Boundedly rational backward induction

Shaowei Ke*

Department of Economics, University of Michigan

June 29, 2018

Abstract

This paper proposes simple axioms that characterize a generalization of backward induction. At any node of a decision tree, the decision maker looks forward a fixed number of stages perfectly. Beyond that, the decision maker aggregates continuation values according to a function that captures reasoning under unpredictability. The model is uniquely identified from the decision maker's preference over decision trees. Confronting a decision tree, the decision maker iteratively revises her plan for the future as she moves forward in the decision tree. A comparative measure of unpredictability aversion and several examples are discussed.

1 Introduction

Backward induction has been used to analyze decision makers' behavior in dynamic decision problems. In a typical dynamic decision problem, fully rational backward induction begins by identifying the optimal choice for the last stages of the problem, and then rolls back to

*Shaowei Ke, shaoweik@umich.edu; I am deeply indebted to Faruk Gul, Stephen Morris, and Wolfgang Pesendorfer for their invaluable advice, guidance, and encouragement. I thank the editor, three anonymous referees, Dilip Abreu, Tilman Börgers, Kfir Eliaz, Andrew Ellis, Larry Epstein, Shengwu Li, Xiao Liu, Marco Mantovani, Yusufcan Masatlioglu, Efe Ok, Doron Ravid, Collin Raymond, Ariel Rubinstein, Gabriel Ziegler, Peio Zuazo-Garin, and numerous seminar participants for their helpful comments.

the first stage. The solution is taken as a prediction of how decision makers behave in such a context. It is well known, however, that decision makers cannot perform fully rational backward induction in reality.[1]

To accommodate deviations from fully rational backward induction, many economists have considered the idea that perhaps the decision maker can only see a few stages ahead perfectly.[2] The difficulty, however, is to understand how the decision maker evaluates the part of the decision problem that is beyond her perfect foresight. Suppose a decision maker can only see one stage ahead perfectly, and is confronted with the following decision problem depicted by a *decision tree* (Figure 1). Without knowing the value the decision maker assigns to the *subtree* beyond the first stage, we do not know how the decision maker would roll back the first stage to evaluate the decision tree.

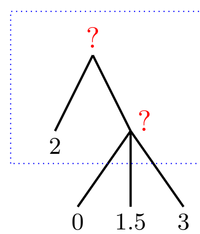

Figure 1: The value of the second-stage subtree that is beyond the decision maker's perfect foresight determines how she compares 2 and $\{0, 1.5, 3\}$, and hence determines the backward induction solution for the entire decision tree. Since the decision maker no longer fully rationally backward inducts beyond the first stage, the value of the second-stage subtree may not be $\max\{0, 1.5, 3\} = 3$.

This paper takes a revealed-preference approach to analyze how the decision maker may

[1] A simple way to see this is to note that if both players in chess perform fully rational backward induction, then either white can force a win, or black can force a win, or both sides can force at least a draw, according to the classic theorem by Zermelo (1913). This prediction is far from what we observe. For more examples, see Güth and Tietz (1990); Rubinstein (1990); McKelvey and Palfrey (1992); Camerer, Johnson, Rymon, and Sen (1993); Binmore (1999); Binmore, McCarthy, Ponti, Samuelson, and Shaked (2002); Asheim and Dufwenberg (2003); Palacios-Huerta and Volij (2009); Levitt, List, and Sadoff (2011); Mantovani (2014); and Rampal (2018).

[2] See, for example, Jéhiel (1995, 1998, 2001); Gabaix, Laibson, Moloche, and Weinberg (2006); and Rampal (2018).

evaluate subtrees beyond her perfect foresight. The decision maker's preference over *individual* decision trees is taken as the primitive. We impose simple and testable axioms that describe how the preference may vary with the *presentation/frame* of the decision problem, that is, the structure of the decision tree. Then, we characterize the class of models that are consistent with the axioms. Note that in previous literature on backward induction, most studies take place in extensive-form games. However, there exist many confounders in extensive-form games.[3] To avoid this, we focus on the simplest choice environment that allows us to study backward induction: the set of individual deterministic finite decision trees.

