Dimensionality in Physics*

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Abstract

This paper is the first of a two-part series which re-interprets relativistic length contraction and time dilation in terms of concepts argued to be more fundamental, broadly construed to mean: concepts which point to the next paradigm. In this paper, Lorentz contraction is re-interpreted in terms of the concept of *dimensional abatement*, and three overarching arguments are given that the latter is more fundamental: Dimensional abatement (*1*) focuses attention on two fundamental spacetime principles the significance of which is unappreciated under the current paradigm, *(2)* permits an understanding of speed of light invariance in terms of dimensionally reduced objects and coordinate frames, and (*3*) leads to the formulation of a principle, called *dimensional superposition* which permits a deeper and more unified conceptual understanding of Maxwell's equations in terms of two metaprinciples.

Keywords: Dimensional abatement, invariance of absolute dimensionality, homodimensionality, heterodimensionality, dimensional superposition, leaf vector, metaprinciple, merge operator

1 Introduction: What is fundamental?

For those who contemplate fundamental aspects of nature, one of the first considerations is to sort out what is fundamental from what is not. Here it is important to keep in mind that fundamentality as a criterion can be applied to different kinds of things, which in each case may well yield different parameters. For instance, when the consideration of fundamentality is applied to physical objects, we tend to regard those as fundamental which, as a set, are building blocks of all others. Indeed, the standard model of particle physics is organized according to this principle, listing an array of particles (or, more accurately, field quanta), including those which model forces, as fundamental.
Already when we consider fundamentality with respect to *quantities*, it becomes less clear what is fundamental and what isn't. It is certainly true that distance, time and mass are commonly and intuitively regarded as the most fundamental physical quantities. This is especially so because it is intuitive to us that, analogously to elementary particles, distance, time and mass can also be used as "building blocks" for constructing all other physical quantities. But ambiguities remain.
Imagine, for example, that there was an intelligent alien culture which had a much stronger intuition for the fundamentality of the concepts of momentum, force and energy; so strong that it considered *those* quantities to be fundamental. Within the worldview of that culture, distance, time and mass would be considered derived i.e. *non-fundamental* quantities.

$$[d] = \frac{[E]}{[F]} \quad [t] = \frac{[p]}{[F]} \quad [m] = \frac{[p^2]}{[E]} \tag{1}$$

Figure 1: Distance, time and mass in "alien fundamental units"

This way of conceptualizing fundamental quantities may not seem intuitive to us, but it is mathematically equivalent to our intuitive notions because the systems of units are expressible in terms

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of each other. Thus, the difference in fundamentality between these two sets of quantities is not an intrinsic one, but is determined by the context which singles out one or the other as fundamental. What I just called the "context" is really the same as what Kuhn, in his *Structure of Scientific Revolutions*, called a *paradigm* [1]: the entire network of linked and generally coherent concepts and ideas which constitute the worldview within which any given theory is embedded.

The mathematical equivalence of the fundamentality of the two sets of quantities suggests that, in general, fundamentality cannot be determined in an absolute sense, i.e. independent of the paradigm within which it is considered. And this gives us a clue for identifying the most fundamental elements of a theory within any given paradigm: it has to be those things which point to, or at least hint at, the *next* paradigm.

The goal of this two-part series of papers is to examine and discuss a concrete example which exemplifies this idea: two physics concepts which are under our current paradigm regarded as fairly fundamental, namely relativistic length contraction and time dilation, will be re-interpreted in terms of other concepts argued to be more fundamental because they lead to novel insights which may cause us to view familiar subjects in a profoundly new way.

2 Lorentz Contraction as Dimensional Abatement

The Lorentz coordinate transformations are the mathematical heart of Einstein's special theory of relativity. In a standard set of coordinate systems, they can be written as [2]

$$\begin{aligned} t'_B - t'_A &= \gamma\left((t_B - t_A) - \frac{\beta}{c}(x_B - x_A)\right) \\ x'_B - x'_A &= \gamma\left((x_B - x_A) - \beta c\,(t_B - t_A)\right) \\ y'_B - y'_A &= y_B - y_A \\ z'_B - z'_A &= z_B - z_A \end{aligned} \tag{2}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is, of course, the Lorentz factor, $\beta = \frac{v}{c}$, v is the relative speed between the primed and unprimed coordinate frames, c is the speed of light, and the primed coordinates belong to a coordinate frame which is in inertial motion relative to the unprimed frame along the positive x-axis. In fact, these equations were already around nearly two decades before special relativity. Within the prevailing paradigm prior to 1905, the transformations were thought to represent the influence of the luminiferous aether, an all-pervading highly rigid yet inviscid transparent and massless substance, on bodies moving relative to some absolute rest frame. In particular, consider a body which has length $L' = x'_B - x'_A$ in the primed frame in which it is at rest, and which therefore in the unprimed frame moves along the positive x-axis at speed v. If we undertake in the unprimed frame an instantaneous measurement of its length $L = x_B - x_A$, we will by the second line of equation (2) find that

$$\frac{L'}{\gamma} = L \tag{3}$$

since an instantaneous measurement in the unprimed frame implies that $t_B - t_A = 0$. As $\gamma > 1$ for a moving body, we have $L < L'$. Within the aether paradigm, Lorentz contraction was conceptualized as a 'squeezing' of a body by the aether as as it moves relative to it.

In 1905, Special Relativity replaced the aether paradigm by focusing on the invariance of the form of the laws of physics in different inertial frames and the independence of the speed of light from the speed of its source, which was soon recast in terms of the invariance of the speed of light[1]. Within the paradigm of special relativity, the transformations are a direct consequence of these invariances: If one wants the laws of physics to 'look the same' in all inertial frames (arguably an intuitive requirement) and the speed of light to be invariant (not at all intuitive, but dictated by empirical observations), then physical bodies *must* transform according to these equations[2].

Over a century later, this is still where we currently stand. I would like to now propose a different conceptualization of the Lorentz contraction which I will attempt to demonstrate in the following sections to be more fundamental. Before I do so, I need to define the relevant terminology.

Definition 1. *Absolute Dimensionality:* The absolute dimensionality of an object is the number of independent length dimensions which characterize it.

[1] See [3] for an interesting discussion of the evolution of the conception of c invariance

[2] neglecting gravity, of course

Example 1. The absolute dimensionality of a 3-dimensional body is 3.

Note that when the context makes it clear, the "absolute" qualifier may be omitted, as in standard usage.

Definition 2. *Volume-Boundary ratio:* The Volume-Boundary ratio of a compact (i.e. closed and bounded) object with absolute dimensionality $n > 1$ is the ratio of its n-dimensional volume to its $n-1$-dimensional boundary.

Example 2. The Volume-Boundary ratio of a cube with sides L is the ratio of its volume to surface area (V/S): $\frac{V}{S} = \frac{1\times1\times1}{6\times(1\times1)}L = \frac{1}{6}\ L$.

Definition 3. *Relative Dimensionality:* Relative Dimensionality is the ratio of the Volume-Boundary ratio of a compact object with absolute dimensionality $n > 1$ to that of a compact reference object, also with absolute dimensionality n.

Example 3. The relative dimensionality of a hemisphere to a sphere, both of radius r, is $\frac{2/3\times\pi r^3}{\pi r^2+1/2\times4\pi r^2} \div \frac{4/3\times\pi r^3}{4\pi r^2} = \frac{2}{3}$.

Note that absolute dimensionality and relative dimensionality are dimensionless numbers whereas the Volume-boundary ratio has units of length. Relative Dimensionality parameterizes the dimensional character of an $n-$dimensional object relative to an n-dimensional reference object: If the relative dimensionality lies in the open interval $(1,\infty)$, then the object has a stronger n-dimensional character than the reference object, and if it lies in the open interval $(0,1)$ it has a weaker n-dimensional (or, equivalently, stronger $n-1$-dimensional) character relative to that of the reference object.

Definition 4. *Dimensional Diminution:* For an $n-$dimensional compact object, dimensional diminution is the decrease of its *relative* dimensionality compared to its original state to a number in the open interval $(0,1)$.

Example 4. A cube of sides L is dimensionally diminished by a factor of $3/4$ when its sides along one of its main directions are contracted by half: $\frac{V}{S} = \frac{(1)\times(1)\times(1/2)}{2\times(1\times1)+4\times(1\times1/2)}L = \frac{1}{8}L = \frac{3}{4}\times\frac{1}{6}L$ i.e. the relative dimensionality of the contracted cube to the original cube is $\frac{3}{4}$. The number quantifies by how much the contracted cube has a weaker $3-$dimensional character compared to the reference object, the uncontracted cube. Its inverse quantifies by how much the contracted cube has a stronger $2-$dimensional character than the original cube.

