A Tale of Valuation Rings in Prime Characteristic

by

Rankeya Datta

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Doctoral Committee:

Professor Karen E. Smith, Chair
Associate Professor Bhargav Bhatt
Professor Melvin Hochster
Professor Mattias Jonsson
Associate Professor James Tappenden

Rankeya Datta

rankeya@gmail.com

ORCID iD: 0000-0001-7726-7289

To my family and teachers

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ABSTRACT

We examine valuation rings in prime characteristic from the lens of singularity theory defined using the Frobenius map. We show that valuation rings are always F-pure, while the question of Frobenius splitting is more mysterious. Using a characteristic-independent local monomialization result of Knaf and Kuhlmann [KK05], we are able to prove that Abhyankar valuations of functions fields over perfect ground fields are always Frobenius split. At the same time, we construct discrete valuation rings of function fields that do not admit any Frobenius splittings. Connections between F-singularities of valuation rings and the notion of defect of an extension of valuations are established. Our examination reveals that there is an intimate relationship between defect and Abhyankar valuations. We study tight closure of ideals of valuation rings, establishing a link between tight closure and Huber's notion of f-adic valued fields. Tight closure turns out to be an interesting closure operation only for those valued fields that are f-adic in the valuation topology. We also introduce a variant of Hochster and Huneke's notion of strong F-regularity [HH89], calling it F-pure regularity. F-pure regularity is a better notion of singularity in the absence of finiteness hypotheses, and we use it to recover an analogue of Aberbach and Enescu's splitting prime [AE05] in the valuative setting. We show that weak F-regularity and F-pure regularity coincide for a valuation ring, and both notions are equivalent to the ring being Noetherian. Thus, the various variants of F-regularity are perhaps reasonable notions of singularity only in the world

of Noetherian rings. In the final chapter, we prove a prime characteristic analogue of a result of Ein, Lazarsfeld and Smith [ELS03] on uniform approximation of valuation ideals associated to real-valued Abhyankar valuations. As a consequence, we deduce a prime characteristic Izumi theorem for real-valued Abhyankar valuations that admit a common smooth center.

CHAPTER I

Introduction

Notions of singularities defined using Frobenius—F-purity, Frobenius splitting and the various variants of F-regularity—have played a central role in commutative algebra and algebraic geometry over the last five decades. The primary goal of this thesis is to systematically describe these so-called F-singularities in the novel, but increasingly important *non-Noetherian* setting of *valuation rings*.

Valuation rings have a long history going back, at least, to the work of Hensel on p-adic numbers. Later Zariski popularized the use of valuations in algebraic geometry through his work on *local uniformization*, which is a local analogue of resolution of singularities [Zar40, Zar42, Zar44]. Moreover, although Hironaka in his ground-breaking work [Hir64a, Hir64b] did not use valuations to resolve singularities in characteristic 0, the only partial results on the resolution problem over fields of prime characteristic rely heavily on valuation-theoretic techniques [Abh56a, Abh66, CP08, CP09]. Valuations have been widely applied in number theory, model theory, birational algebraic geometry [Cut04, FJ04, dBP12, Bou12, JM12, Blu18], differential geometry [LL16, Liu16, Li17], tropical geometry [GRW], and various types of rigid geometries such as Tate's rigid analytic spaces [Tat71], Berkovich spaces [Ber90, Ber93] and Huber's adic spaces [Hub93, Hub94]. More recently, Berkovich and Huber's deep valuation-theoretic techniques have served as foundations for Kedlaya and Liu's relative p-adic Hodge theory [KL15] and Scholze's perfectoid spaces [Sch12]. The latter is already enjoying spectacular success in solving long-standing conjectures in geometry and algebra [Sch12, And16, Bha16, HM17, MS17]. Thus it is not an exaggeration to say that valuations are at the forefront of contemporary research.

In this thesis, we are going to examine valuation rings through the lens of prime characteristic singularity theory. Suppose R is a commutative ring of prime characteristic $p > 0$. The Frobenius map is the ring homomorphism

$$F : R \to R$$

sending each element to its p-th power. While simple enough, this map reveals deep structural properties of a Noetherian ring of prime characteristic, and it is a powerful tool for proving theorems about rings containing an arbitrary field (or varieties, say, over $\mathbb{C}$) by standard reduction to characteristic p techniques. Theories such as Frobenius splitting [MR85] and tight closure [HH90] are well-developed in the Noetherian setting. Since classically most motivating problems were inspired by algebraic geometry and representation theory, this assumption seemed natural and not very restrictive. Now, however, good reasons are emerging to study F-singularities in certain non-Noetherian settings as well. For example, one such setting is cluster algebras [FZ02]. An upper cluster algebra over $\mathbb{F}_p$ need not be Noetherian, but it was shown that it is always Frobenius split, and indeed, admits a "cluster canonical" Frobenius splitting [BMRS15].

The starting point of the use of the Frobenius map to study singularities in prime characteristic is the amazing discovery by Kunz that a Noetherian ring R is regular precisely when $R \xrightarrow{F} R$ is a flat map [Kun69]. In other words, the Frobenius map is able to completely detect regularity of a Noetherian ring. Kunz's result is also the main inspiration behind our thesis, since we show that

Theorem IV.2. *The Frobenius map is always flat for a valuation ring of prime characteristic.*

Thus, a valuation ring of characteristic p might be interpreted as a "non-Noetherian regular local ring".

One can weaken the demand that Frobenius is flat and instead require only that the Frobenius map is *pure* (see section 3.4). Hochster and Roberts observed that this condition, which they called *F-purity*, is often sufficient for controlling singularities of a Noetherian local ring, an observation at the heart of their famous theorem on the Cohen-Macaulayness of invariant rings [HR74, HR76]. Flatness of Frobenius implies that valuation rings of prime characteristic are *always* F-pure.

The most fundamental valuations in geometry, arising as orders of vanishing along prime divisors on normal varieties (called *divisorial valuations*), have valuation rings that are local rings of regular points of varieties. More generally, even though arbitrary valuation rings of prime characteristic behave like regular local rings (as evidenced by flatness of Frobenius), there are some that are decidedly more like local rings of regular points of varieties than others. These are the valuation rings associated to valuations ν of a function field K/k, with value group Γ_ν and residue field κ_ν, such that

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k = \operatorname{tr.deg} K/k.$$

For such a valuation ν, called an *Abhyankar valuation of K/k*, the value group Γ_ν is a free abelian group of finite rank and κ_ν is a finitely generated extension of k. A divisorial valuation is a special case of an Abhyankar valuation, and the non-divisorial

Abhyankar valuations are precisely those (see Example II.65(1)) for which

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) > 1.$$

In other words, Abhyankar valuations are analogues of divisorial valuations whose value groups can have higher rational rank. We will spend a considerable effort understanding Abhyankar valuations, proving, for example, that[1]

Theorem IV.30. *The valuation ring of an Abhyankar valuation of function field over a perfect ground field of prime characteristic is always Frobenius split.*

Frobenius splitting has well-known deep local and global consequences for algebraic varieties; see subsection 3.5.1 for some global consequences. In the local case, Frobenius splitting is said to be a "characteristic p analog" of log canonical singularities for complex varieties, whereas related properties correspond to other singularities in the minimal model program [HW02, Sch09b, Smi97, Tak08]. For projective varieties, Frobenius splitting is related to positivity of the anticanonical bundle; see [BK05, MR85, Smi00, SS10].

Although Abhyankar valuation rings of function fields are Frobenius split, the question of Frobenius splitting of valuation rings in general is quite subtle. For example, it is not difficult construct Noetherian valuation rings that are *not* Frobenius split (Example III.57). The obstruction to Frobenius splitting, at least in the Noetherian case, is tied to Grothendieck's notion of excellent rings (Definition III.13). We show that

Corollary III.56. *A Frobenius split Noetherian domain R with fraction field K such that $[K : K^p] < \infty$ must be excellent.*

[1] Results in the introduction are often stated with simpler hypotheses than in the main body for better readability.

When $[K : K^p] < \infty$, which is almost always satisfied in geometric situations, we say K is *F-finite*, that is, the Frobenius map of K is finite. Thus Corollary III.56 can be rephrased as saying that a generically F-finite, Frobenius split Noetherian domain has to be excellent. More generally, we are able to establish the following

Theorem III.50. *Let R be a generically F-finite Noetherian domain of characteristic p. The following are equivalent:*

1. *R is excellent.*

2. *R is F-finite.*

3. *The module $\operatorname{Hom}_R(F_*R, R)$ is non-trivial.*

4. *For all $e > 0$, $\operatorname{Hom}_R(F^e_*R, R)$ is non-trivial.*

5. *There exists $e > 0$ such that $\operatorname{Hom}_R(F^e_*R, R)$ is non-trivial.*

Here F^e_*R denotes the ring R with R-module structure obtained by restriction of scalars via the e-th iterate of Frobenius, $F^e : R \to R$. [2]

Finiteness of Frobenius is itself a very interesting constraint on valuation rings. For example,

Proposition IV.4. *A valuation ring V is F-finite if and only if F_*V is a free V-module of finite rank.*

As a consequence,

Corollary IV.5. *F-finite valuation rings are always Frobenius split.*

Proposition IV.4 would follow formally from flatness of Frobenius if finitely generated modules over valuation rings were finitely presented. But this is not the case –

[2] Using this notation, a Frobenius splitting is just an R-linear map $F_*R \to R$ that sends $1 \mapsto 1$.

the residue field of a non-Noetherian valuation ring of Krull dimension 1 is finitely generated as a module over the ring, but not finitely presented because the maximal ideal is not finitely generated. Thus Proposition IV.4 is not routine. In fact, its proof easily adapts to yield a result that is even valid in mixed characteristic.

Proposition IV.9. *A valuation ring of* arbitrary *characteristic is a direct summand of every module finite ring extension.*

Thus valuation rings, regardless of their characteristic, satisfy the conclusion of Hochster's direct summand conjecture (now a theorem by work of [And16, Bha16, HM17, Hoc73]) which states that a regular ring is a direct summand of every module finite ring extension. This further illustrates how similar valuation rings are to regular rings.

Using the theory of extensions of valuations, we are able to prove other interesting properties satisfied by F-finite valuation rings (see subsection 4.2.2). As an illustration of the type of results obtained, we have the following:

Theorem IV.15. *Let ν be a valuation of an F-finite field K of characteristic p with value group Γ_ν and residue field κ_ν. If the valuation ring R_ν is F-finite, then:*

1. *Γ_ν is p-divisible or $[\Gamma_\nu : p\Gamma_\nu] = p$.*
2. *If Γ_ν is finitely generated and non-trivial, then ν is a discrete valuation (i.e. Γ_ν is isomorphic to $\mathbb{Z}$).*

As a partial converse, if $[\kappa_\nu : \kappa_\nu^p] = [K : K^p]$, then R_ν is F-finite.

There is also a close relationship between the notions of F-finiteness and *defect* of an extension of valuations (see Definition IV.11 and [Kuh11] for a more general discussion). Specializing to our situation, if ν is a valuation of an F-finite field K of characteristic p and ν^p denotes the restriction of ν to the subfield $K^p \subset K$, then the extension ν/ν^p always satisfies

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] \leq [K : K^p].$$

If equality holds in the above inequality, we say ν/ν^p is *defectless*, and otherwise ν/ν^p *has defect*.

Proposition IV.10. *For a valuation ν of a field K, if the valuation ring of ν is F-finite then the extension ν/ν^p is defectless. That is, the following equality holds:*

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p].$$

Defect of the extension ν/ν^p also detects when a valuation is Abhyankar. Moreover, the relationship between defect and the Abhyankar condition even generalizes to a non-function field setting. When a valuation ν of an *arbitrary* field K is centered on a Noetherian, local domain $(R, \mathfrak{m}_R, \kappa_R)$ such that $\mathrm{Frac}(R) = K$, one has the following beautiful inequality established by Abhyankar [Abh56b, Theorem 1]:

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \mathrm{tr.deg}\, \kappa_\nu/\kappa_R \leq \dim R. \tag{1.1}$$

When equality holds in (1.1), ν behaves a lot like an Abhyankar valuation of a function field. For example, the value group Γ_ν is then again a free abelian group of finite rank, and the residue field κ_ν is finitely generated over κ_R. However, whether a valuation of a function field is Abhyankar is intrinsic to the valuation, while equality in (1.1) with respect to a center depends, unsurprisingly, on the center as well (see Example II.57(4) for an illustration). Bearing this difference in mind, we call a Noetherian center R an *Abhyankar center* of ν, if ν satisfies equality in (1.1) with respect to R.

In practice one is often interested in centers satisfying additional restrictions. For example, in the local uniformization problem for valuations of function fields, one seeks centers that are regular. Similarly, in geometric applications centers are usually local rings of varieties, and consequently essentially of finite type over the ground field. Although satisfying equality in (1.1) is not intrinsic to a valuation, the property of possessing Abhyankar centers from a more restrictive class of local rings may become independent of the center. For example, when K/k is a function field and $\mathcal{C}$ is the collection of local rings that are essentially of finite type over k with fraction field K, then a valuation ν admits an Abhyankar center from the collection $\mathcal{C}$ precisely when ν is an Abhyankar valuation of K/k, and consequently all centers of ν from $\mathcal{C}$ are Abhyankar centers of ν (Proposition II.64). In other words, the property of possessing Abhyankar centers that are essentially of finite type over k is intrinsic to valuations of function fields over k.

Our investigation reveals that even in a non-function field setting, one can find a broad class of Noetherian local domains such that the property of admitting an Abhyankar center from this class is independent of the choice of the center.

Theorem IV.19. *If a valuation ν of an F-finite field K of characteristic p is centered on an excellent local domain R, then a necessary and sufficient condition for R to be an Abhyankar center of ν is for ν/ν^p to be defectless.*

Since the defect of ν/ν^p is intrinsic to ν, this implies

Corollary IV.22. *For valuations of F-finite fields, the property of admitting excellent Abhyankar centers is independent of the choice of the excellent center.*

The analogue of Corollary IV.22 is false when K has characteristic 0, that is, the property of admitting excellent Abhyankar centers for valuations of fields of characteristic 0 is *not* independent of the choice of excellent center; see Remark IV.28(2).

Theorem IV.19 *does not* claim that a valuation of an F-finite field K is necessarily centered on an excellent local ring. In fact, the exact opposite situation is true since we are able to use Theorem IV.19 to systematically construct valuations of F-finite fields K that are not centered on any excellent domains with fraction field K.

Corollary IV.26. *Suppose ν is a valuation of an F-finite field K with valuation ring R_ν that satisfies either of the following conditions:*

1. *R_ν is F-finite, but not Noetherian.*

2. *$\dim(R_\nu) > s$, where $[K : K^p] = p^s$.*

Then ν is not centered on any excellent local domain whose fraction field is K.

Specializing to the case of function fields, Theorem IV.19 shows that

Corollary IV.23. *A valuation ν of a function field K/k of characteristic p is Abhyankar if and only if ν/ν^p is defectless.*

This has the following surprising consequence:

Corollary IV.25. *If ν is a valuation of a function field K/k such that the valuation ring R_ν is F-finite, then ν is divisorial. Consequently, R_ν is Noetherian.*

Perhaps the most intriguing aspect of the defect of ν/ν^p is its relation to Frobenius splitting. Since the valuation ring of an Abhyankar valuation of a function field K over perfect ground field k of prime characteristic is Frobenius split, it follows by Corollary IV.23 that defectless valuations of K/k are Frobenius split. On the other hand, when ν/ν^p has maximal defect, then the valuation ring of ν is *never* Frobenius split.

Proposition IV.29. *Let K be a non-perfect field of characteristic p and ν be a valuation of K such that*

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = 1,$$

that is, ν/ν^p has maximal defect. Then the valuation ring R_ν is not Frobenius split.

Nevertheless, Frobenius splitting is *not* well-understood when the defect of ν/ν^p is not one of two possible extremes.

Closely related to Frobenius splitting and F-purity are the various variants of F-regularity. Strong F-regularity was introduced by Hochster and Huneke [HH89] as a proxy for weak F-regularity — the property that all ideals are tightly closed — because strong F-regularity is easily shown to pass to localizations. Whether or not a weakly F-regular ring remains so after localization is a long-standing open question in tight closure theory, as is the equivalence of weak F-regularity and strong F-regularity. Strong F-regularity has found many applications beyond tight closure [AL, BMRS15, Bli08, BK05, GLP$^+$15, HX15, Pat08, ST12, Sch09a, SS10, SVdB97, SZ15, Smi00], and is closely related to Ramanathan's notion of "Frobenius split along a divisor" [Ram91, Smi00].

Traditionally, strong F-regularity has been defined only for Noetherian F-finite rings. To clarify the situation for valuation rings, we introduce a definition which we call *F-pure regularity* (see Definition III.68) requiring *purity* rather than *splitting* of certain maps. We show that F-pure regularity is better suited for arbitrary rings, in the absence of finiteness conditions. Even in the world of Noetherian rings, regular local rings are *always* F-pure regular, although there exists non-F-finite regular rings that are not strongly F-regular. For example, any generically F-finite, non-excellent regular local ring is not strongly F-regular (Theorem III.50).

F-pure regularity also agrees with another, more technical, generalization of strong F-regularity proposed by Hochster [Hoc07], using tight closure, in the local Noetherian case (Proposition III.76). Given the natural interplay between tight closure and F-regularity, we first examine tight closure in the setting of ideals of valuation rings. Tight closure tends to be quite a lossy operation for valuation rings.

Lemma IV.40. *Let V be a valuation ring of characteristic p, and I an ideal of V. If Q is a non-zero prime ideal of V such that $Q \subsetneq I$, then the tight closure I^* of I equals V.*

The existence of ideals of a valuation ring that do not properly contain a non-zero prime ideal is closely related to Huber's notion of an *f-adic* ring (Definition II.32). Just as commutative rings are the local algebraic objects in scheme theory, f-adic rings are the local algebraic objects in Huber's approach to rigid analytic geometry, which is witnessing a resurgence of interest because of its applications in Scholze's ground-breaking work on perfectoid spaces [Sch12].

By definition, f-adic rings are topological rings satisfying some additional natural hypotheses. Any field K equipped with a valuation ν, henceforth called a *valued field*, has a *valuation topology* induced by ν under which K becomes a topological field. It is then natural to ask if K in its valuation topology is an f-adic field. In prime characteristic, the f-adic valued fields are those fields for which tight closure is an interesting operation.

Proposition IV.41. *Let ν be a non-trivial valuation of a field K of prime characteristic. The following are equivalent:*

1. *K is f-adic in the valuation topology induced by ν.*

2. *There exists a non-zero ideal I of the valuation ring R_ν such that $I^* \neq R_\nu$.*

A non-trivially valued field is f-adic in the valuation topology when its corresponding valuation ring has a height 1 prime ideal (Theorem II.39), a condition which is automatically satisfied for valuation rings of finite Krull dimension. In terms of this height 1 prime, tight closure can be characterized as follows:

Theorem IV.42. *Let ν be a non-trivial valuation of a field K of characteristic p such that K is f-adic in the topology induced by ν. Let I be ideal of R_ν and $\mathfrak{p}$ be the unique height 1 prime of R_ν.*

1. *If $\mathfrak{p} \subsetneq I$, then $I^* = R_\nu$.*

2. *If $I \subsetneq \mathfrak{p}$, then $I^* \neq R_\nu$.*

3. *$\mathfrak{p}^* \neq R_\nu$ if and only if $(R_\nu)_\mathfrak{p}$ is a discrete valuation ring. In this case $\mathfrak{p}^* = \mathfrak{p}$.*

For the expert we note that $(R_\nu)_\mathfrak{p}$ is the ring of power bounded elements of the f-adic valued field K, and $\mathfrak{p}$ is the collection of topologically nilpotent elements of K.

As a consequence of Theorem IV.42, one can precisely say when a valuation ring satisfies the defining property of weak F-regularity.

Corollary IV.44. *Let ν be a valuation of a field K of characteristic p. The following are equivalent:*

1. *All ideals of R_ν are tightly closed.*

2. *The maixmal ideal $\mathfrak{m}_\nu$ is tightly closed.*

3. *R_ν is Noetherian.*

Despite a relatively simple definition, tight closure is devilishly difficult to compute in practice for ideals of Noetherian rings. Valuation rings behave differently in this aspect.

Proposition IV.45. *Let ν be a non-trivial valuation of a field K of characteristic p such that K is f-adic in the valuation topology induced by ν. Let $\mathfrak{p}$ be the height 1 prime ideal of the valuation ring R_ν of ν, and*

$$w : K^\times \to \mathbb{R}$$

be a valuation whose corresponding valuation ring is $(R_\nu)_\mathfrak{p}$. For an ideal $I \subseteq \mathfrak{p}$, if

$$\mathfrak{a} := \inf\{w(i) : i \in I - \{0\}\}.$$

then

$$I^* = \{x \in R_\nu : w(x) \geq \mathfrak{a}\} \cup \{0\}.$$

In particular, $I^* = I$ if $\mathfrak{a} \notin \Gamma_w$.

As a pleasing outcome of the previous proposition, we find that tight closure is indeed a closure operation for ideals of valuation rings, a fact that is not obvious in the non-Noetherian case.

Corollary IV.46. *For any ideal I of a valuation ring V of prime characteristic, $(I^*)^* = I^*$, that is, I^* is tightly closed.*

Having obtained a fairly satisfactory picture of tight closure of ideals of valuation rings, we turn our attention to F-pure regularity in the valuative setting. Our prior considerations reveal that an analogue of Aberbach and Enescu's splitting prime in the Noetherian setting [AE05] exists for valuation rings of prime characteristic, provided splitting of certain maps is replaced by purity.

Theorem IV.50. *The set of elements c along which a valuation ring $(V, \mathfrak{m}_V, \kappa_V)$ of characteristic p fails to be F-pure is the prime ideal*

$$\bigcap_{e \in \mathbb{N}} \mathfrak{m}_V^{[p^e]}.$$

Combined with observations on powers of ideals of valuation rings (see Section 2.6), we are able to show that

Theorem IV.49. *A valuation ring is F-pure regular if and only if it is Noetherian.*

An amusing consequence of Theorem IV.49 and Corollary IV.44 is that weak F-regularity and F-pure regularity coincide for valuation rings of prime characteristic, and both are equivalent to the valuation ring being Noetherian.

We finally end our investigation of F-singularities of valuation rings by comparing our generalization of strong F-regularity with the obvious competing generalization, in which the standard definition in terms of splitting certain maps is naively extended without assuming any finiteness conditions. To avoid confusion with the existing definition of strong F-regularity, we call this *split F-regularity.* We characterize split F-regular valuation rings of F-finite fields as precisely those that are Noetherian and Frobenius split, or equivalently *excellent*; see Corollary IV.58.

Following our study of valuation rings in prime characteristic, we switch gears to study the effect of valuation rings on Noetherian rings, concentrating, in particular, on the interaction between real-valued Abhyankar valuations and their centers. A real-valued valuation ν of a function field K/k centered on a variety X of K/k determines, for any $m \in \mathbb{R}_{\geq 0}$, quasi-coherent ideal sheaves $\mathfrak{a}_m$, consisting of local sections f of $\mathcal{O}_X$ such that $\nu(f) \geq m$. When $X = \operatorname{Spec}(A)$, we use $\mathfrak{a}_m(A)$ to denote the ideal $\{a \in A : \nu(a) \geq m\}$ of A.

For a natural number ℓ, clearly

$$\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m}.$$

Ein, Lazarsfeld and Smith proved the surprising fact that when X is a smooth variety in characteristic 0, the ideal $\mathfrak{a}_{\ell m}$ is also contained in the ℓ-th power of a shift, $\mathfrak{a}_{m-e}$, of $\mathfrak{a}_m$, where the shift e can be chosen independent of m or ℓ [ELS03]. In this thesis, we prove the prime characteristic analogue of this result.

Theorem V.1. *Let X be a regular (equivalently smooth) variety over a perfect field k of prime characteristic with function field K. For any non-trivial, real-valued Abhyankar valuation ν of K/k centered on X, there exists $e \geq 0$, such that for all $m \in \mathbb{R}_{\geq 0}$ and $\ell \in \mathbb{N}$,*

$$\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m} \subseteq \mathfrak{a}_{m-e}^\ell.$$

The proof of the characteristic 0 analogue of Theorem V.1 uses embedded resolution of singularities. Since resolution of singularities is still open in prime characteristic, we use a local monomialization result of Knaf and Kuhlmann, valid for Abhyankar valuations of *arbitrary* characteristic.

Theorem II.69. [KK05] *Let K be a finitely generated field extension of any field k, and ν an Abhyankar valuation of K/k with valuation ring $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$. Suppose $d := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ and κ_ν is separable over k. Then given any finite subset $Z \subset R_\nu$, there exists a variety X of K/k, and a center x of ν on X satisfying the following properties:*

1. *x is a smooth point of X/k and $\mathcal{O}_{X,x}$ is a regular local ring of dimension d.*
2. *$Z \subseteq \mathcal{O}_{X,x}$, and there exists a regular system of parameters $x_1, \ldots, x_d$ of $\mathcal{O}_{X,x}$ such that every $z \in Z$ admits a factorization*

$$z = u x_1^{a_1} \ldots x_d^{a_d},$$

for some $u \in \mathcal{O}_{X,x}^\times$ and $a_i \in \mathbb{N} \cup \{0\}$.

When the ground field k is perfect, the residue field κ_ν of any Abhyankar valuation of K/k is always separable over k because finitely generated field extensions of perfect fields are separable. Thus *every* Abhyankar valuation over a perfect ground field satisfies the conclusion of Theorem II.69.

The other key ingredient in the proof of the characteristic 0 version of Theorem V.1 is the machinery of *multiplier ideals*, whose properties require deep vanishing theorems that are not known in positive characteristic. More precisely, Ein, Lazarsfeld and Smith employ an asymptotic version of multiplier ideals, which was first used by them in [ELS01] in order to prove a uniformity statement about symbolic powers of ideals on smooth varieties. Over the years it has become clear that in prime characteristic a *test ideal* is an analogue of a multiplier ideal. Introduced by Hochster and Huneke in their work on tight closure [HH90], the first link between test and multiplier ideals was forged by Smith [Smi00] and Hara [Har01], following which Hara and Yoshida introduced the notion of test ideals of pairs [HY03]. Even in the absence of vanishing theorems in positive characteristic, test ideals of pairs were shown to satisfy many of the usual properties of multiplier ideals of pairs that make the latter such an effective tool in birational geometry [HY03, HT04, Tak06] (see also Theorem V.23).

Drawing inspiration from [ELS03], we use an asymptotic version of the test ideal of a pair to prove Theorem V.1. However, instead of utilizing tight closure machinery, our approach to asymptotic test ideals is based on Schwede's dual and simpler reformulation of test ideals using *p^{-e}-linear maps*, which are like maps inverse to Frobenius [Sch10, Sch11] (see also [Smi95, LS01]).

Asymptotic test ideals are associated to graded families of ideals (Definition V.25), an example of the latter being the family of valuation ideals $\mathfrak{a}_\bullet := \{\mathfrak{a}_m\}_{m \in \mathbb{R}_{\geq 0}}$. For each $m \geq 0$, one constructs the *m-th asymptotic test ideal* $\tau_m(A, \mathfrak{a}_\bullet)$ of the family $\mathfrak{a}_\bullet$, and then Theorem V.1 is deduced using

Theorem V.2. *Let ν be a non-trivial real-valued Abhyankar valuation of K/k, centered on a regular local ring $(A, \mathfrak{m})$, where A is essentially of finite type over the perfect field k of prime characteristic with fraction field K. Then there exists $r \in A - \{0\}$ such that for all $m \in \mathbb{R}_{\geq 0}$,*

$$r \cdot \tau_m(A, \mathfrak{a}_\bullet) \subseteq \mathfrak{a}_m(A).$$

In other words, $\bigcap_{m \in \mathbb{R}_{\geq 0}} (\mathfrak{a}_m : \tau_m(A, \mathfrak{a}_\bullet)) \neq (0)$.

Finally, as in [ELS03], Theorem V.2 also gives a new proof of a prime characteristic version of Izumi's theorem for arbitrary real-valued Abhyankar valuations with a common regular center (see also the more general work of [RS14]).

Corollary V.3. (Izumi's Theorem for Abhyankar valuations in prime characteristic) *Let ν and w be non-trivial real-valued Abhyankar valuations of K/k, centered on a regular local ring $(A, \mathfrak{m})$, as in Theorem V.2. Then there exists a real number $C > 0$ such that for all $x \in A - \{0\}$,*

$$\nu(x) \leq Cw(x).$$

Corollary V.3 implies that the valuation topologies on A induced by two non-trivial real-valued Abhyankar valuations are *linearly* equivalent.

The use of F-singularity techniques to study valuation rings in prime characteristic began in work of the author and Karen Smith [DS16, DS17a]. Although a sizable portion of this thesis will highlight our joint work, recent results obtained by the author reveal that F-singularities of valuation rings are often best understood by analyzing F-singularities of the Noetherian centers of such rings [Dat17a]. We will focus more on describing this new perspective, often obtaining considerable generalizations of prior results in [DS16, DS17a] in the process. In addition, there is a lot of new material – the discussion of tight closure in the valuative setting in Chapter 4 has not appeared in published form. Moreover the work on uniform approximation of Abhyankar valuation ideals in Chapter 5 is independent of Chapter 4.

CHAPTER II

A glimpse of valuation theory

In this chapter we review those notions of valuation theory that will be used in the rest of the thesis. Stated results will usually not be accompanied by proofs, but appropriate references will be given. The material in Sections 2.6 and 2.8 are somewhat non-standard. The basic reference for this chapter is [Bou89, Chapter VI].

2.1 Local rings

By a **local ring** we mean a ring with a unique maximal ideal which is *not necessarily Noetherian.* Local rings will often be denoted $(A, \mathfrak{m}_A, \kappa_A)$. Here A is the local ring, $\mathfrak{m}_A$ is its maximal ideal and $\kappa_A = A/\mathfrak{m}_A$ is the residue field.

A homomorphism of local rings $\varphi : A \to B$ is called a **local homomorphism** if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. Note that a local homomorphism induces a map of residue fields $\kappa_A \hookrightarrow \kappa_B$.

Given local rings A and B, we say B **dominates** A if A is a subring of B, and the inclusion $A \hookrightarrow B$ is a local homomorphism of local rings, that is, if $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.

If K is a field, then the relation of domination induces a partial ordering among the collection of local subrings of K.

2.2 Valuation rings

Throughout this section let K denote a field of arbitrary characteristic.

Definition II.1. A subring V of K is called a **valuation ring of** K if for all $x \in K$, $x \in V$ or $x^{-1} \in V$.

Note that K is trivially a valuation ring of itself, called the **trivial valuation ring**. We collect some basic properties of valuation rings.

Proposition II.2. *Let V be a valuation ring of a field K. Then we have the following:*

1. *K is the fraction field of V.*

2. *V is a local domain.*

3. *V is integrally closed in K.*

4. *V is a maximal element of the collection of local subrings of K partially ordered by the relation of domination.*

5. *The collection of ideals of V is totally ordered by inclusion.*

6. *There exists an algebraically closed field L and a ring homomorphism $f : V \to L$ which is maximal among the collection of ring homomorphisms from subrings of K to L ordered by the relation of extension of homomorphisms.*

7. *If A is a subring of K with fraction field K, then the integral closure $\overline{A}$ of A in K is the intersection of all valuation rings of K that contain A. If A is local, then $\overline{A}$ equals the intersection of those valuation rings of K that dominate A.*

Proof. See [Bou89, Chapter VI] for proofs of these assertions. □

Remarks II.3.

(a) Conditions $(4), (5)$ and (6) in Proposition II.2 are equivalent to the defining property of a valuation ring.

(b) Condition (5) in Proposition II.2 implies that if x and y are two elements of K, then $xV \subseteq yV$ or $yV \subseteq xV$. Thus finitely generated ideals of a valuation ring are principal, and so, the only non-trivial Noetherian valuation rings are local principal ideal domains, also known as **discrete valuation rings**.

(c) If V is a valuation ring of a field K, then any subring B of K such that $V \subseteq B \subseteq K$ is also clearly a valuation ring of K. Thus B is a local ring. If $\mathfrak{p}$ is the prime ideal $\mathfrak{m}_B \cap V$, then B dominates the local ring $V_{\mathfrak{p}}$. However, $V_{\mathfrak{p}}$ is also a valuation ring of K for the same reason B is. By part (4) of Proposition II.2, we then get

$$B = V_{\mathfrak{p}}.$$

Thus localization at prime ideals induces a bijection between the underlying set of $\mathrm{Spec}(V)$ and the collection of subrings of K that contain V. In particular, if V has Krull dimension 1, then V is maximal (with respect to inclusion) among the collection of proper subrings of K.

(d) If A is a Noetherian subring of a field K, then the integral closure of A in K equals the intersection of discrete valuation rings of K that contain A (c.f. Proposition II.2, part 7).

2.3 Valuations

The simplest way to construct valuation rings is via the notion of a valuation, which we now introduce. For this, recall that a **totally ordered abelian group**

Γ is an ordered abelian group equipped with a total ordering $\leq$ such that for all $\alpha, \beta, \gamma \in \Gamma$, $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$. In other words, the total ordering on Γ is compatible with the group structure. It is easy to verify that totally ordered abelian groups are torsion-free.

Definition II.4. A **valuation** ν of a field K is a group homomorphism

$$\nu : K^\times \to \Gamma,$$

where Γ is a totally ordered abelian group, such that for all $x, y \in K^\times$, if $x + y \neq 0$, then $\nu(x+y) \geq \inf\{\nu(x), \nu(y)\}$. The subgroup $\nu(K^\times)$ is called the **value group** of ν, and denoted Γ_ν. We say ν is **trivial** if Γ_ν is the trivial group. If K is a field extension of k, then ν is a **valuation of** K/k if ν is a valuation of K such that $\nu(k^\times) = \{0\}$. A field equipped with a valuation will be often called a **valued field**.

If ν is a valuation of K, then the set

$$R_\nu \coloneqq \{x \in K^\times : \nu(x) \geq 0\} \cup \{0\}$$

is a valuation ring of K with maximal ideal

$$\mathfrak{m}_\nu \coloneqq \{x \in K^\times : \nu(x) > 0\} \cup \{0\}.$$

The units of R_ν are precisely those elements $x \in K^\times$ such that $\nu(x) = 0$. Thus valuations of a field give rise to valuation rings in a natural way. Note that if ν is a valuation of K/k, then the valuation ring R_ν and the residue field κ_ν are k-algebras.

Conversely, if V is a valuation ring of a field K, then one can give the group $K^\times/V^\times$ ($V^\times$ is the group of units of V) a total ordering as follows: for $x, y \in K^\times$, define $xV^\times \leq yV^\times$ if and only if $yV \subseteq xV$. It is then straightforward to verify that the projection map

$$\pi : K^\times \twoheadrightarrow K^\times/V^\times$$

is a valuation of K whose associated valuation ring R_π is precisely V. Hence there is a canonical way to construct a valuation from a valuation ring.

Notation II.5. If ν is a valuation of K, then its valuation ring will always be denoted $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$. The value group of ν will be denoted Γ_ν.

Valuation rings of K are in one-to-one correspondence with valuations of K up to equivalence of valuations. We say two valuations ν, w of a field K are **equivalent** if there exists an ordered isomorphism of value groups $\varphi : \Gamma_\nu \xrightarrow{\sim} \Gamma_w$ such that $w = \varphi \circ \nu$.

Lemma II.6. *Let ν be a valuation of a field K. If $x, y \in K^\times$ such that $\nu(x) \neq \nu(y)$, then*

$$\nu(x + y) = \inf\{\nu(x), \nu(y)\}.$$

Proof. Assume without loss of generality that $\nu(x) < \nu(y)$. Then $x+y = x(1+y/x)$, where $y/x \in \mathfrak{m}_\nu$. Thus, $1+y/x$ is a unit in R_ν, and so $\nu(x+y) = \nu(x)$, as desired. □

Remark II.7. In this thesis, valuations will be written additively instead of multiplicatively in the sense that the binary operation on the value group will be written as $+$ instead $\cdot$. The use of multiplicative notation, even for valuations of rank > 1 (see Section 2.4 for a definition of rank), is common in rigid geometry.

2.4 Rank of a valuation

Definition II.8. The **rank** of a valuation ν of a field K, denoted $\operatorname{rank}(\nu)$, is the Krull dimension of the associated valuation ring R_ν.

Remark II.9. The rank of ν equals the cardinality of the collection of non-trivial *convex/isolated* subgroups of Γ_ν [Bou89, Chapter VI, §4].

Proposition II.10. *Let ν be a valuation on a field K with value group Γ_ν. If $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) < \infty$, then*

$$\operatorname{rank}(\nu) \leq \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu).$$

Proof. See [Bou89, Chapter VI, §10.2, Corollary]. □

Definition II.11. The number $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ is usually called the **rational rank of** ν.

Please note that despite the similar sounding terminology, the rank of a valuation is usually very different from the rational rank of the valuation.

One has the following characterization of valuations of rank 1, which is a good illustration of how small the rank of a valuation can be compared to its rational rank.

Proposition II.12. *Let ν be a valuation of a field K with value group Γ_ν. Then ν has rank 1 if and only if Γ_ν is order isomorphic to a non-trivial additive subgroup of $\mathbb{R}$.*

Proof. This is proved in [Bou89, Chapter VI, §4.5, Proposition 8]. □

Thus the value group Γ_ν of a rank 1 valuation ν is **Archimedean**, that is, for any $\alpha, \beta \in \Gamma_\nu$, there exists $n \in \mathbb{N}$ such that

$$n\alpha > \beta.$$

Recall that a non-trivial Noetherian valuation ring of a field K is precisely a local principal ideal domain. We can completely characterize Noetherian valuation rings in terms of the value groups of their corresponding valuations.

Lemma II.13. *Let ν be a non-trivial valuation of a field K with value group Γ_ν and valuation ring R_ν. Then R_ν is Noetherian if and only if Γ_ν is order isomorphic to $\mathbb{Z}$.*

Proof. If R_ν is Noetherian and π is a generator of its maximal ideal, then $\Gamma_\nu = \mathbb{Z}\nu(\pi)$. So Γ_ν is clearly order-isomorphic to $\mathbb{Z}$. Conversely, if Γ_ν is order isomorphic to $\mathbb{Z}$, let $\gamma \in \Gamma_\nu$ be a generator of Γ_ν such that $\gamma > 0$. If $x \in R_\nu$ such that $\nu(x) = \gamma$, then one can verify that every ideal of R_ν is generated by some power of x, and so R_ν is Noetherian. □

2.5 Torsion-free modules over a valuation ring

The next result will be crucial in our study of valuation rings in prime characteristic.

Proposition II.14. *Let V be a valuation ring and M be a finitely generated, torsion-free V-module. Then M is a free V-module.*

Proof. The proof is similar to the proof of Nakayama's lemma. Since M is finitely generated, let $d \in \mathbb{N} \cup \{0\}$ be the smallest non-negative integer such that M is generated by a set $\{m_1, \ldots, m_d\}$ of cardinality d. We claim that $\{m_1, \ldots, m_d\}$ is linearly independent. If not, then there exists a non-trivial relation

$$x_1 m_1 + \cdots + x_d m_d = 0,$$

where the $x_i \in V$ are not all 0. By re-arranging the m_i, we may assume without loss of generality that $x_1 \neq 0$ and $x_1 | x_i$, for all $i \geq 1$. Since M is torsion free, this means that m_1 is in the linear span of $\{m_2, \ldots, m_d\}$, contradicting our choice of d. □

As a consequence, for torsion-free modules that are not necessarily finitely generated, we obtain the following result:

Corollary II.15. *Any torsion-free module over a valuation ring is flat.*

Proof. Any torsion-free module is a filtered direct limit of its torsion-free, finitely generated submodules which are all free, hence flat by Proposition II.14. But a filtered direct limit of flat modules is flat [Bou89, Chapter I, §2.3, Proposition 2]. $\square$

Example II.16. Let R be a Dedekind domain. Using the results of this section we recover the well-known fact that torsion free R-modules are flat. Indeed, flatness can be checked locally, and if M is a torsion-free R-module, then for any prime ideal $\mathfrak{p}$ of R, $M_{\mathfrak{p}}$ is a torsion-free module over the Noetherian valuation ring $R_{\mathfrak{p}}$. Hence $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module.

2.6 Ideals of valuation rings

Recall that any two ideals of a valuation ring are comparable under inclusion. This property makes valuation rings special from algebraic and geometric viewpoints. For instance, algebraically we see that any finitely generated ideal of a valuation ring is principal, which from the geometric perspective means that any quasi-compact open subset of the spectrum of a valuation ring is a *distinguished open set* (a set of the form $D(f)$).

Another fun observation is that for ideals of a valuation ring, the axiom of being closed under addition is redundant.

Lemma II.17. *Let V be a valuation ring. Suppose I is a non-empty subset of V such that for all $x \in V$ and $i \in I$, $xi \in I$. Then I is an ideal of V.*

Proof. The hypothesis implies $0 \in I$ (taking $x = 0$). Thus it suffices to show I is closed under addition. Suppose $i, j \in I$, and assume that they are not both 0. Without loss of generality we may suppose $i|j$. Then $i + j = (1 + j/i)i \in I$ by hypothesis because $1 + j/i$ is an element of V and $i \in I$. $\square$

In this remainder of this section we highlight some other interesting properties of valuation rings.

2.6.1 Generators of prime ideals of valuation rings

We have seen that valuation rings are usually not Noetherian. A non-Noetherian valuation ring must have a prime ideal which is not finitely generated, because Cohen proved that when all prime ideals of a ring are finitely generated, then the ring is Noetherian [Mat89, Theorem 3.4].

The next result shows that a non-maximal, non-zero prime ideal of a valuation ring is never finitely generated.

Lemma II.18. *Let $(V, \mathfrak{m}_V, \kappa_V)$ be a valuation ring and $\mathfrak{p}$ be a prime ideal of V. If $\mathfrak{p}$ is finitely generated, then $\mathfrak{p} = (0)$ or $\mathfrak{p} = \mathfrak{m}_V$.*

Proof. Suppose $\mathfrak{p} \neq (0)$, and let $x \in V - \mathfrak{p}$. To show that $\mathfrak{p} = \mathfrak{m}_V$, it suffices to show that x is a unit. Since any two ideals of V are comparable, we get $\mathfrak{p} \subsetneq (x)$. As finitely generated ideals of valuation rings are principal, let $t \neq 0$ be a generator of $\mathfrak{p}$. Then there exists $a \in V$ such that

$$t = ax.$$

But $x \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime. Thus, $a \in \mathfrak{p}$, that is, $a = tu$, for some $u \in V$, and so

$$1 = ux,$$

proving that x is a unit. $\square$

Valuation rings with finitely generated maximal ideals can be characterized in terms of properties of their value groups.

Proposition II.19. *Let ν be a non-trivial valuation of a field K with valuation ring $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$ and value group Γ_ν. Then the following are equivalent:*

1. $\mathfrak{m}_\nu$ *is finitely generated.*

2. Γ_ν *has a smallest element* > 0.

3. $\mathfrak{m}_\nu \neq \mathfrak{m}_\nu^2$.

