RATIONAL SOLUTIONS OF THE PAINLEVÉ-III EQUATION

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ABSTRACT. All of the six Painlevé equations except the first have families of rational solutions, which are frequently important in applications. The third Painlevé equation in generic form depends on two parameters m and n, and it has rational solutions if and only if at least one of the parameters is an integer. We use known algebraic representations of the solutions to study numerically how the distributions of poles and zeros behave as $n \in \mathbb{Z}$ increases and how the patterns vary with $m \in \mathbb{C}$. This study suggests that it is reasonable to consider the rational solutions in the limit of large $n \in \mathbb{Z}$ with $m \in \mathbb{C}$ being an auxiliary parameter. To analyze the rational solutions in this limit, algebraic techniques need to be supplemented by analytical ones, and the main new contribution of this paper is to develop a Riemann-Hilbert representation of the rational solutions of Painlevé-III that is amenable to asymptotic analysis. Assuming further that m is a half-integer, we derive from the Riemann-Hilbert representation a finite dimensional Hankel system for the rational solution in which $n \in \mathbb{Z}$ appears as an explicit parameter.

1. INTRODUCTION

This paper is the first in a series concerned with the large degree asymptotic analysis of rational solutions $u_n(x; m)$ to the generic Painlevé-III equation parametrized by $n \in \mathbb{Z}$ and $m \in \mathbb{C}$. The six Painlevé equations are best known for their transcendental solutions, and indeed their general solutions are frequently referred to as *Painlevé transcendents*. These transcendental solutions are modern special functions that have appeared in numerous applications, most famously in similarity solutions of nonlinear partial differential equations and in integrable probability. However, all of the Painlevé equations except the first are actually families of ordinary differential equations indexed by complex parameters, and it is well-known that if the parameters take on certain special values, then the Painlevé equation admits particular solutions that are either finitely constructed from elementary special functions or rational functions.

For example, the Painlevé-II equation $u'' = 2u^3 + xu + m$ has a complex parameter m, and it is elementary that if $m = 0$ then the equation admits the trivial rational solution $u(x) \equiv 0$. With this solution in hand for $m = 0$, one can apply the *Bäcklund transformation*

$$u(x) \mapsto \hat{u}(x) := -u(x) - \frac{2m+1}{2u(x)^2 + 2u'(x) + x}$$

taking a solution of the equation with parameter m into another solution of the same equation but with parameter $m \mapsto \hat{m} := m+1$. The Bäcklund transformation obviously preserves rationality and with its help one quickly obtains a rational solution of the Painlevé-II equation for each integer value of m. It turns out that the integral values of m are the only ones for which the equation admits a rational solution, and for each $m \in \mathbb{Z}$ there is exactly one rational solution, denoted $u_m(x)$, $m \in \mathbb{Z}$. Motivated by applications, the family of functions $\{u_m(\cdot)\}_{m\in\mathbb{Z}}$ has recently been studied from the analytic perspective, i.e., from the point of view of asymptotic analysis in the limit of large integer m [2, 4, 5, 18].

1.1. The Painlevé-III equation, its symmetries and its rational solutions.

The generic Painlevé-III equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{4\Theta_0 u^2 + 4(1-\Theta_\infty)}{x} + 4u^3 - \frac{4}{u}, \tag{1.1}$$

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is the simplest of the Painlevé equations having a fixed singular point ($x = 0$), and it involves two[1] distinct complex parameters Θ_0 and Θ_∞. As we shall see, both of these features introduce new phenomena into the behavior of even the most elementary, rational solutions.

In order to study the rational solutions of (1.1), it will be convenient to represent the constant parameters Θ_0 and Θ_∞ in the form

$$\Theta_0 = n + m \quad \text{and} \quad \Theta_\infty = m - n + 1. \tag{1.2}$$

Equation (1.1) has many symmetries, including the following elementary ones:

- *Inversion*: if $u(x)$ satisfies (1.1)–(1.2), then $u(x) \mapsto I[u](x) := 1/u(x)$ satisfies (1.1) with modified parameters $I : \Theta_0 \mapsto \Theta_\infty - 1 = m - n$ and $I : \Theta_\infty \mapsto \Theta_0 + 1 = m + n + 1$ (corresponding to changing the sign of n while holding m fixed). The mapping $I : (u(x), \Theta_0, \Theta_\infty) \mapsto (1/u(x), \Theta_\infty - 1, \Theta_0 + 1)$ is an involution.
- *Rotation*: if $u(x)$ satisfies (1.1)–(1.2), then $u(\cdot) \mapsto R[u](x) := -\mathrm{i}u(-\mathrm{i}x)$ satisfies (1.1) with modified parameters $R : \Theta_0 \mapsto \Theta_0 = n + m$ and $R : \Theta_\infty \mapsto 2 - \Theta_\infty = n - m + 1$ (corresponding to swapping m and n). The mapping $R : (u(x), \Theta_0, \Theta_\infty) \mapsto (-\mathrm{i}u(-\mathrm{i}x), \Theta_0, 2 - \Theta_\infty)$ is the generator of a cyclic symmetry group of order 4. Note that R^2 fixes the parameters $(\Theta_0, \Theta_\infty)$ in (1.1) but maps the solution $u(x)$ to its odd reflection $-u(-x)$.

A nontrivial symmetry is the following Bäcklund transformation $u(x) \mapsto \hat{u}(x)$, which was discovered by Gromak [13]:

$$\hat{u}(x) := \frac{xu'(x) + 2xu(x)^2 + 2x - 2(1 - \Theta_\infty)u(x) - u(x)}{u(x) \cdot (xu'(x) + 2xu(x)^2 + 2x + 2\Theta_0 u(x) + u(x))} \tag{1.3}$$

solves (1.1) for modified parameters $\Theta_0 \mapsto \widehat{\Theta}_0 := \Theta_0 + 1 = (n + 1) + m$ and $\Theta_\infty \mapsto \widehat{\Theta}_\infty := \Theta_\infty - 1 = m - (n + 1) + 1$, which amounts to incrementing n for fixed m.

Proposition 1. *Suppose now that* (1.1) *has a solution* $u(x)$ *that is rational. Then either* $m \in \mathbb{Z}$ *or* $n \in \mathbb{Z}$ *or both.*

Proof. Indeed, assuming $u(x) = ax^p + O(x^{p-1})$ as $x \to \infty$ for $p \in \mathbb{Z}$ and $a \neq 0$, from (1.1) we obtain a dominant balance only for $p = 0$, yielding (from the last two terms on the right-hand side) $a^4 = 1$. Continuing the Laurent expansion to the next order by writing $u(x) = a + bx^{-1} + O(x^{-2})$ as $x \to \infty$ with $a^4 = 1$, the calculation of b only brings in the remaining terms in (1.1) that are not proportional to derivatives of u, and we find $b = a^2(\Theta_\infty - 1)/4 - \Theta_0/4$. Therefore, the sum of all finite residues of the assumed rational solution $u(x)$ must equal b as well. If $x = 0$ is a pole of $u(x)$, then a similar dominant balance argument involving the terms $u''(x)$, $u'(x)^2/u(x)$, $u'(x)/x$, $u(x)^2/x$, and $4u(x)^3$ shows that it must be a simple pole of residue $-\Theta_0$. Finally, if $x_0 \neq 0$ is a pole of $u(x)$, then it must be a simple pole and a dominant balance involving $u''(x)$, $u'(x)^2/u(x)$, and $4u(x)^3$ shows that the residue is either $\frac{1}{2}$ or $-\frac{1}{2}$. Letting $k \in \mathbb{Z}$ denote the difference between the number of nonzero poles of $u(x)$ with residues $\frac{1}{2}$ and $-\frac{1}{2}$, we therefore arrive at the identities

$$\frac{1}{2}k \mp \frac{1}{4}(\Theta_\infty - 1) + \frac{1}{4}\Theta_0 = \begin{cases} \Theta_0, & \text{if } x = 0 \text{ is a pole of } u \\ 0, & \text{if } x = 0 \text{ is not a pole of } u, \end{cases} \tag{1.4}$$

where $a^2 = \pm 1$. Using (1.2) then shows that, if $x = 0$ is not a pole of u, then $a^2 = 1$ implies $n = k \in \mathbb{Z}$, while $a^2 = -1$ implies $m = -k \in \mathbb{Z}$. On the other hand, if $x = 0$ is a pole of u, then by inversion symmetry $I[u](x) = 1/u(x)$ is a rational solution of (1.1) analytic at the origin and corresponding to the modified parameters $I : \Theta_0 \mapsto m - n$ and $I : \Theta_\infty \mapsto m + n + 1$. Applying (1.4) to $I[u]$ with parameters replaced by their modified values then yields the same conclusion as in the case that u is analytic at the origin, namely that $n = k \in \mathbb{Z}$ if $a^2 = 1$ and $m = -k \in \mathbb{Z}$ if $a^2 = -1$. □

This argument shows that each rational solution of (1.1) tends to one of four nonzero limits, ± 1 or $\pm\mathrm{i}$, as $x \to \infty$ and hence cannot be an odd function of x. Furthermore, it follows from odd reflection symmetry $R^2 : u(x) \mapsto -u(-x)$ that for given parameters (1.2) with $m \in \mathbb{Z}$ or $n \in \mathbb{Z}$, the rational solutions come in distinct pairs permuted by odd reflection.

[1] In the most general form of the Painlevé-III equation one replaces the terms $4u^3 - 4u^{-1}$ on the right-hand side by $\gamma u^3 + \delta u^{-1}$ for arbitrary parameters $(\gamma, \delta) \in \mathbb{C}^2$. Under the generic assumption that $\gamma\delta \neq 0$, a suitable rescaling of the dependent and independent variables results in the form (1.1). There are two singular reductions: one in which either $\gamma = 0$ or $\delta = 0$ but not both, which can be reduced by scaling to a one-parameter family of equations (or in the more special case that either Θ_0 or $1 - \Theta_\infty$ vanishes to an equation whose general solution is known in closed form), and one in which $\gamma = \delta = 0$, which can be reduced by scaling to a unique form if $\Theta_0(1 - \Theta_\infty) \neq 0$. See [22, §32.2.2] and [12, Section 2.2].

It turns out that if $m \in \mathbb{Z}$ or $n \in \mathbb{Z}$ there indeed exists a rational solution of (1.1)–(1.2). If only one of m and n is integral, then there are exactly two rational solutions, while if both are integral there are exactly four rational solutions. The existence and precise number of the rational solutions can be established by iterated Bäcklund transformations once the cases of $m = 0$ or $n = 0$ are analyzed.

Suppose[2] $n = 0$ and $m \notin \mathbb{Z}$. Then it is obvious that (1.1)–(1.2) has at least the two distinct rational (equilibrium) solutions $u(x) = \pm 1$. It is easy to see that there are no other rational solutions in this case. Indeed, if we consider the rational solutions that tend to ± 1 as $x \to \infty$ and take $n = 0$ in (1.1)–(1.2), a simple dominant balance argument shows that these solutions satisfy $u(x) = \pm 1 + O(x^{-p})$ as $x \to \infty$ for every positive integer p and hence as $u(x)$ is rational the error terms vanish identically so the exact solutions $u(x) = \pm 1$ are the only ones recovered. On the other hand, if we consider the rational solutions that tend to $\pm \mathrm{i}$ as $x \to \infty$ and take $n = 0$ in (1.4) we find that for some $k \in \mathbb{Z}$ we have $m = k$ if $x = 0$ is a pole of u and $m = -k$ otherwise, both of which contradict the assumption that $m \notin \mathbb{Z}$. Similarly if $m = 0$ and $n \notin \mathbb{Z}$, then (1.1)–(1.2) has the pair $u(x) = \pm \mathrm{i}$ as its only rational solutions (this also follows directly using the rotation symmetry generator $\boldsymbol{R}$). Finally if $m = n = 0$ there are precisely four rational solutions: $u(x) = \pm 1$ and $u(x) = \pm \mathrm{i}$. In Section 5.3 we use these facts to determine the precise number of rational solutions of (1.1) for non-integral m.

The rational solutions of (1.1) have been known at least since the paper of Gromak [13]. The paper [20] is an exhaustive survey of special solutions of the Painlevé-III equation that describes the effect of iterating transformations such as (1.3), including cataloguing the exact numbers of poles and zeros of the iterates. This paper also includes complete references on applications of the Painlevé-III equation accurate to the date of publication. Since rational functions are naturally presented as ratios of polynomials, it is compelling to ask whether the polynomials themselves have a simple recurrence formula like (1.3). Such a result was first found for the Painlevé-II equation by Yablonskii [26] and Vorob'ev [24], and since then many algebraic representations of these polynomials have been discovered. For the Painlevé-III equation, a representation of rational solutions in terms of special polynomials was first obtained by Umemura [23, Section 9]. Clarkson further developed Umemura's scheme; in [7] a sequence of functions is defined by setting

$$s_{-1}(x; m) \equiv s_0(x; m) \equiv 1 \tag{1.5}$$

and then using the recurrence relation

$$s_{n+1}(x; m) := \frac{(4x + 2m + 1)\, s_n(x; m)^2 - s_n(x; m) s_n'(x; m) - x\left(s_n(x; m) s_n''(x; m) - s_n'(x; m)^2\right)}{2 s_{n-1}(x; m)}, \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.6}$$

It turns out that the denominator is always a factor of the numerator, so the functions $\{s_n(x; m)\}_{n=0}^{\infty}$ are all *polynomials* in x. Note that comparing with the notation of [7, 8], we have $\mu = m + \frac{1}{2}$, $z = 2x$, $\beta = 2(1 - \Theta_\infty)$, and $\alpha = 2\Theta_0$. The result of the scheme is the following.

Proposition 2 (Umemura [23], Clarkson [7], Clarkson, Law, and Lin [8])**.** *The result of applying the Bäcklund transformation* (1.3) *n times to the seed solution $u(x) \equiv 1$ is the function*

$$u(x) = u_n(x; m) := \frac{s_n(x; m-1) s_{n-1}(x; m)}{s_n(x; m) s_{n-1}(x; m-1)}, \quad n \in \mathbb{Z}_{\geq 0}, \tag{1.7}$$

defined in terms of polynomials $\{s_n(x; m)\}_{n=0}^{\infty}$ determined by (1.5)–(1.6)*. Furthermore, $u_n(x; m)$ is the unique rational solution of* (1.1) *for parameters* (1.2) *for which $u_n(x; m) \to 1$ as $x \to \infty$.*

The family of rational solutions $u_n(x; m)$ can be extended to negative integral values of n through the inversion symmetry $\boldsymbol{I}$:

$$u_{-n}(x; m) := \boldsymbol{I} u_n(x; m) = \frac{1}{u_n(x; m)}, \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.8}$$

It obviously holds that $u_{-n}(x; m) \to 1$ as $x \to \infty$, so the family captures every rational solution of the Painlevé-III equation (1.1) that tends to 1 as $x \to \infty$. It is clearly sufficient to study the family for integers $n \geq 0$. Without loss of

[2] Taking $n = 0$ in (1.1)–(1.2) yields the so-called *sine-Gordon reduction*: writing $u(x) = \mathrm{e}^{-\mathrm{i}\varphi(x)}$ and setting $n = 0$ in (1.1)–(1.2) gives

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{8m}{x}\sin(\varphi) + 8\sin(2\varphi).$$

generality we may also restrict attention to values of m in the closed right half-plane: $\mathrm{Re}(m) \geq 0$; indeed, composing inversion $\boldsymbol{I}$ with two rotations,

$$u_n(x; -m) = \boldsymbol{R}\circ\boldsymbol{I}\circ\boldsymbol{R}u_n(x; m) = \frac{1}{u_n(-x; m)}. \tag{1.9}$$

Moreover, unless $m \in \mathbb{Z}$, studying the family $\{u_n(x; m)\}$ of rational solutions tending to 1 as $x \to \infty$ captures *all* rational solutions of (1.1) because $\boldsymbol{R}^2 u_n(x; m) = -u_n(-x; m)$ is the rational solution of exactly the same Painlevé-III equation (1.1) tending to -1 as $x \to \infty$, and if $m \notin \mathbb{Z}$ this yields two and hence all rational solutions. If both n and m are integers, we may access the rotation symmetry generator $\boldsymbol{R}$ to finally exhaust all rational solutions of (1.1).

Remark 1. *It has been proven by Clarkson, Law, and Lin* [8, Theorem 4.6] *that if $m + \frac{1}{2} \in \mathbb{Z}$, then for $n > |m + \frac{1}{2}|$, s_n has $\frac{1}{2}n(n+1)$ roots, s_n vanishes to order $\frac{1}{2}(n - |m + \frac{1}{2}|)(n - |m + \frac{1}{2}| + 1)$ at the origin, and all remaining roots are simple and nonzero. This shows that when m is a half-integer and n is large, s_n has a root of order $O(n^2)$ at the origin and merely $O(n)$ simple nonzero roots. This result implies that when $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, $u_n(x; m)$ has a simple zero at the origin, while when $m = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$, $u_n(x; m)$ has a simple pole at the origin.*

1.2. **Riemann-Hilbert problem formulation and main result.** The purpose of this paper is to take the first steps toward understanding the family $\{u_n(x; m)\}_{n=0}^{\infty}$ of rational solutions of the Painlevé-III equation (1.1) from the perspective of mathematical analysis, a goal which essentially addresses the question of how $u_n(x; m)$ behaves when n is large and how the result depends on $(x, m) \in \mathbb{C}^2$. In Section 2 we present the results of several plots of poles and zeros of $u_n(x; m)$ set in the context of a formal scaling analysis of the Painlevé-III equation in the limit of large (integral) n. These results suggest numerous remarkable phenomena that can occur in this limit, but whose proofs would require other methods. The issue at hand is that the methods described above for constructing the rational function $u_n(x; m)$ all involve some sort of iteration, producing formulæ that generally become more complicated as n increases. The recurrence (1.6) is preferable to iteration of the Bäcklund transformation (1.3) in the sense that it takes advantage of explicit factorization of the numerator and denominator polynomials in the rational function $u_n(x; m)$, but it is a recurrence nonetheless. Kajiwara and Masuda [17] found a way to express (essentially) the polynomial $s_n(x; m)$ in closed form via Wronskian determinants of polynomials obtained from an elementary generating function. However, unlike certain determinantal representations of Hankel type appearing in the theory of the rational solutions of the Painlevé-II [2] and (for the "generalized Hermite" rational solutions) Painlevé-IV [6] equations, the determinants of Kajiwara and Masuda do not appear to be amenable to asymptotic analysis in the limit of large n (in which the size of the determinant grows without bound). The lack of an analytically tractable formula for $u_n(x; m)$ is the main problem that we address and solve in this paper. After a review of the isomonodromy theory of the Painlevé-III equation in Section 3, in Sections 4 and 5 we construct a Riemann-Hilbert representation of the function $u_n(x; m)$ that can be used [3] to successfully analyze the rational solution for large n. To formulate this problem here in the introduction, given a nonzero $x \in \mathbb{C}$ with $-\pi < \mathrm{Arg}(x) < \pi$, let $L = L_{◥}^{\infty} \cup L_{◥}^{0} \cup L_{◣}^{\infty} \cup L_{◣}^{0}$ be a contour in the complex λ-plane consisting of four arcs with the following properties. There is an intersection point p such that:

- $L_{◥}^{\infty}$ originates from $\lambda = \infty$ in such a direction that $\mathrm{i}x\lambda$ is negative real and terminates at $\lambda = p$, $L_{◥}^{0}$ begins at $\lambda = p$ and terminates at $\lambda = 0$ in a direction such that $-\mathrm{i}x\lambda^{-1}$ is negative real, and the net increment of the argument of λ along $L_{◥}^{\infty} \cup L_{◥}^{0}$ is

$$\Delta\arg(◥) = 2\mathrm{Arg}(x) - 2\pi\,\mathrm{sgn}(\mathrm{Im}(x)). \tag{1.10}$$

- $L_{◣}^{\infty}$ originates from $\lambda = \infty$ in such a direction that $-\mathrm{i}x\lambda$ is negative real and terminates at $\lambda = p$, $L_{◣}^{0}$ begins at $\lambda = p$ and terminates at $\lambda = 0$ in a direction such that $\mathrm{i}x\lambda^{-1}$ is negative real, and the net increment of the argument of λ along $L_{◣}^{\infty} \cup L_{◣}^{0}$ is

$$\Delta\arg(◣) = 2\mathrm{Arg}(x). \tag{1.11}$$

- The arcs $L_{◥}^{\infty}$, $L_{◥}^{0}$, $L_{◣}^{\infty}$, and $L_{◣}^{0}$ do not otherwise intersect.

See Figure 14 below for an illustration. Consider now the following problem.

Riemann-Hilbert Problem 1. *Given parameters $m \in \mathbb{C}$ and $n \in \mathbb{Z}$ as well as $x \in \mathbb{C} \setminus \{0\}$ with $-\pi < \mathrm{Arg}(x) < \pi$, let L denote an x-dependent contour as above, and seek a 2×2 matrix function $\mathbf{Y}(\lambda) = \mathbf{Y}^{(n)}(\lambda; x, m)$ with the following properties:*

(1) ***Analyticity:*** *$\mathbf{Y}(\lambda)$ is analytic in λ in the domain $\lambda \in \mathbb{C} \setminus L$. It takes continuous boundary values on $L \setminus \{0\}$ from each maximal domain of analyticity.*

(2) ***Jump conditions:*** *The boundary values $\mathbf{Y}_\pm(\lambda)$ are related on each arc of L by the following formulæ:*

$$\mathbf{Y}_+(\lambda) = \mathbf{Y}_-(\lambda)\begin{bmatrix} 1 & -\dfrac{\sqrt{2\pi}\lambda_{\boxbslash}^{-(m+1)}}{\Gamma(\frac{1}{2}-m)}\lambda^n \mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})} \\ 0 & 1 \end{bmatrix}, \quad \lambda \in L_{\boxbslash}^0 \tag{1.12}$$

$$\mathbf{Y}_+(\lambda) = \mathbf{Y}_-(\lambda)\begin{bmatrix} 1 & \dfrac{\sqrt{2\pi}\lambda_{\boxbslash}^{-(m+1)}}{\Gamma(\frac{1}{2}-m)}\lambda^n \mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})} \\ 0 & 1 \end{bmatrix}, \quad \lambda \in L_{\boxbslash}^\infty \tag{1.13}$$

$$\mathbf{Y}_+(\lambda) = \mathbf{Y}_-(\lambda)\begin{bmatrix} 1 & 0 \\ \dfrac{\sqrt{2\pi}(\lambda_{\boxbslash}^{(m+1)/2})_+(\lambda_{\boxbslash}^{(m+1)/2})_-}{\Gamma(\frac{1}{2}+m)}\lambda^{-n} \mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})} & 1 \end{bmatrix}, \quad \lambda \in L_{\boxbslash}^\infty \tag{1.14}$$

$$\mathbf{Y}_+(\lambda) = \mathbf{Y}_-(\lambda)\begin{bmatrix} -\mathrm{e}^{2\pi \mathrm{i}m} & 0 \\ \dfrac{\sqrt{2\pi}(\lambda_{\boxbslash}^{(m+1)/2})_+(\lambda_{\boxbslash}^{(m+1)/2})_-}{\Gamma(\frac{1}{2}+m)}\lambda^{-n} \mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})} & -\mathrm{e}^{-2\pi \mathrm{i}m} \end{bmatrix}, \quad \lambda \in L_{\boxbslash}^0. \tag{1.15}$$

(3) ***Asymptotics:*** *$\mathbf{Y}(\lambda) \to \mathbb{I}$ as $\lambda \to \infty$. Also, the matrix function $\mathbf{Y}(\lambda)\lambda_{\boxbslash}^{-(\Theta_0+\Theta_\infty)\sigma_3/2} = \mathbf{Y}(\lambda)\lambda_{\boxbslash}^{-(m+\frac{1}{2})\sigma_3}$ has a well-defined limit as $\lambda \to 0$ (the same limit from each side of L).*

Here, $\lambda_{\boxbslash}^p$ is notation for a certain well-defined (see Section 4.2 below) branch of the power function with its branch cut on the contour $L_{\boxbslash}^0 \cup L_{\boxbslash}^\infty$, $\sigma_3 := \mathrm{diag}[1, -1]$ denotes a standard Pauli spin matrix, and subscripts $+/-$ refer to boundary values taken on the indicated contour from the left/right. We introduce the expansions

$$\mathbf{Y}(\lambda) = \mathbb{I} + \mathbf{Y}_1^\infty(x)\lambda^{-1} + O(\lambda^{-2}), \quad \lambda \to \infty; \quad \mathbf{Y}_1^\infty(x) = \left[Y_{1,jk}^\infty(x)\right]_{j,k=1}^2 \tag{1.16}$$

and

$$\mathbf{Y}(\lambda)\lambda_{\boxbslash}^{-(m+\frac{1}{2})\sigma_3} = \mathbf{Y}_0^0(x) + O(\lambda), \quad \lambda \to 0; \quad \mathbf{Y}_0^0(x) = \left[Y_{0,jk}^0(x)\right]_{j,k=1}^2. \tag{1.17}$$

Note that the matrix coefficients $\mathbf{Y}_1^\infty(x)$ and $\mathbf{Y}_0^0(x)$ depend parametrically on both n and m, as well as x. Then we have the following result.

Theorem 1. *The rational solution $u_n(x; m)$ of the Painlevé-III equation* (1.1) *with parameters m and $n \in \mathbb{Z}$ defined in Proposition 2 and extended to negative integral n by inversion $\mathcal{I}$ is given equivalently in terms of the solution $\mathbf{Y}^{(n)}(\lambda; x, m)$ of Riemann-Hilbert Problem 1 by*

$$u_n(x; m) = \frac{-\mathrm{i}Y_{1,12}^\infty(x)}{Y_{0,11}^0(x)Y_{0,12}^0(x)} \tag{1.18}$$

where we have suppressed the parametric dependence on $n \in \mathbb{Z}$ and $m \in \mathbb{C}$ on the right-hand side.