Definition 5. *Dimensional Reduction:* For an $n-$dimensional object (n>1), dimensional reduction is the decrease of its absolute dimensionality to $n-1$. Equivalently, it is the decrease of its relative dimensionality compared to its original state to 0.

Example 5. A cube of sides L is dimensionally reduced to a *square* of side L when its sides along one of its main directions are contracted to zero. We have $\frac{V}{S} = \frac{(1)\times(1)\times(0)}{2\times(1\times1)+4\times(1\times0)}L = 0$.

Definition 6. *Dimensional Abatement:* Dimensional Abatement is a less specific umbrella term which can either refer to Dimensional Diminution or to Dimensional Reduction.

I can now formally state my proposition:

Proposition. Lorentz contraction, conceptualized in a more fundamental way, signifies *dimensional abatement.* More specifically, it signifies *dimensional diminution for* $0 < v < c$ and *dimensional reduction for* $v = c$.

Let us first perform a sanity check: Is it always the case that a Lorentz-contracted compact body is dimensionally diminished for $0 < v < c$? A little thought should convince us that the answer is yes: since the extent of a compact body perpendicular to its direction of motion remains unchanged, and the only change along the direction of motion is always a decrease, its volume, as well as its surface area, must decrease. If we imagine a compact body of arbitrary shape as being made up of cubical elements oriented along the coordinate axes and sufficiently small that the difference between this imagined approximation and the real shape does not matter, we can deduce the relative dimensionality of the contracted body by considering the relative dimensionality of its contracted constituents. Since every cubical element of the body becomes dimensionally diminished under the contraction of its sides along one of its main directions (see example 4), so will the body as a whole. In the case of $v = c$, $\gamma = \infty$ and hence (3) reduces to $L = 0$. Thus, I find that dimensional abatement is consistent with Lorentz contraction. But is it more *fundamental*? In the next four sections I will offer a number of arguments in support of an affirmative answer.

3 Two Fundamental Spacetime Principles

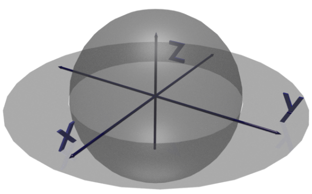
A major innovation relativity introduced into physics was an appreciation for the importance of invariance and symmetry principles. Recall that special relativity was introduced in a formulation based on the invariance of the form of the laws of physics in different inertial frames. Furthermore, we recognize that the four-dimensional interval constituted out of the spatial and temporal interval is more fundamental than they are because, unlike them, it is invariant. In short, invariance imbues a quantity or principle with fundamentality.
Conceptualizing Lorentz contraction as dimensional abatement focuses attention on dimensionality, which in turn leads to the recognition of a fundamental invariance principle which has been implicitly used in physics all along but so far, as far as I can tell, not really appreciated:

Principle 1. *The absolute dimensionality of any compact body is invariant under spacetime coordinate transformations.*

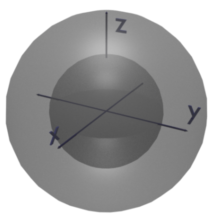
For example, the principle says that if a body is 3-dimensional in one spacetime observer frame, it will be 3-dimensional in all such frames. Surely, it makes sense that the dimensionality of a body should not depend on the coordinate system within which it is considered (as long as the dimensionality of the *coordinate system* does not change), and this may even be alleged to be so obvious as to be trivial. Yet, I have never seen an explicit statement of this principle.
The global property of Minkowski spacetime which ensures that the invariance of absolute dimensionality of a compact body holds everywhere in it is determined by a symmetry principle which I will call the *homodimensionality of space*:

Principle 2. *The dimensionality of every space-like hypersurface of Minkowski spacetime is everywhere the same.*

For example, the principle says there is no spacelike hypersurface of spacetime which in any region becomes, say, 4-dimensional. I will call a space which fails to be homodimensional a *heterodimensional space*. *As a matter of mathematics*, we can, of course, join spacelike hypersurfaces of different dimensionality together to create a heterodimensional space, but *as a matter of physics*, the homodimensionality of space seems to be a fact of nature. Indeed, it seems so obvious that we take it for granted, as seen by the fact that we take this to be part of the definition of "spacelike hypersurface". Explicitly recognizing that in modeling reality we have invoked this principle may permit us to see relationships which are not obvious under the current paradigm.
This global symmetry is arguably more fundamental than the other two known symmetries of space, homogeneity and isotropy, in that *a space can be characterized by it without being characterized by them, but not vice versa*: homogeneity of space asserts that the laws of physics are the same at every point in space, and isotropy asserts that they are the same in every direction. Since a change in the dimensionality of space in some region changes at least some laws of physics there (such as the inverse square law), it identifies special regions and directions in space. This is easy to see for homogeneity but less obvious for isotropy. Isotropy is always violated in a heterodimensional space because one can distinguish the set of directions in a lower-dimensional region from those in its complement in higher-dimensional regions (see figs. (2a) and (2b)).



(a)



(b)

Figure 2: Two compact heterodimensional spaces symmetric around the center: (a) is a space with an inner 3-D and outer 2-D region, and (b) represents a space with an inner 2-D (notice the truncated $z-$axis) and outer $3-D$ region. In both, isotropy is violated because directions in the xy-plane are distinguishable.

4 The Dimensionality of Speed-of-Light Objects

Consider an object characterized by $v = c$ in some spacetime frame. Just assuming the length contraction formula in (3) *without* assuming the full Lorentz transformations, and the invariance of absolute dimensionality is enough to deduce the invariance of the speed of light!

To see this, consider that since $v = c$ for an object in that frame yields $L = 0$ by equation (3), the object will in that frame be observed to be dimensionally reduced in the direction of motion. But by the invariance of absolute dimensionality, it must be observed to be dimensionally reduced in *every* spacetime frame in the direction of motion. Hence, $L = 0$ in every spacetime frame, which means $\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} = 0$ in every such frame, which means $v = c$ in every such frame.

Since there is, in fact, no spacetime frame or coordinate system in which speed-of-light objects are not observed to be dimensionally reduced, this result deserves a stronger interpretation: it is not only the case that bodies associated with $v = c$ are *observed to be* dimensionally reduced, but that they *intrinsically are* dimensionally reduced. There are other 'hints' in special relativity that speed-of-light objects are dimensionally reduced:

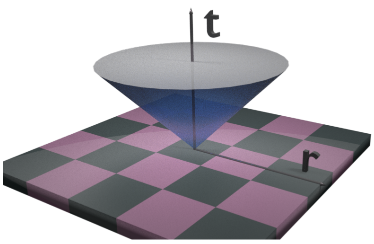
- **Null Four-vectors have only three independent components**. By this, I just mean that in component form, a null four-vector can be rewritten as $(\sqrt{(a^1)^2 + (a^2)^2 + (a^3)^2}, a^1, a^2, a^3)$, eliminating an independent term for the timelike component entirely.

- **In the hypothetical frame of a speed of-light-object, space is compressed into a plane.** This follows from the symmetry of motion, applied to everything in space.

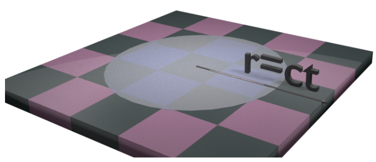
- **In the hypothetical frame of a speed-of-light object, spacetime becomes linearly dependent.** In such frames, both the timelike direction and the direction of motion become lightlike. This can be interpreted as spacetime having *redundant dimensionality* in such a hypothetical frame.

The latter two 'hints' may be regarded with skepticism because assigning a frame to a speed-of-light object does not make sense under the current paradigm. Indeed, it is impossible for spacetime observers to assign coordinate frames to speed-of-light objects, but in order for this to mean that such objects intrinsically cannot be associated with coordinate frames, it must be the case that the only valid coordinate systems for physical objects are *4-dimensional*. Is this true?

As $v \to c$, a moving frame undergoes a 'distortion' that in the limit $v = c$ corresponds precisely to the dimensional reduction of spacetime itself when the linear dependence of the radial and time directions in that limit is interpreted as a kind of *merging* of the two dimensions into a single one (see the appendix for a mathematical implementation of this in terms of a *merge operator*). Indeed, if we wish to impose a basis on an *individual upper light-cone* containing no massive objects, it is not $4-$dimensional (fig. 3a) but 3-dimensional (fig 3b)!