Proof. Since ν is a non-trivial valuation of K, $\mathfrak{m}_\nu$ is not the zero ideal. Therefore $1 \Rightarrow 3$ follows by Nakayama's lemma. Suppose Γ_ν has a smallest element > 0, say γ. If $t \in R_\nu$ such that $\nu(t) = \gamma$, then $\mathfrak{m}_\nu = (t)$. This shows that $2 \Rightarrow 1$. Thus to finish the proof it suffices to show that $3 \Rightarrow 2$. Assume for contradiction that Γ_ν does not have a smallest element > 0. Let $x \in \mathfrak{m}_\nu$ be a non-zero element. Then by our assumption, there exists $\alpha \in \Gamma_\nu$ such that

$$0 < \alpha < \nu(x).$$

Furthermore, there must then also exist $\beta \in \Gamma_\nu$ such that

$$0 < \beta < \inf\{\alpha, \nu(x) - \alpha\}.$$

Let $y \in \mathfrak{m}_\nu$ such that $\nu(y) = \beta$. Then $\nu(y^2) = 2\beta < \alpha + (\nu(x) - \alpha) = \nu(x)$. Thus $y^2 | x$, that is, $x \in \mathfrak{m}_\nu^2$. This shows $\mathfrak{m}_\nu = \mathfrak{m}_\nu^2$, a contradiction. □

Examples II.20.

(a) The maximal ideal of any discrete valuation ring is finitely generated.

(b) Let $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\pi \subset \mathbb{R}$. Let ν be the unique valuation of $\mathbb{F}_p(X, Y)$ with value group Γ such that $\nu(X) = 1$ and $\nu(Y) = \pi$. Then ν has rank 1, but R_ν is not Noetherian since Γ cannot be order isomorphic to $\mathbb{Z}$. Therefore the maximal ideal of R_ν is not finitely generated. More generally, the maximal ideal of any non-Noetherian valuation ring of Krull dimension 1 cannot be finitely generated. If it is, then all prime ideals of such a valuation ring is finitely generated, and the latter implies that the ring is Noetherian by [Mat89, Theorem 3.4].

(c) Let $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ be ordered lexicographically. Let ν_{lex} be the unique valuation of $\mathbb{F}_p(X, Y)$ with value group Γ such that $\nu_{lex}(X) = (1, 0)$ and $\nu_{lex}(Y) = (0, 1)$. Then $R_{\nu_{lex}}$ is not Noetherian, but its maximal ideal is finitely generated since Γ has a smallest element $> (0, 0)$, namely $(0, 1)$. Using (b) we must then have $\operatorname{rank}(\nu_{lex}) \geq 2$. On the other hand $\operatorname{rank}(\nu_{lex}) \leq 2$ by Proposition II.10. Therefore ν_{lex} has rank 2, and the unique height 1 prime ideal of $R_{\nu_{lex}}$ is the only non-finitely generated prime ideal of $R_{\nu_{lex}}$.

2.6.2 Powers of ideals of valuation rings

The goal of this section is to prove the following result:

Proposition II.21. *Let V be a valuation ring and I be a proper ideal of V. Then $\bigcap_{n\in\mathbb{N}} I^n$ is a prime ideal of V.*

We will show that $\bigcap_{n\in\mathbb{N}} I^n$ is a radical ideal of V, hence also a prime ideal because of the following lemma:

Lemma II.22. *Any radical ideal of a valuation ring is either the unit ideal or a prime ideal.*

Proof of Lemma II.22. Suppose J is a radical ideal of a valuation ring V, and assume that J is not the unit ideal. Then J is the intersection of the prime ideals containing it, and the collection of such prime ideals is totally ordered by inclusion. It is easy to verify that the intersection of a chain of prime ideals is prime. $\square$

Remark II.23. Lemma II.22 implies that any closed subset of the spectrum of a valuation ring is irreducible.

Proof of Proposition II.21. Let $\mathcal{I} := \bigcap_{n\in\mathbb{N}} I^n$. Since I is a proper ideal of V, $\mathcal{I}$ is also a proper ideal of V. By the previous lemma, it suffices to show that $\mathcal{I}$ is a radical ideal. Assume ν is a valuation of $\mathrm{Frac}(V)$ whose associated valuation ring is V.

Let $x \in V$ such that $x^m \in \mathcal{I}$, for some $m \in \mathbb{N}$. We may assume $x \neq 0$. Then for all $n \in \mathbb{N}$, $x^m \in (I^n)^m$. Hence for all $n \in \mathbb{N}$, there exists a finitely generated ideal $J \subseteq I^n$ such that $x^m \in J^m$. Since finitely generated ideals of valuation rings are principal, we see that $i^m | x^m$, for some $i \in I^n$. Then $i|x$ because $m\nu(i) \leq m\nu(x)$ implies $\nu(i) \leq \nu(x)$. Thus for all $n \in \mathbb{N}$, $x \in I^n$, that is, $x \in \mathcal{I}$. $\square$

Corollary II.24. *Let $(V, \mathfrak{m}_V, \kappa_V)$ be a valuation ring and $\mathfrak{M} := \bigcap_{n\in\mathbb{N}} \mathfrak{m}_V^n$. Then $\mathfrak{M} = \mathfrak{m}_V$ (i.e. $V/\mathfrak{M} = \kappa_V$) or $V/\mathfrak{M}$ is a discrete valuation ring.*

Proof. By Proposition II.21, $\mathfrak{M}$ is a prime ideal of V. If $\mathfrak{M} \neq \mathfrak{m}_V$, then $\mathfrak{m}_V^2 \neq \mathfrak{m}_V$ and $V/\mathfrak{M}$ is a non-trivial valuation ring of its fraction field. In particular, $\mathfrak{m}_V$ is a non-zero, finitely generated ideal by Proposition II.19. So suppose π is a generator of $\mathfrak{m}_V$. The maximal ideal of $V/\mathfrak{M}$ is generated by the class of π in $V/\mathfrak{M}$. Hence to prove that $V/\mathfrak{M}$ is a discrete valuation ring, it suffices to show that it has Krull dimension 1.

Let P be a non-maximal prime ideal of V. Then $\pi \notin P$, and so for all $n \in \mathbb{N}$, $\pi^n \notin P$. Since ideals of V are comparable, for all $n \in \mathbb{N}$,

$$P \subsetneq (\pi^n) = \mathfrak{m}_V^n.$$

Thus, $P \subseteq \mathfrak{M}$. This shows that there are no prime ideals P of V such that

$$\mathfrak{M} \subsetneq P \subsetneq \mathfrak{m}_V.$$

Therefore $V/\mathfrak{M}$ has Krull dimension 1. $\square$

Notation II.25. For an ideal I of a ring A and a non-negative integer n, $I^{[n]}$ will denote the ideal of A generated by n-th powers of elements of I. Thus $I^{[0]} = A$, $I^{[1]} = I$ and $I^{[n]} \subseteq I^n$, for $n \geq 2$. If I is principal, then $I^{[n]} = I^n$, for all $n \geq 0$.

Lemma II.26. *If I is an ideal of a valuation ring V, then for all $n \geq 0$, $I^{[n]} = I^n$.*

Proof. It suffices to show that $I^n \subseteq I^{[n]}$. However, as observed in the proof of Lemma II.22, if $i \in I^n$, then there exists a finitely generated ideal J of I such that $i \in J^n$. Since finitely generated ideals of a valuation ring are principal, this implies that i must be an element of $J^{[n]} \subseteq I^{[n]}$. $\square$

2.7 The valuation topology

Let ν be a valuation of a field K with value group Γ_ν. The **valuation topology on K induced by ν** is the unique topology, making K into a topological field, such that a basis of open neighborhoods of $0 \in K$ is given by sets of the form

$$K_{>\gamma} \coloneqq \{x \in K : \nu(x) > \gamma\} \cup \{0\},$$

for $\gamma \in \Gamma_\nu$. For instance, the topology induced by the trivial valuation is the discrete topology.

The axioms of a valuation ring imply that sets of the form

$$K_{\geq\gamma} \coloneqq \{x \in K : \nu(x) \geq \gamma\} \cup \{0\}$$

are also open under the valuation topology. Thus the valuation ring R_ν, which equals $K_{\geq 0}$, is an open subring of K in the topology induced by ν. It is easy to verify that for $\gamma \in \Gamma_\nu$, $K_{\geq\gamma}$ is the principal fractional ideal of R_ν generated by any $x \in K$ such that $\nu(x) = \gamma$.

Lemma II.27. *Let K be a field equipped with a valuation ν.*

1. *The valuation topology induced by ν is Hausdorff.*

2. *If ν is not trivial, then the collection of non-zero principal ideals of R_ν form a basis of open neighborhoods of $0 \in K$ for the valuation topology induced by ν.*

Proof. 1 follows from the fact that $\bigcap_{\gamma \in \Gamma_\nu} K_{>\gamma} = \{0\}$.

2. A non-zero principal ideal of R_ν is a principal fractional ideal, hence open by our above discussion. Because ν is not trivial (i.e. Γ_ν is not the trivial group), for any $\gamma \in \Gamma_\nu$ there exists $\gamma' \in \Gamma_\nu$ such that $\gamma, 0 < \gamma'$. Then

$$K_{\geq \gamma'} \subseteq K_{>\gamma},$$

and $K_{\geq \gamma'}$ is a principal ideal of R_ν, generated by any element whose valuation equals γ'. Thus the collection of non-zero principal ideals of R_ν is a collection of open sets cofinal to the sets of the form $K_{>\gamma}$, completing the proof. $\square$

2.8 f-adic valued fields

Just as commutative rings are the local algebraic objects in the theory of schemes, f-adic rings are the local algebraic objects in Huber's theory of adic spaces [Hub93, Hub94]. The theory of adic spaces forms the foundation for Scholze's work on perfectoid spaces, which has been applied with great success to resolve long-standing open questions in algebra and geometry [Sch12, And16, Bha16, HM17, MS17, And18]. In this thesis, we will develop a connection between valued fields that are f-adic in the valuation topology and the theory of *tight closure* for valuation rings in prime characteristic.

2.8.1 Some topological algebra

In order to define f-adic valued fields, and more generally f-adic rings, we first discuss the notion of adic rings.

Definition II.28. Suppose A is a topological ring and I is an ideal of A. Then A is **adic** with **ideal of definition** I (or briefly, I**-adic**) if the set $\{I^n : n \geq 0\}$ is a basis of open neighborhoods of 0.

Examples II.29.

1. One can give any commutative ring A the discrete topology, and A is then adic with ideal of definition (0).

2. Given any commutative ring A and ideal I, there exists a unique topology on A (making A into a topological ring) such that $\{I^n : n \geq 0\}$ is a neighborhood basis of 0.

3. Let K be a field equipped with a valuation of rank 1. Consider R_ν as a topological ring with topology induced by the valuation topology on K (recall R_ν is an open subring of K). Then R_ν is adic, and any principal ideal generated by a non-zero element of the maximal ideal $\mathfrak{m}_\nu$ (such elements are called **pseudo-uniformizers**) is an ideal of definition of R_ν.

4. The completion of a Noetherian local ring $(R, \mathfrak{m})$ with respect to the maximal ideal $\mathfrak{m}$, although admitting a purely algebraic definition, can also be interpreted as the topological completion of R equipped with the $\mathfrak{m}$-adic topology.

Lemma II.30. *Let I and J be two ideals of definition of an adic ring A. Then $\sqrt{I} = \sqrt{J}$. The converse holds if I and J are finitely generated.*

Proof. The proof is straightforward, and we omit it. □

Remarks II.31.

1. An adic ring A is Hausdorff if and only if for any ideal of definition I,

$$\bigcap_{n \geq 0} I^n = (0).$$

2. The converse of Lemma II.30 fails if the ideals are not finitely generated. For instance, suppose ν is a rank 1 valuation of a field K such that the valuation

ring R_ν is not Noetherian. This means that $\mathfrak{m}_\nu$ is not finitely generated, and so $\mathfrak{m}_\nu^n = \mathfrak{m}_\nu$ for all $n > 0$ (Proposition II.19). For any non-zero element $x \in R_\nu$, $\sqrt{(x)} = \mathfrak{m}_\nu$. However, $\mathfrak{m}_\nu$ is not an ideal of definition of R_ν because the topology on R_ν is Hausdorff but $\bigcap_{n\geq 0} \mathfrak{m}_\nu^n \neq (0)$. On the other hand, (x) *is* an ideal of definition of R_ν.

Definition II.32. A topological ring A is **f-adic** or **Huber** if there exists an open subring A_0 of A (called a **ring of definition of** A) such that A_0 in its induced topology is adic and has a finitely generated ideal of definition (this is an ideal of A_0, not of A).

Remarks II.33.

1. The 'f' in f-adic stands for finite because an f-adic ring has a ring of definition which is adic with respect to a *finitely* generated ideal.

2. Following Scholze's work on perfectoid spaces, the terminology 'Huber rings' is becoming more common than 'f-adic rings'. We prefer the latter terminology.

3. Any adic ring with a finitely generated ideal of definition is an f-adic ring. Conversely, one can show that if an adic ring is f-adic, then it must have a finitely generated ideal of definition (see Corollary II.38).

We introduced the notion of f-adic rings because we want to characterize those valued fields that are f-adic in the valuation topology.

Definition II.34. Let A be a topological ring. A subset $B \subseteq A$ is **bounded** if for every open neighborhood U of 0, there exists an open neighborhood of V of 0 such that the set $VB \coloneqq \{vb : v \in V, b \in B\}$ is contained in U.

Said differently, if $\ell_a : A \to A$ denotes left-multiplication by an element $a \in A$, then a subset B of A is bounded if for every open neighborhood U of 0, the set

$$\bigcap_{b \in B} \ell_b^{-1}(U)$$

contains an open neighborhood of 0. Note that each $\ell_b^{-1}(U)$ is open in A because ℓ_b is continuous. However, the above intersection may be infinite, and so not may not be open. The reinterpretation of the concept of boundedness in terms of left-multiplication maps also shows that any finite subset of a topological ring is bounded.

Examples II.35.

1. The valuation ring of a valued field is bounded in the valuation topology on the field.

2. Any adic ring is bounded in its topology. Thus any ring of definition of an f-adic ring is bounded.

Boundedness is easy to check on f-adic rings.

Lemma II.36. *If A is a Huber ring with ring of definition A_0, and I is an ideal of definition of A_0, then a subset $B \subseteq A$ is bounded if and only if there exists $n > 0$ such that $I^n B \subset I$.*

Proof. The proof follows by observing that $\{I^n : n > 0\}$ is a collection of open subgroups of A that is a neighborhood basis of 0. $\square$

The notion of boundedness clarifies which open subrings of a Huber ring are rings of definition.

Proposition II.37. [Hub93] *Let A be a Huber ring and A_0 a subring of A. Then A_0 is a ring of definition of A if and only if A_0 is an open and bounded subring of A.*

Proof. Using Lemma II.36 it is clear that a ring of definition of A is an open and bounded subring of A. Conversely, suppose A_0 is an open and bounded subring of A. Since A is a Huber ring, let B be a ring of definition with ideal of definition I. As A_0 is an open neighborhood of 0, there exists $m > 0$ such that

$$I^m \subset A_0.$$

Of course this does not imply that I^m is an ideal of A_0. However, the collection $\{I^n : n \geq m\}$ is a basis of open neighborhoods of 0 contained in A_0.

Since A_0 is bounded, there exists $n > 0$ such that

$$I^n A_0 \subset I^m.$$

Suppose I^n is generated as an ideal of B by the set $\{x_1, \ldots, x_n\}$. The x_i are also elements of A_0, so let J be the ideal of A_0 generated by $\{x_1, \ldots, x_n\}$. Obviously J is a finitely generated ideal of A_0 and $J \subseteq I^m$. Then J is an ideal of definitiion of the induced topology on A_0, if there exists some power of I which is contained in J. But

$$I^{m+n} = I^m(Bx_1 + \ldots Bx_n) = I^m x_1 + \ldots I^m x_n \subseteq A_0 x_1 + \ldots A_0 x_n = J,$$

and so the proof is complete. □

This proposition has many useful applications in the theory of f-adic rings. For example, it can be used to prove the following result, claimed in Example II.35(2).

Corollary II.38. *Suppose A is a topological ring which is adic. If A is f-adic, then A has a finitely generated ideal of definition.*

Proof. The underlying set of an adic ring is always bounded. Thus if A is adic as well as f-adic, then A is an open and bounded subring of itself. Then Proposition

II.37 implies that A is a ring of definition of itself, and so has a finitely generated ideal of definition. $\square$

2.8.2 When are valued fields f-adic?

Throughout this subsection, we fix a valuation ν on a field K. We will always view K as a topological field with topology induced by ν. Our goal will be to attempt to give characterizations of when K is an f-adic ring in the valuation topology.

Regardless of whether K is f-adic, its valuation ring R_ν is always an open and bounded subring in the valuation topology. Therefore a necessary condition for K to be f-adic is for R_ν to be a ring of definition of K (Proposition II.37).

The case of the trivial valuation can be disposed immediately because if ν is trivial that $R_\nu = K$ has the discrete topology and (0) is a finitely generated ideal of definition. When ν is not trivial, we have the following result:

Theorem II.39. *Let ν be a non-trivial valuation of a field K with valuation ring R_ν. Equip K with the valuation topology induced by ν and let R_ν have the induced topology as an open subset of K. The following conditions are equivalent:*

1. *K is f-adic in the valuation topology.*
2. *R_ν is a ring of definition of K.*
3. *R_ν is an adic ring in the induced topology.*
4. *There exists a non-zero element $a \in R_\nu$ such that $\bigcap_{n\geq 0}(a^n) = (0)$.*
5. *R_ν has a prime ideal of height 1.*
6. *If Σ is the set of non-zero prime ideals of R_ν, then $\bigcap_{\mathfrak{p}\in\Sigma}\mathfrak{p} \neq (0)$.*

Proof. We have already shown the equivalence of (1) and (2), and (2) implies (3) by definition of a ring of definition.

Now assume (3) and suppose I is an ideal of definition of R_ν. Then

$$\bigcap_{n\geq 0} I^n = (0) \tag{2.1}$$

because the induced topology on R_ν is Hausdorff. Since ν is not the trivial valuation, the induced topology on R_ν has a basis of open neighborhoods of 0 given by the collection of non-zero principal ideals of R_ν (Lemma II.27). As I is an open neighborhood of 0 in R_ν, this shows there exists non-zero $a \in R_\nu$ such that $(a) \subseteq I$. Moreover, $\bigcap_{n\geq 0}(a^n) = (0)$ because of (2.1), which proves (3) $\Rightarrow$ (4). At the same time, there must exist $n > 0$ such that $I^n \subseteq (a)$. Otherwise, for all $n > 0$, $(a) \subseteq I^n$ (ideals of a valuation ring are always comparable), and so, $(0) \neq (a) \subseteq \bigcap_{n\geq 0} I^n$, contradicting (2.1). Thus (a) is also an ideal of definition of R_ν, and consequently (3) $\Rightarrow$ (2). Therefore (2) and (3) are equivalent.

Assuming (4), another comparability of ideals argument shows that the collection of ideals $\{(a^n) : n \geq 0\}$ and the collection of non-zero principal ideals of R_ν are cofinal with respect to inclusion. Thus (4) $\Rightarrow$ (3), which establishes the equivalence of (3) and (4).

The equivalence of (5) and (6) is straightforward. Indeed, if (6) holds then $\bigcap_{\mathfrak{p}\in\Sigma} \mathfrak{p}$ is the unique height 1 prime of R_ν (the intersection is a prime ideal because Σ is totally ordered by inclusion). Conversely, since every non-zero prime ideal of R_ν will contain the height 1 prime if it exists, (5) $\Rightarrow$ (6).

To finish the proof, it suffices to show the equivalence of (4) and (5). Assume R_ν has a prime ideal $\mathfrak{q}$ of height 1. Let $a \in \mathfrak{q}$ be a non-zero element. Then $\bigcap_{n\geq 0}(a^n)$ is a prime ideal of R_ν (Proposition II.21) which is contained in $\mathfrak{q}$. Thus $\bigcap_{n\geq 0}(a^n) = (0)$ of $\bigcap_{n\geq 0}(a^n) = \mathfrak{q}$. If the latter equality holds, then $\mathfrak{q} = (a)$, and then $(a^2) \neq (a)$ since a

is not a unit. This contradicts $(a) = \mathfrak{q} = \bigcap_{n\geq 0}(a^n)$. So we must have $\bigcap_{n\geq 0}(a^n) = (0)$, that is, (5) $\Rightarrow$ (4). Finally, if 4 holds, then the element a is not a unit. Thus $\sqrt{(a)}$ is a prime ideal of R_ν since proper radical ideals are prime in a valuation ring (Lemma II.22). Let $\mathfrak{p}$ be a non-zero prime ideal of R_ν. Then $a \in \mathfrak{p}$. Otherwise, for all $n > 0$, $a^n \notin \mathfrak{p}$ and so $\mathfrak{p} \subsetneq (a^n)$. But this means $(0) \neq \mathfrak{p} \subseteq \bigcap_{n\geq 0}(a^n)$, contradicting (4). Thus $a \in \mathfrak{q}$, and so $\sqrt{(a)} \subseteq \mathfrak{p}$. This shows that $\sqrt{(a)}$ is the smallest non-zero prime ideal of R_ν with respect to inclusion, that is, it is the unique height 1 prime of R_ν. $\square$

The following corollary is obvious from the proof of the above theorem.

Corollary II.40. *Let ν be a non-trivial valuation of a field K such that K is f-adic in the valuation topology. A finitely generated ideal I of R_ν is an ideal of definition of R_ν if and only if I is generated by a non-zero element contained in the height* 1 *prime of R_ν.*

Remark II.41. The localization of R_ν at its height 1 prime is the ring of power-bounded elements of K, where an element a of a topological ring A is **power-bounded** if $\{a^n : n > 0\}$ is a bounded set. Moreover, the elements of the height 1 prime are precisely the **topologically nilpotent elements** of K, that is, these are the elements $x \in K$ such that $x^n \to 0$ (in the topology) as $n \to \infty$.

2.9 Extensions of valuations

Let $K \subseteq L$ be an extension of fields. If w is a valuation of L, then its restriction to $K^\times$ is a valuation of K. This leads to the following definition.

Definition II.42. Let $K \subseteq L$ be an extension of fields, and ν be a valuation of K and w a valuation of L. Then w is an **extension of** ν if $w|_{K^\times} = \nu$.

Here are some basic properties of extensions of valuations that are all straightforward to verify.

Lemma II.43. *Let $K \subseteq L$ be a field extension, ν be a valuation of K and w be an extension of ν to L. Then we have the following:*

1. *$R_\nu = K \cap R_w$ and $\mathfrak{m}_\nu = K \cap \mathfrak{m}_w$. In other words, R_w dominates R_ν.*

2. *Γ_ν is a subgroup of Γ_w.*

3. *κ_ν is a subfield of κ_w.*

Proof. For (1), we have $x \in R_\nu$ if and only if $\nu(x) \geq 0$, and the latter holds if and only if $w(x) \geq 0$ and $x \in K$ (because w extends ν). Therefore $R_\nu = R_w \cap K$. One can similarly show that $\mathfrak{m}_\nu = \mathfrak{m}_w \cap K$. The proof of (2) is obvious, while (3) follows from (1) since (1) implies that R_w dominates R_ν. □

In light of the previous lemma, we introduce the following invariants associated to extensions of valuations.

Definition II.44. If w/ν is an extension of valuations, then the **ramification index of** w/ν, denoted $e(w/\nu)$, is the order of the quotient group Γ_w/Γ_ν. The **residue degree of** w/ν, denoted $f(w/\nu)$, is the degree of the extension of residue fields $\kappa_\nu \hookrightarrow \kappa_w$.

2.9.1 Finite field extensions

We have the following fundamental inequality relating the ramification index and residue degree of extensions of valuations to the degree of the field extension, when the extension of fields is finite.

Proposition II.45. *Let $K \subseteq L$ be a finite extension of fields, and ν be a valuation of K. Suppose $\mathcal{S}$ is a collection of mutually inequivalent valuations that extend ν to L such that any valuation of L that extends ν is equivalent to a valuation in $\mathcal{S}$. Then*

$$\sum_{w \in \mathcal{S}} e(w/\nu) f(w/\nu) \leq [L : K].$$

In particular, there are only finitely many valuations of L that extend ν up to equivalence of valuations.

Proof. See [Bou89, Chapter VI, §8.3, Theorem 1]. □

Definition II.46. A collection $\mathcal{S}$, as in the statement of Proposition II.45, is called a **complete system of extensions of ν to L.**

Corollary II.47. *Let $K \subseteq L$ be a finite extension of fields, and w be a valuation on L that extends a valuation ν on K. Then $e(w/\nu)f(w/\nu) \leq [L : K]$. In particular, $e(w/\nu)$ and $f(w/\nu)$ are both finite.*

Proof. This is obvious from Proposition II.45 because ramification index and residue degree of extensions is invariant under equivalence of extensions of valuations. □

Remark II.48. More generally, if $K \subseteq L$ is an algebraic extension and w is an extension of ν to L, then one can show that Γ_w/Γ_ν is a torsion abelian group and $\kappa_\nu \hookrightarrow \kappa_w$ is an algebraic extension [Bou89, Chapter VI, §8.1, Proposition 1].

Definition II.49. An extension of valuations w/ν is **unramified** if $e(w/\nu) = 1$, that is, if w and ν have the same value groups. The extension is **totally unramified** if $e(w/\nu) = 1 = f(w/\nu)$, that is, if the value groups and residue fields of w and ν coincide.

Let ν be a valuation of K, and $\mathcal{S}$ be a complete system of extensions of ν to a finite field extension L of K. For our investigation of F-finiteness of valuation rings, we need to understand when equality holds in the inequality

$$\sum_{w \in \mathcal{S}} e(w/\nu) f(w/\nu) \leq [L : K].$$

This is the content of the next result.

Theorem II.50. *Let $K \subseteq L$ be a finite extension of fields, and ν a valuation of K. Let $\mathcal{S}$ be a complete system of extensions of ν to L. If A be the integral closure of R_ν in L, then the following are equivalent:*

1. *A is a finitely generated R_ν-module.*

2. *A is a free R_ν-module.*

3. $\dim_{\kappa_\nu} A \otimes_{R_\nu} \kappa_\nu = [L : K]$.

If these equivalent conditions hold, then

$$\sum_{w \in \mathcal{S}} e(w/\nu) f(w/\nu) = [L : K].$$

Proof. See [Bou89, Chapter VI, §8.5, Theorem 2]. □

2.9.2 Transcendental field extensions

So far we have mainly discussed the behavior of extensions of valuations under finite field extensions. We will also need to understand how valuations extend over transcendental field extensions. Although we will not embark on an in-depth description of transcendental extensions of valuations, the basic case to consider is how a valuation of a field K extends to a purely transcendental extension $K(X)$ of transcendence degree 1. A few obvious ways of extending valuations in this special case are described in the next result.

Proposition II.51. *Let ν be a valuation of a field K with value group Γ_ν. Suppose Γ_ν is a subgroup of a totally ordered abelian group Γ, and ξ is an element of the larger group Γ.*

1. *There exists a unique valuation w of $K(X)$ extending ν such that for all $\sum_i a_i X^i \in K[X]$,*
$$w(\sum_i a_i X^i) = \inf_i \{\nu(a_i) + i\xi\}.$$
2. *Suppose the image of ξ in the quotient group Γ/Γ_ν is torsion-free element. Then there exists a unique valuation w of $K(X)$ extending ν such that*
$$w(X) = \xi.$$
Moreover, the residue field of w equals the residue field of ν and the value group of w is the ordered subgroup $\Gamma_\nu \oplus \mathbb{Z}\xi \subseteq \Gamma$.
3. *There exists a unique valuation w of $K(X)$ extending ν such that*
$$w(X) = 0,$$
and the image t of X in the residue field κ_w is transcendental over κ_ν. In this case w and ν have the same value groups, and κ_w equals $\kappa_\nu(t)$.

Proof. For (1) see [Bou89, Chapter VI, §10.1, Lemma 1]. The existence and uniqueness of the valuations in (2) and (3) follows from (1). For a proof of the other properties of (2) and (3) we refer the reader to [Bou89, Chapter VI, §10.1, Propositions 1 & 2]. □

2.10 Centers of valuations

Definition II.52. Let ν be a valuation of a field K, with valuation ring $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$. Given a local subring $(A, \mathfrak{m}_A, \kappa_A)$ of K such that the fraction field of A is K, we say that ν is **centered on** A if R_ν dominates A. In other words, ν is centered on A if $\mathrm{Frac}(A) = K$ and for all $a \in A$, $\nu(a) \geq 0$, while $\nu(a) > 0$, if $a \in \mathfrak{m}_A$.

Globally, if X is an integral scheme with function field K, then ν is **centered on** X if the canonical morphism $\mathrm{Spec}(K) \to X$ extends to a morphism $\mathrm{Spec}(R_\nu) \to X$. The image of the closed point of $\mathrm{Spec}(R_\nu)$ in X is called a **center of** ν **on** X.

Remarks II.53.

(a) ν is centered on X if and only if there exists a point $x \in X$ such that ν is centered on the local ring $\mathcal{O}_{X,x}$. A center of a valuation on X need not be a closed point of X.

(b) Suppose X is an integral scheme which is locally of finite type over a field k. If K is the function field of X and ν is a valuation of K centered on X, then ν is necessarily a valuation of K/k.

(c) If X is as in (b), then a center of ν on X, if it exists, is *unique* provided that X is separated over k. This follows from the valuation criterion of separatedness [Har77, Chapter II, Theorem 4.3]. Moreover, any valuation of K/k will always admit a center on X provided X is proper over k. This follows from the valuation criterion of properness [Har77, Chapter II, Theorem 4.7]. Since proper schemes are separated, if X is proper over k, then any valuation of K/k admits a unique center on X.

2.10.1 Valuations centered on Noetherian local domains

For valuations centered on Noetherian local domains, we have the following fundamental inequality due to Abhyankar.

Theorem II.54 (Abhyankar's inequality). *Let ν be a valuation of a field K centered on a Noetherian local domain $(A, \mathfrak{m}_A, \kappa_A)$. Then*

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/\kappa_A \leq \dim(A).$$

If equality holds in the above inequality, then Γ_ν is a free abelian group and κ_ν is a finitely generated field extension of κ_A.

Proof. See [Abh56b, Theorem 1] for the proof. □

Corollary II.55. *Any valuation centered on a Noetherian local domain has finite rank, that is, its valuation ring has finite Krull dimension.*

Proof. By the previous theorem, $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ is finite. Therefore the result follows from Proposition II.10. □

Definition II.56. We will refer to the inequality appearing in Theorem II.54 as **Abhyankar's inequality**. Furthermore, if equality holds in Abhyankar's inequality for a Noetherian local center A, we will call A an **Abhyankar center of** ν.

Examples II.57. In the following examples, we have chosen our base field to be $\mathbb{F}_p$. However, the examples work over any base field of prime characteristic.

1. Let ν be a discrete valuation of a field K (i.e. ν has value group $\mathbb{Z}$). Then the valuation ring R_ν is an Abhyankar center of ν.

2. If ν_{lex} is the lexicographical valuation of $\mathbb{F}_p(X, Y)$ with value group $\mathbb{Z} \oplus \mathbb{Z}$ (see Example II.20(c)), then the local ring of the origin of $\mathbb{A}^2_{\mathbb{F}_p}$ is an Abhyankar center of ν_{lex}.

3. There exists a valuation ν of $\mathbb{F}_p(X, Y, Z)$ with value group $\Gamma_\nu = \mathbb{Z} \oplus \mathbb{Z}\pi \subset \mathbb{R}$ such that $\nu(X) = 1 = \nu(Y)$ and $\nu(Z) = \pi$. Clearly ν is centered on $\mathbb{A}^3_{\mathbb{F}_p}$ at the origin.

The Krull dimension of the local ring at the origin is 3, $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) = 2$ and $\operatorname{tr.deg} \kappa_\nu/\mathbb{F}_p$ is at least 1 since the class of X/Y in κ_ν is transcendental over $\mathbb{F}_p$. Therefore Abhyankar's inequality implies that the local ring of the origin of $\mathbb{A}^3_{\mathbb{F}_p}$ must be an Abhyankar center of ν. Hence we also see that $\operatorname{tr.deg} \kappa_\nu/\mathbb{F}_p = 1$.

4. We want to emphasize that the property of a valuation admitting an Abhyankar center *depends* on the center. To illustrate our claim, we construct a valuation ν admitting two Noetherian local centers, only one of which is an Abhyankar center of ν. Consider the Laurent series field $\mathbb{F}_p((t))$ in one variable, with its canonical t-adic valuation ν_t, whose corresponding valuation ring is the power series ring $\mathbb{F}_p[[t]]$. Since $\mathbb{F}_p((t))$ is uncountable while the function field of $\mathbb{A}^2_{\mathbb{F}_p}$ is countable, one can choose an embedding

$$F_p(X,Y) \hookrightarrow \mathbb{F}_p((t))$$

that maps $X \mapsto t$ and $Y \mapsto q(t)$, where $q(t) \in \mathbb{F}_p[[t]]$ such that $\{t, q(t)\}$ are algebraically independent over $\mathbb{F}_p$. Furthermore, we may assume that $t|q(t)$.

The composition $\mathbb{F}_p(X,Y)^\times \hookrightarrow \mathbb{F}_p((t))^\times \xrightarrow{\nu_t} \mathbb{Z}$ is a valuation of $\mathbb{F}_p(X,Y)$. Let us call this valuation $\nu_{q(t)}$ (the subscript is meant to indicate the dependence on the trascendental power series $q(t)$). Then $\nu_{q(t)}$ is a discrete valuation, and the maximal ideal of the discrete valuation ring $R_{\nu_{q(t)}}$ is generated by X. By our discussion in example (1), $R_{\nu_{q(t)}}$ is an Abhyankar center of $\nu_{q(t)}$.

Since $\nu_{q(t)}(X) = 1$ and $\nu_{q(t)}(Y) = \nu_t(q(t)) \geq 1$, $\nu_{q(t)}$ is also centered on the origin of $\mathbb{A}^2_{\mathbb{F}_p}$. However, the local ring $\mathbb{F}_p(X,Y)_{(X,Y)}$ is *not* an Abhyankar center of $\nu_{q(t)}$. To see this note that by construction, the power series ring $\mathbb{F}_p[[t]]$ dominates the valuation ring $R_{\nu_{q(t)}}$. This induces a map of residue fields $\kappa_{\nu_{q(t)}} \hookrightarrow \mathbb{F}_p$, which shows that $\kappa_{\nu_{q(t)}} = \mathbb{F}_p$. Thus,

$$\operatorname{tr.deg} \kappa_{\nu_{q(t)}}/\mathbb{F}_p = 0,$$

and so, $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_{\nu_{q(t)}}/\mathbb{F}_p = 1 < 2 = \dim\left(\mathbb{F}_p(X,Y)_{(X,Y)}\right)$.

2.11 Valuations of function fields

Throughout this section, we fix a ground field k of *arbitrary* characteristic.

Definition II.58. A field extension K of k is called a **function field over** k if K is a finitely generated field extension of k.

By a **variety over** k or a k**-variety** we will always mean an integral, separated scheme of finite type over k. The field of rational functions of a variety is a function field over k in the above sense, called the **function field of the variety**. If K/k is a function field, then by a **variety of** K/k we mean a k-variety whose function field is K.

Note that if K/k is a function field, then there always exists a projective variety X/k whose function field is K. Moreover, X can be chosen to be normal.

Lemma II.59. *Let X be an integral scheme of finite type over a field k with function field K. Then for any $x \in X$,*

$$\dim(\mathcal{O}_{X,x}) + \operatorname{tr.deg} \kappa(x)/k = \operatorname{tr.deg} K/k.$$

Proof. The proof follows by choosing an affine open neighborhood $\operatorname{Spec}(A)$ of x, and using the well-known fact that for a prime ideal $\mathfrak{p}$ of A, $\dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = \operatorname{tr.deg} K/k$. □

We will next prove a function field analogue of Theorem II.54.

Proposition II.60 (**Abhyankar's inequality for function fields**)**.** *Let K/k be a function field and ν be a valuation of K/k with value group Γ_ν and residue field κ_ν. Then*

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k \leq \operatorname{tr.deg} K/k. \tag{2.2}$$

If equality holds in the above inequality then Γ_ν is a free abelian group, and κ_ν is a finitely generated extension of k.

Proof. Let X be a projective variety with function field K. Then ν admits a center x on X, and consequently ν is centered on the local ring $\mathcal{O}_{X,x}$. Therefore Abhyankar's inequality (Theorem II.54) implies that

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/\kappa(x) \leq \dim(\mathcal{O}_{X,x}). \tag{2.3}$$

However, since

$$\operatorname{tr.deg} \kappa_\nu/\kappa(x) = \operatorname{tr.deg} \kappa_\nu/k - \operatorname{tr.deg} \kappa(x)/k,$$

making this substitution in (2.3), we get

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k \leq \dim(\mathcal{O}_{X,x}) + \operatorname{tr.deg} \kappa(x)/k = \operatorname{tr.deg} K/k, \tag{2.4}$$

as desired. Here we are also using Lemma II.59 for the last equality.

If equality holds in (2.2), then using (2.3) we see that $\mathcal{O}_{X,x}$ is an Abhyankar center of ν. Hence Γ_ν is free and κ_ν is a finitely generated extension of $\kappa(x)$ by another application of Theorem II.54. Since $\kappa(x)/k$ is finitely generated, it follows that κ_ν is a finitely generated extension of k, completing the proof. □

2.11.1 Abhyankar valuations

Definition II.61. Let K/k be a function field. A valuation ν of K/k is called an **Abhyankar valuation of** K/k if

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k = \operatorname{tr.deg} K/k,$$

that is if equality holds in Abhyankar's inequality for valuations of function fields (Proposition II.60).

Remarks II.62.

(a) The notion of an Abhyankar valuation of function field is intrinsic to the valuation, while the notion of an Abhyankar center of a valuation depends on the center (see Example II.57(4)).

(b) The value group of an Abhyankar valuation of a function field is a free abelian group of finite rank, and its residue field is finitely generated field extension of the ground field.

There is a close relationship between Abhyankar valuations and valuations admitting Abhyankar centers. To highlight this relationship, we recall that

Definition II.63. An A-algebra B is **essentially of finite type over** A if there exists a finitely generated A-algebra C and a multiplicative set $S \subset C$ such that $B \cong S^{-1}C$.

Proposition II.64. *Suppose ν is a valuation of a function field K/k. Then the following are equivalent:*

1. *ν is an Abhyankar valuation of K/k.*

2. *ν admits an Abhyankar center which is essentially of finite type over k.*

3. *Any center of ν which is essentially of finite type over k is an Abhyankar center of ν.*

Proof. Note there is always a Noetherian, local center of ν which is essentially of finite type over k (just pick the local ring of the center of ν on a projective model of K/k). Furthermore, while proving Proposition II.60, we showed that if ν admits a center x on a variety X/k with function field K, then ν is an Abhyankar valuation of K/k if and only if $\mathcal{O}_{X,x}$ is an Abhyankar center of ν. This shows the equivalence of (1) and (2). Since any Noetherian local ring which is essentially of finite type over k with fraction field K is always the local ring of a variety of K/k, the equivalence of (1) and (3) also follows. □

Examples II.65. Suppose ν is a valuation of a function field K/k.

1. The quintessential example of an Abhyankar valuation is a *divisorial valuation*, a notion that we now introduce.

   Definition II.66. ν is a **divisorial valuation of** K/k if there exists a normal variety X of K/k and a prime divisor E on X such that ν is equivalent to the valuation ord_E, the order of vanishing along E.

   Thus divisorial valuations are discrete. If ν is divisorial, then it is an Abhyankar valuation of K/k because $\dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma_\nu) = 1$ and $\operatorname{tr.deg} \kappa_\nu/k = \operatorname{tr.deg} K/k - 1$. Alternatively, a divisorial valuation is Abhyankar because it admits an Abhyankar center which is essentially of finite type over k, namely its own valuation ring (Proposition II.64).

2. Divisorial valuations are Abhyankar valuations with value groups of rational rank 1. Conversely, Zariski showed that if $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) = 1$ and ν is Abhyankar, then ν is a divisorial valuation [ZS60, Chapter VI, §14, Theorem 31].

Hence Abhyankar valuations are higher rational rank analogues of divisorial valuations.

3. The valuations of examples (2) and (3) of II.57 are Abhyankar valuations of their respective function fields since they admit Abhyankar centers that are locally of finite type over the ground field. The value groups of both these Abhyankar valuations have rational rank > 1, illustrating the philosophy that Abhyankar valuations are higher rational rank analogues of divisorial valuations.

4. The discrete valuation $\nu_{q(t)}$ of Example II.57 (4) is *not* an Abhyankar valuation of its fraction field $\mathbb{F}_p(X, Y)$. There are multiple ways to verify this. For instance, we showed in II.57 that even though $\nu_{q(t)}$ is centered on the local ring of the origin of $\mathbb{A}^2_{\mathbb{F}_p}$, the latter ring is not an Abhyankar center of $\nu_{q(t)}$. Therefore $\nu_{q(t)}$ is not an Abhyankar valuation of $\mathbb{F}_p(X, Y)/\mathbb{F}_p$ by Proposition II.64.

   Alternatively, one can also use the result of Zariski mentioned in the second example above to conclude that $\nu_{q(t)}$ is not Abhyankar. For if $\nu_{q(t)}$ is Abhyankar, then Zariski's result implies that $\nu_{q(t)}$ must be divisorial. However, the residue field $\kappa_{\nu_{q(t)}}$ was shown to equal $\mathbb{F}_p$ in II.57, while we know that a divisorial valuation of a function field of a surface must have a residue field of transcendence degree 1 over the ground field.

Definition II.67. The **transcendence degree** of a valuation ν of a function field K/k is the transcendence degree of the residue field κ_ν/k.

The transcendence degree of ν is easily verified to be the maximal transcendence degree of the residue field of a center of ν on some model of K/k.

Proposition II.68. *Let $K \subseteq L$ be a finite extension of finitely generated field extensions of k, and suppose that w is valuation on L/k extending a valuation v on K/k.*

Then w is Abhyankar if and only if v is Abhyankar.

Proof. Since L/K is finite, L and K have the same transcendence degree over k. On the other hand, the extension $\kappa(v) \subseteq \kappa(w)$ is also finite by Corollary II.47, and so $\kappa(v)$ and $\kappa(w)$ also have the same transcendence degree over k. Again by Corollary II.47, since Γ_w/Γ_v is a finite abelian group, $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_w/\Gamma_v = 0$. By exactness of

$$0 \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_v \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_w \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_w/\Gamma_z \to 0$$

we conclude that Γ_w and Γ_v have the same rational rank. The result is now clear from the definition of an Abhyankar valuation. $\square$

2.11.2 Local monomialization of Abhyankar valuations

Throughout this section, we fix a function field K/k and a valuation ν of K/k. The problem of *local uniformization* asks if one can always find a Noetherian local center of ν which is essentially of finite type over k and regular. In other words, does there exists a variety X of K/k such that ν is centered on a regular point of X? Local uniformization is the local analogue of resolution of singularities. Indeed it is easy to see that resolution of singularities implies local uniformization.

