Social Discounting and Intergenerational Pareto

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Abstract

The most critical issue in evaluating policies and projects that affect generations of individuals is the choice of social discount rate. This paper shows that there exist social discount rates such that the planner can simultaneously be (i) an exponential discounting expected utility maximizer; (ii) intergenerationally Pareto—i.e., if all individuals from all generations prefer one policy/project to another, the planner agrees; and (iii) strongly non-dictatorial—i.e., no individual from any generation is ignored. Moreover, to satisfy (i)–(iii), if the time horizon is long enough, it is generically sufficient and necessary for social discounting to be more patient than the most patient individual's long-run discounting, independent of the social risk attitude.

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1 Introduction

Many economic decisions are inherently dynamic and affect multiple generations, such as corporate and household long-term investment decisions, intertemporal taxation, durable public good provision, environmental policies, etc. These decisions crucially depend on one parameter, the *social discount rate*, which encapsulates the trade-off between the current benefit and future benefit from the society's point of view. Unfortunately, there is no consensus on which social discount rate should be used. This disagreement has sparked debate, for example, about the cost-benefit analysis of environmental projects that affect many, if not all, future generations. Moreover, the evaluation of those projects is sensitive to the choice of social discount rate. The famous Stern review uses a near-zero social discount rate (pure rate of time preference), and suggests that we should take strong and immediate action on climate change (see Stern (2007)).[1] Nordhaus (2007) argues that Stern's conclusion does not hold if a market rate is used instead. Many economists, however, believe that using a high discount rate (such as a market rate) is ethically indefensible.

In the social discounting literature, some economists have argued that social discounting should be more patient than individual discounting (for example, see Caplin and Leahy (2004) and Farhi and Werning (2007)). The idea is that if social discounting takes into account how future generations will feel about their consumption, then because future generations will value future consumption relatively more than the current generation values future consumption, social discounting will also value future consumption more than the current generation does.[2] However, these studies usually assume that only one (representative) individual is in the society. How their insight carries over to a society with heterogeneous individuals—and which individual's discounting social discounting should be more patient than—remains unanswered.

[1]The consumption discount rate derived from the Ramsey formula used in the Stern review depends on the pure rate of time preference, the elasticity of the marginal utility of consumption, and the growth rate of per capita consumption.

[2]Some economists have also argued that individuals' altruistic discounting for future generations should be excluded from the planner's aggregation. See Hammond (1987) and Boadway (2012).

Let us explain what will go wrong with heterogeneous individuals. What is common among these dynamic economic decisions is that there is a benevolent planner who must make choices from risky alternatives for generations of individuals. In such a setting, first, economists often assume that the planner's objective is an *exponential discounting (expected) utility function*. This assumption is widely used and normatively appealing, because it is equivalent to assuming that the planner's preference is time-consistent, time-invariant, and stationary.[3] Second, it is often assumed that a benevolent planner respects individuals' preferences. In other words, some notion of the *Pareto* property should hold: If "all" individuals agree that one policy/project is better than another, the planner should agree that the former is better.

Despite the fact that these two assumptions are fundamental to economics, they cannot be satisfied simultaneously (see Gollier and Zeckhauser (2005), Zuber (2011), and Jackson and Yariv (2015)). Even if every individual has an exponential discounting utility function, a planner must be dictatorial to ensure that her exponential discounting utility function satisfies some Pareto property. The negative result also challenges the conclusion that social discounting should be more patient than individual discounting. In light of the negative result, with heterogeneous individuals, perhaps we can only conclude that the planner is more patient than the only individual (dictator) she cares about.

This paper addresses these issues using a classic approach. We introduce a new Pareto property, and characterize the range of (pure-time-preference) social discount rates that are compatible with the new property. In models that generate the negative result, there is often only one generation of individuals. The Pareto property they use, which we call *current-generation Pareto*, is the key to the negative result. Current-generation Pareto requires that whenever a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$ by every current-generation individual, then the planner prefers $\mathbf{p}$ to $\mathbf{q}$. In many problems we consider,

[3] A version of the definition of time consistency, time invariance, and stationarity can be found in Halevy (2015). Under the assumption that the utility function is a time-additively separable expected utility function, Halevy's version of the three properties is equivalent to assuming an exponential discounting expected utility function.

multiple generations of individuals are involved. As Pigou (1920) argues, the planner should not only respect how the current generation discounts the future, but also care about the actual well-being of future generations—that is, how future generations will feel about their consumption and how they will discount the future. The Pareto property we introduce, *intergenerational Pareto*, captures this. It requires that whenever a consumption sequence $\mathbf{p}$ is preferred to $\mathbf{q}$ by every individual from every generation, then the planner prefers $\mathbf{p}$ to $\mathbf{q}$.

Specifically, each generation-t individual i lives for one period, and has a discount function $\delta_i(\tau - t)$ to discount period-τ consumption.[4] The planner is intergenerationally Pareto and has an exponential discounting utility function. To contrast with the negative result, we require that the planner be *strongly non-dictatorial* in the sense that she never ignores the preference of any individual from any generation. Under these assumptions, we show how the range of social discount factors depends on (a) individual relative discounting, average discounting, and long-run discounting, and (b) the linear dependency of individual instantaneous utility functions.[5]

We first characterize the range of social discount factors assuming that individual discount functions are exponential. This allows us to compare our results to the negative result directly. We examine two cases. In the first case, individuals share the same instantaneous utility function. In this way, we focus on aggregating individual discount functions. The negative result is avoided: We find that the planner is intergenerationally Pareto and strongly non-dictatorial if and only if the social discount factor is higher than the *least patient* individual's discount factor.

Since the least patient individual's discount factor could be quite low, a wide range of social discount factors can be supported by the first result. The result will be rather different in our second case in which individual instantaneous utility functions are linearly independent. When there are many consumption goods, individual instantaneous utility

[4]Individuals *altruistically* care about future generations' consumption.
[5]The discount rate is equal to one minus the discount factor.

functions are generically linearly independent. Under this assumption, we find that the planner is intergenerationally Pareto and strongly non-dictatorial if and only if the social discount factor is higher than the *most patient* individual's discount factor, *independent of the planner's instantaneous utility function*. This result thus provides a new justification for the use of a near-zero social discount rate.

In general, individual discount functions are not exponential. One challenge that comes with general individual discount functions is that when we say that social discounting is more patient than individual discounting, it is not even clear what individual discounting refers to. We show that when individuals share the same instantaneous utility function, there exist two cutoffs for the social discount factor. One is related to the least patient individual's maximal relative discount factor, and the other to the least patient individual's asymptotic average discount factor. If the social discount factor is above the first cutoff, the planner is intergenerationally Pareto and strongly non-dictatorial. Conversely, if the social discount factor is below the second cutoff, the planner must violate intergenerational Pareto as long as the time horizon is long enough; that is, there exist two consumption sequences such that every individual from every generation thinks that one is better than the other, but the planner disagrees. The two cutoffs are tight.

The two cutoffs merge into one cutoff when individuals exhibit *present bias*. The unique cutoff is equal to the least patient individual's *long-run discount factor*. Each individual's long-run discount factor is defined as the asymptotic relative discount factor and the asymptotic average discount factor.

Lastly, if individual instantaneous utility functions are linearly independent, the cutoff for the social discount factor jumps from the least patient individual's long-run discount factor to the most patient one's, again independent of the planner's instantaneous utility function. We also characterize how the cutoff for the social discount factor changes gradually from the least patient individual's long-run discount factor to the most patient one's, as the number of types of individual instantaneous utility functions increases.

1.1 Related Literature

This paper is not the first to aggregate the preferences of multiple generations of individuals. Indeed, there is a long-running debate on whether future generations should be aggregated. Among others, Pigou (1920), Ramsey (1928), Sen (1961), Feldstein (1964), Solow (1974), Arrow (1999), Caplin and Leahy (2004), and Farhi and Werning (2007) are in favor, and Eckstein (1957), Bain (1960), and Marglin (1963) believe that the government's or the policy maker's decision should only reflect the preferences of the current generation. Our approach is closer to Caplin and Leahy and Farhi and Werning, who show that assuming there is only one individual in each generation, social discounting should be more patient than the sole individual's discounting. Our results show that having multiple heterogeneous individuals in each generation makes an important difference.

Many papers have analyzed the aggregation of one generation of heterogeneous individuals. Weitzman (2001) conducts a survey of economists' discount rates to motivate a gamma discounting model. Gollier and Zeckhauser (2005) study a dynamic efficient allocation problem with heterogeneous individuals and show that even when individuals have constant discount rates, the representative agent has a decreasing discount rate. Zuber (2011) establishes that a planner cannot have an exponential discounting utility function and be (current-generation) Pareto when individuals have private consumption. Jackson and Yariv (2015) present a similar negative result, in which consumption is public. Millner and Heal (2018) show that the negative result goes away if we only require that the planner's objective be time-consistent. A key difference between these papers and ours is that they aggregate only one generation of individuals, whereas we aggregate multiple generations. This distinction is important in economic decisions that have long-term impact, such as environmental policies and intertemporal taxation.

There are other approaches to the study of social discounting. Our paper emphasizes the relation between social discounting and individual discounting implied by intergenerational Pareto. Chambers and Echenique (2018) study three models of discount rates. One aggregates exponential individual discount functions in a utilitarian way, which is similar to Weitzman (2001), Zuber (2011), and Jackson and Yariv (2014, 2015). The other two aggregate exponential individual discount functions by selecting the most pessimistic utilitarian weight and discount function, respectively. Millner (2016) shows that if heterogeneous individuals are not fully paternalistic, they will agree on parameters for the long-run social discount rate. Zuber and Asheim (2012), Asheim and Zuber (2014), Fleurbaey and Zuber (2015), and Piacquadio (2017) study models in which social discounting is due to intergenerational inequality aversion. Jonsson and Voorneveld (2018) study a welfare criterion for multiple generations. Each generation has one individual, and in the limit of the criterion, different generations are treated equally.

In the first part of Drugeon and Wigniolle (2017), they characterize what exponential discounting utility functions can be written as weighted sums of individuals' current selves' and future selves' quasi-hyperbolic discounting utility functions; their result is related to our Propositions 4 and S1, and Proposition 5 of Galperti and Strulovici (2017). Drugeon and Wigniolle (2016) and the second part of Drugeon and Wigniolle (2017) study time-consistent solutions for consumption-saving problems with heterogeneous exponential and quasi-hyperbolic discounting individuals, respectively. The planner in period t maximizes the weighted sum of period-t individuals' utility, and the solution is the subgame perfect Nash equilibrium of the game between the planner's multiple selves.