(a)



(b)

Figure 3: Lightcones are 4-D structures (a), but in the absence of massive objects, a lightcone can be represented just in 3-D space (b), as vectors in the r and t directions are linearly dependent for spherical surfaces expanding at the speed of light, and timelike curves are absent here. Note that in this situation, it is not possible to transform to a frame moving at $v < c$ because doing so presupposes the existence of at least one timelike curve. There is no coordinate frame which falls within the domain of applicability of the Lorentz transformations when all that is left of Minkowski spacetime are spacelike hypersurfaces.

A fictitious observer characterized by $v = c$ would therefore not 'experience' our $3+1$ dimensional spacetime, but a space which has one dimension fewer. Consequently, such an observer could not use the Lorentz transformations because they apply to $3 + 1$ dimensions. This opens up the possibility that the reason spacetime observers cannot assign coordinate frames to speed-of-light objects is not because such objects cannot be intrinsically associated with coordinate frames, but because *dimensionally reduced objects in a dimensionally reduced space are outside the domain of applicability of the Lorentz transformations in* $3+1$ *spacetime*. Mathematically, the inapplicability of the Lorentz transformations manifests itself through $\gamma = \infty$ for $\beta = 1$. If we think of this merely as a limitation imposed on spacetime observers to describe dimensionally reduced objects, then it leaves the possibility open that hypothetical dimensionally reduced observers could describe *themselves* in terms of dimensionally reduced coordinate transformations, even if these are inaccessible to spacetime observers. If we take this possibility seriously, then it opens up a way to understand speed-of-light invariance as a *consequence* of dimensional reduction, by the following chain of reasoning:

1. Assume that there are entities which are dimensionally reduced and intrinsically associable with dimensionally reduced coordinate frames (but inaccessible to spacetime observers because they lie outside the domain of the Lorentz coordinate transformations)[3].

2. In a dimensionally reduced frame, the vector quantities which spacetime observers associate with r and t point in the same direction.

3. The direction in which they point is the lightlike or null direction in spacetime, which imposes the additional constraint that the rates of change of r and t are proportional to each other. This follows from the definition of the null interval, $0 = c^2dt^2 - dr^2$, where the angular terms were omitted because lightcone boundaries in Minkowski spacetime change uniformly along the radial direction.

4. This proportionality implies that their relative rate of change (i.e. $\frac{dr}{dt}$) is *a constant*.

5. By the homogeneity of space, all dimensionally reduced inertial frames are equivalent, making that constant *the same constant* for every 3-dimensional spacelike hypersurface.

6. Hence, that constant is *invariant* in Minkowski spacetime

That the invariance of the speed of light can be intimately linked to dimensional reduction seems to have gone unappreciated because, as best as I can tell, the tending of $L \to 0$ and that of the r and t directions toward the lightlike direction as $v \to c$ is under the current paradigm regarded as a feature of special relativity with practically no deeper significance. The closest that current conceptualizations of, say, photons seem to come to this link is via the recognition of their reduced physical degrees of freedom relative to those of massive particles.

Since photons are characterized by wavelengths, not lengths, it may be questioned whether applying the length contraction concept directly to photons is really legitimate. However, under the re-interpretation of total length contraction as dimensional reduction, the application no longer refers to the length of a photon but to its absolute dimensionality. The dimensional abatement concept is more general than length contraction because the former can be applied where the latter cannot. The next section will show this concretely.

5 The Dimensional Superposition Principle

This section will introduce an application of the dimensional abatement concept outside of special relativity to discover a novel principle. Let us indulge our imagination for a moment and suppose that we impose a foliation on space (i.e. we subdivide 3-dimensional space into mutually disjoint 2-dimensional surfaces called *leaves*), and suppose that on each leaf of the foliation (which, for simplicity, we take to be in the shape of a Euclidean plane), we define what I will call a *plane central force field*: It is essentially a 2-dimensional analog of a central force field in space, with the positive force vectors pointing outward in all directions within the leaf from a central point a in the leaf, and negative force vectors pointing inward, but no force vector pointing in a direction

[3]This assumption will be supported by further arguments in the second part of this series.

which has a nonzero component normal to the leaf (see fig (4a))

(a) (b)

Figure 4: (a) represents a plane central force field imposed on each leaf of a foliation of space. (b) represents an orientation of a vector $\mathbf{v}$ relative to a leaf such that it points to the center a of the central force field at a normal direction. The problem is to express $\mathbf{F}$ pointing towards the center a in terms of $\mathbf{v}$.

Consider now a particular leaf $\mathfrak{L}$ and a point in space not on that leaf, and associate the latter with a vector which I will suggestively label as $\mathbf{v}$. At the moment, no meaning is assigned to $\mathbf{v}$ other than that it is a quantity with magnitude and direction and that it is free (i.e. its initial point is not fixed or localized in the coordinate system). Suppose $\mathbf{v}$ is oriented relative to the foliation such that a ray along $\mathbf{v}$ intersects $\mathfrak{L}$ at a such that it is normal to $\mathfrak{L}$. That, of course, implies that $\mathbf{v}$ is normal to $\mathfrak{L}$ and points to a when its initial point is at the location depicted in fig. (4b). Is there a way to relate $\mathbf{v}$ to the central plane force $\mathbf{F}$ at some particular point on $\mathfrak{L}$?
It turns out that this is not difficult to do (see fig. (5)):

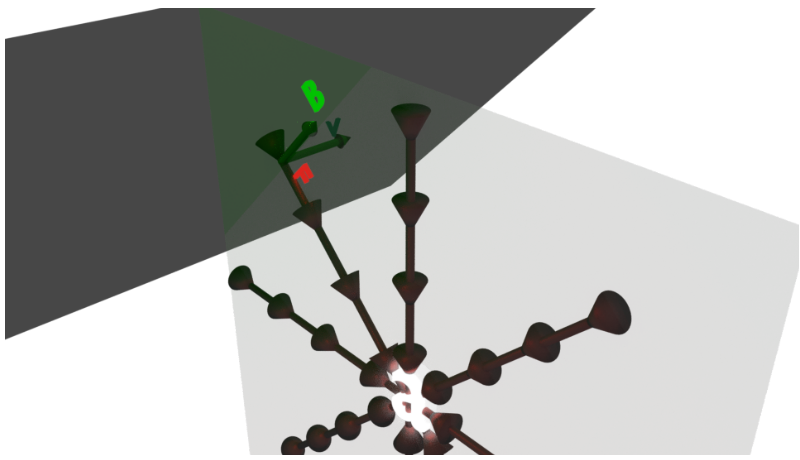


Figure 5: This figure shows how to express $\mathbf{F}$ pointing towards the center a in terms of $\mathbf{v}$: We intersect the leaf in which $\mathbf{F}$ lies with a plane to which $\mathbf{F}$ is normal (darkly shaded) at its initial location in order to identify a vector $\mathbf{B}$ *in the leaf and co-initial with but perpendicular to* $\mathbf{F}$ ($\mathbf{v}$, being a free vector, is moved to also be co-inital with $\mathbf{F}$, and because it is normal to the leaf, it lies in the intersecting plane at that location). This only permits two possible choices for $\mathbf{B}$, and we choose the one which yields a right-handed system for $\mathbf{v}$, $\mathbf{B}$, $\mathbf{F}$ and thus makes $\mathbf{v} \times \mathbf{B}$ proportional to $\mathbf{F}$.

First, we need to define a vector quantity co-initial with and perpendicular to $\mathbf{F}$, but which lies inside $\mathfrak{L}$. I will suggestively label this vector $\mathbf{B}$, but not assign any meaning to it other than that it is a quantity with magnitude and direction which lies in $\mathfrak{L}$. We can identify the direction of $\mathbf{B}$ by intersecting our leaf with the plane which contains the initial point of $\mathbf{F}$ and is normal to it (darkly shaded in fig. (5)). Since $\mathfrak{L}$ is a two-dimensional space, there are two possible directions perpendicular to $\mathbf{F}$ along which $\mathbf{B}$ could point: One in which the system of directions of $\mathbf{v}$, $\mathbf{B}$, $\mathbf{F}$ is right-handed and one in which it is left-handed (notice that since $\mathbf{v}$ is normal to the leaf and

$\mathbf{B}$ is perpendicular to $\mathbf{F}$ while lying in the leaf, all three are orthogonal to each other). I will by convention choose the first.