Long before Hironaka's seminal work on resolution of singularities [Hir64a, Hir64b], Zariski showed that valuations of function fields over ground fields of characteristic 0 can always be locally uniformized [Zar40, Theorem U_3]. Later, de Jong's work on alterations revealed that local uniformization of a valuation is always possible up to a finite extension of the function field K, regardless of the characteristic of the ground field [dJ96] (see also [KK09] for a purely valuation theoretic proof). Moreover, the finite extension of K can even be chosen to be purely inseparable [Tem13].

At present, local uniformization remains wide open when $\operatorname{tr.deg} K/k > 3$ and k has positive characteristic. However, Knaf and Kuhlmann showed that Abhyankar valuations admit a strong form of local uniformization in any characteristic. Their result will be crucial in our exploration of Frobenius splitting of valuation rings and uniform approximation of valuation ideals associated to rank 1 Abhyankar valuations.

Theorem II.69 (**Local monomialization**). *Let K be a finitely generated field extension of any field k, and ν be an Abhyankar valuation of K/k with valuation ring $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$. Suppose $d := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ and κ_ν is separable over k. Then given any finite subset $Z \subset R_\nu$, there exists a variety X of K/k, and a center x of ν on X satisfying the following properties:*

1. *x is a smooth point of X/k and $\mathcal{O}_{X,x}$ is a local ring of dimension d.*
2. *$Z \subseteq \mathcal{O}_{X,x}$, and there exists a regular system of parameters $x_1, \ldots, x_d$ of $\mathcal{O}_{X,x}$ such that every $z \in Z$ admits a factorization*

$$z = u x_1^{a_1} \ldots x_d^{a_d},$$

for some $u \in \mathcal{O}_{X,x}^{\times}$ and $a_i \in \mathbb{N} \cup \{0\}$.

Proof. See [KK05, Theorem 1]. $\square$

Remark II.70. If the ground field k is perfect, then *any* Abhyankar valuation of K/k admits a local monomialization. This is because the residue field κ_ν is then automatically separable over k since κ_ν/k is finitely generated by Proposition II.60.

The presence of the finite set Z in the statement of Theorem II.69 allows us to draw the following conclusion that will be important in the sequel.

Corollary II.71. [Dat17b, Proposition 2.3.3] *Assume k is perfect, and ν is a non-trivial Abhyankar valuation of K/k centered on an affine variety* $\operatorname{Spec}(R)$ *of K/k. Suppose $d = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$. Then there exists a variety* $\operatorname{Spec}(S)$ *of K/k, along with an inclusion of rings $R \hookrightarrow S$ satisfying the following properties:*

(a) Spec(S) *is regular (equivalently smooth since* k *is perfect), and* ν *is centered at* $x \in \mathrm{Spec}(S)$ *such that* $\mathcal{O}_{\mathrm{Spec}(S),x}$ *is a regular local ring of Krull dimension* d*. Moreover, the sinduced map of residue fields* $\kappa(x) \hookrightarrow \kappa_\nu$ *is an isomorphism.*

(b) There exists a regular system of parameters $\{x_1, \dots, x_d\}$ *of* $\mathcal{O}_{\mathrm{Spec}(S),x}$ *such that* $\nu(x_1), \dots, \nu(x_d)$ *freely generate the value group* Γ_ν*.*

Proof. Since the value group Γ_ν is free of rank d (Theorem II.54), one can choose $r_1, \dots, r_d \in R_\nu$ such that $\nu(r_1), \dots, \nu(r_d)$ freely generate Γ_ν. Also, because κ_ν is a finitely generated field extension of k, there exist $y_1, \dots, y_j \in R_\nu$ whose images in κ_ν generate κ_ν over k. Let $t_1, \dots, t_n \in K$ be generators for R over k. Then $t_1, \dots, t_n \in R_\nu$ because ν is centered on Spec(R). Defining

$$Z := \{t_1, \dots, t_n, y_1, \dots, y_j, r_1, \dots, r_d\},$$

by Theorem II.69 there exists a variety X over k with function field K such that ν is centered on a regular point $x \in X$ of codimension d, $Z \subseteq \mathcal{O}_{X,x}$, and there exists a regular system of parameters $\{x_1, \dots, x_d\}$ of $\mathcal{O}_{X,x}$ with respect to which every $z \in Z$ can be factorized as

$$z = u x_1^{a_1} \dots x_d^{a_d},$$

for some $u \in \mathcal{O}_{Y,y}^\times$, and integers $a_i \geq 0$. In particular, each $\nu(r_i)$ is a $\mathbb{Z}$-linear combination of $\nu(x_1), \dots, \nu(x_d)$, which shows that $\{\nu(x_1), \dots, \nu(x_d)\}$ also freely generates Γ_ν. Moreover, by our choice of Z, $\kappa(x) \hookrightarrow \kappa_\nu$ is an isomorphism.

Since $t_1, \dots, t_n \in \mathcal{O}_{X,x}$, we have an inclusion $R \subseteq \mathcal{O}_{X,x}$. Now restricting to an affine neighborhood of x, we may assume $X = \mathrm{Spec}(S)$, where S is regular and $t_1, \dots, t_n \in S$. By construction, $R \subseteq S$ and parts (a) and (b) of the corollary are satisfied. $\square$

Remarks II.72.

1. Corollary II.71 holds more generally for non-perfect ground fields k as long as the residue field of the valuation ring is separable over the ground field.

2. Any valuation of K/k is always centered on an affine variety of K/k. Hence Corollary II.71 implies that when k is perfect, an Abhyankar valuation of K/k is always centered on a regular local ring A which is essentially of finite type over k such that A has a regular system of parameters whose valuations freely generated Γ_ν, and the residue field κ_A of A coincides with the residue field of the valuation.

3. If ν is an *arbitrary* valuation of K/k centered on a variety X at a point x of codimension equal to $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$, then ν is necessarily an Abhyankar valuation of K/k. Indeed, we then have

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k = \dim(\mathcal{O}_{X,x}) + \operatorname{tr.deg} \kappa_\nu/k$$
$$\geq \dim(\mathcal{O}_{X,x}) + \operatorname{tr.deg} \kappa(x)/k = \operatorname{tr.deg} K/k,$$

and so,

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k = \operatorname{tr.deg} K/k$$

by Abhyankar's inequality for function fields (Proposition II.60).

CHAPTER III

Singularities in prime characteristic

3.1 The Frobenius endomorphism

Throughout this chapter, we fix a prime number $p > 0$. If R is a ring of characteristic p, then the map of sets

$$F : R \to R,$$

mapping $r \mapsto r^p$, is a ring homomorphism called the **(absolute) Frobenius map of** R. For a positive integer e, we also have the e-th iterate F^e of the Frobenius map. The image of F^e is a subring of R that will be denoted R^{p^e}.

If I is an ideal of R, then the expansion of I along F^e is the ideal $I^{[p^e]}$ generated by p^e-th powers of elements of I (see Notation II.25). The ideal $I^{[p^e]}$ is called a **Frobenius power of** I. If I is finitely generated, then every Frobenius power of I contains an ordinary power of I, and so the I-adic topology on R coincides with the topology on R generated by the filtered collection of Frobenius powers of I.

A ring of characteristic p is always an $\mathbb{F}_p$-algebra. In particular, a local ring of characteristic p is *equicharacteristic* (i.e. the ring and its residue field have the same characteristic).

Globally, if X is a scheme over $\mathbb{F}_p$, then the **(absolute) Frobenius morphism**

of X, also denoted

$$F : X \to X,$$

is the identity map on the underlying topological space of X, with the induced morphism of sheaves $\mathcal{O}_X \to F_*\mathcal{O}_X$ given by raising local sections to their p-th powers. Note that the endomorphism of $\mathrm{Spec}(R)$ induced by the Frobenius map of R is precisely the Frobenius morphism of $\mathrm{Spec}(R)$. The Frobenius morphism of X is an integral morphism.

Taking inspiration from notation for morphisms of schemes, if $F^e : R \to R$ is the e-th interate of the Frobenius map, then the target copy of R with R-module structure induced by restriction of scalars via F^e is denoted F^e_*R. In other words, F^e_*R has the same underlying group as R, but the action of R is as follows: for $r \in R$ and $x \in F^e_*R$, $r \cdot x = r^{p^e}x$. If $X = \mathrm{Spec}(R)$, then the sheaf $\widetilde{F^e_*R}$ associated to the R-algebra F^e_*R is precisely $F^e_*\mathcal{O}_X$.

Definition III.1. A scheme X over $\mathbb{F}_p$ is **perfect** if the Frobenius morphism of X is an isomorphism. A ring R is **perfect** if $\mathrm{Spec}(R)$ is perfect.

Remark III.2. It is not difficult to show that the only perfect Noetherian rings of prime characteristic are finite direct products of perfect fields. Hence the notion of a perfect ring is not very interesting in the Noetherian world.

3.2 Miracles of Frobenius

3.2.1 Regularity vs. smoothness

The notion of regularity is defined under Noetherian hypotheses. Recall that a Noetherian local ring $(R, \mathfrak{m}_R, \kappa_R)$ is **regular** if $\dim(R) = \dim_{\kappa_R} \mathfrak{m}_R/\mathfrak{m}_R^2$, that is, if the maximal ideal of R can be generated by $\dim(R)$ elements, called a **regular**

system of parameters of R. A regular system of parameters forms a regular sequence on R, and any regular local ring is a unique factorization domain.

Globally, a locally Noetherian scheme X is **regular** if for all $x \in X$, $\mathcal{O}_{X,x}$ is a regular local ring. Regular schemes are reduced. We say X is **singular** if it is not regular, that is, if there exists $x \in X$ such that $\mathcal{O}_{X,x}$ is *not* a regular local ring. A Noetherian ring R (not necessarily local) is **regular** (resp. **singular**) if $\mathrm{Spec}(R)$ is regular (resp. singular).

Regularity of a locally Noetherian scheme is an absolute notion. There is also the related notion of smoothness for morphisms of schemes. To define smoothness, we do not need any Noetherian hypotheses. We will see that the absolute notion of regularity and the relative notion of smoothness often coincide for finite type schemes over a field (Proposition III.6).

There are many equivalent ways to define smoothness. Here is one using the Jacobian criterion.

Definition III.3. Let $f : X \to S$ be a morphism of schemes and $x \in X$. Then f is **smooth of relative dimension** n **at** x if there exists an affine open neighborhood $U = \mathrm{Spec}(B)$ of x and an affine open neighborhood $V = \mathrm{Spec}(A)$ of $f(y)$ such that $f(U) \subset V$ and B is a quotient of a polynomial ring of the form

$$A[X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+r}]/(f_1, \ldots, f_r)$$

such that the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial X_j}(x) \right) \in M_{r \times (n+r)}(\kappa(x))$$

has full rank r. We say f is **étale at** x if it is smooth of relative dimension 0 at x, and f is **smooth** (resp. **étale**) if it is smooth (resp. étale) at all $x \in X$.

Smoothness and regularity are intimately related for schemes locally of finite type over a field. To state this relation, recall

Definition III.4. A locally Noetherian scheme X over a field k is **geometrically regular over** k if for all finite field extensions K of k, $X \otimes_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ is regular.

Remark III.5. If X is locally of finite type over k, then X is geometrically regular over k if and only if $X \otimes_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is regular for an algebraic closure $\overline{k}$ of k. However, the latter condition cannot be taken as a definition of geometric regularity for arbitrary locally Noetherian schemes over fields, because for such schemes $X \otimes_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ may not be locally Noetherian!

Proposition III.6. *Let X be a scheme which is locally of finite type over a field k.*

1. *X is smooth if and only if X is geometrically regular.*
2. *If X is smooth then X is regular. The converse holds when k is perfect.*
3. *For a closed point $x \in X$ with $\kappa(x)/k$ separable, X is smooth at x if and only if $\mathcal{O}_{X,x}$ is regular.*

Proof. See [BLR90, §2.2, Proposition 15] and [Poo17, Proposition 3.5.22]. □

Let X be a smooth variety of dimension n over a field k. The sheaf of Kähler differentials $\Omega_{X/k}$ is locally free of rank n, and so the top exterior power

$$\omega_X := \wedge^n(\Omega_{X/k})$$

is a line bundle on X called the **canonical bundle of** X. A divisor K_X on X such that $\omega_X \cong \mathcal{O}_X(K_X)$ is called a **canonical divisor** (Weil and Cartier divisors coincide on X since X is locally factorial when it is smooth). The canonical bundle is a dualizing sheaf in the sense of Grothendieck-Serre duality. We will have more to say about this later (see subsection 3.6.3).

3.2.2 Kunz's theorem

An amazing fact is that the Frobenius map can detect if a ring is regular. Indeed, it can already detect one of the most basic singularities of a ring.

Lemma III.7. *Let R be a ring of characteristic p. Then R is reduced if and only if the Frobenius map of R is injective.*

Proof. An element $r \in R$ is nilpotent if and only if there exists $e > 0$ such that $r^{p^e} = 0$. The lemma now follows by using Frobenius and its iterates. □

The following result, proved by Kunz, is the starting point of using the Frobenius map to study how far a ring or locally Noetherian scheme is from being regular.

Theorem III.8 (Kunz's theorem on regularity). *Let R be a Noetherian ring of characteristic p. Then R is regular if and only if the Frobenius map of R is a flat ring map.*

Proof. [Kun69, Theorem 2.1]. □

The various notions of singularities that have been proposed and studied since Kunz's result (such as F-purity, Frobenius splitting, F-regularity, F-rationality, etc.) systematically weaken the flatness of the Frobenius map in order to study singular rings in prime characteristic.

3.3 F-finiteness

In this section all rings have prime characteristic p and all schemes are $\mathbb{F}_p$-schemes. Usually these hypotheses will be repeated in the statements of results and definitions.

Definition III.9. A scheme X over $\mathbb{F}_p$ is ***F*-finite** if the Frobenius morphism of X is a finite morphism. A ring R is F-finite if $\operatorname{Spec}(R)$ is F-finite.

Remark III.10. The Frobenius morphism $F : X \to X$ is finite if and only if some iterate F^e is a finite morphism. A ring R is F-finite if and only if there exists $e > 0$ such that R is a finitely generated R^{p^e}-module.

F-finiteness is preserved under localization, quotients, finite type ring maps and completions of Noetherian local rings (this follows using Cohen's structure theorem). Thus F-finite rings and schemes are ubiquitous. For example, any ring which is essentially of finite type over an F-finite field (for instance a perfect field) is F-finite. Therefore a scheme which is locally of finite type over an F-finite field is F-finite since F-finiteness for schemes can be checked affine locally. Moreover, a quotient of a power series ring over an F-finite field is also F-finite. This shows that most schemes one is likely to encounter in geometric applications are going to be F-finite.

Kunz's theorem has the following nice re-interpretation for F-finite, locally Noetherian schemes.

Proposition III.11. *Let X be an F-finite, locally Noetherian $\mathbb{F}_p$-scheme. Then X is regular if and only if $F_*\mathcal{O}_X$ is locally free $\mathcal{O}_X$-module of finite rank. In particular, the regular locus of an F-finite, locally Noetherian $\mathbb{F}_p$-scheme is always open.*

Proof. The equivalence follows from Kunz's theorem on regularity (Theorem III.8) and the fact that flatness coincides with being locally free for finitely presented modules over a ring. The second assertion follows from the equivalence since the locus of points at which $F_*\mathcal{O}_X$ is locally free is always open. $\square$

Like varieties, many F-finite schemes also have finite Krull dimension.

Theorem III.12. *An F-finite, Noetherian $\mathbb{F}_p$-scheme has finite Krull dimension.*

Proof. On reducing to the affine case, the result follows by [Kun76, Proposition 1.1]. $\square$

3.3.1 F-finiteness and excellence

Another reason why F-finiteness is robust from a geometric point-of-view is because of its close relation to Grothendieck's notion of an excellent ring. A Noetherian ring is excellent if it satisfies a list of axioms that ensures it behaves much like a finitely generated algebra over a field (see Definition III.13 below). An arbitrary Noetherian ring can be quite pathological. For instance, the integral closure of a Noetherian domain in a finite extension of its fraction field can fail to be Noetherian, and Noetherian rings can have saturated chains of prime ideals of different lengths. But the class of excellent rings was introduced by Grothendieck to rule out such pathologies. Excellent rings are also supposed to be the most general setting to which one can expect the deep ideas of algebraic geometry, such as resolution of singularities, to extend.

Before we explain the relationship between F-finiteness and excellence, we recall the definition of an excellent ring.

Definition III.13. [DG65, IV_2, Définition (7.8.2)] A Noetherian ring A is **excellent** if it satisfies the following properties:

1. A is *universally catenary.* This means that every finitely generated A-algebra has the property that for any two prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$, all saturated chains of prime ideals from $\mathfrak{p}$ to $\mathfrak{q}$ have the same length.

2. All *formal fibers* of A are *geometrically regular.* This means that for every $\mathfrak{p} \in \mathrm{Spec}(A)$, the fibers of the natural map $\mathrm{Spec}(\widehat{A_{\mathfrak{p}}}) \to \mathrm{Spec}(A_{\mathfrak{p}})$ induced by completion along $\mathfrak{p}$ are geometrically regular in the sense of Definition III.4.

3. For every finitely generated A algebra B, the regular locus of $\mathrm{Spec}(B)$ is open;

that is, the set

$$\{\mathfrak{q} \in \mathrm{Spec}(B) : B_{\mathfrak{q}} \text{ is a regular local ring}\}$$

is open in $\mathrm{Spec}(B)$.

Just like F-finite rings, the class of excellent rings is closed under localizations, homomorphic images and finite type ring maps. Moreover, a relatively recent (unpublished) result of Gabber shows that ideal adic completions of excellent rings are also excellent. In particular, power series rings over excellent rings are excellent [KS16, Corollary 5.5].

The following result of Kunz connects the notions of F-finiteness and excellence.

Theorem III.14 (**Kunz's theorem on excellence**). *A Noetherian F-finite ring of characteristic p is excellent.*

Proof. See [Kun76, Theorem 2.5]. □

3.3.2 Finiteness of module of absolute Kähler differentials

The difficult part of Theorem III.14 is to show that a Noetherian F-finite ring is universally catenary. For this, Kunz exploits the observation that when R is F-finite, the module of absolute Kähler differentials $\Omega_{R/\mathbb{Z}}$ is a finitely generated R-module. The latter holds because

$$\Omega_{R/\mathbb{Z}} = \Omega_{R/\mathbb{F}_p} = \Omega_{R/R^p}$$

using the Leibniz rule.

Module finiteness of $\Omega_{R/\mathbb{Z}}$ has nice consequences. For example, it allows Kunz to establish the following identity which will later play a key role in our investigation of valuations of fields of prime characteristic that admit Abhyankar centers (Theorem IV.19).

Proposition III.15. *Suppose* $(R, \mathfrak{m}_R, \kappa_R)$ *is a Noetherian local domain of characteristic* p. *If* R *is* F*-finite, then*

$$[\mathrm{Frac}(R) : \mathrm{Frac}(R)^p] = p^{\dim(R)}[\kappa_R : \kappa_R^p].$$

Sketch of proof. This is proved in [Kun76, Proposition 2.1]. Kunz uses the analogue of Noether normalization for complete rings and finite generation of $\Omega_{R/\mathbb{Z}}$ to establish that when R is F-finite, then for any minimal prime ideal $\mathfrak{P}$ of the $\mathfrak{m}_R$-adic completion $\widehat{R}$,

$$[\mathrm{Frac}(R) : \mathrm{Frac}(R)^p] = p^{\dim(\widehat{R}/\mathfrak{P})}[\kappa_R : \kappa_R^p].$$

The identity shows that $\dim(\widehat{R}/\mathfrak{P})$ is independent of $\mathfrak{P}$, or in other words that $\widehat{R}$ is equidimensional. Since $\mathfrak{P}$ is minimal, we then have

$$\dim(\widehat{R}/\mathfrak{P}) = \dim(\widehat{R}) = \dim(R),$$

completing the proof. $\square$

Under mild additional hypotheses, finite generation of the module of absolute Kähler differentials actually implies F-finiteness.

Theorem III.16. *Let* R *be a Noetherian ring of characteristic* p. *Then the following are equivalent:*

1. R *is* F*-finite.*
2. *The module of absolute Kähler differentials* $\Omega_{R/\mathbb{Z}}$ *is a finitely generated* R*-module and for each maximal ideal* $\mathfrak{m}$ *of* R, $R_\mathfrak{m}$ *is universally Japanese (Definition III.22).*

Proof. See [Sey80, Théorème (1.1)]. $\square$

For a regular local ring, one can drop the universally Japanese hypothesis from the previous theorem; finiteness of $\Omega_{R/\mathbb{Z}}$ is sufficient for F-finiteness.

Proposition III.17. *Let R be a regular local ring of characteristic p. Then R is F-finite if and only if $\Omega_{R/\mathbb{Z}}$ is a finitely generated R-module.*

Proof. [Sey80, Proposition (3.1)]. □

Remark III.18. When a regular local ring R satisfies the equivalent conditions of Proposition III.17, then $\Omega_{R/\mathbb{Z}}$ is a free R-module. If $x_1, \ldots, x_n \in R$ such that $dx_1, \ldots, dx_n$ is a free R-basis of $\Omega_{R/\mathbb{Z}}$, then $\{x_1, \ldots, x_n\}$ is a *p-basis* of R. This means $R = R^p[x_1, \ldots, x_n]$ and R is a free R^p-module with basis

$$\{x_1^{\alpha_1} \ldots x_1^{\alpha_n} : 0 \leq \alpha_i \leq p-1\}.$$

3.3.3 A partial converse of Kunz's theorem on excellence

In [Kun76], Kunz proved a partial converse of Theorem III.14 in the local case.

Proposition III.19. *Let R be a Noetherian local ring of characteristic p with F-finite residue field. Then R is excellent if and only if R is F-finite.*

Proof. [Kun76, Corollary 2.6]. □

Remark III.20. The hypothesis of Proposition III.19 ensures that the completion $\widehat{R}$ is F-finite, because by Cohen's structure theorem a complete Noetherian local ring of equal characteristic p is F-finite if and only if its residue field is F-finite. One implication of Proposition III.19 is essentially Theorem III.14. Therefore the new assertion of the proposition is that when R is excellent, F-finiteness descends from $\widehat{R}$ to R. Heuristically this should not be surprising because when R is excellent, the completion map $R \to \widehat{R}$ is very well-behaved. For instance, this map has geometrically regular fibers (by the very definition of excellence) and can be expressed as a filtered colimit of smooth maps by Neron-Popescu desingularization [Nér64, Pop90, Swa98].

In this subsection, we would like to highlight a partial converse of Theorem III.14 even in the non-local case. In other words, we want to show that an excellent ring is also F-finite under relatively mild hypotheses. This result is probably well-known to experts, but the precise statement is difficult to locate in the literature.

In order to state the converse, we need the following property satisfied by excellent rings, often called the *Japanese* or *N2* property.

Proposition III.21. [DG65, IV_2, 7.8.3 (vi)] *Let A be a Noetherian excellent domain. The integral closure of A in any finite extension of its fraction field is finite as an A-module.*

Definition III.22. [DG64, IV_0, Définition 23.1.1] A domain R is **Japanese** if it satisfies the conclusion of Proposition III.21 with A replaced by R. A ring R (not necessarily a domain) is **universally Japanese** if every finite type R-algebra which is a domain is Japanese.

Remark III.23. Since excellence is preserved under finite type ring maps, excellent rings are universally Japanese. Moreover, one can show that a ring R is universally Japanese and Noetherian if and only if R is **Nagata**, which means that R is Noetherian and for every prime ideal $\mathfrak{p}$ of R, $R/\mathfrak{p}$ is Japanese [Sta18, Tag 0334].

Armed with the knowledge that excellent rings are Japanese, we have

Theorem III.24. (c.f. [DS17b]) *Let R be a Noetherian domain of characteristic p and fraction field K. Suppose R is generically F-finite, that is, K is F-finite. Then the following are equivalent:*

1. *R is F-finite.*

2. The integral closure of R^p in K is a finitely generated R^p-module.

3. R is Japanese.

4. R is excellent.

Proof. (1) $\Rightarrow$ (4) follows from Theorem III.14 and (4) $\Rightarrow$ (3) from Proposition III.21. Suppose (3) holds. Since R^p is isomorphic to R (Frobenius is injective because R is reduced), R^p is also Japanese. The fraction field of R^p is K^p, and K is a finite extension of K^p. Therefore by definition of the Japanese property, we see (3) $\Rightarrow$ (2).

To finish the proof, it suffices to show (2) $\Rightarrow$ (1). Assume (2), and let S be the integral closure of R^p in K. Then S is a Noetherian R^p-module by hypothesis, hence since R is an R^p-submodule of S, it is also a finitely generated R^p-module. $\square$

Corollary III.25. [DS17b] *Let R be a reduced, Noetherian ring of characteristic p whose total quotient ring K is F-finite. Then R is excellent if and only if R is F-finite.*

Proof. The backward implication is again a consequence of Kunz's Theorem III.14. So assume that R is excellent. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be the minimal primes of R. We denote the corresponding minimal primes of R^p by $\mathfrak{q}_i^p$. Let K_i be the fraction field of $R/\mathfrak{q}_i$, so that K_i^p is the fraction field of $R^p/\mathfrak{q}_i^p$. Then we have a commutative diagram

$$\begin{tikzcd}
R \arrow[r, hook] & R/\mathfrak{q}_1 \times \cdots \times R/\mathfrak{q}_n \arrow[r, hook] & K_1 \times \cdots \times K_n \cong K \\
R^p \arrow[u, hook] \arrow[r, hook] & R^p/\mathfrak{q}_1^p \times \cdots \times R^p/\mathfrak{q}_n^p \arrow[u, hook] \arrow[r, hook] & K_1^p \times \cdots \times K_n^p \cong K^p \arrow[u, hook]
\end{tikzcd}$$

where all rings involved are R^p-modules, and the horizontal maps are injections because R is reduced. Since R is excellent, so is each quotient $R/\mathfrak{q}_i$, and F-finiteness of K implies that each K_i is also a finitely generated K_i^p-module. Thus, Theorem

III.24 implies that each $R/\mathfrak{q}_i$ is F-finite, that is, $R/\mathfrak{q}_i$ a finitely generated $(R/\mathfrak{q}_i)^p = R^p/\mathfrak{q}_i^p$-module. As a consequence,

$$R^p/\mathfrak{q}_1^p \times \cdots \times R^p/\mathfrak{q}_n^p \hookrightarrow R/\mathfrak{q}_1 \times \cdots \times R/\mathfrak{q}_n$$

is a finite map, and so is the map $R^p \hookrightarrow R^p/\mathfrak{q}_1^p \times \cdots \times R^p/\mathfrak{q}_n^p$. This shows that $R/\mathfrak{q}_1 \times \cdots \times R/\mathfrak{q}_n$ is a finitely generated R^p-module, and being a submodule of the Noetherian R^p-module $R/\mathfrak{q}_1 \times \cdots \times R/\mathfrak{q}_n$, R is also a finitely generated R^p-module. Thus, R is F-finite. □

Theorem III.24 offers a simple way to think about excellence in prime characteristic, at least for domains in function fields over F-finite ground fields.

Remark III.26. The results of this subsection may give the impression that it is difficult to come up with examples of excellent rings that are not F-finite. But this is not the case. Indeed, if k is a field of prime characteristic that is not F-finite, then the polynomial ring $k[x]$ is an excellent ring which is not F-finite. What Theorem III.24 does demonstrate is that it is impossible to construct generically F-finite excellent domains that are not F-finite.

3.3.4 Example of a generically F-finite, non-excellent regular ring

It is easy to construct examples of non-excellent rings, even among regular local rings of dimension 1, also known as discrete valuation rings. A non-excellent discrete valuation ring *must* have prime characteristic because all Dedekind domains whose fraction fields have characteristic 0 are excellent [Sta18, Tag 07QW]. One of the first examples of a prime characteristic non-excellent discrete valuation ring was given by Nagata. He proceeds by taking a field k of prime characteristic such that $[k : k^p] = \infty$, and then forming the discrete valuation ring $k \otimes_{k^p} k^p[[t]]$, which he shows is not excellent. However, his example is not generically F-finite, that is, the fraction field of $k \otimes_{k^p} k^p[[t]]$ is not F-finite.

Based on the material we have introduced so far, we show that one can also construct examples of non-excellent discrete valuation rings even in the function field of $\mathbb{P}^2_{\mathbb{F}_p}$.

Recall that in Example II.57(4), we constructed a discrete valuation $\nu_{q(t)}$ of $\mathbb{F}_p(X,Y)$ which is centered on two different Noetherian local domains, only one of which is an Abhyankar center of $\nu_{q(t)}$. We later saw that the same valuation is also not an Abhyankar valuation of $\mathbb{F}_p(X,Y)/\mathbb{F}_p$ (see Examples II.65). The residue field $\kappa_{\nu_{q(t)}}$ coincides with $\mathbb{F}_p$. Then

$$p^{\dim(R_{\nu_{q(t)}})}[\kappa_{\nu_{q(t)}} : \kappa^p_{\nu_{q(t)}}] = p \neq p^2 = [\mathbb{F}_p(X,Y) : \mathbb{F}_p(X,Y)^p],$$

and therefore $R_{\nu_{q(t)}}$ is not F-finite by Proposition III.15, hence also not excellent since F-finiteness and excellence coincide for generically F-finite Noetherian domains (Theorem III.24). In particular, $\Omega_{R_{\nu_{q(t)}}/\mathbb{Z}}$ is not a finitely generated $R_{\nu_{q(t)}}$-module by Proposition III.17.

3.3.5 A curiosity

Non-excellent regular local rings exhibit other very interesting behavior. In this subsection we highlight one such phenomenon, which also ties nicely with our discussion of finite generation of the module of absolute Kähler differentials for F-finite rings (subsection 3.3.2).

A flat and unramified [1] ring homomorphism is always étale (and vice-versa) [Sta18, Tag 08WD]. However, if we replace 'unramified' by 'formally unramified' and 'étale'

[1] An unramified ring map for us has finite presentation by convention. Sometimes this is also called G-**unramified** in the literature [Sta18].

by 'formally étale' then the assertion no longer holds, even for extensions of regular local rings. Indeed, if $R_{\nu_{q(t)}}$ is the non-excellent discrete valuation ring from subsection 3.3.4 above, then the canonical completion map

$$\mathrm{can} : R_{\nu_{q(t)}} \to \widehat{R_{\nu_{q(t)}}}$$

is faithfully flat and formally unramified, but not formally étale, as we now explain.

Suppose, more generally, that $(A, \mathfrak{m}_A, \kappa_A)$ is a regular local ring of characteristic p which is *not* excellent. Also suppose κ_A is F-finite. Note that $R_{\nu_{q(t)}}$ satisfies all of these properties. We claim that the canonical map

$$A \to \widehat{A}$$

is faithfully flat and formally unramified, but not formally étale. Faithful flatness is obvious, and formal unramifiedness follows if $\Omega_{\widehat{A}/A} = 0$ [Sta18, Tag00UO].

By our assumptions, A is not F-finite (Theorem III.24), hence

$$\Omega_{A/\mathbb{Z}}$$

is *not* a finitely generated A-module (Proposition III.17). On the other hand, the completion $\widehat{A}$ is F-finite because κ_A is F-finite (use Cohen's structure theorem). Thus

$$\Omega_{\widehat{A}/\mathbb{Z}}$$

is a finitely generated $\widehat{A}$-module. Using the fundamental exact sequence

$$\Omega_{A/\mathbb{Z}} \otimes_A \widehat{A} \to \Omega_{\widehat{A}/\mathbb{Z}} \to \Omega_{\widehat{A}/A} \to 0 \tag{3.1}$$

we conclude that $\Omega_{\widehat{A}/A}$ must also be finitely generated $\widehat{A}$-module. Tensoring the above exact sequence by $\kappa_{\widehat{A}} = \kappa_A$, in order to show that $\Omega_{\widehat{A}/A} = 0$, it suffices to establish by Nakayama's lemma that

$$\Omega_{A/\mathbb{Z}} \otimes_A \kappa_{\widehat{A}} \to \Omega_{\widehat{A}/\mathbb{Z}} \otimes_{\widehat{A}} \kappa_{\widehat{A}}$$

is surjective. But this follows from the 4-lemma applied to the following diagram with exact rows

$$\begin{tikzcd}
\mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r] \arrow[d, "="] & \Omega_{A/\mathbb{Z}} \otimes_A \kappa_A \arrow[r] \arrow[d] & \Omega_{\kappa_A/\mathbb{Z}} \arrow[r] \arrow[d, "="] & 0 \\
\mathfrak{m}_{\widehat{A}}/\mathfrak{m}_{\widehat{A}}^2 \arrow[r] & \Omega_{\widehat{A}/\mathbb{Z}} \otimes_{\widehat{A}} \kappa_{\widehat{A}} \arrow[r] & \Omega_{\kappa_{\widehat{A}}/\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

To summarize, we have proved so far that the canonical map $A \to \widehat{A}$ is faithfully flat and formally unramified. However, this map cannot be formally étale. If it is, then the exact sequence from (3.1) is also exact on the left, that is,

$$0 \to \Omega_{A/\mathbb{Z}} \otimes_A \widehat{A} \to \Omega_{\widehat{A}/\mathbb{Z}} \to \Omega_{\widehat{A}/A} \to 0$$

is exact [Sta18, 031K]. Then $\Omega_{A/\mathbb{Z}} \otimes_A \widehat{A}$ is a finitely generated $\widehat{A}$-module since it is a submodule of the finitely generated $\widehat{A}$-module $\Omega_{\widehat{A}/\mathbb{Z}}$. But this is impossible because $\Omega_{A/\mathbb{Z}}$ is not a finitely generated A-module and finite generation of modules descends over faithfully flat base change [Bou89, Chapter I, §3.6, Proposition 11].

3.4 F-purity

Since regularity of a Noetherian ring of prime characteristic is characterized by flatness of the Frobenius map of the ring, a natural way to study singular rings is by replacing flatness of Frobenius by some weaker property, and examining the resulting class of rings satisfying this weaker property. For example, a necessary condition for Frobenius to be flat is for Frobenius to be injective since faithfully flat maps are injective. Therefore as a weakening of flatness of Frobenius, one may choose to study rings for which Frobenius is injective. However, injectivity of Frobenius is too general a notion of singularity since it characterizes reduced rings (Lemma III.7). Hence we want to restrict our attention to studying those rings for which Frobenius satisfies a property that is weaker than flatness, but not as general as injectivity.

This leads naturally to the notion of F-purity, which is based on the notion of *pure map* of modules. Therefore we discuss pure maps first.

3.4.1 Pure maps of modules

Definition III.27. Let A be a ring (of arbitrary characteristic). A map of A-modules $\varphi : M \to N$ is **pure** if for all A-modules P,

$$\varphi \otimes \mathrm{id}_P : M \otimes_A P \to N \otimes_A P$$

is injective. A ring homomorphism $A \to B$ is **pure**, if it is pure as a map of A-modules. Here B is considered as an A-module by restriction of scalars.

Remark III.28. By taking $P = A$ in the definition of purity, we see that a pure map of modules is always injective.

We gather some basic properties of pure maps of modules for the convenience of the reader.

Lemma III.29. *Let A be an arbitrary commutative ring A, not necessarily Noetherian nor of characteristic p.*

(a) If $M \to N$ and $N \to Q$ are pure maps of A-modules, then the composition $M \to N \to Q$ is also pure.

(b) If a composition $M \to N \to Q$ of A-modules is pure, then $M \to N$ is pure.

(c) If B is an A-algebra and $M \to N$ is pure map of A-modules, then $B \otimes_A M \to B \otimes_A N$ is a pure map of B-modules.

(d) Let B be an A-algebra. If $M \to N$ is a pure map of B-modules, then it is also pure as a map of A-modules.

(e) An A-module map $M \to N$ is pure if and only if for all prime ideals $\mathfrak{p} \subset A$, $M_\mathfrak{p} \to N_\mathfrak{p}$ is pure.

(f) A faithfully flat map of rings is pure.

(g) If $(\Lambda, \leq)$ is a directed set with a least element λ_0, $\{N_\lambda\}_{\lambda\in\Lambda}$ is a direct limit system of A-modules indexed by Λ and $M \to N_{\lambda_0}$ is an A-linear map, then $M \to \varinjlim_\lambda N_\lambda$ is pure if and only if $M \to N_\lambda$ is pure for all λ.

(h) A map of modules $A \to N$ over a Noetherian local ring $(A, \mathfrak{m}_A, \kappa_A)$ is pure if and only if $E \otimes_A A \to E \otimes_A N$ is injective where E is the injective hull of the residue field of R.

Proof. Properties (a)-(d) follow easily from the definition of purity and elementary properties of tensor product. As an example, let us prove (d). If P is an A-module, we want to show that $P \otimes_A M \to P \otimes_A N$ is injective. The map of B-modules

$$(P \otimes_A B) \otimes_B M \to (P \otimes_A B) \otimes_B N$$

is injective by purity of $M \to N$ as a map of B-modules. Using the natural A-module isomorphisms $(P \otimes_A B) \otimes_B M \cong P \otimes_A M$ and $(P \otimes_A B) \otimes_B N \cong P \otimes_A N$, we conclude that $P \otimes_A M \to P \otimes_A N$ is injective in the category of A-modules.

Property (e) follows from (c) by tensoring with $A_\mathfrak{p}$ and the fact that injectivity of a map of modules is a local property. Property (f) follows from [Bou89, Chapter I, §3.5, Proposition 9(c)]. Properties (g) and (h) are proved in [HH95, Lemma 2.1]. □

Example III.30. If $(R, \mathfrak{m}_R, \kappa_R)$ is a Noetherian local ring, and $\widehat{R}$ is the $\mathfrak{m}_R$-adic completion of R, then $R \to \widehat{R}$ is faithfully flat, hence pure.

3.4.2 Definition of F-purity

Definition III.31. A ring R of characteristic p is F**-pure** if the Frobenius map $F : R \to F_*R$ is a pure map of R-modules.

Remark III.32. If R is F-pure, then the Frobenius map of R is injective. Thus F-pure rings are reduced. If the Frobenius map of R is flat, then it is faithfully flat, hence pure (Lemma III.29). Therefore F-purity sits in-between injectivity and flatness of Frobenius. In particular, a regular ring of prime characteristic is always F-pure.

The notion of F-purity first appeared in Hochster and Roberts's work in invariant theory on the Cohen-Macaulayness of rings of invariants of linearly reductive groups acting on regular rings [HR74]. Further evidence that F-purity is a good notion of singularity stems from the fact that it implies nice cohomological properties. For instance, by studying the action of Frobenius on local cohomology modules, Hochster and Roberts showed that F-purity greatly simplifies the structure of local cohomology modules of Noetherian rings [HR76].

The following criterion, established by Fedder, allows one to construct non-regular examples of F-pure rings.

Theorem III.33 (**Fedder's criterion**)**.** *Let $(R, \mathfrak{m}_R, \kappa_R)$ be a regular local ring of characteristic p and let $S := R/I$, for an ideal I of R. Then S is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}_R^{[p]}$.*

In particular, if I is generated by a single element f, then S is F-pure if and only if $f^{p-1} \notin \mathfrak{m}_R^{[p]}$.

Proof. [Fed83, Theorem 1.12]. $\square$

Example III.34. Let k be a field of characteristic p and $R = k[X, Y, Z]_{(X,Y,Z)}$. Suppose $f = XY - Z^2$. Then one can use Fedder's criterion to see that $S = R/(f)$

is F-pure. Indeed, it is not difficult to verify that

$$(XY - Z^2)^{p-1} \notin (X^p, Y^p, Z^p).$$

Remark III.35. For a non-local version of Fedder's criterion, we refer the reader to [Fed83, Theorem 1.13]. Also, contrast Fedder's criterion with Remark III.66.

3.5 Frobenius splitting

Strengthening purity of Frobenius leads one to the notion of *Frobenius splitting*, a term first coined by Mehta and Ramanathan in [MR85]. Recall that a map of A-modules $M \to N$ is **split** if it admits a left inverse in the category of A-modules.

Definition III.36. A ring R of prime characteristic is **Frobenius split** if the Frobenius map $F : R \to F_*R$ admits a left inverse, called a **Frobenius splitting**, in the category of R-modules.

In other words, R is Frobenius split if there exists an R-linear map $F_*R \to R$ that maps $1 \mapsto 1$. Note that a Frobenius splitting exists if there is a surjective R-linear map $F_*R \to R$.

Since split maps of R-modules are clearly pure, a Frobenius split ring is F-pure. However, the converse is false even for regular local rings. Indeed, we will see later that the discrete valuation ring constructed in Example II.57(4) is *not* Frobenius split, even though every regular ring of prime characteristic is F-pure (Remark III.32). This will follow from the more general observation that a Frobenius split, generically F-finite Noetherian domain has to be excellent, whereas the aforementioned discrete valuation ring is not excellent (see subsection 3.3.4).

Despite this cautionary observation, Frobenius splitting and F-purity are equivalent for most rings which arise in geometry, which is why they are often used synonymously in the literature. This follows from the following beautiful result of Hochster and Roberts.

Theorem III.37. *Let A be a ring, not necessarily Noetherian or of prime characteristic. Suppose $\varphi : M \to N$ is a map of A-modules such that* $\operatorname{coker}(\varphi)$ *is finitely presented. Then φ is pure if and only if it splits.*

Proof. [HR76, Corollary 5.2] □

Corollary III.38. *Let R be a Noetherian F-finite ring of characteristic p. Then R is F-pure if and only if R is Frobenius split.*

Proof. The hypothesis implies that the cokernel of the Frobenius map is always a finitely presented R-module. □

3.5.1 Global Frobenius splitting and consequences

Definition III.39. Let X be a scheme over $\mathbb{F}_p$. Then we say X is **(globally) Frobenius split** if the morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ has a left-inverse in the category of $\mathcal{O}_X$-modules.

The existence of a global Frobenius splitting has strong consequences for the geometry of X. In order to highlight some of these consequences, we will repeatedly use the following two results.

Lemma III.40. *Let $X/\mathbb{F}_p$ be a scheme. Then the following are equivalent:*

1. *X is Frobenius split.*
2. *There exists $e > 0$ such that the morphism $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits in the category of $\mathcal{O}_X$-modules.*
3. *For all $e > 0$, the morphism $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits in the category of $\mathcal{O}_X$-modules.*

Proof. This is a simple consequence of the observation that the morphism $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ factors as $\mathcal{O}_X \to F_*\mathcal{O}_X \to F^e_*\mathcal{O}_X$, for any $e > 0$. Here the morphism $F_*\mathcal{O}_X \to F^e_*\mathcal{O}_X$ is obtained by applying the functor F_* to the morphism $\mathcal{O}_X \to F^{e-1}_*\mathcal{O}_X$, induced by the $(e-1)$-th iterate of Frobenius. □

Proposition III.41 (Projection formula). *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then there exists a canonical isomorphism*

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}).$$

Proof. This is [Har77, Exercise II.5.1], so we omit the proof. □

Applying the projection formula when f is an iterate of Frobenius and $\mathcal{E}$ is a line bundle gives us the following

Corollary III.42. *If X is a scheme over $\mathbb{F}_p$, $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{L}$ is a line bundle on X, then*

$$F^e_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} \cong F^e_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes p^e}).$$

Proof. By the projection formula, $F^e_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} = F^e_*(\mathcal{F} \otimes_{\mathcal{O}_X} (F^e)^*\mathcal{L})$. Thinking of line bundles in terms of transition functions, we see that the pullback $(F^e)^*$ raises transition functions of $\mathcal{L}$ to their p^e-th powers. Therefore $(F^e)^*\mathcal{L} \cong \mathcal{L}^{\otimes p^e}$. □

The first consequence of the existence of a Frobenius splitting we want to highlight is a strong form of Serre-vanishing.