Confronting a decision tree, the decision maker makes a sequence of choices until she reaches a lottery. Decision trees are defined recursively: A depth-1 decision tree is a finite set of lotteries, a depth-2 decision tree is a finite set of lotteries and depth-1 decision trees, and so on. Thus, a decision tree is represented by a set $a = \{a_1, \ldots, a_n\}$ consisting of subtrees $a_1, \ldots, a_n$. Each subtree may be a lottery or yet another decision tree; that is, another set of subtrees and lotteries.

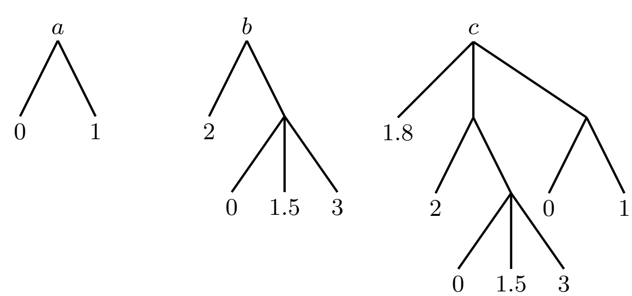

Figure 2: Think of the numbers at the end of decision trees as the utility of lotteries. The first decision tree a is a depth-1 decision tree. The second decision tree b is a depth-2 decision tree. The last decision tree c is a depth-3 decision tree, and $c = \{1.8, a, b\}$.

[3]For example, it is often found that players do not end the centipede game immediately, unlike what fully rational backward induction predicts. However, this could happen because, although players have backward inducted rationally, they believe that other players will make mistakes, or they have social preferences for fairness, etc.

The axioms we impose on the preference yield a new class of models that generalize fully rational backward induction. In the resulting model, the decision maker performs fully rational backward induction to evaluate the first $0 \leq \kappa \leq +\infty$ stages of a decision tree. For subtrees beyond κ stages, it is *as if* the decision maker uses a general aggregator to aggregate the subtree values. For a subtree $a = \{a_1, \ldots, a_n\}$ that is beyond κ stages, the aggregator takes the following form:

$$U(a) = f^{-1}\left(\frac{1}{n}\sum f(U(a_i))\right), \tag{1}$$

in which $U(\cdot)$ is the utility function. Depending on f, the aggregator ranges from the maximum function to the minimum function. Both κ and f are uniquely identified from the decision maker's preference. We call this representation the *κ-Boundedly-Rational-Backward-Induction* (κ-BRBI) representation.

We offer two interpretations of (1). First, note that the aggregator in fully rational backward induction is the maximum function. In our model, instead of the maximum, (1) is a general notion of *average*. Thus, (1) captures the idea that beyond the decision maker's perfect foresight, she evaluates subtrees according to some "foggy" overall impression.

In the second interpretation, it is *as if* the decision maker knows the structure of the decision tree, but is unable to predict or does not trust her future selves' choices beyond κ stages. First, she envisions that she will choose uniformly randomly after κ stages, captured by $1/n$ in (1). Such a belief is called *Laplacian*, as Laplace (1824) suggests that the uniform prior should be applied to unknown events from "the principle of insufficient reason." Second, the decision maker's attitude toward the Laplacian belief is captured by $f \circ U$, in which f is a *second-order expected utility function*. We show that the concavity of f is a comparative measure of *unpredictability aversion*; that is, the extent to which the decision maker avoids unpredictable situations.

The example in Figure 3 illustrates how the model assigns values to decision trees. Suppose $\kappa = 1$ and the aggregator is the simple average function ($f(u) = u$). For the degenerate decision tree 1.8 (the left-hand side) and the depth-1 decision tree $\{0, 1\}$ (the right-hand side), the decision maker's evaluation is identical to fully rational backward induction. For the depth-2 decision tree, the decision maker uses (1) to aggregate the subtree beyond the first stage, and then rolls back the first stage using fully rational backward induction.

