpha'}^n g &= \partial_{\alpha'}^m \int \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}\partial_{\beta'}^n g(\beta')\, d\beta' \\
&= \partial_{\alpha'}^m \iint_0^1 f'((1-s)\beta' + s\alpha')\partial_{\beta'}^n g(\beta')\, ds\, d\beta' \\
&= (-1)^n \int_0^1 s^m(1-s)^n \left(\int f^{(m+n+1)}((1-s)\beta' + s\alpha')g(\beta')\, d\beta'\right) ds
\end{aligned}$$

The estimate follows by using Cauchy Schwartz. □

Proposition A.0.9. *Let $f, g, h \in \mathcal{S}(\mathbb{R})$. Then we have the following estimates*

1. $\|[f, \mathbb{H}]g\|_2 \lesssim \|f'\|_2\|g\|_1$
2. $\|[f, g; h]\|_2 \lesssim \|f'\|_2\|g'\|_2\|h\|_2$
3. $\|\partial_{\alpha'}[f, [g, \mathbb{H}]]h\|_2 \lesssim \|f'\|_2\|g'\|_2\|h\|_2$
4. $\|[f, g; h']\|_2 \lesssim \|f'\|_\infty\|g'\|_\infty\|h\|_2$
5. $\|[f, g; h']\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|f'\|_\infty\|g'\|_\infty\|h\|_{\dot{H}^{\frac{1}{2}}}$
6. $\|[f, g; h]\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \lesssim \|f'\|_\infty\|g'\|_2\|h\|_2$

Proof. 1) We see that

$$|[f, \mathbb{H}]g| \lesssim \int \left|\frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}\right| |g(\beta')|^{\frac{1}{2}}|g(\beta')|^{\frac{1}{2}}\, d\beta' \lesssim \left(\int \left|\frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}\right|^2 |g(\beta')|\, d\beta'\right)^{\frac{1}{2}} \|g\|_1$$

The estimate now follows from Hardy's inequality.

2) We see that

$$
\begin{aligned}
|[f,g;h]| &\lesssim \int \left|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right|\left|\frac{g(\alpha')-g(\beta')}{\alpha'-\beta'}\right||h(\beta')|\,d\beta' \\
&\lesssim \|f'\|_2\left(\int\left|\frac{g(\alpha')-g(\beta')}{\alpha'-\beta'}\right|^2|h(\beta')|^2\,d\beta'\right)^{\frac{1}{2}}
\end{aligned}
$$

The estimate now follows from Hardy's inequality.

3) We see that

$$
\begin{aligned}
&\partial_{\alpha'}[f,[g,\mathbb{H}]]h \\
&= \partial_{\alpha'}(f[g,\mathbb{H}]h - [g,\mathbb{H}]fh) \\
&= \frac{1}{i\pi}\partial_{\alpha'}\int \frac{(g(\alpha')-g(\beta'))(f(\alpha')-f(\beta'))}{\alpha'-\beta'}h(\beta')\,d\beta' \\
&= -\frac{1}{i\pi}\int \frac{g(\alpha')-g(\beta')}{\alpha'-\beta'}\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}h(\beta')\,d\beta' + g'(\alpha')\left(\frac{1}{i\pi}\int\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}h(\beta')\,d\beta'\right) \\
&\quad + f'(\alpha')\left(\frac{1}{i\pi}\int\frac{g(\alpha')-g(\beta')}{\alpha'-\beta'}h(\beta')\,d\beta'\right)
\end{aligned}
$$

The estimate now follows by previous estimates.

4) This is a special case of Proposition A.0.3

5) We observe that the operator T defined by the action $h \mapsto [f,g;h']$ is bounded on L^2. Also we clearly see that $T(1) = 0$. It is also easy to see that the kernel of this operator is a Calderon Zygmund kernel and hence satisfies the conditions for Proposition A.0.5. Hence the operator T is bounded on $\dot{H}^{\frac{1}{2}}$.