Our paper is also related to Mongin (1998), who establishes that under a standard form of Pareto, as long as individuals' subjective probabilities are linearly independent or their instantaneous utility functions are affinely independent, the planner must be dictatorial. Related results can be found in Mongin (1995) and Chambers and Hayashi (2006). In our model, if we view periods as states and discount factors as subjective probabilities, Mongin's result seems to apply. Nonetheless, our planner is not dictatorial. The technical reason why our Theorem 1 can bypass Mongin's negative result is the assumption that all individuals share the same instantaneous utility function. As for Theorem 2, we first aggregate individual utility functions with identical instantaneous utility functions into an EDU function whose discount factor is equal to the social discount factor. Then, we aggregate utility functions with identical discount factors (subjective probabilities). Both steps bypass Mongin's negative result.

The paper proceeds as follows. In Section 2, we describe individuals' and the planner's preferences. Section 3 introduces a variant of the negative result and two key assumptions of the paper, intergenerational Pareto and strong non-dictatorship. We characterize the range of social discount factors under the assumption that individuals have exponential discount functions in Section 4, and under the assumption that individuals have general discount functions in Section 5. Section 6 concludes.

2 Preferences

There are $2 < T < +\infty$ generations/periods. In each generation, $1 < N < +\infty$ individuals live for one period. With an abuse of notation, let $N := \{1, \ldots, N\}$ and $T := \{1, \ldots, T\}$. The generation-t individual i is the parent of the generation-$(t+1)$ individual i, in which $t, t+1 \in T$ and $i \in N$. In each period, there is a public risky consumption good denoted by $\Delta(X)$, in which $\Delta(X)$ is the set of probability measures on a compact set $X \subset \mathbb{R}^m$.[6] A typical consumption sequence is denoted by $\mathbf{p} = (p_1, \ldots, p_T) \in \Delta(X)^T$.[7]

Although individuals live for one period, they altruistically care about future generations' consumption. We assume throughout the paper that the generation-t individual i has the following *discounting utility* function:

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i(\tau - t) u_i(p_\tau), \tag{1}$$

[6] All results we derive apply to the case in which each individual has his own consumption. We only need to view public consumption as an N-tuple of individual consumption, and let each individual care only about his own component.

[7] We discuss what may change if we allow uncertainty to resolve over time in Section S4 in the Supplemental Material.

in which $\delta_i : \{0, \ldots, T-1\} \to \mathbb{R}_{++}$ with $\delta_i(0) = 1$ is called the *discount function*, and the *instantaneous utility function* $u_i : \Delta(X) \to \mathbb{R}$ is a continuous expected utility function. The generation-t individual i's discounting utility function induces a preference, denoted by $\succsim_{i,t}$, over consumption sequences $\Delta(X)^T$.

We have assumed that the generation-$(t+1)$ individual i inherits the generation-t individual i's discount function and instantaneous utility function. This assumption does not imply that a parent and his offspring have the same preference, because the generation-$(t+1)$ individuals' discount functions are shifted one period forward. This assumption simplifies our analysis and can be relaxed (see Section S1.1 in the Supplemental Material).

In each period $t \in T$, the planner's objective is an *exponential discounting utility* (EDU) function:

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau), \tag{2}$$

in which $\delta > 0$ is the *social discount factor*, and u, a continuous expected utility function on $\Delta(X)$, is the planner's instantaneous utility function. In each period $t \in T$, U_t induces the planner's preference, denoted by $\succsim_t$, over consumption sequences $\Delta(X)^T$.

It is well known that if the planner's objective is a discounting utility function, the planner is time-consistent if and only if the planner's discount function is exponential.[8] More generally, (2) holds if and only if the planner's preference is time-consistent, time-invariant, and stationary (see footnote 3). Also note that (2) holds for every $t \in T$; that is, the social discount factor and the planner's instantaneous utility function never change.

Lastly, to rule out uninteresting cases and simplify the statement of our results, we assume that there are some fixed consequences $x_*, x^* \in X$ such that for any $i \in N$, $u_i(x_*) = u(x_*) = 0$ and $u_i(x^*) = u(x^*) = 1$ throughout the paper. A similar assumption, called the *minimum agreement condition*, also appears in De Meyer and Mongin (1995). Our main findings do not rely on this assumption, and we provide a more detailed discussion following Lemma 1.

[8]Since individuals only live for one period, time consistency may have a nonstandard interpretation for them. In contrast, the planner is a long-lived entity who tries to stick to an objective function that exhibits nice properties. The interpretation of time consistency for the planner is similar to the standard one.

More generally, for any continuous expected utility function v defined on $\Delta(X)$, we say that it is *normalized* if $v(x^*) = 1$ and $v(x_*) = 0$. One may think of x^* as one dollar and x_* as zero dollars, or x^* as the best consumption good and x_* as the worst.

3 Intergenerational Pareto and Dictatorship

We want to assume that the planner's preference $(\succsim_t)_{t\in T}$ satisfies some Pareto property. In a dynamic setting, however, there are multiple ways to define the Pareto property. Different notions of Pareto have different implications. For example, Zuber (2011) and Jackson and Yariv (2015) show that if a planner has an EDU function and follows their Pareto property, the planner must be dictatorial. To motivate our new Pareto property, it is useful to first understand the negative result. Below, we introduce a version of the negative result.

3.1 A Variant of the Negative Result

Below is a variant of the Pareto property used by Zuber (2011) and Jackson and Yariv (2015) that fits our setting.

Definition 1 *The planner's preference $(\succsim_t)_{t\in T}$ is current-generation Pareto if for any consumption sequences $\mathbf{p}, \mathbf{q} \in \Delta(X)^T$, in each period $t \in T$, $\mathbf{p} \succsim_{i,t} \mathbf{q}$ for all $i \in N$ implies $\mathbf{p} \succsim_t \mathbf{q}$, and $\mathbf{p} \succ_{i,t} \mathbf{q}$ for all $i \in N$ implies $\mathbf{p} \succ_t \mathbf{q}$.*

This notion of Pareto says that in any period t, if all current-generation individuals agree that a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$, the planner should agree that $\mathbf{p} \succsim_t \mathbf{q}$. The same applies when the preferences are all strict.

Consider a situation in which every generation-t individual i has an EDU function; that is, for some discount factor $\delta_i > 0$,

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i^{\tau-t} u_i(p_\tau).$$

Let us present below a variant of the negative result.

Proposition 1 *Suppose each generation-t individual i has an EDU function with discount factor δ_i and instantaneous utility function u_i such that δ_i's are distinct. The planner is current-generation Pareto if and only if there exists some $i \in N$ such that for any $t \in T$, $U_t = U_{i,t}$.*

The result says that if we require that the planner be current-generation Pareto and have an EDU function, the planner's preference must be identical to some individual's preference in every period. Since consumption is public, our setting is closer to Jackson and Yariv (2015). However, Jackson and Yariv's result is different from the above proposition. For example, they require that instantaneous utility functions be defined on a one-dimensional space and twice continuously differentiable, and we require that instantaneous utility functions be expected utility functions.

The intuition is as follows. First, the planner is current-generation Pareto if and only if her EDU function is equal to a weighted sum of the individuals' EDU functions; because we consider expected utility functions, this is an implication of Harsanyi (1955). Next, for simplicity, suppose there are only two individuals with identical instantaneous utility functions $u_1 = u_2$. The planner attaches a weight $\omega \in [0, 1]$ to the first individual and $1 - \omega$ to the second individual. Now, for the planner to not be dictatorial, there must be some $\omega \in (0, 1)$ and $\delta > 0$ such that

$$\omega\delta_1 + (1 - \omega)\delta_2 = \delta,$$

and

$$\omega\delta_1^2 + (1 - \omega)\delta_2^2 = \delta^2.$$

However, one cannot find such a δ, unless $\omega = 0$ or 1.

3.2 Intergenerational Pareto

A key feature of environmental policies and many other economic policies is that such decisions affect multiple generations. Current-generation Pareto only takes into account the preferences of the current generation. Although current-generation individuals altruistically care about future consumption and the planner should respect how they discount the future, how they think about the future may well differ from how future generations will think. Since future generations will be affected by the planner's decision, the planner should take into account their actual well-being (including how they will discount their own future). The following Pareto property captures this idea.

Definition 2 *The planner's preference* $(\succsim_t)_{t\in T}$ *is intergenerationally Pareto if for any consumption sequences* $\mathbf{p}, \mathbf{q} \in \Delta(X)^T$*, in each period* $t \in T$*,* $\mathbf{p} \succsim_{i,s} \mathbf{q}$ *for all* $i \in N$ *and all* $s \geq t$ *implies* $\mathbf{p} \succsim_t \mathbf{q}$*, and* $\mathbf{p} \succ_{i,s} \mathbf{q}$ *for all* $i \in N$ *and all* $s \geq t$ *implies* $\mathbf{p} \succ_t \mathbf{q}$*.*

Intergenerational Pareto says that in any period t, if all current- and future-generation individuals agree that a consumption sequence is preferred to another sequence, the planner should agree. For example, suppose all current-generation individuals are extremely selfish: They are willing to sacrifice the environment to increase their own consumption. If the planner is current-generation Pareto, the planner must agree with them, and let them destroy the environment. However, if the planner is intergenerationally Pareto, the planner is allowed to disagree with them, because what they prefer hurts future generations. Note that if the planner is current-generation Pareto, she is also intergenerationally Pareto. Therefore, intergenerational Pareto is weaker than current-generation Pareto.

Our model considers expected utility functions. This enables us to apply the classic result from Harsanyi (1955) and Fishburn (1984) to characterize the consequence of intergenerational Pareto.

Lemma 1 *(Harsanyi (1955)) The planner's preference* $(\succsim_t)_{t\in T}$ *is intergenerationally Pareto if and only if in each period* $t \in T$*, there exists a finite sequence of nonnegative numbers*

$(\omega_t(i,s))_{i\in N, s\geq t}$ *such that*

$$U_t = \sum_{i=1}^{N} \sum_{s=t}^{T} \omega_t(i,s) U_{i,s}.$$

The lemma above follows from Harsanyi (1955) and Fishburn (1984), and shows that intergenerational Pareto is equivalent to intergenerational utilitarianism in our setting; that is, the planner is intergenerationally Pareto if and only if in each period, her utility function is equal to a weighted sum of all the current- and future-generation individuals' utility functions. In contrast, current-generation Pareto is equivalent to current-generation utilitarianism. We omit the proof of this lemma.

Instantaneous utility functions are normalized. In general, it is possible that there do not exist two consumption sequences such that all individuals strictly prefer one to the other; in that case, if the planner is indifferent to all consumption sequences, the planner will be intergenerational Pareto trivially. If the planner is always indifferent, her instantaneous utility function is constant and her discount function can be arbitrary. The normalization assumption rules out this uninteresting case.