Given this definition of $\mathbf{B}$, it only takes a little thought to realize that $\mathbf{F}$ must be proportional to the cross-product $\mathbf{v} \times \mathbf{B}$. The reason is that the vector $\mathbf{v} \times \mathbf{B}$ is by definition normal to the plane spanned by $\mathbf{v}$ and $\mathbf{B}$, which defines a family of directions parallel to $\mathbf{F}$ (since $\mathbf{F}$ is perpendicular both to the $\mathbf{v}$ and the $\mathbf{B}$ directions), and because $\mathbf{v}$ and $\mathbf{B}$ intersect in $\mathfrak{L}$, $\mathbf{v} \times \mathbf{B}$ must also lie in $\mathfrak{L}$. The only freedom left is the magnitude of $\mathbf{v} \times \mathbf{B}$, but this can be easily fixed by defining a scalar proportionality constant q, such that we have:

$$\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B}) \tag{4}$$

Note that while the left side of this equation was assigned the meaning of a force, none of the terms on the right were assigned a physical meaning. The only assumptions I made were that

1. $\mathbf{F}$ is a plane central force with origin a in a planar leaf $\mathfrak{L}$ of a foliation imposed on space.
2. $\mathbf{v}$ is a free vector in space.
3. The foliation is imposed such that the direction of $\mathbf{v}$ is normal to $\mathfrak{L}$, pointing to a in $\mathfrak{L}$ in the initial configuration.

The rest, and in particular $\mathbf{B}$, was a mathematical construction to express $\mathbf{F}$ in terms of $\mathbf{v}$. Is the fact that equation (4) is formally identical to the magnetic Lorentz force a coincidence?

We can check this by calculating the speed at which a Coulomb field is transformed (counterfactually, as it turns out) into a pure magnetic field. The calculation is given in the appendix and suggests that the speed necessary to change the observation of a purely electric field into a purely magnetic field is *the speed of light*[4]. But in the previous section we saw that $v = c$ implies dimensional reduction! Thus, it is not a coincidence that we obtained equation (4) by subdividing space into 2-dimensional leaves: the field on each individual leaf is a dimensionally reduced analog of the electric force field, and integrating it over the line which connects the centers of all plane central force fields yields an infinitely extended three-dimensional object. Hence,

The physical structures represented by magnetic force fields are line integrals of dimensionally reduced analogs of the physical structures represented by electric force fields.

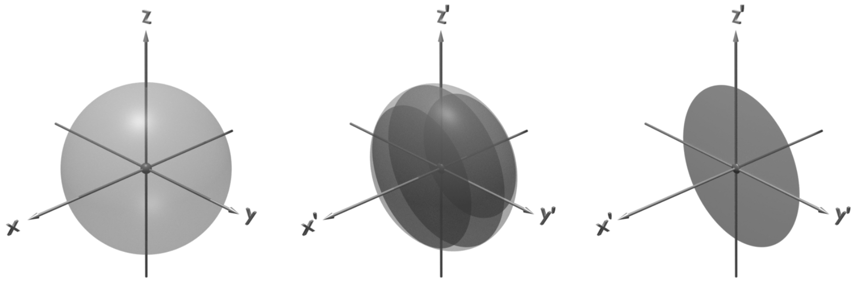
The definition of relative dimensionality given in section 2 applied to compact objects, but a field, being defined over all of space, is not compact. Thus, dimensional diminution in a field, when the field source is in motion relative to the observer, has to be implemented by a completely different mechanism than that for compact bodies. This mechanism deserves to be elevated to a principle, which I will call *dimensional superposition*:

Principle 3. *An infinitely extended field is dimensionally diminished by putting it at every point in space in a superposition with the line integral of its dimensionally reduced analog such that the relative contribution of each is in proportion to the dimensional diminution of the field*[5].

Thus, for $v = 0$, where v is the motion the point-like field source in a frame, the field of the electric force, which I will label by $\mathbf{F_E}$, is dimensionally undiminished (fig. (6a)) and therefore the contribution of the field of the magnetic force, labeled by $\mathbf{F_B}$, to the superposition is zero. In a frame moving at $0 < v < c$ and momentarily coinciding with the rest frame in the standard configuration, the $\mathbf{F_E}$-field becomes dimensionally diminished through the appearance of $\mathbf{F_B}$ (fig. (6b)), in addition to changing its spherical symmetry to a flattened ellipsoidal symmetry. The greater the motion of the field source in a particular frame, the more dimensionally diminished the electric force field it sets up, which means the stronger the contribution of the magnetic force field to the superposition. For a fictitious charge moving at $v = c$, the electric force field becomes dimensionally *reduced* to its 2-dimensional analog (fig. (6c)).

[4]While the calculation in the appendix does show that imposing the condition of transforming a purely electric field into a purely magnetic field yields c (equation (34)), it also reveals pathological features of the transformed field which reflect the impossibility of such a transformation. I attempted to circumvent these pathologies heuristically via what I call a *merge operator* to implement the action of the Lorentz transformation at the limit $v = c$. The resulting fields are the basis for the discussion that follows, but because the merge operator models an unphysical process, to wit: acceleration to the speed of light, the result of that calculation is not indicative but counterfactual.

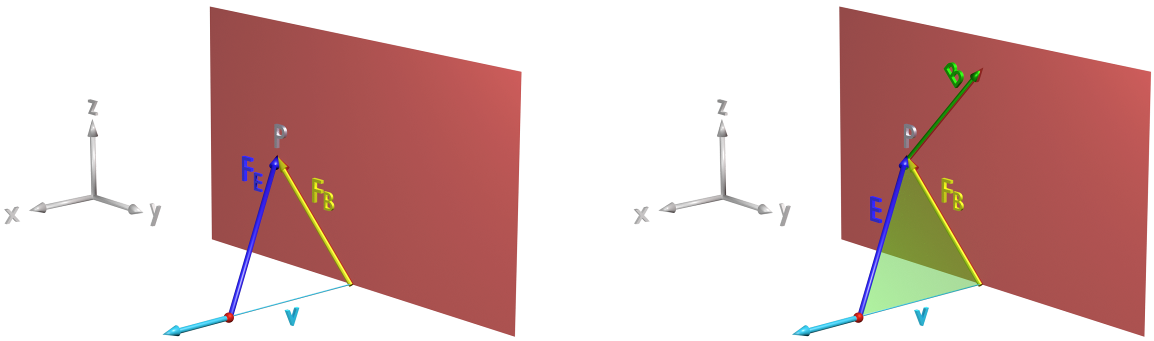
[5]Note that dimensional superposition applies to the electromagnetic *force* fields, rather than the $\mathbf{E}$ and $\mathbf{B}$ fields.



(a) (b) (c)

Figure 6: A graphic representation of *dimensional superposition*: The force field vectors are abstracted out, leaving only their symmetries (spherical, ellipsoidal+cylindroidal and circular, respectively), in order to focus on the appearance of the magnetic field as an indicator of dimensional diminution. An infinitely extended Coulomb *force* field (a), shown here only for a finite spherical region with the field source at the center is dimensionally diminished by putting it at every point in space in superposition with the line integral (incompletely represented in (b) by a few disks centered around a line which represents the direction of motion of the point charge) of its dimensionally reduced analog (c) (see equations (43) and (44)). The field in (a) is taken to be observed in the rest frame of the source while the field in (b) is observed in a moving frame the origin of which momentarily coincides with that of the rest frame in the standard configuration. Note that in addition to the appearance of the magnetic field, the electric force field in (b) is also dimensionally diminished in that its symmetry has changed from that of a sphere to that of a flattened ellipsoid. The magnetic force field in (c) of a fictitious charge traveling at $v = c$ is all that remains as the electric force field disappears to spacetime observers. It is important to remember that the dimensional superposition principle expresses a relationship between the symmetries of the *force fields* $\mathbf{F_E}$ and $\mathbf{F_B}$, not of the $\mathbf{E}$ and $\mathbf{B}$ fields.

To see how dimensional superposition can be applied, I will show how to use it to quickly express, to within a proportionality factor, the magnetic field of a point-like charge moving at constant velocity in terms of its velocity and electric field. First, consider that for a point-like charge moving at a constant velocity $\mathbf{v}$, the leaf in which the magnetic force vector $\mathbf{F_B}$ lies is uniquely fixed by the direction of the particle's velocity $\mathbf{v}$ and the point in space $\mathbf{P}$ at which the magnetic field is to be determined. After the leaf has been determined, find $\mathbf{F_B}$ by projecting $\mathbf{F_E}$ onto the leaf, taking care to match the orientation of $\mathbf{F_B}$ to the direction of $\mathbf{v}$ relative to the leaf and the charge sign (fig (7a)). Then, using the same method as in fig. (5), determine the magnetic field $\mathbf{B}$ at $\mathbf{P}$ in terms of $\mathbf{v}$ and $\mathbf{F_B}$ (fig. (7b)).