6) The L^∞ estimate is obtained easily by an application of Cauchy Schwartz and Hardy's inequality. Now we use $\|f\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\frac{f(\alpha')-f(\beta')}{\alpha'-\beta'}\right\|_{L^2(\mathbb{R}\times\mathbb{R},d\alpha'\,d\beta')}$ and see that

$$
\begin{aligned}
\frac{[f,g;h](\alpha')-[f,g;h](\beta')}{\alpha'-\beta'} &= \frac{1}{i\pi}\int\frac{h(s)}{\alpha'-\beta'}\left[\frac{f(\alpha')-f(s)}{\alpha'-s}-\frac{f(\beta')-f(s)}{\beta'-s}\right]\frac{g(\alpha')-g(s)}{\alpha'-s}\,ds \\
&\quad + \frac{1}{i\pi}\int\frac{h(s)}{\alpha'-\beta'}\left[\frac{g(\alpha')-g(s)}{\alpha'-s}-\frac{g(\beta')-g(s)}{\beta'-s}\right]\frac{f(\beta')-f(s)}{\beta'-s}\,ds
\end{aligned}
$$

Now we use the following notation to simplify the calculation

$$
F(a,b) = \frac{f(a)-f(b)}{a-b} \quad \text{and } G(a,b) = \frac{g(a)-g(b)}{a-b}
$$

Hence we have

$$
\begin{aligned}
&\frac{[f,g;h](\alpha')-[f,g;h](\beta')}{\alpha'-\beta'} \\
&= -\frac{1}{i\pi}\int \frac{h(s)}{\beta'-s}F(\alpha',s)G(\alpha',s)\,ds + \frac{1}{i\pi}\int \frac{h(s)}{\beta'-s}F(\alpha',\beta')G(\alpha',s)\,ds \\
&\quad + \frac{1}{i\pi}\int \frac{h(s)}{\alpha'-s}F(\beta',s)G(\alpha',\beta')\,ds - \frac{1}{i\pi}\int \frac{h(s)}{\alpha'-s}F(\beta',s)G(\beta',s)\,ds \\
&= -\mathbb{H}(F(\alpha',\cdot)G(\alpha',\cdot)h(\cdot))(\beta') + F(\alpha',\beta')\mathbb{H}(G(\alpha',\cdot)h(\cdot))(\beta') \\
&\quad + G(\alpha',\beta')\mathbb{H}(F(\beta',\cdot)h(\cdot))(\alpha') - \mathbb{H}(F(\beta',\cdot)G(\beta',\cdot)h(\cdot))(\alpha')
\end{aligned}
$$

and we see that

$$
\begin{aligned}
\|\mathbb{H}(F(\alpha',\cdot)G(\alpha',\cdot)h(\cdot))(\beta')\|_{L^2(\mathbb{R}\times\mathbb{R},d\alpha'\,d\beta')} &\lesssim \Big\| \|F(\alpha',\beta')G(\alpha',\beta')h(\beta')\|_{L^2(\mathbb{R},d\beta')}\Big\|_{L^2(\mathbb{R},d\alpha')} \\
&\lesssim \|f'\|_\infty\|h\|_2\Big\|\|G(\alpha',\beta')\|_{L^\infty(\mathbb{R},d\beta')}\Big\|_{L^2(\mathbb{R},d\alpha')} \\
&\lesssim \|f'\|_\infty\|g'\|_2\|h\|_2
\end{aligned}
$$

The other terms are handled similarly. □

Proposition A.0.10. *Let $f\in\mathcal{S}(\mathbb{R})$ and let w be a smooth non-zero weight with $w,\frac{1}{w}\in L^\infty(\mathbb{R})$ and $w'\in L^2(\mathbb{R})$. Then*

1. $\|f\|_\infty^2 \lesssim \left\|\frac{f}{w}\right\|_2\|w(f')\|_2$
2. $\|f\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}}^2 \lesssim \left\|\frac{f}{w}\right\|_2\|(wf)'\|_2 + \left\|\frac{f}{w}\right\|_2^2\|w'\|_2^2$