3.3 Dictatorship

In the negative result, a planner can have an EDU function and be current-generation Pareto as long as she is dictatorial. To rule out dictatorship, we introduce a strong notion of non-dictatorship such that not only is the planner not dictatorial, but also every individual from every generation has a say.[9]

Definition 3 *We say that the planner is strongly non-dictatorial if for each* $t \in T$*,*

$$U_t(\mathbf{p}) = f_t\left(U_{1,t}(\mathbf{p}), \ldots, U_{1,T}(\mathbf{p}), U_{2,t}(\mathbf{p}), \ldots, U_{2,T}(\mathbf{p}), \ldots, U_{N,T}(\mathbf{p})\right)$$

for some (strictly) increasing function f_t*.*

[9] When the planner is not dictatorial, we only know that at least two individuals' preferences are taken into account by the planner.

In light of Lemma 1, under intergenerational Pareto, this means that the planner's utility function can be written as a weighted sum of individual utility functions with positive weights.

Intergenerational Pareto is weaker than current-generation Pareto. According to Lemma 1, the planner has more utilitarian weights to assign under intergenerational Pareto, which makes it easier for the planner to aggregate individuals' utility functions into an EDU function. The strongly non-dictatorial property, on the other hand, makes the aggregation problem harder, because it requires that all weights be positive.

4 Individuals with Exponential Discount Functions

We address two aspects of social discounting. First, can we bypass the negative result? If so, which social discount factors are reasonable? In particular, which social discount factors, under our assumptions, are compatible with intergenerational Pareto? Second, recall that in the social discounting literature, economists have argued that the social discount factor should be higher than the individual discount factor. Accordingly, with heterogeneous individuals, which individual's discount factor should the social discount factor be higher than?

To contrast with the negative result, we first examine a special case of our model in which individual discount functions are exponential.

4.1 Aggregating Individual Discount Functions

To focus on discounting, suppose that all individual instantaneous utility functions are identical; that is, there is some continuous expected utility function $u : \Delta(X) \to \mathbb{R}$ such that each generation-t individual i's utility function is

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i^{\tau-t} u(p_\tau).$$

This assumption will be relaxed soon, and we will use the result established under this assumption to highlight how the range of reasonable social discount factors is affected by individual instantaneous utility functions. An alternative interpretation of this assumption is that the planner only wants to aggregate individual discount functions. Therefore, it is without loss of generality to replace the (possibly heterogeneous) individual instantaneous utility functions with the planner's instantaneous utility function u.[10]

Proposition 2 *Suppose each generation-t individual i has an EDU function with discount factor* δ_i *and instantaneous utility function* u*. Let the planner's instantaneous utility function be* u*. The planner is intergenerationally Pareto and strongly non-dictatorial if and only if* $\delta > \min_i \delta_i$*.*

When individuals share the same instantaneous utility function, according to Lemma 1, the planner must use the same instantaneous utility function in order to satisfy the Pareto property.

Proposition 2 shows that under intergenerational Pareto rather than current-generation Pareto, a positive result can be established. Moreover, under the current set of assumptions, it is the least patient individual's discount factor that the social discount factor should be higher than.

Because discount functions are exponential and consumption is public, Proposition 2 can be directly compared to Jackson and Yariv (2014, 2015). In Jackson and Yariv, adding more current-generation exponential discounting individuals to the aggregation cannot help eliminate the negative result. In contrast, we add future-generation exponential discounting individuals to the aggregation, and this helps.

To see why, first recall that when $u_i = u$, Jackson and Yariv (2014) show that utilitarian aggregation of the current generation leads to a social discount function that exhibits present bias. The fact that future generations will not care about past consumption as much as past

[10]In this interpretation, however, each individual i's preference in the definition of Pareto properties must be replaced with another preference induced by a discounting utility function with a discount function δ_i and an instantaneous utility function u chosen by the planner.

generations did helps us remove the present bias. In our model, past consumption does not enter future generations' utility functions; that is, $\delta_i(\tau) = 0$ for any $\tau < 0$. This implies that, for example, generation-t individual i's relative discount factor applied to period-t consumption (relative to period-$(t-1)$ consumption) is equal to "$\delta_i(0)/\delta_i(-1) = +\infty$." Thus, generation-$t$ is "infinitely patient" between period $t-1$ and period t. The infinite patience can be used in the aggregation to offset the present bias generated by aggregating the current generation alone. In fact, the same result continues to hold even if individuals backward discount past consumption exponentially (see Section S3 in the Supplemental Material).[11]

To understand how the proposition is proved, consider the planner in the first period. According to Lemma 1, the planner is intergenerationally Pareto if and only if her objective function U_1 satisfies

$$U_1(\mathbf{p}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \omega(i,t) U_{i,t}(\mathbf{p}), \tag{3}$$

for any consumption sequence $\mathbf{p}$, in which $\omega(i,t) \geq 0$ is the weight the planner assigns to the generation-t individual i. Consider how the planner discounts period-τ consumption. Since the planner and individuals have EDU functions, and their instantaneous utility functions are identical, equation (3) becomes

$$\delta^{\tau-1} = \sum_{t=1}^{\tau} \sum_{i=1}^{N} \omega(i,t) \delta_i^{\tau-t}. \tag{4}$$

To prove the "if" part of the proposition, we let all individuals' weights be equal to some small numbers, except for the least patient individuals. We show that if those weights are sufficiently small, there exist positive weights for the least patient individuals such that the weighted sum of all individuals' utility functions is an EDU function with the social discount factor $\delta > \min_i \delta_i$. For example, suppose $N = T = 2$ and $\delta_1 < \delta_2$.[12] Let $\omega(i,t) = \varepsilon$ whenever

[11] See Caplin and Leahy (2004) and Ray et al. (2017) for models that allow backward discounting for past consumption.

[12] We have assumed $T > 2$ because when $T \leq 2$, there will be no negative result (such as Proposition 1)

$i = 2$. Equation (4) implies that

$$\omega(1,1) = 1 - \omega(2,1) = 1 - \varepsilon$$

and

$$\omega(1,2) = \delta - \omega(1,1)\delta_1 - \omega(2,1)\delta_2 - \omega(2,2) = \delta - \delta_1 + \varepsilon(\delta_1 - \delta_2 - 1).$$

Since $\delta > \delta_1$, $\omega(1,1)$ and $\omega(1,2)$ are positive when $\varepsilon = 0$. Therefore, when $\varepsilon > 0$ is sufficiently small, $\omega(i,t)$'s can all be positive.

To understand the "only-if" part, suppose individual 1's discount factor is the lowest. By letting $\tau = 1$, equation (4) implies that $\sum_{i=1}^{N} \omega(i,1) = 1$. Since the planner is strongly non-dictatorial, we can assume that $\omega(i,t)$'s are positive, and hence $\sum_{t=1}^{\tau} \sum_{i=1}^{N} \omega(i,t) > 1$. Then, equation (4) implies that

$$\delta^{\tau-1} = \sum_{t=1}^{\tau} \sum_{i=1}^{N} \omega(i,t)\delta_i^{\tau-t} \geq \delta_1^{\tau-1} \sum_{t=1}^{\tau} \sum_{i=1}^{N} \omega(i,t) > \delta_1^{\tau-1}, \tag{5}$$

which means $\delta > \delta_1$.

Note that Proposition 2 is not very helpful in pinning down social discount factors, because the least patient individual's discount factor can be quite low. Thus, many social discount factors can satisfy our requirements. However, as will be shown below, this is no longer the case once we relax the unrealistic assumption that individuals share the same instantaneous utility function.

4.2 Social Discounting and Individual Instantaneous Utility Functions

The assumption that individuals share the same instantaneous utility function is clearly unrealistic. As long as $|X| \geq N$ (i.e., the number of deterministic consumption goods is

trivially. However, to illustrate the idea of the proof here, we only need an example with $T = 2$.

higher than the number of individuals in each generation), generically, the instantaneous utility functions should not only be different, but also linearly independent.[13]

Definition 4 *An N-tuple of continuous expected utility functions $(u_i)_{i\in N}$ is linearly independent if there are no constants $\alpha_1, \ldots, \alpha_N$ that are not all zero, and $\sum_{i\in N} \alpha_i u_i(p) = 0$ for all $p \in \Delta(X)$.*

It turns out that when individual instantaneous utility functions are linearly independent, the cutoff for the social discount factor jumps from $\min_i \delta_i$ to $\max_i \delta_i$; that is, generically, the social discount factor should be higher than the *most* patient individual's discount factor.

Proposition 3 *Suppose each generation-t individual i has an EDU function with discount factor δ_i and instantaneous utility function u_i such that $(u_i)_{i\in N}$ is linearly independent. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $(u_i)_{i\in N}$.*[14] *The planner is intergenerationally Pareto and strongly non-dictatorial if and only if $\delta > \max_i \delta_i$.*

To understand why we assume that the planner's instantaneous utility function is a strict convex combination of individual instantaneous utility functions, note that Lemma 1 implies that the intergenerationally Pareto and strongly non-dictatorial planner's utility function is equal to a weighted sum of individual discounting utility functions with positive weights. Thus, the planner's instantaneous utility function must also be a positively weighted sum of individual instantaneous utility functions. Since instantaneous utility functions are normalized, the weights sum up to 1.

Notice that the planner's instantaneous utility function—in other words, her risk attitude—is independent of the cutoff for the social discount factor. This is somewhat surprising.

[13] However, for example, if individual instantaneous utility functions are drawn from some fixed small set of continuous expected utility functions rather than the set of all continuous expected utility functions, or the number of consumption goods is lower than N, individual instantaneous utility functions need not be linearly independent. See Theorem 3 for results without assuming linear independence.

[14] By a strict convex combination of $(u_i)_{i\in N}$, we mean that u is in the interior of the convex hull of $u_1, \ldots, u_N$.

Suppose there are two individuals, 1 and 2, and individual 2 is more patient. The above result says that even if the social discount factor is close to individual 2's discount factor, it is not necessarily the case that the planner's risk attitude is also close to individual 2's risk attitude. We can have a planner whose risk attitude is close to individual 1's, but the social discount factor is close to individual 2's.

If there are many individuals with a wide range of discount factors, this result may imply that the planner must be very patient in order to be intergenerationally Pareto and strongly non-dictatorial. This provides a new justification for the use of the near-zero social discount rate by Stern (2007). If one thinks that a market rate is higher than the lowest individual discount rate, this result also rules out the use of a market rate as the social discount rate.

This result shows that the cutoff for the social discount factor in Proposition 2 is not robust. When $u_i = u_j$ for any $i, j \in N$, the cutoff is $\min_i \delta_i$. If we introduce a small perturbation to u_i's, generically, the cutoff jumps discontinuously to $\max_i \delta_i$.

One may wonder whether there is any intermediate case that yields a cutoff for the social discount factor between $\min_i \delta_i$ and $\max_i \delta_i$. In Section 5.4, under a more general assumption about individual instantaneous utility functions, we explain the intermediate cases.