The proof of this theorem will be completed at the end of Section 5. Finally, in Section 6 we study how the Riemann-Hilbert representation degenerates when $m \in \mathbb{Z} + \frac{1}{2}$.

2. Numerical Observations and Formal Scaling Theory

2.1. **Scaling analysis.** Eliminating Θ_0 and Θ_∞ in favor of m and n by (1.2), the Painlevé-III equation (1.1) becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{4(n+m)u^2 + 4(n-m)}{x} + 4u^3 - \frac{4}{u}. \tag{2.1}$$

Considering m fixed and n large, we introduce a new independent variable by the scaling $x = ny$, and then to further zoom in on the neighborhood of a particular point y_0 we set $y = y_0 + w/n$. A simple calculation then shows that if we set $p(w) := -\mathrm{i}u(x) = -\mathrm{i}u(ny_0 + w)$, (2.1) becomes

$$\frac{\mathrm{d}^2 p}{\mathrm{d}w^2} = \frac{1}{p}\left(\frac{\mathrm{d}p}{\mathrm{d}w}\right)^2 + \frac{4\mathrm{i}}{y_0}(p^2 - 1) - 4p^3 + \frac{4}{p} + O(n^{-1})$$

where the final term combines several others all of which are proportional to n^{-1}. Neglecting this formally small term and replacing p with the symbol $\dot{p}$ indicating a formal approximation yields an autonomous nonlinear equation parametrized by $y_0 \in \mathbb{C} \setminus \{0\}$:

$$\frac{\mathrm{d}^2 \dot{p}}{\mathrm{d}w^2} = \frac{1}{\dot{p}}\left(\frac{\mathrm{d}\dot{p}}{\mathrm{d}w}\right)^2 + \frac{4\mathrm{i}}{y_0}(\dot{p}^2 - 1) - 4\dot{p}^3 + \frac{4}{\dot{p}}. \tag{2.2}$$

This model equation admits a first integral: multiply (2.2) through by $\dot{p}'/\dot{p}^2$ ($\prime = \mathrm{d}/\mathrm{d}w$) and rearrange to obtain

$$\frac{\dot{p}'\dot{p}''}{\dot{p}^2} - \frac{(\dot{p}')^3}{\dot{p}^3} = 4\left[\frac{\mathrm{i}}{y_0}(1 - \dot{p}^{-2}) - \dot{p} + \dot{p}^{-3}\right]\dot{p}'$$

which is easily integrated to yield

$$\frac{(\dot{p}')^2}{2\dot{p}^2} = 4\left[\frac{\mathrm{i}}{y_0}(\dot{p} + \dot{p}^{-1}) - \frac{1}{2}\dot{p}^2 - \frac{1}{2}\dot{p}^{-2}\right] + \frac{8C}{y_0^2},$$

where C is a constant of integration. Therefore,

$$\left(\frac{\mathrm{d}\dot{p}}{\mathrm{d}w}\right)^2 = \frac{16}{y_0^2}P(\dot{p}; y_0, C), \quad P(\dot{p}; y_0, C) := -\frac{y_0^2}{4}\dot{p}^4 + \frac{\mathrm{i}y_0}{2}\dot{p}^3 + C\dot{p}^2 + \frac{\mathrm{i}y_0}{2}\dot{p} - \frac{y_0^2}{4}. \tag{2.3}$$

Suppose that y_0 and C are such that the quartic $P(\dot{p}; y_0, C)$ has a double root $\dot{p} = p_0$; eliminating C between the equations $P(p_0; y_0, C) = 0$ and $P'(p_0; y_0, C) = 0$ shows that p_0 is a solution of the quartic equation

$$y_0 p_0^4 - \mathrm{i}p_0^3 + \mathrm{i}p_0 - y_0 = 0. \tag{2.4}$$

Obviously, $p_0^2 - 1$ is a factor of the left-hand side: $y_0 p_0^4 - \mathrm{i}p_0^3 + \mathrm{i}p_0 - y_0 = (p_0^2 - 1)(y_0(p_0^2 + 1) - \mathrm{i}p_0)$, so there are four possibilities for double roots of $P(\dot{p}; y_0, C)$, namely:

$$p_0 = 1, \quad p_0 = -1, \quad p_0 = p_0^+(y_0) := \frac{\mathrm{i}}{2y_0} - \mathrm{i}\sqrt{\frac{1}{4y_0^2} + 1}, \quad p_0 = p_0^-(y_0) := \frac{\mathrm{i}}{2y_0} + \mathrm{i}\sqrt{\frac{1}{4y_0^2} + 1}. \tag{2.5}$$

Note that since the quartic equation (2.4) is the same equation as arises upon setting $\dot{p} = p_0$ and neglecting derivatives of $\dot{p}$ in (2.2), the four values (2.5) are precisely the equilibrium solutions of the differential equation (2.2). The corresponding values of C are then obtained explicitly from the equation $P'(p_0; y_0, C) = 0$, which is linear in C (and the coefficient of C is nonzero in each case):

$$C = -\frac{\mathrm{i}y_0}{4p_0} - \frac{3\mathrm{i}y_0}{4}p_0 + \frac{y_0^2}{2}p_0^2. \tag{2.6}$$

Thus, whenever C is given by (2.6) and p_0 is a root of the quartic equation (2.4) (equivalently, an equilibrium solution of (2.2)),

$$P(\dot{p}; y_0, C) = -\frac{y_0^2}{4}(\dot{p} - p_0)^2(\dot{p}^2 + b\dot{p} + c), \quad \text{where} \quad b := 2p_0 - \frac{2\mathrm{i}}{y_0}, \quad c := \frac{1}{p_0^2}.$$

For each fixed (y_0, C) pair, the root locus of $P(\dot{p}; y_0, C)$ is invariant under $\dot{p} \mapsto 1/\dot{p}$. Since ± 1 are individually fixed by this involution while the other two possible double roots listed in (2.5) are permuted by this involution, we see that if there exists a double root distinct from 1 or -1, then there are two distinct double roots and hence $P(\dot{p}; y_0, C)$ factors as a perfect square of a quadratic with distinct roots. If one of the points ± 1 is a double root, then either all four roots coincide, the two remaining roots coalesce at ∓ 1, or the two remaining roots are distinct simple roots that are permuted by the involution.

2.2. **Experiments and conjectures.** To begin to assess the validity of predictions following from the above formal large-n scaling arguments, we may try to examine a finite number of the functions $u_n(x;m)$, say for $n = 0, 1, 2, \dots, N$, and plot their poles and zeros in x. Since according to Proposition 2, $u_n(x;m) \to 1$ as $x \to \infty$ and $u_n(x;m)$ is rational in x with simple poles and zeros only, such plots actually convey complete information. In practice, it is substantially more efficient for large n to implement the polynomial recurrence scheme of Umemura/Clarkson than to directly iterate the Bäcklund transformation (1.3). Therefore, we symbolically compute a sufficient number of the polynomials s_n, which have coefficients rational in m. Then by using rational values[3] for the real and imaginary parts of m, we may apply the *Mathematica*[4] routine `NSolve` with the option `WorkingPrecision->30` to obtain accurate approximations of the roots. We then plot separately the roots of the four polynomial factors in the representation (1.7). As long as the roots of the factors are simple and distinct, no information is lost in making such a plot; this is known to be the case [7, 8] unless $m \in \mathbb{Z} + \frac{1}{2}$, in which case for large enough n there is a common root of high order at the origin in all four factors, leading to a high degree of cancellation. We restrict our numerical calculations of poles and zeros to nonnegative values of n and to $\mathrm{Re}(m) \geq 0$ without loss of generality, compare (1.8) and (1.9).

Since the scaling formalism is based at first on the scaling $x = ny$, it is useful to initially view the plots of poles/zeros of $u_n(x;m)$ in the y-plane. Figures 1–4 study the convergence properties of the pole/zero patterns in the y-plane as n increases for several values of $m \in \mathbb{C}$. The key feature evident in the plots of Figures 1, 2, and 3 is that while there

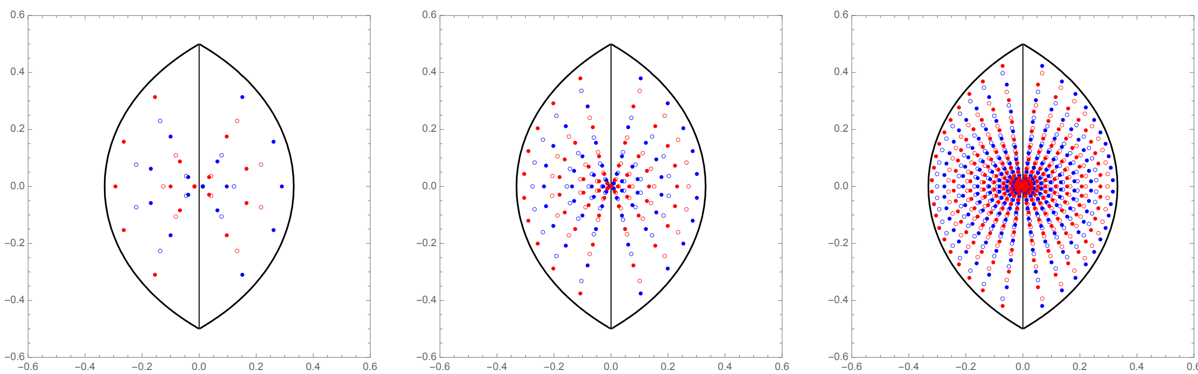

FIGURE 1. Poles of $u_n(x;m)$ (red dots, filled for the roots of $s_n(x;m)$ and unfilled for the roots of $s_{n-1}(x;m-1)$) and zeros of $u_n(x;m)$ (blue dots, filled for the roots of $s_n(x;m-1)$ and unfilled for the roots of $s_{n-1}(x;m)$) rendered in the $y = x/n$-plane for $m = 0$. Left: $n = 5$, center: $n = 10$, right: $n = 20$. The black curves are independent of n and m and form the boundaries of two half-eye-shaped regions known to contain the poles and zeros of $u_n(x;m)$ for large n [3].

is some variability with the value of $m \in \mathbb{C}$, as n increases the region of the y-plane that contains the poles and zeros of $u_n(ny;m)$ appears to stabilize to an eye-shaped domain E that is independent of both n and m. Figure 4 shows a similar convergence study, here for a half-integral value of m. While the poles and zeros seem to move toward the same eye-shaped domain E as n increases, the distribution of poles and zeros within E appears to be completely different than in Figures 1–3, with poles and zeros concentrating only along one "eyebrow" of the eye E.

Taken together, these figures suggest that $u_n(ny;m)$ may have a well-defined limit as $n \to \infty$ as long as y is restricted to the exterior of E. We are led to formulate the following conjecture.

Conjecture 1. *Assume that y lies outside of a certain eye-shaped bounded domain $E \subset \mathbb{C}$. Then*

$$\lim_{n\to\infty} u_n(ny;m) = \mathrm{i}p_0^+(y), \tag{2.7}$$

where $p_0^+(y)$ is defined by (2.5) *in which the square root refers to the principal branch.*

This conjecture asserts that for y outside of E, the quartic $P(\dot{p};y,C)$ has a distinct pair of double roots at $\dot{p} = p_0^{\pm}(y)$, and that the equilibrium $\dot{p} = p_0^+(y)$ (we are identifying y with the constant y_0) is the relevant solution of the autonomous

[3]We observed that if the real or imaginary part of m is irrational then `NSolve` performs poorly for moderately large n.

[4]We used *Mathematica* version 11.

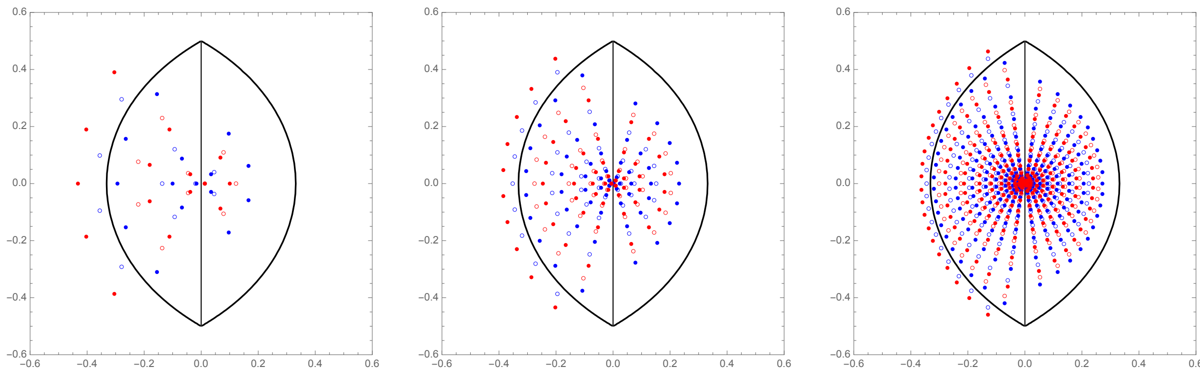

FIGURE 2. As in Figure 1 but for $m = 1$.

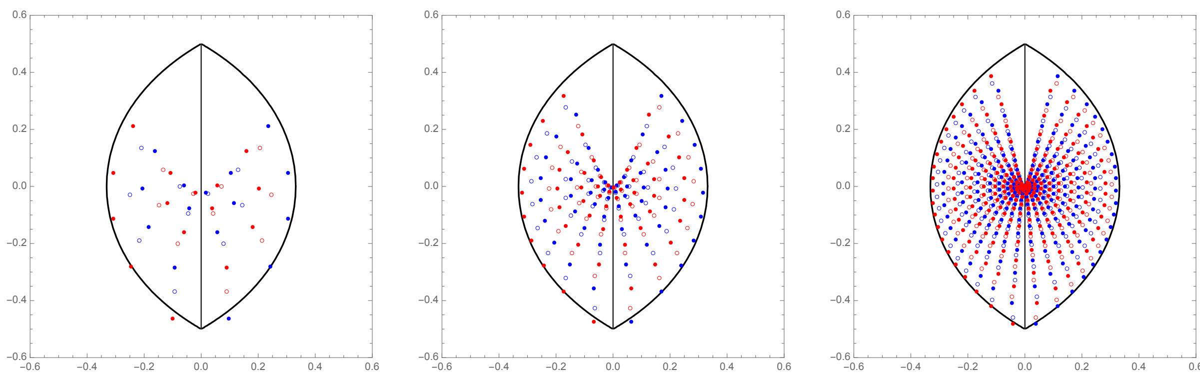

FIGURE 3. As in Figure 1 but for $m = \frac{4}{5}\mathrm{i}$.

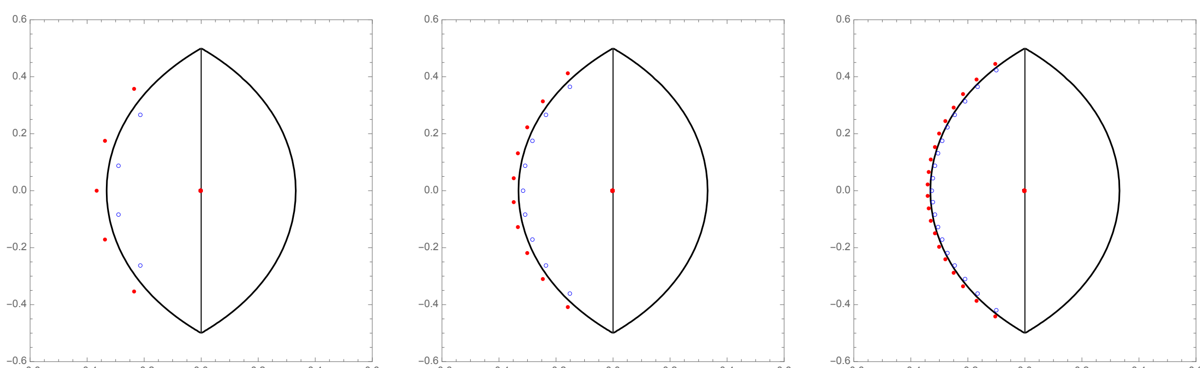

FIGURE 4. As in Figure 1 but for $m = \frac{1}{2}$. Here we know from [8] that the apparent pole near the origin in the plots is an artifact of our method of plotting separately the roots of the polynomial factors in (1.7); in fact $u_n(x; \frac{1}{2})$ has a simple zero at $x = 0$.

model differential equation (2.2). Note that $\mathrm{i}p_0^+(y)$ is independent of the second parameter m, and $\mathrm{i}p_0^+(y) \to 1$ as $y \to \infty$, which is consistent with the fact that for each fixed n, $u_n(x; m) \to 1$ as $x \to \infty$. A suitably precise version of Conjecture 1 is proven in [3] using the Riemann-Hilbert representation of $u_n(x; m)$ presented in Theorem 1 formulated in Section 1.2; part of the proof is to correctly specify the domain E and characterize its boundary ∂E as branches of a certain zero locus in the complex y-plane. Indeed, the black curves shown in Figures 1–4 are fully described in [3]; in particular the top and bottom corners of the domain E lie at the points $y = \pm\frac{1}{2}\mathrm{i}$.

The asymptotic pattern of poles and zeros of $u_n(x; m)$ is qualitatively similar to that shown in Figure 4 whenever $m \in \mathbb{Z} + \frac{1}{2}$, but different details emerge as m is increased through half-integers as illustrated in Figure 5. From these

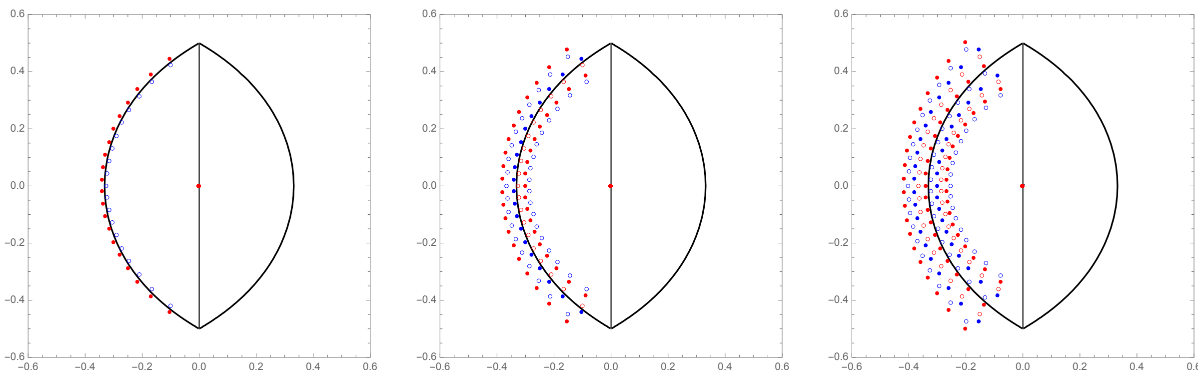

FIGURE 5. As in Figure 4 but for $n = 20$ and $m = \frac{1}{2}$ (left), $m = \frac{3}{2}$ (center), and $m = \frac{5}{2}$ (right).

plots we may formulate a second conjecture.

Conjecture 2. *Suppose that $m = \frac{1}{2} + k$, $k \in \mathbb{Z}_{\geq 0}$. Then as $n \to \infty$, the poles and zeros of $u_n(ny, m)$ accumulate near the left boundary arc of the domain E in the y-plane. In more detail, the poles and zeros are arranged along $4k + 2$ non-intersecting arcs roughly parallel to and $o(1)$ distance from the left boundary arc of E. The outermost curve contains n poles of $u_n(ny; m)$ coming from roots of $s_n(ny; m)$ and moving inwards the next curve contains $n - 1$ zeros of $u_n(ny; m)$ coming from roots of $s_{n-1}(ny; m)$. If $k > 0$ there are then k families of four nested curves each; the j^{th} family lies to the outside of the $j + 1^{\text{st}}$ and consists of (in order from outside to inside, $j = 1, \dots, k$):*

- *A curve containing $n - j + 1$ zeros of $u_n(ny; m)$ coming from roots of $s_n(ny; m - 1)$.*
- *A curve containing $n - j$ poles of $u_n(ny; m)$ coming from roots of $s_{n-1}(ny; m - 1)$.*
- *A curve containing $n - j$ poles of $u_n(ny; m)$ coming from roots of $s_n(ny; m)$.*
- *A curve containing $n - j - 1$ zeros of $u_n(ny; m)$ coming from roots of $s_{n-1}(ny; m)$.*

A suitably precise form of Conjecture 2 is proven in [3] using classical steepest descent analysis for certain Hankel systems with Bessel function coefficients derived from Riemann-Hilbert Problem 1 in Section 6 below.

Comparing Figures 1–3 with Figures 4–5 makes clear that the asymptotic behavior of $u_n(x; m)$ cannot possibly be uniform with respect to m in any neighborhood of a half-integral value. It appears to therefore be compelling to investigate how $u_n(x; m)$ behaves if n is large while simultaneously m is close to a given half-integer. Such an experiment is reproduced in Figure 6. This figure suggests that if m is taken to be very close to a half-integer, the majority of the poles and zeros of $u_n(x; m)$ are captured in the midst of a process in which they are collapsing toward the origin, leaving just a small fraction of them near the left (for positive half-integer m) "eyebrow". In this situation, the domain containing the majority of the poles and zeros appears to be smaller than the full domain E. This collapse process can be studied [3] with the help of Theorem 1 and asymptotic analysis in a double-scaling limit in which n is large and m differs from a half-integer by an exponentially small amount. The green curve plotted in Figure 6 is one of the outcomes of this analysis. The same analysis shows that the convergence claimed in Conjecture 1 also holds for y in the annular region between the boundary of E and the green curve, as well as near the right "eyebrow" (but something more like Conjecture 2 occurs near the left "eyebrow").

Taking now $m \notin \mathbb{Z} + \frac{1}{2}$, an interesting question suggested by the scaling analysis above is whether $u_n(ny_0 + w; m)$ behaves asymptotically (as a function of w for fixed $y_0 \in E$) like an elliptic function solving (2.3) for a suitable choice of integration constant C such that the quartic P has four distinct roots. To investigate this, we select a point y_0 in the domain E and display in Figure 7 the poles and zeros of $u_n(ny_0 + w; m)$ in the w-plane. This figure suggests that

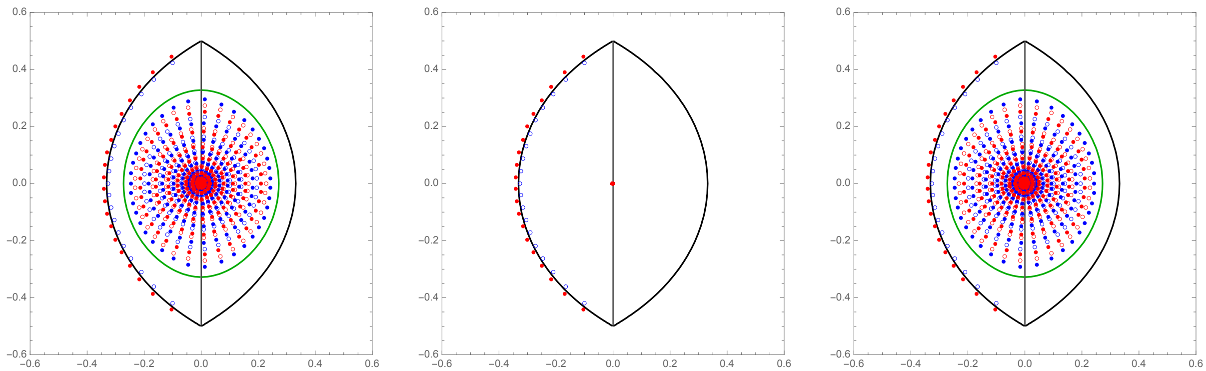

FIGURE 6. As in Figure 1 but for $n = 20$ and $m = \frac{1}{2} - 10^{-4}$ (left), $m = \frac{1}{2}$ (center), and $m = \frac{1}{2} + 10^{-4}$ (right). Superimposed in green is another curve that better approximates the central pole/zero region in a double-scaling limit where n grows while m approaches a half-integer [3].

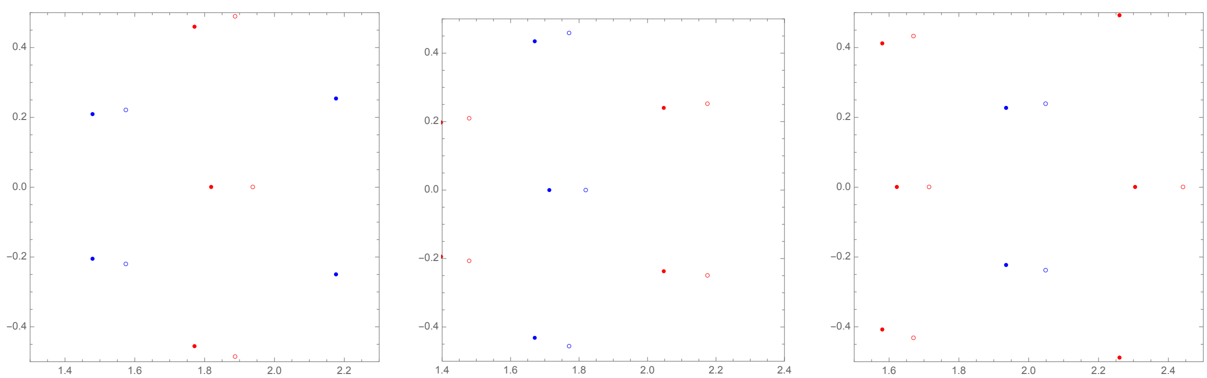

FIGURE 7. As in Figure 1 but plotted in the w-plane for $m = 0$ and $y_0 = 0.1$ with $n = 18$ (left), $n = 19$ (center), and $n = 20$ (right).

indeed for given large n, the poles and zeros are arranged roughly in a doubly-periodic lattice, with the lattice becoming more rigid as n increases. An important observation is that the lattice does not appear to become fixed as n increases, although its lattice vectors do. To the contrary, there appears to be a strong fluctuation of the offset of the lattice as n is increased in integer increments. These observations are consistent with the approximation of $u_n(ny_0 + w; m)$ by a family of solutions of the autonomous elliptic function differential equation (2.3) differing by an n-dependent shift in the argument w. We formulate this as a conjecture.