Proof. 1) We see that

$$
\partial_{\alpha'}(f^2) = 2\left(\frac{f}{w}\right)(wf')
$$

Now we integrate and use Cauchy Schwartz to get the estimate.
2) The L^∞ estimate is obtained from the first estimate by observing that

$$
\|f\|_\infty^2 \lesssim \left\|\frac{f}{w}\right\|_2\|w(f')\|_2 \lesssim \left\|\frac{f}{w}\right\|_2\|(wf)'\|_2 + \left\|\frac{f}{w}\right\|_2\|w'\|_2\|f\|_\infty
$$

Now use the inequality $ab\le\frac{a^2}{2\epsilon}+\frac{\epsilon b^2}{2}$ on the last term to obtain the estimate. For the $\dot{H}^{\frac{1}{2}}$ estimate, using $|\partial_{\alpha'}| = i\mathbb{H}\partial_{\alpha'}$ we see that

$$\|f\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \left| \int \left(\frac{\bar{f}}{w} \right) (w\mathbb{H}f') \, d\alpha' \right| \lesssim \left\| \frac{f}{w} \right\|_2 \|w\mathbb{H}f'\|_2$$

Now as $w\mathbb{H}f' = [w, \mathbb{H}]f' + \mathbb{H}(wf')$ we have

$$\|f\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \left\| \frac{f}{w} \right\|_2 (\|w'\|_2 \|f\|_\infty + \|wf'\|_2) \lesssim \left\| \frac{f}{w} \right\|_2 \|w'\|_2 \|f\|_\infty + \left\| \frac{f}{w} \right\|_2 \|(wf)'\|_2$$

Hence using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we see that

$$\|f\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \left\| \frac{f}{w} \right\|_2 \|(wf)'\|_2 + \left\| \frac{f}{w} \right\|_2^2 \|w'\|_2^2 + \|f\|_\infty^2 \lesssim \left\| \frac{f}{w} \right\|_2 \|(wf)'\|_2 + \left\| \frac{f}{w} \right\|_2^2 \|w'\|_2^2$$

□

Proposition A.0.11. *Let $f, g \in \mathcal{S}(\mathbb{R})$ and let $w, h \in L^\infty(\mathbb{R})$ be smooth functions with w being real valued. Also assume that $w', h' \in L^2(\mathbb{R})$. Then*

1. $\|fwh\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|fw\|_{\dot{H}^{\frac{1}{2}}} \|h\|_\infty + \|f\|_2 \|(wh)'\|_2 + \|f\|_2 \|w'\|_2 \|h\|_\infty$
2. $\|fgw\|_2 \lesssim \|fw\|_{\dot{H}^{\frac{1}{2}}} \|g\|_2 + \|gw\|_{\dot{H}^{\frac{1}{2}}} \|f\|_2 + \|f\|_2 \|g\|_2 \|w'\|_2$

Proof. 1) We see that

$$|\partial_{\alpha'}|^{\frac{1}{2}}(fwh) = [|\partial_{\alpha'}|^{\frac{1}{2}}, h]fw + h|\partial_{\alpha'}|^{\frac{1}{2}}(fw) = [|\partial_{\alpha'}|^{\frac{1}{2}}, hw]f + h[|\partial_{\alpha'}|^{\frac{1}{2}}, w]f + h|\partial_{\alpha'}|^{\frac{1}{2}}(fw)$$

The estimate now follows from the estimate $\|[|\partial_{\alpha'}|^{\frac{1}{2}}, g]f\|_2 \lesssim \||\partial_{\alpha'}|^{\frac{1}{2}} g\|_{BMO} \|f\|_2 \lesssim \|g'\|_2 \|f\|_2$
2) We observe that

$$\begin{aligned} fgw &= (\mathbb{P}_H f)(\mathbb{P}_H g)w + (\mathbb{P}_H f)(\mathbb{P}_A g)w + (\mathbb{P}_A f)(\mathbb{P}_H g)w + (\mathbb{P}_A f)(\mathbb{P}_A g)w \\ &= (\mathbb{P}_H f)\overline{(\mathbb{P}_A \bar{g})}w + (\mathbb{P}_H f)(\mathbb{P}_A g)w + (\mathbb{P}_A f)(\mathbb{P}_H g)w + (\mathbb{P}_A f)\overline{(\mathbb{P}_H \bar{g})}w \end{aligned}$$