To understand how this proposition is proved, consider again the planner in the first period. To prove the "if" part of Proposition 3, we want to find positive weights $\omega(i,t)$'s such that the weighted sum of all individuals' EDU functions is equal to the planner's EDU function. Focus on one arbitrary $j \in N$. We show that we can find positive weights $\tilde{\omega}(j,1), \ldots, \tilde{\omega}(j,T)$ such that $\sum_{t \in T} \tilde{\omega}(j,t) U_{j,t}$ is equal to an EDU function with any discount factor that is higher than δ_j. In particular, we can find positive weights $\tilde{\omega}(i,1), \ldots, \tilde{\omega}(i,T)$ for each $i \in N$ such that

$$\sum_{t=1}^{T} \tilde{\omega}(i,t) U_{i,t}(\mathbf{p}) = \sum_{\tau=1}^{T} \delta^{\tau-1} u_i(p_\tau),$$

for any consumption sequence $\mathbf{p}$, because $\delta > \max_i \delta_i$. Now, since the planner's instantaneous utility function $u = \sum_{i \in N} \lambda_i u_i$ for some positive numbers λ_i's, we only need to let $\omega(i,t) = \lambda_i \tilde{\omega}(i,t)$.

The "only-if" part of Proposition 3 may be surprising. Note that when $(u_i)_{i \in N}$ is linearly independent and u is in the interior of $\text{co}(\{u_i\}_{i \in N})$, there is a unique way to write u as a strict convex combination of $(u_i)_{i \in N}$.[15] Suppose $\sum_{i \in N} \lambda_i u_i = u$ and $\sum_{i \in N} \lambda_i = 1$ for some positive numbers λ_i's. The planner's period-1 EDU function satisfies

$$U_1(\mathbf{p}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \omega(i,t) U_{i,t}(\mathbf{p}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \omega(i,t) \sum_{\tau=t}^{T} \delta_i^{\tau-t} u_i(p_\tau), \tag{6}$$

in which $\omega(i,t) > 0$ is the weight the planner assigns to the generation-t individual i. This implies that the planner's instantaneous utility function for period-1 consumption satisfies

$$u(p_1) = \sum_{i=1}^{N} \omega(i,1) u_i(p_1)$$

for any p_1. Because $u = \sum_{i \in N} \lambda_i u_i$ and $(u_i)_{i \in N}$ is linearly independent,

$$\omega(i,1) = \lambda_i \tag{7}$$

must hold for any $i \in N$. Similarly, for period-2 consumption, equation (6) implies that

$$\delta u(p_2) = \sum_{i=1}^{N} [\omega(i,1)\delta_i + \omega(i,2)] u_i(p_2)$$

for any p_2. Since instantaneous utility functions do not change over time, the unique way to write u as a strict convex combination of $(u_i)_{i \in N}$ does not change; that is, $\delta u(p_2) = \delta \sum_{i \in N} \lambda_i u_i(p_2)$ for any p_2. Then, for any $i \in N$,

$$\lambda_i \delta = \omega(i,1)\delta_i + \omega(i,2). \tag{8}$$

[15] We use $\text{co}(\cdot)$ to denote the convex hull of a set.

Equations (7) and (8), together with the strongly non-dictatorial property, imply that $\delta > \delta_i$ for any $i \in N$. Hence, $\delta > \max_i \delta_i$.

5 Individuals with General Discount Functions

Individual discount functions are often not exponential (see Strotz (1955), Laibson (1997), and Frederick et al. (2002)).[16] Allowing individuals to have general discount functions, as in (1), raises a challenge to our previous findings: When we say that the social discount factor should be higher than some individual's discount factor, it is not clear how individual discount factors should be defined. Our analysis below shows how the range of reasonable social discount factors depends on some asymptotic characteristic of general individual discount functions, and how our positive results in Section 4 can be generalized.

5.1 Aggregating Individual Discount Functions

Again, we begin with the case in which individual instantaneous utility functions are identical; that is, there is some continuous expected utility function $u : \Delta(X) \to \mathbb{R}$ such that each generation-t individual i's utility function is

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i(\tau - t)u(p_\tau).$$

Because we will need to vary T in part of the results below, we assume that individual discount functions are well defined for natural numbers. Starting from a set of individual discount functions δ_i's defined over natural numbers $\mathbb{N}$, whenever a finite T is chosen, we restrict the domain of δ_i's to $\{0, \ldots, T-1\}$. For instance, suppose individuals have quasi-hyperbolic discount functions. We first define $\delta_i(\tau) = \beta_i \delta_i^{\tau-1}$ for any $\tau > 0$. Then, we choose T and focus on $\delta_i(0), \ldots, \delta_i(T-1)$.

[16] In contrast, for normative reasons, we may prefer to require that the planner have an EDU function. Such a requirement also makes our positive results sharper.

For each individual discount function δ_i, we call $\sqrt[\tau]{\delta_i(\tau)}$ the *average discount function*, and $\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ the *relative discount function*. The average discount function measures the equivalent exponential discount factor for τ-period-ahead consumption. The relative discount function captures the instantaneous discounting for consumption that is $\tau+1$ periods ahead relative to consumption that is τ periods ahead.

We make two assumptions about the individual discount functions. The first assumption says that average discounting has a limit; that is,

$$\lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)} \text{ exists.} \tag{A1}$$

The second assumption says that the relative discount function is bounded; that is,

$$\text{there exists some } \alpha > 0 \text{ such that } \frac{\delta_i(\tau+1)}{\delta_i(\tau)} < \alpha \text{ for all } \tau \geq 0. \tag{A2}$$

The following theorem characterizes the set of social discount factors that are compatible with intergenerational Pareto under these assumptions.

Theorem 1 *Suppose each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function δ_i such that (A1) and (A2) hold. Let the planner's instantaneous utility function be u. Then,*

1. *for each $\delta > \min_i \max_{\tau\in\{0,\dots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *for each $\delta < \min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

The theorem shows how social discounting depends on individual discounting when individuals have heterogeneous general discount functions. We can find two cutoffs for the social discount factor. If the social discount factor is above the *least patient* individual's maximal relative discount factor, the planner's preference must be intergenerationally Pareto and strongly non-dictatorial. If the social discount factor is below the *least patient* individual's asymptotic average discount factor, the planner's preference must have violated the intergenerationally Pareto property *as long as T is large enough*.

The first part of the theorem confirms that positive results can still be established when individuals have arbitrary discount functions. Given a social discount factor, we can also apply this result to check whether intergenerational Pareto holds. The second part of the theorem says that if the social discount factor is too low, there must be two consumption sequences such that all individuals from all generations prefer one over the other, but the planner disagrees. A reasonable social discount factor should not allow this to happen.

Note that for any fixed T, $\max\limits_{\tau\in\{0,\dots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} \geq \sqrt[T-1]{\delta_i(T-1)}$, because

$$\sqrt[T-1]{\delta_i(T-1)} = \sqrt[T-1]{\frac{\delta_i(T-1)}{\delta_i(T-2)} \cdot \dots \cdot \frac{\delta_i(1)}{\delta_i(0)}}; \tag{9}$$

that is, $\sqrt[T-1]{\delta_i(T-1)}$ is the geometric mean of $\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$'s. Therefore, $\max\limits_{\tau\in\{0,\dots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ will not be lower than $\lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$ when T is large enough, and hence the first cutoff for the social discount factor will eventually be higher than the second cutoff.

Although the first cutoff may be strictly higher than the second, the two cutoffs in the theorem are "tight" in the following sense. For the first cutoff, there exist some individual discount functions δ_i's and T such that if the social discount factor $\delta \leq \min_i \max_{\tau\in\{0,\dots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, the planner cannot be both intergenerationally Pareto and strongly non-dictatorial. This happens, for example, when δ_i's are exponential. Similarly, for the second cutoff, we can find individual discount functions δ_i's such that for any finite T, if the social discount factor $\delta \geq \min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$, the planner is intergenerationally Pareto. This happens, for example, when δ_i's are quasi-hyperbolic (see Section S2 in the Supplemental Material).

To prove the first part of this theorem, we first focus on one arbitrary $i \in N$. The key step is to show that for each $t \in T$, we can find positive weights $(\tilde{\omega}(i,t,s))_{s=t}^{T}$ such that $\sum_{s=t}^{T} \tilde{\omega}(i,t,s) U_{i,s}$ is equal to an EDU function with any discount factor that is higher than

$\max_{\tau\in\{0,\dots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. Therefore, the planner can use utilitarian aggregation to transform each generation-t individual i's discounting utility function into an EDU function with a discount factor that is higher than $\max_{\tau\in\{0,\dots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. Then, we only need to let the planner use utilitarian weights to aggregate these EDU functions, as in Proposition 2.

The proof of the second part is similar to that of Proposition 2. Suppose individual 1's asymptotic average discount factor is the lowest strictly. When τ is large enough (and hence T must be large enough), we know that $\delta_i(\tau - s) \geq \delta_1(\tau - s)$. Hence, (5) becomes

$$\delta^{\tau-1} = \sum_{t=1}^{\tau}\sum_{i=1}^{N} \omega(i,t)\delta_i(\tau - t) \geq \delta_1(\tau - 1)\sum_{t=1}^{\tau}\sum_{i=1}^{N} \omega(i,t) \geq \delta_1(\tau - 1).$$

Therefore, $\delta \geq \lim_{\tau\to\infty} \sqrt[\tau]{\delta_1(\tau)}$ when τ is large enough.

5.2 Individual Long-Run Discounting

It turns out that for many widely used classes of individual discount functions, the two cutoffs in Theorem 1 merge into one. This is not a coincidence, and will help us identify an important characteristic of the individual discount function that determines the cutoff for the social discount factor. Let us introduce the following assumption:

$$\text{the relative discount function } \frac{\delta_i(\tau+1)}{\delta_i(\tau)} \text{ is nondecreasing in } \tau. \tag{A3}$$

In the literature of time inconsistency, when an individual has a nondecreasing relative discount function, the individual has (weak) *present bias*. A discount function δ_i is hyperbolic if for some $\alpha_i, \beta_i > 0$, $\delta_i(\tau) = (1+\alpha_i\tau)^{-\beta_i}$, and is quasi-hyperbolic if for some $\beta_i \in (0,1]$ and $\delta_i > 0$, $\delta_i(\tau) = \beta_i\delta_i^{\tau}$ for any $\tau > 0$. Exponential, hyperbolic, and quasi-hyperbolic discount functions all satisfy (A3).

Under (A2) and (A3), $\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ is nondecreasing and bounded. Hence, $\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ exists. Whenever $\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ exists, because the average discount factor is the geometric mean of relative discount factors (see equation (9)), the average discount factor also has a limit. Therefore, assumptions (A2) and (A3) imply (A1).