Theorem III.43 (Strong Serre-vanishing). *Let X be a Frobenius split projective variety over a field k of prime characteristic. If $\mathcal{L}$ is an ample line bundle on X, then for all $i > 0$,*

$$H^i(X, \mathcal{L}) = 0.$$

Proof. Ordinary Serre-vanishing implies that for $n \gg 0$, $H^i(X, \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$. Since $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits, so does the map

$$\mathcal{L} \to F^e_* \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L},$$

upon tensoring by $\mathcal{L}$. By Corollary III.42, $F^e_* \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L} = F^e_*(\mathcal{L}^{\otimes p^e})$. Therefore $H^i(X, \mathcal{L})$ is a direct summand of $H^i(X, F^e_*(\mathcal{L}^{\otimes p^e}))$ for all $i \geq 0$. Since F^e is an affine morphism, $H^i(X, F^e_*(\mathcal{L}^{\otimes p^e})) = H^i(X, \mathcal{L}^{\otimes p^e})$, and this latter cohomology group vanishes when $e \gg 0$ and $i > 0$. Hence $H^i(X, \mathcal{L})$ must also vanish for $i > 0$. □

Remark III.44. The proof of Theorem III.43 shows, more generally, that if X is a Frobenius split scheme and $\mathcal{L}$ is a line bundle on X such that for some $i \geq 0$ and all $n \gg 0$, $H^i(X, \mathcal{L}^{\otimes n}) = 0$, then $H^i(X, \mathcal{L}) = 0$.

The other surprising consequence of Frobenius splitting is that Kodaira vanishing holds for Frobenius split smooth projective varieties, even though Kodaira vanishing is known to fail in general in prime characteristic [Ray78].

Theorem III.45 (**Kodaira vanishing**). *Let X be a smooth projective variety of dimension d over a field k of prime characteristic with canonical bundle ω_X. If X is Frobenius split, then for any ample line bundle $\mathcal{L}$ on X and for all $i > 0$,*

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = 0.$$

Proof. By Serre duality we know that $h^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = h^{d-i}(X, \mathcal{L}^{-1})$. Since $\mathcal{L}$ is ample and ω_X is coherent, Serre vanishing implies that for all $n \gg 0$ and $i > 0$,

$$h^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0.$$

Therefore for all $n \gg 0$ and for all $i > 0$,

$$h^{d-i}(X, \mathcal{L}^{\otimes -n}) = 0.$$

Since X is Frobenius split, Remark III.44 implies that for all $i > 0$,

$$h^{d-i}(X, \mathcal{L}^{-1}) = 0.$$

A second application of Serre duality then shows that for all $i > 0$, $h^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = 0$. $\square$

Remark III.46. The reader will notice that the proof of Kodaira vanishing holds more generally for a Frobenius split projective k-scheme which is Cohen-Macaulay and equidimensional. Indeed, Serre duality holds in this setting.

3.6 p^{-e}-linear maps

The previous section shows that the existence of a non-trivial $\mathcal{O}_X$-linear map $F_*\mathcal{O}_X \to \mathcal{O}_X$ that maps $1 \mapsto 1$ has strong consequences for the geometry of a scheme X over $\mathbb{F}_p$. In this section we will study more general maps of this type, so we give them a special name.

Definition III.47. [BB11, BS13] Let X be a scheme over $\mathbb{F}_p$. A **p^{-e}-linear map** is an additive map

$$\varphi : \mathcal{O}_X \to \mathcal{O}_X$$

such that for local sections $r, s \in \mathcal{O}_X(U)$,

$$\varphi(r^{p^e} s) = r\varphi(s).$$

Equivalently, a p^{-e}-linear map can be specified by a map of $\mathcal{O}_X$-modules $F^e_*\mathcal{O}_X \to \mathcal{O}_X$. If $X = \mathrm{Spec}(R)$, then we will also call an R-linear map $F^e_* R \to R$ a **p^{-e}-linear map**.

Example III.48. A Frobenius splitting is a p^{-1}-linear map that maps $1 \mapsto 1$.

Remark III.49. One can define the notion of p^{-e}-linear maps more generally (in an obvious way) for maps between sheaves of $\mathcal{O}_X$-modules. However, in this thesis by a p^{-e}-linear map we always means a p^{-e}-linear *self* map of the structure sheaf $\mathcal{O}_X$, or equivalently, an $\mathcal{O}_X$-linear map $F^e_*\mathcal{O}_X \to \mathcal{O}_X$. Thus when we say X has no non-trivial p^{-e}-linear maps, we mean it has no non-trivial $\mathcal{O}_X$-linear maps $F^e_*\mathcal{O}_X \to \mathcal{O}_X$. We are also going to be sloppy and frequently refer to $\mathcal{O}_X$-linear maps $F^e_*\mathcal{O}_X \to \mathcal{O}_X$ as p^{-e}-linear maps.

Our first goal in this section will be to explain why the existence of non-trivial p^{-e}-linear maps is closely related to excellence and F-finiteness. This is based on joint work with Karen Smith [DS17b].

3.6.1 p^{-e}-linear maps, excellence and F-finiteness

The main result of this subsection is the following:

Theorem III.50. [DS17b] *Let R be a Noetherian domain of characteristic p whose fraction field is F-finite. The following are equivalent:*

1. *R is excellent.*
2. *R is F-finite.*
3. *The module $\mathrm{Hom}_R(F_*R, R)$ is non-trivial.*
4. *For all $e > 0$, $\mathrm{Hom}_R(F^e_*R, R)$ is non-trivial.*
5. *There exists $e > 0$ such that $\mathrm{Hom}_R(F^e_*R, R)$ is non-trivial.*

Conditions (3)-(5) in Theorem III.50 can be stated using Hochster's notion of a **solid algebra**.

Definition III.51. An R-module M is **solid** if there exists a non-trivial R-module map $M \to R$. An R-algebra A is a **solid algebra** if it is solid as an R-module.

Thus condition (3) above precisely states that F_*R is a solid R-algebra via Frobenius, or equivalently, that R is a solid R^p-algebra. Similarly conditions (4) and (5) deal with the solidity of R over R^{p^e}. The theorem states that if R is a domain whose fraction field is F-finite, then R is a solid algebra via Frobenius if and only if R is excellent.

The proof of Theorem III.50 requires the following lemma, which is independent of the characteristic of rings.

Lemma III.52. [DS17b] *Let $R \xrightarrow{f} S$ be an injective ring homomorphism of Noetherian domains such that the induced map of fraction fields* $\operatorname{Frac}(R) \hookrightarrow \operatorname{Frac}(S)$ *is finite. If the canonical map*

$$S \to \operatorname{Hom}_R(\operatorname{Hom}_R(S, R), R)$$

is injective, then f is also a finite map.

Proof. Note that if M is a finitely generated R-module, then so is $\operatorname{Hom}_R(M, R)$. Thus the lemma follows by Noetherianity if we can show that $\operatorname{Hom}_R(S, R)$ is a finitely generated R-module. Let n be the degree of the field extension $\operatorname{Frac}(S)/\operatorname{Frac}(R)$. Then there exists a basis $x_1, \ldots, x_n$ of $\operatorname{Frac}(S)$ over $\operatorname{Frac}(R)$ such that $x_i \in S$ [AM69, 5.1.7].

Let T be the free R-submodule of S generated by the x_i. It is clear that S/T is a torsion R-module. Then applying $\operatorname{Hom}_R(-, R)$ to the short exact sequence

$$0 \to T \to S \to S/T \to 0$$

we get the exact sequence

$$0 \to \operatorname{Hom}_R(S/T, R) \to \operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(T, R).$$

Since S/T is a torsion R-module and R is a domain, $\mathrm{Hom}_R(S/T, R) = 0$. Thus, $\mathrm{Hom}_R(S, R)$ is a submodule of $\mathrm{Hom}_R(T, R)$, and the latter is free of rank n. But R is a Noetherian ring, and so $\mathrm{Hom}_R(S, R)$ is also finitely generated. $\square$

A necessary condition for the injectivity of $S \to \mathrm{Hom}_R(\mathrm{Hom}_R(S, R), R)$ in the situation of the previous lemma is for the module $\mathrm{Hom}_R(S, R)$ to be non-trivial. If only non-triviality of this module is assumed, injectivity of $S \to \mathrm{Hom}_R(\mathrm{Hom}_R(S, R), R)$ follows for a large class of examples as shown in the following result:

Proposition III.53. [DS17b] *Let $R \xhookrightarrow{f} S$ be an injective ring homomorphism of arbitrary domains such that the induced map $\mathrm{Frac}(R) \hookrightarrow \mathrm{Frac}(S)$ is algebraic. If S is a solid R-algebra, then the canonical map $S \to \mathrm{Hom}_R(\mathrm{Hom}_R(S, R), R)$ is injective. If, in addition, R and S are Noetherian and f is generically finite, then f is a finite map.*

Proof. By non-triviality of $\mathrm{Hom}_R(S, R)$, there exists an R-linear map $S \xrightarrow{\phi} R$ such that $\phi(1) \neq 0$, and so, for all non-zero $r \in R$, $\phi(r) = r\phi(1) \neq 0$. For the injectivity of

$$S \to \mathrm{Hom}_R(\mathrm{Hom}_R(S, R), R),$$

it suffices to show that for each non-zero $s \in S$, there exists $\varphi \in \mathrm{Hom}_R(S, R)$ such that $\varphi(s) \neq 0$. Now since s is algebraic over $\mathrm{Frac}(R)$, there exists $\sum_{i=0}^{n} a_i T^i \in R[T]$ such that $a_0 \neq 0$, and

$$a_n s^n + a_{n-1}s^{n-1} + \dots a_1 s + a_0 = (a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1)s + a_0 = 0.$$

Suppose ℓ_λ is left multiplication by λ, where $\lambda := a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1 \in S$. Then $\phi \circ \ell_\lambda \in \mathrm{Hom}_R(S, R)$, and

$$\phi \circ \ell_\lambda(s) = \phi(-a_0) = -a_0\phi(1) \neq 0,$$

which proves injectivity of $S \to \mathrm{Hom}_R(\mathrm{Hom}_R(S,R),R)$.

If $R \xrightarrow{f} S$ is a generically finite map of Noetherian domains, then f is a finite map by Lemma III.52 and what we just proved. $\square$

Remark III.54. As a special case of Proposition III.53, we obtain the following result: Let R be any domain and K be any field containing R. If the integral closure $\overline{R}$ of R in K is a solid R-algebra, then the canonical map $\overline{R} \to \mathrm{Hom}_R(\mathrm{Hom}_R(\overline{R},R),R)$ is injective. In particular, a Noetherian domain R is Japanese precisely when the integral closure of R in any finite extension of its fraction field is a solid R-algebra.

Proof of Theorem III.50. We already know (1) and (2) are equivalent from Theorem III.24.

For (2) implies (3), assume F_*R is a finitely generated R-module. Let K be the fraction field of R, and denote by F_*K the fraction field of F_*R, again emphasizing the K-vector space structure via Frobenius. Note $F_*K = F_*R \otimes_R K$. Since

$$\mathrm{Hom}_R(F_*R,R) \otimes_R K \cong \mathrm{Hom}_K(F_*K,K) \neq 0,$$

it follows that $\mathrm{Hom}_R(F_*R,R) \neq 0$.

We now show (3) implies (4). If $\mathrm{Hom}_R(F_*R,R)$ is non-trivial, then there exists $\phi : F_*R \to R$ such that

$$\phi(1) = c \neq 0.$$

By induction, suppose there exists $\varphi \in \mathrm{Hom}_R(F_*^{e-1}R,R)$ such that $\varphi(1) \neq 0$. Then the p^{-e}-linear map

$$F_*^eR \xrightarrow{F_*^{e-1}(\phi)} F_*^{e-1}R \xrightarrow{\varphi} R$$

maps $c^{(p^{e-1}-1)p} \mapsto c\varphi(1) \neq 0$, showing that $\mathrm{Hom}_R(F_*^eR,R)$ is non-trivial.

Obviously, (4) implies (5). We finish the proof by proving that (5) implies (2). By assumption, F_*^eK is a finite extension of K. We now apply Proposition III.53, taking

taking $S = F^e_* R$ and $f = F^e$. The proposition implies that F^e is a finite map. Thus, also F is a finite map, and we have proved (5) implies (2). $\square$

Corollary III.55. [DS17b] *If R is a non-excellent domain of characteristic $p > 0$ which is generically F-finite, then* $\mathrm{Hom}_R(F^e_* R, R) = 0$ *for all $e \in \mathbb{N}$.*

Corollary III.56. *A generically F-finite, Frobenius split Noetherian domain is F-finite (equivalently excellent).*

Example III.57. Since the discrete valuation ring of $\mathbb{F}_p(X, Y)$ constructed in Example II.57(4) is not excellent (subsection 3.3.4), this ring has no non-trivial p^{-e}-linear maps by Theorem III.50. On the other hand, the ring is F-pure since it is regular. This provides an example of an F-pure ring that is not Frobenius split, which shows that F-purity is a more general notion of singularity in prime characteristic than Frobenius splitting.

3.6.2 Some open questions

Theorem III.50 and Example III.57 raise the following interesting questions.

- Do excellent domains of prime characteristic admit non-trivial p^{-e}-linear (self) maps? Stated differently, if R is an excellent domain of prime characteristic, is $F^e_* R$ always a solid R-algebra?
- Is every excellent regular ring of prime characteristic Frobenius split? More generally, is every excellent F-pure ring also Frobenius split?

The results of this section provide affirmative answers to both questions for Noetherian domains whose fraction fields are F-finite, that is, in the generically F-finite setting. Moreover, as we now explain, the questions also have affirmative answers for complete local rings (such rings are always excellent). For complete rings, Hochster gave a very useful criterion to check if a module is solid using local cohomology.

Proposition III.58. [Hoc94, Corollary 2.4] *Let $(R, \mathfrak{m}_R, \kappa_R)$ be a complete local Noetherian ring of Krull dimension d. Then an R-module M is solid if and only if $H^d_{\mathfrak{m}_R}(M) \neq 0$.*

When R is a complete local Noetherian ring of characteristic p,

$$H^d_{\mathfrak{m}_R}(F^e_* R) = H^d_{\mathfrak{m}_R^{[p^e]}}(R) = H^d_{\mathfrak{m}_R}(R),$$

for any $e > 0$. A result of Grothendieck implies that $H^d_{\mathfrak{m}_R}(R) \neq 0$ [ILL+07, Theorem 9.3], therefore allowing us to conclude using Proposition III.58 that $F^e_* R$ is a solid R-algebra. In other words, complete local Noetherian rings always have non-trivial p^{-e}-linear maps for any $e > 0$.

The fact that complete local Noetherian F-pure rings are Frobenius split follows from the following result:

Proposition III.59. *Let $(R, \mathfrak{m}_R, \kappa_R)$ be a Noetherian local ring of characteristic p and $\widehat{R}$ denote its $\mathfrak{m}_R$-adic completion. The following are equivalent:*

1. *R is F-pure.*
2. *There exists an R-linear map $F_* R \to \widehat{R}$ that maps $1 \mapsto 1$.*

Proof. Assume (2) and let $\varphi : F_* R \to \widehat{R}$ be an R-linear map that maps $1 \mapsto 1$. The composition

$$R \xrightarrow{F} F_* R \xrightarrow{\varphi} \widehat{R}$$

is then easily verified to be the canonical map from a Noetherian local ring to its completion. Since this canonical map is faithfully flat, hence pure, Lemma III.29(b) implies that $F : R \to F_*R$ is also pure. In other words, R is F-pure, and so, (2) $\Rightarrow$ (1).

Conversely, suppose R is F-pure and let $E = E_R(\kappa_R)$ denote the injective hull of the residue field of R. Then $F \otimes_R \mathrm{id}_E : E \to F_*R \otimes_R E$ is injective. Applying the Matlis dual $\mathrm{Hom}_R(\ , E)$ to this injective map gives a surjection

$$\mathrm{Hom}_R(F_*R \otimes_R, E, E) \twoheadrightarrow \mathrm{Hom}_R(E, E).$$

Matlis duality implies that $\mathrm{Hom}_R(E, E) \cong \widehat{R}$, while Hom-$\otimes$ adjunction shows

$$\mathrm{Hom}_R(F_*R \otimes_R, E, E) \cong \mathrm{Hom}_R(F_*R, \mathrm{Hom}_R(E, E)) \cong \mathrm{Hom}_R(F_*R, \widehat{R}).$$

Therefore purity of Frobenius induces a surjection

$$\chi : \mathrm{Hom}_R(F_*R, \widehat{R}) \twoheadrightarrow \widehat{R},$$

which using the canonical isomorphisms from above can be verified to be evaluation at 1, that is, if $\varphi \in \mathrm{Hom}_R(F_*R, \widehat{R})$, then

$$\chi(\varphi) = \varphi(1).$$

But surjectivity of χ is equivalent to (2). $\square$

Corollary III.60. *A complete local Noetherian ring of characteristic p is F-pure if and only if it is Frobenius split.*

Proof. Apply Proposition III.59 with $R = \widehat{R}$. $\square$

Despite these partial results, the relationship between F-purity and Frobenius splitting and the existence of non-trivial p^{-e}-linear maps remain mysterious for arbitrary excellent rings. In fact, to the best of our knowledge, it is not known if an excellent discrete valuation ring of prime characteristic is always Frobenius split.

3.6.3 Grothendieck duality and the divisor correspondence

Throughout this subsection, we assume X is a smooth variety over an F-finite field k of characteristic p. Our aim is to show that a p^{-e}-linear map $\varphi : F^e_*\mathcal{O}_X \to \mathcal{O}_X$ roughly corresponds to an effective divisor Δ_φ on X such that

$$\Delta_\varphi \sim (1-p^e)K_X.$$

In other words, p^{-e}-linear maps correspond to global sections of the invertible sheaf $\mathcal{O}_X((1-p^e)K_X) = \omega_X^{\otimes(1-p^e)}$ (Corollary III.65), which should further convince the reader of the geometric nature of such maps.

The divisor correspondence is a formal consequence of Grothendieck duality for proper morphisms, so we briefly review what we need from duality first.

Theorem III.61 (**Grothendieck duality for proper morphisms**)**.** *Let $g : Y \to Z$ be a proper morphism of Noetherian schemes.*

1. *There exists a functor*

$$g^! : D^+_{\mathrm{Coh}}(Z) \to D^+_{\mathrm{Coh}}(Y)$$

such that if $\mathcal{D}$ is a dualizing complex of Z then $g^!(\mathcal{D})$ is a dualizing complex of Y.

2. *There is a a natural transformation*

$$\mathrm{Tr}_g : \mathbf{R}g_* \circ g^! \to \mathrm{Id},$$

called the ***trace of*** *g, which induces an isomorphism*

$$\Theta_g : \mathbf{R}g_*\mathbf{R}\mathscr{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{F}, g^!(\mathcal{G})) \to \mathbf{R}\mathscr{H}om^\bullet_{\mathcal{O}_Z}(\mathbf{R}g_*(\mathcal{F}), \mathcal{G}),$$

for all $\mathcal{F} \in D^b_{\mathrm{Coh}}(Y)$ and for all $\mathcal{G} \in D^b_{\mathrm{Coh}}(Z)$.[2]

[2] $\mathscr{H}om$ here means sheaf-Hom.

3. *If g is finite,* Tr_g *induces an isomorphism*

$$\Theta_g : g_* \mathbf{R}\mathscr{H}om^{\bullet}_{\mathcal{O}_Y}(\mathcal{F}, g^!(\mathcal{G})) \to \mathbf{R}\mathscr{H}om^{\bullet}_{\mathcal{O}_Z}(g_*(\mathcal{F}), \mathcal{G}),$$

for all $\mathcal{F} \in D^b_{\mathrm{Coh}}(Y)$ *and for all* $\mathcal{G} \in D^b_{\mathrm{Coh}}(Z)$.

Proof. The first and second assertions follow from [Har66, Chapter VI, Corollary 3.5] and [Har66, Chapter VII, Corollary 3.4]. The third assertion follows from the second because when g is finite, $g_* : \mathrm{QCoh(Y)} \to \mathrm{QCoh(Z)}$ is exact since g is affine, and so g_* and $\mathbf{R}g_*$ are naturally isomorphic as functors from $D_{\mathrm{QCoh}}(Y) \to D_{\mathrm{QCoh}}(Z)$ [Sta18, Tag 08D7]. □

Specializing to the case of interest for us, suppose X is a smooth variety of dimension n over an F-finite field k, and let $f : X \to \mathrm{Spec}(k)$ be the the structure morphism. Then X has a *normalized dualizing complex* [Har66, Chapter V, Theorem 8.3]

$$\omega^{\bullet}_X = f^!(\mathcal{O}_{\mathrm{Spec}(k)}[0]) = (\wedge^n \Omega_{X/k})[n] = \omega_X[n],$$

where $\omega_X = \wedge^n \Omega_{X/k}$ is the canonical bundle of X introduced in subsection 3.2.1. By assumption, the Frobenius map F of X is a finite morphism. Therefore $F^!(\omega^{\bullet}_X)$ is also a dualizing complex of X by the above duality theorem. In this situation $F^!(\omega^{\bullet}_X)$ and $\omega^{\bullet}_X$ are actually isomorphic in $D^+_{\mathrm{Coh}}(X)$, as we now show.

Proposition III.62. *Let X be a smooth variety over an F-finite field k with structure morphism $f : X \to \mathrm{Spec}(k)$. Let $\omega^{\bullet}_X = f^!(\mathcal{O}_{\mathrm{Spec}(k)}[0])$ be the normalized dualizing complex of X.*

1. *If F is the Frobenius map of X, then $F^!(\omega^{\bullet}_X) \cong \omega^{\bullet}_X$ in $D^+_{\mathrm{Coh}}(X)$.*

2. *For any coherent sheaf $\mathcal{F}$ on X, $F_* \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \cong \mathscr{H}om_{\mathcal{O}_X}(F_*(\mathcal{F}), \omega_X)$.*

Proof. (1) We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "F_k"] & \mathrm{Spec}(k) \end{tikzcd}$$

where the horizontal maps are Frobenius (subscripts are chosen to distinguish the Frobenius of X and the Frobenius of $\mathrm{Spec}(k)$ for the reader's convenience). Now $F_X^!(\omega_X^\bullet) = F_X^!(f^!(\mathcal{O}_{\mathrm{Spec}(k)}[0])) \cong (f \circ F_X)^!(\mathcal{O}_{\mathrm{Spec}(k)}[0]) = (F_k \circ f)^!(\mathcal{O}_{\mathrm{Spec}(k)}[0]) \cong f^!(F_k^!(\mathcal{O}_{\mathrm{Spec}(k)}[0]))$. However, k is F-finite, and using the definition of $(\ )^!$ for a finite morphism and duality for finite morphisms [Har66, Chapter III, §6], we get

$$F_k^!(\mathcal{O}_{\mathrm{Spec}(k)}[0]) \cong \mathcal{O}_{\mathrm{Spec}(k)}[0].$$

Thus, $F_X^!(\omega_X^\bullet) \cong f^!(F_k^!(\mathcal{O}_{\mathrm{Spec}(k)}[0])) \cong f^!(\mathcal{O}_{\mathrm{Spec}(k)}[0]) = \omega_X^\bullet$.

(2) Suppose $\mathcal{F}$ is a coherent sheaf on X. Since Frobenius of X is a finite morphism, Grothendieck duality for proper/finite morphisms (Theorem III.61(3)) gives an isomorphism in the derived category

$$\Theta_F : F_* \mathbf{R}\mathscr{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}[0], F^!(\omega_X^\bullet)) \to \mathbf{R}\mathscr{H}om^\bullet_{\mathcal{O}_X}(F_*(\mathcal{F}[0]), \omega_X^\bullet).$$

From (1) we have

$$F^!(\omega_X^\bullet) \cong \omega_X^\bullet = \omega_X[n],$$

and we know F_* is exact. Taking cohomology in degree $-n$ and using the fact that

$$\mathbf{R}^i \mathscr{H}om^\bullet_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}) \cong \mathscr{H}om_{D_{QCoh}(X)}(\mathcal{G}, \mathcal{H}[i])$$

gives the desired isomorphism

$$F_* \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(F_*(\mathcal{F}), \omega_X),$$

which completes the proof of (2). $\square$

Remarks III.63.

1. Proposition III.62 clearly also holds for the iterates F^e of Frobenius.

2. Let X be a smooth variety over an F-finite field. Taking $\mathcal{F} = \omega_X$, Proposition III.62 gives an isomorphism of sheaves

$$F_* \mathscr{H}om_{\mathcal{O}_X}(\omega_X, \omega_X) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(F_*\omega_X, \omega_X).$$

   Passing to global sections under the above isomorphism, the image of the identity morphism $\mathrm{id} : \omega_X \to \omega_X$ corresponds to an $\mathcal{O}_X$-linear map $F_*\omega_X \to \omega_X$. This map is called the **trace of Frobenius**. It features prominently in F-singularity theory (see [BST15, ST14]).

We now have all the tools needed to prove the correspondence between p^{-e}-linear maps on X and global sections of $\mathcal{O}_X((1-p^e)K_X) = \omega_X^{\otimes(1-p^e)}$ alluded to in the beginning of this subsection. In fact, we are able to prove a more general result.

Theorem III.64. [MR85, SS10, BS13] *Let X be a smooth variety over an F-finite field k of characteristic p. Then for any divisor Weil D on X,*

$$\mathscr{H}om_{\mathcal{O}_X}(F_*^e\mathcal{O}_X(D), \mathcal{O}_X) \cong F_*^e\mathcal{O}_X((1-p^e)K_X - D).$$

Thus, $\mathrm{Hom}_{\mathcal{O}_X}(F_*^e\mathcal{O}_X(D), \mathcal{O}_X)$ *is in one-to-one correspondence with the global sections of* $\mathcal{O}_X((1-p^e)K_X - D)$.

Proof. We have

$$\begin{aligned}
\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) &\cong \mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \omega_X, \omega_X) \\
&\cong \mathscr{H}om_{\mathcal{O}_X}\left(F^e_*(\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \omega_X^{\otimes p^e}), \omega_X\right) \\
&\cong F^e_* \mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{O}_X(D + p^e K_X), \mathcal{O}_X(K_X)\right) \\
&\cong F^e_* \mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{O}_X, \mathcal{O}_X((1-p^e)K_X - D))\right) \\
&\cong F^e_*\mathcal{O}_X((1-p^e)K_X - D).
\end{aligned}$$

Here the first and fourth isomorphisms follow from elementary properties of invertible sheaves, the second isomorphism follows from Corollary III.42 and the third isomorphism from Proposition III.62 applied to the iterate F^e instead of F (the proof is exactly the same). □

Corollary III.65. *Let X be a smooth variety over an F-finite field k of characteristic p with function field K. Then there exists a one-to-one correspondence*

$$\left\{ \begin{array}{c} p^{-e} - \textit{linear maps} \\ F^e_*\mathcal{O}_X \to \mathcal{O}_X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{rational functions } f \in K \textit{ such that} \\ \operatorname{div}(f) + (1-p^e)K_X \geq 0 \end{array} \right\}$$

Proof. Apply Theorem III.64 with $D = 0$. □

Remark III.66. Corollary III.65 puts restrictions on when smooth varieties over F-finite fields can have non-trivial p^{-e}-linear maps. For instance, if X is a smooth projective variety over an F-finite k and X has a non-trivial p^{-e}-linear map for $e > 0$, then the canonical bundle ω_X cannot be ample because $\omega_X^{\otimes(1-p^e)}$ has non-trivial global sections. Thus a smooth hypersurface X of $\mathbb{P}^n_k$ defined by a homogeneous polynomial of degree $d > n+1$ has no non-trivial p^{-e}-linear maps because its canonical bundle

$$\omega_X \cong \mathcal{O}_{\mathbb{P}^n_k}(d - n - 1)|_X$$

is ample. In particular, smooth hypersurfaces of large degree in $\mathbb{P}^n_k$ are *never* (globablly) Frobenius split (see [Smi00]).

3.7 F-regularity

An important class of Frobenius split rings are the strongly F-regular rings. Originally, strongly F-regular rings were defined only in the Noetherian F-finite case.

Definition III.67. A Noetherian F-finite ring R of characteristic p is **strongly F-regular** if for every non-zero-divisor c, there exists $e > 0$ such that the map

$$R \to F^e_* R \quad \text{sending } 1 \mapsto c$$

splits in the category of R-modules [HH89].

In this section, we show that by replacing the word "splits" with the words "is pure" in the above definition, we obtain a well-behaved notion of F-regularity in a broader setting. Hochster and Huneke themselves suggested, but never pursued, this possibility in [HH94, Remark 5.3].

Strong F-regularity first arose as a technical tool in the theory of *tight closure*; Hochster and Huneke made use of it in their deep proof of the existence of test elements [HH94]. Indeed, the original motivation for (and the name of) strong F-regularity was born of a desire to better understand **weak F-regularity**, the property of a Noetherian ring that all ideals are tightly closed. In many contexts, strong and weak F-regularity are known to be equivalent (see e.g. [LS99] for the graded case, [HH89] for the Gorenstein case) but it is becoming clear that at least for many applications, strong F-regularity is the more useful and flexible notion. Applications beyond tight closure include commutative algebra more generally [AL, Bli08, ST12, Sch09a, SZ15], algebraic geometry [GLP+15, HX15, Pat08, SS10, Smi00], representation theory [BK05, MR85, Ram91, SVdB97] and combinatorics [BMRS15].

3.7.1 F-pure regularity

We propose the following definition, intended to be a generalization of strong F-regularity to arbitrary commutative rings of characteristic p, not necessarily F-finite or Noetherian.

Definition III.68. [DS16] Let c be an element in a ring R of prime characteristic p. Then R is said to be **F-pure along** c if there exists $e > 0$ such that the R-linear map

$$\lambda_c^e : R \to F_*^e R \text{ sending } 1 \mapsto c$$

is a pure map of R-modules. We say R is **F-pure regular** if it is F-pure along every non-zerodivisor.

A ring R is F-pure if and only if it is F-pure along the element 1. Thus F-pure regularity is a substantial strengthening of F-purity, requiring F-purity along *all* non-zerodivisors (for sufficiently large iterates of Frobenius) instead of just along the unit.

Remarks III.69.

(i) If R is Noetherian and F-finite, then the map $\lambda_c^e : R \to F_*^e R$ is pure if and only if it splits (by Theorem III.37). So F-pure regularity for a Noetherian F-finite ring is the same as strong F-regularity.

(ii) If c is a zerodivisor, then the map λ_c^e is never injective for any $e \geq 1$. In particular, a ring is never F-pure along a zerodivisor.

(iii) The terminology "F-pure along c" is chosen to honor Ramanathan's closely related notion of "Frobenius splitting along a divisor" [Ram91]. See [Smi00].

The following proposition gathers some basic properties of F-pure regularity for arbitrary commutative rings.

Proposition III.70. [DS16] *Let R be a commutative ring of characteristic p, not necessarily Noetherian or F-finite.*

(a) If R is F-pure along some element, then R is F-pure. More generally, if R is F-pure along a product cd, then R is F-pure along the factors c and d.

(b) If R is F-pure along some element, then R is reduced.

(c) If R is an F-pure regular ring with finitely many minimal primes, and $S \subset R$ is a multiplicative set, then $S^{-1}R$ is F-pure regular. In particular, F-pure regularity is preserved under localization in Noetherian rings, as well as in domains.

(d) Let $\varphi : R \to T$ be a pure ring map which maps non-zerodivisors of R to non-zerodivisors of T. If T is F-pure regular, then R is F-pure regular. In particular, if $\varphi : R \to T$ is faithfully flat and T is F-pure regular, then R is F-pure regular.

(e) Let $R_1, \ldots, R_n$ be rings of characteristic p. If $R_1 \times \cdots \times R_n$ is F-pure regular, then each R_i is F-pure regular.

Proof. (a) Multiplication by d is an an R-linear map, so by restriction of scalars also

$$F_*^e R \xrightarrow{\times d} F_*^e R$$

is R-linear. Precomposing with λ_c^e we have

$$R \xrightarrow{\lambda_c^e} F_*^e R \xrightarrow{\times d} F_*^e R \quad \text{sending } 1 \mapsto cd,$$

which is λ_{cd}^e. Our hypothesis that R is F-pure along cd means that there is some e for which this composition is pure. So by Lemma III.29(b), it follows also that λ_c^e is pure. That is, R is F-pure along c (and by symmetry, also along d). The second statement follows since F-purity along the product $c \times 1$ implies R is F-pure along 1. So some iterate of Frobenius is a pure map, and so F-purity follows from Lemma III.29(b).

(b) By (a) we see that R is F-pure. In particular, the Frobenius map is pure and hence injective, so R is reduced.

(c) Note R is reduced by (b). Let $\alpha \in S^{-1}R$ be a non-zerodivisor. Because R has finitely many minimal primes, a standard prime avoidance argument shows that there exists a non-zerodivisor $c \in R$ and $s \in S$ such that $\alpha = c/s$ (a minor modification of [Hoc, Proposition on Pg 57]). By hypothesis, R is F-pure along c. Hence there exists $e > 0$ such that the map $\lambda_c^e : R \to F_*^e R$ is pure. Then the map

$$\lambda_{c/1}^e : S^{-1}R \longrightarrow F_*^e(S^{-1}R) \text{ sending } \quad 1 \mapsto c/1$$

is pure by III.29(e) and the fact that $S^{-1}(F_*^e R) \cong F_*^e(S^{-1}R)$ as $S^{-1}R$-modules (the isomorphism $S^{-1}(F_*^e R) \cong F_*^e(S^{-1}R)$ is given by $r/s \mapsto r/s^{p^e}$). Now the $S^{-1}R$-linear map

$$\ell_{1/s} : S^{-1}R \to S^{-1}R \text{ sending } \quad 1 \mapsto 1/s$$

is an isomorphism. Applying F_*^e, we see that

$$F_*^e(\ell_{1/s}) : F_*^e(S^{-1}R) \to F_*^e(S^{-1}R) \text{ sending } 1 \mapsto 1/s$$

is also an isomorphism of $S^{-1}R$-modules. In particular, $F_*^e(\ell_{1/s})$ is a pure map of $S^{-1}R$-modules. So purity of

$$F_*^e(\ell_{1/s}) \circ \lambda_{c/1}^e$$

follows by III.29(a). But $F_*^e(\ell_{1/s}) \circ \lambda_{c/1}^e$ is precisely the map

$$\lambda_{c/s}^e : S^{-1}R \to F_*^e(S^{-1}R) \text{ sending } 1 \mapsto c/s.$$

(d) Let $c \in R$ be a non-zerodivisor. Then $\varphi(c)$ is a non-zero divisor in T by hypothesis. Pick $e > 0$ such that the map $\lambda^e_{\varphi(c)} : T \to F^e_* T$ is a pure map of T-modules. By III.29(f) and III.29(a),

$$R \xrightarrow{\varphi} T \xrightarrow{\lambda^e_{\varphi(c)}} F^e_* T$$

is a pure map of R-modules. We have commutative diagram of R-linear maps

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\lambda^e_c"] & T \arrow[d, "\lambda^e_{\varphi(c)}"] \\ F^e_* R \arrow[r, "F^e_*(\varphi)"] & F^e_* T \end{tikzcd}$$

The purity of λ^e_c follows by III.29(b). Note that if φ is faithfully flat, then it is pure by III.29(f) and maps non-zerodivisors to non-zerodivisors.

(e) Let $R := R_1 \times \cdots \times R_n$. Consider the multiplicative set

$$S := R_1 \times \cdots \times R_{i-1} \times \{1\} \times R_{i+1} \times \cdots \times R_n.$$

Since $S^{-1}R \cong R_i$, it suffices to show that $S^{-1}R$ is F-pure regular. So let $\alpha \in S^{-1}R$ be a non-zerodivisor. Note that we can select $u \in R$ and $s \in S$ such that u is a non-zerodivisor and $\alpha = u/s$. So we can now repeat the proof of (c) verbatim to see that $S^{-1}R$ must be pure along α. $\square$

Remark III.71. It is worth observing in Definition III.68, that if the map λ^e_c is a pure map, then λ^f_c is also a pure map for all $f \geq e$. Indeed, to see this note that it suffices to show that λ^{e+1}_c is pure. We know R is F-pure by III.70(a). So Frobenius

$$F : R \to F_* R$$

is a pure map of R-modules. By hypothesis,

$$\lambda^e_c : R \to F^e_* R$$

is pure. Hence III.29(d) tell us that

$$F_*(\lambda_c^e) : F_*R \to F_*(F_*^e R)$$

is a pure map of R-modules. Hence the composition

$$R \xrightarrow{F} F_*R \xrightarrow{F_*(\lambda_c^e)} F_*(F_*^e R) \ \text{ sending } \ 1 \mapsto c$$

is a pure map of R-modules by III.29(a). But $F_*(F_*^e R)$ as an R-module is precisely $F_*^{e+1}R$. So

$$\lambda_c^{e+1} : R \to F_*^{e+1}R.$$

is pure.

Example III.72. The polynomial ring over $\mathbb{F}_p$ in infinitely many variables (localized at the obvious maximal ideal) is an example of a F-pure ring which is not Noetherian.

3.7.2 Relationship of F-pure regularity to other singularities

We show that our generalization of strong F-regularity continues to enjoy many important properties of the more restricted version.

Theorem III.73. (C.f. [HH94, Theorem 3.1(c)]) *A regular local ring, not necessarily F-finite, is F-pure regular.*

Proof. Let (R, m) be a regular local ring. By Krull's intersection theorem we know that

$$\bigcap_{e>0} m^{[p^e]} = 0.$$

Since R is a domain, the non-zerodivisors are precisely the non-zero elements of R. So let $c \in R$ be a non-zero element. Choose e such that $c \notin m^{[p^e]}$. We show that the map

$$\lambda_c^e : R \to F_*^e R; \ \ 1 \mapsto c$$

is pure.

By Lemma III.29, it suffices to check that for the injective hull E of the residue field of R, the induced map

$$\lambda_c^e \otimes id_E : R \otimes_R E \to F_*^e R \otimes_R E$$

is injective, and for this, in turn, we need only check that the socle generator is not in the kernel.

Recall that E is the direct limit of the injective maps

$$R/(x_1, \ldots, x_n) \xrightarrow{x} R/(x_1^2, \ldots, x_n^2) \xrightarrow{x} R/(x_1^3, \ldots, x_n^3) \xrightarrow{x} R/(x_1^4, \ldots, x_n^4) \longrightarrow \cdots$$

where $x_1, \ldots, x_n$ is a minimal set of generators for m, and the maps are given by multiplication by $x = \Pi_{i=1}^d x_i$ [HK71]. So the module $F_*^e R \otimes_R E$ is the direct limit of the maps

$$R/(x_1^{p^e}, \ldots, x_n^{p^e}) \xrightarrow{x^{p^e}} R/(x_1^{2p^e}, \ldots, x_n^{2p^e}) \xrightarrow{x^{p^e}} R/(x_1^{3p^e}, \ldots, x_n^{3p^e}) \xrightarrow{x^{p^e}} \cdots$$

which remains injective by the faithful flatness of $F_*^e R$. The induced map $\lambda_c^e \otimes id_E$: $E \to F_*^e R \otimes E$ sends the socle (namely the image of 1 in R/m) to the class of c in $R/m^{[p^e]}$, so it is non-zero provided $c \notin m^{[p^e]}$. Thus for every non-zero c in a regular local (Noetherian) ring, we have found an e, such that the map λ_c^e is pure. So regular local rings are F-pure regular. $\square$

Proposition III.74. [DS16] *An F-pure regular ring is normal, that is, it is integrally closed in its total quotient ring.*

Proof. Take a fraction r/s in the total quotient ring integral over R. On clearing denominators in an equation of integral dependence, we have $r \in \overline{(s)}$, the integral closure of the ideal (s). This implies that there exists an h such that $(r, s)^{n+h} =$

$(s)^n(r,s)^h$ for all n [Mat89, page 64]. Setting $c = s^h$, this implies $cr^n \in (s)^n$ for all large n. In particular, taking $n = p^e$, we see that class of r modulo (s) is in the kernel of the map induced by tensoring the map

$$R \to F^e_* R \quad \text{sending } 1 \mapsto c \tag{3.2}$$

with the quotient module $R/(s)$. By purity of the map (3.2), it follows that $r \in (s)$. We conclude that r/s is in R and that R is normal. □

3.7.3 Connections with Tight Closure

In his lecture notes on tight closure [Hoc07], Hochster suggests another way to generalize strong F-regularity to non-F-finite (but Noetherian) rings using tight closure. We show here that his generalized strong F-regularity is the same as F-pure regularity for local Noetherian rings.

Although Hochster and Huneke introduced tight closure only for Noetherian rings, we can make the same definition in general for an arbitrary ring of prime characteristic p. Let $N \hookrightarrow M$ be R-modules. The **tight closure** of N in M is an R-module N^*_M containing N. By definition, an element $x \in M$ is in N^*_M if there exists $c \in R$, not in any minimal prime, such that for all sufficiently large e, the element $c \otimes x \in F^e_* R \otimes_R M$ belongs to the image of the module $F^e_* R \otimes_R N$ under the natural map $F^e_* R \otimes_R N \to F^e_* R \otimes_R M$ induced by tensoring the inclusion $N \hookrightarrow M$ with the R-module F^e_R. We say that N is tightly closed in M if $N^*_M = N$.

Definition III.75. Let R be a Noetherian ring of characteristic p. We say that R is **strongly F-regular in the sense of Hochster** if, for any pair of R-modules $N \hookrightarrow M$, $N^*_M = N$.

The next result compares F-pure regularity with strong F-regularity in the sense of Hochster:

Proposition III.76. [DS16] *Let R be an arbitrary commutative ring of prime characteristic. If R is F-pure regular, then N is tightly closed in M for any pair of R modules $N \subset M$. The converse also holds if R is Noetherian and local.*

Proof. Suppose $x \in N^*_M$. Equivalently the class $\overline{x}$ of x in M/N is in $0^*_{M/N}$. So there exists c not in any minimal prime such that $c \otimes \overline{x} = 0$ in $F^e_* R \otimes_R M/N$ for all large e. But this means that the map

$$R \to F^e_* R \quad \text{sending} \quad 1 \mapsto c$$

is not pure for any e, since the naturally induced map

$$R \otimes M/N \to F^e_* R \otimes M/N$$

has $1 \otimes \overline{x}$ in its kernel.

For the converse, let $c \in R$ be not in any minimal prime. We need to show that there exists some e such that the map $R \to F^e_* R$ sending 1 to c is pure. Let E be the injective hull of the residue field of R. According to Lemma III.29(i), it suffices to show that there exists an e such that after tensoring E, the induced map

$$R \otimes E \to F^e_* R \otimes E$$

is injective. But if not, then a generator η for the socle of E is in the kernel for every e, that is, for all e, $c \otimes \eta = 0$ in $F^e_* R \otimes E$. In this case, $\eta \in 0^*_E$, contrary to our hypothesis that all modules are tightly closed. □

Remarks III.77.