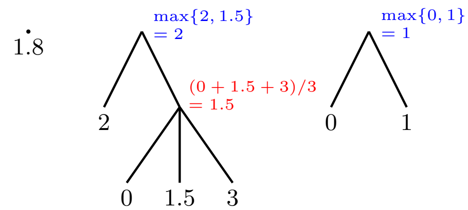

Figure 3: The decision maker with $\kappa = 1$ and $f(u) = u$ evaluates subtrees 1.8, $\{2, \{0, 1.5, 3\}\}$, and $\{0, 1\}$.

The preference also describes how the decision maker actually chooses in a decision tree. Confronting a decision tree $a = \{a_1, \ldots, a_n\}$, the decision maker chooses her most preferred subtree a_i. If the chosen subtree a_i is also a decision tree, she continues to choose her most preferred subtree from $a_i = \{b_1, \ldots, b_m\}$. By using this choice procedure, we have made a *history-independence* assumption: The decision maker's preference over subtrees does not depend on how she reaches the current decision tree. Such a choice procedure implies that the decision maker's perfect foresight moves forward as she makes choices; that is, if the decision maker can see the κ^{th} stage of a perfectly when choosing from $\{a_1, \ldots, a_n\}$, she can now see the $(\kappa + 1)^{\text{th}}$ stage of a perfectly when choosing from $\{b_1, \ldots, b_m\}$. This procedure continues iteratively until she reaches a lottery.

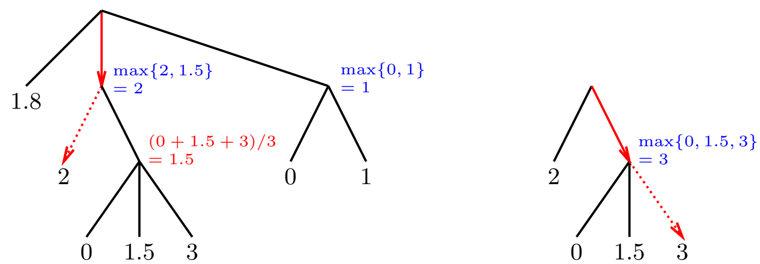

Figure 4: Suppose $\kappa = 1$ and $f(u) = u$. First, the decision maker chooses one of the subtrees from 1.8, $\{2, \{0, 1.5, 3\}\}$, and $\{0, 1\}$. Since the value of $\{2, \{0, 1.5, 3\}\}$ is 2, which is the highest among the three subtrees, the decision maker chooses it from the original (the left-hand) decision tree. Once she moves on to the chosen subtree $\{2, \{0, 1.5, 3\}\}$ (the right-hand tree), the subtree $\{0, 1.5, 3\}$ is no longer beyond the decision maker's perfect foresight. The value of $\{0, 1.5, 3\}$ becomes 3 instead of 1.5, and the decision maker chooses $\{0, 1.5, 3\}$. She will end up with 3.

Figure 4 shows how the decision maker with $\kappa = 1$ and $f(u) = u$ chooses in a decision tree. Notice that according to the subtree values that the decision maker assigns when making the first choice (in the left-hand-side decision tree), the decision maker would have assumed that she would choose 2 from $\{2, \{0, 1.5, 3\}\}$ next. However, as she moves on to the next stage, she ends up noticing that the value of $\{0, 1.5, 3\}$ is 3 instead of 1.5. Thus, the decision maker may iteratively revise her plan as she makes choices.