More importantly, when $\lim_{\tau\to\infty}\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ exists, the asymptotic relative discount factor and the asymptotic average discount factor coincide:

$$\lim_{\tau\to\infty}\frac{\delta_i(\tau+1)}{\delta_i(\tau)}=\lim_{\tau\to\infty}\sqrt[\tau]{\delta_i(\tau)}.$$

Definition 5 *When* $\lim_{\tau\to\infty}\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ *exists, we call* $\delta_i^*:=\lim_{\tau\to\infty}\frac{\delta_i(\tau+1)}{\delta_i(\tau)}=\lim_{\tau\to\infty}\sqrt[\tau]{\delta_i(\tau)}$ *individual i's long-run discount factor.*

The following corollary of Theorem 1 has only one cutoff for the social discount factor, and shows how social discounting is related to individual long-run discounting.

Corollary 1 *Suppose each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function* δ_i *such that (A2) and (A3) hold. Let the planner's instantaneous utility function be u. Then,*

1. *for each* $\delta>\min_i\delta_i^*$*, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *for each* $\delta<\min_i\delta_i^*$*, there exists some* $T^*>0$ *such that if* $T\geq T^*$*, the planner is not intergenerationally Pareto.*

If individuals have hyperbolic discount functions, the cutoff for the social discount factor is $\min_i\delta_i^*=1$; if individuals have quasi-hyperbolic discount functions with $\delta_i(\tau)=\beta_i\delta_i^\tau$, the cutoff is $\min_i\delta_i^*=\min_i\delta_i$.

In Section S2 in the Supplemental Material, we reinterpret the generation-$(t+s)$ individual i (with $s>0$) as a future self of the generation-t individual i, which offers a new interpretation of intergenerational Pareto and allows us to discuss how our findings are related to the time-inconsistency literature. In addition, we provide a stronger result similar to Corollary 1 for the case in which individuals have quasi-hyperbolic discount functions.

From here on, to simplify the statement of our results, we focus on the case in which long-run discount factors δ_i^*'s are well defined.

5.3 Social Discounting and Individual Instantaneous Utility Functions

Corollary 1 shows that if all individuals share the same instantaneous utility function, the social discount factor only has to be higher than the lowest individual long-run discount factor. As one may expect, when individual instantaneous utility functions are linearly independent, the cutoff for the social discount factor jumps from $\min_i \delta_i^*$ to $\max_i \delta_i^*$. Thus, generically, if social discounting is more patient than the *most* patient individual's long-run discounting, the planner is intergenerationally Pareto and strongly non-dictatorial; otherwise, if the time horizon is long enough, intergenerational Pareto is violated.

Theorem 2 *Suppose each generation-t individual i's discounting utility function has an instantaneous utility function u_i and a discount function δ_i such that (A2) and (A3) hold and $(u_i)_{i\in N}$ is linearly independent. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $(u_i)_{i\in N}$. Then,*

1. *for each $\delta > \max_i \delta_i^*$, the planner is intergenerationally Pareto and strongly non-dictatorial;*
2. *for each $\delta < \max_i \delta_i^*$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

Again, the cutoff is independent of the planner's risk attitude. Theorem 2 assumes (A2) and (A3). If we replace (A3) with (A1), as in Theorem 1, the only change in the statement of Theorem 2 will be that instead of one cutoff, we will have two cutoffs, as in Theorem 1.

The proof of this theorem is similar to Proposition 3, with some new elements taken from the proof of Theorem 1. Similar to Theorem 1, the second part of Theorem 2 requires that the time horizon be long enough.

5.4 Transition of the Cutoff

Let us further illustrate how the cutoff changes from the least patient individual's long-run discount factor to the most patient individual's. An individual's instantaneous utility function describes his risk attitude. Let Θ be some positive integer between 1 and N. Suppose there is a linearly independent Θ-tuple of instantaneous utility functions $(u^\theta)_{\theta=1}^{\Theta}$ representing Θ generic types of risk attitude. Assume that individual i's instantaneous utility function $u_i \in \{u^\theta\}_{\theta=1}^{\Theta}$, and there is no redundant type in $\{u^\theta\}_{\theta=1}^{\Theta}$; that is, for each type u^θ, at least one individual's instantaneous utility function is equal to u^θ. If $\Theta = 1$, we are in the case of Theorem 1. If $\Theta = N$, we are in the case of Theorem 2. Define $\delta_\theta^* := \min_{k \in \{i \in N : u_i = u^\theta\}} \delta_k^*$; that is, for each θ, let δ_θ^* be the least patient individual's long-run discount factor whose type is u^θ. Define

$$\delta_{\text{maxmin}}^* := \max_\theta \delta_\theta^*.$$

Theorem 3 *Suppose for some linearly independent Θ-tuple of instantaneous utility functions $(u^\theta)_{\theta=1}^{\Theta}$, each generation-$t$ individual i's discounting utility function has an instantaneous utility function $u_i \in \{u^\theta\}_{\theta=1}^{\Theta}$ and a discount function δ_i such that (A2) and (A3) hold and $\{u_i\}_{i \in N} = \{u^\theta\}_{\theta=1}^{\Theta}$. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $(u_i)_{i \in N}$. Then,*

1. *for each $\delta > \delta_{maxmin}^*$, the planner is intergenerationally Pareto and strongly non-dictatorial;*
2. *for each $\delta < \delta_{maxmin}^*$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

Note that as Θ increases from 1 to N, the cutoff may not increase monotonically. The idea of the proof of the theorem is as follows. For each type of risk attitude u^θ, we can apply Theorem 1 to show that the cutoff for the social discount factor implied by aggregating type-u^θ individuals is δ_θ^*. When aggregating across types, we apply Theorem 2 to show that the maximal δ_θ^* is the cutoff for the social discount factor.

6 Conclusion

The value of a policy or a public project that affects generations of individuals often crucially depends on which social discount rate is used for the evaluation. However, there is no consensus on which social discount rate is the right one to use. This paper considers a few important and widely used assumptions in economics, and characterizes the set of social discount rates that are compatible with those assumptions. The key assumptions are (i) individuals discount future consumption heterogeneously, (ii) the planner has an exponential discounting expected utility function, (iii) the planner is intergenerationally Pareto, which means that if all individuals from the current and future generations agree that one consumption sequence is better than another, the planner must agree, and (iv) the planner never completely ignores any individual's preference.

We show that for a generic set of individual instantaneous utility functions, social discounting should be more patient than the most patient individual's long-run discounting, as long as the time horizon is long enough, independent of the planner's instantaneous utility function. Therefore, using a near-zero social discount rate is justifiable in our framework.

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A Appendix

A.1 Proof of Proposition 1

Proof. If Part If there exists some $i \in N$ such that $U_t = U_{i,t}$ for any $t \in T$, the planner only takes individual i into account in period t. The corresponding weights in period t are $\omega_i = 1$ and $\omega_j = 0$ for all $j \neq i$. According to Lemma 1, the planner's preference $(\succsim_t)_{t \in T}$ is current-generation Pareto.

Only-If Part Suppose the planner is current-generation Pareto. We only prove the only-if part for the first period. According to Lemma 1, there exists an N-tuple of nonnegative weights $(\omega_i)_{i \in N}$, such that

$$\sum_{i=1}^{N} \omega_i \sum_{\tau=1}^{T} \delta_i^{\tau-1} u_i(p_\tau) = \sum_{\tau=1}^{T} \delta^{\tau-1} u(p_\tau);$$

that is, for $\tau = 1, \ldots, T-1$,

$$\sum_{i=1}^{N} \omega_i \delta_i^{\tau-1} u_i(p_\tau) = \delta^{\tau-1} u(p_\tau).$$

Let $\tau = 1$, 2, and 3. We have

$$\begin{cases} \sum_{i=1}^{N} \omega_i u_i(p) = u(p), \\ \sum_{i=1}^{N} \omega_i \delta_i u_i(p) = \delta u(p), \\ \sum_{i=1}^{N} \omega_i \delta_i^2 u_i(p) = \delta^2 u(p), \end{cases}$$

for any $p \in \Delta(X)$. Let $p = x^*$. The first equation shows that $\sum_{i \in N} \omega_i = 1$. Combining the second and the third equations above,

$$\left(\sum_{i=1}^{N} \omega_i \delta_i \right)^2 = \sum_{i=1}^{N} \omega_i \delta^2. \tag{10}$$

Since $\sum_{i \in N} \omega_i = 1$ and δ_i's are distinct, by Jensen's inequality, equation (10) holds if and only if there is some $i \in N$ such that $\omega_i = 1$ ($\omega_j = 0$ for any $j \neq i$). Thus, $U_1 = U_{i,1}$. Since the planner's instantaneous utility function and the social discount factor do not change over time, $U_t = U_{i,t}$ for any $t \in T$. ■

A.2 Proof of Proposition 2

Proof. The following lemma will be useful in proving Proposition 2.

Lemma 2 *Given a positive N-tuple $(\delta_i)_{i\in N}$, for any $t \in T$, there exists a finite sequence of positive numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that*

$$\sum_{i=1}^{N}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i^{\tau-s} = \delta^{\tau-t} \tag{11}$$

for any $\tau \geq t$ if and only if $\delta > \min_i \delta_i$.

Proof. If Part Without loss of generality, we assume that $\delta_1 = \min_i \delta_i$. First, we fix all the weights other than individual 1's. Let $\omega_t(i,s) = \epsilon_t(s) > 0$ for any $i \geq 2$, $t \geq 1$, and $s \geq t$. The remaining part is to find $(\omega_t(1,s))_{t\in T, s\geq t}$ such that

1. equation (11) holds;
2. $\omega_t(1,s) > 0$, for any $t \geq 1$ and $s \geq t$.

Construct $(\omega_t(1,s))_{t\in T, s\geq t}$ by the following recursive formula:

$$\omega_t(1,s) = \begin{cases} 1 - \sum_{i=2}^{N}\omega_t(i,s), & \text{if } s = t, \\ \delta^{s-t} - \sum_{i=1}^{N}\omega_t(i,t)\delta_i^{s-t} - \cdots - \sum_{i=1}^{N}\omega_t(i,s-1)\delta_i - \sum_{i=2}^{N}\omega_t(i,s), & \text{if } s > t. \end{cases} \tag{12}$$

It can be verified that (12) ensures that equation (11) holds for any $t \in T$ and $\tau \geq t$. The remaining part is to show that $(\omega_{1,t}(s))_{t\in T, s\geq t}$ derived from (12) are strictly greater than zero, if $(\epsilon_t(s))_{t\in T, s\geq t}$ are small enough. We prove it in two steps.

Step 1 Setting $\epsilon_t(s) = 0$, the recursive formula (12) becomes

$$\omega_t(1,s) = \begin{cases} 1, & \text{if } s = t, \\ \delta^{s-t-1}(\delta - \delta_1), & \text{if } s > t, \end{cases}$$

for each $t \in T$. This can be proved by induction. Since $\delta > \delta_1$, we have $\omega_t(1,s) > 0$.