1. We do not know whether Proposition III.76 holds in the non-local case. Indeed, we do not know if F-pure regularity is a local property: if $R_{\mathfrak{m}}$ is F-pure regular

for all maximal ideals $\mathfrak{m}$ of R, does it follow that R is F-pure regular? If this were the case, then our argument above extends to arbitrary Noetherian rings.

2. A Noetherian ring of characteristic p is **weakly F-regular** if N is tightly closed in M for any pair of *Noetherian* R modules $N \subset M$. Clearly F-pure regular implies weakly F-regular. The converse is a long standing open question in the F-finite Noetherian case. For valuation rings, however, we will show that weak and F-pure regularity are equivalent (and both are equivalent to the valuation ring being Noetherian); see Corollary IV.53.

3.7.4 Elements along which F-purity fails

We now observe an analog of the splitting prime of Aberbach and Enescu [AE05]; See also [Tuc12, 4.7].

Proposition III.78. [DS16] *Let R be a ring of characteristic p, and consider the set*

$$\mathcal{I} := \{c \in R : \text{ } R \textit{ is not F-pure along } c\}.$$

Then $\mathcal{I}$ is closed under multiplication by R, and $R - \mathcal{I}$ is multiplicatively closed. In particular, if $\mathcal{I}$ is closed under addition, then $\mathcal{I}$ is a prime ideal (or the whole ring).

Proof. We first note that $\mathcal{I}$ is closed under multiplication by elements of R. Indeed, suppose that $c \in \mathcal{I}$ and $r \in R$. Then if $rc \notin \mathcal{I}$, we have that R is F-pure along rc, but this implies R is F-pure along c by Proposition III.70(a), contrary to $c \in \mathcal{I}$.

We next show that the complement $R \setminus \mathcal{I}$ is a multiplicatively closed set (if non-empty). To wit, take $c, d \notin \mathcal{I}$. Because R is F-pure along both c and d, we have that there exist e and f such such the maps

$$R \xrightarrow{\lambda_c^e} F_*^e R \text{ sending } 1 \mapsto c, \quad \text{and } R \xrightarrow{\lambda_d^f} F_*^f R \text{ sending } 1 \mapsto d$$

are both pure. Since purity is preserved by restriction of scalars (Lemma III.29(d)), we also have that

$$F^e_* R \xrightarrow{F^e_*(\lambda^f_d)} F^e_* F^f_* R = F^{e+f}_* R$$

is pure. Hence the composition

$$R \xrightarrow{\lambda^e_c} F^e_* R \xrightarrow{\lambda^f_d} F^e_* F^f_* R \text{ sending } 1 \mapsto c^{p^e} d$$

is pure as well (Lemma III.29(a)). This means that $c^{p^e} d$ is not in $\mathcal{I}$, and since $\mathcal{I}$ is closed under multiplication, neither is cd. Note also that if $R \setminus \mathcal{I}$ is non-empty, then $1 \in R \setminus \mathcal{I}$ by Proposition III.70(a). Thus $R \setminus \mathcal{I}$ is a multiplicative set.

Finally, if $\mathcal{I}$ is closed under addition (and $\mathcal{I} \neq R$), we conclude that $\mathcal{I}$ is a prime ideal since it is an ideal whose complement is a multiplicative set. □

Remarks III.79.

1. If R is a Noetherian local ring, then the set $\mathcal{I}$ of Proposition III.78 can be checked to be closed under addition. Indeed, suppose $c_1, c_2 \in \mathcal{I}$. Then for any $e > 0$, the maps

   $$\lambda^e_{c_i} : R \to F^e_* R \text{ sending } 1 \mapsto c_i$$

   are not pure for $i = 1, 2$. In particular, if E is the injective hull of the residue field of R, then

   $$\lambda^e_{c_i} \otimes_R \mathrm{id}_E : E \to F^e_* R \otimes_R E$$

   is not injective for $i = 1, 2$. However, any two non-zero submodules of E have a non-empty intersection since the submodules must contain the residue field κ_R of R. This shows that

   $$\lambda^e_{c_1+c_2} \otimes_R \mathrm{id}_E = (\lambda^e_{c_1} + \lambda^e_{c_2}) \otimes_R \mathrm{id}_E = (\lambda^e_{c_1} \otimes_R \mathrm{id}_E) + (\lambda^e_{c_2} \otimes_R \mathrm{id}_E)$$

is not injective since the kernel of this map contains $\ker(\lambda^e_{c_1} \otimes_R \mathrm{id}_E) \cap \ker(\lambda^e_{c_2} \otimes_R \mathrm{id}_E)$. Therefore $\lambda^e_{c_1+c_2}$ is not pure for any $e > 0$, that is, $c_1 + c_2 \in \mathcal{I}$. Thus when R is a Noetherian local ring, the set $\mathcal{I}$ of elements along which R is not F-pure is a prime ideal (if R is F-pure) or the whole ring.

2. Likewise, we will see in the next chapter that for valuation rings, the set $\mathcal{I}$ is also an ideal (Theorem IV.50). However, for an arbitrary ring, $\mathcal{I}$ can fail to be an ideal. For example, under suitable hypothesis, the set $\mathcal{I}$ is also the union of the centers of F-purity in the sense of Schwede. Hence, in this case $\mathcal{I}$ is a finite union of ideals but not necessarily an ideal in the non-local case; see [Sch10].

CHAPTER IV

F-singularities of valuation rings

In this chapter we study valuation rings through the lens of F-singularity theory introduced in Chapter III. Thus, we work with valuation rings of prime characteristic p, unless specified otherwise, and frequently switch between the language of valuations and valuation rings. Many of the results in this chapter were obtained in joint work with Karen Smith [DS16, DS17a]. Chapter II contains a fairly detailed account of the necessary background from valuation theory.

Before embarking on a discussion of F-singularity theory in the setting of valuation rings, we make some preliminary observations.

Let ν be a valuation of a field K of characteristic p. We denote the restriction of ν to the subfield K^p by ν^p. The following properties of the extension ν/ν^p are straightforward to verify.

Lemma IV.1. *The extension of valuations ν/ν^p satisfies the following properties:*

1. *The valuation ring R_{ν^p} of ν^p is the subring R_ν^p of R_ν.*
2. *ν is the unique extension (up to equivalence) of ν^p to K.*
3. *R_ν is the integral closure of R_{ν^p} in K.*
4. *$\mathfrak{m}_{\nu^p} R_\nu = \mathfrak{m}_\nu^{[p]}$.*

5. *The map of residue fields $\kappa_{\nu^p} \hookrightarrow \kappa_\nu$ maps κ_{ν^p} isomorphically onto κ_ν^p. Thus the residue degree $f(\nu/\nu^p)$ equals $[\kappa_\nu : \kappa_\nu^p]$.*

6. *If Γ_ν is the value group of ν, then the value group of ν^p is $p\Gamma_\nu$. Thus the ramification index $e(\nu/\nu^p)$ of the extension ν/ν^p equals $[\Gamma_\nu : p\Gamma_\nu]$.*

Proof. Property (1) follows from the observation that

$$R_{\nu^p} = K^p \cap R_\nu.$$

For (2), let w be a valuation of K that extends ν^p. Then using (1),

$$R_\nu^p = R_{\nu^p} = K^p \cap R_w. \tag{4.1}$$

Since R_w is integrally closed in K, it is in particular closed under taking p-th roots. Hence (4.1) implies $R_\nu = R_w$, which is another way of saying that ν and w are equivalent. The remaining properties follow from (1), and we leave their verification to the reader. □

4.1 Flatness of Frobenius

The starting point of my joint work with Karen Smith on the use of F-singularity techniques in valuation theory was the observation that, like for regular local rings, Frobenius is always flat for a valuation ring of prime characteristic.

Theorem IV.2. [DS16] *Let V be a valuation ring of characteristic p. Then the Frobenius map $F : V \to F_*V$ is a flat map. Hence V is always F-pure.*

Proof. Clearly F_*V is a torsion-free V-module, hence flat since torsion-free modules over valuation rings are flat (Corollary II.15). Since the Frobenius map induces the identity map on Spec, it follows that F_*V is a faithfully flat V-module. Thus F is pure (that is, V is F-pure) by Lemma III.29(f). □

Remarks IV.3.

1. The above theorem shows that all valuation rings of prime characteristic are 'close' to being Frobenius split, since F-purity satisfies many of the good properties of Frobenius splitting.

2. In the world of Noetherian rings, flatness of Frobenius characterizes regularity (Theorem III.8). Thus one may think of valuation rings, at least in prime characteristic, as non-Noetherian analogues of regular local rings.

3. The V-algebra F_*V is a filtered colimit of its finitely generated V-subalgebras (with V itself being the minimal such subalgebra with respect to inclusion). Any such subalgebra B is a free V-module of finite rank (Proposition II.14), hence has a V-basis containing the element 1. In particular, the ring homomorphism $V \to B$ then splits. Thus $F : V \to F_*V$ is a filtered direct limit of split ring homomorphisms. This provides another proof of the F-purity of V using Lemma III.29(g).

4. There is no reason for the Frobenius map of a valuation ring to be split, even though this map is a filtered direct limit of split maps. Indeed, Example III.57 shows that Frobenius splitting of valuation rings can fail even for generically F-finite discrete valuation rings.

4.2 F-finite valuation rings

A necessary condition for a domain of prime characteristic to be F-finite is for its fraction field to also be F-finite. Hence in our investigation of F-finiteness of valuation rings, we assume that rings are generically F-finite to begin with. Note that if ν is a valuation of an F-finite field K, the residue field of ν is also F-finite.

This follows from the inequality (see Corollary II.47)

$$e(\nu/\nu^p)f(\nu/\nu^p) \leq [K : K^p],$$

because the residue degree $f(\nu/\nu^p)$ coincides with the degree of the field extension κ_ν/κ_ν^p according to Lemma IV.1.

4.2.1 The basics

The basic result on F-finiteness of valuation rings is the following.

Proposition IV.4. [DS16] *Let K be an F-finite field. A valuation ring V of K is F-finite if and only if F_*V is a free V-module.*

Proof. First assume F_*V is free over V. Since $K \otimes_R F_*V \cong F_*K$ as K-vector spaces, the rank of F_*V over V must be the same as the rank of F_*K over K, namely the degree $[F_*K : K] = [K : K^p]$. Since K is F-finite, this degree is finite, and so F_*V is a free V-module of finite rank. In particular, V is F-finite.

Conversely, suppose that V is F-finite. Then F_*V is a finitely generated, torsion-free V-module. Hence it is free by Proposition II.14. □

Corollary IV.5. [DS16] *An F-finite valuation ring is Frobenius split.*

Proof. One of the rank one free summands of F_*V is the copy of V under F, so this copy of V splits off F_*V. Alternatively, since $V \to F_*V$ is pure, we can use Theorem III.37: the cokernel of $V \to F_*V$ is finitely presented because it is finitely generated (being a quotient of the finitely generated V-module F_*V) and the module of relations is finitely generated (by $1 \in F_*V$). □

Remark IV.6. Since a valuation ring of prime characteristic is always F-pure, Theorem III.37 implies that any valuation ring whose Frobenius endomorphism has a finitely presented cokernel is also Frobenius split. However, since the Frobenius map of a valuation ring is injective, its cokernel is finitely presented if and only if the valuation ring F-finite. Thus Theorem III.37 gives no extra information about Frobenius splitting of valuation rings over Corollary IV.5.

An argument similar to that of Corollary IV.5 can be used to establish a stronger characteristic independent result. To state this result, we introduce the notion of a splinter.

Definition IV.7. [Bha12, Definition 1.2] A ring R (of arbitrary characteristic) is a **splinter** if given a finite ring homomorphism $\varphi : R \to S$ such that the induced map $\mathrm{Spec}(\varphi)$ is surjective, φ admits a left inverse in Mod_R.

Remark IV.8. If R is reduced (in particular, a domain), then R is a splinter if and only if any finite ring extension $R \hookrightarrow S$ splits in Mod_R. This is because a ring homomorphism from a reduced ring induces a surjective map on Spec precisely when the homomorphism is injective.

Hochster's famous direct summand conjecture, now a theorem by work of [Hoc73, And16, Bha16] (see also [HM17]), may be rephrased as saying that all regular rings are splinters. It turns out that like regular local rings, valuation rings in all characteristics are also splinters, providing further evidence that valuation rings behave like regular rings.

Proposition IV.9. *A valuation ring of arbitrary characteristic (including mixed) is a splinter.*

Proof. Let V be a valuation ring (of any characteristic). Suppose $\varphi : V \to S$ is a ring homomorphism such that $\mathrm{Spec}(\varphi)$ is surjective. Choose a prime ideal $\mathfrak{p}$ of S

which contracts to the zero ideal of V. Then the induced homomorphism

$$V \xrightarrow{\varphi} S \twoheadrightarrow S/\mathfrak{p}$$

is also finite and injective. Moreover, $S/\mathfrak{p}$ is a torsion-free V-module. Again using Proposition II.14 one may then conclude that $S/\mathfrak{p}$ is a free V-module, and Nakayama's lemma shows that there exists a free V-basis of $S/\mathfrak{p}$ containing the element $1 \in S/\mathfrak{p}$. Therefore there exists a V-linear map

$$\tau : S/\mathfrak{p} \to V$$

that maps $1 \mapsto 1$. The composition $S \twoheadrightarrow S/\mathfrak{p} \xrightarrow{\tau} V$ now gives a splitting of φ. □

4.2.2 A numerical criterion and consequences

Proposition IV.10 (Numerical criterion for F-finiteness). *Suppose ν is a valuation of an F-finite field K of characteristic p with valuation ring $(R_\nu, \mathfrak{m}_\nu, \kappa_\nu)$. Then the following are equivalent:*

1. *R_ν is F-finite.*
2. *$\dim_{\kappa_\nu^p} R_\nu/\mathfrak{m}_\nu^{[p]} = [K : K^p]$*

If these equivalent conditions hold, then $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p]$.

Proof. Consider the extension of fields $K^p \subset K$. Lemma IV.1 shows that R_ν is the integral closure of $R_{\nu^p} = R_\nu^p$ in K, ν is the *unique extension of* ν^p to K up to equivalence, κ_ν^p is isomorphic to the residue field κ_{ν^p}, and the expansion of the maximal ideal of R_{ν^p} in R_ν is $\mathfrak{m}_\nu^{[p]}$. Moreover, the ramification index of the extension ν/ν^p equals $[\Gamma_\nu : p\Gamma_\nu]$ and the residue degree of ν/ν^p equals $[\kappa_\nu : \kappa_\nu^p]$. Thus the present proposition follows upon applying Theorem II.50 to the field extension K/K^p and the (unique) extension of valuations ν/ν^p. □

Definition IV.11. [Kuh11, Page 281] Let $K \subset L$ be a finite field extension of fields of characteristic p. Suppose ν is a valuation of K that admits a unique extension w (up to equivalence) to L (for example, ν could be Henselian). Then the **defect of** w/ν is the integer $\delta \in \mathbb{N} \cup \{0\}$ such that

$$p^\delta e(w/\nu) f(w/\nu) = [L : K].$$

The extension w/ν is **defectless** if

$$e(w/\nu) f(w/\nu) = [L : K],$$

that is, if $\delta = 0$, and w/ν has **maximal defect** if $e(w/\nu) = f(w/\nu) = 1$, that is, if w/ν is totally unramified.

Remarks IV.12.

1. In the language of defect of unique extensions of valuations, Proposition IV.10 implies that if a valuation ν has an F-finite valuation ring, then the extension of valuations ν/ν^p is defectless. The converse is false in general – any non-discrete Abhyankar valuation of an F-finite function field gives a counter-example (this will be established in Corollaries IV.23 and IV.25). Nevertheless, the converse does hold for discrete valuation rings as we will see soon (Corollary IV.14).

2. The non-trivial implication in the equivalence of Proposition IV.10 is (2) $\Rightarrow$ (1). Here we provide a direct proof of this fact, independent of the proof in [Bou89] of Theorem II.50. Suppose

$$\dim_{\kappa^p}(R_\nu/\mathfrak{m}_\nu^{[p]}) = [K : K^p] = n.$$

Choose $x_1, \ldots, x_n \in R_\nu$ such that the images of x_i in $R_\nu/\mathfrak{m}_\nu^{[p]}$ form a κ_ν^p-basis, and let

$$L \coloneqq R_\nu^p x_1 + \cdots + R_\nu^p x_n.$$

The module L is a finitely generated, torsion free R_ν^p-module, hence free over R_ν^p since finitely generated torsion-free modules over valuation rings are free. To prove (1), it suffices to show that

$$L = R_\nu.$$

The rank of the free R_ν^p-module L equals $\dim_{\kappa_\nu^p} L/\mathfrak{m}_{\nu^p}L$, and it is easy to see that the images of $x_1, \ldots, x_n$ in $L/\mathfrak{m}_{\nu^p}L$ form a κ_ν^p-basis of $L/\mathfrak{m}_{\nu^p}L$. Thus, L is a free R_ν^p-module of rank n with basis $\{x_1, \ldots, x_n\}$.

Observe that the K^p-linearly independent set $\{x_1, \ldots, x_n\}$ is also a K^p-basis of K. Let $s \in R_\nu$ be a non-zero element, and $r_1, \ldots, r_n \in K^p$ such that

$$s = r_1x_1 + \cdots + r_nx_n.$$

Clearly $L = R_\nu$ if we can show that all the r_i are elements of R_ν^p. By renumbering the x_i, we may assume without loss of generality (because R_ν^p is a valuation ring) that $r_1 \neq 0$ and

$$r_ir_1^{-1} \in R_\nu^p,$$

for all $i \geq 2$. If $r_1 \in R_\nu^p$, then the r_i are already in V^p. If not, r_1^{-1} is an element of the maximal ideal of R_ν^p, and then the equation

$$r_1^{-1}s = x_1 + r_2r_1^{-1}x_2 + \cdots + r_nr_1^{-1}x_n,$$

contradicts κ^p-linear independence of the images of $x_1, \ldots, x_n$ in $R_\nu/\mathfrak{m}_\nu^{[p]}$. Hence all the r_i are elements of R_ν^p, showing that $s \in L$. □

The previous proposition demonstrates that the dimension of the κ_ν^p-vector space $R_\nu/\mathfrak{m}_\nu^{[p]}$ reflects F-finiteness of R_ν. A closer analysis of $\dim_{\kappa_\nu^p} R_\nu/\mathfrak{m}_\nu^{[p]}$ reveals the following:

Lemma IV.13. [DS17a] *Let $(V, \mathfrak{m}_V, \kappa_V)$ be valuation ring of characteristic p. Then the dimension of $V/\mathfrak{m}_V^{[p]}$ over κ_V^p equals*

(a) $[\kappa_V : \kappa_V^p]$ if $\mathfrak{m}_V$ is not finitely generated.

(b) $p[\kappa_V : \kappa_V^p]$ if $\mathfrak{m}_V$ is finitely generated.

Proof. Consider the short exact sequence of κ_V^p-vector spaces

$$0 \to \mathfrak{m}_V/\mathfrak{m}_V^{[p]} \to V/\mathfrak{m}_V^{[p]} \to \kappa_V \to 0. \tag{4.2}$$

If $\mathfrak{m}_V$ is not finitely generated, then Proposition II.19 and Lemma II.26 imply that

$$\mathfrak{m}_V/\mathfrak{m}_V^{[p]} = \mathfrak{m}_V/\mathfrak{m}_V^p = 0,$$

and (a) follows. Otherwise, $\mathfrak{m}_V$ is principal, and we have a filtration

$$\mathfrak{m}_V \supsetneq \mathfrak{m}_V^2 \supsetneq \cdots \supsetneq \mathfrak{m}_V^{p-1} \supsetneq \mathfrak{m}_V^{[p]} = \mathfrak{m}_V^p.$$

Since $\mathfrak{m}_V^i/\mathfrak{m}_V^{i+1} \cong \kappa_V$, we see that

$$\dim_{\kappa_V^p}(\mathfrak{m}_V/\mathfrak{m}_V^{[p]}) = (p-1)[\kappa_V : \kappa_V^p].$$

From the short exact sequence (4.2), $\dim_{\kappa_V^p}(V/\mathfrak{m}_V^{[p]}) = p[\kappa_V : \kappa_V^p]$, proving (b). □

Corollary IV.14. *Let ν be a discrete valuation of an F-finite field K of characteristic p (i.e. Γ_ν is order isomorphic to $\mathbb{Z}$). Then R_ν is F-finite if and only if ν/ν^p is defectless.*

Proof. The 'only if' assertion follows readily from Proposition IV.10 and the definition of defect. Coversely, suppose ν/ν^p is defectless. Since the maximal ideal of R_ν is finitely generated, Lemma IV.13 shows that

$$\dim_{\kappa_\nu^p} R_\nu/\mathfrak{m}^{[p]} = p[\kappa_\nu : \kappa_\nu^p] = [\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p].$$

Here the final equality follows by our assumption that ν/ν^p is defectless. Proposition IV.10 can again be used to conclude that R_ν is F-finite. □

Theorem IV.15. [Dat17a] *Let ν be a valuation of an F-finite field K of characteristic p. If the valuation ring R_ν is F-finite, then the following hold:*

1. $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p]$.

2. Γ_ν *is p-divisible or* $[\Gamma_\nu : p\Gamma_\nu] = p$.

3. *If Γ_ν is finitely generated and non-trivial, then ν is a discrete valuation.*

As a partial converse, if $[\kappa_\nu : \kappa_\nu^p] = [K : K^p]$, then R_ν is F-finite.

Proof. Let us first prove the three properties assuming R_ν is F-finite. (1) was already mentioned in Proposition IV.10, and the same proposition also implies that

$$\dim_{\kappa_\nu^p} R_\nu/\mathfrak{m}_\nu^{[p]} = [K : K^p] = [\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p].$$

Lemma IV.13 shows that $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [\kappa_\nu : \kappa_\nu^p]$ or $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = p[\kappa_\nu : \kappa_\nu^p]$ depending on whether $\mathfrak{m}_\nu$ is finitely generated. Thus,

$$[\Gamma_\nu : p\Gamma] = 1 \text{ or } [\Gamma_\nu : p\Gamma_\nu] = p,$$

proving (2).

For (3) note that a non-trivial finitely generated ordered abelian group is free, hence never p-divisible. Then (2) shows that $[\Gamma_\nu : p\Gamma_\nu] = p$, and if

$$\Gamma_\nu \cong \mathbb{Z}^{\oplus s},$$

we get $p^s = p$, that is, $s = 1$. This implies Γ_ν is order isomorphic to $\mathbb{Z}$, as desired.

In order to prove the second assertion of the theorem, if $f(\nu/\nu^p) = [\kappa_\nu : \kappa_\nu^p] = [K : K^p]$, then Corollary II.47 implies that

$$[\Gamma_\nu : p\Gamma_\nu] = e(\nu/\nu^p) = 1.$$

In other words, the value group is p-divisible, and so, the group does not possess a smallest element > 0. This means that the maximal ideal $\mathfrak{m}_\nu$ is not finitely generated (Proposition II.19). By Lemma IV.13, we then have

$$\dim_{\kappa_\nu^p} R_\nu/\mathfrak{m}_\nu^{[p]} = [\kappa_\nu : \kappa_\nu^p] = [K : K^p].$$

Therefore R_ν is F-finite using Proposition IV.10. $\square$

Examples IV.16.

1. The perfection $\mathbb{F}_p[[t^{1/p^\infty}]] := \bigcup_{e\in\mathbb{N}} \mathbb{F}_p[[t^{1/p^e}]]$ of the power series ring $\mathbb{F}_p[[t]]$ is a non-Noetherian, F-finite valuation ring of its fraction field $\mathbb{F}_p((t^{1/p^\infty}))$. More generally, a non-trivial valuation ring of any perfect field of prime characteristic is not Noetherian, but F-finite because Frobenius is an isomorphism for such a ring. Rings of prime characteristic for which Frobenius is an isomorphism are called *perfect rings*. Such rings have been extensively investigated of late since finding applications in Scholze's work on perfectoid spaces [Sch12].

2. While perfect rings are trivially F-finite, there exist non-Noetherian, F-finite valuation rings that are *not* perfect. Suppose L is a perfect field of prime characteristic equipped with a non-trivial valuation ν with value group Γ_ν. For instance L can be a perfectoid field, or the algebraic closure of a field which has non-trivial valuations. Then the residue field κ_ν of the associated valuation ring is also perfect. Now consider the group

$$\Gamma' := \Gamma_\nu \oplus \mathbb{Z}$$

ordered lexicographically, and the field $L(X)$, where X is an indeterminate. There exists a *unique* extension w of the valuation ν to $L(X)$ with value group

Γ' such that for any polynomial $f = \sum_{i=0}^{n} a_i X^i$ in $L[X]$, we have

$$w(f) = \inf\{(\nu(a_i), i) : i = 0, \ldots, n\}.$$

The residue field κ_w of w equals the residue field κ_ν (see Proposition II.51), hence is also perfect. Also, Γ' has a smallest element > 0 in the lexicographical order, namely $(0, 1)$. Thus, if $(R_w, \mathfrak{m}_w)$ is the valuation ring of w, the maximal ideal $\mathfrak{m}_w$ is principal, and in fact generated by X. Using Proposition IV.10 we see that

$$\dim_{\kappa_w^p}(R_w/\mathfrak{m}_w^{[p]}) = p[\kappa_w : \kappa_w^p] = p = [L(X) : L(X)^p]. \tag{4.3}$$

Then R_w is F-finite by Proposition IV.10, not Noetherian because $\Gamma' = \Gamma_\nu \oplus \mathbb{Z}$ has rational rank at least 2, and not perfect because the field $L(X)$ is not perfect.

Curiously, if instead of taking $\Gamma' = \Gamma_\nu \oplus \mathbb{Z}$ ordered lexicographically we take $\Gamma' = \mathbb{Z} \oplus \Gamma_\nu$ ordered lexicographically in the above construction, the resulting extension w of ν to $L(X)$ (with obvious modifications to the definition of w) does not have an F-finite valuation ring R_w. Indeed, then the maximal ideal of R_w is not finitely generated, while the residue field κ_w still coincides with κ_ν, which is perfect. Thus $\dim_{\kappa_w^p}(R_w/\mathfrak{m}_w^{[p]}) = [\kappa_w : \kappa_w^p] = 1 \neq [L(X) : L(X)^p]$.

3. We will later see that if K is a function field over an F-finite ground field k, then the only F-finite valuation rings of K/k are those associated to divisorial valuations (Corollary IV.25).

4.2.3 Behavior under finite extensions

We have observed that the property of a valuation of a function field being Abhyankar is preserved under finite field extensions (Proposition II.68). The goal of this subsection is to prove an analogous result for F-finiteness.

Proposition IV.17. [DS16, DS17a] *Let $K \hookrightarrow L$ be a finite extension of F-finite fields of characteristic p. Let ν be a valuation of K and w be an extension of ν to L. Then:*

(i) The ramification indices $e(\nu/\nu^p)$ and $e(w/w^p)$ are equal.

(ii) The residue degrees $f(\nu/\nu^p)$ and $f(w/w^p)$ are equal.

(iii) R_ν is F-finite if and only if R_w is F-finite.

For the proof of this proposition, we will need the following lemma about behavior of maximal ideals of valuation rings under finite extensions.

Lemma IV.18. [DS17a] *With the hypothesis of Proposition IV.17, the maximal ideal of the valuation ring of ν is finitely generated if and only if the maximal ideal of the valuation ring of w is finitely generated.*

Proof of Lemma IV.18. For ideals of a valuation ring, finite generation is the same as being principal, and principality of the maximal ideal is equivalent to the value group having a smallest element > 0 (Proposition II.19). Thus, it suffices to show that the value group Γ_ν of ν has this property if and only if Γ_w does.

Assume Γ_w has a smallest element $g > 0$. We claim that for each $t \in \mathbb{N}$, the only positive elements of Γ_w less than tg are $g, 2g, \ldots, (t-1)g$. Indeed, suppose $0 < h < tg$. Since g is smallest, $g \leq h < tg$, whence $0 \leq h - g < (t-1)g$. So by induction, $h - g = ig$ for some $i \in \{0, 1, \ldots, t-2\}$, and hence h is among $g, 2g, \ldots, (t-1)g$.

Now, because $e(w/\nu) = [\Gamma_w : \Gamma_\nu] \leq [L : K] < \infty$ by Corollary II.47, every element of Γ_w/Γ_ν is torsion. Let n be the smallest positive integer such that $ng \in \Gamma_\nu$. We claim that ng is the smallest positive element of Γ_ν. Indeed, the only positive elements smaller than ng in Γ_w are $g, 2g, \ldots, (n-1)g$, and none of these are in Γ_ν by our choice of n.

Conversely, if Γ_ν has a smallest element $h > 0$, then the set

$$S := \{g \in \Gamma_w : 0 < g < h\}$$

is finite because for distinct g_1, g_2 in this set, their classes in Γ_w/Γ_ν are also distinct, while Γ_w/Γ_ν is a finite group. Then the smallest positive element of Γ_w is the smallest element of S, or h if S is empty. $\square$

Proof of Proposition IV.17. By Corollary II.47, we have

$$e(w/\nu)f(w/\nu) = [\Gamma_w : \Gamma_\nu][\kappa_w : \kappa_\nu] \leq [L : K],$$

so both $e(w/\nu)$ and $f(w/\nu)$ are finite. Of course, we also know that the ramification indices $e(w/w^p) = [\Gamma_w : p\Gamma_w]$ and $e(\nu/\nu^p) = [\Gamma_\nu : p\Gamma_\nu]$ are finite, as are the residue degrees $f(w/w^p) = [\kappa_w : \kappa_w^p]$ and $f(\nu/\nu^p) = [\kappa_\nu : \kappa_\nu^p]$.

(i) We need to show that $[\Gamma_w : p\Gamma_w] = [\Gamma_\nu : p\Gamma_\nu]$. Since Γ_w is torsion-free, multiplication by p induces an isomorphism $\Gamma_w \cong p\Gamma_w$, under which the subgroup Γ_ν maps to $p\Gamma_\nu$. Thus $[p\Gamma_w : p\Gamma_\nu] = [\Gamma_w : \Gamma_\nu]$. Using the commutative diagram of finite index abelian subgroups

$$\begin{tikzcd} \Gamma_w & \Gamma_\nu \arrow[l, hook'] \\ p\Gamma_w \arrow[u, hook] & p\Gamma_\nu, \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

we see that $[\Gamma_w : p\Gamma_w][p\Gamma_w : p\Gamma_\nu] = [\Gamma_w : \Gamma_\nu][\Gamma_\nu : p\Gamma_\nu]$. Whence $[\Gamma_w : p\Gamma_w] = [\Gamma_\nu : p\Gamma_\nu]$.

(ii) We need to show that $[\kappa_w : \kappa_w^p] = [\kappa_\nu : \kappa_\nu^p]$. We have $[\kappa_w^p : \kappa_\nu^p] = [\kappa_w : \kappa_\nu]$, so the result follows from computing the extension degrees in the commutative diagram of finite field extensions

$$\begin{tikzcd} \kappa_w & \kappa_\nu \arrow[l, hook'] \\ \kappa_w^p \arrow[u, hook] & \kappa_\nu^p. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

(iii) By Proposition IV.10, a necessary and sufficient condition for the F-finiteness of a valuation ring $(V, \mathfrak{m}_V, \kappa_V)$ with F-finite fraction field K is that

$$\dim_{\kappa_V^p} V/\mathfrak{m}_V^{[p]} = [K : K^p]. \tag{4.4}$$

Lemma IV.13 gives a formula for $\dim_{\kappa_V^p} V/\mathfrak{m}_V^{[p]}$ in terms of $[\kappa_V : \kappa_V^p]$ that depends on whether the maximal ideal is finitely generated, which is the same for ν and w by Lemma IV.18. Also (ii) tell us that $[\kappa_\nu : \kappa_\nu^p] = [\kappa_w : \kappa_w^p]$, and similarly $[K : K^p] = [L : L^p]$. Thus Lemma IV.13 and equation (4.4) guarantee that the valuation ring of ν is F-finite if and only if the valuation ring of w is F-finite. $\square$

4.3 Valuations centered on prime characteristic Noetherian local domains

Recall that if ν is a valuation of a field K centered on a Noetherian local ring $(R, \mathfrak{m}_R, \kappa_R)$ such that $\mathrm{Frac}(R) = K$, then

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \mathrm{tr.deg}\, \kappa_\nu/\kappa_R \leq \dim(R),$$

and if equality holds in the above inequality then R is called an *Abhyankar center of* ν (Theorem II.54 and Definition II.56).

We have verified (Example II.57(iv)) that the property that a valuation admits an Abhyankar center depends on the choice of the center, that is, it is not an intrinsic property of a valuation. However, if additional restrictions are imposed on the class of centers, then the property of possessing these more restrictive Abhyankar centers becomes intrinsic to ν. This happens, for example, if we require centers to be essentially of finite type over a field k; ν admits an Abhyankar center that is essentially of finite type over k if and only if ν is an *Abhyankar valuation* of the corresponding function field K/k (see Proposition II.64), in the sense that

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k = \operatorname{tr.deg} K/k.$$

Note that the above equality is independent of any properties of a center, which is why we can conclude that admitting Abhyankar centers that are essentially of finite type over a field is intrinsic to a valuation.

The interplay between Abhyankar valuations and valuations admitting Abhyankar centers raises the natural question of whether there is a class of admissible centers, even in a non-function field setting, such that the property of a valuation admitting an Abhyankar center from this class is independent of the choice of the center. The next result provides an affirmative answer for a broad class of Noetherian centers in prime characteristic.

Theorem IV.19. [Dat17a] *Let $(R, \mathfrak{m}_R, \kappa_R)$ be an excellent local domain of characteristic p. Let K be the fraction field of R, and assume $[K : K^p] < \infty$. Suppose ν is a non-trivial valuation of K centered on R with value group Γ_ν and valuation ring $(V, \mathfrak{m}_\nu, \kappa_\nu)$. Then R is an Abhyankar center of ν if and only if*

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p],$$

that is ν/ν^p is defectless.

We will prove Theorem IV.19 by first developing a connection between the inequality

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/\kappa_R \leq \dim(R)$$

and the quantities $[\Gamma_\nu : p\Gamma_\nu]$ and $[\kappa_\nu : \kappa_\nu^p]$. This will also shed light on precisely where F-finiteness is used in the proof of Theorem IV.19.

In order to achieve the above goal, we recall some general facts about torsion-free abelian groups and F-finite fields.

Lemma IV.20. [DS16, Dat17a] *Let p be a prime number, K an F-finite field of characteristic p, and Γ a torsion-free abelian group such that $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma)$ is finite. We have the following:*

1. *If L is an algebraic extension of K, then*
$$[L : L^p] \leq [K : K^p],$$
with equality if $K \subseteq L$ is a finite extension. In particular, L is then also F-finite.

2. *If L is field extension of K of transcendence degree t, then*
$$[L : L^p] \leq p^t [K : K^p],$$
with equality if L is finitely generated over K.

3. *If $s = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma)$, then*
$$[\Gamma : p\Gamma] \leq p^s,$$
with equality if Γ is finitely generated.

Proof of Lemma IV.20. (1) To show that $[L : L^p] = [K : K^p]$ when $K \subseteq L$ is finite, one may repeat the argument of the proof of Proposition IV.17(ii) verbatim by replacing κ_w by L and κ_ν by K. So suppose $K \subseteq L$ is algebraic, and $[K : K^p] < \infty$. It suffices to show that if $a_1, \dots, a_n \in L$ are linearly independent over L^p, then
$$n \leq [K : K^p].$$

Let

$$\widetilde{L} \coloneqq K(a_1, \ldots, a_n).$$

Since L is algebraic over K, $\widetilde{L}$ is a finite extension K, and so by what we already established,

$$[\widetilde{L} : \widetilde{L}^p] = [K : K^p].$$

On the other hand, since $a_1, \ldots, a_n$ are linearly independent over L^p, and $\widetilde{L}^p \subseteq L^p$, it follows that $a_1, \ldots, a_n$ are also linearly independent over $\widetilde{L}^p$. Thus,

$$n \leq [\widetilde{L} : \widetilde{L}^p] = [K : K^p],$$

as desired.

(2) By hypothesis, L is algebraic over a purely transcendental extension $F \coloneqq K(X_1, \ldots, X_t)$. Then (1) shows that

$$[L : L^p] \leq [F : F^p] = p^t[K : K^p],$$

with equality when L is finitely generated over K because then L is a finite extension of F.

(3) If Γ is finitely generated, then $\Gamma \cong \mathbb{Z}^{\oplus s}$, where $s = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma)$. Then

$$[\Gamma : p\Gamma] = [\mathbb{Z}^{\oplus s} : p\mathbb{Z}^{\oplus s}] = p^s.$$

To finish that proof it suffices to show that, even if Γ is not necessarily finitely generated, $\Gamma/p\Gamma$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$ of dimension $\leq s$. So let $t_1, \ldots, t_n$ be elements of Γ whose classes modulo $p\Gamma$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. Then we claim that the t_i are $\mathbb{Z}$-independent elements of Γ. Assume on the contrary that there is some non-trivial relation $a_1t_1 + \cdots + a_nt_n = 0$, for some integers a_i. Since Γ is torsion-free, we can assume without loss of generality, that at least one a_j is not divisible by p. But now modulo $p\Gamma$, this relation produces a non-trivial relation on classes of the t_i in $\Gamma/p\Gamma$, contrary to the fact that these are linearly independent. This shows that any $\mathbb{Z}/p\mathbb{Z}$-linearly independent subset of $\Gamma/p\Gamma$ must have cardinality at most s. $\square$

Using the previous lemma, we can now relate the ramification index (i.e. $[\Gamma_\nu : p\Gamma_\nu]$) and residue degree (i.e. $[\kappa_\nu : \kappa_\nu^p]$) of the extension of valuations ν/ν^p to the inequality $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/\kappa_R \leq \dim(R)$.

Proposition IV.21. [Dat17a] *Let ν be a valuation of a field K of characteristic p with valuation ring $(V, \mathfrak{m}_\nu, \kappa_\nu)$, centered on Noetherian local domain $(R, \mathfrak{m}_R, \kappa_R)$. Suppose*

$$[\kappa_R : \kappa_R^p] < \infty.$$

We have the following:

1. $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] \leq p^{\dim(R)}[\kappa_R : \kappa_R^p]$.
2. *R is an Abhyankar center of ν if and only if* $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = p^{\dim(R)}[\kappa_R : \kappa_R^p]$.

Proof of Proposition IV.21. Throughout the proof, let

$$s := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) \text{ and } t := \operatorname{tr.deg} \kappa_\nu/\kappa_R.$$

(1) Abhyankar's inequality (II.54) implies

$$s + t \leq \dim(R).$$

In particular, s and t are both finite. Using Lemma IV.20(3), we get

$$[\Gamma_\nu : p\Gamma_\nu] \leq p^s.$$

On the other hand, since κ_R is F-finite by hypothesis, and κ_ν has transcendence degree t over κ_R, Lemma IV.20(2) shows

$$[\kappa_\nu : \kappa_\nu^p] \leq p^t[\kappa_R : \kappa_R^p].$$

Thus,

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] \leq p^{s+t}[\kappa_R : \kappa_R^p] \leq p^{\dim(R)}[\kappa_R : \kappa_R^p]. \tag{4.5}$$

(2) Suppose R is an Abhyankar center of ν, that is,

$$s + t = \dim(R).$$

By Theorem II.54, Γ_ν is a free abelian group of rank s, and κ_ν is a finitely generated field extension of κ_R of transcendence degree t. Again using Lemma IV.20, we get

$$[\Gamma_\nu : p\Gamma_\nu] = p^s \text{ and } [\kappa_\nu : \kappa_\nu^p] = p^t[\kappa_R : \kappa_R^p],$$

and so

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = p^{s+t}[\kappa_R : \kappa_R^p] = p^{\dim(R)}[\kappa_R : \kappa_R^p],$$

proving the forward implication.

Conversely, if

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = p^{\dim(R)}[\kappa_R : \kappa_R^p]$$

then

$$p^{\dim(R)}[\kappa_R : \kappa_R^p] = [\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] \leq p^{s+t}[\kappa_R : \kappa_R^p] \leq p^{\dim(R)}[\kappa_R : \kappa_R^p],$$

where the inequalities follow from (4.5). Thus, $\dim(R) = s + t$, which by definition means that R is an Abhyankar center of ν. $\square$

Theorem IV.19 now follows readily from Proposition IV.21.

Proof of Theorem IV.19. Assume R is an excellent local domain with fraction field K such that$[K : K^p] < \infty$. Then R is F-finite (Theorem III.24), and as a consequence,

$$[\kappa_R : \kappa_R^p] < \infty.$$

In particular, R satisfies the hypotheses and conclusion of Proposition IV.21. Therefore it suffices to show that

$$[K : K^p] = p^{\dim(R)}[\kappa_R : \kappa_R^p]. \tag{4.6}$$

But this follows from Proposition III.15. □

Theorem IV.19 has many interesting consequences.

Corollary IV.22. [Dat17a] *Let ν be a valuation of an F-finite field K of characteristic p. If ν admits an excellent center which is Abhyankar, then any other excellent center of ν is also an Abhyankar center of ν.*

In other words, the property of possessing excellent Abhyankar centers is intrinsic to a valuation.

Proof. The proof follows easily from Theorem IV.19 using the observation that the identity $[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p]$ (that is, whether ν/ν^p is defectless) is independent of the choice of a center. □

Moreover, we also obtain a significant generalization of Proposition II.64.

Corollary IV.23. [DS16, Dat17a] *Let ν be a valuation of a function field K/k over an F-finite ground field k of characteristic p. The following are equivalent:*

(1) ν is an Abyankar valuation of K/k.

(2) ν admits an Abhyankar center which is an excellent local ring.

(3) ν/ν^p is defectless.

If the equivalent conditions hold, then Γ_ν is a free abelian group of finite rank and κ_ν is a finitely generated extension of k.

Proof. The hypotheses imply that K is F-finite. The final assertion is a consequence of ν being an Abhyankar valuation (Proposition II.60), and assertions (2) and (3) are equivalent because of Theorem IV.19. For (1) $\Rightarrow$ (2), any center of an Abhyankar valuation ν which is essentially of finite type over k (hence excellent) is an Abhyankar center of ν by Proposition II.64. It remains to show that (3) $\Rightarrow$ (1). Suppose ν/ν^p is defectless, that is,

$$[\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] = [K : K^p].$$

Let $n \coloneqq \operatorname{tr.deg} K/k$, $s \coloneqq \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ and $t \coloneqq \operatorname{tr.deg} \kappa_\nu/k$. Then n is finite by hypothesis, and s, t are finite because

$$s + t \leq n \tag{4.7}$$

according to Abhyankar's inequality for valuations of function fields (2.2). Lemma IV.20 implies

$$[K : K^p] = p^n[k : k^p], [\Gamma_\nu : p\Gamma_\nu] \leq p^s \text{ and } [\kappa_\nu : \kappa_\nu^p] \leq p^t[k : k^p].$$

Therefore

$$p^n[k : k^p] = [\Gamma_\nu : p\Gamma_\nu][\kappa_\nu : \kappa_\nu^p] \leq p^{s+t}[k : k^p],$$

and hence $n \leq s + t$. Combining this inequality with (4.7) gives $n = s + t$, that is,

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k = \operatorname{tr.deg} K/k.$$

But this precisely means ν is an Abhyankar valuation of K/k (Definition II.61). □

Another surprising consequence is that non-Noetherian F-finite valuation rings are not very common in geometric situations.