Conjecture 3. *Assume that $m \notin \mathbb{Z} + \frac{1}{2}$ is fixed, and fix $y_0 \in E$. Then there is a solution $\dot{p} = \dot{p}_n(w; y_0)$ (an elliptic function of w) of the differential equation* (2.3) *for suitable $C = C(y_0)$ such that the quartic P has distinct roots, for which*

$$\lim_{n\to\infty} \left(u_n(ny_0 + w; m) - \mathrm{i}\dot{p}_n(w; y_0)\right) = 0. \tag{2.8}$$

This conjecture is proved in [3] using Theorem 1. Part of the proof involves isolating the correct value of the integration constant C given $y_0 \in E$. It is also important in the proof that y_0 not lie on the imaginary axis, which is excluded from E as shown in Figures 1–6. Also, w should be restricted to a bounded domain that excludes arbitrarily small fixed neighborhoods of certain lattice points.

We have already pointed out that the two "corner points" of the eye-shaped domain E occur at the values $y = y_0 = \pm\frac{1}{2}\mathrm{i}$. These values are the only ones for which the quartic P can have only one four-fold root. This particularly severe degeneration of the quartic suggests that the rational solution $u_n(x; m)$ may behave in a special way for large n when

$x \approx \pm\frac{1}{2}\mathrm{i}n$, a notion that is reinforced by another suitable rescaling of (2.1). Indeed, to localize $y = x/n$ near $y_0 = \pm\frac{1}{2}\mathrm{i}$, we set $x = \pm\mathrm{i}(\frac{1}{2}n + (\frac{1}{32}n)^{1/3}\xi^{\pm})$ and consider $\xi^{\pm}$ to be bounded. Similarly, since $p_0^+(\pm\frac{1}{2}\mathrm{i}n) = \pm 1$, we wish to localize u near $\pm\mathrm{i}$ so we set $u = \pm\mathrm{i}(1 - (\frac{1}{4}n)^{-1/3}W^{\pm})$ and consider $W^{\pm}$ to be bounded. (The exponents of $\pm\frac{1}{3}$ are chosen to achieve a dominant balance, and the numerical coefficients of $\frac{1}{32}$ and $\frac{1}{4}$ are chosen for convenience.) Making these substitutions, we multiply (2.1) through by $\mp\frac{1}{8}\mathrm{i}xu(x)$ and obtain

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\xi^2} = 2W^3 + \xi W + m + O(n^{-1/3}), \quad \xi = \xi^{\pm}, \quad W = W^{\pm},$$

where again the final term combines several others all proportional to $n^{-1/3}$ or more negative powers of n. Neglecting the error terms and relabeling W as $\dot{W}$ yields as a model equation

$$\frac{\mathrm{d}^2 \dot{W}}{\mathrm{d}\xi^2} = 2\dot{W}^3 + \xi\dot{W} + m \tag{2.9}$$

which is the Painlevé-II equation with parameter m. Based on this calculation, we may expect that when n is large and m is held fixed, the rational Painlevé-III functions behave near the points $x = \pm\frac{1}{2}\mathrm{i}n$ like certain solutions of the Painlevé-II equation (2.9); moreover, the dependence on the fixed parameter m becomes apparent at leading order in this approximation. To explore this possibility, we plot the poles and zeros of $u_n(x;m)$ in the $\xi^{\pm}$ planes for two fixed values of m and for increasing n in Figures 8–11.

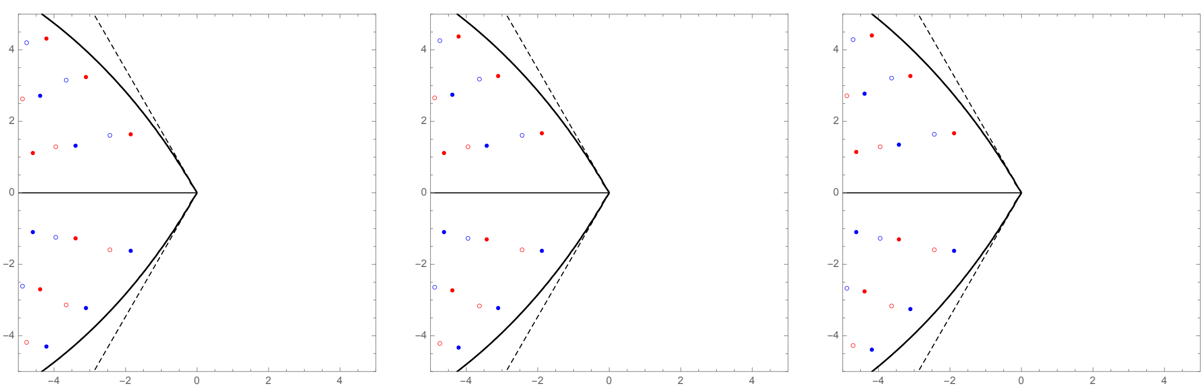

FIGURE 8. As in Figure 1 but plotted in the ξ^+-plane for $m = 0$ and $n = 18$ (left), $n = 19$ (center), and $n = 20$ (right). Also shown with dashed lines are the rays $\mathrm{Arg}(\xi^+) = \pm\frac{2}{3}\pi$, which are the tangents to the boundary of E at the upper corner.

In each of these figures, the three plots for consecutive reasonably large values of n are nearly indistinguishable to the eye, suggesting convergence to a particular solution of (2.9) independent of n. To try to identify the relevant particular solutions, we may start with the outer approximation given in Conjecture 1 and re-express it in terms of the recentered and rescaled independent variables $\xi^{\pm}$, taking careful account of the principal branch interpretation of the square root in (2.5). Thus, $u_n(x;m) \approx \mathrm{i}p_0^+(y) = \mathrm{i}p_0^+(n^{-1}x) = \pm\mathrm{i}2^{1/6}n^{-1/3}(\xi^{\pm})^{1/2} + O(n^{-2/3}\xi^{\pm})$ assuming that Conjecture 1 holds and that $\xi^{\pm}$ is small compared to $n^{2/3}$. If this expression is to agree in some overlap domain with an approximation based on the Painlevé-II equation (2.9), we should express $W = W^{\pm}$ in terms of $u_n(x;m) \approx \mathrm{i}p_0^+(y)$. Thus, $W^{\pm} = (\frac{1}{4}n)^{1/3}(1 \pm \mathrm{i}u_n(x;m)) \approx (\frac{1}{4}n)^{1/3}(1 \mp p_0^+(y)) = \pm\mathrm{i}(\frac{1}{2}\xi^{\pm})^{1/2} + O(n^{-1/3}\xi^{\pm})$ if also $\xi^{\pm}$ is small compared to $n^{1/3}$. Assumption of an overlap domain then suggests that the relevant solutions of the Painlevé-II equation (2.9) should satisfy $\dot{W}^{\pm} \sim \pm\mathrm{i}(\frac{1}{2}\xi^{\pm})^{1/2}$ as $\xi^{\pm} \to \infty$ in the exterior domain where the outer approximation is valid. In the limit $n \to \infty$, this region corresponds to the sector $\mathrm{Arg}(\xi^{\pm}) \in (-\frac{2}{3}\pi, \frac{2}{3}\pi)$. It is known that [11, Chapter 11] for each complex m there are two and only two solutions of the Painlevé-II equation (2.9) denoted $\dot{W} = \dot{W}^{\pm}(\xi;m)$ with the asymptotic behavior $\dot{W}^{\pm}(\xi;m) \sim \pm\mathrm{i}(\frac{1}{2}\xi)^{1/2}$ as $\xi \to \infty$ with $|\mathrm{Arg}(\xi)| \leq \frac{2}{3}\pi - \epsilon$ for $\epsilon > 0$ sufficiently small, where the one-half power denotes the principal branch. These are known as (increasing) *tritronquée* solutions of (2.9). We are led to formulate the following conjecture.

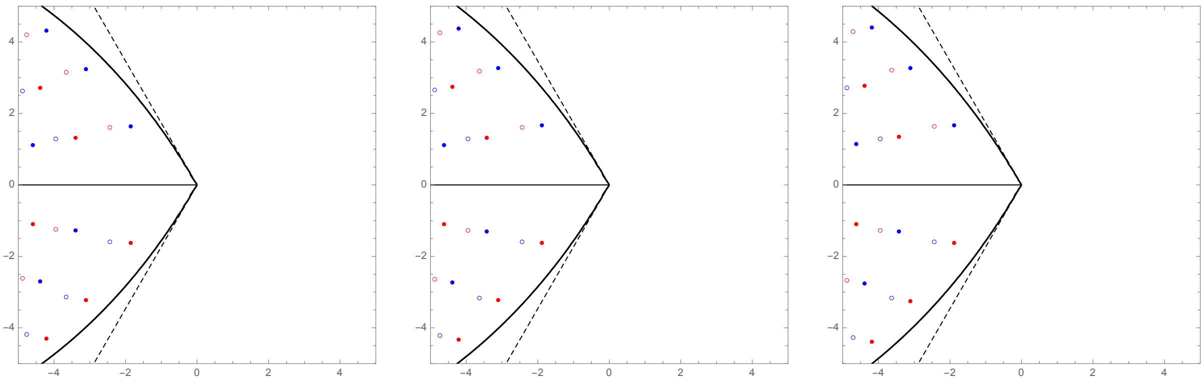

FIGURE 9. As in Figure 1 but plotted in the ξ^{-}-plane for $m = 0$ and $n = 18$ (left), $n = 19$ (center), and $n = 20$ (right). Also shown with dashed lines are the rays $\mathrm{Arg}(\xi^{-}) = \pm\frac{2}{3}\pi$, which are the tangents to the boundary of E at the lower corner.

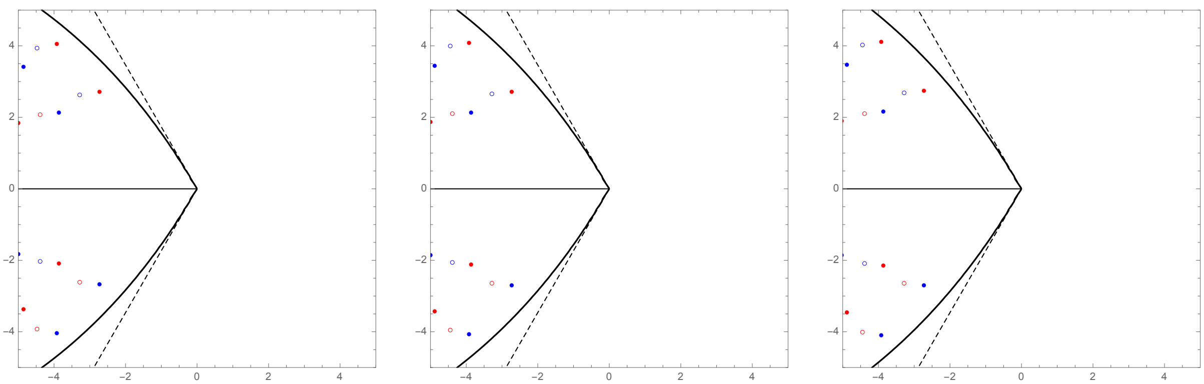

FIGURE 10. As in Figure 8 (zooming into the upper corner of the domain E) but for $m = \frac{4}{5}\mathrm{i}$.

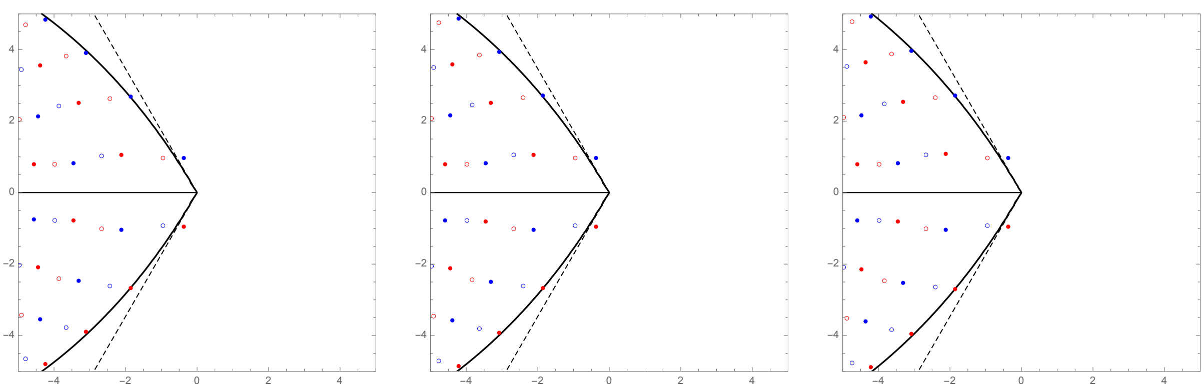

FIGURE 11. As in Figure 9 (zooming into the lower corner of the domain E) but for $m = \frac{4}{5}\mathrm{i}$.

Conjecture 4. *Let $m \in \mathbb{C}$ be fixed. Then,*

$$\lim_{n\to\infty} \left(\frac{1}{4}n\right)^{1/3} (1 \pm \mathrm{i}u_n(\pm\mathrm{i}(\tfrac{1}{2}n + (\tfrac{1}{32}n)^{1/3}\xi); m) = \dot{W}^{\pm}(\xi; m), \tag{2.10}$$

where $\dot{W} = \dot{W}^{\pm}(\xi; m)$ are the aforementioned increasing tritronquée solutions of the Painlevé-II equation (2.9).

The convergence might be expected to be uniform on compact subsets of the ξ-plane from which arbitrarily small open disks centered at the poles of the tritronquée solution in question have been excised. The assertion that the particular solutions of (2.9) should be of tritronquée type means that they are asymptotically analytic in a sector of the complex ξ-plane of opening angle $\frac{4}{3}\pi$, consistent with the plots in Figures 8–11. Tronquée and tritronquée solutions of the Painlevé-II equation (2.9) were originally studied long ago by Boutroux; see also Joshi and Mazzocco [16] and Novokshenov [21]. When $m = 0$, the Painlevé-II equation (2.9) has the obvious symmetry $\dot{W}(\xi) \mapsto -\dot{W}(\xi)$, and by uniqueness of the two tritronquée solutions this means that $\dot{W}^-(\xi; 0) = -\dot{W}^+(\xi; 0)$. Comparing Figures 8–9 we therefore expect a sign change while the figures clearly show instead some sort of reciprocation, with poles and zeros of $u_n(x; m)$ being exchanged. The explanation for this lies in the relation $u = \pm\mathrm{i}(1 - (\frac{1}{4}n)^{-1/3}W^\pm)$, which shows that both poles and zeros of u correspond to $W^\pm$ becoming very large; in other words, both the red and the blue dots in Figures 8–11 should be attracted in the limit $n \to +\infty$ toward the fixed simple poles of the corresponding tritronquée solution of the Painlevé-II equation (2.9). More to the point, assuming the validity of Conjecture 4 with the suggested nature of convergence, one may apply the argument principle to the rational function $u_n(\pm\mathrm{i}(\frac{1}{2}n + (\frac{1}{32}n)^{1/3}\xi); m)$ about a Jordan curve C in the ξ-plane that encloses exactly one pole of the corresponding tritronquée solution of (2.9). The index (increment of the argument) of u_n about C is zero for sufficiently large n because u_n converges uniformly on C to $\pm\mathrm{i}$ as $\dot{W}^\pm$ is analytic and therefore bounded on C. This means that in fact *each pole of the Painlevé-II tritronquée would be expected to attract (in the ξ-plane) an equal number of poles and zeros of u_n in the large-n limit*. One can see the indicated pairing of poles with zeros in Figures 8–11, although with larger values of n the phenomenon should become even more obvious to the eye.

Remark 2. *While tritronquée solutions are by definition asymptotically (i.e., for large $|\xi|$) pole-free in a certain sector of the complex plane, the pole-free property is not a priori guaranteed in any bounded region of the complex-plane. However, recently it was shown* [9] *that all tritronquée solutions of the Painlevé-I equation are actually analytic down to the origin in the asymptotically pole-free sector, proving a conjecture of Dubrovin. See* [1, 14] *for related results on certain solutions of the Painlevé-II equation* (2.9). *It is not known whether the tritronquée solutions $\dot{W}^\pm(\xi; m)$ of the Painlevé-II equation are exactly pole-free in the sector $-\frac{2}{3}\pi < \mathrm{Arg}(\xi) < \frac{2}{3}\pi$. Because we expect pole/zero pairs of u_n to converge toward fixed poles of $\dot{W}^\pm$ in the ξ-plane, in our opinion the plots shown in Figures 8–11 are not sufficiently resolved (i.e., n is not sufficiently large) to provide convincing evidence one way or the other, even though Figure 11 shows some poles and zeros of u_n lying in the asymptotic pole-free sector for $\dot{W}^-(\xi; \frac{4}{5}\mathrm{i})$ near the origin.*

The origin $x = 0$ is a fixed singular point of the Painlevé-III equation (1.1) and its presence appears to affect the pattern of poles and zeros of $u_n(x; m)$ close to the origin if $m \notin \mathbb{Z} + \frac{1}{2}$, as can be seen in Figures 1–3. In particular, the density of the regular distribution of poles and zeros within the domain E seems to blow up as $y_0 \to 0$, a phenomenon that is confirmed by the asymptotic analysis in [3]. However, this accumulation phenomenon cannot be uniformly valid in any neighborhood of the origin because $u_n(x; m)$ is rational. Our numerical computations suggest that the x-distance of the smallest poles and zeros of $u_n(x; m)$ to the origin scales as n^{-1} when n is large, which suggests introducing into (2.1) the scaling $x = n^{-1}z$ and considering n large for m bounded. Then (2.1) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{4u^2 + 4}{z} + O(n^{-1}), \tag{2.11}$$

which is a perturbation of the parameter-free PIII_3 equation

$$\frac{\mathrm{d}^2 \dot{u}}{\mathrm{d}z^2} = \frac{1}{\dot{u}}\left(\frac{\mathrm{d}\dot{u}}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}\dot{u}}{\mathrm{d}z} + \frac{4\dot{u}^2 + 4}{z} \tag{2.12}$$

(arising from the general Painlevé-III equation in the special case $\gamma = \delta = 0$, see [12, Section 2.2]). We may therefore expect that $u_n(n^{-1}z; m)$ should behave like a particular solution (or possibly a family of particular solutions parametrized by m and/or n) of this limiting equation when n is large and z is bounded. To explore this possibility, we plotted the poles and zeros of $u_n(n^{-1}z; m)$ in the complex z-plane for two different fixed values of m and increasing large n in Figures 12 and 13. Noting the alternation in the pattern of poles and zeros with increasing n in each case and taking into account the symmetry $\dot{u} \mapsto -\dot{u}^{-1}$ of (2.12) leads to the following conjecture.

Conjecture 5. *Let $m \in \mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$ be given. Then there exists a corresponding particular solution $\dot{u}(z; m)$ of the m-independent model equation* (2.12) *such that*

$$\lim_{j\to\infty} u_{2j}((2j)^{-1}z; m) = \dot{u}(z; m) \quad \textit{and} \quad \lim_{j\to\infty} u_{2j+1}((2j+1)^{-1}z; m) = -\dot{u}(z; m)^{-1}. \tag{2.13}$$

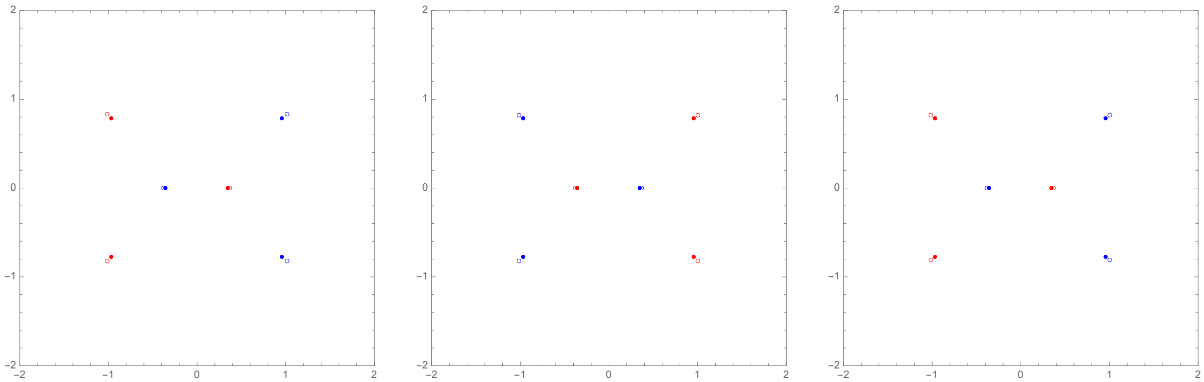

FIGURE 12. As in Figure 1 but plotted in the z-plane for $m = 0$ and $n = 18$ (left), $n = 19$ (center), and $n = 20$ (right).

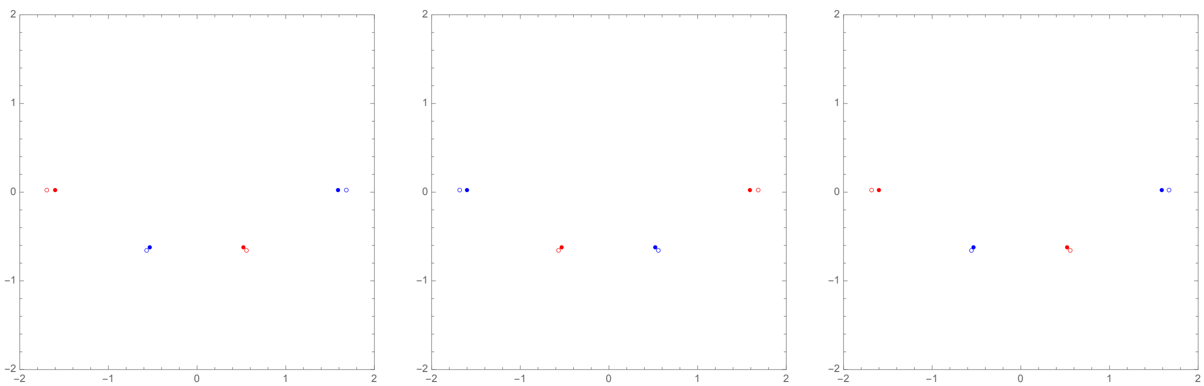

FIGURE 13. As in Figure 1 but plotted in the z-plane for $m = \frac{4}{5}\mathrm{i}$ and $n = 18$ (left), $n = 19$ (center), and $n = 20$ (right).

The reason for excluding half-integral values of m from this statement is that $u_n(x;m)$ has either a simple pole or a simple zero at the origin [8] for such m and asymptotic analysis [3] shows convergence to a function of $y = x/n$ (the analytic continuation of $\mathrm{i}p_0^+(y)$ to the complement of the "eyebrow"), which would correspond under rescaling either to $\dot{u} \equiv 0$ or $\dot{u} \equiv \infty$; moreover, this limit is independent of whether n is odd or even. Naturally, this discrepancy raises again the question of how the solution behaves near the origin in a double-scaling limit of large n and m close to a half-integer.

The asymptotic analysis to establish Conjectures 4 and 5 using Theorem 1 is work in progress. The proof of Conjecture 5 is expected to be particularly challenging because Riemann-Hilbert Problem 1 cannot even be formulated for $x = 0$.