Step 2 Plugging $\epsilon_t(s)$ into formula (12), we have,

$$\begin{cases} \omega_t(1,t) = 1 - (N-1)\epsilon_t(t), \\ \omega_t(1,t+1) = \delta - \delta_1 - \left[\sum_{i=2}^{N}(\delta_i - \delta_1)\right]\epsilon_t(t) - (N-1)\epsilon_t(t+1), \\ \omega_t(1,t+2) = \delta(\delta - \delta_1) - \left[\sum_{i=2}^{N}\delta_i(\delta_i - \delta_1)\right]\epsilon_t(t) - \left[\sum_{i=2}^{N}(\delta_i - \delta_1)\right]\epsilon_t(t+1) - (N-1)\epsilon_t(t+2), \\ \vdots \end{cases}$$

Then, we know that $\omega_t(1,s) = F_t^{(s)}(\epsilon_t(t), \ldots, \epsilon_t(s)|\delta, \delta_1, \ldots, \delta_n)$, in which $F_t^{(s)}$ is an affine (and hence continuous) function of $\epsilon_t(t), \ldots, \epsilon_t(s)$. Since $F_t^{(s)}$ is continuous, the weights $\omega_t(1,s)$'s are strictly greater than zero, if $\epsilon_t(s)$'s are small enough.

Only-If Part For any $t < T$, let $\tau = t, t+1$ in (11). We have

$$\begin{cases} \sum_{i=1}^{N}\omega_t(i,t) = 1, \\ \sum_{i=1}^{N}\omega_t(i,t)\delta_i + \sum_{i=1}^{N}\omega_t(i,t+1) = \delta. \end{cases}$$

Combining the above two equations,

$$\sum_{i=1}^{N}\omega_t(i,t)\delta = \sum_{i=1}^{N}\omega_t(i,t)\delta_i + \sum_{i=1}^{N}\omega_t(i,t+1).$$

Rearranging the above equation, we have

$$\delta = \frac{\sum_{i=1}^{N}\omega_t(i,t)\delta_i + \sum_{i=1}^{N}\omega_t(i,t+1)}{\sum_{i=1}^{N}\omega_t(i,t)} > \frac{\sum_{i=1}^{N}\omega_t(i,t)\delta_i}{\sum_{i=1}^{N}\omega_t(i,t)} > \frac{\sum_{i=1}^{N}\omega_t(i,t)\min_i\delta_i}{\sum_{i=1}^{N}\omega_t(i,t)} = \min_{i\in N}\delta_i.$$

■

Now we prove Proposition 2.

If Part Taking the weights from the if part of Lemma 2, since the planner's instantaneous utility function u is identical to individual instantaneous utility function u, we immediately know that the planner has the desired EDU function, because

$$\begin{aligned} U_t(\mathbf{p}) &= \sum_{s=t}^{T}\sum_{i=1}^{N}\omega_t(i,s)U_{i,s}(\mathbf{p}) = \sum_{s=t}^{T}\sum_{i=1}^{N}\omega_t(i,s)\sum_{\tau=s}^{T}\delta_i^{\tau-s}u(p_\tau) \\ &= \sum_{\tau=t}^{T}\sum_{s=t}^{\tau}\sum_{i=1}^{N}\omega_t(i,s)\delta_i^{\tau-s}u(p_\tau) = \sum_{\tau=t}^{T}\delta^{\tau-t}u(p_\tau). \end{aligned}$$

Only-If Part Suppose the planner's preference is intergenerationally Pareto and strongly non-dictatorial. For each $t \in T$, there exists a finite sequence of positive numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that

$$U_t(\mathbf{p}) = \sum_{s=t}^{T}\sum_{i=1}^{N}\omega_t(i,s)U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{T}\sum_{s=t}^{\tau}\sum_{i=1}^{N}\omega_t(i,s)\delta_i^{\tau-s}u(p_\tau).$$

Then, for any $t \in T$ and $\tau \geq t$, the following equality holds:

$$\sum_{s=t}^{\tau}\sum_{i=1}^{N}\omega_t(i,s)\delta_i^{\tau-s}u(p_\tau) = \delta^{\tau-t}u(p_\tau).$$

Let $p_\tau = x^*$. Lemma 2 implies that $\delta > \min_i \delta_i$. ■

A.3 One-Individual Intergenerational Aggregation

We prove the following lemma for intergenerational aggregation when each generation only has one individual.

Lemma 3 *Assume that $N = \{i\}$. Suppose each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function δ_i such that (A1) and (A2) hold. For any $\delta > \hat{\delta}_i := \max_{\tau\in\{0,\ldots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, the planner is intergenerationally Pareto and strongly non-dictatorial.*

Proof. We want to show that for any $\delta > \hat{\delta}_i$ and $t \in T$, there exists a finite sequence of positive numbers $(\omega_t(i,s))_{s \geq t}$ such that

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau) = \sum_{s=t}^{T} \omega_t(i,s) U_{i,s}(\mathbf{p}).$$

Given any $\delta > \hat{\delta}_i$, for each $t \in T$, we can construct $(\omega_t(i,s))_{s \geq t}$ according to the following formula:

$$\omega_t(i,s) = \begin{cases} 1, & \text{if } s = t, \\ \delta^{s-t-1}\left(\delta - \hat{\delta}_i\right) + \sum_{\tau=t}^{s-1} \left[\hat{\delta}_i \delta_i(s-1-\tau) - \delta_i(s-\tau)\right] \omega_t(i,\tau), & \text{if } s > t. \end{cases} \tag{13}$$

Note that by assuming $\delta > \hat{\delta}_i$, for $s > t$, the first term of $\omega_t(i,s)$ is strictly greater than 0. According to the definition of $\hat{\delta}_i$, the second term of $\omega_t(i,s)$ is greater than 0. Hence, $\omega_t(i,s) > 0$ for any $s \geq t$. Then,

$$U_t(\mathbf{p}) = \sum_{s=t}^{T} \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{s=t}^{T} \omega_t(i,s) \left[\sum_{\tau=s}^{T} \delta_i(\tau - s) u(p_\tau)\right] = \sum_{\tau=t}^{T} \left[\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s)\right] u(p_\tau).$$

We want to prove that $U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau)$ by induction. Consider $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s)$. When $\tau = t$, $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s) = \omega_t(i,t) = 1 = \delta^0$. Suppose for some $\tau \geq t$, we have proven that $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s) = \delta^{\tau-t}$. We want to prove that for $\tau + 1$,

$$\sum_{s=t}^{\tau+1} \delta_i(\tau + 1 - s)\omega_t(i,s) = \delta^{\tau-t+1}. \tag{14}$$

To prove (14), we only need to notice that according to (13),

$$\begin{aligned}
&\sum_{s=t}^{\tau+1} \delta_i(\tau+1-s)\omega_t(i,s) = \omega_t(i,\tau+1) + \sum_{s=t}^{\tau} \delta_i(\tau+1-s)\omega_t(i,s) \\
= \quad & \omega_t(i,\tau+1) + \hat{\delta}_i \left[\delta^{\tau-t} + \sum_{s=t}^{\tau} \frac{\delta_i(\tau+1-s)}{\hat{\delta}_i}\omega_t(i,s) - \delta^{\tau-t}\right] \\
= \quad & \omega_t(i,\tau+1) + \hat{\delta}_i \left[\delta^{\tau-t} + \sum_{s=t}^{\tau} \frac{\delta_i(\tau+1-s)}{\hat{\delta}_i}\omega_t(i,s) - \sum_{s=t}^{\tau} \delta_i(\tau-s)\omega_t(i,s)\right] \\
= \quad & \omega_t(i,\tau+1) + \hat{\delta}_i\delta^{\tau-t} + \sum_{s=t}^{\tau} \left[\delta_i(\tau+1-s) - \hat{\delta}_i\delta_i(\tau-s)\right]\omega_t(i,s) = \delta^{\tau-t+1}.
\end{aligned}$$

By induction, we know that $\sum_{s=t}^{\tau} \delta_i(\tau-s)\omega_t(i,s) = \delta^{\tau-t}$ for any $\tau \geq t$, and hence $U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u_i(p_\tau)$. ■

A.4 Proof of Proposition 3

Proof. If Part Lemma 3 states that the planner can find a sequence of positive numbers $(\tilde{\omega}_t(i,s))_{t\in T, i\in N, s\geq t}$ such that

$$\sum_{s=1}^{T} \tilde{\omega}_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{\tau=1}^{T} \delta^{\tau-1} u_i(p_\tau),$$

for any $i \in N$ and consumption sequence $\mathbf{p}$, because $\delta > \max_i \hat{\delta}_i = \max_i \delta_i$. Now, since the planner's instantaneous utility function $u = \sum_{i\in N} \lambda_i u_i$, we only need to let $\omega(i,t) = \lambda_i \tilde{\omega}(i,t)$. Then,

$$\sum_{i=1}^{N}\sum_{s=1}^{T} \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{i=1}^{N}\sum_{s=1}^{T} \lambda_i \tilde{\omega}_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{i=1}^{N} \lambda_i \left[\sum_{\tau=1}^{T} \delta^{\tau-1} u_i(p_\tau)\right] = \sum_{\tau=1}^{T} \delta^{\tau-1} u(p_\tau).$$

Only-If Part Note that when $(u_i)_{i\in N}$ is linearly independent and u is in the interior of $\text{co}(\{u_i\}_{i\in N})$, there is a unique way to write u as a strict convex combination of $(u_i)_{i\in N}$. Suppose $\sum_{i\in N} \lambda_i u_i = u$, in which $\lambda_i > 0$ and $\sum_{i\in N} \lambda_i = 1$. We only need to consider the period-1 planner. The planner's discounting utility function satisfies

$$U_1(\mathbf{p}) = \sum_{s=1}^{T}\sum_{i=1}^{N}\omega_1(i,s)U_{i,s}(\mathbf{p}) = \sum_{s=1}^{T}\sum_{i=1}^{N}\omega_1(i,s)\sum_{\tau=s}^{T}\delta_i^{\tau-s}u_i(p_\tau), \tag{15}$$

in which $\omega_1(i,s) > 0$ is the weight the period-1 planner assigns to the generation-s individual i. Since $\mathbf{p}$ is arbitrary, this implies that the planner's instantaneous utility function for period-1 consumption satisfies

$$u(p_1) = \sum_{i=1}^{N}\omega_1(i,1)u_i(p_1)$$

for any p_1. Because $u = \sum_{i\in N}\lambda_i u_i$ and $(u_i)_{i\in N}$ is linearly independent,

$$\omega_1(i,1) = \lambda_i \tag{16}$$

must hold for any $i \in N$. Similarly, equation (15) implies that for period-2 consumption,

$$\delta u(p_2) = \sum_{i=1}^{N}[\omega_1(i,1)\delta_i + \omega_1(i,2)]u_i(p_2)$$

for any p_2. Since instantaneous utility functions do not depend on time, the unique way to write u as a strict convex combination of $(u_i)_{i\in N}$ does not change; that is, $\delta u(p_2) = \delta\sum_{i\in N}\lambda_i u_i(p_2)$ for any p_2. Then, for any $i \in N$,

$$\lambda_i\delta = \omega_1(i,1)\delta_i + \omega_1(i,2). \tag{17}$$

Combining equations (16) and (17) gives us

$$\delta = \delta_i + \frac{\omega_1(i,2)}{\omega_1(i,1)} > \delta_i,$$

for any $i \in N$. The last strict inequality follows from the fact that $\omega_1(i,s) > 0$. Therefore,

$\delta > \max_i \delta_i$. ■

A.5 Proof of Theorem 1

Proof. Part I We prove the first part in three steps.