Proposition IV.24. [Dat17a] *Let ν be a non-trivial valuation of an F-finite field K centered on an excellent local domain A. Then R_ν is F-finite if and only if R_ν is a discrete valuation ring and A is an Abhyankar center of ν.*

Proof. If R_ν is F-finite, then ν/ν^p is defectless (Remark IV.12(1)). Thus Corollary IV.22 shows that A is an Abhyankar center of ν, and so, the value group Γ_ν is a finitely generated abelian group (Theorem II.54). Since ν is non-trivial, using Theorem IV.15 we conclude that R_ν must be a discrete valuation ring. This proves the forward implication.

Conversely, if R_ν is a discrete valuation ring and A is an Abhyankar center of ν, then Theorem IV.19 shows that ν/ν^p is defectless. But for Noetherian valuation rings with F-finite fraction fields, lack of defect of ν/ν^p is equivalent to R_ν being F-finite (see Corollary IV.14). □

Corollary IV.25. [DS17a] *Let ν be a non-trivial valuation of a function field K over an F-finite ground field k of characteristic p. Then R_ν is F-finite if and only if ν is divisorial.*

Proof. The backward implication is trivial because when ν is divisorial, R_ν is essentially of finite type over k, hence F-finite. For the forward implication, note that ν is always centered on some excellent local domain A of K/k. Then Proposition IV.24 shows that R_ν is a discrete valuation ring and A is an Abhyankar center of ν. Moreover, Corollary IV.23 implies that ν is an Abhyankar valuation of K/k. However, any rank 1 Abhyankar valuation of a function field is divisorial (see Example II.65(2)). □

We are also able to easily construct valuations that are *not* centered on any excellent domains.

Corollary IV.26. *Suppose ν is a valuation of an F-finite field K with valuation ring R_ν that satisfies either of the following conditions:*

1. *R_ν is F-finite, but not Noetherian.*

2. $\dim(R_\nu) > s$*, where* $[K : K^p] = p^s$*.*

Then ν *is not centered on any excellent local domain whose fraction field is* K*.*

Proof. Suppose ν satisfies (1). As R_ν is not Noetherian, Proposition IV.24 implies that ν cannot be centered on any excellent local domain with fraction field K.

If A is an excellent local domain with fraction field K, then recall that we have the identity (Proposition III.15)

$$p^{\dim(A)}[\kappa_A : \kappa_A^p] = [K : K^p] = p^s,$$

because A is F-finite. In particular, $\dim(A) \leq s$, where s is as above. If ν is centered on A, then Abhyankar's inequality (Theorem II.54) shows in particular that

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) \leq \dim(A) \leq s.$$

However, the Krull dimension of R_ν is at most $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ (Proposition II.10). Thus $\dim(R_\nu) \leq s$, which contradicts the hypothesis of (2). Hence ν cannot be centered on any excellent local domain with fraction field K. $\square$

Example IV.27. Let w be the valuation of $L(X)$ (where L is a perfect field) constructed in Example IV.16(ii). The valuation ring R_w satisfies conditions (1) and (2) of Corollary IV.26. We have already observed that R_w satisfies (1). To see that R_w satisfies (2), observe that the value group of w has a proper, non-trivial isolated/convex subgroup because it is constructed as a direct sum of two ordered groups with lexicographical order. Thus R_w has Krull dimension at least 2 [Bou89, §4.5], while $[L(X) : L(X)^p] = p$.

Although R_w is a valuation ring of a function field, it does not contain the ground field L. So even though w/w^p is defectless, this example does not contradict Corollary IV.23, or the problem of local uniformization in prime characteristic.

Remarks IV.28.

1. The analogue of Corollary IV.23 is false for valuations of function fields over algebraically closed ground fields of characteristic 0. That is, whether such valuations admit excellent Abhyankar centers depend on the excellent centers. For instance, by imitating the construction of Example II.57(4) using the fields $\mathbb{C}(X,Y)$ and $\mathbb{C}((t))$ instead, one can show that there exists a discrete valuation ν of $\mathbb{C}(X,Y)/\mathbb{C}$ centered on $\mathbb{C}[X,Y]_{(X,Y)}$ such that the latter is not an Abhyankar center of ν (see [ELS03, Example 1(iv)] for more details). However, ν is also trivially centered on its own valuation ring that is an excellent Abhyankar center of ν, because any discrete valuation ring whose fraction field has characteristic 0 is excellent [Sta18, Tag 07QW].

2. The pathologies highlighted in Corollary IV.26 do not arise for valuations of function fields that are trivial on the ground field. Indeed, if K/k is an F-finite function field, and ν is a valuation of K/k, then Corollary IV.25 shows that R_ν cannot simultaneously be F-finite and non-Noetherian, while Abhyankar's inequality (for function fields)

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) + \operatorname{tr.deg} \kappa_\nu/k \leq \operatorname{tr.deg} K/k$$

shows that R_ν cannot satisfy the second part Corollary IV.26 because

$$\dim(R_\nu) \leq \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) \leq \operatorname{tr.deg} K/k \leq \log_p([K : K^p]).$$

Here the last inequality follows from Lemma IV.20(2).

4.4 Frobenius splitting

Valuation rings of prime characteristic are *always* F-pure, hence very close to being Frobenius split. Nevertheless, it is natural to ask which valuation rings admit a Frobenius splitting. This very question, which arose in conversations of Karl Schwede, Zsolt Patakfalvi and Karen Smith, inspired the author's joint work with Karen Smith on using F-singularity techniques to probe the structure of valuation rings.

We have seen that F-finite valuation rings are always Frobenius split (Corollary IV.5). However, as is evident from the results of the previous sections, F-finiteness imposes strong restrictions on valuation rings, and there are many non-F-finite valuation rings even in function fields. This makes Frobenius splitting a significantly more difficult notion of singularity to penetrate in the non-Noetherian and, usually, non-F-finite world of valuation rings. Nevertheless, our work indicates that Frobenius splitting is related to the defect of the extension of valuations ν/ν^p, and it is this relationship that we hope to highlight in this section.

We begin by proving a negative result in the case that the extension of valuations ν/ν^p has maximal defect. Note that if R is a domain of characteristic p, then the existence of a Frobenius splitting $F_*R \to R$ is equivalent to the existence of an R^p-linear map $R \to R^p$ that maps $1 \mapsto 1$. We will also call a map of the latter type a Frobenius splitting of R.

Proposition IV.29. [DS17a] *Let K be a non-perfect field of characteristic p and ν a valuation of K. If ν/ν^p is totally unramified (i.e. ν/ν^p has maximal defect), then the valuation ring R_ν is never Frobenius split.*

Proof. Recall that ν/ν^p is totally unramified if $e(\nu/\nu^p) = [\Gamma_\nu : p\Gamma_\nu] = 1$ and $f(\nu/\nu^p) = [\kappa_\nu : \kappa_\nu^p] = 1$ (Definition II.49). This means that the value group of ν is p-divisible and the residue field of ν is perfect. The p-divisibility of Γ_ν shows that

$$\mathfrak{m}_\nu = \mathfrak{m}_\nu^{[p]}.$$

Then any Frobenius splitting

$$\varphi : R_\nu \to R_\nu^p$$

maps the maximal ideal $\mathfrak{m}_\nu$ of ν into the maximal ideal of R_ν^p, thereby inducing a Frobenius splitting of residue fields

$$\widetilde{\varphi} : \kappa_\nu \to \kappa_\nu^p.$$

However, κ_ν is perfect, so that $\widetilde{\varphi}$ is just the identity map. Since K is not perfect, φ has a non-trivial kernel, that is, some non-zero $x \in R_\nu$ gets mapped to 0. By p-divisibility of Γ_ν, one can write

$$x = uy^p,$$

for a unit u in V, and $y \neq 0$. Then $0 = \varphi(x) = y^p\varphi(u)$, which shows that $\varphi(u) = 0$. But this contradicts injectivity of $\widetilde{\varphi}$, proving that no Frobenius splitting of R_ν exists. □

Our main result of this section is that in contrast to Proposition IV.29, when K is a function field over a perfect ground field k and ν is a valuation of K/k such that ν/ν^p is defectless (equivalently ν is Abhyankar by Corollary IV.23), then R_ν is Frobenius split. In fact, we prove a more general result.

Theorem IV.30. [Dat17a] *Let K be a function field of an F-finite field k of characteristic p. If ν is an Abhyankar valuation of K/k such that κ_ν is separable over k, then R_ν is Frobenius split.*

The key ingredient in our proof of Theorem IV.30 is the local monomialization result of Knaf and Kuhlmann for Abhyankar valuations (Theorem II.69). A consequence of local monomialization yields the following 'special' regular local center of any Abhyankar valuation with separable residue field.

Lemma IV.31. *Let ν be an Abhyankar valuation as in Theorem IV.30. Suppose $d \coloneqq \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$. Then there exists a regular local ring $(A, \mathfrak{m}_A, \kappa_A)$ which is essentially of finite type over k with fraction field K satisfying the following properties:*

1. *R_ν is centered on A, and $\kappa_A \hookrightarrow \kappa_\nu$ is an isomorphism.*
2. *A has Krull dimension d, and there exist a regular system of parameters $\{x_1, \ldots, x_d\}$ of A such that $\{\nu(x_1), \ldots, \nu(x_d)\}$ freely generates the value group Γ_ν.*

Proof of Lemma IV.31. This is a special case of Corollary II.71. □

Remark IV.32. For a valuation ν of K/k, the existence of a center which is an essentially of finite type k-algebra of Krull dimension equal to $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu)$ implies that ν is Abhyankar (see Remark II.72(3)). Thus, only Abhyankar valuations admit centers satisfying the hypothese of Lemma IV.31.

From now on, A will denote a choice of a regular local center of ν that satisfies Lemma IV.31, and $\{x_1, \ldots, x_d\}$ a regular system of parameters of A whose valuations freely generate Γ_ν. Observe that A is F-finite since it is essentially of finite type over an F-finite field. Then Theorem III.8 implies that A is free over its p-th power subring A^p of rank equal to $[K : K^p] = [k : k^p]p^n$, where

$$n \coloneqq \operatorname{tr.deg} K/k.$$

For

$$f \coloneqq [\kappa_\nu : \kappa_\nu^p] = [k : k^p]p^{n-d},$$

if we choose

$$1 = y_1,\ y_2,\ \ldots,\ y_f \in A,$$

such that the images of y_i in $\kappa_A = \kappa_\nu$ form a basis of κ_ν over κ_ν^p, then it is well-known that

$$\mathcal{B} := \{y_j x_1^{\beta_1} \dots x_d^{\beta_d} : 1 \leq j \leq f,\ 0 \leq \beta_i \leq p-1\},$$

is a free basis of A over A^p. Note the elements y_j are units in A.

With respect to the basis $\mathcal{B}$, A has a natural Frobenius splitting

$$\eta_{\mathcal{B}} : A \to A^p,$$

given by mapping $1 = y_1 x_1^0 \dots x_d^0 \mapsto 1$, and all the other basis elements to 0. Extending $\eta_{\mathcal{B}}$ uniquely to a K^p-linear map

$$\widetilde{\eta_{\mathcal{B}}} : K \to K^p$$

of the fraction fields, we will show that the restriction of $\widetilde{\eta_{\mathcal{B}}}$ to R_ν yields a Frobenius splitting of R_ν, or in other words, $\widetilde{\eta_{\mathcal{B}}}|_{R_\nu}$ maps into R_ν^p.

Claim IV.33. *For any $a \in A$, either $\eta_{\mathcal{B}}(a) = 0$ or $\nu(\eta_{\mathcal{B}}(a)) \geq \nu(a)$.*

Theorem IV.30 follows from the claim using the following general observation.

Lemma IV.34. *Let ν be a valuation of a field K of characteristic p with valuation ring R_ν, and A a subring of R_ν such that* $\mathrm{Frac}(A) = K$. *Suppose $\varphi : A \to A^{p^e}$ is an A^{p^e}-linear map, for some $e \geq 1$. Consider the following statements:*

(i) For all $a \in A$, $\varphi(a) = 0$ or $\nu(\varphi(a)) \geq \nu(a)$.

(ii) For all $a, b \in A$ such that $\nu(a) \geq \nu(b)$, if $\varphi(ab^{p^e-1}) \neq 0$, then $\nu(\varphi(ab^{p^e-1})) \geq \nu(b^{p^e})$.

(iii) φ extends to an $R_\nu^{p^e}$-linear map $R_\nu \to R_\nu^{p^e}$.

(iv) φ extends uniquely to an $R_\nu^{p^e}$-linear map $R_\nu \to R_\nu^{p^e}$.

Then (ii), (iii) and (iv) are equivalent, and (i) $\Rightarrow$ (ii). Moreover, if φ is a Frobenius splitting of A satisfying (i) or (ii), then φ extends to a Frobenius splitting of R_ν.

Proof of Lemma IV.34. For the final assertion on Frobenius splitting, note that the extension of a Frobenius splitting remains a Frobenius splitting since $1 \mapsto 1$ also in the extension.

(i) $\Rightarrow$ (ii): If $\varphi(ab^{p^e-1}) \neq 0$, we have

$$\nu(\varphi(ab^{p^e-1})) \geq \nu(ab^{p^e-1}) \geq \nu(b^{p^e}),$$

where the first inequality follows from (i), and the second inequality follows from $\nu(a) \geq \nu(b)$.

(ii) $\Rightarrow$ (iii): Extending φ to a K^{p^e}-linear map $\widetilde{\varphi} : K \to K^{p^e}$, it suffices to show that $\widetilde{\varphi}|_{R_\nu}$ maps into $R_\nu^{p^e}$. Let $r \in R_\nu$ be a non-zero element. Since K is the fraction field of A and R_ν, one can express r as a fraction a/b, for non-zero $a, b \in A$. Note

$$\nu(a) \geq \nu(b).$$

Then

$$\widetilde{\varphi}(r) = \widetilde{\varphi}\left(\frac{a}{b}\right) = \frac{1}{b^{p^e}}\varphi(ab^{p^e-1}). \tag{4.8}$$

If $\varphi(ab^{p^e-1}) = 0$, then $\widetilde{\varphi}(r) = 0$, and r maps into $R_\nu^{p^e}$. Otherwise by assumption,

$$\nu(\varphi(ab^{p^e-1})) \geq \nu(b^{p^e}),$$

and so,

$$\nu(\widetilde{\varphi}(r)) = \nu(\varphi(ab^{p^e-1})) - \nu(b^{p^e}) \geq 0,$$

that is $\widetilde{\varphi}(r)$ is an element of $K^{p^e} \cap R_\nu = R_\nu^{p^e}$.

(iii) $\Rightarrow$ (iv): Since A and R_ν have the same fraction field, any extension of φ to R_ν is obtained as a restriction to R_ν of the unique extension of φ to a K^{p^e}-linear map $\widetilde{\varphi} : K \to K^{p^e}$. Thus, uniqueness follows. See (4.8) above for a concrete description of how φ extends to R_ν.

To finish the proof of the lemma, it suffices to show (iv) $\Rightarrow$ (ii). But this also follows from (4.8). $\square$

Proof of Claim IV.33. Recall that

$$\mathcal{B} = \{y_j x_1^{\beta_1} \dots x_d^{\beta_d} : 1 \le j \le f,\ 0 \le \beta_i \le p-1\}$$

is a basis of A over A^p, where the x_i and y_j are chosen such that $\{\nu(x_1), \dots, \nu(x_d)\}$ freely generates the value group Γ_ν, and the images of $1 = y_1,\ y_2,\ \dots,\ y_f$ in κ_ν form a basis of κ_ν over κ_ν^p. The A^p-linear Frobenius splitting $\eta_{\mathcal{B}}$ is given by

$$\eta_{\mathcal{B}}\left(\sum_{j=1}^{f} \sum_{0 \le \beta_i \le p-1} c^p_{j,\beta_1,\dots,\beta_d} y_j x_1^{\beta_1} \dots x_d^{\beta_d}\right) = c^p_{1,0,0,\dots,0}.$$

Thus, we need to show that either $c^p_{1,0,0,\dots,0} = 0$ or

$$\nu(c^p_{1,0,0,\dots,0}) \ge \nu\left(\sum_{j=1}^{f} \sum_{0 \le \beta_i \le p-1} c^p_{j,\beta_1,\dots,\beta_d} y_j x_1^{\beta_1} \dots x_d^{\beta_d}\right).$$

Assuming without loss of generality that $\sum_{j=1}^{f} \sum_{0 \le \beta_i \le p-1} c^p_{j,\beta_1,\dots,\beta_d} y_j x_1^{\beta_1} \dots x_d^{\beta_d} \neq 0$, we will prove the stronger fact that

$$\nu\left(\sum_{j=1}^{f} \sum_{0 \le \beta_i \le p-1} c^p_{j,\beta_1,\dots,\beta_d} y_j x_1^{\beta_1} \dots x_d^{\beta_d}\right) = \inf\{\nu(c^p_{j,\beta_1,\dots,\beta_d} y_j x_1^{\beta_1} \dots x_d^{\beta_d}) : c^p_{j,\beta_1,\dots,\beta_d} \neq 0\}. \tag{4.9}$$

For two non-zero terms $c^p_{j,\alpha_1,\dots,\alpha_d} y_j x_1^{\alpha_1} \dots x_d^{\alpha_d}$ and $c^p_{k,\beta_1,\dots,\beta_d} y_k x_1^{\beta_1} \dots x_d^{\beta_d}$ in the above sum,

$$\nu(c^p_{j,\alpha_1,\dots,\alpha_d} y_j x_1^{\alpha_1} \dots x_d^{\alpha_d}) = \nu(c^p_{k,\beta_1,\dots,\beta_d} y_k x_1^{\beta_1} \dots x_d^{\beta_d}) \tag{4.10}$$

if and only if

$$p\nu(c_{j,\alpha_1,\dots,\alpha_d}) + \alpha_1\nu(x_1) + \dots + \alpha_d\nu(x_d) = p\nu(c_{k,\beta_1,\dots,\beta_d}) + \beta_1\nu(x_1) + \dots + \beta_d\nu(x_d). \tag{4.11}$$

By $\mathbb{Z}$-linear independence of $\nu(x_1),\ \dots,\ \nu(x_d)$, for all $i = 1, \dots, d$, we get

$$p | (\alpha_i - \beta_i).$$

Since $0 \leq \alpha_i, \beta_i \leq p-1$, this means that $\alpha_i = \beta_i$ for all i. Moreover, then

$$\nu(c^p_{j,\alpha_1,\ldots,\alpha_d}) = \nu(c^p_{k,\beta_1,\ldots,\beta_d}).$$

Thus, (4.10) holds precisely when $\nu(c^p_{j,\alpha_1,\ldots,\alpha_d}) = \nu(c^p_{k,\beta_1,\ldots,\beta_d})$ and $\alpha_i = \beta_i$, for all $i = 1, \ldots, d$.

For ease of notation, let us use $\underline{\alpha}$ as a shorthand for $\alpha_1, \ldots, \alpha_d$, and $\underline{x}^{\underline{\alpha}}$ for $x_1^{\alpha_1} \ldots x_d^{\alpha_d}$. Then for a fixed non-zero term $c^p_{j_1,\underline{\alpha}} y_{j_1} \underline{x}^{\underline{\alpha}}$, consider the set

$$\{c^p_{j_1,\underline{\alpha}} y_{j_1} \underline{x}^{\underline{\alpha}},\ c^p_{j_2,\underline{\alpha}} y_{j_2} \underline{x}^{\underline{\alpha}}, \ldots,\ c^p_{j_i,\underline{\alpha}} y_{j_i} \underline{x}^{\underline{\alpha}}\}$$

of all non-zero terms of $\sum_{j=1}^{f} \sum_{0 \leq \beta_i \leq p-1} c^p_{j,\beta_1,\ldots,\beta_d} y_j x_1^{\beta_1} \ldots x_d^{\beta_d}$ having the same valuation as $c^p_{j_1,\underline{\alpha}} y_{j_1} \underline{x}^{\underline{\alpha}}$. In particular, by the above reasoning we also have

$$\nu(c^p_{j_1,\underline{\alpha}}) = \nu(c^p_{j_2,\underline{\alpha}}) = \cdots = \nu(c^p_{j_i,\underline{\alpha}}).$$

Adding these terms of equal valuation, in the valuation ring R_ν one can write

$$c^p_{j_1,\underline{\alpha}} y_{j_1} \underline{x}^{\underline{\alpha}} + c^p_{j_2,\underline{\alpha}} y_{j_2} \underline{x}^{\underline{\alpha}} + \cdots + c^p_{j_i,\underline{\alpha}} y_{j_i} \underline{x}^{\underline{\alpha}} =$$
$$\left(y_{j_1} + \left(\frac{c_{j_2,\underline{\alpha}}}{c_{j_1,\underline{\alpha}}} \right)^p y_{j_2} + \cdots + \left(\frac{c_{j_i,\underline{\alpha}}}{c_{j_1,\underline{\alpha}}} \right)^p y_{j_i} \right) c^p_{j_1,\underline{\alpha}} \underline{x}^{\underline{\alpha}},$$

where

$$y_{j_1} + \left(\frac{c_{j_2,\underline{\alpha}}}{c_{j_1,\underline{\alpha}}} \right)^p y_{j_2} + \cdots + \left(\frac{c_{j_i,\underline{\alpha}}}{c_{j_1,\underline{\alpha}}} \right)^p y_{j_i}$$

is a unit in R_ν by the κ_ν^p-linear independence of the images of $y_{j_1}, \ldots, y_{j_i}$ in κ_ν and the fact that $(c_{j_2,\underline{\alpha}}/c_{j_1,\underline{\alpha}})^p, \ldots, (c_{j_i,\underline{\alpha}}/c_{j_1,\underline{\alpha}})^p$ are units in R_ν^p. Thus, the valuation of the sum

$$c^p_{j_1,\underline{\alpha}} y_{j_1} \underline{x}^{\underline{\alpha}} + \cdots + c^p_{j_i,\underline{\alpha}} y_{j_i} \underline{x}^{\underline{\alpha}}$$

equals the valuation of any of its terms. Now rewriting

$$\sum_{j=1}^{f} \sum_{0 \leq \beta_i \leq p-1} c^p_{j,\beta_1,\ldots,\beta_d} y_j x_1^{\beta_1} \ldots x_d^{\beta_d}$$

by collecting non-zero terms having the same valuation, (4.9), hence also the claim, follows. □

Corollary IV.35. *Valuation rings of Abhyankar valuations of function fields over perfect ground fields of prime characteristic are always Frobenius split.*

Proof. Knaf and Kuhlmann's local monomialization result holds unconditionally under these hypotheses because the residue field of the Abhyankar valuation is automatically separable over the perfect ground field by Proposition II.60. □

Examples IV.36.

(a) A valuation ring of a function field of a curve over an F-finite ground field is always Frobenius split. Indeed, such a valuation ring is an F-finite discrete valuation ring since it is always centered on some normal affine model of dimension 1 of the function field.

(b) For a positive integer n, consider $\mathbb{Z}^{\oplus n}$ with the lexicographical order. That is, if $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{Z}^{\oplus n}$, then

$$e_1 > e_2 > \cdots > e_n.$$

There exists a unique valuation ν_{lex} on $\mathbb{F}_p(X_1, \ldots, X_n)/\mathbb{F}_p$ such that for all $i \in \{1, \ldots, n\}$,

$$\nu_{lex}(X_i) = e_i.$$

The valuation ν_{lex} is clearly Abhyakar since $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^{\oplus n}) = n$, which coincides with the transcendence degree of $\mathbb{F}_p(X_1, \ldots, X_n)/\mathbb{F}_p$. One can also show that the valuation ring $R_{\nu_{lex}}$ has Krull dimension n and residue field $\mathbb{F}_p$. The valuation is centered on the regular local ring $\mathbb{F}_p[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ such that the valuations of the obvious regular system of parameters of this center freely generate

$\mathbb{Z}^{\oplus n}$ and the residue field coincides with the residue field of ν_{lex}. Then a Frobenius splitting of $R_{\nu_{lex}} \to R^p_{\nu_{lex}}$ is obtained by extending the canonical splitting on $\mathbb{F}_p[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ with respect to the basis

$$\{X_1^{\beta_1} \ldots X_n^{\beta} : 0 \leq \beta_i \leq p-1\}.$$

This splitting of $\mathbb{F}_p[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ maps

$$X_1^{\alpha_1} \ldots X_n^{\alpha_n} \mapsto \begin{cases} X_1^{\alpha_1} \ldots X_n^{\alpha_n} & \text{if } p|\alpha_i \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\pi \subset \mathbb{R}$. Consider the valuation ν (Example II.57(2)) of $\mathbb{F}_p(X, Y, Z)/\mathbb{F}_p$ given by

$$\nu(X) = \nu(Y) = 1, \nu(Z) = \pi.$$

As verified before, $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma) = 2$ and $\operatorname{tr.deg} \kappa_\nu/\mathbb{F}_p = 1$ and so ν is an Abhyankar valuation. Although ν is centered on the regular local ring $\mathbb{F}_p[X, Y, Z]_{(X,Y,Z)}$, no regular system of parameters of this center can freely generate the value group because the center has dimension 3, whereas the value group is free of rank 2. However, blowing up the origin in $\mathbb{A}^3_{\mathbb{F}_p}$, we see that ν is now centered on the regular local ring

$$\mathbb{F}_p\left[X, \frac{Y}{X}, \frac{Z}{X}\right]_{(X, Z/X)},$$

and the valuations of the regular system of paramaters $\{X, Z/X\}$ freely generate Γ_ν. Furthermore, the residue field of $\mathbb{F}_p[X, Y/X, Z/X]_{(X,Z/X)}$ can be checked to coincide with the residue field of the valuation ring. Relabelling Y/X and Z/X as U, W respectively, a Frobenius splitting of R_ν is obtained by extending the Frobenius splitting of $\mathbb{F}_p[X, U, W]_{(X,W)}$ given by the same rule as in example (a) with respect to the transcendental elements X, U, W over $\mathbb{F}_p$.

Remarks IV.37.

1. We expect any Abhyankar valuation ring of an F-finite function field to be Frobenius split. However, at present we do not know how to remove the separability hypothesis on the residue field of the valuation since we use Knaf and Kuhlmann's local monomialization result which also requires this additional assumption.

2. As is the case in algebraic geometry, our investigation reveals that Frobenius splitting is quite mysterious for valuation rings. We have made some headway into understanding this notion of singularity when the defect of ν/ν^p is one of two possible extremes, that is, when ν/ν^p is totally unramified and when ν/ν^p is defectless. However, we do not really understand how Frobenius splitting behaves for intermediate defect.

4.5 Tight closure of ideals

A preliminary investigation of tight closure in the setting of valuation rings was started in [DS16]. The few tight closure related results obtained in [DS16] stem organically from the authors' focus on understanding F-regularity for valuation rings. In this section, we present a more systematic treatment of tight-closure of ideals of valuation rings. In doing so we discover that tight closure is intimately related to valued fields which are f-adic in the valuation topology.

Recall that if R is domain of prime characteristic (not necessarily Noetherian), then the **tight closure** of an ideal I or R, denoted I^*, consists of elements $r \in R$ for which there exist $c \neq 0$ such that for all $e \gg 0$,

$$cr^{p^e} \in I^{[p^e]}.$$

An ideal I is **tightly closed** if $I^* = I$.

Here are some basic properties of tight closure.

Proposition IV.38. *Let I be an ideal of a domain R of characteristic p.*

1. *I^* is an ideal of R that contains I.*

2. *$I^* \neq R$ if and only if $\bigcap_{e \in \mathbb{N}} I^{[p^e]} = (0)$.*

3. *If $\varphi : R \hookrightarrow S$ is an injective ring homomorphism of domains, then $I^*S \subseteq (IS)^*$.*

Proof. The proof of (1) is easy, so we omit it. For the proof of (2), note that $I^* = R$ if and only if $1 \in I^*$. Looking back at the definition of tight closure, this is equivalent to the existence of a non-zero $c \in R$ such that for all $e \gg 0$,

$$c = c \cdot 1 \in I^{[p^e]}.$$

But such a c is precisely a non-zero element in the intersection $\bigcap_{e \in \mathbb{N}} I^{[p^e]}$. For (3) note that $I^{[p^e]}S = (IS)^{[p^e]}$ for all $e \in \mathbb{N}$, and if $i \in I^*$, then using the injectivity of φ it follows that $\varphi(i) \in (IS)^*$. Since I^*S is generated as an ideal of S by elements of the form $\varphi(i)$ for $i \in I^*$, we get the desired inclusion. □

Remark IV.39*.* It is not clear if I^* is tightly closed in the setting of arbitrary domains of prime characteristic. The proof in the Noetherian case uses finite generation of I^* in an essential way. We will later see that although valuation rings are highly non-Noetherian, tight closure is a closure operation on ideals of such rings (Corollary IV.46).

Proposition IV.38 implies that valuation rings of Krull dimension > 1 have many proper ideals whose tight closure is the whole ring.

Lemma IV.40. *Let V be a valuation ring of characteristic p, and I an ideal of V. If Q is a non-zero prime ideal of V such that $Q \subsetneq I$, then $I^* = V$.*

Proof. By hypothesis there exists $i \in I$ such that $i \notin Q$. Since Q is prime, for all $e > 0$,

$$i^{p^e} \notin Q.$$

Because any two ideals of a valuation ring are comparable, it follows that for all $e > 0$,

$$0 \neq Q \subsetneq I^{[p^e]},$$

and therefore $I^* = V$ by Proposition IV.38(2). □

Tight closure is an interesting operation only for those valued fields that are f-adic in the valuation topology.

Proposition IV.41. *Let ν be a non-trivial valuation of a field K of prime characteristic. The following are equivalent:*

1. *K is f-adic in the valuation topology induced by ν (see Definition II.32).*

2. *There exists a non-zero ideal I of R_ν such that $I^* \neq R_\nu$.*

Proof. Theorem II.39 shows that K is f-adic in the valuation topology if and only if there exists a non-zero element $a \in R_\nu$ such that

$$\bigcap_{e \in \mathbb{N}} (a^{p^e}) = \bigcap_{n \in \mathbb{N}} (a^n) = (0).$$

The existence of such a non-zero element is clearly equivalent to the existence of an ideal I of R_ν such that $I^* \neq R_\nu$ by Proposition IV.38(2). □

In light of Proposition IV.41, we will assume in our discussion of tight closure that valued fields are f-adic in the valuation topology. Recall, this implies that the corresponding valuation rings then have a height 1 prime ideal (when the valuation is non-trivial) by Theorem II.39.

Theorem IV.42. *Let ν be a non-trivial valuation of a field K of characteristic p such that K is f-adic in the topology induced by ν. Let I be ideal of R_ν and $\mathfrak{p}$ be the unique height 1 prime of R_ν.*

1. *If $\mathfrak{p} \subsetneq I$, then $I^* = R_\nu$.*
2. *If $I \subsetneq \mathfrak{p}$, then $I^* \neq R_\nu$.*
3. *$\mathfrak{p}^* \neq R_\nu$ if and only if $(R_\nu)_\mathfrak{p}$ is a discrete valuation ring.*

Proof. We already proved (1) in Lemma IV.40. To prove (2) we need to show that if $I \subsetneq \mathfrak{p}$, then $\bigcap_{e\in\mathbb{N}} I^{[p^e]} = (0)$. Lemma II.26 implies that for all $e > 0$,

$$I^{[p^e]} = I^{p^e},$$

and so,

$$\bigcap_{e\in\mathbb{N}} I^{[p^e]} = \bigcap_{n\in\mathbb{N}} I^n$$

is a prime ideal of R_ν by Proposition II.21. However $\bigcap_{e\in\mathbb{N}} I^{[p^e]}$ is properly contained in $\mathfrak{p}$, which is the height 1 of R_ν. This forces $\bigcap_{e\in\mathbb{N}} I^{[p^e]}$ to be the zero ideal.
(3) Suppose $(R_\nu)_\mathfrak{p}$ is a discrete valuation ring. Consider the injective localization map $R_\nu \hookrightarrow (R_\nu)_\mathfrak{p}$. By part 3 of Proposition IV.38, we have

$$\mathfrak{p}^*(R_\nu)_\mathfrak{p} \subseteq (\mathfrak{p}(R_\nu)_\mathfrak{p})^*.$$

If $(R_\nu)_\mathfrak{p}$ is a discrete valuation ring, $(\mathfrak{p}(R_\nu)_\mathfrak{p})^* = \mathfrak{p}(R_\nu)_\mathfrak{p} \neq (R_\nu)_\mathfrak{p}$, and so $\mathfrak{p}^*$ cannot equal R_ν.

For the converse, if $\mathfrak{p}^* \neq R_\nu$, then

$$(0) = \bigcap_{e\in\mathbb{N}} \mathfrak{p}^{[p^e]} = \bigcap_{n\in\mathbb{N}} \mathfrak{p}^n.$$

This shows that $\mathfrak{p} \neq \mathfrak{p}^2$, and so there exists $a \in \mathfrak{p} - \mathfrak{p}^2$. Since ideals of R_ν are comparable, we get $\mathfrak{p}^2 \subsetneq aR_\nu$. Thus for all non-zero $x \in \mathfrak{p}$,

$$\nu(x^2) = 2\nu(x) > \nu(a). \tag{4.12}$$

The local ring $(R_\nu)_\mathfrak{p}$ is a valuation ring of K of Krull dimension 1. Thus there exists a real-valued valuation (see Proposition II.12)

$$w : K^\times \to \mathbb{R},$$

whose corresponding valuation ring is $(R_\nu)_\mathfrak{p}$. The localization map

$$R_\nu \hookrightarrow (R_\nu)_\mathfrak{p}$$

induces an *order-preserving* group homomorphism $\varphi : \Gamma_\nu \to \mathbb{R}$ such that the following diagram commutes

$$\begin{tikzcd} K^\times \arrow[r, "\nu"] \arrow[dr, "w"'] & \Gamma_\nu \arrow[d, "\varphi"] \\ & \mathbb{R} \end{tikzcd}$$

Then (4.12) shows that for all $x \in \mathfrak{p}$,

$$2w(x) = \varphi(2\nu(x)) \geq \varphi(\nu(a)) = w(a) > 0,$$

where the last inequality holds because a is a non-zero element of the maximal ideal of $(R_\nu)_\mathfrak{p}$. Hence for all $x \in \mathfrak{p}$,

$$w(x) \geq \frac{w(a)}{2} > 0,^{1} \tag{4.13}$$

and so $(R_\nu)_\mathfrak{p}$ is a discrete valuation ring using the following group-theoretic observation.

[1]Division by 2 makes sense because w takes values in $\mathbb{R}$.

Lemma IV.43. *Let Γ be a non-trivial subgroup of $\mathbb{R}$. If*

$$\inf\{\gamma \in \Gamma : 0 < \gamma\} > 0,$$

then Γ is order isomorphic to $\mathbb{Z}$.

Proof of Lemma IV.43. Let $\ell := \inf\{\gamma \in \Gamma : 0 < \gamma\}$. By hypothesis, $\ell > 0$. We first show that $\ell \in \Gamma$. If not, then there exists $\alpha, \beta \in \Gamma$ such that

$$\ell < \beta < \alpha < \ell + \epsilon,$$

for $\epsilon = \ell/2$. Then $\alpha - \beta \in \Gamma$ (since Γ is a group) and

$$0 < \alpha - \beta < \ell,$$

contradicting the definition of ℓ. Therefore $\ell \in \Gamma$, and then a similar argument shows that $\Gamma = \mathbb{Z}\ell$. $\square$

The proof of the theorem follows because we know that every element of the maximal ideal $\mathfrak{p}(R_\nu)_\mathfrak{p}$ has valuation at least $w(a)/2$ by (4.13), which means that

$$0 < w(a)/2 \leq \inf\{\gamma \in \Gamma_w : 0 < \gamma\}.$$

The lemma then allows us to conclude that Γ_w is order isomorphic to $\mathbb{Z}$. $\square$

Corollary IV.44. *Let ν be a valuation of a field K of characteristic p. The following are equivalent:*

1. *All ideals of R_ν are tightly closed.*
2. *$\mathfrak{m}_\nu$ is tightly closed.*
3. *R_ν is Noetherian.*

Proof. A Noetherian valuation ring is either a field or a discrete valuation ring, hence always regular. Thus (3) $\Rightarrow$ (1) is well-known. For the converse, we may assume ν is non-trivial as otherwise the implication is trivial. Then Proposition IV.41 implies that K is f-adic in the valuation topology induced by ν. Let $\mathfrak{p}$ be the unique height 1 prime of R_ν. Since all ideals of R_ν are tightly closed, Theorem IV.42 implies that there does not exist an ideal $I \neq R_\nu$ such that

$$\mathfrak{p} \subsetneq I.$$

Therefore $\mathfrak{p}$ must be the maximal ideal of R_ν. Moreover since $\mathfrak{p}^* = \mathfrak{p} \neq R_\nu$, Theorem IV.42 again implies that $R_\nu = (R_\nu)_\mathfrak{p}$ is a discrete valuation ring. Thus (1) and (3) are equivalent.

The implication (3) $\Rightarrow$ (2) is also clear. To finish the proof, it suffices to show (2) $\Rightarrow$ (3). If $\mathfrak{m}_\nu^* = \mathfrak{m}_\nu$, then Proposition IV.38(2) implies

$$\bigcap_{e \in \mathbb{N}} \mathfrak{m}_\nu^{[p^e]} = 0.$$

Hence by Lemma IV.40, there is no non-zero prime ideal $\mathcal{Q}$ of R_ν such that

$$\mathcal{Q} \subsetneq \mathfrak{m}_\nu.$$

Therefore $\dim(R_\nu) \leq 1$, and a similar argument as in the previous paragraph again shows that R_ν is Noetherian. $\square$

4.5.1 Probing deeper

The above results imply that only ideals contained in the height 1 prime of a valuation ring (if such a prime exists) have interesting tight closure. We now provide a more precise characterization of the tight closure of such ideals.

Proposition IV.45. *Let ν be a non-trivial valuation of a field K of characteristic p such that K is f-adic in the valuation topology induced by ν. Let $\mathfrak{p}$ be the height 1 prime ideal of R_ν, and*

$$w : K^\times \to \mathbb{R}$$

be a valuation whose corresponding valuation ring is $(R_\nu)_\mathfrak{p}$. For an ideal $I \subseteq \mathfrak{p}$, if

$$\mathfrak{a} := \inf\{w(i) : i \in I - \{0\}\},$$

then

$$I^* = \{x \in R_\nu : w(x) \geq \mathfrak{a}\} \cup \{0\}.$$

In particular, $I^* = I$ if $\mathfrak{a} \notin \Gamma_w$.

Proof. As in the proof of Theorem IV.42, there exists an ordered group homomorphism

$$\varphi : \Gamma_\nu \to \mathbb{R}$$

such that $\varphi \circ \nu = w$.

Since $I^{[p^e]}$ is generated by p^e-th powers of elements of I, it follows that

$$\inf\{w(j) : j \in I^{[p^e]} - \{0\}\} = p^e\mathfrak{a}. \tag{4.14}$$

Let $x \in I^*$. Then there exists a non-zero element $c \in R_\nu$ such that for $e \gg 0$,

$$cx^{p^e} \in I^{[p^e]}.$$

Assume for contradiction that $w(x) < \mathfrak{a}$. Then for $e \gg 0$,

$$p^e(\mathfrak{a} - w(x)) = p^e\mathfrak{a} - w(x^{p^e}) > w(c).$$

Thus for $e \gg 0$,

$$p^e\mathfrak{a} > w(cx^{p^e}),$$

which contradicts $cx^{p^e} \in I^{[p^e]}$ because of (4.14). Thus,

$$I^* \subseteq \{x \in R_\nu : w(x) \geq \mathfrak{a}\} \cup \{0\}.$$

Conversely, if $x \in R_\nu$ such that $w(x) \geq \mathfrak{a}$, then for any non-zero element $c \in \mathfrak{p}$ and $e > 0$,

$$w(cx^{p^e}) > p^e \mathfrak{a},$$

because $w(c) > 0$. By definition of infimum, there exists $i \in I$ such that

$$\frac{w(c)}{p^e} + w(x) > w(i) \geq a,$$

and so $w(cx^{p^e}) > w(i^{p^e})$. Since $w = \varphi \circ \nu$, we see that

$$\varphi(\nu(cx^{p^e})) > \varphi(\nu(i^{p^e})),$$

which, because φ is order preserving, implies that

$$\nu(cx^{p^e}) > \nu(i^{p^e}).$$

Hence, for all $e > 0$, there exists $i \in R_\nu$ such that

$$cx^{p^e} \in i^{p^e} R_\nu \subseteq I^{[p^e]},$$

that is, $x \in I^*$. This proves

$$\{x \in R_\nu : w(x) \geq \mathfrak{a}\} \cup \{0\} \subseteq I^*,$$

thereby demonstrating that $I^* = \{x \in R_\nu : w(x) \geq \mathfrak{a}\} \cup \{0\}$.

For the second part of the proposition, it suffices to show $I^* \subseteq I$, if $\mathfrak{a} \notin \Gamma_w$. Using what we just proved,

$$I^* = \{x \in R_\nu : w(x) \geq \mathfrak{a}\} \cup \{0\} = \{x \in R_\nu : w(x) > \mathfrak{a}\} \cup \{0\}.$$

However, for any $x \in R_\nu$ such that $w(x) > \mathfrak{a}$, there exists $i \in I - \{0\}$ satisfying

$$w(x) > w(i) > \mathfrak{a},$$

by the definition of infimum. Again using $w = \varphi \circ \nu$ this shows $\nu(x) > \nu(i)$, that is, $x \in iR_\nu \subseteq I$. Thus $I^* \subseteq I$. □

Proposition IV.45 confirms that tight closure is indeed a closure operation for ideals of a valuation ring (c.f. Remark IV.39).

Corollary IV.46. *Let ν be a valuation of field K of characteristic p. Then for any ideal I of R_ν, I^* is tightly closed.*

Proof. We may assume K is f-adic in the topology induced by ν as otherwise the tight closure of every non-zero ideal of R_ν is R_ν itself (Proposition IV.41), and R_ν is clearly tightly closed. If w is the valuation as defined as in Proposition IV.45, then for any ideal I of R_ν, $I^* = \{x \in R_\nu : w(x) \geq \mathfrak{a}\} \cup \{0\}$, where

$$\mathfrak{a} := \inf\{w(i) : i \in I - \{0\}\}.$$

Clearly $\mathfrak{a}$ is also the infimum of $\{w(j) : j \in I^* - \{0\}\}$, which shows $(I^*)^* = I^*$. □

Corollary IV.47. *If V is a valuation ring of Krull dimension* 1*, then for any ideal I of V, $I^* = I$ or I^* is a principal ideal.*

Proof. The associated valuation of V can be chosen to be real-valued, and so calling this valuation w is consistent with the notation of Proposition IV.45. Let I be an ideal of V, and suppose $\mathfrak{a}$ is the infimum as in Proposition IV.45. If $\mathfrak{a} \in \Gamma_w$, then I^* is the principal ideal generated by any element $x \in V$ such that $w(x) = \alpha$. Otherwise $I^* = I$ by the second assertion of Proposition IV.45. □

Corollary IV.48. *Let ν be a non-trivial valuation of a field K of characteristic p such that K is f-adic in the valuation topology. Let $\mathfrak{p}$ be the height 1 prime of R_ν. Then the following are equivalent:*

1. $\mathfrak{p}^* \neq R_\nu$.
2. $\bigcap_{e \in \mathbb{N}} \mathfrak{p}^{[p^e]} = (0)$.
3. $(R_\nu)_\mathfrak{p}$ *is a discrete valuation ring.*
4. $\mathfrak{p}^* = \mathfrak{p}$.