Our model has four main axioms. One states that if the decision maker can solve all depth-k decision trees correctly, she can also solve depth-j trees correctly whenever $j \leq k$. This allows us to identify κ—the boundary of the decision maker's perfect foresight. Next, we consider three axioms for subtrees beyond the perfect foresight. Suppose b is a subtree that is beyond κ stages. First, if we replace a subtree of b with a better one, b becomes better. Second, combining a good subtree with a bad subtree yields a new subtree that is ranked in between. The last main axiom is built on a simple idea: When a depth-1 tree contains fewer lotteries, each of its lotteries commands more attention. Therefore, swapping a better lottery from a subtree of b that has more branches for a worse lottery from a subtree of b that has fewer branches accentuates the better lottery and hides the worse lottery, which makes subtree b better. Figure 5 illustrates this idea.

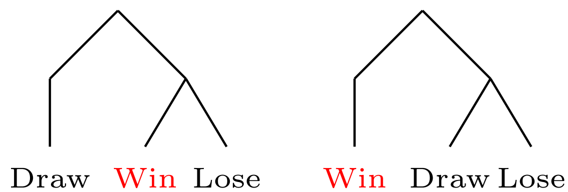

Figure 5: Suppose both subtrees are beyond κ stages. Compared to the left-hand tree, the right-hand tree accentuates "Win" and hides "Draw." The decision maker prefers the right-hand tree.

1.1 Related Literature

Our work belongs to the bounded rationality literature on choices in complex situations. Several papers have examined specific heuristics or reasoning processes. Jéhiel (2001) considers a stochastic value function beyond a player's imperfect foresight, and studies an equilbrium notion in which the players' forecasts within the imperfect foresight are correct. Gabaix et al. (2006) study a reasoning procedure in which the decision maker evaluates continuation problems as if they end right away. Based on the heuristic, the reasoning procedure endogenously determines the optimal number of stages the decision maker should look forward. Rampal (2018) assumes that the decision maker uses the simple average of the maximum and the minimum payoffs to evaluate actions beyond κ stages. He introduces an equilibrium notion that features uncertainty over the opponents' numbers of stages of perfect foresight. Our paper does not start with a specific heuristic or reasoning process. We propose simple and testable axioms on the decision maker's preference over decision trees and provide the representation theorem.

Fudenberg and Strzalecki (2015) also adopt the revealed-preference approach to study the decision maker's choices in individual deterministic finite decision trees. They propose a dynamic extension of the logit model in which the decision maker is averse to large decision trees. Their measure of choice aversion is based on the size of the decision tree (recursively) and takes a specific functional form. Our decision maker chooses deterministically and does not necessarily avoid large decision trees. She avoids decision trees with inferior subtrees, and the extent to which she avoids them is captured by the concavity of a general second-order expected utility function.

Our *Disjoint Set Betweenness* axiom is related to several existing axioms. Bolker (1966) is the first to use this type of axiom. He studies a generalization of the concept of expected value in mathematics. The axiom is not imposed on the preference. Bolker's resulting formula is different from ours, but similar to Ahn (2008), who uses a similar axiom to study ambiguity. Gul and Pesendorfer (2001) use a related axiom to model temptation and self-control. In their model, the decision maker may prefer a smaller choice set to a larger one because the larger one contains tempting bad lotteries. Their axiom is stronger than ours, as it applies to the case in which choice sets have nonempty intersections.

The rest of the paper is organized as follows. The setup is introduced in Section 2. A special case of the model is presented in Section 3 and the general case in Section 4. Section 5 discusses a few behavioral predictions of the model.

2 Setup

We consider a decision maker who makes a series of choices until a lottery is reached. A *decision tree* describes this choice situation. Let X be a compact separable metric space and $D_0 := \Delta(X)$ be the set of lotteries (Borel probability measures) on X, endowed with the Prokhorov metric.[4] Generic lotteries are denoted by p, q, r, s.