3. Lax Pair and Isomonodromy Theory for the Painlevé-III Equation

The representation of the Painlevé-III equation (1.1) as the compatibility condition for a Lax pair of first-order linear systems was discovered by Jimbo and Miwa [15]. Consider the linear differential equations

$$\frac{\partial \mathbf{\Psi}}{\partial \lambda}(\lambda;x) = \mathbf{A}(\lambda;x)\mathbf{\Psi}(\lambda;x), \quad \mathbf{A}(\lambda;x) := \frac{\mathrm{i}x}{2}\sigma_3 + \frac{1}{\lambda}\begin{bmatrix} -\frac{1}{2}\Theta_\infty & y \\ v & \frac{1}{2}\Theta_\infty \end{bmatrix} + \frac{1}{\lambda^2}\begin{bmatrix} \frac{1}{2}\mathrm{i}x - \mathrm{i}st & \mathrm{i}s \\ -\mathrm{i}t(st-x) & -\frac{1}{2}\mathrm{i}x + \mathrm{i}st \end{bmatrix}, \tag{3.1}$$

and

$$\frac{\partial \mathbf{\Psi}}{\partial x}(\lambda;x)=\mathbf{B}(\lambda;x)\mathbf{\Psi}(\lambda;x),\quad \mathbf{B}(\lambda;x):=\frac{\mathrm{i}\lambda}{2}\sigma_3+\frac{1}{x}\begin{bmatrix}0 & y\\ v & 0\end{bmatrix}-\frac{1}{\lambda x}\begin{bmatrix}\frac{1}{2}\mathrm{i}x-\mathrm{i}st & \mathrm{i}s\\ -\mathrm{i}t(st-x) & -\frac{1}{2}\mathrm{i}x+\mathrm{i}st\end{bmatrix}. \tag{3.2}$$

Here, Θ_∞ is a constant parameter and $y=y(x)$, $v=v(x)$, $s=s(x)$, and $t=t(x)$ are coefficient functions[5] (potentials). The matrix coefficient of λ^{-2} in (3.1) and of $-(\lambda x)^{-1}$ in (3.2) looks complicated, but it simply represents the most general matrix having $\pm\frac{1}{2}\mathrm{i}x$ as its eigenvalues (all such matrices depend on two parameters whose roles are played by $s(x)$ and $t(x)$). The compatibility condition $\mathbf{A}_x-\mathbf{B}_\lambda+[\mathbf{A},\mathbf{B}]=\mathbf{0}$ for the simultaneous equations (3.1)–(3.2) is the first-order system of nonlinear differential equations

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x}&=-2xs+\Theta_\infty y, \qquad & x\frac{\mathrm{d}v}{\mathrm{d}x}&=-2xt(st-x)-\Theta_\infty v,\\ x\frac{\mathrm{d}s}{\mathrm{d}x}&=(1-\Theta_\infty)s-2xy+4yst, \qquad & x\frac{\mathrm{d}t}{\mathrm{d}x}&=\Theta_\infty t-2yt^2+2v. \end{aligned} \tag{3.3}$$

This system admits an integral of motion:

$$I:=\frac{2\Theta_\infty}{x}st-\Theta_\infty-\frac{2}{x}yt(st-x)+\frac{2}{x}vs \tag{3.4}$$

is a conserved quantity, i.e, (3.3) implies that $\mathrm{d}I/\mathrm{d}x=0$ holds identically. Using (3.3) one can show that the combination

$$u(x):=-\frac{y(x)}{s(x)} \tag{3.5}$$

satisfies the differential equation

$$x\frac{\mathrm{d}u}{\mathrm{d}x}=2x-(1-2\Theta_\infty)u+4stu^2-2xu^2. \tag{3.6}$$

Taking another x-derivative and letting Θ_0 denote the constant value of the integral I one then obtains the Painlevé-III equation in the form (1.1). (For some details of these calculations, see the last lines of the proof of Lemma 2 in Section 5.2 below.) The isomonodromy method algorithm for solving the initial-value problem for (1.1) with initial conditions $u(x_0)=u_0$ and $u'(x_0)=u_0'$ is then the following [11]. Given constants $(\Theta_0,\Theta_\infty,x_0,u_0,u_0')\in\mathbb{C}^5$ with $x_0u_0\neq 0$,

(1) Choose an arbitrary nonzero initial value of y: $y(x_0)=y_0\neq 0$. Then from (3.5) at $x=x_0$ one obtains the initial value of s: $s_0:=s(x_0)=-y_0/u_0$, which is well-defined and nonzero. Next, since $s_0u_0^2=-u_0y_0\neq 0$, $t_0:=t(x_0)$ is well-defined from (3.6) at $x=x_0$:

$$t_0=\frac{1}{4u_0y_0}\left(2x_0-(1-2\Theta_\infty)u_0-2x_0u_0^2-x_0u_0'\right). \tag{3.7}$$

Finally, from (3.4) using $I=\Theta_0$ and substituting for s_0 and t_0 we get the initial value of v: $v_0:=v(x_0)$ where

$$v_0=\frac{1}{16y_0u_0^2}\left(4x_0^2+(1-4\Theta_\infty^2)u_0^2-4x_0u_0-8\Theta_0x_0u_0^3-4x_0^2u_0^4-4x_0^2u_0'+2x_0u_0u_0'+x_0^2u_0'^2\right). \tag{3.8}$$

Note that s_0 is proportional, while t_0 and v_0 are inversely proportional, to the arbitrary[6] nonzero constant y_0.

(2) Taking $y=y_0$, $v=v_0$, $s=s_0$, $t=t_0$, and $x=x_0\neq 0$, seek four specific fundamental solution matrices of (3.1) called *canonical solutions*, namely two satisfying the normalization condition

$$\mathbf{\Psi}\lambda^{\Theta_\infty\sigma_3/2}\mathrm{e}^{-\mathrm{i}x\lambda\sigma_3/2}\to\mathbb{I},\quad \lambda\to\infty \tag{3.9}$$

in two different abutting sectors with opening angle π and bisected by directions in which the factors $\mathrm{e}^{\pm\mathrm{i}x\lambda}$ are oscillatory; and two satisfying the normalization condition

$$\begin{bmatrix}a(x) & b(x)s(x)\\ a(x)t(x) & b(x)(s(x)t(x)-x)\end{bmatrix}^{-1}\mathbf{\Psi}\lambda^{-\Theta_0\sigma_3/2}\mathrm{e}^{\mathrm{i}x\lambda^{-1}\sigma_3/2}\to\mathbb{I},\quad \lambda\to 0, \tag{3.10}$$

[5]Our parametrization of the Lax system (3.1)–(3.2) differs from that of Jimbo and Miwa [15], who instead of $s(x)$ and $t(x)$ worked with the combinations (in the notation of [11]) $U(x):=s(x)t(x)$ and $w(x):=t(x)^{-1}$. The parametrization (3.1)–(3.2) has the advantage that the singularities of the potentials y, v, and s are exactly the singularities of the simultaneous solution $\mathbf{\Psi}$ with respect to the parameter x.

[6]Given any constant $\alpha\neq 0$, the system of equations (3.3) is obviously invariant under the substitution $(y(x),v(x),s(x),t(x))\mapsto(\alpha y(x),\alpha^{-1}v(x),\alpha s(x),\alpha^{-1}t(x))$, which also leaves $u(x)$ defined by (3.5) invariant.

in two different abutting sectors with opening angle π and bisected by directions in which the factors $e^{\pm ix\lambda^{-1}}$ are oscillatory. In (3.10), $a(x)$ and $b(x)$ are arbitrary except that the determinant of the matrix factor on the left should be equal to 1 and therefore $a(x)b(x) = -x^{-1}$. The two fundamental matrices near $\lambda = 0$ are obviously related by right-multiplication by one λ-independent *Stokes matrix* for each of the two sector boundary arcs; similarly for the fundamental solution matrices near $\lambda = \infty$. A fifth *connection matrix* relates the solution in one sector near $\lambda = 0$ to that in one sector near $\lambda = \infty$. The four Stokes matrices and the connection matrix constitute the solution of the *direct monodromy problem*.

(3) The equation (3.2) implies that the Stokes matrices and the connection matrix are independent of x when y, v, s, and t evolve according to (3.3); this is the isomonodromy property of the representation (3.1)–(3.2). Hence, letting $x \in \mathbb{C}$ be arbitrary, solve the *inverse monodromy (Riemann-Hilbert) problem* of determining the four fundamental solution matrices from the jump conditions relating them via right-multiplication by the Stokes matrices and the connection matrix and from the asymptotic normalization conditions (3.9)–(4.4). From the solution of this problem the coefficients (y, v, s, t) of equation (3.1) can then be extracted and from them u is obtained for $x \neq x_0$ from (3.5).

4. Monodromy Data for $u(x) = u_0(x; m) = 1$

In the special case that $\Theta_0 = \Theta_\infty - 1$, i.e., $n = 0$ for arbitrary $m \in \mathbb{C}$, the Painlevé-III equation (1.1) has the rational (constant) solutions $u(x) = \pm 1$. Our aim in this section is to calculate the necessary monodromy data so that the solution $u(x) = 1$ can be obtained from an appropriate Riemann-Hilbert problem. Although this appears to involve the study of the direct problem (3.1) alone, our approach will be to leverage the compatibility with the isomonodromic deformation (3.2) to solve the latter equation instead and then build in additional dependence on λ via integration constants to satisfy (3.1) as well. With these results in hand, in Section 5 we will apply Schlesinger transformations to increment/decrement by 2 the value of the difference $\Theta_\infty - \Theta_0 = 1 - 2n$ and thus obtain a Riemann-Hilbert representation for the Bäcklund chain of rational solutions with seed solution $u(x) = 1$.

4.1. **The Lax pair for $\Theta_0 = \Theta_\infty - 1$ and $u(x) = 1$.** Since we will be exploiting the differential equation (3.2) to construct the monodromy data, we need to know how the coefficients (y, v, s, t) depend on x. From (3.5) with $u(x) \equiv 1$ we find that $s(x) \equiv -y(x)$, so the differential equation for $y(x)$ in (3.3) closes as a linear equation with solution

$$y(x) = -\frac{1}{4}Ke^{2x}x^{\Theta_\infty} \quad \text{and hence also} \quad s(x) = \frac{1}{4}Ke^{2x}x^{\Theta_\infty}, \tag{4.1}$$

where $K \neq 0$ is an arbitrary constant of integration. Using this result and $u(x) \equiv 1$ in (3.6) we obtain $t(x)$:

$$t(x) = (1 - 2\Theta_\infty)K^{-1}e^{-2x}x^{-\Theta_\infty}.$$

Finally, using these along with $I = \Theta_0 = \Theta_\infty - 1$ in (3.4), we solve for $v(x)$:

$$v(x) = -\frac{1}{4}(1 - 2\Theta_\infty)(4x + 1 + 2\Theta_\infty)K^{-1}e^{-2x}x^{-\Theta_\infty}.$$

In order that the coefficients in the Lax pair are well-defined, we assume for the purposes of this calculation that $x \in \mathbb{C} \setminus \mathbb{R}_-$ and agree to label the argument of x as being in the interval $(-\pi, \pi)$, i.e., we use the principal branch $\arg(x) = \operatorname{Arg}(x)$. The arbitrary constant K plays a similar role as the arbitrary nonzero initial value $y_0 = y(x_0)$ in the solution of the initial-value problem for (1.1) by the isomonodromy method. Next, introducing into (3.2) the well-defined substitution

$$\mathbf{\Psi} = e^{x\sigma_3}x^{\Theta_\infty\sigma_3/2}x^{-1/2}\mathbf{W},$$

one finds that the first-row matrix entries W_{1j} are solutions W of the confluent hypergeometric equation (cf., [22, Eq. 13.14.1])

$$\frac{d^2W}{d\zeta^2} + \left[-\frac{1}{4} + \frac{\kappa}{\zeta} + \frac{1 - 4\mu^2}{4\zeta^2}\right]W = 0, \quad \mu = \frac{1}{4}, \quad \kappa = \frac{1}{2}(\Theta_\infty - 1), \tag{4.2}$$

where $\zeta := ix(\lambda + 2i - \lambda^{-1})$. The elements W_{2j} of the second row are obtained from those in the first row by the formula

$$W_{2j} = -\frac{4\zeta(W_{1j}'(\zeta) - \frac{1}{2}W_{1j}(\zeta)) + (4\kappa - i(1 - 2\Theta_\infty)\lambda^{-1})W_{1j}(\zeta)}{K(1 + i\lambda^{-1})}.$$

If we fix a fundamental pair of solutions of (4.2) that depend on λ only through the variable ζ as the first row of the matrix $\mathbf{W}$, then the general solution of (3.2) can be written in the form

$$\mathbf{\Psi} = \mathrm{e}^{x\sigma_3} x^{\Theta_\infty \sigma_3/2} x^{-1/2} \mathbf{W}\mathbf{C}(\lambda), \tag{4.3}$$

where $\mathbf{C}(\lambda)$ cannot depend on x but might depend on λ. Having found the general solution of the "x-equation" (3.2) in the Lax pair for the Painlevé-III equation, we can now determine $\mathbf{C}(\lambda)$ such that the expression (4.3) is simultaneously a solution of both (compatible, because $y(x)$, $v(x)$, $w(x)$, and $U(x)$ satisfy (3.3)) equations (3.1)–(3.2). Upon substitution of (4.3) into (3.1) one easily finds that

$$\mathbf{C}(\lambda) = (\lambda + \mathrm{i})^{-1/2}\mathbf{C},$$

where $\mathbf{C}$ is a matrix independent of both x and λ.

4.2. **Normalized simultaneous solutions for** $\mathrm{Im}(x) \neq 0$**.** For the moment, we assume that $\mathrm{Im}(x) \neq 0$ and define x^p (e.g., in (4.3)) by taking $\arg(x) = \mathrm{Arg}(x) \in (-\pi, \pi)$. Later in Section 4.4 we will consider the exceptional cases $\arg(\pm x) = 0$. Our goal now is to determine the values of the matrix $\mathbf{C}$ in order to define the four canonical fundamental solution matrices satisfying the normalization conditions (3.9)–(3.10). Note that (3.10) here takes the form

$$\begin{bmatrix} a(x) & b(x)\frac{1}{4}K\mathrm{e}^{2x}x^{\Theta_\infty} \\ a(x)K^{-1}(1-2\Theta_\infty)\mathrm{e}^{-2x}x^{-\Theta_\infty} & b(x)\frac{1}{4}(1-2\Theta_\infty - 4x)\end{bmatrix}^{-1} \mathbf{\Psi}\lambda^{-\Theta_0\sigma_3/2}\mathrm{e}^{\mathrm{i}x\lambda^{-1}\sigma_3/2} \to \mathbb{I}, \quad \lambda \to 0 \tag{4.4}$$

where

$$a(x)b(x) = -\frac{1}{x}. \tag{4.5}$$

To specify these four solutions carefully, we should make sure that the power functions λ^p for various p appearing in the normalization conditions, as well as the scalar factor $(\lambda+\mathrm{i})^{-1/2}$ and the solutions W of the confluent hypergeometric equation (4.2) that are chosen for the first row of the matrix $\mathbf{W}$ are all unambiguous. We do this as follows. Firstly, we note that according to the Wronskian identity [22, Eq. 13.14.30], we may choose as a fundamental pair of solutions of (4.2) the two Whittaker functions $W_{11} := W_{-\kappa,\mu}(-\zeta)$ and $W_{12} := W_{\kappa,\mu}(\zeta)$. Now, $W_{\pm\kappa,\mu}(z)$ are multi-valued functions, and to be completely unambiguous we select in both cases the principal branches, whose argument z lies in the domain $\arg(z) \in (-\pi, \pi)$. These solutions are related by the identity (cf., [22, Eq. 13.14.13])

$$\lim_{\epsilon\downarrow 0} W_{\pm\kappa,\mu}(-z+\mathrm{i}\epsilon) = \mathrm{e}^{\pm 2\pi\mathrm{i}\kappa}\lim_{\epsilon\downarrow 0} W_{\pm\kappa,\mu}(-z-\mathrm{i}\epsilon) + \frac{2\pi\mathrm{i}\mathrm{e}^{\pm\mathrm{i}\pi\kappa}}{\Gamma(\frac{1}{2}+\mu\mp\kappa)\Gamma(\frac{1}{2}-\mu\mp\kappa)}W_{\mp\kappa,\mu}(z), \quad z > 0, \tag{4.6}$$

and its (negative) derivative

$$\lim_{\epsilon\downarrow 0} W'_{\pm\kappa,\mu}(-z+\mathrm{i}\epsilon) = \mathrm{e}^{\pm 2\pi\mathrm{i}\kappa}\lim_{\epsilon\downarrow 0} W'_{\pm\kappa,\mu}(-z-\mathrm{i}\epsilon) - \frac{2\pi\mathrm{i}\mathrm{e}^{\pm\mathrm{i}\pi\kappa}}{\Gamma(\frac{1}{2}+\mu\mp\kappa)\Gamma(\frac{1}{2}-\mu\mp\kappa)}W'_{\mp\kappa,\mu}(z), \quad z > 0, \tag{4.7}$$

which express jump conditions for $W_{\pm\kappa,\mu}(z)$ and its derivative across the branch cut on the negative real z-axis. We also have the asymptotic behavior (cf., [22, Eq. 13.14.21])

$$W_{\pm\kappa,\mu}(z) = \mathrm{e}^{-z/2}z^{\pm\kappa}(1+O(z^{-1})), \quad z \to \infty, \quad \arg(z) \in (-\pi,\pi),$$

as well as

$$\zeta(W'_{11}(\zeta) - \tfrac{1}{2}W_{11}(\zeta)) = -\kappa\mathrm{e}^{\zeta/2}(-\zeta)^{-\kappa}(1+O(\zeta^{-1})), \quad \zeta\to\infty, \quad \arg(-\zeta)\in(-\pi,\pi),$$

and

$$\zeta(W'_{12}(\zeta) - \tfrac{1}{2}W_{12}(\zeta)) = -\mathrm{e}^{-\zeta/2}\zeta^{\kappa+1}(1+O(\zeta^{-1})), \quad \zeta\to\infty, \quad \arg(\zeta)\in(-\pi,\pi),$$

and in these last three relations the indicated power functions all have their principal values. Now, with the principal branches selected, given $\mathrm{Arg}(x) \in (-\pi,\pi)$, the matrix $\mathbf{W}$ becomes a well-defined analytic function of λ, henceforth denoted $\mathbf{W} = \mathbf{W}(x,\lambda)$, defined in the complement of the preimage under ζ of the real axis. This x-dependent preimage is therefore the jump contour L for $\mathbf{W}$, and it takes different forms for $-\pi < \mathrm{Arg}(x) < 0$ and $0 < \mathrm{Arg}(x) < \pi$; see Figure 14. Given a value of x with $\mathrm{Im}(x) \neq 0$ and a corresponding jump contour L as illustrated in this figure, we will now define the multivalued functions λ^p and $(\lambda+\mathrm{i})^{-1/2}$ precisely as follows. For λ^p, we take as a branch cut $L^{\infty}_{\boxbslash} \cup L^{0}_{\boxbslash}$. Furthermore, noting that as x varies in the upper half-plane $L^{\infty}_{\boxbslash}$ sweeps through the left half λ-plane, we define $\arg(\lambda) = 0$ for sufficiently large positive λ when $\mathrm{Im}(x) > 0$. Similarly, as x varies in the lower half-plane $L^{\infty}_{\boxbslash}$ sweeps through the right half λ-plane and we therefore define $\arg(\lambda) = \pi$ for $\lambda < 0$ of sufficiently large magnitude when $\mathrm{Im}(x) < 0$. This choice of branch along with the cut $L^{\infty}_{\boxbslash} \cup L^{0}_{\boxbslash}$ unambiguously determines $\arg(\lambda)$ and hence λ^p

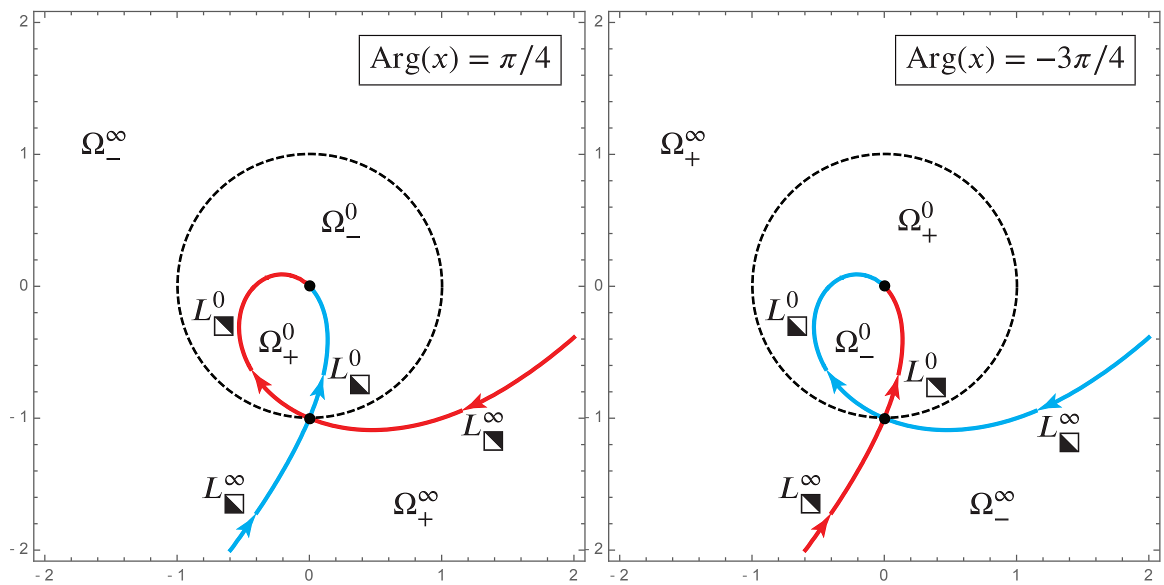

FIGURE 14. The jump contour L for the Whittaker matrix $\mathbf{W}(x,\lambda)$ takes a different form depending on whether $0 < \mathrm{Arg}(x) < \pi$ (left) or $-\pi < \mathrm{Arg}(x) < 0$ (right). The arcs $L^{\infty}_{\text{⬔}}$ and $L^{0}_{\text{⬔}}$ (red) are where $\zeta < 0$, and the arcs $L^{\infty}_{\text{⬕}}$ and $L^{0}_{\text{⬕}}$ (cyan) are where $\zeta > 0$. All four contour arcs meet at the only zero of ζ, namely $\lambda = -\mathrm{i}$. Together with the unit circle (dotted), the contour arcs divide the complex λ-plane into four disjoint domains as indicated, $\Omega^{0}_{\pm}$ adjacent to $\lambda = 0$ and where $\pm\mathrm{Im}(\zeta) > 0$ holds, and unbounded domains $\Omega^{\infty}_{\pm}$ where $\pm\mathrm{Im}(\zeta) > 0$ holds. The subscript notation ⬕/⬔ on the contour arcs is a mnemonic for the lower/upper triangular structure of jump matrices defined below (cf., (4.34)–(4.35)) that will be carried by the corresponding contour arcs.

for any $p \in \mathbb{C}$ given x with $\mathrm{Im}(x) \neq 0$. We use the notation $\lambda^{p}_{\text{⬕}}$ to indicate this branch. Note that if $\arg_{\text{⬕}}(\lambda)$ denotes the value of the argument corresponding to this choice of branch we have

$$-\frac{\pi}{2} - \mathrm{Arg}(x) < \arg_{\text{⬕}}(\lambda) < \frac{3\pi}{2} - \mathrm{Arg}(x), \quad |\lambda| \to \infty, \tag{4.8}$$

while

$$\mathrm{Arg}(x) - \frac{\pi}{2} < \arg_{\text{⬕}}(\lambda) < \mathrm{Arg}(x) + \frac{3\pi}{2}, \quad |\lambda| \to 0. \tag{4.9}$$

Then, to define $(\lambda + \mathrm{i})^{-1/2}$, we select $L^{\infty}_{\text{⬕}}$ as the branch cut and for $\mathrm{Im}(x) > 0$ we take $(\lambda + \mathrm{i})^{-1/2}$ to be positive for sufficiently positive values of $\lambda + \mathrm{i}$, while for $\mathrm{Im}(x) < 0$ we take $(\lambda + \mathrm{i})^{-1/2}$ to be negative imaginary for sufficiently negative values of $\lambda + \mathrm{i}$. We denote the resulting well-defined function as $(\lambda + \mathrm{i})^{-1/2}_{\text{⬕}}$. With this choice, we have in particular that

$$(\lambda + \mathrm{i})^{-1/2}_{\text{⬕}} = \mathrm{e}^{-\mathrm{i}\pi/4} + O(\lambda), \quad \lambda \to 0, \tag{4.10}$$

and

$$(\lambda + \mathrm{i})^{-1/2}_{\text{⬕}} = \lambda^{-1/2}_{\text{⬕}}(1 + O(\lambda^{-1})), \quad \lambda \to \infty. \tag{4.11}$$

With these definitions in hand, we now construct the four normalized solutions for $u(x) = 1$ as analytic functions of λ in the four disjoint domains $\Omega^{\infty}_{\pm}$ and $\Omega^{0}_{\pm}$. We will denote the resulting piecewise-analytic simultaneous matrix solution of (3.1)–(3.2) by $\mathbf{\Psi}(\lambda; x)$.

4.2.1. *Defining* $\mathbf{\Psi}(\lambda; x)$ *for* $\lambda \in \Omega^{\infty}_{+}$. We define $\mathbf{\Psi}(\lambda; x)$ for $\lambda \in \Omega^{\infty}_{+}$ by the formula

$$\mathbf{\Psi}(\lambda; x) = \mathrm{e}^{x\sigma_3} x^{\Theta_\infty \sigma_3/2} x^{-1/2} (\lambda + \mathrm{i})^{-1/2}_{\text{⬕}} \mathbf{W}(x,\lambda)\mathbf{C}^{\infty}_{+}, \quad \lambda \in \Omega^{\infty}_{+}, \tag{4.12}$$

and we determine the constant matrix $\mathbf{C}_+^\infty$ so that $\mathbf{\Psi} = \mathbf{\Psi}(\lambda; x)$ satisfies (3.9) (with $\lambda^{\Theta_\infty\sigma_3/2}$ defined carefully as $\lambda_{\blacksquare}^{\Theta_\infty\sigma_3/2}$) in the limit $\lambda \to \infty$ in Ω_+^∞. Note that the precisely-defined factor $(\lambda + \mathrm{i})_{\blacksquare}^{-1/2}$ satisfies (4.11), and that when $\lambda \to \infty$ the Whittaker matrix $\mathbf{W}(x, \lambda)$ takes the following asymptotic form:

$$\mathbf{W}(x,\lambda) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 4K^{-1}\zeta \end{bmatrix} + O(\lambda^{-1})\right)\begin{bmatrix} \mathrm{e}^{\zeta/2}(-\zeta)^{-\kappa} & 0 \\ 0 & \mathrm{e}^{-\zeta/2}\zeta^{\kappa} \end{bmatrix}, \quad \lambda \to \infty. \tag{4.13}$$

This can be further simplified by recalling that $\zeta = \mathrm{i}x(\lambda + 2i - \lambda^{-1})$ is large when λ is large, and making use of the fact that the expressions $(\pm\zeta)^{\pm\kappa}$ refer to the principal branch. Indeed, by definition $\mathrm{Im}(\zeta) > 0$ and $\mathrm{Im}(-\zeta) < 0$ hold for λ in the domain Ω_+^∞. Therefore to define $(-\zeta)^{-\kappa}$ by the principal branch we need to have $-\pi < \arg(-\zeta) < 0$ or, for large λ, $-\pi < \arg(-\mathrm{i}x\lambda(1+O(\lambda^{-1})) < 0$. Writing $\arg(-\mathrm{i}x\lambda(1+O(\lambda^{-1}))) = -\frac{1}{2}\pi + \mathrm{Arg}(x) + \arg_{\blacksquare}(\lambda) + \mathrm{Arg}(1+O(\lambda^{-1})) + 2\pi\ell$, $\ell \in \mathbb{Z}$, where $\arg_{\blacksquare}(\lambda)$ satisfies (according to Figure 14 and (4.8) for large $\lambda \in \Omega_1^\infty$) $\arg_{\blacksquare}(\lambda) + \mathrm{Arg}(x) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we see that $\ell = 0$, and therefore $(-\zeta)^{-\kappa} = \mathrm{e}^{\mathrm{i}\pi\kappa/2}x^{-\kappa}\lambda_{\blacksquare}^{-\kappa}(1 + O(\lambda^{-1}))$ as $\lambda \to \infty$ in Ω_+^∞, where $x^{-\kappa}$ refers to the principal branch. Similarly, to define ζ^κ by the principal branch we need to have $0 < \arg(\zeta) < \pi$ or for large λ, $0 < \arg(\mathrm{i}x\lambda(1 + O(\lambda^{-1}))) < \pi$. Writing $\arg(\mathrm{i}x\lambda(1 + O(\lambda^{-1}))) = \frac{1}{2}\pi + \mathrm{Arg}(x) + \arg_{\blacksquare}(\lambda) + \mathrm{Arg}(1 + O(\lambda^{-1})) + 2\pi\ell$ and again using $\arg_{\blacksquare}(\lambda) + \mathrm{Arg}(x) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ gives $\ell = 0$ so that $\zeta^\kappa = \mathrm{e}^{\mathrm{i}\pi\kappa/2}x^\kappa\lambda_{\blacksquare}^\kappa(1 + O(\lambda^{-1}))$ as $\lambda \to \infty$ in Ω_+^∞, where again x^κ is the principal branch. Putting these results together gives

$$\mathbf{\Psi}(\lambda; x)\mathrm{e}^{-\mathrm{i}x\lambda\sigma_3/2}\lambda_{\blacksquare}^{\Theta_\infty\sigma_3/2} = \left(\begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi\kappa/2} & 0 \\ 0 & 4K^{-1}\mathrm{e}^{\mathrm{i}\pi(\kappa+1)/2} \end{bmatrix} + O(\lambda^{-1})\right) \cdot \lambda_{\blacksquare}^{-\Theta_\infty\sigma_3/2}\mathrm{e}^{\mathrm{i}x\lambda\sigma_3/2}\mathbf{C}_+^\infty\mathrm{e}^{-\mathrm{i}x\lambda\sigma_3/2}\lambda_{\blacksquare}^{\Theta_\infty\sigma_3/2}, \quad \lambda \to \infty, \quad \lambda \in \Omega_+^\infty. \tag{4.14}$$

Since Ω_+^∞ contains directions in which both exponential factors $\mathrm{e}^{\pm\mathrm{i}x\lambda}$ are exponentially large as $\lambda \to \infty$, this can only have a finite limit if $\mathbf{C}_+^\infty$ is a diagonal matrix, in which case the correct normalization requires that

$$\mathbf{C}_+^\infty := \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\kappa/2} & 0 \\ 0 & -\frac{\mathrm{i}}{4}K\mathrm{e}^{-\mathrm{i}\pi\kappa/2} \end{bmatrix}. \tag{4.15}$$

Using this formula for $\mathbf{C}_+^\infty$ in (4.12) completes the precise definition of $\mathbf{\Psi}(\lambda; x)$ for $\lambda \in \Omega_+^\infty$.