Proof. (1) is equivalent to (2) by Proposition IV.38, and (1) is equivalent to (3) by Theorem IV.42. Thus it suffices to show that (1) and (4) are equivalent. Clearly (4) implies (1). Conversely, if $\mathfrak{p}^* \neq R_\nu$, then $\mathfrak{p}^*$ cannot properly contain $\mathfrak{p}$. Otherwise the tight closure of $\mathfrak{p}^*$ would equal R_ν by Theorem IV.42(1), contradicting $(\mathfrak{p}^*)^* = \mathfrak{p}^*$. □

4.6 F-regularity of valuation rings

In our discussion of F-regularity in Chapter 3, we introduced the notion of F-pure regularity which mimics the definition of strong F-regularity, but replaces the splitting of certain maps by purity. The present chapter exhibits that purity is a more tractable notion in the non-Noetherian world than splitting. Indeed, valuation rings are always F-pure, while there exist even Noetherian valuation rings that are not Frobenius split. Thus it is natural to focus on F-pure regularity when studying the various variants of F-regularity in the valuative setting.

4.6.1 F-pure regularity and valuations

The main result is:

Theorem IV.49. [DS16] *A valuation ring of characteristic p is F-pure regular if and only if it is Noetherian. Equivalently, a valuation ring is F-pure regular if and only if it is a field or a discrete valuation ring.*

A key ingredient in the proof is the following characterization of the set of elements along which a valuation ring fails to be F-pure (see Definition III.68):

Theorem IV.50. [DS16] *The set of elements c along which a valuation ring $(V, \mathfrak{m}_V, \kappa_V)$ of characteristic p fails to be F-pure is the prime ideal*

$$\bigcap_{e \in \mathbb{N}} \mathfrak{m}_V^{[p^e]}.$$

Proof of Theorem IV.50. Recall $\bigcap_{e \in \mathbb{N}} \mathfrak{m}_V^{[p^e]}$ is a prime ideal because it equals

$$\bigcap_{n \in \mathbb{N}} \mathfrak{m}_V^n$$

(see Lemma II.26), and the latter is prime by Proposition II.21.

Let $\mathcal{I}$ be the set of elements along which V fails to be F-pure[2]. First, take any $c \in \bigcap_{e \in \mathbb{N}} \mathfrak{m}_V^{[p^e]}$. We need to show that V is not F-pure along c, that is, the map

$$\lambda_c^e : V \to F_*^e V \quad \text{sending } 1 \mapsto c$$

is not pure for any e. Because $c \in \mathfrak{m}^{[p^e]}$,

$$\lambda_c^e \otimes \mathrm{id}_{\kappa_V}$$

is the zero map. Therefore λ_c^e is not pure for any e, which means V is not F-pure along c, that is, $c \in \mathcal{I}$.

[2]One can show independently of establishing the equality

$$\mathcal{I} = \bigcap_{e \in \mathbb{N}} \mathfrak{m}_V^{[p^e]}$$

that $\mathcal{I}$ is a prime ideal. Indeed, $\mathcal{I}$ is an ideal of V since $\mathcal{I}$ is closed under multiplication by elements of V (Proposition III.78), and any subset of a valuation ring which is closed under multiplication by elements of the ring is an ideal (Lemma II.17). Moreover, since V is F-pure, $1 \notin \mathcal{I}$ and so $\mathcal{I}$ is a prime ideal by again applying Proposition III.78.

For the other inclusion, let $c \notin \mathfrak{m}^{[p^e]}$ for some $e > 0$. We claim that $\lambda_c^e : V \to F_*^e V$ is pure. Apply Lemma III.29(g) to the set Σ of finitely generated submodules of $F_*^e V$ which contain c. Note Σ is a directed set under inclusion with a least element, namely the V-submodule of $F_*^e V$ generated by c, and $F_*^e V$ is the colimit of the elements of Σ. It suffices to show that if $T \in \Sigma$, then

$$\lambda_T : V \to T \;\; \text{sending} \;\; 1 \mapsto c$$

is pure. But T is free since it is a finitely generated, torsion-free module over a valuation ring (Proposition II.14). Since $c \notin \mathfrak{m}^{[p^e]} = \mathfrak{m}F_*^e V$, by the V module structure on T, we get $c \notin \mathfrak{m}T$. By Nakayama's Lemma, we know c is part of a free basis for T. So λ_T splits, and is pure in particular. $\square$

Remark IV.51. The prime ideal $\bigcap_{e\in\mathbb{N}} \mathfrak{m}_V^{[p^e]}$ is a valuation theoretic analogue of Aberbach and Enescu's splitting prime [AE05] in the Noetherian setting.

Proof of Theorem IV.49. A Noetherian valuation ring is regular and so F-pure regular (Theorem III.73). Conversely, suppose V is F-pure regular. We may assume V is not a field, that is, $\mathfrak{m}_V \neq (0)$. Let

$$\mathfrak{M} := \bigcap_{e\in\mathbb{N}} \mathfrak{m}_V^{[p^e]}.$$

By Theorem IV.50, $\mathfrak{M} = 0$. In particular,

$$\mathfrak{M} \neq \mathfrak{m}_V,$$

and so $V = V/\mathfrak{M}$ is a discrete valuation ring by Corollary II.24. $\square$

Corollary IV.52. [DS16] *For a valuation ring $(V, \mathfrak{m}_V, \kappa_V)$ of characteristic p, let $\mathfrak{M} := \bigcap_{e\in\mathbb{N}} m_V^{[p^e]}$. Then the quotient $V/\mathfrak{M}$ is a F-pure regular valuation ring. Furthermore, V is F-pure regular if and only if $\mathfrak{M}$ is zero.*

Proof. The second statement follows immediately from Theorem IV.50. For the first, observe that $V/\mathfrak{M}$ is a domain since $\mathfrak{M}$ is prime. Thus ideals of $V/\mathfrak{M}$ inherit a total ordering under inclusion from V, and so $V/\mathfrak{M}$ is a valuation ring whose maximal ideal η (which is the image of $\mathfrak{m}_V$) satisfies $\bigcap_{e\in\mathbb{N}} \eta^{[p^e]} = 0$. So $V/\mathfrak{M}$ is F-pure regular by Theorem IV.50. □

Corollary IV.53. [DS16] *For a valuation ring V of prime characteristic, the following are equivalent:*

1. *V is F-pure regular.*
2. *All ideals of V are tightly closed.*
3. *The maximal ideal of V is tightly closed.*
4. *V is Noetherian.*

Proof. The equivalence of (2), (3) and (4) is precisely the content of Corollary IV.44, and (1) and (4) are equivalent by Theorem IV.49. □

Remark IV.54. An outstanding open problem in tight closure theory of Noetherian rings is whether strong F-regularity (more generally F-pure regularity) is equivalent to all ideals being tightly closed, also known as *weak F-regularity.* Corollary IV.53 confirms this conjecture in the setting of valuation rings.

Remark IV.55. Theorem IV.49 indicates that F-regularity is perhaps a useful notion of singularity only for Noetherian rings. Nevertheless, there do exist non-Noetherian rings that are F-pure regular. For example, a polynomial ring in infinitely many variables over $\mathbb{F}_p$ is F-pure regular, but in this example the fraction field is *not* F-finite. Perhaps a reasonable conjecture is that any F-pure regular domain with F-finite fraction field has to be Noetherian.

4.6.2 Split F-regularity

Of course, there is another obvious way[3] to adapt Hochster and Huneke's definition of strong F-regularity to arbitrary rings of prime characteristic p.

Definition IV.56. A ring R is **split F-regular** if for all non-zero divisors c, there exists $e \in \mathbb{N}$ such that the map $R \to F_*^e R$ sending 1 to c splits as a map of R-modules.

Since split maps are pure, a split F-regular ring is F-pure regular. Thus a split F-regular valuation ring *must* be Noetherian. Split F-regular rings are also clearly Frobenius split. On the other hand, Example III.57 shows that a discrete valuation ring need not be Frobenius split, so split F-regularity is strictly stronger than F-pure regularity. In particular, not every regular local ring is split F-regular, so split F-regularity is perhaps not the correct notion of singularity even for Noetherian rings in a non-F-finite setting.

Remark IV.57. Nevertheless, split F-regularity usually coincides with F-pure regularity in geometric situations. For example, if R is an F-pure regular Noetherian domain whose fraction field is F-finite, then the only obstruction to split F-regularity is the splitting of Frobenius. This is a consequence of Corollary III.56, which tells us that R is F-finite if it is Frobenius split, and Theorem III.37, which implies that splitting and purity are the same in F-finite Noetherian rings.

Corollary IV.58. *For a discrete valuation ring V of characteristic p whose fraction field K is F-finite, the following are equivalent:*

(i) V is split F-regular;

(ii) V is Frobenius split;

(iii) V is F-finite;

[3] This generalization is used for cluster algebras in [BMRS15] for example.

(iv) V *is free over* V^p*;*

(v) V *is excellent.*

(vi) $p[\kappa_V : \kappa_V^p] = [K : K^p]$.

(vii) $\dim_{\kappa_V^p} V/\mathfrak{m}_V^p = [K : K^p]$.

Moreover, if K *is a function field over an F-finite ground field* k*, and* V *is a valuation of* K/k*, then (i)-(vii) are equivalent to* V *being a divisorial valuation ring.*

Proof. All this has been proved already. Recall that a discrete valuation ring is a regular local ring, so it is always F-pure regular and hence split F-regular if it is F-finite. Also, the final statement is equivalent to the others by Corollary IV.25. $\square$

CHAPTER V

Uniform approximation of Abhyankar valuation ideals in prime characteristic

We have seen so far that Abhyankar valuations of function fields, which are higher rational rank analogues of divisorial valuations, satisfy many desirable properties. For example, the value group of an Abhyankar valuation is a free abelian group of finite rank and the residue field is a finitely generated extension of the ground field. Moreover, under a mild hypothesis on the residue field, an Abhyankar valuation always admits a local monomialiation in *any* characteristic (Theorem II.69). Using this local monomialization result, we even established in Chapter IV that the valuation rings associated to Abhyankar valuations over perfect ground fields of prime characteristic are always Frobenius split. In this chapter we provide further evidence in favor of the geometric nature of these valuations. We begin by introducing the main result (Theorem V.1) and providing an indication of our strategy of proving it.

5.1 The main result

Let X be a variety over a field k of prime characteristic, with function field K. Suppose ν is a real-valued valuation of K/k centered on X. Then for all $m \in \mathbb{R}$, we have the *valuation ideals*

$$\mathfrak{a}_m(X) \subseteq \mathcal{O}_X,$$

consisting of local sections f such that $\nu(f) \geq m$. When $X = \mathrm{Spec}(A)$, we use $\mathfrak{a}_m(A)$ to denote the ideal $\{a \in A : \nu(a) \geq m\}$ of A.

The goal of this chapter is to use the theory of asymptotic test ideals in positive characteristic to prove the following uniform approximation result for Abhyankar valuation ideals established in the characteristic 0 setting by Ein, Lazarsfeld and Smith [ELS03].

Theorem V.1. *Let X be a regular (equivalently smooth) variety over a perfect field k of prime characteristic with function field K. For any non-trivial, real-valued Abhyankar valuation ν of K/k centered on X, there exists $e \geq 0$, such that for all $m \in \mathbb{R}_{\geq 0}$ and $\ell \in \mathbb{N}$,*

$$\mathfrak{a}_m(X)^\ell \subseteq \mathfrak{a}_{\ell m}(X) \subseteq \mathfrak{a}_{m-e}(X)^\ell.$$

Thus, the theorem says that the valuation ideals $\mathfrak{a}_{\ell m}$ associated to a real-valued Abhyankar valuation are uniformly approximated by powers of $\mathfrak{a}_m$. Thus even though the associated graded ring

$$\bigoplus_{m \in \mathbb{R}} \mathfrak{a}_m$$

is usually very far from being finitely generated, Theorem V.1 provides some measure of control over it.

In [ELS03] (see also [Blu18]), Theorem V.1 is proved over a ground field of characteristic 0 using the machinery of *asymptotic multiplier ideals*, first defined in [ELS01] in order to prove a uniformity statement about symbolic powers of ideals on regular varieties. It has since become clear that in prime characteristic a *test ideal* is an analogue of a multiplier ideal. Introduced by Hochster and Huneke in their work on tight closure [HH90], the first link between test and multiplier ideals was forged by Smith [Smi00] and Hara [Har01], following which Hara and Yoshida introduced the notion of test ideals of pairs [HY03]. Even in the absence of vanishing theorems in positive characteristic, test ideals of pairs were shown to satisfy many of the usual properties of multiplier ideals of pairs that make the latter such an effective tool in birational geometry [HY03, HT04, Tak06] (see also Theorem V.23).

We employ an asymptotic version of the test ideal of a pair to prove Theorem V.1, drawing inspiration from the asymptotic multiplier ideal techniques in [ELS03]. However, instead of utilizing tight closure machinery, our approach to asymptotic test ideals is based on Schwede's dual and more global reformulation of test ideals using p^{-e}*-linear maps*, which are like maps inverse to Frobenius [Sch10, Sch11] (see also [Smi95, LS01]).

Asymptotic test ideals are associated to graded families of ideals (Definition V.25), an example of the latter being the family of valuation ideals $\mathfrak{a}_\bullet \coloneqq \{\mathfrak{a}_m(A)\}_{m\in\mathbb{R}_{\geq 0}}$. For each $m \geq 0$, one constructs the *m-th asymptotic test ideal* $\tau_m(A, \mathfrak{a}_\bullet)$ of the family $\mathfrak{a}_\bullet$, and then Theorem V.1 is deduced using

Theorem V.2. *Let ν be a non-trivial real-valued Abhyankar valuation of K/k, centered on a regular local ring $(A, \mathfrak{m})$, where A is essentially of finite type over the perfect field k of prime characteristic with fraction field K. Then there exists $r \in A-\{0\}$ such that for all $m \in \mathbb{R}_{\geq 0}$,*

$$r \cdot \tau_m(A, \mathfrak{a}_\bullet) \subseteq \mathfrak{a}_m(A).$$

In other words, $\bigcap_{m\in\mathbb{R}_{\geq 0}}(\mathfrak{a}_m : \tau_m(A, \mathfrak{a}_\bullet)) \neq (0)$.

The proof of the characteristic 0 analogue of Theorem V.2 in [ELS03] uses resolution of singularities, which is not known in prime characteristic. For our purpose, Knaf and Kuhlmann's local monomialization of Abhyankar valuations suffices instead. Local monomialization allows us to reduce Theorem V.2 to the case where the valuation ideals are monomial ideals in a polynomial ring, allowing us to use a very concrete characterization of asymptotic test ideals of monomial ideals (Example V.24). Thus we can bypass the otherwise difficult problem of computing asymptotic test ideals of graded families.

Finally, as in [ELS03], Theorem V.2 also gives a new proof of a prime characteristic version of Izumi's theorem for arbitrary real-valued Abhyankar valuations with a common regular center (see also the more general work of [RS14]).

Corollary V.3 (**Izumi's Theorem for Abhyankar valuations in prime characteristic**)**.** *Let ν and w be non-trivial real-valued Abhyankar valuations of K/k, centered on a regular local ring $(A, \mathfrak{m})$, as in Theorem V.2. Then there exists a real number $C > 0$ such that for all $x \in A - \{0\}$,*

$$\nu(x) \leq Cw(x).$$

Thus, Corollary V.3 implies that the valuation topologies on A induced by two non-trivial real-valued Abhyankar valuations are *linearly* equivalent. We also show that Theorem V.1, Theorem V.2 and Corollary V.3 fail in general when the real-valued valuations are *not* Abhyankar (Examples V.39)

5.2 Valuation ideals

We are primarily interested in valuations whose value groups are ordered subgroups of $\mathbb{R}$, a condition that is equivalent to the valuation rings having Krull dimension 1 (Proposition II.12). For any such real-valued valuation ν with *center* x on X and any $m \in \mathbb{R}$, one has the **valuation ideal** $\mathfrak{a}_m(X) \subseteq \mathcal{O}_X$, where locally

$$\Gamma(U, \mathfrak{a}_m(X)) = \begin{cases} \{f \in \mathcal{O}_X(U) : \nu(f) \geq m\}, & \text{if } x \in U, \\ \mathcal{O}_X(U), & \text{if } x \notin U. \end{cases}$$

Note $\mathfrak{a}_m(X) = \mathcal{O}_X$ when $m \leq 0$. If $X = \operatorname{Spec}(A)$, we use $\mathfrak{a}_m(A)$ to denote the ideal $\{a \in A : \nu(a) \geq m\}$ of A, and when X or A is clear from context, we just write $\mathfrak{a}_m$.

An important feature of valuation ideals implicitly used in the rest of the chapter is the following:

Lemma V.4. [Dat17b] *Given an affine variety* $\operatorname{Spec}(A)$*, if* $\mathfrak{p}$ *is the prime ideal of* A *corresponding to the center of a real-valued valuation* ν *on* $\operatorname{Spec}(A)$*, then for all real numbers* $m > 0$*, the ideal* $\mathfrak{a}_m(A)$ *is* $\mathfrak{p}$*-primary. Moreover,* $\mathfrak{a}_m(A_\mathfrak{p}) = \mathfrak{a}_m(A)A_\mathfrak{p}$*.*

Proof. For $b \in A$, if $\nu(b) > 0$, then by the Archimedean property, $n\nu(b) = \nu(b^n) \geq m$, for some $n \in \mathbb{N}$. This shows that $\mathfrak{p}$ is the radical of $\mathfrak{a}_m(A)$. If $ab \in \mathfrak{a}_m(A)$ and $a \notin \mathfrak{a}_m(A)$, then $\nu(b) > 0$, so that for some n, $b^n \in \mathfrak{a}_m(A)$, as we just showed. Hence $\mathfrak{a}_m(A)$ is $\mathfrak{p}$-primary.

Note if $s \notin A - \mathfrak{p}$, $\nu(s) = 0$. Thus, the inclusion $\mathfrak{a}_m(A)A_\mathfrak{p} \subseteq \mathfrak{a}_m(A_\mathfrak{p})$ is clear. Conversely, if $a/s \in \mathfrak{a}_m(A_\mathfrak{p})$, since $\nu(a/s) = \nu(a) - \nu(s) = \nu(a)$, we get $a \in \mathfrak{a}_m(A)$, proving $\mathfrak{a}_m(A_\mathfrak{p}) \subseteq \mathfrak{a}_m(A)A_\mathfrak{p}$. □

Remark V.5. The argument in Lemma V.4 can be easily modified to see that valuation ideals are quasicoherent. One can extend the definition of valuation ideals to valuations that are not necessarily real-valued. However, when the Archimedean property of real numbers does not hold for the value group, these ideal sheaves may no longer be quasicoherent.

We now show that as a consequence of local monomialization of Abhyankar valuations (Theorem II.69) one can always choose a regular center of any real-valued Abhyankar valuation whose valuation ideals are monomial in an appropriate sense.

Theorem V.6. [DS17b] *Assume* k *is perfect, and* ν *is a non-trivial, real-valued Abhyankar valuation of* K/k *of rational rank* d*, centered on an affine variety* $\operatorname{Spec}(R)$

of K/k. Then there exists an affine variety $\mathrm{Spec}(S)$ *of K/k, along with an inclusion of rings $R \hookrightarrow S$ such that*

1. *S is regular and ν is centered at a point $x \in \mathrm{Spec}(S)$ d of codimension d.*

2. *The valuation ideals of $\mathcal{O}_{\mathrm{Spec}(S),x}$ are generated by monomials in a regular system of parameters of $\mathcal{O}_{\mathrm{Spec}(S),x}$.*

Proof. Recall that our hypotheses imply that the value group Γ_ν is a free abelian group of rank d. By Corollary II.71, there exists S satisfying (1) and a regular system of parameters $\{x_1, \dots, x_d\}$ of $\mathcal{O}_{\mathrm{Spec}(S),x}$ such that $\nu(x_1), \dots, \nu(x_d)$ freely generate Γ_ν. Note this implies that distinct monomials in $x_1, \dots, x_d$ have distinct valuations.

Suppose $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_{\mathrm{Spec}(S),x}$. We want to show that the valuation ideals $\mathfrak{a}_m$ of $\mathcal{O}_{\mathrm{Spec}(S),x}$ are monomial in $\{x_1, \dots, x_d\}$. For $m > 0$, since $\mathfrak{a}_m$ is $\mathfrak{p}$-primary, we know that $\mathfrak{p}^n \subseteq \mathfrak{a}_m$ for some $n \in \mathbb{N}$. Note $\mathfrak{p}^n$ has a monomial generating set $\{x_1^{\alpha_1} \dots x_d^{\alpha_d} : \alpha_1 + \dots + \alpha_d = n\}$. Modulo $\mathfrak{p}^n$, any non-zero element $t \in \mathfrak{a}_m$ can be expressed as a finite sum s of monomials of the form $x_1^{\beta_1} \dots x_d^{\beta_d}$, with

$$0 < \beta_1 + \dots + \beta_d \leq n - 1,$$

and where the coefficients of the monomials are units in $\mathcal{O}_{\mathrm{Spec}(S),x}$. Then expressing

$$t = s + u,$$

for $u \in \mathfrak{p}^n$, we see that

$$\nu(s) \geq m$$

because $\nu(t), \nu(u) \geq m$. However, $\nu(s)$ equals the smallest valuation of the monomials $x_1^{\beta_1} \dots x_d^{\beta_d}$ appearing in the sum since monomials have distinct valuations. Thus, each such $x_1^{\beta_1} \dots x_d^{\beta_d} \in \mathfrak{a}_m$, completing the proof. $\square$

Example V.7. Let ν_π be the valuation on $\mathbb{F}_p(X,Y,Z)/\mathbb{F}_p$ with value group $\mathbb{Z}\oplus\mathbb{Z}\pi\subset\mathbb{R}$ such that $\nu_\pi(X)=1=\nu_\pi(Y)$, $\nu_\pi(Z)=\pi$, and for any polynomial $\sum b_{\alpha\beta\gamma}X^\alpha Y^\beta Z^\gamma\in\mathbb{F}_p[X,Y,Z]$,

$$\nu_\pi(\sum b_{\alpha\beta\gamma}X^\alpha Y^\beta Z^\gamma)=\inf\{\alpha+\beta+\pi\gamma : b_{\alpha\beta\gamma}\neq 0\}.$$

One can verify that ν_π is Abhyankar with $\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma_{\nu_\pi}=2$ and $\operatorname{tr.deg}\nu_\pi=1$. For example Y/X is a unit in the valuation ring R_{ν_π} whose image in the residue field is transcendental over $\mathbb{F}_p$. Note ν_π is centered on $\mathbb{A}^3_{\mathbb{F}_p}=\operatorname{Spec}(\mathbb{F}_p(X,Y,Z))$ at the origin. However, the system of parameters X,Y,Z of the local ring at the origin do not freely the generate the value group. On the other hand, blowing up the origin and considering the affine chart $\operatorname{Spec}(\mathbb{F}_p[X,\frac{Y}{X},\frac{Z}{X}])$, we see that ν_π is centered on $\mathbb{F}_p[X,\frac{Y}{X},\frac{Z}{X}]$ with center $(X,Z/X)$, and now the regular system of parameters $X,Z/X$ of the local ring $\mathbb{F}_p[X,\frac{Y}{X},\frac{Z}{X}]_{(X,\frac{Z}{X})}$ do indeed freely generate the value group. Thus the valuation ideals of $\mathbb{F}_p[X,\frac{Y}{X},\frac{Z}{X}]_{(X,\frac{Z}{X})}$ are monomials in X and Z/X.

5.3 Test Ideals

Beginning with a review of test ideals, the goal is to construct an asymptotic version that plays a role similar to asymptotic multiplier ideals in characteristic 0. We also examine how asymptotic test ideals transform under étale and birational ring maps. We will work with a dual reformulation of the theory of test ideals due to Schwede using p^{-e}-linear maps, rather than using tight closure theory. An excellent source describing this dual approach to test ideals and its relation to tight closure theory is the survey [ST12].

5.3.1 Uniformly F-compatible ideals

The construction of test ideals is based on the existence of certain distinguished ideals that are 'compatible' with respect to p^{-e}-linear maps in the following sense:

Definition V.8. Let R be a ring of characteristic p and $\varphi : F^e_* R \to R$ a p^{-e}-linear map. An ideal I of R is **φ-compatible** if

$$\varphi(F^e_*(I)) \subseteq I.$$

In other words, φ maps elements of J back into J, or equivalent, φ induces a p^{-e}-linear map

$$\overline{\varphi} : F^e_*(R/I) \to R/I$$

such that the following diagram commutes

$$\begin{tikzcd} F^e_* R \arrow[r, "\varphi"] \arrow[d, "F^e_*(\pi)"] & R \arrow[d, "\pi"] \\ F^e_*(R/I) \arrow[r, "\overline{\varphi}"] & R/I \end{tikzcd}$$

where the vertical maps are the obvious projections. An ideal $I \subseteq R$ is **uniformly F-compatible** if for all $e \in \mathbb{N}$ and for all $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$,

$$\varphi(F^e_*(I)) \subseteq I.$$

We collect some basic properties of compatible ideals.

Proposition V.9. *Let R be a ring of characteristic p and $\varphi : F^e_* R \to R$ an R-linear map, for $e > 0$.*

1. *Arbitrary sums and intersections of φ-compatible ideals are φ-compatible.*

2. *Finite products of φ-compatible ideals are φ-compatible.*

3. *If I is a φ-compatible ideal, then any prime ideal associated to I (i.e. an element of $Ass_R(R/I)$) is uniformly φ-compatible.*

4. If R is Noetherian and I is a φ-compatible ideal, then so is its radical $\sqrt{I}$.

All the above properties hold when φ-compatible is replaced by uniformly F-compatible.

Proof. Properties (1) and (2) are clearly from the definition of φ-compatibility and the fact that φ is an additive map. For (3), suppose $\mathfrak{p}$ is an associated prime of I. Then there exists an element $a \notin I$ such that

$$(I : a) = \mathfrak{p}.$$

Thus

$$I \supseteq \varphi(F^e_*(I)) \supseteq \varphi(F^e_*(a^{p^e}\mathfrak{p})) = a\varphi(F^e_*(\mathfrak{p})),$$

and so $\varphi(F^e_*(\mathfrak{p})) \subseteq (I : a) = \mathfrak{p}$, as desired. Assertion (4) follows from (1) and (3) because in the Noetherian case $\sqrt{I}$ is the intersection of the prime ideals associated to I.

Finally all four assertions also hold for uniformly F-compatible ideals because φ is an arbitrary p^{-e}-linear map in this proposition. □

Lemma V.10. *Let R be a Frobenius split ring. Then any ideal of R which is compatible with respect to a Frobenius splitting is a radical ideal. Hence all uniformly F-compatible ideals of a Frobenius split ring are radical.*

Proof. Let $\varphi : F_*R \to R$ be a Frobenius splitting, and I be an ideal of R which is φ-compatible. Then φ induces a Frobenius splitting

$$\overline{\varphi} : F^e_*(R/I) \to R/I$$

of R/I which means that the Frobenius map of R/I is injective. Thus R/I is reduced, and so I is a radical ideal. The second assertion follows easily from the first and the definition of uniform F-compatibility. □

Remark V.11. If R is a split F-regular domain, then the only uniformly F-compatible ideals of R are the zero ideal and the unit ideal. Indeed if I is uniformly F-compatible and non-zero, then for an non-zero element $i \in I$, there exists $\varphi : F^e_* R \to R$ that maps $i \mapsto 1$. Then $I \supseteq \varphi(F^e_*(I)) = R$.

5.3.2 F-compatible ideals in valuation rings

Drawing inspiration from our considerations in Chapter IV, let us try to figure out which ideals of valuation rings are uniformly F-compatible. In fact, Lemma V.10 immediately implies

Proposition V.12. *If V is a Frobenius split valuation ring, then any ideal of V which is compatible with respect to a Frobenius splitting is prime ideal. In particular, all uniformly F-compatible ideals of V are prime or the whole ring.*

Proof. Apply Lemma V.10 along with the fact that radical ideals of valuation rings are prime or the whole ring. □

Corollary V.13. *Let K/k be a function field over an F-finite ground field k, and ν be an Abhyankar valuation of K/k such that κ_ν is separable over k. Then any uniformly F-compatible ideal of R_ν is a prime ideal or the whole ring.*

Proof. R_ν is Frobenius split by Theorem IV.30 and so we may apply the previous proposition. □

For an ideal $\mathfrak{a}$ of a valuation ring V, the ideal

$$\bigcap_{e \in \mathbb{N}} \mathfrak{a}^{[p^e]}$$

featured prominently during our investigation of tight closure and F-regularity for valuation rings in the previous chapter. Unsurprisingly, these intersections are also related to uniform F-compatibility.

Lemma V.14. *Let R be a ring of characteristic p and $\mathfrak{a}$ be an ideal of R. Then the ideal*

$$\bigcap_{e\in\mathbb{N}} \mathfrak{a}^{[p^e]}$$

is uniformly F-compatible.

Proof. Let φ be a p^{-e}-linear map. Then for any $f > 0$,

$$\varphi(F^e_*(\mathfrak{a}^{[p^{e+f}]})) \subseteq \mathfrak{a}^{[p^f]}$$

because $\mathfrak{a}^{[p^{e+f}]}$ is generated by elements of the form $(x^{p^f})^{p^e}$, for $x \in \mathfrak{a}$, and

$$\varphi((x^{p^f})^{p^e}) = x^{p^f} \in \mathfrak{a}^{[p^f]}.$$

Therefore,

$$\varphi\big(F^e_*(\bigcap_{f\in\mathbb{N}} \mathfrak{a}^{[p^f]})\big) = \varphi\big(F^e_*(\bigcap_{f\in\mathbb{N}} \mathfrak{a}^{[p^{e+f}]})\big) \subseteq \bigcap_{f\in\mathbb{N}} \mathfrak{a}^{[p^f]},$$

as desired. □

Theorem V.15. *If $(V, \mathfrak{m}_V, \kappa_V)$ is a valuation ring of characteristic p, then any non-maximal prime ideal of V is uniformly F-compatible. In addition, $\mathfrak{m}_V$ is uniformly F-compatible when it is not finitely generated.*

Proof. Let $\mathfrak{p}$ be a non-maximal prime ideal of V and define

$$\Sigma := \{a \in V : a \in \mathfrak{m}_V - \mathfrak{p}\}.$$

Note Σ is non-empty because $\mathfrak{p}$ is not maximal. We claim that

$$\mathfrak{p} = \bigcap_{a\in\Sigma} \bigcap_{e\in\mathbb{N}} a^{p^e} V.$$

First observe that the claim shows that $\mathfrak{p}$ is uniformly F-compatible since $\bigcap_{e\in\mathbb{N}} a^{p^e} V$ is uniformly F-compatible (Lemma V.14), and arbitrary intersections of uniformly F-compatible ideals are uniformly F-compatible by Proposition V.9(1).

If $a \notin \mathfrak{p}$, then for all $e \in \mathbb{N}$, $a^{p^e} \notin \mathfrak{p}$. By comparability of ideals of a valuation ring, it follows that $\mathfrak{p} \subseteq \bigcap_{e \in \mathbb{N}} a^{p^e} V$, and hence,

$$\mathfrak{p} \subseteq \bigcap_{a \in \Sigma} \bigcap_{e \in \mathbb{N}} a^{p^e} V.$$

The ideal

$$\bigcap_{a \in \Sigma} \bigcap_{e \in \mathbb{N}} a^{p^e} V$$

is prime because it is an intersection of the prime ideals $\bigcap_{e \in \mathbb{N}} a^{p^e} V$ (Proposition II.21). To finish the proof of the claim it suffices to show that for any prime ideal $\mathfrak{q}$ such that $\mathfrak{p} \subsetneq \mathfrak{q}$, we have

$$\mathfrak{q} \neq \bigcap_{a \in \Sigma} \bigcap_{e \in \mathbb{N}} a^{p^e} V. \tag{5.1}$$

Now by hypothesis, there exists $a \in \mathfrak{q} - \mathfrak{p}$. Thus $a \in \Sigma$, and in order to establish (5.1) it is enough to prove that

$$\bigcap_{e \in \mathbb{N}} a^{p^e} V \neq \mathfrak{q}.$$

If $\mathfrak{q}$ is not generated by a then this is obvious. If $\mathfrak{q}$ is generated by a (this is impossible unless $\mathfrak{q}$ is the maximal ideal by Lemma II.18), then

$$a^p V \neq aV = \mathfrak{q},$$

and we are again done.

For the second assertion of the theorem, if $\mathfrak{m}_V$ is not finitely generated, then for all $n \in \mathbb{N}$,

$$\mathfrak{m}_V^{[n]} = \mathfrak{m}_V^n = \mathfrak{m}_V,$$

where the first equality follows from Lemma II.26 and the second equality from Proposition II.19 (see also Notation II.25 for the meaning of $\mathfrak{m}_V^{[n]}$) . Thus,

$$\mathfrak{m}_V = \bigcap_{e \in \mathbb{N}} \mathfrak{m}_V^{p^e} = \bigcap_{e \in \mathbb{N}} \mathfrak{m}_V^{[p^e]}$$

is uniformly F-compatible using Lemma V.14. $\square$

5.3.3 Absolute test ideals and test ideals of pairs

From now we assume rings are Noetherian and F-finite. Without these hypotheses it is not clear if test ideals, as we will define them, exist (Remark V.22). For simplicity all definitions will be made for domains, since this is the only setting we will need. Moreover, although test ideals can be patched affine locally to give global test ideals of Noetherian, F-finite schemes, we will work exclusively in the affine setting.

The notion of the (absolute) test ideal of a Noetherian, F-finite domain ties in naturally with our discussion of F-compatible ideals in the previous subsection.

Definition V.16. If R is a Noetherian, F-finite domain of characteristic p, then the **(absolute) test ideal of** R, denoted $\tau(R)$, is the unique minimal element (with respect to inclusion) of the collection of non-zero, uniformly F-compatible ideals of R.

It is *not* obvious why the collection of non-zero, uniformly F-compatible ideals of R has a *unique* minimal element with respect to inclusion. The existence of this minimal element is a consequence of a deep result of Hochster and Huneke on the existence of (completely stable) test elements in tight closure theory [HH94, Theorem 5.10] (see also [ST12] and Theorem V.21).

Before introducing completely stable test elements, we define the more general notion of test ideals of pairs. The absolute test ideal $\tau(R)$ can be interpreted as a test ideal of a suitable pair.

Definition V.17. Let R be an F-finite Noetherian domain, $\mathfrak{a} \subseteq R$ a non-zero ideal, and $t > 0$ a real number. The test ideal[1] of the **pair** $(R, \mathfrak{a}^t)$ is defined to be the

[1] In tight closure literature, this is usually called the *big* or *non-finitistic* test ideal of the pair $(R, \mathfrak{a}^t)$.

smallest *non-zero* ideal I of R such that for all $e \in \mathbb{N}$, and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$,

$$\phi(F^e_*(I\mathfrak{a}^{\lceil t(p^e-1)\rceil})) \subseteq I.$$

It is denoted $\tau(R, \mathfrak{a}^t)$, or $\tau(\mathfrak{a}^t)$ when R is clear from context.

Remark V.18. The absolute test ideal $\tau(R)$ is the test ideal of the pair (R, R).

We now explain why test ideals of pairs (hence absolute test ideals) exist in our setting.

Definition V.19. [TW15, Definition 5.1] A non-zero element c of a domain R of characteristic p is called a **test element**[2] if for all non-zero $d \in R$, there exists $e \in \mathbb{N}$, and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, such that $\phi(F^e_*(d)) = c$.

The following result demonstrates that test elements exist in geometric settings.

Proposition V.20. *Let R be a Noetherian F-finite domain of characteristic p.*

1. *Suppose c is a non-zero element of R such that the localization R_c is strongly F-regular*[3]*. Then some power of c is a test element. Thus, test elements always exist for F-finite Noetherian domains.*

2. *If R is essentially of finite type over an F-finite field k, and $\mathfrak{J}(R/k)$ is the Jacobian ideal of R, then every non-zero element of $\mathfrak{J}(R/k)$ is a test element.*

Indication of proof. (1) follows from the proof of [HH89, Theorem 3.4] and (2) from [Hoc04, Corollary 8.2]. □

Armed with the knowledge that test elements exist for Noetherian F-finite domains, one can verify that test ideals of pairs exist via the following observation:

[2]This is usually called a *completely stable test element* in tight closure literature.

[3]Such a c always exists since the regular locus of R is non-empty (R is generically regular) and open by Proposition III.11.

Theorem V.21. *Let R be a F-finite Noetherian domain of characteristic p. If $c \in R$ is a test element, then*

$$\tau(R, \mathfrak{a}^t) = \sum_{e \in \mathbb{N}} \sum_{\phi} \phi(F^e_*(c\mathfrak{a}^{\lceil t(p^e-1) \rceil})),$$

where ϕ ranges over all elements of $\operatorname{Hom}_R(F^e_* R, R)$.

Proof. See [HT04, Lemma 2.1]. □

Remark V.22. Definition V.17 is a reformulation, due to Schwede, of a notion that was originally defined via tight closure theory. Despite the myriad applications of test ideals defined via F-compatible ideals, it should be emphasized that Schwede's approach relies crucially on the existence of non-trivial p^{-e}-linear maps. While such maps are always guaranteed in the F-finite setting, the author's work with Karen Smith (see Section 3.6) demonstrates that we cannot expect to develop a theory of test ideals for non-excellent rings that uses the ideas of uniform F-compatibility.

5.3.4 Properties of test ideals of pairs

Having addressed the issue of the existence of test ideals, we now collect most of their basic properties, in part to highlight their similarity with multiplier ideals.

Theorem V.23. *Suppose R is an F-finite Noetherian domain of characteristic p with non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$. Let $t > 0$ be a real number.*

1. *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^t) \subseteq \tau(\mathfrak{b}^t)$.*

2. *If the integral closures of $\mathfrak{a}$ and $\mathfrak{b}$ coincide, then $\tau(\mathfrak{a}^t) = \tau(\mathfrak{b}^t)$.*

3. *If $s > t$, then $\tau(\mathfrak{a}^s) \subseteq \tau(\mathfrak{a}^t)$.*

4. *For any $m \in \mathbb{N}$, $\tau((\mathfrak{a}^m)^t) = \tau(\mathfrak{a}^{mt})$.*

5. *There exists some $\epsilon > 0$ depending on t such that for all $s \in [t, t+\epsilon]$, $\tau(\mathfrak{a}^s) = \tau(\mathfrak{a}^t)$.*

6. *$\tau(R)$ defines the closed locus of prime ideals $\mathfrak{p}$ such that $R_{\mathfrak{p}}$ is not strongly F-regular. Thus, $\tau(R) = R$ if and only if R is strongly F-regular.*

7. *We have $\tau(R)\mathfrak{a} \subseteq \tau(\mathfrak{a})$. Hence, if R is strongly F-regular (in particular regular), $\mathfrak{a} \subseteq \tau(\mathfrak{a})$.*

8. *If $W \subset R$ is a multiplicative set, then $\tau(W^{-1}R, (\mathfrak{a}W^{-1}R)^t) = \tau(R, \mathfrak{a}^t)W^{-1}R$.*

9. *If $(R, \mathfrak{m})$ is local and $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, then $\tau(\widehat{R}, (\mathfrak{a}\widehat{R})^t) = \tau(R, \mathfrak{a}^t)\widehat{R}$.*

10. ***(Briançon–Skoda)** If R is regular and $\mathfrak{a}$ can be generated by r elements, then for all integers $m \geq r$, $\tau(\mathfrak{a}^m) = \mathfrak{a}\tau(\mathfrak{a}^{m-1})$.*

11. *If R is regular, $x \in R$ a regular parameter and $\overline{R} := R/xR$, then $\tau(\overline{R}, (\mathfrak{a}\overline{R})^t) \subseteq \tau(R, \mathfrak{a}^t)\overline{R}$.*

12. ***(Subadditivity)** If R is regular and essentially of finite type over a perfect field, then for all $n \in \mathbb{N}$, $\tau(\mathfrak{a}^{nt}) \subseteq \tau(\mathfrak{a}^t)^n$.*

Indication of proof. For proofs and precise references for all statements, please consult [ST12, Section 6], or [SZ15, Theorem 4.6] when the ring is regular (the setting of this paper). □

Example V.24 (**Test ideals of monomial ideals**). Let $\mathfrak{a}$ be a non-zero monomial ideal of the polynomial ring $R = k[x_1, \dots, x_n]$, where k is an F-finite field characteristic $p > 0$. For any real number $t > 0$, consider the convex hull $P(t\mathfrak{a})$ in $\mathbb{R}^n$ of the set

$$\{(ta_1, \dots, ta_n) \colon x_1^{a_1} \dots x_n^{a_n} \in \mathfrak{a}\},$$

and let $\operatorname{Int}(P(t\mathfrak{a}))$ be the points in the topological interior of this convex hull. Then Hara and Yoshida show [HY03, Theorem 4.8] that test and multiplier ideals of $\mathfrak{a}$ coincide, and so using a computation by Howald [How01]

$$\tau(\mathfrak{a}^t) = \langle x_1^{b_1} \dots x_n^{b_n} : b_i \in \mathbb{N} \cup \{0\}, (b_1+1, \dots, b_n+1) \in \operatorname{Int}(P(t\mathfrak{a}))\rangle.$$

5.3.5 Asymptotic test ideals

Asymptotic test ideals are defined for graded families of ideals, which we introduce first.

Definition V.25. Let Φ be an additive sub-semigroup of $\mathbb{R}$, and R be a ring. A **graded family of ideals of R indexed by Φ** is a family of ideals $\{\mathfrak{a}_s\}_{s\in\Phi}$ such that for all $s, t \in \Phi$,

$$\mathfrak{a}_s \cdot \mathfrak{a}_t \subseteq \mathfrak{a}_{s+t}.$$

We also assume $\mathfrak{a}_s \neq 0$, for all s.

Examples V.26.

1. If $\mathfrak{a}$ is a non-zero ideal of a domain R, then $\{\mathfrak{a}^n\}_{n\in\mathbb{N}\cup\{0\}}$ is a graded family of ideals.

2. If R is a Noetherian domain, the symbolic powers $\{\mathfrak{a}^{(n)}\}_{n\in\mathbb{N}\cup\{0\}}$ of a fixed non-zero ideal $\mathfrak{a}$ is an example of a graded family that was studied extensively in [ELS01, HH02].

3. Let ν be a non-trivial real-valued valuation of K/k centered on a domain R over k with fraction field K. Then the collection of valuation ideals $\{\mathfrak{a}_m(R)\}_{m\in\mathbb{R}_{\geq 0}}$ is a graded family of ideals by properties of a valuation (since v is non-trivial, the ideals $\mathfrak{a}_m$ are all non-zero).