We will control only the first term and the other terms are controlled similarly. Now see that

$$\left\| (\mathbb{P}_H f)\overline{(\mathbb{P}_A \bar{g})}w \right\|_2 = \|(\mathbb{P}_H f)(\mathbb{P}_A \bar{g})w\|_2$$

Hence we have

$$\begin{aligned} 2(\mathbb{P}_H f)(\mathbb{P}_A \bar{g})w &= (\mathbb{I} - \mathbb{H})\{(\mathbb{P}_H f)(\mathbb{P}_A \bar{g})w\} + (\mathbb{I} + \mathbb{H})\{(\mathbb{P}_H f)(\mathbb{P}_A \bar{g})w\} \\ &= [w\mathbb{P}_A \bar{g}, \mathbb{H}]\mathbb{P}_H f - [w\mathbb{P}_H f, \mathbb{H}]\mathbb{P}_A \bar{g} \end{aligned}$$

Now observe that as w is real valued we have

$$\begin{aligned}\|[w\mathbb{P}_A\bar{g},\mathbb{H}]\mathbb{P}_H f\|_2 &\lesssim \|[w\mathbb{P}_A\bar{g}]\|_{\dot{H}^{\frac{1}{2}}}\|\mathbb{P}_H f\|_2 \lesssim \left(\|[w,\mathbb{H}]\bar{g}\|_{\dot{H}^{\frac{1}{2}}} + \|w\bar{g}\|_{\dot{H}^{\frac{1}{2}}}\right)\|f\|_2 \\ &\lesssim \|w'\|_2\|g\|_2\|f\|_2 + \|wg\|_{\dot{H}^{\frac{1}{2}}}\|f\|_2\end{aligned}$$

Similarly we have

$$\begin{aligned}\|[w\mathbb{P}_H f,\mathbb{H}]\mathbb{P}_A\bar{g}\|_2 &\lesssim \|w\mathbb{P}_H f\|_{\dot{H}^{\frac{1}{2}}}\|\mathbb{P}_A\bar{g}\|_2 \lesssim \left(\|[w,\mathbb{H}]f\|_{\dot{H}^{\frac{1}{2}}} + \|wf\|_{\dot{H}^{\frac{1}{2}}}\right)\|g\|_2 \\ &\lesssim \|w'\|_2\|f\|_2\|g\|_2 + \|wf\|_{\dot{H}^{\frac{1}{2}}}\|g\|_2\end{aligned}$$

□

Proposition A.0.12. *Let $\mathbb{H}$ be the Hilbert transform and let $\mathcal{H},\widetilde{\mathcal{H}}$ be defined as in §5 and let $f, f_1, f_2, f_3, g \in \mathcal{S}(\mathbb{R})$. Let M be defined as in Lemma 5.3.3 and we will suppress the dependence of M i.e. we write $a \lesssim b$ instead of $a \leq C(M)b$. With this notation we have the following estimates*

1. $\|(\mathbb{H}-\mathcal{H})f\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f\|_2$
2. $\|(\mathbb{H}-\mathcal{H})f\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f\|_{\dot{H}^{\frac{1}{2}}}$
3. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]g\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f'\|_2\|g\|_1$
4. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]g\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f\|_{\dot{H}^{\frac{1}{2}}}\|g\|_2$
5. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f'\|_\infty\|g\|_2$
6. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f'\|_2\|g\|_\infty$
7. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f'\|_2\|g\|_{\dot{H}^{\frac{1}{2}}}$
8. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\left\{\|f'\|_{\dot{H}^{\frac{1}{2}}}\|g\|_2 + \|f'g\|_2\right\}$
9. $\|\partial_{\alpha'}[f_1,[f_2,\mathbb{H}-\widetilde{\mathcal{H}}]]\partial_{\alpha'}f_3\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f_1'\|_2\|f_2'\|_2\|f_3'\|_2$
10. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]g\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f'\|_2\|g\|_2$
11. $\|[f,\mathbb{H}-\widetilde{\mathcal{H}}]\partial_{\alpha'}(g)\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f'\|_\infty\|g\|_{\dot{H}^{\frac{1}{2}}}$
12. $\left\|[f_1,f_2;f_3]-[f_1,f_2;f_3]_{\widetilde{h}}\right\|_{L^\infty\cap\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f_1'\|_\infty\|f_2'\|_2\|f_3\|_2$
13. $\left\|[f_1,f_2;\partial_{\alpha'}f_3]-[f_1,f_2;\partial_{\alpha'}f_3]_{\widetilde{h}}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty\|f_1'\|_\infty\|f_2'\|_\infty\|f_3\|_{\dot{H}^{\frac{1}{2}}}$