4.2.2. *Defining* $\mathbf{\Psi}(\lambda; x)$ *for* $\lambda \in \Omega_-^\infty$. In a similar way, we define $\mathbf{\Psi}(\lambda; x)$ for $\lambda \in \Omega_-^\infty$ by the formula

$$\mathbf{\Psi}(\lambda; x) = \mathrm{e}^{x\sigma_3}x^{\Theta_\infty\sigma_3/2}x^{-1/2}(\lambda + \mathrm{i})_{\blacksquare}^{-1/2}\mathbf{W}(x,\lambda)\mathbf{C}_-^\infty, \quad \lambda \in \Omega_-^\infty \tag{4.16}$$

and we determine $\mathbf{C}_-^\infty$ so that $\mathbf{\Psi} = \mathbf{\Psi}(\lambda; x)$ satisfies (3.9) with $\lambda^{\Theta_\infty\sigma_3/2}$ interpreted as $\lambda_{\blacksquare}^{\Theta_\infty\sigma_3/2}$ in the limit $\lambda \to \infty$ with $\lambda \in \Omega_-^\infty$. Again we may use both (4.11) and (4.13), and it remains to interpret the principal branch power functions $(\pm\zeta)^{\pm\kappa}$ appearing in (4.13). Now by definition, $\mathrm{Im}(\zeta) < 0$ and $\mathrm{Im}(-\zeta) > 0$ hold for $\lambda \in \Omega_-^\infty$, so for the principal branch powers we have $-\pi < \arg(\zeta) < 0$ and $0 < \arg(-\zeta) < \pi$. Writing $\arg(\zeta) = \arg(\mathrm{i}x\lambda(1 + O(\lambda^{-1}))) = \frac{1}{2}\pi + \mathrm{Arg}(x) + \arg_{\blacksquare}(\lambda) + \mathrm{Arg}(1 + O(\lambda^{-1})) + 2\pi\ell$, $\ell \in \mathbb{Z}$, and taking into account that $\arg_{\blacksquare}(\lambda) + \mathrm{Arg}(x) \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$ according to Figure 14 and (4.8) we find that $\ell = -1$ and so $\zeta^\kappa = \mathrm{e}^{-3\pi\mathrm{i}\kappa/2}x^\kappa\lambda_{\blacksquare}^\kappa(1 + O(\lambda^{-1}))$ as $\lambda \to \infty$ from Ω_-^∞ where x^κ is the principal branch. Similarly writing $\arg(-\zeta) = \arg(-\mathrm{i}x\lambda(1 + O(\lambda^{-1}))) = -\frac{1}{2}\pi + \mathrm{Arg}(x) + \arg_{\blacksquare}(\lambda) + \mathrm{Arg}(1 + O(\lambda^{-1})) + 2\pi\ell$ we get that $\ell = 0$ and so $(-\zeta)^{-\kappa} = \mathrm{e}^{\mathrm{i}\pi\kappa/2}x^{-\kappa}\lambda_{\blacksquare}^{-\kappa}(1 + O(\lambda^{-1}))$ as $\lambda \to \infty$ from Ω_-^∞ where $x^{-\kappa}$ is the principal branch. Using this information and imposing the normalization condition (3.9) on the formula (4.16) we learn that the matrix $\mathbf{C}_-^\infty$ must again be diagonal for the required limit to exist, and then

$$\mathbf{C}_-^\infty = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\kappa/2} & 0 \\ 0 & -\frac{\mathrm{i}}{4}K\mathrm{e}^{3\pi\mathrm{i}\kappa/2} \end{bmatrix}.$$

Combining this with (4.16) completes the definition of $\mathbf{\Psi}(\lambda; x)$ for $\lambda \in \Omega_-^\infty$.

4.2.3. *Defining* $\mathbf{\Psi}(\lambda;x)$ *for* $\lambda\in\Omega_-^0$. We write $\mathbf{\Psi}(\lambda;x)$ for $\lambda\in\Omega_-^0$ in the form

$$\mathbf{\Psi}(\lambda;x)=\mathrm{e}^{x\sigma_3}x^{\Theta_\infty\sigma_3/2}x^{-1/2}(\lambda+\mathrm{i})_{\boxslash}^{-1/2}\mathbf{W}(x,\lambda)\mathbf{C}_-^0,\quad \lambda\in\Omega_-^0, \tag{4.17}$$

and try to determine the constant matrix $\mathbf{C}_-^0$ such that (4.4) holds (with $\lambda^{-\Theta_0\sigma_3/2}$ carefully interpreted as $\lambda_{\boxslash}^{-\Theta_0\sigma_3/2}$) for some appropriate a and b in the limit $\lambda\to 0$ from Ω_-^0. Note that the precisely-defined factor $(\lambda+\mathrm{i})_{\boxslash}^{-1/2}$ is analytic near $\lambda=0$ and satisfies (4.10), while in the limit $\lambda\to 0$, the Whittaker matrix $\mathbf{W}(x,\lambda)$ takes the following asymptotic form:

$$\mathbf{W}(x,\lambda)=\left(\begin{bmatrix}1 & 1\\ K^{-1}(1-2\Theta_\infty) & K^{-1}(1-2\Theta_\infty-4x)\end{bmatrix}+O(\lambda)\right)\begin{bmatrix}\mathrm{e}^{\zeta/2}(-\zeta)^{-\kappa} & 0\\ 0 & \mathrm{e}^{-\zeta/2}\zeta^{\kappa}\end{bmatrix},\quad \lambda\to 0. \tag{4.18}$$

We carefully interpret the principal branch powers appearing in (4.18) by noting that $\lambda\in\Omega_-^0$ means by definition that $\mathrm{Im}(\zeta)<0$ so we need to have $-\pi<\arg(\zeta)<0$ and $0<\arg(-\zeta)<\pi$. Writing $\arg(\zeta)=\arg(-\mathrm{i}x\lambda^{-1}(1+O(\lambda)))=-\frac{1}{2}\pi+\mathrm{Arg}(x)-\arg_{\boxslash}(\lambda)+\mathrm{Arg}(1+O(\lambda))+2\pi\ell$, $\ell\in\mathbb{Z}$, and observing from Figure 14 and (4.9) that λ small and in Ω_-^0 means $\arg_{\boxslash}(\lambda)-\mathrm{Arg}(x)\in(-\frac{1}{2}\pi,\frac{1}{2}\pi)$, we see that $\ell=0$ and so $\zeta^\kappa=\mathrm{e}^{-\mathrm{i}\pi\kappa/2}x^\kappa\lambda_{\boxslash}^{-\kappa}(1+O(\lambda))$ as $\lambda\to 0$ from Ω_-^0 where x^κ is the principal branch. Similarly, writing $\arg(-\zeta)=\arg(\mathrm{i}x\lambda^{-1}(1+O(\lambda))=\frac{1}{2}\pi+\mathrm{Arg}(x)-\arg_{\boxslash}(\lambda)+\mathrm{Arg}(1+O(\lambda))+2\pi\ell$ and again using $\arg_{\boxslash}(\lambda)-\mathrm{Arg}(x)\in(-\frac{1}{2}\pi,\frac{1}{2}\pi)$ we find that $\ell=0$ and so $(-\zeta)^{-\kappa}=\mathrm{e}^{-\mathrm{i}\pi\kappa/2}x^{-\kappa}\lambda_{\boxslash}^{\kappa}(1+O(\lambda))$ as $\lambda\to 0$ from Ω_-^0 where $x^{-\kappa}$ denotes the principal branch. Using this information in (4.4) we see that for a similar reason that $\mathbf{C}_+^\infty$ as given by (4.15) is diagonal, $\mathbf{C}_-^0$ must be a diagonal matrix, say

$$\mathbf{C}_-^0=\begin{bmatrix}c & 0\\ 0 & d\end{bmatrix} \tag{4.19}$$

with c and d independent of both x and λ, and then $\mathbf{\Psi}=\mathbf{\Psi}(\lambda;x)$ indeed satisfies (4.4) provided that

$$\begin{aligned}a(x)&=\mathrm{e}^{-\mathrm{i}\pi\kappa/2}\mathrm{e}^{-\mathrm{i}\pi/4}c\\ b(x)&=4K^{-1}\mathrm{e}^{-\mathrm{i}\pi\kappa/2}\mathrm{e}^{-\mathrm{i}\pi/4}x^{-1}d.\end{aligned} \tag{4.20}$$

Note that $a(x)$ is independent of x. The unimodularity condition (4.5) is then equivalent to the following condition on the constants c and d:

$$\det(\mathbf{C}_-^0)=cd=-\frac{1}{4}\mathrm{i}K\mathrm{e}^{\mathrm{i}\pi\kappa}. \tag{4.21}$$

Therefore, to completely define $\mathbf{\Psi}(\lambda;x)$ we should simply choose convenient values for c and d consistent with (4.21) and then combine (4.19) with (4.17).

4.2.4. *Defining* $\mathbf{\Psi}(\lambda;x)$ *for* $\lambda\in\Omega_+^0$. We write $\mathbf{\Psi}(\lambda;x)$ for $\lambda\in\Omega_+^0$ in the form

$$\mathbf{\Psi}(\lambda;x)=\mathrm{e}^{x\sigma_3}x^{\Theta_\infty\sigma_3/2}x^{-1/2}(\lambda+\mathrm{i})_{\boxslash}^{-1/2}\mathbf{W}(x,\lambda)\mathbf{C}_+^0,\quad \lambda\in\Omega_+^0, \tag{4.22}$$

for a constant matrix $\mathbf{C}_+^0$ to be determined from the normalization condition (4.4) in which $\lambda^{-\Theta_0\sigma_3/2}$ is interpreted as $\lambda_{\boxslash}^{-\Theta_0\sigma_3/2}$. We may again use (4.10) and (4.18) and it remains to interpret the principal branch power functions ζ^κ and $(-\zeta)^{-\kappa}$ for $\lambda\in\Omega_+^0$. By definition, $\lambda\in\Omega_+^0$ means $\mathrm{Im}(\zeta)>0$, so $0<\arg(\zeta)<\pi$ and $-\pi<\arg(-\zeta)<0$. Writing $\arg(\zeta)=\arg(-\mathrm{i}x\lambda^{-1}(1+O(\lambda)))=-\frac{1}{2}\pi+\mathrm{Arg}(x)-\arg_{\boxslash}(\lambda)+\mathrm{Arg}(1+O(\lambda))+2\pi\ell$, $\ell\in\mathbb{Z}$, and noting from Figure 14 and (4.9) that λ small in Ω_+^0 means that $\arg_{\boxslash}(\lambda)-\mathrm{Arg}(x)\in(\frac{1}{2}\pi,\frac{3}{2}\pi)$, we obtain $\ell=1$ and therefore $\zeta^\kappa=\mathrm{e}^{3\pi\mathrm{i}\kappa/2}x^\kappa\lambda_{\boxslash}^{-\kappa}(1+O(\lambda)))$ as $\lambda\to 0$ from Ω_+^0 where x^κ is the principal branch. Likewise writing $\arg(-\zeta)=\arg(\mathrm{i}x\lambda^{-1}(1+O(\lambda)))=\frac{1}{2}\pi+\mathrm{Arg}(x)-\arg_{\boxslash}(\lambda)+\mathrm{Arg}(1+O(\lambda))+2\pi\ell$ we see that $\ell=0$ and therefore $(-\zeta)^{-\kappa}=\mathrm{e}^{-\mathrm{i}\pi\kappa/2}x^{-\kappa}\lambda_{\boxslash}^{\kappa}(1+O(\lambda))$ as $\lambda\to 0$ from Ω_+^0 where $x^{-\kappa}$ is the principal branch. Using this information in (4.4) we see that the matrix $\mathbf{C}_+^0$ must be diagonal (again, for similar reasons that $\mathbf{C}_+^\infty$ as given by (4.15) is diagonal):

$$\mathbf{C}_+^0=\begin{bmatrix}g & 0\\ 0 & h\end{bmatrix} \tag{4.23}$$

where the constants g and h are related to $a(x)$ and $b(x)$ by

$$\begin{aligned}a(x)&=\mathrm{e}^{-\mathrm{i}\pi\kappa/2}\mathrm{e}^{-\mathrm{i}\pi/4}g\\ b(x)&=4K^{-1}\mathrm{e}^{3\pi\mathrm{i}\kappa/2}\mathrm{e}^{-\mathrm{i}\pi/4}x^{-1}h.\end{aligned} \tag{4.24}$$

Once again, $a(x)$ is independent of x, and the unimodularity condition (4.5) is then equivalent to

$$\det(\mathbf{C}_+^0) = gh = -\frac{1}{4}\mathrm{i}K\mathrm{e}^{-\mathrm{i}\pi\kappa}. \tag{4.25}$$

Choosing any constants g and h consistent with (4.25) therefore determines $\mathbf{\Psi}(\lambda; x)$ for $\lambda \in \Omega_+^0$ by combining (4.23) with (4.22).

4.3. **Jump matrices for** $\mathrm{Im}(x) \neq 0$**.** Before computing the jump matrices, we will remove the ambiguity of the constants c, d, g, h still present in the definition of $\mathbf{\Psi}(\lambda; x)$ for $\lambda \in \Omega_\pm^0$ in the following way:

- If $\mathrm{Im}(x) > 0$, we choose c and d so that $\mathbf{C}_-^0 = \mathbf{C}_-^\infty$. This is allowed because the diagonal elements of $\mathbf{C}_-^\infty$ obviously also satisfy (4.21) because $2\kappa + 1 = \Theta_\infty$. Similarly, if $\mathrm{Im}(x) < 0$, we choose g and h such that $\mathbf{C}_+^0 = \mathbf{C}_+^\infty$, which is consistent because the diagonal elements of $\mathbf{C}_+^\infty$ satisfy (4.25).
- We then insist that the normalization factors $a(x)$ and $b(x)$ appearing in (4.4) are exactly the same regardless of whether $\lambda \to 0$ from Ω_-^0 or from Ω_+^0.

The first choice implies that at every point $\lambda \neq -\mathrm{i}$ of the unit circle forming the common boundary of Ω_-^∞ and Ω_-^0 (for $\mathrm{Im}(x) > 0$) or the common boundary of Ω_+^∞ and Ω_+^0 (for $\mathrm{Im}(x) < 0$), the boundary values taken by $\mathbf{\Psi}(\lambda; x)$ agree, i.e., *the jump matrix for* $\mathbf{\Psi}(\lambda; x)$ *across the unit circle* $S^1 \setminus \{-\mathrm{i}\}$ *is exactly the identity matrix*. The second choice together with the first implies, in light of (4.20) and (4.24), that the matrices $\mathbf{C}_\pm^0$ are necessarily given by

$$\mathbf{C}_-^0 = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\kappa/2} & 0 \\ 0 & -\frac{1}{4}\mathrm{i}K\mathrm{e}^{3\pi\mathrm{i}\kappa/2} \end{bmatrix} \quad \text{and} \quad \mathbf{C}_+^0 = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\pi\kappa/2} & 0 \\ 0 & -\frac{1}{4}\mathrm{i}K\mathrm{e}^{-\mathrm{i}\pi\kappa/2} \end{bmatrix}.$$

Note that these formulæ do not depend on the sign of $\mathrm{Im}(x)$. Thus, the matrix function $\mathbf{\Psi}(\lambda; x)$ has been determined modulo only the value of the constant $K \neq 0$, as an analytic function of $\lambda \in \mathbb{C} \setminus L$ where $L = L_{\text{⬔}}^\infty \cup L_{\text{⬔}}^0 \cup L_{\text{⬕}}^\infty \cup L_{\text{⬕}}^0$ is the jump contour for the Whittaker matrix $\mathbf{W}$ illustrated with red and cyan curves in Figure 14.

The jump conditions satisfied by $\mathbf{\Psi}(\lambda; x)$ across the four arcs of L oriented as shown in Figure 14 are computed by comparing the formulæ for $\mathbf{\Psi}(\lambda; x)$ on either side using the identities (4.6)–(4.7) together with the fact that $\zeta < 0$ along $L_{\text{⬔}}^0$ and $L_{\text{⬔}}^\infty$ while $\zeta > 0$ along $L_{\text{⬕}}^0$ and $L_{\text{⬕}}^\infty$. One also has to take into account that the factor $(\lambda + \mathrm{i})_{\text{⬕}}^{-1/2}$ changes sign across $L_{\text{⬕}}^\infty$ by definition, but otherwise is analytic. The jump conditions are as follows:

- The arc $L_{\text{⬔}}^\infty$ separates the domain Ω_+^∞ on its left from Ω_-^∞ on its right. Using $\zeta < 0$ for $\lambda \in L_{\text{⬔}}^\infty$ we deduce that

$$\mathbf{\Psi}_+(\lambda; x) = \mathbf{\Psi}_-(\lambda; x)\mathbf{V}_{\text{⬔}}^\infty, \quad \lambda \in L_{\text{⬔}}^\infty \tag{4.26}$$

  where

$$\mathbf{V}_{\text{⬔}}^\infty := \begin{bmatrix} 1 & \frac{1}{4}K\mathrm{e}^{\mathrm{i}\pi\kappa} \cdot \dfrac{2\pi}{\Gamma(\frac{1}{2} + \mu - \kappa)\Gamma(\frac{1}{2} - \mu - \kappa)} \\ 0 & 1 \end{bmatrix}. \tag{4.27}$$

- The arc $L_{\text{⬔}}^0$ separates the domain Ω_-^0 on its left from Ω_+^0 on its right. Using $\zeta < 0$ we get

$$\mathbf{\Psi}_+(\lambda; x) = \mathbf{\Psi}_-(\lambda; x)\mathbf{V}_{\text{⬔}}^0, \quad \lambda \in L_{\text{⬔}}^0 \tag{4.28}$$

  where

$$\mathbf{V}_{\text{⬔}}^0 := \begin{bmatrix} 1 & -\frac{1}{4}K\mathrm{e}^{\mathrm{i}\pi\kappa} \cdot \dfrac{2\pi}{\Gamma(\frac{1}{2} + \mu - \kappa)\Gamma(\frac{1}{2} - \mu - \kappa)} \\ 0 & 1 \end{bmatrix}. \tag{4.29}$$

- The arc $L_{\text{⬕}}^0$ separates the domain Ω_+^0 on its left from Ω_-^0 on its right. Using $\zeta > 0$ we arrive at

$$\mathbf{\Psi}_+(\lambda; x) = \mathbf{\Psi}_-(\lambda; x)\mathbf{V}_{\text{⬕}}^0, \quad \lambda \in L_{\text{⬕}}^0 \tag{4.30}$$

  where

$$\mathbf{V}_{\text{⬕}}^0 := \begin{bmatrix} \mathrm{e}^{2\pi\mathrm{i}\kappa} & 0 \\ (\frac{1}{4}K\mathrm{e}^{\mathrm{i}\pi\kappa})^{-1} \cdot \dfrac{2\pi}{\Gamma(\frac{1}{2} + \mu + \kappa)\Gamma(\frac{1}{2} - \mu + \kappa)} & \mathrm{e}^{-2\pi\mathrm{i}\kappa} \end{bmatrix}. \tag{4.31}$$

- Finally, the arc $L^{\infty}_{\text{⬕}}$ separates the domain Ω^{∞}_{-} on its left from Ω^{∞}_{+} on its right. Using $\zeta > 0$ and taking into account that $(\lambda + \mathrm{i})^{-1/2}_{\text{⬕}}$ changes sign across $L^{\infty}_{\text{⬕}}$ we obtain

$$\mathbf{\Psi}_{+}(\lambda; x) = \mathbf{\Psi}_{-}(\lambda; x)\mathbf{V}^{\infty}_{\text{⬕}}, \quad \lambda \in L^{\infty}_{\text{⬕}}, \tag{4.32}$$

where

$$\mathbf{V}^{\infty}_{\text{⬕}} := \begin{bmatrix} -\mathrm{e}^{-2\pi\mathrm{i}\kappa} & 0 \\ (\frac{1}{4}K\mathrm{e}^{\mathrm{i}\pi\kappa})^{-1} \cdot \dfrac{2\pi}{\Gamma(\frac{1}{2} + \mu + \kappa)\Gamma(\frac{1}{2} - \mu + \kappa)} & -\mathrm{e}^{2\pi\mathrm{i}\kappa} \end{bmatrix}. \tag{4.33}$$

These formulæ may be simplified further by recalling the definitions $\mu = \frac{1}{4}$ and $\kappa = \frac{1}{2}(\Theta_{\infty} - 1)$ (so $\Theta_{\infty} = m + 1$ for $n = 0$ implies $\kappa = \frac{1}{2}m$), using the duplication formula [22, Eq. 5.5.5] $\Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2})$, and choosing

$$K = 2^{m+2}\mathrm{e}^{-\mathrm{i}\pi m/2}.$$

Thus we find

$$\mathbf{V}^{\infty}_{\text{⬔}} = \mathbf{V}^{\infty}_{\text{⬔}}(m) := \begin{bmatrix} 1 & \dfrac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - m)} \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}^{0}_{\text{⬔}} = \mathbf{V}^{0}_{\text{⬔}}(m) := \begin{bmatrix} 1 & -\dfrac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - m)} \\ 0 & 1 \end{bmatrix}, \tag{4.34}$$

$$\mathbf{V}^{0}_{\text{⬕}} = \mathbf{V}^{0}_{\text{⬕}}(m) := \begin{bmatrix} \mathrm{e}^{\mathrm{i}\pi m} & 0 \\ \dfrac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + m)} & \mathrm{e}^{-\mathrm{i}\pi m} \end{bmatrix}, \quad \mathbf{V}^{\infty}_{\text{⬕}} = \mathbf{V}^{\infty}_{\text{⬕}}(m) := \begin{bmatrix} -\mathrm{e}^{-\mathrm{i}\pi m} & 0 \\ \dfrac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + m)} & -\mathrm{e}^{\mathrm{i}\pi m} \end{bmatrix}. \tag{4.35}$$

In the general theory [11] of the direct monodromy problem for (1.1), the Stokes constants are subject to an identity known as the *cyclic relation*. In this setting, the cyclic relation is simply equivalent to the statement that for consistency, the ordered product of the jump matrices around the self-intersection point $\lambda = -\mathrm{i}$ must be the identity:

$$\mathbf{V}^{\infty}_{\text{⬕}}(m)^{-1}\mathbf{V}^{\infty}_{\text{⬔}}(m)^{-1}\mathbf{V}^{0}_{\text{⬕}}(m)\mathbf{V}^{0}_{\text{⬔}}(m) = \mathbb{I}. \tag{4.36}$$

While it is straightforward to check directly that (4.36) holds, this identity is in fact a simple consequence of the way the jump matrices were computed, namely by comparing four functions, each of which admits analytic continuation to a full neighborhood of the self-intersection point $\lambda = -\mathrm{i}$ and that differ only by right-multiplication by constant matrices.