[4]The main results of the paper will go through if we replace D_0 with a convex set of real numbers representing monetary payoffs. The difference is that we will replace the utility of lotteries with monetary

A depth-1 decision tree is a nonempty finite subset of lotteries. When the decision maker confronts a depth-1 decision tree $a \subset D_0$, she chooses one of its lotteries. For any set Z, let $K(Z)$ denote the collection of all nonempty finite subsets of Z. Then, $D_1 := K(D_0)$ denotes the set of depth-1 decision trees. A depth-2 decision tree is a nonempty finite subset of lotteries and depth-1 decision trees. We require that a depth-2 decision tree be different from a depth-1 decision tree. Let

$$D_2 := K(D_0 \cup D_1) \setminus D_1$$

denote the set of depth-2 decision trees. By definition, in Figure 6, decision tree $a = \{p, q\}$ is a depth-1 decision tree, and decision tree $b = \{r, a\} = \{r, \{p, q\}\}$ is a depth-2 decision tree.

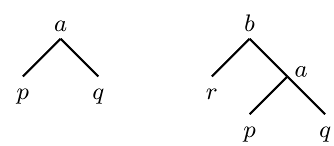

Figure 6: The depth-2 decision tree $b = \{r, a\} = \{r, \{p, q\}\} \in D_2$ consists of a lottery r and a depth-1 decision tree $a = \{p, q\} \in D_1$.

Recursively, we define the set of depth-k decision trees as

$$D_k := K\left(\bigcup_{j=0}^{k-1} D_j\right) \Big\backslash \left(\bigcup_{j=0}^{k-1} D_j\right).$$

The definition has two implications: When the decision maker confronts a depth-k decision tree, (i) she makes at most k choices to reach a lottery, and (ii) there exists some lottery that takes exactly k choices to reach. Let $D := \bigcup_{j=1}^{\infty} D_j$ be the set of all decision trees.

According to the construction, a typical decision tree $a = \{a_1, \ldots, a_n\} \in D$ is represented as a finite set of *subtrees*, $a_1, \ldots, a_n$. For example, in Figure 6, the depth-2 decision tree b

payoffs. Considering lotteries allows us to discuss decision trees with nonmonetary payoffs, and allows us to separately identity the decision maker's attitude toward objective uncertainty and her attitude toward subjective uncertainty about her own future choices, as will be shown below.

consists of subtrees $a \in D_1$ and $r \in D_0$. A subtree could either be another decision tree or a lottery. Let $\mathcal{D} := D \cup D_0$ denote the set of all subtrees. It can be verified that $D = K(\mathcal{D})$.

The decision maker has a binary relation/preference $\succsim$ on $\mathcal{D}$, the set of decision trees and lotteries. We say that $\succsim$ is *nontrivial* if there exist some subtrees $a, b \in \mathcal{D}$ such that $a \succ b$. We assume throughout the paper that X is *rich*; that is, for any $x \in X$, there exist countably many distinct consequences that are indifferent to x. We need richness to ensure that it is always possible to have an arbitrary number of indifferent lotteries in a decision tree. For example, since we use sets to construct decision trees, if there is a unique best consequence in X, it cannot appear more than once in a depth-1 decision tree.[5] Lastly, when restricted to D_0, $\succsim$ is a preference over lotteries. We say that $\succsim$ on D_0 has an *expected utility representation* if there is a continuous function $U : D_0 \to \mathbb{R}$ representing $\succsim$ on D_0 such that

$$U(p) = \int_X U \; dp \tag{2}$$

for each $p \in D_0$.[6] We call a function $\hat{U} : \mathcal{D} \to \mathbb{R}$ an expected utility function if, when restricted to D_0, $\hat{U}$ is an expected utility representation of $\succsim$ on D_0.

Throughout the paper, we impose two axioms on $\succsim$.

Axiom 1 *(Weak Order) $\succsim$ is complete and transitive.*

Axiom 2 *(vNM) $\succsim$ on D_0 has an expected utility representation.*

The first axiom is standard. The second axiom is equivalent to the three well-known von Neumann–Morgenstern axioms in expected utility theory.