Proof. To simplify the calculations we define

$$F(a,b) = \frac{f(a)-f(b)}{a-b} \qquad F_h(a,b) = \frac{f(a)-f(b)}{\widetilde{h}(a)-\widetilde{h}(b)}$$
$$G(a,b) = \frac{g(a)-g(b)}{a-b} \qquad G_h(a,b) = \frac{g(a)-g(b)}{\widetilde{h}(a)-\widetilde{h}(b)}$$
$$F_i(a,b) = \frac{f_i(a)-f_i(b)}{a-b} \qquad F_{ih}(a,b) = \frac{f_i(a)-f_i(b)}{\widetilde{h}(a)-\widetilde{h}(b)}$$
$$H(a,b) = \frac{(\widetilde{h}(a)-a)-(\widetilde{h}(b)-b)}{a-b} \qquad H_h(a,b) = \frac{(\widetilde{h}(a)-a)-(\widetilde{h}(b)-b)}{\widetilde{h}(a)-\widetilde{h}(b)}$$

We have the identities

$$\frac{F(\alpha',s)-F(\beta',s)}{\alpha'-\beta'} = \frac{F(\alpha',\beta')-F(\beta',s)}{\alpha'-s}$$
$$\frac{H_h(\alpha',s)-H_h(\beta',s)}{\alpha'-\beta'} = \frac{1}{\widetilde{h}(\alpha')-\widetilde{h}(s)}\left\{H(\alpha',\beta')-H_h(\beta',s)\left(\frac{\widetilde{h}(\alpha')-\widetilde{h}(\beta')}{\alpha'-\beta'}\right)\right\}$$

1. We see that

$$(\widetilde{\mathcal{H}}-\mathcal{H})f = \frac{1}{i\pi}\int \frac{1-\widetilde{h}_{\beta'}(\beta')}{\widetilde{h}(\alpha')-\widetilde{h}(\beta')} f(\beta')\,d\beta' = \widetilde{\mathcal{H}}((1-\widetilde{h}_{\alpha'})f)$$

   Hence as $\widetilde{\mathcal{H}}$ is bounded on L^2 we have $\|(\widetilde{\mathcal{H}}-\mathcal{H})f\|_2 \lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty \|f\|_2$. Now we have

$$(\mathbb{H}-\widetilde{\mathcal{H}})f = \frac{1}{i\pi}\int\left(\frac{1}{\alpha'-\beta'} - \frac{1}{\widetilde{h}(\alpha')-\widetilde{h}(\beta')}\right) f(\beta')\,d\beta' = \frac{1}{i\pi}\int \frac{H_h(\alpha',\beta')}{\alpha'-\beta'} f(\beta')\,d\beta'$$

   Now using corollary (A.0.4) we see that $\|(\mathbb{H}-\widetilde{\mathcal{H}})f\|_2 \lesssim \|\widetilde{h}'-1\|_\infty \|f\|_2$. Hence the required estimate follows.

2. Observe that $(\mathbb{H}-\mathcal{H})(1) = 0$ and that the kernel of this operator is

$$K(\alpha',\beta') = \frac{1}{\alpha'-\beta'} - \frac{\widetilde{h}_{\beta'}(\beta')}{\widetilde{h}(\alpha')-\widetilde{h}(\beta')}$$

   Hence this kernel satisfies

$$|K(\alpha'.\beta')| \lesssim \frac{\|\widetilde{h}_{\alpha'}-1\|_\infty}{|\alpha'-\beta'|} \qquad |\nabla_{\alpha'}K(\alpha'.\beta')| \lesssim \frac{\|\widetilde{h}_{\alpha'}-1\|_\infty}{|\alpha'-\beta'|^2}$$

   and by the first estimate we also have boundedness on L^2 with operator norm $\lesssim \|\widetilde{h}_{\alpha'}-1\|_\infty$.