(a) (b)

Figure 7: (a) shows that the magnetic force vector of a moving point-like charge can be thought of as a projection of the electric force vector onto the leaf normal to its direction of motion and containing the field point $\mathbf{P}$. For a positive charge, $\mathbf{v}$ must point out of the leaf, otherwise the direction of the projection (i.e. of $\mathbf{F_B}$) is flipped. (b) shows that once $\mathbf{B}$ is determined through $\mathbf{v}$ and $\mathbf{F_B}$, it turns out to be be normal to the plane containing $\mathbf{v}$ and $\mathbf{E}$ (notice the replacement of $\mathbf{F_E}$ in (a) by $\mathbf{E}$ in (b).)

Noticing that since $\mathbf{B}$ is perpendicular to $\mathbf{F_B}$ and to $\mathbf{v}$, and $\mathbf{E}$ lies in the same plane as the latter two, that $\mathbf{B}$ is normal to the plane containing $\mathbf{v}$ and $\mathbf{E}$, we can immediately determine that

$$\mathbf{B} \propto (\mathbf{v} \times \mathbf{E}) \tag{5}$$

In units in which $1 = c = \frac{1}{c^2} = \epsilon_0\mu_0$, the proportionality becomes an equality [4].
There is no closed loop in the leaf containing the point $\mathbf{P}$ which encloses a current at the point at which a ray along $\mathbf{v}$ pierces the leaf. Yet, the magnetic force field at $\mathbf{P}$ is nonetheless non-zero, which means that dimensional superposition implies the presence of something 'like' a current but not an actual current crossing the leaf there, which we might as well designate by the symbol $\mathbf{J}_d$, a 'virtual' current per unit area of the leaf. At this time, it is not obvious to me whether the exact form of $\mathbf{J}_d$ can be deduced from dimensional superposition alone.
Considering the opposite extreme of an infinite straight line charge, we find that since the symmetry of the situation causes the electric field to, in effect, 'lose' a dimension, the electric force and the magnetic force of any current in it both point radially, becoming more alike.
These examples illustrate a general point: The concept of dimensional superposition, itself arising out of the relativistic concept of dimensional abatement, permits us to conceive of magnetic fields *directly* as a relativistic effect. Usually, when magnetic fields are explained as a relativistic effect, the explanation requires switching to moving frames and back (e.g. [5],[2],[4]), but with dimensional superposition, magnetic fields can be thought of as a relativistic effect in a single frame due to the dimensional diminution of the electric force field of moving sources.
Finally, consider that if we think of magnetic field vectors as being defined in 3-space, then their transformation properties as axial vectors under space inversion seem counterintuitive. But if we think of them as being defined in the 2-dimensional leaf of a foliation, and always transforming together with their leaf, these properties become at once natural and intuitive: A *leaf vector* is inverted either when the coordinate directions in the leaf are inverted or the direction perpendicular to the leaf is. But if both inversions are carried out, it inverts twice and is thereby restored to its original state. A space inversion amounts to both inversions at once and hence leaves a leaf vector unchanged, as shown in figs. (8a) and (8b). This supplies a novel conceptual understanding of magnetic field vectors as axial or pseudovectors.

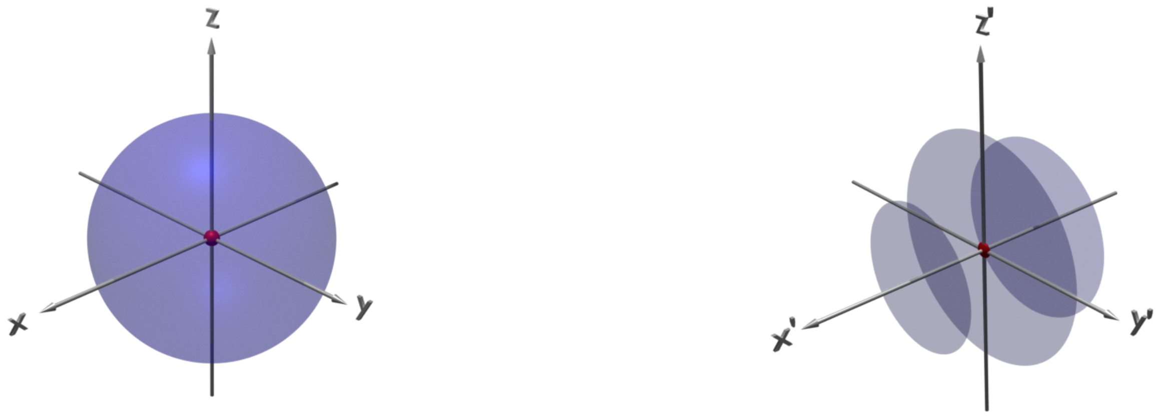
(a) (b)

Figure 8: In (a), two vectors are represented, both lying in the xy-plane: $\mathbf{a} = (a_1, a_2)$ is a leaf vector defined in the 2-dimensional leaf at $z_{\text{leaf}} = 0$, and $\mathbf{b} = (b_1, b_2, b_3) = (b_1, b_2, 0)$ is defined in 3-dimensional space. An inversion of all 3 components of $\mathbf{b}$ to $(-b_1, -b_2, -b_3)$ flips its direction, a shown in (b). On the other hand, $\mathbf{a}$ is unaffected by a sign flip of its components if this is combined with a flip of the orientation of the leaf with respect to the z-coordinate axis. The latter in effect inverts the xy-coordinate system in the leaf back to its original parity. Combining the two 'intrinsic' components of $\mathbf{a}$ with $z = z_{\text{leaf}}$ permits the definition of a vector-like object $(a_1, a_2, z_{\text{leaf}})$ that has identical transformation properties to those of an axial vector. In an arbitrary coordinate system, the normal direction of the leaf of $\mathbf{B}$ is always along the direction of $\mathbf{v}$ and its orientation along that direction is determined by the sign of the moving charges.

If magnetic field vectors are axial because they are defined in a leaf, then why aren't magnetic *force* vectors similarly axial? Because the magnetic force field, unlike the magnetic field, is a *physical structure* in space. The foliation of space into leaves on which to define a plane central force field, *along with the magnetic field itself*, is a convenient *mathematical construction* to analyze this inherently 3-dimensional physical structure. If it were possible to accelerate an electrical charge to the speed of light, then its force field would be defined exclusively on a plane. Since this is impossible, *there are no plane central force fields as such in 3-D space*. Nevertheless, it is highly useful to think of the magnetic force field as being composed of constituent central force fields on separate leaves because this allows us to easily construct the magnetic field, and thereby gives deeper insight into the laws of electromagnetism, as will be seen in the next section.

6 Two Metaprinciples Upholding Four Laws

This section will use dimensional superposition to show that Maxwell's equations are not independent of each other, and can be thought of as special cases of two more general principles. Let us begin by considering Gauss's law and, in the venerable tradition of unphysical Gedankenexperiments which have served to illustrate important physical insights, let us imagine a situation in which it *fails*. Fig. (9a) represents a situation in which a point charge is enclosed by a Gaussian surface, yet fails to produce any flux across the surface, so that the integral of the electric force field $\mathbf{F_E}$, and therefore also of $\mathbf{E}$, over the closed surface is zero[6]. If we now consider the same events in a momentarily coinciding moving frame, then dimensional superposition would superpose any field with a structure that is the line integral of its one-dimension reduced analog, depicted in fig (9b) (without including the transformed electric force field).



(a) (b)

Figure 9: A thought experiment is depicted in (a): We suppose that the shaded region contains a charge, yet that the surface integral of the electric force field over a Gaussian surface enclosing the region is zero. By dimensional superposition, this implies that for an observer observing the same situation in a moving frame, the flux of magnetic force across the closed line integral in each leaf enclosing either the moving charge or the surface integral of $\mathbf{J}_d$ will also be zero. Since $\mathbf{B}$ is perpendicular to the magnetic force, this means that $\oint \mathbf{B} \cdot d\mathbf{s} = 0$ in each leaf even though it encloses either the moving charge or a current due to $\mathbf{J}_d$. This also holds in converse: By dimensional superposition, a current which fails to produce a non-zero flux of magnetic force across an enclosing closed loop in the leaves of a foliation will in the rest frame of the sources manifest itself as a set of charges which fail to produce a non-zero flux of electric force, and therefore $\mathbf{E}$, across an enclosing Gaussian surface. Thus, dimensional superposition implies that the Gauss and Ampère-Maxwell laws are not independent: if one fails, then so does the other.