The remaining axioms will pin down a representation of the decision maker's preference in which the decision maker looks forward κ stages perfectly. Beyond κ stages, to evaluate subtrees, the decision maker will use an aggregator that is different from the one used by fully rational backward induction. To understand how the aggregator is revealed from the

[5] Alternatively, we can use multisets to define decision trees without assuming richness.

[6] As usual, we identify X with the set of degenerate lotteries. For each $x \in X$, the degenerate lottery that gives the decision maker x with probability 1 is denoted by δ_x. Equation (2) means that $U(p) = \int_X U(\delta_x)\, dp$.

decision maker's preference, in the next section we first investigate a special case of the model in which $\kappa = 0$. Then, we will characterize the general case with $0 \leq \kappa \leq +\infty$.

3 0-Stage Perfect Foresight

We consider the following testable axioms on the decision maker's preference, which encapsulate how the decision maker's preference is allowed to deviate from full rationality when the decision maker cannot even evaluate depth-1 trees perfectly.

Axiom 3 *(Monotonicity) For any $a = \{a_1, a_2, \ldots, a_n\}, a' = \{a'_1, a_2, \ldots, a_n\} \in D$, $a_1 \succsim a'_1$ implies $a \succsim a'$, and $a_1 \succ a'_1$ implies $a \succ a'$.*

A fully rational decision maker satisfies the first part of *Monotonicity* ($a_1 \succsim a'_1$ implying $a \succsim a'$), but violates the second part ($a_1 \succ a'_1$ implying $a \succ a'$). To see this, suppose $a = \{p, q\}$, $a' = \{p', q\}$, and $q \succ p \succ p'$. A fully rational decision maker is indifferent between a and a', since they have the same best lottery q. *Monotonicity* requires that $a \succ a'$; that is, the decision maker avoids decision trees with inferior subtrees.

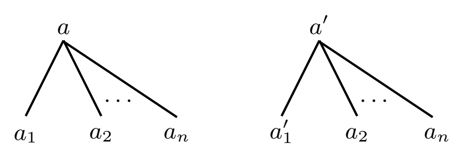

Figure 7: *Monotonicity* requires that $a_1 \succsim a'_1$ if and only if $a \succsim a'$.

The next axiom, *Disjoint Set Betweenness*, considers two decision trees a and b that have no subtree in common.[7] For example, suppose $a = \{\text{Win}\}$, $b = \{\text{Draw,Lose}\}$, and the decision maker prefers $\{\text{Win}\}$ to $\{\text{Draw,Lose}\}$. *Disjoint Set Betweenness* states that $\{\text{Win,Draw,Lose}\}$ is ranked in between; that is, $\{\text{Win}\}$ is preferred to $\{\text{Win,Draw,Lose}\}$, which in turn is preferred to $\{\text{Draw,Lose}\}$.

[7]However, two subtrees with no subtree in common may have common (terminal) lotteries.

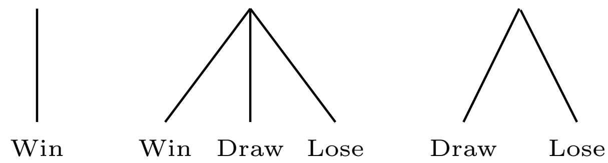

Figure 8: The decision maker should have the following preference: {Win} $\succsim$ {Win,Draw,Lose} $\succsim$ {Draw,Lose}.

Axiom 4 *(Disjoint Set Betweenness) For any $a, b \in D$ such that $a \cap b = \emptyset$, $a \succsim b$ implies $a \succsim a \cup b \succsim b$.*

When $a \succsim b$, a fully rational decision maker should be indifferent between a and $a \cup b$, since they both contain the same best subtree from a. *Disjoint Set Betweenness* allows the decision maker to strictly prefer a to $a \cup b$ because $a \cup b$ contains inferior subtrees that a does not have. Similarly, $a \cup b$ may be strictly preferred to b because $a \cup b$ contains better subtrees that b does not have. Thus, as in *Monotonicity*, the decision maker in our model is averse to decision trees with inferior subtrees. This means that *Disjoint Set Betweenness* may be violated if the decision maker simply dislikes large decision trees. For example, $\{p\} \sim \{q\}$ and the decision maker prefers $\{p\}, \{q\}$ to $\{p, q\}$.