4.4. **The limiting cases of $x > 0$ and $x < 0$.** The jump contour L for the Whittaker matrix $\mathbf{W}(x, \lambda)$ undergoes a bifurcation when x crosses either the positive or negative real axes. The bifurcation that occurs as $\mathrm{Arg}(x)$ passes through zero is illustrated in Figure 15. Clearly, the arcs $L^{0}_{\text{⬕}}$ and $L^{\infty}_{\text{⬕}}$ depend continuously on $\mathrm{Arg}(x)$ near $\mathrm{Arg}(x) = 0$, but the parts of $L^{0}_{\text{⬔}}$ and $L^{\infty}_{\text{⬔}}$ close to the unit circle become interchanged as $\mathrm{Arg}(x)$ passes through zero. However, noting that the matrices $\mathbf{V}^{\infty}_{\text{⬔}}(m)$ and $\mathbf{V}^{0}_{\text{⬔}}(m)$ as defined in (4.34) are inverse to each other, we easily conclude that the jump conditions satisfied by the matrix $\mathbf{\Psi}(\lambda; x)$ actually depend continuously on $\mathrm{Arg}(x)$ near $\mathrm{Arg}(x) = 0$. This makes it possible to define the jump conditions by continuity for $\mathrm{Arg}(x) = 0$. Note also that not only are the branch cuts of the functions $\lambda^{p}_{\text{⬕}}$ and $(\lambda + \mathrm{i})^{-1/2}_{\text{⬕}}$ continuous with respect to $\mathrm{Arg}(x)$ near $\mathrm{Arg}(x) = 0$, but so also are the functions themselves.

On the other hand, as x approaches the negative real axis from above and below, the bifurcation as illustrated in Figure 16 is apparently more serious. Indeed, the arcs of $L^{\infty}_{\text{⬕}}$ and $L^{0}_{\text{⬕}}$ near the unit circle are now interchanged while $L^{\infty}_{\text{⬔}}$ and $L^{0}_{\text{⬔}}$ depend continuously on $\mathrm{Arg}(-x)$. Since, according to (4.35), $\mathbf{V}^{0}_{\text{⬕}}(m)\mathbf{V}^{\infty}_{\text{⬕}}(m) = -\mathbb{I}$, it is not hard to see that in the limit $\mathrm{Arg}(-x) \to 0$ the limiting jump conditions from $\mathrm{Im}(x) > 0$ and $\mathrm{Im}(x) < 0$ differ precisely on the unit circle, by a sign. In terms of the matrix $\mathbf{\Psi}(\lambda; x)$ itself,

$$\lim_{\epsilon \downarrow 0} \mathbf{\Psi}(\lambda; x + \mathrm{i}\epsilon) = \mathrm{sgn}(\ln|\lambda|) \lim_{\epsilon \downarrow 0} \mathbf{\Psi}(\lambda; x - \mathrm{i}\epsilon), \quad x < 0.$$

Naturally, both limiting values correspond to simultaneous solutions of the Painlevé-III Lax pair (3.1)–(3.2) for exactly the same solution $u(x) = 1$; the apparent monodromy in the function $\mathbf{\Psi}(\lambda; x)$ about $x = 0$ can be absorbed into a sign change in the arbitrary constants a and b appearing in (4.4). For practical calculations one has to be careful about the values of the power functions $\lambda^{p}_{\text{⬕}}$ for $|\lambda| < 1$ in taking the limit of $\mathbf{\Psi}(\lambda; x)$ as x approaches a negative real value from

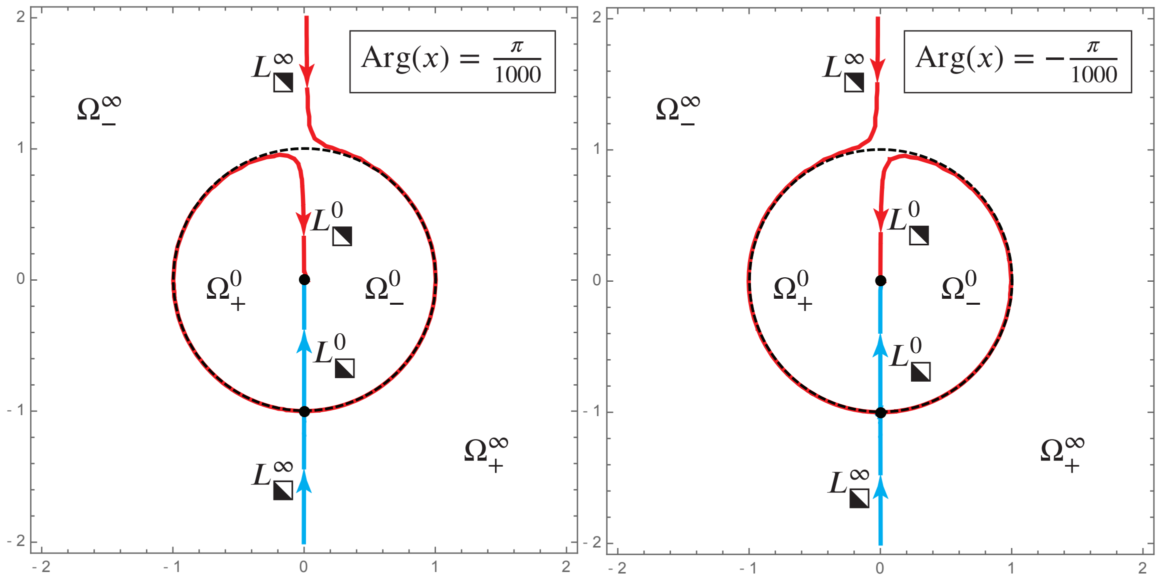

FIGURE 15. As in Figure 14 except for values of x close to the positive real axis.

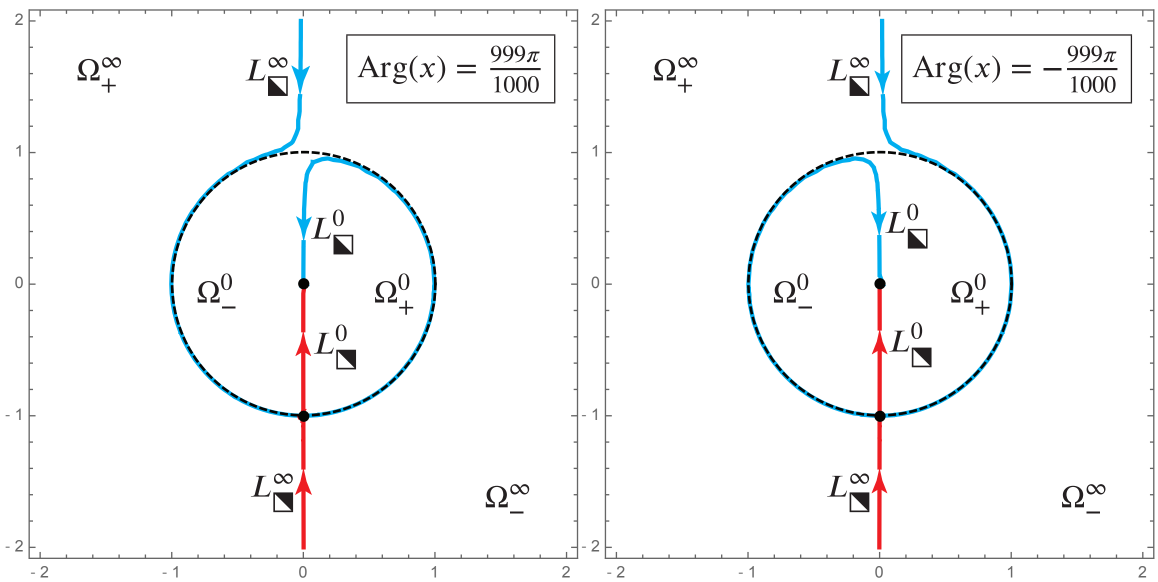

FIGURE 16. As in Figure 14 except for values of x close to the negative real axis.

the upper/lower half-planes. Indeed, keeping track of the dependence of $\arg_{\blacksquare}(\lambda)$ on x with the augmented notation $\arg_{\blacksquare}(\lambda; x)$, we have the identity

$$\lim_{\epsilon\downarrow 0} \arg_{\blacksquare}(\lambda; x + \mathrm{i}\epsilon) = \lim_{\epsilon\downarrow 0} \arg_{\blacksquare}(\lambda; x - \mathrm{i}\epsilon) - 2\pi \mathrm{sgn}(\ln|\lambda|), \quad x < 0.$$

While the bifurcation for $\arg(-x) = 0$ is of a merely technical nature for the rational solutions of (1.1) at hand, typical solutions of Painlevé-III are multivalued in a neighborhood of the fixed singularity $x = 0$, and for such solutions the bifurcation leads to a jump of $u(x)$ across a branch cut on the negative real axis.

5.1. **Schlesinger transformations to increment/decrement n.** Now suppose that $\mathbf{V}^{\infty}_{\text{⬕}}$, $\mathbf{V}^{\infty}_{\text{⬔}}$, $\mathbf{V}^{0}_{\text{⬕}}$, and $\mathbf{V}^{0}_{\text{⬔}}$ are any unimodular 2×2 matrices satisfying the cyclic relation (4.36), and that $\mathbf{\Psi}(\lambda; x)$ is an analytic function of λ in the domain $\mathbb{C} \setminus L$, $L := L^{\infty}_{\text{⬕}} \cup L^{\infty}_{\text{⬔}} \cup L^{0}_{\text{⬕}} \cup L^{0}_{\text{⬔}}$, satisfying jump conditions of the form (4.26), (4.28), (4.30), (4.32), as well as asymptotic conditions of the form

$$\mathbf{\Psi}(\lambda; x)\lambda_{\text{⬕}}^{\Theta_\infty \sigma_3/2}\mathrm{e}^{-\mathrm{i}x\lambda\sigma_3/2} = \mathbb{I} + \mathbf{\Psi}_1^{\infty}(x)\lambda^{-1} + \cdots, \quad \lambda \to \infty \tag{5.1}$$

and

$$\mathbf{\Psi}(\lambda; x)\lambda_{\text{⬕}}^{-\Theta_0 \sigma_3/2}\mathrm{e}^{\mathrm{i}x\lambda^{-1}\sigma_3/2} = \mathbf{\Psi}_0^{0}(x) + \mathbf{\Psi}_1^{0}(x)\lambda + \cdots, \quad \lambda \to 0. \tag{5.2}$$

Here, $\mathbf{\Psi}_k^{\infty}(x)$, $k \geq 1$ and $\mathbf{\Psi}_k^{0}(x)$, $k \geq 0$, are certain matrix coefficients. Since it necessarily holds that $\det(\mathbf{\Psi}(\lambda; x)) = 1$, it follows that $\det(\mathbf{\Psi}_0^{0}(x)) = 1$ and $\mathrm{tr}(\mathbf{\Psi}_1^{\infty}(x)) = 0$. We define the Pauli-type matrices

$$\hat{\sigma} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \check{\sigma} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and supposing further that the matrix element $\Psi^{0}_{0,11}(x)$ is not identically zero, we consider the Schlesinger transformation (also known as a Darboux transformation) given by

$$\widehat{\mathbf{\Psi}}(\lambda; x) := (\hat{\sigma}\lambda_{\text{⬕}}^{1/2} + \widehat{\mathbf{B}}(x)\lambda_{\text{⬕}}^{-1/2})\mathbf{\Psi}(\lambda; x), \tag{5.3}$$

where

$$\widehat{\mathbf{B}}(x) := \begin{bmatrix} \Psi^{0}_{0,21}(x)\Psi^{\infty}_{1,12}(x)/\Psi^{0}_{0,11}(x) & -\Psi^{\infty}_{1,12}(x) \\ -\Psi^{0}_{0,21}(x)/\Psi^{0}_{0,11}(x) & 1 \end{bmatrix}. \tag{5.4}$$

Note that $\det(\widehat{\mathbf{\Psi}}(\lambda; x)) = \det(\mathbf{\Psi}(\lambda; x))$ by direct calculation. Since $\lambda_{\text{⬕}}^{\pm 1/2}$ are analytic except on $L^{0}_{\text{⬕}} \cup L^{\infty}_{\text{⬕}}$ across which these factors change sign, $\widehat{\mathbf{\Psi}}(\lambda; x)$ is also analytic for $\lambda \in \mathbb{C} \setminus L$, and it is a direct matter to check the following jump conditions:

$$\widehat{\mathbf{\Psi}}_{+}(\lambda; x) = \widehat{\mathbf{\Psi}}_{-}(\lambda; x)\begin{cases} \mathbf{V}^{0}_{\text{⬔}}, & \lambda \in L^{0}_{\text{⬔}}, \\ \mathbf{V}^{\infty}_{\text{⬔}}, & \lambda \in L^{\infty}_{\text{⬔}}, \\ -\mathbf{V}^{0}_{\text{⬕}}, & \lambda \in L^{0}_{\text{⬕}}, \\ -\mathbf{V}^{\infty}_{\text{⬕}}, & \lambda \in L^{\infty}_{\text{⬕}}. \end{cases} \tag{5.5}$$

Next, combining (5.1) and (5.3), observe that in the limit $\lambda \to \infty$ we have

$$\begin{aligned} \widehat{\mathbf{\Psi}}(\lambda; x)\lambda_{\text{⬕}}^{(\Theta_\infty - 1)\sigma_3/2}\mathrm{e}^{-\mathrm{i}x\lambda\sigma_3/2} &= (\hat{\sigma}\lambda_{\text{⬕}}^{1/2} + \widehat{\mathbf{B}}(x)\lambda_{\text{⬕}}^{-1/2})(\mathbb{I} + \mathbf{\Psi}_1^{\infty}(x)\lambda^{-1} + \cdots)\lambda_{\text{⬕}}^{-\sigma_3/2} \\ &= \lambda(\hat{\sigma} + \widehat{\mathbf{B}}(x)\lambda^{-1})(\mathbb{I} + \mathbf{\Psi}_1^{\infty}(x)\lambda^{-1} + \cdots)(\check{\sigma} + \hat{\sigma}\lambda^{-1}) \\ &= \hat{\sigma}\check{\sigma}\lambda + [\hat{\sigma}^2 + \hat{\sigma}\mathbf{\Psi}_1^{\infty}(x)\check{\sigma} + \widehat{\mathbf{B}}(x)\check{\sigma}] + \widehat{\mathbf{\Psi}}_1^{\infty}(x)\lambda^{-1} + \cdots \\ &= \mathbb{I} + \widehat{\mathbf{\Psi}}_1^{\infty}(x)\lambda^{-1} + \cdots, \end{aligned}$$

where

$$\widehat{\mathbf{\Psi}}_1^{\infty} := \hat{\sigma}\mathbf{\Psi}_1^{\infty}(x)\hat{\sigma} + \hat{\sigma}\mathbf{\Psi}_2^{\infty}(x)\check{\sigma} + \widehat{\mathbf{B}}(x)\hat{\sigma} + \widehat{\mathbf{B}}(x)\mathbf{\Psi}_1^{\infty}(x)\check{\sigma}. \tag{5.6}$$

Similarly, combining (5.2) with (5.3) shows that in the limit $\lambda \to 0$ we have

$$\begin{aligned} \widehat{\mathbf{\Psi}}(\lambda; x)\lambda_{\text{⬕}}^{-(\Theta_0 + 1)\sigma_3/2}\mathrm{e}^{\mathrm{i}x\lambda^{-1}\sigma_3/2} &= (\hat{\sigma}\lambda_{\text{⬕}}^{1/2} + \widehat{\mathbf{B}}(x)\lambda_{\text{⬕}}^{-1/2})(\mathbf{\Psi}_0^{0}(x) + \mathbf{\Psi}_1^{0}(x)\lambda + \cdots)\lambda_{\text{⬕}}^{-\sigma_3/2} \\ &= \lambda^{-1}(\widehat{\mathbf{B}}(x) + \hat{\sigma}\lambda)(\mathbf{\Psi}_0^{0}(x) + \mathbf{\Psi}_1^{0}(x)\lambda + \cdots)(\hat{\sigma} + \check{\sigma}\lambda) \\ &= \widehat{\mathbf{B}}(x)\mathbf{\Psi}_0^{0}(x)\hat{\sigma}\lambda^{-1} + \widehat{\mathbf{\Psi}}_0^{0}(x) + \widehat{\mathbf{\Psi}}_1^{0}(x)\lambda + \cdots \\ &= \widehat{\mathbf{\Psi}}_0^{0}(x) + \widehat{\mathbf{\Psi}}_1^{0}(x)\lambda + \cdots, \end{aligned}$$

where

$$\widehat{\mathbf{\Psi}}_0^{0}(x) := \widehat{\mathbf{B}}(x)\mathbf{\Psi}_0^{0}(x)\check{\sigma} + \widehat{\mathbf{B}}(x)\mathbf{\Psi}_1^{0}(x)\hat{\sigma} + \hat{\sigma}\mathbf{\Psi}_0^{0}(x)\hat{\sigma}. \tag{5.7}$$

Thus, the Schlesinger transformation (5.3) results in a simple modification of the jump conditions and preserves the form of the asymptotic conditions (5.1)–(5.2), but with the replacements $\Theta_\infty \mapsto \widehat{\Theta}_\infty := \Theta_\infty - 1$ and $\Theta_0 \mapsto \widehat{\Theta}_0 := \Theta_0 + 1$.

Comparing with (1.2), we see that these replacements have the effect of incrementing the value of n by 1 and holding m fixed. Similarly, assuming that $\Psi^0_{0,22}(x)$ is not identically zero and setting

$$\check{\mathbf{\Psi}}(\lambda; x) := (\check{\sigma}\lambda_{\blacksquare}^{1/2} + \check{\mathbf{B}}(x)\lambda_{\blacksquare}^{-1/2})\mathbf{\Psi}(\lambda; x), \tag{5.8}$$

where

$$\check{\mathbf{B}}(x) := \begin{bmatrix} 1 & -\Psi^0_{0,12}(x)/\Psi^0_{0,22}(x) \\ -\Psi^\infty_{1,21}(x) & \Psi^0_{0,12}(x)\Psi^\infty_{1,21}(x)/\Psi^0_{0,22}(x) \end{bmatrix} \tag{5.9}$$

respectively, one finds that again $\det(\check{\mathbf{\Psi}}(\lambda; x)) = \det(\mathbf{\Psi}(\lambda; x))$ and (5.5) holds with $\check{\mathbf{\Psi}}$ replacing $\widehat{\mathbf{\Psi}}$, but now as $\lambda \to \infty$,

$$\check{\mathbf{\Psi}}(\lambda; x)\lambda_{\blacksquare}^{(\Theta_\infty+1)\sigma_3/2}\mathrm{e}^{-\mathrm{i}x\lambda\sigma_3/2} = \mathbb{I} + \check{\mathbf{\Psi}}^\infty_1(x)\lambda^{-1} + \cdots,$$

where

$$\check{\mathbf{\Psi}}^\infty_1(x) := \check{\sigma}\mathbf{\Psi}^\infty_1(x)\check{\sigma} + \check{\sigma}\mathbf{\Psi}^\infty_2(x)\widehat{\sigma} + \check{\mathbf{B}}(x)\check{\sigma} + \check{\mathbf{B}}(x)\mathbf{\Psi}^\infty_1(x)\widehat{\sigma},$$

and similarly, as $\lambda \to 0$,

$$\check{\mathbf{\Psi}}(\lambda; x)\lambda_{\blacksquare}^{-(\Theta_0-1)\sigma_3/2}\mathrm{e}^{\mathrm{i}x\lambda^{-1}\sigma_3/2} = \check{\mathbf{\Psi}}^0_0(x) + \check{\mathbf{\Psi}}^0_1(x)\lambda + \cdots$$

where

$$\check{\mathbf{\Psi}}^0_0(x) := \check{\mathbf{B}}(x)\mathbf{\Psi}^0_0(x)\widehat{\sigma} + \check{\mathbf{B}}(x)\mathbf{\Psi}^0_1(x)\check{\sigma} + \check{\sigma}\mathbf{\Psi}^0_0(x)\check{\sigma}.$$

Therefore, the Schlesinger transformation (5.8) also results in a simple modification of the jump conditions and preserves the form of the asymptotic conditions (5.1)–(5.2), but now with the replacements $\Theta_\infty \mapsto \check{\Theta}_\infty := \Theta_\infty + 1$ and $\Theta_0 \mapsto \check{\Theta}_0 := \Theta_0 - 1$, replacements having the effect of decrementing the value of n by 1 and holding m fixed. We now show that the transformations (5.3) and (5.8) are in fact inverse to each other:

Lemma 1. $\check{\widehat{\mathbf{\Psi}}}(\lambda; x) = \widehat{\check{\mathbf{\Psi}}}(\lambda; x) = \mathbf{\Psi}(\lambda; x)$.

Proof. Fix $x \in \mathbb{C}$ such that $\mathbf{\Psi}(\lambda; x)$ exists satisfying the appropriate analyticity, jump, and normalization conditions; hence in particular the diagonal elements of $\mathbf{\Psi}^0_0(x)$ are finite. If $\Psi^0_{0,11}(x) \neq 0$ so that $\widehat{\mathbf{\Psi}}(\lambda; x)$ exists, then according to (5.7) with (5.4), the fact that $\det(\mathbf{\Psi}^0_0(x)) = 1$ implies that $\widehat{\Psi}^0_{0,22}(x) = 1/\Psi^0_{0,11}(x) \neq 0$. Therefore, (5.8) can be applied to $\widehat{\mathbf{\Psi}}(\lambda; x)$ with the elements of $\check{\mathbf{B}}(x)$ obtained from $\widehat{\mathbf{\Psi}}^\infty_1(x)$ and $\widehat{\mathbf{\Psi}}^0_0(x)$ rather than $\mathbf{\Psi}^\infty_1(x)$ and $\mathbf{\Psi}^0_0(x)$. Both rows of the latter matrix are proportional to $[1, -\widehat{\Psi}^0_{0,12}(x)/\widehat{\Psi}^0_{0,22}(x)]$, while both columns of $\widehat{\mathbf{B}}(x)$ are proportional to $[-\Psi^\infty_{1,12}(x), 1]^\top$, with the inner product being

$$-\Psi^\infty_{1,12}(x) - \frac{\widehat{\Psi}^0_{0,12}(x)}{\widehat{\Psi}^0_{0,22}(x)} = -\Psi^\infty_{1,12}(x) - \widehat{\Psi}^0_{0,12}(x)\Psi^0_{0,11}(x) = 0,$$

again using (5.7) with (5.4). Therefore, since $\check{\sigma}\widehat{\sigma} = \mathbf{0}$,

$$\check{\widehat{\mathbf{\Psi}}}(\lambda; x) = \begin{bmatrix} 1 & 0 \\ -\Psi^0_{0,21}(x)/\Psi^0_{0,11}(x) - \widehat{\Psi}^\infty_{1,21}(x) & 1 \end{bmatrix} \mathbf{\Psi}(\lambda; x) = \mathbf{\Psi}(\lambda; x),$$

with the help of (5.6) and (5.4). Another proof of this result is simply to note that the matrices $\check{\widehat{\mathbf{\Psi}}}(\lambda; x)$ and $\mathbf{\Psi}(\lambda; x)$ satisfy exactly the same analyticity, jump, and normalization conditions, and therefore since $\det(\mathbf{\Psi}(\lambda; x)) = 1$, Liouville's theorem shows that $\check{\widehat{\mathbf{\Psi}}}(\lambda; x)\mathbf{\Psi}(\lambda; x)^{-1} = \mathbb{I}$. The proof that (5.3) can be applied to $\check{\mathbf{\Psi}}(\lambda; x)$ provided that $\Psi^0_{0,22}(x) \neq 0$ so that the latter exists, with the result that $\widehat{\check{\mathbf{\Psi}}}(\lambda; x) = \mathbf{\Psi}(\lambda; x)$, is completely analogous. □

5.2. **The defining inverse monodromy problem for the rational solution** $u_n(x;m)$**.** Let $\mathbf{\Psi}^{(0)}(\lambda;x,m) := \mathbf{\Psi}(\lambda;x)$ be the matrix function defined in Sections 4.1–4.2, which satisfies (5.1)–(5.2) with $\Theta_0 = m$ and $\Theta_\infty = m+1$, and for which $\Psi^0_{0,11}(x) = a(x) = \mathrm{e}^{-\mathrm{i}\pi\kappa/2}\mathrm{e}^{-\mathrm{i}\pi/4}c \neq 0$ and $\Psi^0_{0,22}(x) \not\equiv 0$ for $b(x) \not\equiv 0$ (note that both inequalities follow from (4.20)–(4.21)). We now apply the Schlesinger transformations (5.3) and (5.8) repeatedly, assuming that after each iteration, the condition $\Psi^0_{0,11}(x)\Psi^0_{0,22}(x) \not\equiv 0$ persists[7] to obtain for each integer $n \in \mathbb{Z}$ a matrix function $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ that satisfies (5.1)–(5.2) as well as the jump conditions

$$\mathbf{\Psi}^{(n)}_+(\lambda;x,m) = \mathbf{\Psi}^{(n)}_-(\lambda;x,m)\begin{cases}\mathbf{V}^0_{\text{⬔}}(m), & \lambda \in L^0_{\text{⬔}},\\ \mathbf{V}^\infty_{\text{⬔}}(m), & \lambda \in L^\infty_{\text{⬔}},\\ (-1)^n\mathbf{V}^0_{\text{⬕}}(m), & \lambda \in L^0_{\text{⬕}},\\ (-1)^n\mathbf{V}^\infty_{\text{⬕}}(m), & \lambda \in L^\infty_{\text{⬕}},\end{cases} \tag{5.10}$$

where now the matrices $\mathbf{V}^0_{\text{⬔}}(m)$ and $\mathbf{V}^\infty_{\text{⬔}}(m)$ are defined in (4.34) and $\mathbf{V}^0_{\text{⬕}}(m)$ and $\mathbf{V}^\infty_{\text{⬕}}(m)$ are defined in (4.35). Since $\det(\mathbf{\Psi}^{(0)}(\lambda;x,m)) = 1$ it follows that $\det(\mathbf{\Psi}^{(n)}(\lambda;x,m)) = 1$ for all $n \in \mathbb{Z}$. The *inverse monodromy problem* consists of fixing $n \in \mathbb{Z}$, $m \in \mathbb{C}$, and $x \in \mathbb{C}\setminus\{0\}$ and attempting to determine $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ from the following conditions only:

- Analyticity: $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ is analytic for $\lambda \in \mathbb{C}\setminus L$ and analyticity extends to the the contour L from each component of its complement.
- Jump conditions: The boundary values taken by $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ on the four oriented arcs of L are to be related by the jump conditions (5.10).
- Behavior for small and large λ: $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ satisfies the two conditions (5.1)–(5.2) in which Θ_0 and Θ_∞ are defined in terms of m and n by (1.2).