The failure of Gauss's law would now manifest itself as a failure of the 'current' (either the moving point charge or the surface integral of $\mathbf{J}_d$ in this case) to produce a magnetic *force* flux across an enclosing closed loop in any of the leaves of the foliation. Since the magnetic field lies in the leaf but is perpendicular to the magnetic force, this implies that the closed line integral around the current $\oint \mathbf{B} \cdot d\mathbf{s}$ will be zero. But that means precisely that Ampère-Maxwell's law has failed!
It can be easily seen that, given dimensional superposition, the converse is also true: If we ever discovered an instance in which Ampère-Maxwell's law failed, then in the frame in which the current becomes a static charge distribution, by dimensional superposition Gauss's law would also fail as in this case, the failure to produce a flux of magnetic force across a closed loop in the leaf transforms to a failure to produce a flux of electric force across a Gaussian surface in space. In

[6] We assume that it was somehow ascertained that there really was an unscreened charge inside, for instance by showing that a smaller Gaussian surface inside did produce a non-zero integral. This setup is of course unphysical, but the argument given here does not depend on this unphysical aspect of the thought experiment.

short, expressed in differential form, if we are given dimensional superposition, then

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \nabla \times \mathbf{B}' = \mu_0(\mathbf{J}' + \mathbf{J}'_d) \tag{6}$$

where the primes on $\mathbf{B}$, $\mathbf{J}$ and $\mathbf{J}_d$ are meant to indicate a switch from the unprimed rest frame to a moving frame observing the same system and $\mathbf{J}'_d$, again, is the term which accounts for magnetic fields deduced from dimensional superposition which do not enclose a real current in a closed loop. We arrived at (6) intuitively, using the dimensional superposition principle. Let us check it mathematically. Consider the rest-frame current density 4-vector $J^\mu = (\rho, 0, 0, 0)$ and boost it in the standard configuration (i.e. in the positive x direction):

$$\Lambda^\nu_\mu J^\mu = J^\nu = (\gamma\rho, -\gamma\beta\rho, 0, 0) = (\gamma\rho, -\gamma J_{x'}, 0, 0) \tag{7}$$

In the last step, I used $\mathbf{v}\rho = \mathbf{J}$ for the $-x'$-direction ($c = 1$). This says that a charge density at rest, observed in a frame moving in the positive x-direction will be observed as a current density in the negative x'-direction. Because $J_{y'} = J_{z'} = 0$, we have $J_{x'} = J$ and I can consider the negative x'-direction as the direction of the current density 3-vector in the primed frame:

$$-\gamma J \hat{\mathbf{x}}' \equiv \mathbf{J}' \tag{8}$$

To have a relationship of the same form as in (6), I need to compare the timelike component of J^μ with the spacelike components of J^ν, which simplifies to a comparison with the first spacelike component of J^ν because I used the standard configuration. In the comparison, I set them equal to Gauss's law and the Ampère-Maxwell law, respectively, so that in three-dimensional notation

$$\rho = \epsilon_0(\nabla \cdot \mathbf{E}) \Longleftrightarrow \mathbf{J}' = \frac{1}{\mu_0}(\nabla \times \mathbf{B}' - \mu_0\epsilon_0 \frac{\partial}{\partial t}\mathbf{E}') \tag{9}$$

Here, the mutual implication between the two sides is warranted by the Lorentz transformation together with Maxwell's equations: in a frame at constant motion relative to a frame in which a static charge density obeying Gauss's law is observed, that charge density will manifest itself as a current density obeying the Ampére-Maxwell law (in addition to a transformed charge density). Conversely, if in a frame a current density obeying the Ampeŕe-Maxwell law is observed, then there exists a frame at rest relative to the source charges in which the current density will manifest itself as a charge density obeying Gauss's law. It is immediately obvious that if we set

$$\mathbf{J}'_d = \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}' \tag{10}$$

and rearrange terms, we recover (6). This confirms that Gauss's law and Ampère's law for electromagnetostatics are not two separate independent laws but two very different-looking manifestations of one and the same underlying idea, linked by dimensional superposition. The fundamental idea can be stated in terms of a *metaprinciple* i.e. a principle which generalizes a set of fundamental principles (or laws) of physics to remove dependence on the dimensionality of the situation in its formulation:

Metaprinciple 1. *A flux-producing object*[7] *producing flux in radial directions only can be identified by its flux through a fully enclosing surface.*

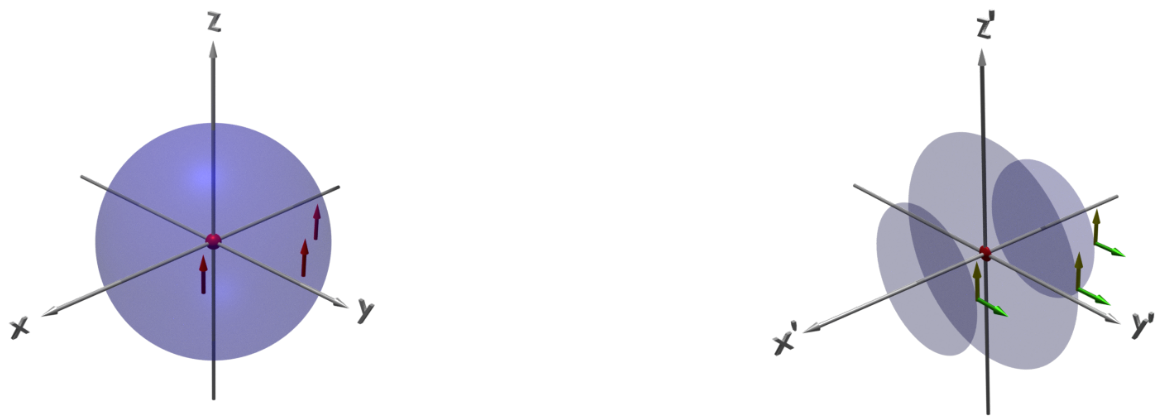
Gauss's law is an instance of this metaprinciple in three spatial dimensions, applied to electric charge such that flux refers to that of the electric *force*, and therefore also the electric field. Ampère-Maxwell's law is an instance of this metaprinciple in two spatial dimensions, applied to electric currents and changing electric fields such that the flux refers to that of the magnetic force, but obscured by the orthogonality of the magnetic field to the magnetic force. The dimensional superposition principle connects them through the dimensional diminution of the electric field in moving frames, and the electromagnetic field tensor $F^{\mu\nu}$ incorporates both instances of the metaprinciple through its 4-divergence:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu \tag{11}$$

[7] I use the term 'flux-producing object' rather than 'charge' because the latter, when used in conjunction with magnetism, is a loaded term.

I interpret the fact that $F^{\mu\nu}$ incorporates both instances of the metaprinciple as being ultimately due to local conservation of charge: applied to 3 dimensions, the metaprinciple permits tracking the change of charge density in a given volume over time, while applied to 2 dimensions, it permits tracking the change of current density in space. That both of these must match for a given charge distribution is a statement of the continuity equation, itself a consequence of local charge conservation.

Consider now Faraday's law, and let us imagine, once again, as an unphysical Gedankenexperiment, a situation in which it fails. Fig. (10a) represents a situation in which a point charge at rest produces an electric field such that some of its field vectors have a tangential component. We can immediately see that if any of those tangential components overlap with a closed loop containing the charge, they will produce a non-zero closed line integral of the electric field. In other words, this unphysical electrostatic field has a non-zero curl. Since the electrostatic field points in the same direction as the electrostatic force, the latter has the same curl as the electrostatic field. If we now consider this situation in a moving frame such that the tangential components are perpendicular to the direction of motion, then, again, by dimensional superposition the electric force field is superposed with a structure that is the line integral of its one-dimension reduced analog. In this configuration, the one-dimension lower analog of the the curl of the electrostatic force is a curl of the magnetic force field in the leaves of the foliation. Now, since the **B**-field is perpendicular in the leaf to the plane central force field, the curl of the magnetic force causes **B** to attain non-zero radial components, as depicted in fig. (10b). These radial components contribute to a non-zero divergence of **B**. But that means precisely that Gauss's law of magnetism has failed!