To rule out some uninteresting deviations from full rationality, we impose the following two axioms that a fully rational decision maker satisfies. The first requires that the decision maker not be fooled by trivial extensions of decision trees.

Axiom 5 *(Indifference to Trivial Extensions) For any $a \in \mathcal{D}$, $a \sim \{a\}$.*

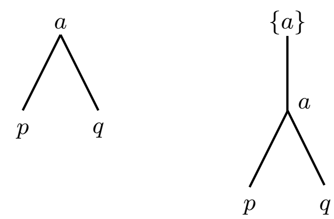

Figure 9: The decision maker is indifferent between the two decision trees, $a = \{p, q\}$ and $\{a\} = \{\{p, q\}\}$.

The second axiom is a continuity condition. Intuitively, we want a decision tree's utility to not change much when its subtree values are slightly perturbed. For example, for depth-1 trees a and b, this means that if a and b share the same size and the utility of a's lotteries is close to that of b's *pairwisely*, a's utility should be close to b's. This idea can be formalized as follows. Let $|\cdot|$ denote the cardinality of a set. Define $D_1^n := \{a \in D_1 : |a| = n\}$ for each positive integer n. Recall that the distance between two lotteries p and q is given by the Prokhorov metric $d(p, q)$. Analogous to the Hausdorff metric, we define the following metric on D_1^n: For any $a, b \in D_1^n$,

$$d^n(a, b) = \max\left\{\max_{p\in a}\min_{q\in b} d(p, q),\ \max_{q\in b}\min_{p\in a} d(p, q)\right\}.$$

We may also continue to define the metric for depth-k trees, but this turns out to be unnecessary.

Axiom 6 *(Continuity) For any $a \in D_1$, $\left\{b \in D_1^{|a|} : b \succ a\right\}$ and $\left\{b \in D_1^{|a|} : a \succ b\right\}$ are open in $D_1^{|a|}$.*

To better understand the implication of the final axiom, we use the following lemma to summarize the behavioral implication of the axioms presented so far. We first define a representation.

Definition 1 *The preference $\succsim$ has a Recursive Average (RA) representation if there exists an expected utility function $U : \mathcal{D} \to \mathbb{R}$ and a sequence of continuously strictly increasing* *symmetric functions* $g_n : U(\mathcal{D})^n \to \mathbb{R}$ *for* $n \in \mathbb{N}$ *such that (i) for any* $a_1, a_2 \in \mathcal{D}$, $a_1 \succsim a_2$ *if and only if* $U(a_1) \geq U(a_2)$*; (ii) for any* $b = \{b_1, \ldots, b_n\} \in D$, $U(b) = g_n(U(b_1), \ldots, U(b_n))$*; and (iii) for any* $u_1, \ldots, u_n \in U(\mathcal{D})$, $\min u_i \leq g_n(u_1, \ldots, u_n) \leq \max u_i$.

The representation is recursive. For example, suppose $a = \{p, q, r\}$ and $b = \{s, a\} = \{s, \{p, q, r\}\}$. Then,

$$U(b) = g_2(U(s), U(a)) = g_2(U(s), g_3(U(p), U(q), U(r))).$$

Moreover, the utility of b is always between the utility of b's best subtree and the utility of b's worst subtree.

Lemma 1 *The preference* $\succsim$ *has an RA representation if and only if* $\succsim$ *satisfies Axioms 1–6.*

Disjoint Set Betweenness ensures that g_n's are between the maximum and the minimum. *Monotonicity* not only ensures that the g_n's are well defined and increasing, but also implies their recursivity.[8] To see this, consider $a = \{p, q, r\}$ and $b = \{s, a\}$. Suppose there is a lottery p' such that $p' \sim a$. According to *Monotonicity*, $p' \sim a$ implies that $b \sim \{s, p'\}$. In other words, $U(b) = U(\{s, p'\}) = g_2(U(s), U(p')) = g_2(U(s), U(a))$.