Now suppose R is an F-finite Noetherian domain of characteristic p, and $\{\mathfrak{a}_m\}_{m\in\Phi}$ is a graded family of ideals of R indexed by some sub-semigroup Φ of $\mathbb{R}$. Then for any real number $t > 0$, $m \in \Phi$ and $\ell \in \mathbb{N}$, we have

$$\tau(\mathfrak{a}_m^t) = \tau((\mathfrak{a}_m^\ell)^{t/\ell}) \subseteq \tau(\mathfrak{a}_{\ell m}^{t/\ell}).$$

Here the first equality follows from Theorem V.23(4), and the inclusion follows from Theorem V.23(1) using the fact that $\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m}$.

Thus, for a fixed $m \in \Phi$, the set $\{\tau(\mathfrak{a}_{\ell m}^{1/\ell})\}_{\ell\in\mathbb{N}}$ is filtered under inclusion ($\tau(\mathfrak{a}_{\ell_1 m}^{1/\ell_1})$ and $\tau(\mathfrak{a}_{\ell_2 m}^{1/\ell_2})$ are both contained in $\tau(\mathfrak{a}_{\ell_1\ell_2 m}^{1/\ell_1\ell_2})$). Since R is a Noetherian ring, this implies that $\{\tau(\mathfrak{a}_{\ell m}^{1/\ell})\}_{\ell\in\mathbb{N}}$ has a unique maximal element under inclusion, which will be the m-th asymptotic test ideal.

Definition V.27. For a graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\Phi}$ of an F-finite Noetherian domain R of characteristic p, and for any $m \in \Phi$, we define the **m-th asymptotic test ideal of the graded system**, denoted $\tau_m(R, \mathfrak{a}_\bullet)$ (or $\tau_m(\mathfrak{a}_\bullet)$ when R is clear from context), as follows:

$$\tau_m(R, \mathfrak{a}_\bullet) := \sum_{\ell\in\mathbb{N}} \tau(\mathfrak{a}_{\ell m}^{1/\ell}).$$

By the above discussion, $\tau_m(R, \mathfrak{a}_\bullet)$ equals $\tau(\mathfrak{a}_{\ell m}^{1/\ell})$ for a *sufficiently divisible* $\ell \gg 0$.

Asymptotic test ideals satisfy appropriate analogues of properties satisfied by test ideals of pairs (Theorem V.23), since they equal test ideals of suitable pairs. We highlight a few properties that will be important for us in the sequel.

Proposition V.28. [Har05, SZ15] *Suppose R is a regular domain, essentially of finite type over a perfect field of characteristic p, with a graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\Phi}$.*

1. *For any $m \in \Phi$, $\mathfrak{a}_m \subseteq \tau(\mathfrak{a}_m) \subseteq \tau_m(\mathfrak{a}_\bullet)$.*

2. For any $m \in \Phi$ and $\ell \in \mathbb{N}$, $\mathfrak{a}_{\ell m} \subseteq \tau_{\ell m}(\mathfrak{a}_\bullet) \subseteq \tau_m(\mathfrak{a}_\bullet)^\ell$.

Proof. We get (1) using Theorem V.23(7), and the definition of asymptotic test ideals.

Property (2) is crucial, and is a consequence of the subadditivity property of test ideals (Theorem V.23(11)). The first inclusion $\mathfrak{a}_{\ell m} \subseteq \tau_{\ell m}(\mathfrak{a}_\bullet)$ follows from (1). For the second inclusion, for a sufficiently divisible $n \gg 0$, we have

$$\tau_{\ell m}(\mathfrak{a}_\bullet) = \tau(\mathfrak{a}_{n\ell m}^{1/n}) = \tau(\mathfrak{a}_{n\ell m}^{\ell/n\ell}),$$

and by subadditivity, $\tau(\mathfrak{a}_{n\ell m}^{\ell/n\ell}) \subseteq \tau(\mathfrak{a}_{n\ell m}^{1/n\ell})^\ell$. But if n is sufficiently divisible, $\tau(\mathfrak{a}_{n\ell m}^{1/n\ell})^\ell = \tau_m(\mathfrak{a}_\bullet)^\ell$, completing the proof. $\square$

5.3.6 (Asymptotic) test ideals and étale maps

We study a transformation law for test ideals under essentially étale maps. Recall that an *essentially étale map of rings $A \to B$* is a formally étale map [DG64, IV_0, Définition 19.10.2] such that B is a localization of a finitely presented A-algebra. Formally étale maps of *Noetherian* rings are automatically flat [DG64, IV_0, Théorème 19.7.1]. The main example of essentially étale maps for us will be a local homomorphism of Noetherian local rings $\varphi : (A, \mathfrak{m}_A, \kappa_A) \to (B, \mathfrak{m}_B, \kappa_B)$ that is flat, unramified ($\mathfrak{m}_A B = \mathfrak{m}_B$, $\kappa_A \hookrightarrow \kappa_B$ is finite separable), and essentially of finite type. Such a φ is essentially étale by [Sta18, Tag 025B].

Proposition V.29. [Stä16] *Let R be a Gorenstein (in particular, regular) domain essentially of finite type over an F-finite field. If $R \to S$ is an essentially étale map, then for any non-zero ideal $\mathfrak{a}$ of R and a real number $t > 0$,*

$$\tau(S, (\mathfrak{a}S)^t) = \tau(R, \mathfrak{a}^t)S.$$

Indication of proof. Note $R \to S$ is injective since R is a domain and $R \to S$ is flat. Therefore $\mathfrak{a}S$ is a non-zero ideal of S, and $\tau((\mathfrak{a}S)^t)$ makes sense. Now for a proof, see [Stä16, Corollary 6.19], where the result is stated in terms of Cartier algebras. $\square$

A key point in the proof of [Stä16, Corollary 6.19] is the fact that for an essentially étale map of rings $A \to B$ of characteristic p, the functor F_*^e commutes with base change. Although this fact is well-known, in F-singularity literature it is often stated with restrictive hypotheses on A on B that are not needed. Thus, we include a proof here of the general version.

Lemma V.30. *Let $A \to B$ be an essentially étale map of rings of characteristic p (A, B are not necessarily Noetherian). Then the relative Frobenius map*

$$F_{B/A} : F_*^e A \otimes_A B \to F_*^e B. \tag{5.2}$$

is an isomorphism.

Proof. The isomorphism (5.2) is well-known when $A \to B$ is étale [Gro77, XV, Proposition 2(c)(2)]. Since we know F_*^e commutes with localization, (5.2) will follow when B is an essentially étale A-algebra if one can show that B is a localization of an étale A-algebra. Let C be a finitely presented A-algebra, and $S \subset C$ a multiplicative set such that

$$B = S^{-1}C.$$

Since $0 = \Omega_{B/A} = S^{-1}\Omega_{C/A}$ and C is finitely presented, there exists $f \in S$ such that

$$\Omega_{C[1/f]/A} = f^{-1}\Omega_{C/A} = 0,$$

that is, $C[1/f]$ is an unramified A-algebra.

For any prime ideal $\mathfrak{q}$ of C that does not intersect S, we know that $C_{\mathfrak{q}} = (S^{-1}C)_{S^{-1}\mathfrak{q}}$ is formally smooth over A. Then the Jacobian criterion of local smoothness shows that there exists

$$g_{\mathfrak{q}} \in C - \mathfrak{q}$$

such that $C[1/g_{\mathfrak{q}}]$ is a smooth A-algebra. Here the main point is that formal smoothness of $C_{\mathfrak{q}}$ ensures $\Omega_{C_{\mathfrak{q}}/A}$ is free of the 'correct' rank for a presentation of C (see for example [Hoc07, Theorem on pg. 33]). Since $\{g_{\mathfrak{q}} : \mathfrak{q} \cap S = \emptyset\}$ generates the unit ideal in $S^{-1}C$, there is some $h \in S$ such that

$$h \in \sum_{\mathfrak{q} \cap S = \emptyset} g_{\mathfrak{q}} C.$$

Then $D(h) \subset \operatorname{Spec}(C)$ is smooth on an open cover, and so, $C[1/h]$ is a smooth A-algebra. This shows

$$C[1/fh]$$

is an étale A-algebra, and because B is a further localization of $C[1/fh]$, we are done. $\square$

Proposition V.29 has the following consequence for asymptotic test ideals:

Corollary V.31. *Let $R \xrightarrow{\varphi} S$ be an essentially étale map, where R is a Gorenstein (in particular, regular) domain, essentially of finite type over an F-finite field. Suppose $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \Phi}$ is a graded family of non-zero ideals of R, and consider the family $\mathfrak{a}_\bullet S = \{\mathfrak{a}_m S\}_{m \in \Phi}$.*

1. *For all $m \in \Phi$, $\tau_m(S, \mathfrak{a}_\bullet S) = \tau_m(R, \mathfrak{a}_\bullet) S$.*

2. *If $\bigcap_{m \in \Phi} (\mathfrak{a}_m : \tau_m(R, \mathfrak{a}_\bullet)) \neq (0)$, then $\bigcap_{m \in \Phi} (\mathfrak{a}_m S : \tau_m(S, \mathfrak{a}_\bullet S)) \neq (0)$.*

Proof. Again, by the injectivity of φ, $\mathfrak{a}_\bullet S$ is a graded family of non-zero ideals of S. Then

$$\tau_m(S, \mathfrak{a}_\bullet S) \coloneqq \sum_{\ell \in \mathbb{N}} \tau\big((\mathfrak{a}_{\ell m} S)^{1/\ell}\big) = \sum_{\ell \in \mathbb{N}} \tau(\mathfrak{a}_{\ell m}^{1/\ell}) S = \left(\sum_{\ell \in \mathbb{N}} \tau(\mathfrak{a}_{\ell m}^{1/\ell}) \right) S = \tau_m(R, \mathfrak{a}_\bullet) S,$$

where the second quality follows from Proposition V.29. This proves (1).

For (2), if r is a non-zero element in $\bigcap_{m \in \Phi} (\mathfrak{a}_m : \tau_m(R, \mathfrak{a}_\bullet))$, then using (1), $\varphi(r)$ is a non-zero element in $\bigcap_{m \in \Phi} (\mathfrak{a}_m S : \tau_m(S, \mathfrak{a}_\bullet S))$. $\square$

5.3.7 (Asymptotic) test ideals and birational maps

We now examine the behavior of test ideals under birational ring maps. The main result (Proposition V.33) is probably known to experts, but we include a proof, drawing inspiration from [HY03, BS13, ST14].

Setup V.32. *Let k be an F-finite field of characteristic p. Fix an extension $R \hookrightarrow S$ of smooth, integral, finitely generated k-algebras such that* $\operatorname{Frac}(R) = \operatorname{Frac}(S) = K$. *Let $Y = \operatorname{Spec}(S)$, $X = \operatorname{Spec}(R)$, and*

$$\pi : Y \to X$$

denote the birational morphism induced by the extension $R \subseteq S$. Choose canonical divisors K_Y and K_X that agree on the locus where π is an isomorphism, and let

$$K_{Y/X} \coloneqq K_Y - \pi^* K_X.$$

Define $\omega_{S/R} \coloneqq \Gamma(Y, \mathcal{O}_Y(K_{Y/X}))$. Then $\omega_{S/R}$ is a locally principal invertible fractional ideal of S, with inverse $\omega_{S/R}^{-1} = \Gamma(Y, \mathcal{O}_Y(-K_{Y/X}))$.

We use the following fact implicitly in the results of this subsection: *Under Setup V.32, if $\mathfrak{I}$ is a non-zero fractional ideal of S, then $R \cap \mathfrak{I}$ is a non-zero ideal of R.*

This follows by clearing denominators because any element of S can be written as a quotient of two elements of R since R and S have the same fraction field.

Proposition V.33. *Under the hypotheses of Setup V.32, if $\mathfrak{a}$ is a non-zero ideal of S and $\widetilde{\mathfrak{a}}$ denotes the contracted ideal $\mathfrak{a} \cap R$, then for any real $t > 0$,*

$$\tau(R, \widetilde{\mathfrak{a}}^t) \subseteq \big(\omega_{S/R} \cdot \tau(S, (\widetilde{\mathfrak{a}}S)^t)\big) \cap R \subseteq \big(\omega_{S/R} \cdot \tau(S, \mathfrak{a}^t)\big) \cap R.$$

Proof. The inclusion $\big(\omega_{S/R} \cdot \tau((\widetilde{\mathfrak{a}}S)^t)\big) \cap R \subseteq \big(\omega_{S/R} \cdot \tau(\mathfrak{a}^t)\big) \cap R$ is a consequence of the containment $\tau((\widetilde{\mathfrak{a}}S)^t) \subseteq \tau(\mathfrak{a}^t)$ (Theorem V.23(1)).

By definition, $\tau(R, \widetilde{\mathfrak{a}}^t)$ is the smallest non-zero ideal (under inclusion) I of R such that for all $e \in \mathbb{N}$, $\phi \in \operatorname{Hom}_R(F^e_* R, R)$,

$$\phi\big(F^e_*(I\widetilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil})\big) \subseteq I. \tag{5.3}$$

Thus to prove

$$\tau(R, \widetilde{\mathfrak{a}}^t) \subseteq \big(\omega_{S/R} \cdot \tau(S, (\widetilde{\mathfrak{a}}S)^t)\big) \cap R,$$

it suffices to show that $I = (\omega_{S/R} \cdot \tau((\widetilde{\mathfrak{a}}S)^t)) \cap R$ satisfies (5.3).

Extending ϕ to a K-linear map

$$\phi_K : F^e_* K \to K,$$

it is enough to show that

$$\phi_K\left(F^e_*\big(\omega_{S/R} \cdot \tau((\widetilde{\mathfrak{a}}S)^t) \cdot \widetilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}\big)\right) \subseteq \omega_{S/R} \cdot \tau((\widetilde{\mathfrak{a}}S)^t). \tag{5.4}$$

Our strategy will be to obtain an S-linear map $F^e_* S \to S$ from ϕ_K, and then use the defining property of $\tau((\widetilde{\mathfrak{a}}S)^t)$ to prove (5.4).

Using the correspondence between divisors and p^{-e}-linear maps (Theorem III.64), ϕ corresponds to a section

$$g \in \Gamma(X, \mathcal{O}_X((1-p^e)K_X)),$$

whose pullback

$$g = \pi^* g$$

is a global section of

$$\mathcal{O}_Y((1-p^e)\pi^* K_X) = \mathcal{O}_Y((1-p^e)(K_Y - K_{Y/X})).$$

Using Theorem III.64 again, $g = \pi^* g$ corresponds to a p^{-e}-linear map of $\mathcal{O}_Y$-modules

$$F^e_* \mathcal{O}_Y\big((1-p^e)K_{Y/X}\big) \to \mathcal{O}_Y,$$

which, taking global sections, induces an S-linear map

$$\varphi_g : F^e_*(\omega_{S/R}^{\otimes 1-p^e}) \to S.$$

Algebraically, the map φ_g can be constructed from ϕ in a natural way. For ease of notation, let

$$M := F^e_*\big(\omega_{S/R}^{\otimes 1-p^e}\big).$$

We claim that φ_g is obtained by restricting ϕ_K to the S-submodule M of $F^e_* K$. This needs some justification because $\phi_K|_M$ is *a priori* an S-linear map from $M \to K$, whereas φ_g maps into S. Choose a non-zero $f \in R$ such that

$$R_f \hookrightarrow S_f$$

is an isomorphism. Localizing at f, the extensions $\varphi_g[f^{-1}]$ of φ_g and $\phi_K|_M[f^{-1}]$ of $\phi_K|_M$ agree on the S-module

$$M_f = F^e_*(S_f) = F^e_*(R_f)$$

with the map $\phi[f^{-1}]$. Since the localization map $M \to M_f$ is injective, it follows that

$$\varphi_g = \phi_K|_M,$$

as desired.

Since the inclusion $\tau(R, \widetilde{\mathfrak{a}}^t) \subseteq \omega_{S/R} \cdot \tau((\widetilde{\mathfrak{a}}S)^t)$ can be checked locally on S, one may assume that $\omega_{S/R}^{-1}$ is principal, say $\omega_{S/R}^{-1} = cS$. Then left-mutiplication by $F^e_*(c^{p^e-1})$ induces an S-linear map $F^e_*S \to M$, yielding the element

$$\widetilde{\phi} := F^e_*S \xrightarrow{F^e_*(c^{p^e-1})\cdot} M \xrightarrow{\phi_K|_M} S$$

of $\operatorname{Hom}_S(F^e_*S, S)$. Finally, we get

$$\phi_K\Big(F^e_*\big(\omega_{S/R} \cdot \tau((\widetilde{\mathfrak{a}}S)^t) \cdot \widetilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}\big)\Big) = c^{-1} \cdot \phi_K\Big(F^e_*\big(c^{p^e-1}\tau((\widetilde{\mathfrak{a}}S)^t) \cdot \widetilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}\big)\Big) =$$
$$c^{-1} \cdot \widetilde{\phi}\Big(F^e_*\big(\tau((\widetilde{\mathfrak{a}}S)^t) \cdot \widetilde{\mathfrak{a}}^{\lceil t(p^e-1)\rceil}\big)\Big) \subseteq c^{-1}\tau((\widetilde{\mathfrak{a}}S)^t) = \omega_{S/R} \cdot \tau((\widetilde{\mathfrak{a}}S)^t),$$

where the inclusion follows by the defining property of $\tau((\widetilde{\mathfrak{a}}S)^t)$, and the fact that $\widetilde{\phi} \in \operatorname{Hom}_S(F^e_*S, S)$. This proves (5.4), hence the proposition. $\square$

Corollary V.34. *Suppose in Setup V.32, we are given a graded family* $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\Phi}$ *of non-zero ideals of* S*. Denote by* $\widetilde{\mathfrak{a}}_\bullet$ *the family* $\{\mathfrak{a}_m \cap R\}_{m\in\Phi}$*. Then*

1. *For all* $m \in \Phi$, $\tau_m(R, \widetilde{\mathfrak{a}}_\bullet) \subseteq (\omega_{S/R} \cdot \tau_m(S, \mathfrak{a}_\bullet)) \cap R$.
2. *If* $\bigcap_{m\in\Phi}(\mathfrak{a}_m : \tau_m(S, \mathfrak{a}_\bullet)) \neq (0)$*, then* $\bigcap_{m\in\Phi}(\mathfrak{a}_m \cap R : \tau_m(R, \widetilde{\mathfrak{a}}_\bullet)) \neq (0)$.

Proof. Clearly $\widetilde{\mathfrak{a}}_\bullet$ is a graded family of non-zero ideals of R. Now (1) follows from Proposition V.33 by choosing a sufficiently divisible $\ell \gg 0$ such that $\tau_m(\widetilde{\mathfrak{a}}_\bullet) = \tau((\mathfrak{a}_{\ell m} \cap R)^{1/\ell})$ and $\tau_m(\mathfrak{a}_\bullet) = \tau(S, \mathfrak{a}_{\ell m}^{1/\ell})$.

For (2), let $\mathfrak{J}$ denote the non-zero ideal $\bigcap_{m\in\Phi}(\mathfrak{a}_m : \tau_m(\mathfrak{a}_\bullet))$ of S. Note $\mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R$ is a non-zero ideal of R because $\mathfrak{J} \cdot \omega_{S/R}^{-1}$ is a non-zero fractional ideal of S, and R and S have the same fraction field. Then for all $m \in \Phi$,

$$(\mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R) \cdot \tau_m(\widetilde{\mathfrak{a}}_\bullet) \subseteq (\mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R) \cdot \big((\omega_{S/R} \cdot \tau_m(\mathfrak{a}_\bullet)) \cap R\big)$$
$$\subseteq \big(\mathfrak{J} \cdot \omega_{S/R}^{-1} \cdot \omega_{S/R} \cdot \tau_m(\mathfrak{a}_\bullet)\big) \cap R = (\mathfrak{J} \cdot \tau_m(\mathfrak{a}_\bullet)) \cap R \subseteq \mathfrak{a}_m \cap R.$$

Thus, $(0) \neq \mathfrak{J} \cdot \omega_{S/R}^{-1} \cap R \subseteq \bigcap_{m \in \Phi} (\mathfrak{a}_m \cap R : \tau_m(\widetilde{\mathfrak{a}}_\bullet))$. $\square$

Remark V.35. The proofs of Proposition V.33 and Corollary V.34 globalize in a straightforward manner. We work at the affine level since this is sufficient for our purposes, and also because we have defined test ideals of pairs only in the affine setting.

5.4 Proof of Theorem V.2

For a ring A of K/k admitting a center of ν, we will say A **satisfies Theorem V.2 for** ν if

$$\bigcap_{m \in \mathbb{R}_{\geq 0}} (\mathfrak{a}_m : \tau_m(A, \mathfrak{a}_\bullet)) \neq (0),$$

where $\mathfrak{a}_m$ are the valuation ideals of A associated to ν.

To prove Theorem V.2 we need the following general fact about primary ideals of a Noetherian domain, which in particular implies that if Theorem V.2 holds for the local ring of the center x of a variety X of K/k, then it also holds on any affine open neighborhood of x.

Lemma V.36. *Let A be a Noetherian domain and $\mathfrak{p}$ be a prime ideal of A.*

1. *For any $\mathfrak{p}$-primary ideal $\mathfrak{a}$ of A, $\mathfrak{a}A_\mathfrak{p} \cap A = \mathfrak{a}$.*

2. *Let $\{\mathfrak{a}_i\}_{i \in I}, \{J_i\}_{i \in I}$ be collections ideals of A such that each $\mathfrak{a}_i$ is $\mathfrak{p}$-primary. Then*

$$\bigcap_{i \in I} (\mathfrak{a}_i A_\mathfrak{p} : J_i A_\mathfrak{p}) = \left(\bigcap_{i \in I} (\mathfrak{a}_i : J_i) \right) A_\mathfrak{p}.$$

Thus, $\bigcap_{i \in I} (\mathfrak{a}_i A_\mathfrak{p} : J_i A_\mathfrak{p}) \neq (0)$ if and only if $\bigcap_{i \in I} (\mathfrak{a}_i : J_i) \neq (0)$.

Proof of Lemma V.36. (1) follows easily from the definition of a primary ideal. For

(2), the containment $\left(\bigcap_{i\in I}(\mathfrak{a}_i : J_i)\right)A_{\mathfrak{p}} \subseteq \bigcap_{i\in I}(\mathfrak{a}_i A_{\mathfrak{p}} : J_i A_{\mathfrak{p}})$ is easy to verify. Now let

$$\tilde{s} \in \bigcap_{i\in I}(\mathfrak{a}_i A_{\mathfrak{p}} : J_i A_{\mathfrak{p}}),$$

and choose $t \in A - \mathfrak{p}$ such that $t\tilde{s} \in A$, noting that $t\tilde{s}$ is also in the ideal $\bigcap_{i\in I}(\mathfrak{a}_i A_{\mathfrak{p}} : J_i A_{\mathfrak{p}})$. Then for all $i \in I$,

$$(t\tilde{s}) \cdot J_i \subseteq (t\tilde{s}) \cdot (J_i A_{\mathfrak{p}} \cap A) \subseteq \mathfrak{a}_i A_{\mathfrak{p}} \cap A = \mathfrak{a}_i,$$

where the last equality comes from (1). Thus, $t\tilde{s} \in \bigcap_{i\in I}(\mathfrak{a}_i : J_i)$, and so $\tilde{s} \in \left(\bigcap_{i\in I}(\mathfrak{a}_i : J_i)\right)A_{\mathfrak{p}}$, establishing the other inclusion. Since $A \to A_{\mathfrak{p}}$ is injective, the final statement is clear. $\square$

Using Lemma V.36, Theorem V.2 is proved as follows:

Proof of Theorem V.2. Let $(A, \mathfrak{m}_A, \kappa_A)$ be the regular local ring ν is centered on, where A is essentially of finite type over the perfect field k with fraction field K. Suppose $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_\nu) = d$ and $\operatorname{tr.deg} K/k = n$. Let R be a finitely generated, regular k-subalgebra of K with a prime ideal $\mathfrak{p}$ such that $A = R_{\mathfrak{p}}$. Using local monomialization of real-valued valuations (Theorem V.6), choose a finitely generated regular k-subalgebra S of K along with an inclusion $R \hookrightarrow S$ such that ν is centered on the prime $\mathfrak{q}$ of S, and $S_{\mathfrak{q}}$ has Krull dimension d and a regular system of parameters $\{x_1, \ldots, x_d\}$ such that $\nu(x_1), \ldots, \nu(x_d)$ freely generate the value group Γ_ν.

Note that if $\{\mathfrak{b}_m\}_{m\in\mathbb{R}_{\geq 0}}$ is the set of valuation ideals of S, then $\{\mathfrak{b}_m \cap R\}_{m\in\mathbb{R}_{\geq 0}}$ is the set of valuation ideals of R. If $S_{\mathfrak{q}}$ satisfies Theorem V.2, then so does S (Lemma V.4 and Lemma V.36), hence R (Corollary V.34), hence also $R_{\mathfrak{p}} = A$ because $\mathfrak{p}$ is the center of ν on R (using Lemma V.36 again). Thus, it suffices to prove Theorem V.2 for $A = S_{\mathfrak{q}}$.

The valuation ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\mathbb{R}_{\geq 0}}$ of A are then monomial in the regular system of parameters $x_1,\ldots,x_d$ (see proof of Proposition V.6). Because A has dimension d, its residue field κ_A has transcendence degree $n-d$ over k. Now using the fact that k is perfect, choose a separating transcendence basis $\{t_1,\ldots,t_{n-d}\}$ of κ_A/k, and pick $y_1,\ldots,y_{n-d}\in A$ such that

$$y_i \equiv t_i \mod \mathfrak{m}_A.$$

By [Bou89, VI, §10.3, Theorem 1], $\{x_1,\ldots,x_d,y_1,\ldots,y_{n-d}\}$ is algebraically independent over k, and we obtain a local extension

$$j : k[x_1,\ldots,x_d,y_1,\ldots,y_{n-d}]_{(x_1,\ldots,x_d)} \hookrightarrow A,$$

of local rings of the same dimension that is unramified by construction. Moreover, j is also flat [Mat89, Theorem 23.1], essentially of finite type, hence essentially étale.

Let

$$\widetilde{A} \coloneqq k[x_1,\ldots,x_d,y_1,\ldots,y_{n-d}]_{(x_1,\ldots,x_d)}.$$

It is easy to see that $\mathfrak{a}_\bullet \cap \widetilde{A} \coloneqq \{\mathfrak{a}_m \cap \widetilde{A}\}_{m\in\mathbb{R}_{\geq 0}}$ is the collection of valuation ideals of $\widetilde{A}$ with respect to the restriction of ν to $\operatorname{Frac}(\widetilde{A})$. Moreover, if S is a set of monomials in $x_1,\ldots,x_d$ generating $\mathfrak{a}_m$, and I_m is the ideal of $\widetilde{A}$ generated by S, then $I_m = I_mA\cap\widetilde{A} = \mathfrak{a}_m\cap\widetilde{A}$, where the first equality follows by faithful flatness of j. Thus, each $\mathfrak{a}_m\cap\widetilde{A}$ is generated by the same monomials in $x_1,\ldots,x_d$ that generate $\mathfrak{a}_m$. Then to prove the theorem, it suffices to show by Corollary V.31 that

$$\bigcap_{m\in\mathbb{R}_{\geq 0}} (\mathfrak{a}_m\cap\widetilde{A} : \tau_m(\mathfrak{a}_\bullet\cap\widetilde{A})) \neq (0).$$

But now we are in the setting of Example V.24 since we are dealing with monomial ideals in the localization of a polynomial ring. We claim that

$$x_1\ldots x_d \in \bigcap_{m\in\mathbb{R}_{\geq 0}} (\mathfrak{a}_m\cap\widetilde{A} : \tau_m(\mathfrak{a}_\bullet\cap\widetilde{A})).$$

Choose $\ell \in \mathbb{N}$ such that $\tau_m(\mathfrak{a}_\bullet \cap \widetilde{A}) = \tau((\mathfrak{a}_{\ell m} \cap \widetilde{A})^{1/\ell})$. Since $\mathfrak{a}_{\ell m} \cap \widetilde{A}$ is generated by $\left\{x_1^{a_1} \dots x_d^{a_d} : \sum a_i \nu(x_i) \geq \ell m\right\}$, and test ideals commute with localization, we conclude using Example V.24 that $\tau_m(\mathfrak{a}_\bullet \cap \widetilde{A}) = \tau((\mathfrak{a}_{\ell m} \cap \widetilde{A})^{1/\ell})$ is generated by monomials $x_1^{b_1} \dots x_d^{b_d}$ such that $(b_1+1, \dots, b_d+1)$ is in the interior of the convex hull of

$$\left\{\left(\frac{a_1}{\ell}, \dots, \frac{a_d}{\ell}\right) : a_i \in \mathbb{N} \cup \{0\}, \sum \frac{a_i}{\ell} \nu(x_i) \geq m\right\}.$$

Then clearly $\sum (b_i+1)\nu(x_i) \geq m$, that is, $(x_1 \dots x_n) \cdot x_1^{b_1} \dots x_d^{b_d} \in \mathfrak{a}_m \cap \widetilde{A}$. This shows that for all $m \in \mathbb{R}_{\geq 0}$,

$$(x_1 \dots x_n) \cdot \tau_m(\mathfrak{a}_\bullet \cap \widetilde{A}) \subseteq \mathfrak{a}_m \cap \widetilde{A},$$

as desired. □

Remark V.37. The transformation law for test ideals under essentially étale maps (Proposition V.29) and its asymptotic version (Corollary V.31) are results of independent interest. However, their use in the proof of Theorem V.2 can be avoided. Indeed, after reducing the proof of Theorem V.2 to the case of a regular local center $(A, \mathfrak{m}_A, \kappa_A)$ with a regular system of parameters $\{r_1, \dots r_d\}$ whose valuations freely generate the value group, the behavior of test ideals under completion gives another way of proving Theorem V.2. Briefly, using the structure theory of complete local rings, identify $\widehat{A}$ with a power-series ring

$$\kappa_A[[x_1, \dots, x_d]],$$

where $r_i \mapsto x_i$ under this identification. Since the graded family of valuation ideals $\mathfrak{a}_\bullet$ of A are monomial in $\{r_1, \dots, r_d\}$ (Proposition V.6), the graded family $\mathfrak{a}_\bullet \widehat{A}$ consists of ideals monomial in $x_1, \dots, x_d$. Explicitly, $\mathfrak{a}_m \widehat{A}$ is generated by

$$\{x_1^{\alpha_1} \dots x_d^{\alpha_d} : \alpha_1 \nu(r_1) + \dots + \alpha_d \nu(r_d) \geq m\}.$$

As the formation of test ideals commutes with completion (Theorem V.23 (9)), for any $m \in \mathbb{R}_{\geq 0}$, $\tau_m(\widehat{A}, \mathfrak{a}_\bullet \widehat{A}) = \tau_m(A, \mathfrak{a}_\bullet)\widehat{A}$, and so by faithful flatness of the canonical map $A \to \widehat{A}$, to prove that Theorem V.2 holds for A, it suffices to show that

$$x_1 \ldots x_d \in \bigcap_{m \in \mathbb{R}_{\geq 0}} (\mathfrak{a}_m \widehat{A} : \tau_m(\widehat{A}, \mathfrak{a}_\bullet \widehat{A})). \tag{5.5}$$

However, $\kappa_A[[x_1, \ldots, x_d]]$ is also the $(x_1, \ldots, x_d)$-adic completion of the local ring

$$\kappa_A[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)},$$

and so we are reduced to analyzing test ideals of monomial ideals in a polynomial ring (Example V.24). Now the argument in the final paragraph of the proof of Theorem V.2 can be repeated verbatim in order to obtain (5.5).

5.5 Consequences of Theorem V.2

Throughout this section k is a perfect field of prime characteristic, X a regular variety over k with function field K, and ν a non-trivial, real-valued Abhyankar valuation of K/k centered on $x \in X$.

5.5.1 Proof of Theorem V.1

Our goal is to show that there exists $e \geq 0$ such that for all $m \in \mathbb{R}_{\geq 0}$, $\ell \in \mathbb{N}$,

$$\mathfrak{a}_m(X)^\ell \subseteq \mathfrak{a}_{\ell m}(X) \subseteq \mathfrak{a}_{m-e}(X)^\ell.$$

From now we also assume $m > 0$, as otherwise all the ideals equal $\mathcal{O}_X$.

Let $(\mathfrak{a}_\bullet)_x \coloneqq \{\mathfrak{a}_m(\mathcal{O}_{X,x})\}_{m \in \mathbb{R}_{\geq 0}}$ denote the graded system of valuation ideals of the center $\mathcal{O}_{X,x}$. Using Theorem V.2, fix a nonzero $\tilde{s} \in \mathcal{O}_{X,x}$ such that

$$\tilde{s} \in \bigcap_{m \in \mathbb{R}_{\geq 0}} \big(\mathfrak{a}_m(\mathcal{O}_{X,x}) : \tau_m((\mathfrak{a}_\bullet)_x)\big).$$

Define $e := \nu(\tilde{s})$.

Since $\mathfrak{a}_\bullet$ is a graded family of ideal sheaves, the inclusion $\mathfrak{a}_m^\ell \subseteq \mathfrak{a}_{\ell m}$ follows. Thus it suffices to show that for the above choice of e,

$$\Gamma(U, \mathfrak{a}_{\ell m}) \subseteq \Gamma(U, \mathfrak{a}_{m-e}^\ell), \tag{5.6}$$

for all $m \in \mathbb{R}_{\geq 0}$, $\ell \in \mathbb{N}$, and affine open $U \subseteq X$. Furthermore, we may assume U contains the center x of ν, as otherwise $\Gamma(U, \mathfrak{a}_{\ell m})$ and $\Gamma(U, \mathfrak{a}_{m-e}^\ell)$ both equal $\mathcal{O}_X(U)$. We use $(\mathfrak{a}_\bullet)_U$ to denote the collection $\{\mathfrak{a}_m(U)\}_{m \in \mathbb{R}_{\geq 0}}$ of valuation ideals of $\mathcal{O}_X(U)$.

Utilizing Lemma V.4 and Lemma V.36(2), express $\tilde{s}$ as a fraction s_U/t, for some non-zero

$$s_U \in \bigcap_{m \in \mathbb{R}_{\geq 0}} \big(\mathfrak{a}_m(U) : \tau_m((\mathfrak{a}_\bullet)_U)\big),$$

and $t \in \mathcal{O}_X(U)$ such that $t_x \in \mathcal{O}_{X,x}^\times$. Then $\nu(s_U) = \nu(\tilde{s}) = e$, and it follows that for all $m \in \mathbb{R}_{\geq 0}$,

$$\tau_m\big((\mathfrak{a}_\bullet)_U\big) \subseteq \mathfrak{a}_{m-e}(U).$$

Proposition V.28(2) implies that $\Gamma(U, \mathfrak{a}_{\ell m}) \subseteq \tau_m\big((\mathfrak{a}_\bullet)_U\big)^\ell$, and we obtain (5.6) by observing that

$$\Gamma(U, \mathfrak{a}_{\ell m}) \subseteq \tau_m\big((\mathfrak{a}_\bullet)_U\big)^\ell \subseteq \mathfrak{a}_{m-e}(U)^\ell = \Gamma(U, \mathfrak{a}_{m-e}^\ell). \quad \square$$

5.5.2 Proof of Corollary V.3

We want to prove that if ν, w are two non-trivial real-valued Abhyankar valuations of K/k, centered on a regular local ring $(A, \mathfrak{m})$ essentially of finite type over k with fraction field K, then there exists $C > 0$ such that for all $x \in A$,

$$\nu(x) \leq Cw(x).$$

Our argument is similar to [ELS03], and is provided for completeness.

We let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \mathbb{R}_{\geq 0}}$ denote the collection of valuation ideals of A associated to ν, and $\mathfrak{b}_\bullet = \{\mathfrak{b}_m\}_{m \in \mathbb{R}_{\geq 0}}$ the collection associated to w. Since A is Noetherian, there exists a non-zero $x \in \mathfrak{m}$ such that for all non-zero y in $\mathfrak{m}$,

$$w(x) \leq w(y).$$

Otherwise, one can find a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathfrak{m}$ such that $w(x_1) > w(x_2) > w(x_3) > \dots$, giving us a strictly ascending chain of ideals $\mathfrak{b}_{w(x_1)} \subsetneq \mathfrak{b}_{w(x_2)} \subsetneq \mathfrak{b}_{w(x_3)} \subsetneq \dots$. For the rest of the proof, let

$$\delta \coloneqq \inf\{\nu(x) : x \in \mathfrak{m} - \{0\}\}.$$

Claim V.38. *There exists $p > 0$ such that for all $\ell \in \mathbb{N}$, $\mathfrak{a}_{\ell p} \subseteq \mathfrak{b}_{\ell\delta}$.*

Assuming the claim, let $C \coloneqq 2p/\delta$, and suppose there exists $x_0 \in \mathfrak{m}$ such that $\nu(x_0) > Cw(x_0)$. Now choose $\ell \in \mathbb{N}$ such that

$$(\ell - 1)\delta \leq w(x_0) < \ell\delta. \tag{5.7}$$

Such an ℓ exists by the Archimedean property of $\mathbb{R}$, and moreover, $\ell \geq 2$ since $w(x_0) \geq \delta$. Clearly, $x_0 \notin \mathfrak{b}_{\ell\delta}$, and multiplying (5.7) by C, we get

$$2(\ell - 1)p \leq Cw(x_0) < 2\ell p.$$

But $\ell \geq 2$ implies $\ell p \leq 2(\ell - 1)p \leq Cw(x_0) < \nu(x_0)$. Then $x_0 \in \mathfrak{a}_{\ell p}$, contradicting $\mathfrak{a}_{\ell p} \subseteq \mathfrak{b}_{\ell\delta}$. This completes the proof of Izumi's theorem (Corollary V.3) modulo the proof Claim V.38.

Proof of Claim V.38: By our choice of δ, $\mathfrak{b}_\delta = \mathfrak{m}$. Thus, for all $\ell \in \mathbb{N}$, $\mathfrak{m}^\ell \subseteq \mathfrak{b}_{\ell\delta}$. Since by Theorem V.2

$$\bigcap_{m \in \mathbb{R}_{\geq 0}} \big(\mathfrak{a}_m : \tau_m(\mathfrak{a}_\bullet)\big) \neq (0), \tag{5.8}$$

there must exist some $p > 0$ such that $\tau_p(\mathfrak{a}_\bullet) \subseteq \mathfrak{m}$. Otherwise, for all $m \in \mathbb{R}_{\geq 0}$, $\tau_m(\mathfrak{a}_\bullet) = A$, which would imply that any $s \in \bigcap_{m \in \mathbb{R}} \left(\mathfrak{a}_m : \tau_m(\mathfrak{a}_\bullet)\right)$ is also an element of $\bigcap_{m \in \mathbb{R}_{\geq 0}} \mathfrak{a}_m = (0)$, contradicting (5.8). Then by Proposition V.28(2), for all $\ell \in \mathbb{N}$,

$$\mathfrak{a}_{\ell p} \subseteq \tau_p(\mathfrak{a}_\bullet)^\ell \subseteq \mathfrak{m}^\ell \subseteq \mathfrak{b}_{\ell\delta}. \qquad \square$$

Examples V.39.

1. Uniform approximation of valuation ideals (Theorem V.1) *fails* in general for real-valued valuations that are not Abhyankar. The discrete valuation $\nu_{q(t)}$ of $\mathbb{F}_p(X,Y)$ constructed in Example II.57, among other things, also provides a counter-example to Theorem V.1. Recall that $\nu_{q(t)}$ is obtained as the composition

$$\mathbb{F}_p(X,Y)^\times \hookrightarrow \mathbb{F}_p((t))^\times \xrightarrow{t-adic} \mathbb{Z},$$

by mapping $X \mapsto t$ and $Y \mapsto q(t)$ such that $t, q(t)$ are algebraically independent over $\mathbb{F}_p$. We can choose

$$q(t) = a_1 t + a_2 t^2 + a_3 t^3 + \dots,$$

such that $a_1 \neq 0$. Then $\nu_{q(t)}$ is centered on

$$A \coloneqq \mathbb{F}_p[X,Y]_{(X,Y)}.$$

Now for any $m \in \mathbb{N}$, the valuation ideal $\mathfrak{a}_m$ of the center A contains the ideal

$$(X^m, Y - a_1X + a_2X^2 + \dots + a_{m-1}X^{m-1}).$$

Therefore $A/\mathfrak{a}_m$ has length $\leq m$.

Suppose there exists e as in Theorem V.1. Fixing $m \in \mathbb{N}$ such that

$$m > e,$$

we see that for all $\ell \in \mathbb{N}$, the length of $A/\mathfrak{a}_{\ell m}$ is $\leq \ell m$. In other words, for a fixed m, the length of $A/\mathfrak{a}_{\ell m}$ grows as a linear function in ℓ. On the other hand,

$$\mathfrak{a}_{m-e}^{\ell} \subseteq (X, Y)^{\ell}.$$

Thus the length of $A/\mathfrak{a}_{m-e}^{\ell}$ is at least the length of $A/(X,Y)^{\ell}$, and the latter grows as a quadratic function in ℓ. Hence $\mathfrak{a}_{\ell m}$ cannot possibly be contained in $\mathfrak{a}_{m-e}^{\ell}$ when $\ell \gg 0$, thereby providing a counter-example to Theorem V.1. Since Theorem V.1 is a formal consequence of Theorem V.2, we also see that Theorem V.2 must be false for non-Abhyankar real-valued valuations.

2. Izumi's theorem (Corollary V.3) also fails in general when the valuations ν and w are not both Abhyankar. To see this, we take one valuation to be the unique valuation ν_π on $\mathbb{F}_p(X, Y)$ such that

$$\nu_\pi(X) = 1 \text{ and } \nu_\pi(Y) = \pi.$$

Note ν_π is an Abhyankar valuation of $\mathbb{F}_p(X, Y)/\mathbb{F}_p$ since

$$\operatorname{tr.deg} \mathbb{F}_p(X, Y)/\mathbb{F}_p = 2 = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_{\nu_\pi}).$$

Choose the other valuation to be of the form $\nu_{q(t)}$, where, specifically,

$$q(t) = \sum_{i=1}^{\infty} t^{i!}.$$

It is not difficult to check that $t, q(t)$ are algebraically independent over $\mathbb{F}_p$ (see also [Bou89, Chapter VI, §3, Exercise 1]), so that the valuation $\nu_{q(t)}$ is indeed well-defined.

Both ν_π and $\nu_{q(t)}$ are centered on $\mathbb{F}_p[X, Y]_{(X,Y)}$. For all $n \in \mathbb{N}$, defining

$$x_n \coloneqq Y - \sum_{i=1}^{n} X^{i!},$$

we see that,

$$\nu_\pi(x_n) = 1 \text{ and } \nu_{q(t)}(x_n) = (n+1)!.$$

Clearly there does not exist a fixed real number $C > 0$ such that for all $n \in \mathbb{N}$,

$$\nu_{q(t)}(x_n) = (n+1)! \leq C = C\nu_\pi(x_n).$$

Thus, Izumi's theorem fails when the real-valued valuations are not Abhyankar.

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