(a) (b)

Figure 10: A second thought experiment is depicted in (a): We imagine that a static point charge produces a field such that at least some electric force vectors have nonzero tangential components (the radial components are abstracted out), giving the electrostatic force field a non-zero curl. By dimensional superposition, this means that an observer in motion relative to the charge in a direction perpendicular to those components will observe magnetic forces which also have tangential components in their leaves. Since the **B**-field is orthogonal to the magnetic force, this implies that the magnetic field she observes has nonzero radial components in the leaf, and hence a non-zero divergence. Thus, Faraday's law and Gauss's law of magnetism are also not independent of each other.

The converse of this, going from a failure of Gauss's law of magnetism to a failure of Faraday's law, has two complications:

First, it is possible for an electric field to have a non-zero curl, namely when it is accompanied by a time-changing magnetic field. Addressing this complication is straightforward: demand that, in addition to imagining a failure of Gauss's law of magnetism, the magnetic field be stationary, so we impose $\frac{\partial}{\partial t}\mathbf{B} = 0$.

Second, if we ever discovered an instance in which Gauss's law of magnetism failed in a magnetostationary context, then this might imply that Faraday's law also fails, but could it not alternatively imply the discovery of a magnetic charge or monopole?

The discussion in the previous section makes it clear that, given the dimensional superposition concept, the closest thing to a magnetic analog of an electric point charge is an electric point charge

traveling at the speed of light in space, an impossibility. This contrasts with Dirac's conception of a magnetic charge [6]. Dirac-type monopoles have features which may cause one to question their physicality (e.g. the Dirac string), but it is possible to overcome such problems by going outside the scope of classical electrodynamics, as in the case of 't Hooft-Polyakov monopoles [7][8].
So, to properly establish the converse relationship, we need to restrict the scope of the discussion to Maxwell's classical abelian gauge theory and exclude Dirac monopoles. Under those restrictions, the discovery of a failure of Gauss's law of magnetism in a magnetostationary context does necessarily imply the existence of a rest frame in which Faraday's law also fails, because an observed non-zero divergence of the magnetic field would manifest itself in the rest frame of the sources as a non-zero curl in the electrostatic field. Thus,

$$\nabla \times \mathbf{E} = 0 \iff \nabla \cdot \mathbf{B}' = 0 \tag{12}$$

Where the prime on **B** once again indicates that the system is observed in a moving frame. In checking the interdependence of the Gauss and Ampère-Maxwell laws mathematically, I took advantage of the ability the reformulate the current density 4-vector in terms of the electromagnetic fields and transform it:

$$J^\mu = (\rho, \mathbf{J}) = (\epsilon_0 \nabla \cdot \mathbf{E}, \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \longrightarrow J^\nu \tag{13}$$

This strategy appears to be less sucessful for the Faraday and Magnetic Gauss laws, as it would require the definition of a magnetic current density 4-vector, which identically yields the zero 4-vector

$$M^\mu = (\rho_m, \mathbf{J}_m) = (\frac{1}{\mu_0} \nabla \cdot \mathbf{B}, -\frac{1}{\mu_0}(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t})) = (0, \mathbf{0}) \tag{14}$$

rendering any transformation on it trivial[8]. Fortunately, the mathematical argument given for the connection between the Gauss and Ampère-Maxwell laws already verifies that aspect of the mutual implication in (12) which depends on the dimensionality of each space in which a flux is considered. The only other aspect left to check is the mathematical verification that vector fields with tangential components have a non-zero curl, and that vector fields with directions orthogonal in a plane to vector fields with a non-zero curl have a non-zero divergence, both basic exercises in vector calculus. Thus, once again, Faraday's law and Gauss's law of magnetism are not two separate independent laws, but two different-looking manifestations of one and the same fundamental idea linked by dimensional superposition. The fundamental idea can be described in the most general terms as follows:

Metaprinciple 2. *A flux-producing object produces flux in radial directions only*[9].

Faraday's law of electro*statics* is an instance of this metaprinciple in 3 spatial dimensions, applied to electric charge, such that, again, flux refers to the electric force and therefore also to the electric field. Gauss's law of magnetism is an instance of this metaprinciple in 2 spatial dimensions, applied to electric currents such that the flux refers to the magnetic force, and again, obscured by the orthogonality of the magnetic field to the magnetic force. The dimensional superposition principle connects them, and the dual electromagnetic field tensor $G^{\mu\nu}$ implements this connection by incorporating both through its 4-divergence

$$\partial_\nu G^{\mu\nu} = 0 \tag{15}$$

An interpretation of this equation that is more geometric than just $(\rho_m, \mathbf{J}_m) = (0, \mathbf{0})$ is in preparation.

7 Conclusion

This paper re-interpreted Lorentz contraction as dimensional abatement and, using this re-interpretation, arrived at a number of results, collected below:

[8] Note however, that the equivalence $0 = 0 \iff 0 = 0$ (the magnetic analog of equation (9)) still does satisfy the mutual implication, even if only trivially.

[9] In a general relativistic setting, the most immediately applicable situation is the Reissner-Nordström metric. It obeys this metaprinciple under a proper redefinition of "radial direction", but it would be interesting to examine whether there exist any solutions to the Einstein Field equations which violate it.

- The re-interpretation focuses attention on two fundamental spacetime principles:
  - The invariance of the absolute dimensionality of a compact body under coordinate transformations in Minkowski spacetime
  - The homodimensionality of spacelike hypersurfaces of Minkowski spacetime
- Speed-of-light objects are dimensionally reduced, and speed of light invariance can be thought of as consequence, rather than as a cause, of this fact.
- Magnetic force fields represent structures which are line integrals of dimensionally reduced analogs of the structures represented by electric force fields.
- The appearance of magnetic fields in frames moving relative to electric field sources is a consequence of the dimensional diminution of the electric field in those frames, and the relationship between electric and magnetic forces follows a dimensional superposition principle.
- The dimensional superposition principle can be used to show that Maxwell's equations are not four independent equations, but in a set of two pairs embody specific instances in different dimensions of two metaprinciples such that the laws constituting each pair imply each other.

Special relativity and classical electrodynamics are mature theories. I can only make sense out of the fact that these results were not obtained long ago by noticing that they rely on ideas which properly belong to the *next* paradigm. And this is, ultimately, what makes dimensional abatement more fundamental than Lorentz contraction.

Acknowledgments

I wish to thank James Liu for many fruitful discussions and John Belcher for getting me started, years ago, on the road to understanding electromagnetism visually.

Appendix: Transforming a Pure E-field Into a Pure B-field

Here we provide the calculation referenced on page 8, for more details see ([9]), Chapter 11. We begin by considering a Lorentz transformation of the electromagnetic field tensor:

$$F'^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\rho}F^{\lambda\rho} \tag{16}$$

where

$$F^{\lambda\rho} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_x & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{17}$$

is the field tensor and

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

is the Lorentz transformation matrix for a boost of βc in the positive x-direction. After simplification, the components can be written as

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - \beta B_z) & B'_y &= \gamma(B_y + \beta E_z) \\ E'_z &= \gamma(E_z + \beta B_y) & B'_z &= \gamma(B_z - \beta E_y) \end{aligned} \tag{19}$$

The two equations on the first line already tell us that there exists no spacetime frame in which an electric field or magnetic field which is non-zero in another frame completely vanishes. That means that what we set out to do is impossible if we confine our consideration to spacetime frames. However, as mentioned in section 4, we are now not only considering spacetime (i.e. 4-dimensional) frames but also the possibility of dimensionally reduced ones, and this justifies proceeding with the calculation. In vector notation, the equations for a general transformation are

$$\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \tag{20}$$

and

$$\mathbf{B}' = \gamma(\mathbf{B} - \beta \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B}) \tag{21}$$

These are the general expressions. I will first consider equation (20) to obtain the speed that it takes for the transformation, then (21) to obtain the fields. For our purposes, we want

$$\mathbf{E}' = \mathbf{B} = 0 \tag{22}$$

Imposing this on (20), I obtain

$$\gamma\mathbf{E} = \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \tag{23}$$

multiplying both sides by $\gamma + 1$ gives

$$(\gamma+1)\gamma\mathbf{E} = \gamma^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \tag{24}$$

or, rearranging,

$$\gamma^2(\mathbf{E} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) + \gamma\mathbf{E} = 0 \tag{25}$$