In the first step, the period-t planner does $T-t+1$ times aggregations for each individual i. The σ^{th} aggregation aggregates individual i from generation σ to generation T (σ starts from t and ends at T), and it assigns weight $\tilde{\omega}_{t,\sigma}(i,s)$ to generation-s individual i for $s \geq \sigma$.

By Lemma 3, we know that for each $t \in T$, $i \in N$ and $\sigma \geq t$, the planner can find a sequence of positive weights $(\tilde{\omega}_{t,\sigma}(i,s))_{s\geq\sigma}$ such that

$$\sum_{s=\sigma}^{T} \tilde{\omega}_{t,\sigma}(i,s) U_{i,s}(\mathbf{p}) = \sum_{\tau=\sigma}^{T} \delta^{\tau-\sigma} u(p_\tau),$$

for any $\delta > \hat{\hat{\delta}}_i$. The σ^{th} aggregation, as if, gives the planner an exponential discounting generation-σ individual i with a discount factor slightly higher than $\hat{\hat{\delta}}_i$.

In the second step, the period-t planner collects all of the weights assigned to generation-s individual i for all $T-t+1$ times step-one aggregations. Then the weight assigned to generation-s individual i by the period-t planner is

$$\tilde{\omega}_t(i,s) = \sum_{\sigma=s}^{T} \tilde{\omega}_{t,\sigma}(i,s).$$

Essentially, in each period t, the step-two aggregation under weights $(\tilde{\omega}_t(i,s))_{t\in T, s\geq t}^{i\in N}$ gives the planner N exponential discounting individuals from the t^{th} generation to the T^{th} generation, and each individual has a discount factor slightly higher than $\hat{\hat{\delta}}_i$.

Lastly, by Proposition 2, the planner can aggregate N exponential discounting individuals one more time, and obtain an EDU function with any social discount factor greater than $\min_i \hat{\hat{\delta}}_i$.

Part II Define $\tilde{\delta}_i := \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$. We assume that $\tilde{\delta}_1$ is the unique minimum of

$\tilde{\delta}_1, \ldots, \tilde{\delta}_N$. The proof can easily be extended to the case with multiple minima. We prove it by contradiction. Suppose the planner is intergenerationally Pareto. For each $t \in T$, there exists a finite sequence of nonnegative numbers $(\omega_t(i,s))_{i \in N, s \geq t}$ such that the following equality holds:

$$\sum_{s=t}^{\tau} \sum_{i=1}^{N} \omega_t(i,s)\delta_i(\tau - s)u(p_\tau) = \delta^{\tau-t}u(p_\tau) \tag{18}$$

for any $t \in T$ and $\tau \geq t$.

By letting $\tau = t$, equation (18) shows that $\sum_{i \in N} \omega_t(i,t) = 1$ for any $t \in T$. Then,

$$\delta^{\tau-t} = \frac{\sum_{s=t}^{\tau}\sum_{i=1}^{N} \omega_t(i,s)\delta_i(\tau - s)}{\sum_{i=1}^{N} \omega_t(i,t)} \geq \frac{\sum_{i=1}^{N} \omega_t(i,t)\delta_i(\tau - t)}{\sum_{i=1}^{N} \omega_t(i,t)}. \tag{19}$$

Since $\tilde{\delta}_1 = \min_i \tilde{\delta}_i$, there exists $T_1 > 0$ such that for each $\tau > T_1$, $\delta_1(\tau - t) = \min_i \delta_i(\tau - t)$. Hence, (19) becomes

$$\delta^{\tau-t} \geq \frac{\sum_{i=1}^{N} \omega_{i,t}(t)\delta_1(\tau - t)}{\sum_{i=1}^{N} \omega_{i,t}(t)} = \delta_1(\tau - t). \tag{20}$$

According to our assumptions, $\delta < \tilde{\delta}_1$. Then, there exists $T_2 > 0$ such that for each $\tau > T_2$,

$$\delta^{\tau-t} < \delta_1(\tau - t). \tag{21}$$

Let $T^* = \max\{T_1, T_2\}$. Then, (20) and (21) contradict each other. ■

A.6 Proof of Theorem 2

Proof. Part I We prove this theorem in two steps. First, we again consider the special case in which there is only one individual i to be aggregated across generations. Since the individual relative discount factor is nondecreasing, $\delta_i^* \geq \hat{\delta}_i := \max_{\tau \in \{0,\ldots,T-2\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. By Lemma 3, because the social discount factor $\delta > \max_i \delta_i^* \geq \delta_i^*$, for any $i \in N$ and $t \in T$, we can find some positive $(\omega_t(i,s))_{s\geq t}$ such that

$$\sum_{s=t}^{T}\omega_t(i,s)U_{i,s}(\mathbf{p})=\sum_{\tau=t}^{T}\delta^{\tau-t}u_i(p_\tau);$$

that is, we can aggregate each individual's utility functions across generations into an EDU function with discount factor δ.

Consider any N-tuple of positive numbers $(\lambda_i)_{i\in N}$ such that $\sum_{i\in N}\lambda_i=1$. Together with the weights $(\omega_t(i,s))_{t\in T,i\in N,s\geq t}$ we have found above, let the planner's utility function satisfy

$$\begin{aligned}U_t(\mathbf{p})&=\sum_{i=1}^{N}\sum_{s=t}^{T}\lambda_i\omega_t(i,s)U_{i,s}(\mathbf{p})=\sum_{i=1}^{N}\sum_{\tau=t}^{T}\delta^{\tau-t}\lambda_i u_i(p_\tau)\\&=\sum_{\tau=t}^{T}\delta^{\tau-t}\sum_{i=1}^{N}\lambda_i u_i(p_\tau)=\sum_{\tau=t}^{T}\delta^{\tau-t}u(p_\tau),\end{aligned}$$

in which $u=\sum_{i\in N}\lambda_i u_i$ is an arbitrary strict convex combination of $(u_i)_{i\in N}$.

Part II We prove it by contradiction. Suppose there exists an intergenerationally Pareto planner with the social discount factor $\delta<\max_i\delta_i^*$. By intergenerational Pareto, for any $t\in T$, there exists nonnegative numbers $(\omega_t(i,s))_{i\in N,s\geq t}$ such that the following equality holds for any $t\in T$:

$$\sum_{\tau=t}^{T}\delta^{\tau-t}u(p_\tau)=\sum_{s=t}^{T}\sum_{i=1}^{N}\omega_t(i,s)\sum_{\tau=s}^{T}\delta_i(\tau-s)u_i(p_\tau)=\sum_{\tau=t}^{T}\sum_{i=1}^{N}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i(\tau-s)u_i(p_\tau).$$

Since $\mathbf{p}$ is arbitrary, the equation above implies that for any $t\in T$ and $\tau\geq t$,

$$\sum_{i=1}^{N}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i(\tau-s)u_i(p_\tau)=\delta^{\tau-t}u(p_\tau). \tag{22}$$

Recall that u is a strict convex combination of $(u_i)_{i\in N}$ and $(u_i)_{i\in N}$ is linearly independent. There is a unique way to write u as a convex combination of $(u_i)_{i\in N}$. Moreover, when $\tau=t$, equation (22) becomes

$$\sum_{i=1}^{N} \omega_t(i,t)u_i(p_t) = u(p_t) \tag{23}$$

for any $t \in T$. Thus, $\omega_t(i,t) > 0$ for any $i \in N$ and $t \in T$. Combining equations (22) and (23), we have

$$\delta^{\tau-t} \sum_{i=1}^{N} \omega_t(i,t)u_i(p_\tau) = \sum_{i=1}^{N}\sum_{s=t}^{\tau} \omega_t(i,s)\delta_i(\tau-s)u_i(p_\tau).$$

Since $(u_i)_{i\in N}$ is linearly independent, for any $i \in N$, $t \in T$, and $\tau \geq t$, the above equation implies

$$\omega_t(i,t)\delta^{\tau-t} = \sum_{s=t}^{\tau} \omega_t(i,s)\delta_i(\tau-s),$$

which in turn implies

$$\begin{aligned} \delta^{\tau-t} &= \frac{\sum_{s=t}^{\tau} \omega_t(i,s)\delta_i(\tau-s)}{\omega_t(i,t)} = \frac{\omega_t(i,t)\delta_i(\tau-t) + \sum_{s=t+1}^{\tau} \omega_t(i,s)\delta_i(\tau-s)}{\omega_t(i,t)} \\ &\geq \frac{\omega_t(i,t)\delta_i(\tau-t)}{\omega_t(i,t)} = \delta_i(\tau-t); \end{aligned}$$

that is,

$$\delta \geq \sqrt[\hat{\tau}]{\delta_i(\hat{\tau})} \tag{24}$$

for any $1 \leq \hat{\tau} < T$.

Without loss of generality, we assume δ_N^* is a maximum of $\{\delta_i^*\}_{i\in N}$. Since $\delta < \delta_N^* = \lim_{\tau\to\infty} \sqrt[\tau]{\delta_N(\tau)}$, there exists T^* such that for any $T \geq T^*$, $\delta < \sqrt[T-1]{\delta_N(T-1)}$, which contradicts (24). ■

A.7 Proof of Theorem 3

Proof. Part I We prove Part I in two steps. First, we aggregate individuals who share the same u^θ. For each $\theta \in \Theta$, $I^\theta := \{i \in N : u_i = u^\theta\}$ is called a "family," which is the set of i's whose instantaneous utility functions are u^θ. By Corollary 1, we know that for each θ and each $\delta > \min_{i\in I_\theta} \delta_i^*$, there exists a sequence of weights $(\omega_t(i,s))_{t\in T, i\in I_\theta, s\geq t}$ such that

$$U_t^\theta(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u^\theta(p_\tau) = \sum_{s=t}^{T} \sum_{i\in I_\theta} \omega_t(i,s) U_{i,s}(\mathbf{p}).$$

for each $t \in T$. Now, we have $|\Theta|$ exponential discounting expected utility functions U_t^θ's with linearly independent instantaneous utility functions u^θ's.