Hence by Proposition A.0.5 we have the boundedness on $\dot{H}^{\frac{1}{2}}$.

3. Note that

$$[f, \mathbb{H} - \widetilde{\mathcal{H}}]g = \frac{1}{i\pi}\int F(\alpha', \beta')H_h(\alpha', \beta')g(\beta')\, d\beta'$$

and hence by Cauchy Schwartz we have

$$|[f, \mathbb{H} - \widetilde{\mathcal{H}}]g| \lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|g\|_1^{\frac{1}{2}} \left(\int |F(\alpha', \beta')|^2 |g(\beta')|\, d\beta'\right)^{\frac{1}{2}}$$

The estimate now follows from Hardy's inequality.

4. We see that

$$|[f, \mathbb{H} - \widetilde{\mathcal{H}}]g| \lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|g\|_2 \left(\int |F(\alpha', \beta')|^2\, d\beta'\right)^{\frac{1}{2}}$$

We now obtain the estimate easily as $\iint |F(\alpha', \beta')|^2\, d\beta'\, d\alpha' \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}}^2$.

5. We observe

$$\begin{aligned}
[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g) &= \frac{1}{i\pi}\int F(\alpha', \beta')H_h(\alpha', \beta')g_{\beta'}(\beta')\, d\beta' \\
&= \frac{1}{i\pi}\int \frac{H_h(\alpha', \beta')}{\alpha' - \beta'} f_{\beta'}(\beta')g(\beta')\, d\beta' + \frac{1}{i\pi}\int \frac{F(\alpha', \beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}\widetilde{h}_{\beta'}(\beta')g(\beta')\, d\beta' \\
&\quad - \frac{1}{i\pi}\int \frac{F(\alpha', \beta')H_h(\alpha', \beta')}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}\widetilde{h}_{\beta'}(\beta')g(\beta')\, d\beta'
\end{aligned}$$

The estimate now follows from corollory (A.0.4).

6. This also follows from the computation above and corollory (A.0.4).

7. We see that

$$\begin{aligned}
[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g) &= \frac{1}{i\pi}\int F(\alpha', \beta')H_h(\alpha', \beta')g_{\beta'}(\beta')\, d\beta' \\
&= \frac{1}{i\pi}\int \partial_{\beta'}(F(\alpha', \beta')H_h(\alpha', \beta'))(g(\alpha') - g(\beta'))\, d\beta'
\end{aligned}$$

Now as $F(\alpha', \beta') = \dfrac{f(\alpha') - f(\beta')}{\alpha' - \beta'}$, if the derivative falls on f then we can use estimate 4) above. All other terms are bounded pointwise by

$$\|\widetilde{h}_{\alpha'} - 1\|_\infty \int \left|\frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}\right|\left|\frac{g(\alpha') - g(\beta')}{\alpha' - \beta'}\right| d\beta'$$

Now use Cauchy Schwartz and Hardy's inequality.

8. We see that

$$\begin{aligned}
&[f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g) \\
&= \frac{1}{i\pi}\int F(\alpha', \beta')H_h(\alpha', \beta')g_{\beta'}(\beta')\, d\beta' \\
&= -\frac{1}{i\pi}\int (\partial_{\beta'}F(\alpha', \beta'))H_h(\alpha', \beta')g(\beta')\, d\beta' - \frac{1}{i\pi}\int \{\partial_{\beta'}H_h(\alpha', \beta')\}f_{\beta'}(\beta')g(\beta')\, d\beta' \\
&\quad - \frac{1}{i\pi}\int \left(\frac{F(\alpha', \beta') - f_{\beta'}(\beta')}{\alpha' - \beta'}\right)\{(\alpha' - \beta')\partial_{\beta'}(H_{\alpha', \beta'})\}g(\beta')\, d\beta'
\end{aligned}$$

Now we use $\partial_{\beta'}F(\alpha', \beta') = \frac{F(\alpha', \beta') - f_{\beta'}(\beta')}{\alpha' - \beta'}$ and the fact that their $L^2(\mathbb{R}\times\mathbb{R}, d\alpha'\, d\beta')$ is bounded by $\|f'\|_{\dot{H}^{\frac{1}{2}}}$, to obtain the required estimate.