I want to dot this into the unit direction vector $\hat{\boldsymbol{\beta}}$, which gives

$$\hat{\boldsymbol{\beta}} \cdot \boldsymbol{\beta} = \beta \tag{26}$$

and

$$\hat{\boldsymbol{\beta}}\cdot\mathbf{E} = Ecos\theta \tag{27}$$

where θ is the angle between the direction of the electric field and the direction of motion. Substituting (26) and (27) into equation (25) and factoring out $Ecos\theta$, I obtain

$$Ecos\theta(\gamma^2(1-\beta^2)+\gamma) = 0 \tag{28}$$

As long as $Ecos\theta \neq 0$ it is legitimate to divide both sides by $Ecos\theta$. I will assume this and treat the case $Ecos\theta = 0$ separately later. So, assuming $Ecos\theta \neq 0$ and dividing out,

$$\gamma^2(1-\beta^2) + \gamma = 0 \tag{29}$$

Using the quadratic formula[10] with $a = (1-\beta^2)$, $b = 1$ and $c = 0$, I obtain

$$\gamma = \frac{-1 \pm \sqrt{(1-0)}}{2(1-\beta^2)} \tag{30}$$

One solution is $\gamma = 0$ and the other is

$$\gamma = \frac{-2}{2(1-\beta^2)} = \frac{1}{\beta^2 - 1} \tag{31}$$

or

$$\frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\beta^2-1} \tag{32}$$

or

$$\beta^2 - 1 = \sqrt{1-\beta^2} \tag{33}$$

which has as solutions $\beta = \pm 1$. The physical solution is $\beta = 1$ or

$$v = c \tag{34}$$

What about the case $Ecos\theta = 0$? This term is zero either if $E = 0$ or $cos\theta = 0$ (or both). Our initial assumptions require that $E \neq 0$, so we must have $cos\theta = 0$, thus $\boldsymbol{\beta}$ and $\mathbf{E}$ are at a right angle. Returning to equation (23) and inserting the special case where the two are at a right angle in $\boldsymbol{\beta}\cdot\mathbf{E} = \beta Ecos\theta = 0$, I obtain

$$\gamma\mathbf{E} = 0 \tag{35}$$

Again, since $E \neq 0$ by assumption, it must be that $\gamma = 0$. This is, of course, a problem. It gets worse when we impose the conditions of (22) on (21) to obtain

$$\mathbf{B}' = -\gamma\boldsymbol{\beta} \times \mathbf{E} \tag{36}$$

When $\mathbf{E}$ and $\boldsymbol{\beta}$ are orthogonal, we have $sin\theta = 1$ and plugging this into (36) gives for the magnitude of $\mathbf{B}'$

$$B' = -\gamma\beta E = -0\infty E = \text{not defined} \tag{37}$$

Because we just found that for a right angle between $\boldsymbol{\beta}$ and $\mathbf{E}$, $\gamma = 0$, hence $\beta = \infty$. Now, when $\mathbf{E}$ and $\boldsymbol{\beta}$ are parallel, we have $sin\theta = 0$ and using the solution we obtained in equation (34) for angles other than a right angle gives $\gamma = \infty$, so that (36) yields for the magnitude of $\mathbf{B}'$

$$B' = -\infty E0 = \text{not defined} \tag{38}$$

Finally, for any angle which is neither parallel nor perpendicular, we obtain from using (34) in (36)

$$B' = -\infty \tag{39}$$

I believe this pathological behavior arises from the impossibility of the requirement of transforming a pure electric field into a pure magnetic field, itself ultimately due to the fact that it is not legitimate to use the Lorentz transformations at $v = c$. However, taking the interpretation that at the speed of light the directions of motion and time merge permits a novel approach to circumventing this problem. I define for the standard configuration what I will call a *merge transformation matrix* or *merge operator* as follows:

$$\lim_{v\to c}\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{40}$$

[10] There is a subtlety at this step: It might be tempting to substitute $1-\beta^2 = \gamma^{-2}$ and cancel out the squared Lorentz factors in (29), but we have to remember that our goal is to find γ as a function of β, not assume it. Further, for $\gamma = 0$ the division is invalid.

This is a non-orthogonal projection operator which, when acting on a timelike 4-vector in the standard configuration, transforms it into a null 4-vector. Note that in order for the operator to perform its action correctly, the coordinate system must be oriented in the standard configuration, otherwise we are not guaranteed that the transformation yields a null 4-vector.
The justification for defining this operator as the limit of the Lorentz transformation matrix at the speed of light is not mathematical but heuristic (Hence ':=' instead of equality): its action is the type of action the Lorentz transformation approaches as v tends to c. The merge operator, in effect, merges the timelike and first spacelike components and thereby projects the 4-vector onto a 2-dimensional plane which moves along the $x^{1\prime}$-direction at $v = c$. The operator in (40) is not unique, in the sense that other merge operators could be defined which by means of non-orthogonal projection transform a timelike vectors into different null vectors. This reflects the absence of a mathematical justification, something which is of course desirable and important to find.
Using this heuristic device, we want to now examine this charge in its rest frame after the operation has been carried out. The obvious caveat is that since it is physically impossible to transform to the rest frame of a light-speed object, we are now venturing into a realm outside of testable physics. My justification for doing so is threefold: first, what is presented below is grounded on a reasonable extrapolation from established physics; second, it leads to more sensible results than equations (37), (38) and (39); and third, the results underpin the dimensional superposition principle which does yield novel insights well within the realm of testable physics.
In the plane moving at $v = c$, the fictitious charge experiences itself to be at rest. We need to account for the loss of one spatial dimension, which we can do by taking the limit of a line charge λ at zero length:

$$\lim_{L\to 0} \lambda := \tilde{q} \tag{41}$$

Where $\tilde{q}$ has dimensional units of Coulomb per meter. Here and below, I will denote any quantities in the plane by a tilde, to distinguish them from the corresponding spatial quantities. The electric field set up by $\tilde{q}$ can likewise be thought of in terms of the cross-section of an infinite line charge:

$$\lim_{L\to 0} \frac{\lambda}{2\pi r \epsilon_0} := \frac{1}{2\pi\epsilon_0}\frac{\tilde{q}}{\tilde{r}} \tag{42}$$

The direction of the field is radially outward in the plane, so we have

$$\tilde{\mathbf{E}} = \frac{1}{2\pi\epsilon_0}\frac{\tilde{q}}{\tilde{r}}\hat{\tilde{\mathbf{r}}} \tag{43}$$

which is a 2-dimensional analog of Coulomb's law, as we might have expected.
A spacetime observer would, however, observe no electric field because he cannot assign to this fictitious charge a rest frame (its associated current density null 4-vector has a current density as a timelike component). Rather, he would observe a pure magnetic field. For an ordinary point charge moving at constant velocity $\mathbf{v}$, the magnetic field is given by $\frac{1}{c^2}(\mathbf{v}\times\mathbf{E})$. Making the substitutions $v = c$ and $\mathbf{E} = \tilde{\mathbf{E}}$ and applying this to equation (43), I obtain

$$\tilde{\mathbf{B}} = \frac{\mu_0}{2\pi}\frac{\tilde{q}c}{\tilde{r}}\hat{\tilde{\boldsymbol{\phi}}} \tag{44}$$

Where $\hat{\tilde{\boldsymbol{\phi}}}$ points in a direction orthogonal both to $\tilde{\mathbf{E}}$ and the direction of motion of the plane. Notice that this is the same result we would have obtained had we defined the fictious charge moving at the speed of light in terms of the of the cross-section of a line current:

$$\lim_{\substack{L\to 0\\ v\to c}} \lambda v := \tilde{q}c \tag{45}$$

and then applied it to the magnetic field of a line charge in the same limits

$$\lim_{\substack{L\to 0\\ v\to c}} \frac{\mu_0 \lambda v}{2\pi r} := \frac{\mu_0}{2\pi}\frac{\tilde{q}c}{\tilde{r}} \tag{46}$$

Because $\hat{\tilde{\mathbf{r}}}$ has no component normal to the plane, the magnetic field is entirely confined within a plane moving at the speed of light, centered at the location of $\tilde{q}$. This is my immediate justification for assigning to the magnetic field of a fictitious charge moving at the speed of light a circular symmetry, as represented in figure (6c). My ultimate justification is that thinking of a fictitious electric charge moving at the speed of light in this way sets the foundation for gleaning deeper insights into Maxwell's equations via dimensional superposition.

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