By its construction in Sections 4.1–4.2, $\mathbf{\Psi}^{(0)}(\lambda;x,m)$ is the simultaneous solution of a Lax pair of linear problems. We now show that this is also true for $\mathbf{\Psi}^{(n)}(\lambda;x,m)$, $\forall n \in \mathbb{Z}$, establishing simultaneously some related important properties.

Lemma 2. *Let $n \in \mathbb{Z}$ and $m \in \mathbb{C}$ be fixed and suppose the above inverse monodromy problem for $\mathbf{\Psi}(\lambda;x) = \mathbf{\Psi}^{(n)}(\lambda;x,m)$ is solvable for x in some domain $D \subset \mathbb{C}\setminus\{0\}$.*

1. *For $\lambda \in \mathbb{C}\setminus L$, the function $\mathbf{\Psi}(\lambda;x) = \mathbf{\Psi}^{(n)}(\lambda;x,m)$ is a simultaneous solution matrix of the Lax system* (3.1)–(3.2) *in which the x-dependent coefficients y, v, s, and t are given in terms of the leading matrix coefficients in the expansions* (5.1)–(5.2) *by*

$$y(x) = -\mathrm{i}x\Psi^\infty_{1,12}(x), \quad v(x) = \mathrm{i}x\Psi^\infty_{1,21}(x), \quad s(x) = -x\Psi^0_{0,11}(x)\Psi^0_{0,12}(x), \quad t(x) = \frac{\Psi^0_{0,21}(x)}{\Psi^0_{0,11}(x)}. \tag{5.11}$$

2. *None of the three matrix elements $\Psi^0_{0,11}(x)$, $\Psi^0_{0,12}(x)$, nor $\Psi^0_{0,22}(x)$ of the leading coefficient in the expansion* (5.2) *of $\mathbf{\Psi}(\lambda;x) = \mathbf{\Psi}^{(n)}(\lambda;x,m)$ vanishes identically on the domain D.*
3. *The combination $u(x) := -y(x)/s(x)$ (cf.,* (3.5)*) is a solution of the Painlevé-III equation* (1.1) *meromorphic on D with parameters Θ_0 and Θ_∞ given by* (1.2)*.*

Proof. It is a standard result based on Liouville's theorem and the fact that the jump matrices are all unimodular that there can be at most one solution of the inverse monodromy conditions and that this solution satisfies $\det(\mathbf{\Psi}^{(n)}(\lambda;x,m)) = 1$. Applying analytic Fredholm theory to a suitable singular integral equation equivalent to the inverse monodromy problem and parametrized analytically by $x \in \mathbb{C}\setminus\{0\}$, existence of a solution for $x \in D$ implies that for each $m \in \mathbb{C}$ and for each fixed λ disjoint from the jump contour L for all $x \in D$, $x \mapsto \mathbf{\Psi}^{(n)}(\lambda;x,m)$ is analytic on D. In particular, in a neighborhood of such fixed λ and any $x \in D$, $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ is jointly differentiable with respect to both λ and x. Because the jump matrices in (5.10) are independent of both λ (on each arc) and x, it follows that the matrices

$$\mathbf{A}^{(n)}(\lambda;x,m) := \frac{\partial\mathbf{\Psi}^{(n)}}{\partial\lambda}(\lambda;x,m)\mathbf{\Psi}^{(n)}(\lambda;x,m)^{-1} \quad \text{and} \quad \mathbf{B}^{(n)}(\lambda;x,m) := \frac{\partial\mathbf{\Psi}^{(n)}}{\partial x}(\lambda;x,m)\mathbf{\Psi}^{(n)}(\lambda;x,m)^{-1}$$

are both analytic functions of (λ,x) in the domain $(\mathbb{C}\setminus\{0\})\times D$. Note that to define $\mathbf{B}^{(n)}(\lambda;x,m)$, we may take the jump contour L to be locally independent of x because the boundary values taken from each sector on L are analytic

[7]See statement 2 of Lemma 2.

functions of λ. From (5.1) we see that in the limit $\lambda \to \infty$,

$$\begin{aligned}\mathbf{A}^{(n)}(\lambda;x,m) &= \frac{\mathrm{i}x}{2}\sigma_3 + \left(\frac{\mathrm{i}x}{2}\left[\mathbf{\Psi}_1^\infty(x),\sigma_3\right] - \frac{\Theta_\infty}{2}\sigma_3\right)\lambda^{-1} \\ &\quad + \left(-\mathbf{\Psi}_1^\infty(x) - \frac{\Theta_\infty}{2}\left[\mathbf{\Psi}_1^\infty(x),\sigma_3\right] + \frac{\mathrm{i}x}{2}\left\{\left[\mathbf{\Psi}_2^\infty(x),\sigma_3\right] - \left[\mathbf{\Psi}_1^\infty(x),\sigma_3\right]\mathbf{\Psi}_1^\infty(x)\right\}\right)\lambda^{-2} + O(\lambda^{-3}), \\ \mathbf{B}^{(n)}(\lambda;x,m) &= \frac{\mathrm{i}}{2}\sigma_3\lambda + \frac{\mathrm{i}}{2}[\mathbf{\Psi}_1^\infty(x),\sigma_3] + \left(\mathbf{\Psi}_1^{\infty\prime}(x) + \frac{\mathrm{i}}{2}\left[\mathbf{\Psi}_2^\infty(x),\sigma_3\right] - \frac{\mathrm{i}}{2}\left[\mathbf{\Psi}_1^\infty(x),\sigma_3\right]\mathbf{\Psi}_1^\infty(x)\right)\lambda^{-1} + O(\lambda^{-2}).\end{aligned} \tag{5.12}$$

Similarly, in the limit $\lambda \to 0$, from (5.2) we get

$$\begin{aligned}\mathbf{A}^{(n)}(\lambda;x,m) &= \frac{\mathrm{i}x}{2}\mathbf{\Psi}_0^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1}\lambda^{-2} \\ &\quad + \left(\frac{\Theta_0}{2}\mathbf{\Psi}_0^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1} + \frac{\mathrm{i}x}{2}\mathbf{\Psi}_1^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1} - \frac{\mathrm{i}x}{2}\mathbf{\Psi}_0^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1}\mathbf{\Psi}_1^0(x)\mathbf{\Psi}_0^0(x)^{-1}\right)\lambda^{-1} \\ &\quad + O(1) \\ \mathbf{B}^{(n)}(\lambda;x,m) &= -\frac{\mathrm{i}}{2}\mathbf{\Psi}_0^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1}\lambda^{-1} + \mathbf{\Psi}_0^{0\prime}(x)\mathbf{\Psi}_0^0(x)^{-1} + \frac{\mathrm{i}}{2}\left[\mathbf{\Psi}_0^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1},\mathbf{\Psi}_1^0(x)\mathbf{\Psi}_0^0(x)^{-1}\right] + O(\lambda).\end{aligned} \tag{5.13}$$

Therefore, Liouville's theorem shows that $\mathbf{A}^{(n)}(\lambda;x,m)$ and $\mathbf{B}^{(n)}(\lambda;x,m)$ are Laurent polynomials:

$$\mathbf{A}^{(n)}(\lambda;x,m) = \frac{\mathrm{i}x}{2}\sigma_3 + \left(\frac{\mathrm{i}x}{2}\left[\mathbf{\Psi}_1^\infty(x),\sigma_3\right] - \frac{\Theta_\infty}{2}\sigma_3\right)\lambda^{-1} + \frac{\mathrm{i}x}{2}\mathbf{\Psi}_0^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1}\lambda^{-2} \tag{5.14}$$

and

$$\mathbf{B}^{(n)}(\lambda;x,m) = \frac{\mathrm{i}}{2}\sigma_3\lambda + \frac{\mathrm{i}}{2}[\mathbf{\Psi}_1^\infty(x),\sigma_3] - \frac{\mathrm{i}}{2}\mathbf{\Psi}_0^0(x)\sigma_3\mathbf{\Psi}_0^0(x)^{-1}\lambda^{-1}. \tag{5.15}$$

Furthermore, the coefficients of different powers of λ in (5.14)–(5.15) are analytic matrix-valued functions of x on D. Since $\mathbf{\Psi}_\lambda^{(n)}(\lambda;x,m) = \mathbf{A}^{(n)}(\lambda;x,m)\mathbf{\Psi}^{(n)}(\lambda;x,m)$ and $\mathbf{\Psi}_x^{(n)}(\lambda;x,m) = \mathbf{B}^{(n)}(\lambda;x,m)\mathbf{\Psi}^{(n)}(\lambda;x,m)$, matching (5.14)–(5.15) with (3.1)–(3.2) using also $\det(\mathbf{\Psi}_0^0(x)) = 1$ yields the expressions (5.11) and proves statement 1.

Suppose $\Psi_{0,11}^0(x) \equiv 0$ holds as an identity on D. From $\det(\mathbf{\Psi}_0^0(x)) \equiv 1$ we then get $\Psi_{0,12}^0(x)\Psi_{0,21}^0(x) \equiv -1$. Therefore $s(x) \equiv 0$ and $\frac{1}{2}\mathrm{i}x - \mathrm{i}s(x)t(x) \equiv -\frac{1}{2}\mathrm{i}x$, so the matrices $\mathbf{A}^{(n)}(\lambda;x,m)$ and $\mathbf{B}^{(n)}(\lambda;x,m)$ can be written in the alternate form

$$\begin{aligned}\mathbf{A} = \mathbf{A}^{(n)}(\lambda;x,m) &= \frac{\mathrm{i}x}{2}\sigma_3 + \frac{1}{\lambda}\begin{bmatrix}-\frac{1}{2}\Theta_\infty & y \\ v & \frac{1}{2}\Theta_\infty\end{bmatrix} + \frac{1}{\lambda^2}\begin{bmatrix}-\frac{1}{2}\mathrm{i}x & 0 \\ -\mathrm{i}V & \frac{1}{2}\mathrm{i}x\end{bmatrix} \\ \mathbf{B} = \mathbf{B}^{(n)}(\lambda;x,m) &= \frac{\mathrm{i}\lambda}{2}\sigma_3 + \frac{1}{x}\begin{bmatrix}0 & y \\ v & 0\end{bmatrix} - \frac{1}{\lambda x}\begin{bmatrix}-\frac{1}{2}\mathrm{i}x & 0 \\ -\mathrm{i}V & \frac{1}{2}\mathrm{i}x\end{bmatrix}\end{aligned} \tag{5.16}$$

with $y(x)$ and $v(x)$ defined as in (5.11), while

$$V(x) := -x\Psi_{0,21}^0(x)\Psi_{0,22}^0(x).$$

Existence of the simultaneous fundamental solution matrix $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ of the Lax system implies that these coefficient matrices satisfy the zero-curvature compatibility condition $\mathbf{A}_x - \mathbf{B}_\lambda + [\mathbf{A},\mathbf{B}] = \mathbf{0}$, which in turn implies that $y(x) \equiv 0$ also, making $\mathbf{A}$ and $\mathbf{B}$ lower-triangular with explicit diagonal entries. Therefore, the elements of the first row are determined from the Lax system up to overall constants c_1 and c_2 by

$$\begin{bmatrix}\Psi_{11}^{(n)}(\lambda;x,m) & \Psi_{12}^{(n)}(\lambda;x,m)\end{bmatrix} = \begin{bmatrix}c_1\mathrm{e}^{\mathrm{i}x(\lambda+\lambda^{-1})/2}\lambda_{\blacksquare}^{-\Theta_\infty/2} & c_2\mathrm{e}^{\mathrm{i}x(\lambda+\lambda^{-1})/2}\lambda_{\blacksquare}^{-\Theta_\infty/2}\end{bmatrix}.$$

Applying the condition (5.1) then forces the choice $c_2 = 0$, so $\Psi_{12}^{(n)}(\lambda;x,m) \equiv 0$ and therefore also $\Psi_{0,12}^0(x) \equiv 0$ on D. But since $\det(\mathbf{\Psi}_0^0(x)) \equiv 1$, this contradicts the assumption that $\Psi_{0,11}^0(x) \equiv 0$.

Suppose next that $\Psi_{0,22}^0(x) \equiv 0$. Then using $\det(\mathbf{\Psi}_0^0(x)) \equiv 0$ shows that the combination $-\mathrm{i}t(x)(s(x)t(x) - x)$ vanishes identically, and then the compatibility condition for the matrices $\mathbf{A}^{(n)}(\lambda;x,m)$ and $\mathbf{B}^{(n)}(\lambda;x,m)$ implies that also $v(x) \equiv 0$. Therefore, the coefficient matrices are upper-triangular in this case, and since also $\frac{1}{2}\mathrm{i}x - \mathrm{i}s(x)t(x) \equiv -\frac{1}{2}\mathrm{i}x$, the second row of $\mathbf{\Psi}^{(n)}(\lambda; x, m)$ takes the form

$$\begin{bmatrix}\Psi_{21}^{(n)}(\lambda; x, m) & \Psi_{22}^{(n)}(\lambda; x, m)\end{bmatrix} = \begin{bmatrix}c_1 \mathrm{e}^{-\mathrm{i}x(\lambda+\lambda^{-1})/2}\lambda_{\blacksquare}^{\Theta_\infty/2} & c_2 \mathrm{e}^{-\mathrm{i}x(\lambda+\lambda^{-1})/2}\lambda_{\blacksquare}^{\Theta_\infty/2}\end{bmatrix} \tag{5.17}$$

where c_1 and c_2 are constants. Applying as before the condition (5.1) now forces $c_1 = 0$, so $\Psi^0_{0,21}(x)$ and $\Psi^0_{0,22}(x)$ both vanish identically in contradiction to $\det(\mathbf{\Psi}^0_0(x)) \equiv 1$.

Finally, suppose that $\Psi^0_{0,12}(x) \equiv 0$ on D. Then also $s(x) \equiv 0$ and $s(x)t(x) \equiv 0$, and the compatibility condition for the Lax system implies that also $y(x) \equiv 0$, making the coefficient matrices lower-triangular. Solving for the first row of $\mathbf{\Psi}^{(n)}(\lambda; x, m)$ now yields

$$\begin{bmatrix}\Psi_{11}^{(n)}(\lambda; x, m) & \Psi_{12}^{(n)}(\lambda; x, m)\end{bmatrix} = \begin{bmatrix}c_1 \mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})/2}\lambda_{\blacksquare}^{-\Theta_\infty/2} & c_2 \mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})/2}\lambda_{\blacksquare}^{-\Theta_\infty/2}\end{bmatrix} \tag{5.18}$$

for constants c_1 and c_2, and applying the normalization condition (5.1) forces $c_1 = 1$ and $c_2 = 0$. For this result to be compatible with (5.2) it is then necessary that $\Theta_0 + \Theta_\infty = 0$, i.e., that $m = -\frac{1}{2}$. But, if $m = -\frac{1}{2}$, the jump condition across the arc $L^\infty_{\blacksquare}$ implies that (using $\Theta_\infty = \frac{1}{2} - n$ for $m = -\frac{1}{2}$)

$$\Psi^{(n)}_{12+}(\lambda; x, -\tfrac{1}{2}) - \Psi^{(n)}_{12-}(\lambda; x, -\tfrac{1}{2}) = \sqrt{2\pi}\Psi^{(n)}_{11-}(\lambda; x, -\tfrac{1}{2}) = \sqrt{2\pi}\mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})/2}\lambda_{\blacksquare}^{n/2-1/4}, \quad \lambda \in L^\infty_{\blacksquare}. \tag{5.19}$$

The right-hand side is nonzero on the indicated contour, which is obviously inconsistent with $\Psi^{(n)}_{12}(\lambda; x, -\frac{1}{2}) \equiv 0$ implied by $c_2 = 0$. All together, since assuming $\Psi^0_{0,11}(x) \equiv 0$, $\Psi^0_{0,22}(x) \equiv 0$, or $\Psi^0_{0,12}(x) \equiv 0$ leads in each case to a contradiction, we have established statement 2.

The potentials $y(x)$, $v(x)$, and $s(x)$ are analytic on D by analytic Fredholm theory, and by statement 2 it also holds that $t(x)$ is meromorphic on D. In general, the compatibility condition $\mathbf{A}_x - \mathbf{B}_\lambda + [\mathbf{A}, \mathbf{B}] = \mathbf{0}$ on the matrices (5.14)–(5.15) implies that these four functions satisfy the coupled nonlinear differential equations (3.3). The system (3.3) has a conserved quantity I defined by (3.4); to determine its constant value, it suffices evaluate it at any $x \in D$ that makes each term in I finite (it is only necessary to avoid the isolated zeros of $\Psi^0_{0,11}(x)$). Note that the direct monodromy problem (3.1) has an irregular singular point of Poincaré rank 1 at $\lambda = 0$ and hence by general theory two fundamental solutions exist in a vicinity of $\lambda = 0$ which are uniquely specified by their asymptotics as $\lambda \to 0$ in the associated Stokes sectors. An explicit computation of the formal expansions directly from the differential equation (3.1) (cf., [25]) yields, upon comparison with the expansion (5.2) the identity $I = \Theta_0$. Now, the expression $u(x) = -y(x)/s(x)$ defines a meromorphic function on D because the zeros of $s(x)$ are isolated by statement 2. Differentiating this expression using (3.3) and eliminating $y(x) = -s(x)u(x)$, one finds that $u(x)$ and the product $s(x)t(x)$ are related by the first order differential equation (3.6). Solving this identity for $s(x)t(x)$ in terms of $u(x)$ and $u'(x)$ and differentiating the result yields a second-order differential expression involving $u(x)$ alone. On the other hand, the product $s(x)t(x)$ can be differentiated directly using (3.3) after which $y(x)$ can be eliminated using $y(x) = -s(x)u(x)$, $v(x)$ can be eliminated using the integral of motion $I = \Theta_0$, and finally the product $s(x)t(x)$ can be eliminated once again using (3.6). Equating these two equivalent expressions for the derivative of $s(x)t(x)$ yields precisely the Painlevé-III equation (1.1) for $u(x)$. This proves statement 3. □

Next, we have the following result.

Lemma 3. *Given $n \in \mathbb{Z}$ and $m \in \mathbb{C}$, there is a finite set $P_n(m)$ such that the inverse monodromy problem is uniquely solvable for $x \in \mathbb{C} \setminus (\mathbb{R}_- \cup P_n(m))$. The corresponding solution $u(x)$ of the Painlevé-III equation* (1.1) *is a rational function.*

Proof. Since existence of a solution implies uniqueness by a Liouville argument, it is sufficient to establish existence for suitable x. To this end we first consider $n = 0$. The explicit solution $\mathbf{\Psi}^{(0)}(\lambda; x, m)$ of the direct monodromy problem constructed in Section 4 obviously satisfies the conditions of the inverse monodromy problem as well, and it is well-defined for $x \in \mathbb{C} \setminus \mathbb{R}_-$. A calculation shows that the leading term $\mathbf{\Psi}^0_0(x)$ takes the form

$$\mathbf{\Psi}^0_0(x) = \begin{bmatrix}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\pi m/2} & 2^m \mathrm{e}^{-3\pi\mathrm{i}/4}\mathrm{e}^{2x}x^m \\ \frac{1}{4}\mathrm{e}^{3\pi\mathrm{i}/4}2^{-m}(2m+1)x^{-1}\mathrm{e}^{-2x}x^{-m} & \frac{1}{4}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}\pi m/2}(2m+1+4x)x^{-1}\end{bmatrix}, \quad n = 0. \tag{5.20}$$

Obviously, $\Psi^0_{0,11}(x)$, $\Psi^0_{0,22}(x)$, $\mathrm{e}^{-2x}x^{-m}\Psi^0_{0,12}(x)$, and $\mathrm{e}^{2x}x^m\Psi^0_{0,21}(x)$ are all rational functions (with poles at $x=0$ only). Similar calculations give

$$\Psi^\infty_{1,12}(x) = -\mathrm{i}2^m\mathrm{e}^{-\mathrm{i}\pi m/2}\mathrm{e}^{2x}x^m \quad\text{and}\quad \Psi^\infty_{1,21}(x) = -\mathrm{i}2^{-(m+4)}\mathrm{e}^{\mathrm{i}\pi m/2}(2m+1)(4x-2m-1)\mathrm{e}^{-2x}x^{-m}, \quad n=0. \quad (5.21)$$

Therefore also $\mathrm{e}^{-2x}x^{-m}\Psi^\infty_{1,12}(x)$ and $\mathrm{e}^{2x}x^m\Psi^\infty_{1,21}(x)$ are rational functions. Clearly, $P_0(m)=\emptyset$ (the pole at $x=0$ is already excluded as $0\in\mathbb{R}_-$), and the corresponding solution $u(x)=-\mathrm{i}\Psi^\infty_{1,12}(x)/(\Psi^0_{0,11}(x)\Psi^0_{0,12}(x))\equiv 1$ is clearly rational. Next, let $k\geq 0$ be an integer, and suppose that $P_k(m)$ is finite, that the inverse monodromy problem for $n=k$ is (uniquely) solvable for $m\in\mathbb{C}$ and $x\in\mathbb{C}\setminus(\mathbb{R}_-\cup P_k(m))$, and that for $n=k$ the expansion coefficients $\Psi^0_{0,11}(x)$, $\Psi^0_{0,22}(x)$, $\mathrm{e}^{-2x}x^{-m}\Psi^0_{0,12}(x)$, $\mathrm{e}^{2x}x^m\Psi^0_{0,21}(x)$, $\mathrm{e}^{-2x}x^{-m}\Psi^\infty_{1,12}(x)$, and $\mathrm{e}^{2x}x^m\Psi^\infty_{1,21}(x)$ are all rational functions. Taking $D=\mathbb{C}\setminus(\mathbb{R}_-\cup P_k(m))$ and applying Lemma 2 we see that $\Psi^0_{0,11}(x)\not\equiv 0$ holds on D, so the Schlesinger transformation (5.3) exists on D except at the finitely-many zeros of the rational function $\Psi^0_{0,11}(x)$ in D. Letting $P_{k+1}(m)$ denote the union of the set of these zeros with $P_k(m)$, the matrix $\mathbf{\Psi}^{(k+1)}(\lambda;x,m) := \widehat{\mathbf{\Psi}}^{(k)}(\lambda;x,m)$ clearly satisfies all of the properties of the inverse monodromy problem for $n=k$, $m\in\mathbb{C}$, and $x\in\mathbb{C}\setminus(\mathbb{R}_-\cup P_k(m))$. Since, according to (5.4) and the inductive hypotheses in force, the matrix $\mathrm{e}^{-x\sigma_3}x^{-m\sigma_3/2}\widehat{\mathbf{B}}(x)x^{m\sigma_3/2}\mathrm{e}^{x\sigma_3}$ is a rational function of x, it then follows that the transformed expansion coefficients are such that $\widehat{\Psi}^0_{0,11}(x)$, $\widehat{\Psi}^0_{0,22}(x)$, $\mathrm{e}^{-2x}x^{-m}\widehat{\Psi}^0_{0,12}(x)$, $\mathrm{e}^{2x}x^m\widehat{\Psi}^0_{0,21}(x)$, $\mathrm{e}^{-2x}x^{-m}\widehat{\Psi}^\infty_{1,12}(x)$, and $\mathrm{e}^{2x}x^m\widehat{\Psi}^\infty_{1,21}(x)$ are all rational functions, as is $\widehat{u}(x)=-\mathrm{i}\widehat{\Psi}^\infty_{1,12}(x)/(\widehat{\Psi}^0_{0,11}(x)\widehat{\Psi}^0_{0,12}(x))$, which by Lemma 2 satisfies the Painlevé-III equation with parameters $n=k+1$ and m. The desired conclusion therefore holds for all integers $n\geq 0$ by induction on n.

For $n\leq 0$, we apply instead the transformation (5.8)–(5.9) to decrease n, making use of the fact that $\Psi^0_{0,22}(x)\not\equiv 0$. A parallel induction argument shows that the desired conclusion holds for all negative integers n as well. □

We remark that the points at which the inverse monodromy problem fails to have a solution need not coincide with the poles or zeros of the rational function $u(x)$.

5.3. **Induced Bäcklund transformations.** The Schlesinger transformation (5.3) implies a corresponding Bäcklund transformation for the potentials $v(x)$, $y(x)$, $s(x)$ and $t(x)$:

$$\begin{aligned}
\widehat{v}(x) &:= -\mathrm{i}xt(x)\\
\widehat{y}(x) &:= \frac{\mathrm{i}}{x}\left(xs(x)-(\Theta_\infty-1)y(x)+y(x)^2t(x)\right)\\
\widehat{s}(x) &:= \frac{\mathrm{i}y(x)}{x^2}\left(x^2+y(x)^2t(x)^2-\Theta_\infty y(x)t(x)-v(x)y(x)\right)\\
\widehat{t}(x) &:= \mathrm{i}x\frac{y(x)t(x)^2-\Theta_\infty t(x)-v(x)}{x^2+y(x)^2t(x)^2-\Theta_\infty y(x)t(x)-v(x)y(x)}.
\end{aligned} \qquad (5.22)$$

It is straightforward to confirm directly that whenever (v,y,s,t) solves (3.3), then so does $(\widehat{v},\widehat{y},\widehat{s},\widehat{t})$ when Θ_∞ is replaced in (3.3) by $\widehat{\Theta}_\infty := \Theta_\infty - 1$. Defining $\widehat{u}(x) := -\widehat{y}(x)/\widehat{s}(x)$ and using (5.22) along with $u(x)=-y(x)/s(x)$, the identity $I=\Theta_0$, and (3.6), one arrives at Gromak's transformation (1.3). This proves the following.