9. We have

$$\begin{aligned}
&\partial_{\alpha'}[f_1, [f_2, \mathbb{H} - \widetilde{\mathcal{H}}]]\partial_{\alpha'}f_3 \\
&= \frac{1}{i\pi}\partial_{\alpha'}\int (f_1(\alpha') - f_1(\beta'))F_2(\alpha', \beta')H_h(\alpha', \beta')f_{3\beta'}(\beta')\, d\beta' \\
&= f_1'(\alpha')\left(\frac{1}{i\pi}\int F_2(\alpha', \beta')H_h(\alpha', \beta')f_{3\beta'}(\beta')\, d\beta'\right) \\
&\quad + f_2'(\alpha')\left(\frac{1}{i\pi}\int F_1(\alpha', \beta')H_h(\alpha', \beta')f_{3\beta'}(\beta')\, d\beta'\right) \\
&\quad + \frac{1}{i\pi}\int F_1(\alpha', \beta')F_2(\alpha', \beta')\big\{(\alpha' - \beta')(\partial_{\alpha'}H_h(\alpha', \beta')) - H_h(\alpha', \beta')\big\}f_{3\beta'}(\beta')\, d\beta'
\end{aligned}$$

Each of the terms are now easily controlled by previous estimates.

10. Note that

$$[f, \mathbb{H} - \widetilde{\mathcal{H}}]g = \frac{1}{i\pi}\int F(\alpha', s)H_h(\alpha', s)g(s)\, ds$$

Hence the L^∞ estimate follows immediately. We now show the $\dot{H}^{\frac{1}{2}}$ estimate. We see that

$$
\begin{aligned}
&\frac{([f, \mathbb{H} - \widetilde{\mathcal{H}}]g)(\alpha') - ([f, \mathbb{H} - \widetilde{\mathcal{H}}]g)(\beta')}{\alpha' - \beta'} \\
&= \frac{1}{i\pi} \int \frac{F(\alpha', s)H_h(\alpha', s) - F(\beta', s)H_h(\beta', s)}{\alpha' - \beta'} g(s)\, ds \\
&= \frac{F(\alpha', \beta')}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} g(s)\, ds - \frac{1}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} F(\beta', s) g(s)\, ds \\
&\quad + \frac{H(\alpha', \beta')}{i\pi} \int \frac{F(\beta', s)}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')} g(s)\, ds \\
&\quad - \frac{1}{i\pi} \left(\frac{\widetilde{h}(\alpha') - \widetilde{h}(\beta')}{\alpha' - \beta'} \right) \int \frac{F(\beta', s)}{\widetilde{h}(\alpha') - \widetilde{h}(s)} H_h(\beta', s) g(s)\, ds
\end{aligned}
$$

We can control each of the terms. The first term is controlled as

$$
\begin{aligned}
&\left\| \frac{F(\alpha', \beta')}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} g(s)\, ds \right\|_{L^2(\mathbb{R}\times\mathbb{R}, d\alpha'\, d\beta')} \\
&\lesssim \left\| \| F(\alpha', \beta') \|_{L^\infty(d\alpha')} \left\| \int \frac{H_h(\alpha', s)}{\alpha' - s} g(s)\, ds \right\|_{L^2(d\alpha')} \right\|_{L^2(d\beta')} \\
&\lesssim \| \widetilde{h}_{\alpha'} - 1 \|_\infty \| f' \|_2 \| g \|_2
\end{aligned}
$$