Next, we apply Proposition 3 to aggregate U_t^θ's. It follows immediately that if $\delta > \max_{\theta\in\Theta} \min_{i\in I_\theta} \delta_i^*$, the planner is intergenerationally Pareto and strongly non-dictatorial.

Part II We prove its contrapositive. Suppose the planner is intergenerationally Pareto. Then, for each $t \in T$, there exists a finite sequence of positive numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau) = \sum_{s=t}^{T} \sum_{\theta\in\Theta} \sum_{i\in I_\theta} \omega_t(i,s) \sum_{\tau=s}^{T} \delta_i(\tau-s) u_i(p_\tau),$$

and hence

$$\delta^{\tau-t} u(p_\tau) = \sum_{\theta\in\Theta} \sum_{i\in I_\theta} \sum_{s=t}^{\tau} \omega_t(i,s) \delta_i(\tau-s) u^\theta(p_\tau) \tag{25}$$

for any $t \in T$ and $\tau \geq t$.

By letting $\tau = t$ in equation (25), we have

$$u(p_t) = \sum_{\theta\in\Theta} \sum_{i\in I_\theta} \omega_t(i,t) u^\theta(p_t). \tag{26}$$

Recall that u is a strict convex combination of $(u_i)_{i\in N}$. Equation (26) shows that $\sum_{i\in I_\theta} \omega_t(i,t) > 0$ for each θ. Combining equations (25) and (26), we have

$$\sum_{\theta\in\Theta} \sum_{i\in I_\theta} \delta^{\tau-t} \omega_t(i,t) u^\theta(p_\tau) = \sum_{\theta\in\Theta} \sum_{i\in I_\theta} \sum_{s=t}^{\tau} \omega_t(i,s) \delta_i(\tau-s) u^\theta(p_\tau).$$

Since $(u^\theta)_{i=1}^{\Theta}$ is linearly independent, the above equation is equivalent to

$$\sum_{i\in I_\theta} \delta^{\tau-t}\omega_t(i,t) = \sum_{i\in I_\theta}\sum_{s=t}^{\tau} \omega_t(i,s)\delta_i(\tau-s)$$

for any $\theta \in \Theta$. Rearranging the above equation, we obtain

$$\delta^{\tau-t} = \frac{\sum_{i\in I_\theta}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i(\tau-s)}{\sum_{i\in I_\theta}\omega_t(i,t)} > \frac{\sum_{i\in I_\theta}\omega_t(i,t)\delta_i(\tau-t)}{\sum_{i\in I_\theta}\omega_t(i,t)}. \tag{27}$$

Letting τ go to infinity, it is easy to see that (27) becomes $\delta \geq \min_{i\in I_\theta} \delta_i^*$ for $\forall\theta \in \Theta$. Hence, $\delta \geq \max_{\theta\in\Theta}\min_{i\in I_\theta}\delta_i^* = \delta^*_{\text{maxmin}}$. ■

A.8 Infinite Time Horizon

Our findings can be extended to the case with $T = +\infty$. When $T = +\infty$, we require that individual discount factors $(\delta_i(\tau))_{\tau=0}^{\infty}$ be an absolutely summable sequence (in ℓ^1) and $\max_i \delta_i^* < 1$, and that the social discount factor $\delta < 1$. The result below will show that even when individual instantaneous utility functions are identical, the cutoff for the social discount factor will jump from $\min_i \delta_i^*$ to $\max_i \delta_i^*$ when T becomes infinite. We first define intergenerational utilitarianism.

Definition 6 *The planner is intergenerationally utilitarian if in each period $t \in T$, there exists a sequence of nonnegative numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that $0 < \sum_{i=1}^{N}\sum_{s=t}^{T}\omega_t(i,s) < \infty$, and*

$$U_t = \sum_{i=1}^{N}\sum_{s=t}^{T}\omega_t(i,s)U_{i,s}.$$

Below, we will assume intergenerational utilitarianism rather than intergenerational Pareto, because the equivalence between intergenerational utilitarianism and intergenerational Pareto for countably infinitely many individuals is not yet established.

Proposition 4 *Suppose $T = +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u_i and a discount function δ_i such that (A2)* *and (A3) hold. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $(u_i)_{i\in N}$. Then,*

1. *for each $\max_i \delta_i^* < \delta < 1$, the planner is intergenerationally utilitarian and strongly non-dictatorial;*

2. *for each $\delta < \max_i \delta_i^*$, the planner is not simultaneously intergenerationally utilitarian and strongly non-dictatorial.*

Proof. Part I Since u is a strict convex combination of $(u_i)_{i\in N}$, suppose $u = \sum_i \lambda_i u_i$ for some $\lambda_1, \dots, \lambda_N > 0$ such that $\sum_i \lambda_i = 1$. For each $i \in N$ and each $t \in T$, we want to construct a sequence of positive and absolutely summable numbers $(\omega_t(i,s))_{s=t}^{\infty}$ such that

$$\sum_{s=t}^{\infty} \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} u_i(p_\tau).$$

If this can be done, then in period t, let $\lambda_i \omega_t(i,s)$ be the planner's utilitarian weight for the generation-s individual i, in which case

$$\sum_{i=1}^{N} \sum_{s=t}^{\infty} \lambda_i \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(p_\tau) = U_t(\mathbf{p}),$$

which means that the planner is intergenerationally utilitarian and strongly non-dictatorial.

Next, we show that the following recursive definition of $(\omega_t(i,s))_{s=t}^{\infty}$ works: For each $s \geq t$,

$$\omega_t(i,s) = \begin{cases} 1, & \text{if } s = t, \\ \sum_{\sigma=t}^{s-1} [\delta \cdot \delta_i(s-\sigma) - \delta_i(s-\sigma+1)] \omega_t(i,\sigma), & \text{if } s > t. \end{cases} \tag{28}$$

First, it can be verified that each $\omega_t(i,s)$ is positive, because $\delta > \max_i \delta_i^*$ and the individual relative discount factor is nondecreasing. Second, it can be verified inductively that for any finite τ,

$$\sum_{s=t}^{\tau} \sum_{i=1}^{N} \omega_t(i,s) \delta_i(\tau - s) u_i(p_\tau) = \delta^{\tau-t} u(p_\tau)$$

for any $p_\tau \in \Delta(X)$. These two steps are similar to the steps in the proof of Lemma 3. Thus, we only have to show that $(\omega_t(i,s))_{s=t}^{\infty}$ is summable. Clearly, $\sum_{s=t}^{n} \omega_t(i,s)$ is nondecreasing in n. If we can show that $\sum_{s=t}^{n} \omega_t(i,s)$ is bounded above and the bound is a constant, this part of the theorem is proven.

Sum up both sides of equation (28) from $s = t$ to n. We obtain that

$$1 = \sum_{s=t}^{n-1} \left((1-\delta) \sum_{\tau=0}^{n-1-s} \delta_i(\tau) + \delta_i(n-s) \right) \omega_t(i,s) + \omega_t(i,n).$$

Because $\sum_{\tau=0}^{n-1-s} \delta_i(\tau) > 1$ and $\delta_i(n-s) > 0$, $(1-\delta) \sum_{\tau=0}^{n-1-s} \delta_i(\tau) + \delta_i(n-s) > 1 - \delta$, which implies that

$$1 > \sum_{s=t}^{n-1} (1-\delta)\omega_t(i,s) + \omega_t(i,n) > (1-\delta) \sum_{s=t}^{n-1} \omega_t(i,s).$$

Therefore, $\sum_{s=t}^{n-1} \omega_t(i,s)$ is bounded above by $1/(1-\delta)$ for any n.

Part II Assume that δ_N^* is the unique maximum of $\{\delta_i^*\}_{i \in N}$. The proof can easily be extended to the case with multiple maxima. We prove the contrapositive of this part. Suppose the planner is intergenerationally utilitarian and strongly non-dictatorial; that is, for each $t \in T$, there exists a sequence of positive numbers $(\omega_t(i,s))_{i \in N, s \geq t}$ such that $U_t = \sum_{i,s} \omega_t(i,s) U_{i,s}$. Hence, for any $t \in T$ and $\tau \geq t$,

$$\sum_{s=t}^{\tau} \sum_{i=1}^{N} \omega_t(i,s) \delta_i(\tau - s) u_i(p_\tau) = \delta^{\tau - t} u(p_\tau).$$

Consider a consumption sequence that yields x^* in every period, $(x^*, x^*, \dots)$. Then, the equation above becomes

$$\sum_{s=t}^{\tau} \sum_{i=1}^{N} \omega_t(i,s) \delta_i(\tau - s) = \delta^{\tau - t}.$$

Since u_i's and u are normalized, we know that for each t, $\sum_{i \in N} \omega_t(i,t) = 1$. Due to the strongly non-dictatorial property, in particular, $\omega_t(N,t) \in (0,1)$. Then,

$$\delta^{\tau-t} = \sum_{s=t}^{\tau} \sum_{i=1}^{N} \omega_t(i,s)\delta_i(\tau - s) > \omega_t(N,t)\delta_N(\tau - t).$$

Therefore, $\delta > \sqrt[\tau-t]{\omega_t(N,t)\delta_N(\tau - t)}$ for every τ implies that $\delta \geq \delta_N^*$. ■

Proposition 4 covers the case in which u_i's are identical. Thus, Proposition 4 says that if $T = +\infty$, the cutoff for the social discount factor again jumps from $\min_i \delta_i^*$ to $\max_i \delta_i^*$, compared to Theorem 1/Corollary 1.

Note that the second part of Proposition 4 is weaker than the second part of Theorem 1, Corollary 1, or Theorem 2. In Proposition 4, if the social discount factor is lower than the highest individual long-run discount factor, the conclusion is that either intergenerational utilitarianism is violated or the planner has ignored some individual from some generation.

Nonetheless, there is still discontinuity between Proposition 4 and Theorem 1/Corollary 1. In Theorem 1/Corollary 1, if the social discount factor is lower than the lowest individual long-run discount factor, we know that intergenerational Pareto is violated, which implies that at least one of the two properties, intergenerational utilitarianism or the strongly non-dictatorial property, is violated, as in Proposition 4.

The intuition for this discontinuity is the following. For simplicity, suppose u_i's are the same. Fixing an arbitrarily large but finite T, the planner can always attach small enough utilitarian weights to individuals with high δ_i^*. In this way, the planner can keep her social discount factor low. However, if T is infinite, fixing any positive weights, as τ increases to infinity, $\delta_i(\tau)$ of the individual with the highest δ_i^* dominates all other individuals' discount factors regardless of his weight. Therefore, the social discount factor cannot be strictly less than $\max_i \delta_i^*$.