Proposition 3. *The rational function $u(x)$ obtained from the inverse monodromy problem with parameters $m\in\mathbb{C}$ and $n\in\mathbb{Z}_{\geq 0}$ coincides with the function $u(x)=u_n(x;m)$ obtained via n iterations of the Bäcklund transformation* (1.3) *starting from the seed $u_0(x;m)\equiv 1$.*

This result establishes the link between the algebraic representation (1.6)–(1.7) of $u_n(x;m)$ and the analytic representation afforded by the inverse monodromy problem. It is easy to check that the Bäcklund transformation (1.3) preserves the property $u(x)\to 1$ as $x\to\infty$, and therefore $u_n(x;m)$ and its odd reflection $R^2u_n(x;m)=-u_n(-x;m)$ are distinct rational solutions of the Painlevé-III equation (1.1) for the same values of $n\in\mathbb{Z}$ and $m\in\mathbb{C}$. Suppose that $m\notin\mathbb{Z}$, but $u(x)$ is a rational solution of (1.1) for parameters (m,n). We may invert the Bäcklund transformation (the corresponding explicit formula for the inverse can be obtained from the n-reducing Schlesinger transformation (5.8) in the same way that Gromak's transformation can be deduced from (5.3)) and apply the inverse n times to $u(x)$, thereby arriving at a rational solution of (1.1) with parameters $(m,0)$. However, it has been shown that when $n=0$ and $m\notin\mathbb{Z}$, the only rational solutions of (1.1) are the constants ± 1. By Lemma 1, the inverse transformation is injective and therefore it follows that either $u(x)=u_n(x;m)$ or $u(x)=R^2u_n(x;m)$, i.e., for $m\notin\mathbb{Z}$ and $n\in\mathbb{Z}$, there are exactly two rational solutions. From this it follows that for general m it is sufficient to study the family of functions $\{u_n(x;m)\}_{n\in\mathbb{Z}}$ to analyze all rational solutions of (1.1). This can be done using the inverse monodromy problem, suitably reformulated in the form of Riemann-Hilbert Problem 1, which we now are in a position to establish.

5.4. **Change of normalization.** To study the asymptotic behavior of the rational solutions for n a large integer and $m \in \mathbb{C}$ fixed, it is useful to study in place of $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ a matrix that is normalized to the identity matrix as $\lambda \to \infty$. Therefore, we consider the matrix $\mathbf{Y}^{(n)}(\lambda;x,m)$ defined by a small modification of the left-hand side of (5.1):

$$\mathbf{Y}^{(n)}(\lambda;x,m) := \mathbf{\Psi}^{(n)}(\lambda;x,m)\lambda_{\boxbslash}^{\Theta_\infty\sigma_3/2}\mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})\sigma_3/2}$$

where Θ_∞ is given by (1.2). It is easy to check that if it exists for a given $x \in \mathbb{C}$, this matrix satisfies the conditions of Riemann-Hilbert Problem 1. Recalling the expansions (1.16)–(1.17), the coefficients $\mathbf{Y}_1^\infty(x)$ and $\mathbf{Y}_0^0(x)$ are related to the expansions of $\mathbf{\Psi}^{(n)}(\lambda;x,m)$ by

$$\mathbf{\Psi}_1^\infty(x) = \mathbf{Y}_1^\infty(x) - \frac{\mathrm{i}x}{2}\sigma_3 \quad \text{and} \quad \mathbf{\Psi}_0^0(x) = \mathbf{Y}_0^0(x), \tag{5.23}$$

and therefore combining (3.5), (5.11), and (5.23), the rational solution $u_n(x;m)$ of the Painlevé-III equation (1.1) is given by (1.18).

It is a consequence of the cyclic relation (4.36) that at this point we may take the contour L to be arbitrary subject to the restrictions indicated in Subsection 1.1. Such a modified form of L can always be connected with the original L by a homotopy that moves the intersection point but maintains the increment of arguments as specified by (1.10)–(1.11), and throughout which the power functions $\lambda_{\boxbslash}^p$ appearing in the jump conditions (1.12)–(1.15) are deformed in a natural way by analytic continuation. This completes the proof of Theorem 1.

6. Algebraic Solution of Riemann-Hilbert Problem 1 for $m \in \mathbb{Z} + \frac{1}{2}$

Note that the jump matrices on $L_{\boxbslash}^\infty \cup L_{\boxbslash}^0$ reduce to the identity if $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$. Likewise, the jump matrices on $L_{\boxbslash}^\infty \cup L_{\boxbslash}^0$ reduce to the identity if $m = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$. This observation results in an algebraic solution technique for half-integer values of m that we will now describe.

Suppose first that $m = \frac{1}{2} + k$, $k \in \mathbb{Z}_{\geq 0}$. Then according to Riemann-Hilbert Problem 1, $\mathbf{Y}^{(n)}(\lambda;x,m)$ is analytic for $\mathbb{C} \setminus L$ where now we may take $L = L_{\boxbslash}^0 \cup L_{\boxbslash}^\infty$ because the jump matrices on $L_{\boxbslash}^0 \cup L_{\boxbslash}^\infty$ reduce to the identity so analyticity follows by Morera's theorem. Moreover, the jump condition on L takes the form

$$\mathbf{Y}_+^{(n)}(\lambda;x,\tfrac{1}{2}+k) = \mathbf{Y}_-^{(n)}(\lambda;x,\tfrac{1}{2}+k)\begin{bmatrix} 1 & 0 \\ \dfrac{\sqrt{2\pi}}{k!}(\lambda_{\boxbslash}^{k/2+3/4})_+(\lambda_{\boxbslash}^{k/2+3/4})_-\lambda^{-n}\mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})} & 1\end{bmatrix}, \quad \lambda \in L, \quad k \in \mathbb{Z}_{\geq 0}. \tag{6.1}$$

A similar Morera argument therefore implies that the second column of $\mathbf{Y}^{(n)}(\lambda;x,\frac{1}{2}+k)$ has no jump across L and hence is analytic for $\lambda \in \mathbb{C}\setminus\{0\}$. Applying the normalization condition at $\lambda = \infty$ yields $Y_{12}^{(n)}(\lambda;x,\frac{1}{2}+k) = O(\lambda^{-1})$ and $Y_{22}^{(n)}(\lambda;x,\frac{1}{2}+k) = 1 + O(\lambda^{-1})$ as $\lambda \to \infty$, while $Y_{j2}^{(n)}(\lambda;x,\frac{1}{2}+k) = O(\lambda^{k+1})$ as $\lambda \to 0$ for $j = 1,2$. It follows by Liouville's theorem that

$$Y_{12}^{(n)}(\lambda;x,\tfrac{1}{2}+k) = \sum_{j=1}^{k+1} a_j^{(n,k)}(x)\lambda^{-j} \quad \text{and} \quad Y_{22}^{(n)}(\lambda;x,\tfrac{1}{2}+k) = 1 + \sum_{j=1}^{k+1} b_j^{(n,k)}(x)\lambda^{-j}$$

where $a_j^{(n,k)}(x)$ and $b_j^{(n,k)}(x)$ are coefficients to be determined. The first column of the jump condition (6.1) can then be used together with the Plemelj formula and the normalization conditions $Y_{11}^{(n)}(\lambda;x,\frac{1}{2}+k) = 1+O(\lambda^{-1})$ and $Y_{21}^{(n)}(\lambda;x,\frac{1}{2}+k) = O(\lambda^{-1})$ as $\lambda \to \infty$ to express $Y_{j1}^{(n)}(\lambda;x,\frac{1}{2}+k)$ explicitly in terms of $Y_{j2}^{(n)}(\lambda;x,\frac{1}{2}+k)$:

$$Y_{11}^{(n)}(\lambda;x,\tfrac{1}{2}+k) = 1 + \frac{1}{\mathrm{i}k!\sqrt{2\pi}}\int_L \frac{Y_{12}^{(n)}(\mu;x,\frac{1}{2}+k)(\mu_{\boxbslash}^{k/2+3/4})_+(\mu_{\boxbslash}^{k/2+3/4})_-\mu^{-n}\mathrm{e}^{-\mathrm{i}x(\mu-\mu^{-1})}}{\mu-\lambda}\,\mathrm{d}\mu$$

and

$$Y_{21}^{(n)}(\lambda; x, \tfrac{1}{2}+k) = \frac{1}{\mathrm{i}k!\sqrt{2\pi}} \int_L \frac{Y_{22}^{(n)}(\mu; x, \frac{1}{2}+k)(\mu_{\text{◪}}^{k/2+3/4})_+(\mu_{\text{◪}}^{k/2+3/4})_- \mu^{-n} \mathrm{e}^{-\mathrm{i}x(\mu-\mu^{-1})}}{\mu - \lambda}\, \mathrm{d}\mu.$$

It only remains to enforce the condition that $Y_{j1}^{(n)}(\lambda; x, \frac{1}{2}+k) = O(\lambda^{k+1})$ as $\lambda \to 0$ for $j = 1, 2$. Expanding $(\mu - \lambda)^{-1}$ for small λ in a geometric series and elimination of the second column elements in favor of $a_j^{(n,k)}(x)$ and $b_j^{(n,k)}(x)$, $j = 1, \dots, k+1$, yields separate $(k+1) \times (k+1)$ linear systems of Hankel type separately for the $a_j^{(n,k)}(x)$ and the $b_j^{(n,k)}(x)$: defining coefficients $I_{n,k,j}^+(x)$ by

$$I_{n,k,j}^+(x) := \int_L (\lambda_{\text{◪}}^{k/2+3/4})_+(\lambda_{\text{◪}}^{k/2+3/4})_- \lambda^{-n-j} \mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})}\, \mathrm{d}\lambda \tag{6.2}$$

the systems are

$$\mathbf{H}_{n,k}^+(x)\mathbf{a}^{(n,k)}(x) = -\mathrm{i}\sqrt{2\pi}k!\mathbf{e}^{(1)} \quad \text{and} \quad \mathbf{H}_{n,k}^+(x)\mathbf{b}^{(n,k)}(x) = -\mathbf{v}_{n,k}^+(x)$$

where $\mathbf{e}^{(1)} := (1, 0, 0, \dots, 0)^\top$ denotes the first coordinate unit vector, the unknowns are arranged in vectors as

$$\mathbf{a}^{(n,k)}(x) := (a_1^{(n,k)}(x), \dots, a_{k+1}^{(n,k)}(x))^\top, \quad \mathbf{b}^{(n,k)}(x) := (b_1^{(n,k)}(x), \dots, b_{k+1}^{(n,k)}(x))^\top,$$

and the Hankel matrix and right-hand side vector for the $\mathbf{b}^{(n,k)}(x)$ system are

$$\mathbf{H}_{n,k}^+(x) := \{I_{n,k,p+q}^+(x)\}_{p,q=1}^{k+1}, \quad \mathbf{v}_{n,k}^+(x) := (I_{n,k,1}^+(x), \dots, I_{n,k,k+1}^+(x))^\top.$$

Therefore, when $m = \frac{1}{2} + k$, $k \in \mathbb{Z}_{\geq 0}$, Riemann-Hilbert Problem 1 has a solution obtained by linear algebra in dimension $k+1$ provided that x is such that the complex Hankel determinant

$$D_{n,k}^+(x) := \det(\mathbf{H}_{n,k}^+(x))$$

is nonzero. From the formula (1.18) we then get the corresponding rational solution $u_n(x; \frac{1}{2}+k)$ of the Painlevé-III equation (1.1) for $k = 0, 1, 2, 3, \dots$ in the form

$$u_n(x; \tfrac{1}{2}+k) = \frac{\sqrt{2\pi}k!a_1^{(n,k)}(x)}{a_{k+1}^{(n,k)}(x) \displaystyle\sum_{j=1}^{k+1} a_j^{(n,k)}(x) I_{n,k,j+k+2}^+(x)}, \quad k \in \mathbb{Z}_{\geq 0}. \tag{6.3}$$

For instance, if $k = 0$, then we obtain

$$a_1^{(n,0)}(x) = -\frac{\mathrm{i}\sqrt{2\pi}}{D_{n,0}^+(x)} \quad \text{and} \quad b_1^{(n,0)}(x) = -\frac{1}{D_{n,0}^+(x)} \int_L (\lambda_{\text{◪}}^{3/4})_+(\lambda_{\text{◪}}^{3/4})_- \lambda^{-n-1} \mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})}\, \mathrm{d}\lambda$$

where

$$D_{n,0}^+(x) := \int_L (\lambda_{\text{◪}}^{3/4})_+(\lambda_{\text{◪}}^{3/4})_- \lambda^{-n-2} \mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})}\, \mathrm{d}\lambda.$$

Therefore, assuming that $D_{n,0}^+(x) \neq 0$, the solution of Riemann-Hilbert Problem 1 has been obtained in closed form for arbitrary integer n and for $m = \frac{1}{2}$. The corresponding rational solution of the Painlevé-III equation (1.1) is

$$u_n(x; \tfrac{1}{2}) = \mathrm{i} \frac{\displaystyle\int_{L_{\text{◪}}^\infty \cup L_{\text{◪}}^0} (\lambda_{\text{◪}}^{3/4})_+(\lambda_{\text{◪}}^{3/4})_- \lambda^{-(n+2)} \mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})}\, \mathrm{d}\lambda}{\displaystyle\int_{L_{\text{◪}}^\infty \cup L_{\text{◪}}^0} (\lambda_{\text{◪}}^{3/4})_+(\lambda_{\text{◪}}^{3/4})_- \lambda^{-(n+3)} \mathrm{e}^{-\mathrm{i}x(\lambda-\lambda^{-1})}\, \mathrm{d}\lambda}. \tag{6.4}$$

Assuming that the integrals in the fraction (6.4) have no common zeros, we see that the zeros of $u_n(x; \frac{1}{2})$ are the points where Riemann-Hilbert Problem 1 has no solution for $m = \frac{1}{2}$, while the poles of $u_n(x; \frac{1}{2})$ are regular points for $\mathbf{Y}^{(n)}(\lambda; x, \frac{1}{2})$.

Next assume that $m=-(\frac{1}{2}+k)$, $k\in\mathbb{Z}_{\geq 0}$. Then according to Riemann-Hilbert Problem 1, the matrix $\mathbf{Y}^{(n)}(\lambda;x,-\frac{1}{2}-k)$ is analytic for $\lambda\in\mathbb{C}\setminus L$, where we may now take L to be the contour $L=L^{\infty}_{\boxslash}\cup L^{0}_{\boxslash}$, across which we may write the jump condition in the form

$$\mathbf{Y}^{(n)}_{+}(\lambda;x,-\tfrac{1}{2}-k)=\mathbf{Y}^{(n)}_{-}(\lambda;x,-\tfrac{1}{2}-k)\begin{bmatrix}1 & \dfrac{\sqrt{2\pi}}{k!}(\lambda_{\boxslash}^{k-1/2})_{\infty}\lambda^{n}\mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})}\\ 0 & 1\end{bmatrix},\quad \lambda\in L,\quad k\in\mathbb{Z}_{\geq 0},$$

where $(\lambda_{\boxslash}^{k-1/2})_{\infty}$ denotes the function

$$(\lambda_{\boxslash}^{k-1/2})_{\infty}:=\begin{cases}\lambda_{\boxslash}^{k-1/2}, & \lambda\in L^{\infty}_{\boxslash},\\ -\lambda_{\boxslash}^{k-1/2}, & \lambda\in L^{0}_{\boxslash}.\end{cases}$$

Note that $(\lambda_{\boxslash}^{k-1/2})_{\infty}$ is continuous at the junction point between $L^{0}_{\boxslash}$ and $L^{\infty}_{\boxslash}$ because $\lambda_{\boxslash}^{k-1/2}$ changes sign across its jump contour of $L^{0}_{\boxslash}\cup L^{\infty}_{\boxslash}$. Obviously, it is now the first column of $\mathbf{Y}^{(n)}(\lambda;x,-\frac{1}{2}-k)$ that is analytic for $\lambda\in\mathbb{C}\setminus\{0\}$, and from the normalization conditions $Y^{(n)}_{11}(\lambda;x,-\frac{1}{2}-k)=1+O(\lambda^{-1})$ and $Y^{(n)}_{21}(\lambda;x,-\frac{1}{2}-k)=O(\lambda^{-1})$ as $\lambda\to\infty$ while $Y^{(n)}_{j1}(\lambda;x,-\frac{1}{2}-k)=O(\lambda^{-k})$ as $\lambda\to 0$, we see that the entries of the first column necessarily take the form

$$Y^{(n)}_{11}(\lambda;x,-\tfrac{1}{2}-k)=1+\sum_{j=1}^{k}c^{(n,k)}_{j}(x)\lambda^{-j}\quad\text{and}\quad Y^{(n)}_{21}(\lambda;x,-\tfrac{1}{2}-k)=\sum_{j=1}^{k}d^{(n,k)}_{j}(x)\lambda^{-j}$$

where $c^{(n,k)}_{j}(x)$ and $d^{(n,k)}_{j}(x)$ are coefficients to be determined. The jump condition together with the normalization condition that $Y^{(n)}_{12}(\lambda;x,-\frac{1}{2}-k)=O(\lambda^{-1})$ and $Y^{(n)}_{22}(\lambda;x,-\frac{1}{2}-k)=1+O(\lambda^{-1})$ as $\lambda\to\infty$ then determines the second column from the first:

$$Y^{(n)}_{12}(\lambda;x,-\tfrac{1}{2}-k)=\frac{1}{\mathrm{i}k!\sqrt{2\pi}}\int_{L}\frac{Y^{(n)}_{11}(\mu;x,-\frac{1}{2}-k)(\mu_{\boxslash}^{k-1/2})_{\infty}\mu^{n}\mathrm{e}^{\mathrm{i}x(\mu-\mu^{-1})}}{\mu-\lambda}\,\mathrm{d}\mu$$

and

$$Y^{(n)}_{22}(\lambda;x,-\tfrac{1}{2}-k)=1+\frac{1}{\mathrm{i}k!\sqrt{2\pi}}\int_{L}\frac{Y^{(n)}_{21}(\mu;x,-\frac{1}{2}-k)(\mu_{\boxslash}^{k-1/2})_{\infty}\mu^{n}\mathrm{e}^{\mathrm{i}x(\mu-\mu^{-1})}}{\mu-\lambda}\,\mathrm{d}\mu.$$

Then demanding that $Y^{(n)}_{j2}(\lambda;x,-\frac{1}{2}-k)=O(\lambda^{k})$ as $\lambda\to 0$ yields two Hankel systems on the coefficients $c^{(n,k)}_{j}(x)$ and $d^{(n,k)}_{j}(x)$. Setting

$$I^{-}_{n,k,j}(x):=\int_{L}(\lambda_{\boxslash}^{k-1/2})_{\infty}\lambda^{n-j}\mathrm{e}^{\mathrm{i}x(\lambda-\lambda^{-1})}\,\mathrm{d}\lambda,$$

these systems take the form

$$\mathbf{H}^{-}_{n,k}(x)\mathbf{c}^{(n,k)}(x)=-\mathbf{v}^{-}_{n,k}(x)\quad\text{and}\quad\mathbf{H}^{-}_{n,k}(x)\mathbf{d}^{(n,k)}(x)=-\mathrm{i}k!\sqrt{2\pi}\mathbf{e}^{(1)}$$

where

$$\mathbf{c}^{(n,k)}(x):=(c^{(n,k)}_{1}(x),\dots,c^{(n,k)}_{k}(x))^{\top},\quad\mathbf{d}^{(n,k)}(x):=(d^{(n,k)}_{1}(x),\dots,d^{(n,k)}_{k}(x))^{\top},$$

and the Hankel matrix and right-hand side vector for the $\mathbf{c}^{(n,k)}(x)$ system are

$$\mathbf{H}^{-}_{n,k}(x):=\{I^{-}_{n,k,p+q}(x)\}_{p,q=1}^{k},\quad\mathbf{v}^{-}_{n,k}(x):=(I^{-}_{n,k,1}(x),\dots,I^{-}_{n,k,k}(x))^{\top}.$$

Therefore, if $k\in\mathbb{Z}_{\geq 1}$ and $m=-\frac{1}{2}-k$, then Riemann-Hilbert Problem 1 has a solution obtained by $k\times k$ linear algebra, provided that the Hankel determinant

$$D^{-}_{n,k}(x):=\det(\mathbf{H}^{-}_{n,k}(x))$$

is nonzero given x. From (1.18) we get the corresponding rational solution of the Painlevé-III equation (1.1) in the form

$$u_n(x; -\tfrac{1}{2} - k) = \frac{\mathrm{i} I^-_{n,k,0}(x) + \mathrm{i} \sum_{j=1}^{k} c_j^{(n,k)}(x) I^-_{n,k,j}(x)}{c_k^{(n,k)}(x) I^-_{n,k,k+1}(x) + c_k^{(n,k)}(x) \sum_{j=1}^{k} c_j^{(n,k)}(x) I^-_{n,k,j+k+1}(x)}, \quad k \in \mathbb{Z}_{\geq 1}. \tag{6.5}$$

Note that if $k = 0$, the linear algebra system is trivial and hence Riemann-Hilbert Problem 1 *always* has a solution when $m = -\frac{1}{2}$:

$$\mathbf{Y}^{(n)}(\lambda; x, -\tfrac{1}{2}) = \begin{bmatrix} 1 & \dfrac{1}{\mathrm{i}\sqrt{2\pi}} \displaystyle\int_L \frac{(\mu_{\boxbslash}^{-1/2})_\infty \mu^n \mathrm{e}^{\mathrm{i}x(\mu - \mu^{-1})}}{\mu - \lambda} \, \mathrm{d}\mu \\ 0 & 1 \end{bmatrix}.$$

The corresponding rational solution of the Painlevé-III equation (1.1) is

$$u_n(x; -\tfrac{1}{2}) = \mathrm{i} \frac{\displaystyle\int_{L^\infty_{\boxbslash} \cup L^0_{\boxbslash}} (\lambda_{\boxbslash}^{-1/2})_\infty \lambda^n \mathrm{e}^{\mathrm{i}x(\lambda - \lambda^{-1})} \, \mathrm{d}\lambda}{\displaystyle\int_{L^\infty_{\boxbslash} \cup L^0_{\boxbslash}} (\lambda_{\boxbslash}^{-1/2})_\infty \lambda^{n-1} \mathrm{e}^{\mathrm{i}x(\lambda - \lambda^{-1})} \, \mathrm{d}\lambda}.$$

Remark 3. *We remark that in both cases the solution becomes more complicated as $|m|$ increases. This is similar to the situation with the explicit solution of the Fokas-Its-Kitaev Riemann-Hilbert problem for orthogonal polynomials* [10]. *Significantly however, the large parameter n appears explicitly in the (algebraic) solution of the Hankel system corresponding to any fixed half-integral value of m. It is this latter feature that enables a direct large-n asymptotic analysis by classical steepest descent methods* [3].

Another observation is that the formula (6.4) can be written in terms of Bessel functions. Indeed, we may write this formula in simplified form as

$$u_n(x; \tfrac{1}{2}) = \mathrm{i} \frac{\displaystyle\int_0^\infty \lambda^{-n-1/2} \mathrm{e}^{-\mathrm{i}x(\lambda - \lambda^{-1})} \, \mathrm{d}\lambda}{\displaystyle\int_0^\infty \lambda^{-n-3/2} \mathrm{e}^{-\mathrm{i}x(\lambda - \lambda^{-1})} \, \mathrm{d}\lambda}$$

where in both integrals the path of integration is the same, chosen (depending on x) so that the integrals are convergent at $\lambda = 0, \infty$, and also the branch of $\lambda^{-n-1/2}$ is arbitrary as long as it is analytic along the contour of integration and taken to be the same in both integrals. By the substitution $\lambda = e^t$ and comparison with [22, Equation 10.9.18] we then find that if $\mathrm{Im}(x) > 0$, then

$$u_n(x; \tfrac{1}{2}) = \mathrm{i} \frac{H^{(2)}_{n-1/2}(-\frac{\mathrm{i}}{2}x)}{H^{(2)}_{n+1/2}(-\frac{\mathrm{i}}{2}x)}$$

where $H^{(2)}_\nu(z)$ denotes a Hankel function. This formula admits meromorphic continuation to the whole complex x-plane. The same formula can then be expressed in terms of spherical Bessel functions of the second kind [22, 10.47(ii)] as

$$u_n(x; \tfrac{1}{2}) = \mathrm{i} \frac{\mathsf{h}^{(2)}_{n-1}(-\frac{\mathrm{i}}{2}x)}{\mathsf{h}^{(2)}_{n}(-\frac{\mathrm{i}}{2}x)}.$$

The functions $\mathrm{e}^{\mathrm{i}z}\mathsf{h}^{(2)}_n(z)$ are explicit polynomials in z^{-1} [22, Equation 10.49.7] and this in turn leads to the explicit formula

$$u_n(x; \tfrac{1}{2}) = \frac{\displaystyle\sum_{j=1}^{n} \frac{(2n-j-1)!}{(n-j)!(j-1)!} x^j}{\displaystyle\sum_{j=0}^{n} \frac{(2n-j)!}{(n-j)!j!} x^j}.$$

The identification of $u(x;\frac{1}{2})$ with ratios of Bessel polynomials was also noted in [8]. More generally, from [22, Equation 10.9.18] it is clear that the integrals $I^{\pm}_{n,k,j}(x)$ are proportional to Hankel functions, and hence the expression for $u_n(x;\pm(\frac{1}{2}+k))$ can always be written in terms of ratios of Hankel-type determinants whose entries are Bessel functions. More important from the point of view of asymptotic analysis in the large-n limit however is the fact that the coefficients are integrals that may be analyzed by classical steepest descent methods; see [3]. This is an effective strategy precisely because the matrices involved have a fixed size as $n \to \infty$.

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