For the second term we have

$$
\begin{aligned}
&\left\| \frac{1}{i\pi} \int \frac{H_h(\alpha', s)}{\alpha' - s} F(\beta', s) g(s)\, ds \right\|_{L^2(\mathbb{R}\times\mathbb{R}, d\alpha'\, d\beta')} \\
&\lesssim \left\| \left\| \int \frac{H_h(\alpha', s)}{\alpha' - s} F(\beta', s) g(s)\, ds \right\|_{L^2(d\alpha')} \right\|_{L^2(d\beta')} \\
&\lesssim \| \widetilde{h}_{\alpha'} - 1 \|_\infty \| g \|_2 \left\| \| F(\beta', \cdot) \|_{L^\infty} \right\|_{L^2(d\beta')} \\
&\lesssim \| \widetilde{h}_{\alpha'} - 1 \|_\infty \| f' \|_2 \| g \|_2
\end{aligned}
$$

Similarly the third term is controlled as

$$
\begin{aligned}
&\left\| \frac{H(\alpha',\beta')}{i\pi} \int \frac{F(\beta',s)}{\widetilde{h}(\alpha') - \widetilde{h}(\beta')} g(s)\, ds \right\|_{L^2(\mathbb{R}\times\mathbb{R}, d\alpha'\, d\beta')} \\
&\lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \left\| \left\| \int \frac{F(\beta',s)}{\widetilde{h}(\alpha') - \widetilde{h}(s)} g(s)\, ds \right\|_{L^2(d\alpha')} \right\|_{L^2(d\beta')} \\
&\lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|g\|_2 \left\| \|F(\beta',\cdot)\|_{L^\infty} \right\|_{L^2(d\beta')} \\
&\lesssim \|\widetilde{h}_{\alpha'} - 1\|_\infty \|f'\|_2 \|g\|_2
\end{aligned}
$$

and the last term is controlled similarly to the second term. Hence we have the required estimate.

11. We see that the operator $T : g \mapsto [f, \mathbb{H} - \widetilde{\mathcal{H}}]\partial_{\alpha'}(g)$ is bounded on L^2 and that $T(1) = 0$. It is also easy to see that its kernel satisfies the conditions of Proposition A.0.5 and hence the estimate follows.

12. This is proved in exactly the same was as we proved estimate 10.

13. We again see that the operator $T : f_3 \mapsto [f_1, f_2; \partial_{\alpha'} f_3] - [f_1, f_2; \partial_{\alpha'} f_3]_{\widetilde{h}}$ is bounded on L^2 and that $T(1) = 0$. Its kernel also satisfies the conditions of Proposition A.0.5 and hence we have the estimate.

□

Proposition A.0.13. *Let $f, \partial_t f, \partial_t^2 f \in C(\mathbb{R}\times[0,T)) \cap L^\infty(\mathbb{R}\times[0,T))$ then for any $t \in [0,T)$ we have*

$$
\limsup_{s\to 0^+} \frac{\|f(\cdot, t+s)\|_\infty - \|f(\cdot,t)\|_\infty}{s} \leq \|\partial_t f(\cdot,t)\|_\infty
$$

Proof. Fix $s > 0$ satisfying $t + s \in [0,T)$ and for every $\epsilon > 0$ we find $a_\epsilon \in \mathbb{R}$ such that $\|f(\cdot, t+s)\|_\infty \leq |f|(a_\epsilon, t+s) + \epsilon$. Observe that $|f|(a_\epsilon, t) \leq \|f(\cdot,t)\|_\infty$ and hence we have

$$
\begin{aligned}
\|f(\cdot,t+s)\|_\infty - \|f(\cdot,t)\|_\infty &\leq |f|(a_\epsilon, t+s) - |f|(a_\epsilon, t) + \epsilon \\
&\leq |f(a_\epsilon, t+s) - f(a_\epsilon, t)| + \epsilon \\
&\leq \sup_{\substack{\alpha'\in\mathbb{R} \\ u\in(0,s)}} |\partial_t f(\alpha', t+u)| s + \epsilon
\end{aligned}
$$

Now let $\epsilon \to 0$ to get

$$
\frac{\|f(\cdot,t+s)\|_\infty - \|f(\cdot,t)\|_\infty}{s} \leq \sup_{u\in(0,s)} \|\partial_t f(\cdot, t+u)\|_\infty
$$

As $\partial_t^2 f \in L^\infty(\mathbb{R} \times [0,T))$, we take the limit as $s \to 0$ to finish the proof. $\square$

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