The RA representation is very general, because there are few restrictions on g_n's. Our last axiom pins down the form of g_n's, which significantly reduces the number of parameters of the model. The last axiom is built on a simple idea: When a depth-1 tree contains fewer lotteries, each of its lotteries commands more attention. To see what attention has to do with choices, let us first introduce a notion of a *swap*.

Definition 2 *For any* $a = \{a_1, a_2, \ldots, a_n\} \in D_2$ *such that* $a_1, a_2 \in D_1$, $|a_1| \geq |a_2|$, $p \in a_1 \backslash a_2$, $q \in a_2 \backslash a_1$, $a_1' := a_1 \backslash \{p\} \cup \{q\} \notin \{a_3, \ldots, a_n\}$*, and* $a_2' := a_2 \backslash \{q\} \cup \{p\} \notin \{a_3, \ldots, a_n\}$*, a swap of* p *for* q *is*

$$\Delta_q^p(a) := a \backslash \{a_1, a_2\} \cup \{a_1', a_2'\}.$$

[8]More discussions about *Monotonicity* and *Disjoint Set Betweenness* can be found in Ke (2018).

See Figure 5 for an example of a swap of "Win" for "Draw." The definition requires that $a \in D_2$ be a depth-2 tree. Hence, a_1, a_2 are two sets of lotteries. In the definition, the lottery p originally belongs to a larger subtree (a_1) than the subtree (a_2) containing q. We assume that lotteries from a smaller subtree command more attention. Therefore, the swap of p for q accentuates p and masks q. If p is preferred to q, we call this swap an *accentuating swap* to emphasize the fact that after the swap, the better lottery p is more salient and the worse lottery q is less. When we write $\Delta_q^p(a)$ to denote the swap of p for q, implicitly we have imposed the assumptions in Definition 2.

Axiom 7 *(Preference for Accentuating Swaps) If $a \in D_2$ and $p \succsim q$, then $\Delta_q^p(a) \succsim a$.*

While the axiom is satisfied by a fully rational decision maker, it allows for departures from full rationality. Consider the example in Figure 5. There is a depth-2 decision tree $a = \{a_1, a_2\}$ where $a_1 = \{\text{Win,Lose}\}$, $a_2 = \{\text{Draw}\}$. For a fully rational decision maker, it does not matter which lottery is presented at which part of the tree; that is, she is indifferent between a and $\{\{\text{Draw,Lose}\}, \{\text{Win}\}\}$. In contrast, when a boundedly rational decision maker looks forward in decision tree a, since $|a_1| > |a_2|$, there are more (degenerate) lotteries competing for attention in a_1 than in a_2. An accentuating swap of "Win" for "Draw" makes "Win" more salient and "Draw" less. Therefore, the swapped decision tree appears to be better. This axiom may be violated if, for example, the decision maker prefers "well-organized" decision trees. Suppose x_1, x_2, x_3 are bags, y_1, y_2 are jackets, and x_1 is better than y_1. The decision maker who prefers well organized decision trees may strictly prefer $\{\{x_1, x_2, x_3\}, \{y_1, y_2\}\}$ to $\{\{x_1, y_2\}, \{y_1, x_2, x_3\}\}$, which violates *Preference for Accentuating Swaps.*

The theorem below identifies the class of aggregators that are consistent with the axioms. We first define the representation.

Definition 3 *The preference $\succsim$ has a 0-Stage Boundedly Rational Backward Induction (0-BRBI) representation if there exists an expected utility function $U : \mathcal{D} \to \mathbb{R}$ and a continu-*

ously strictly increasing function $f : U(\mathcal{D}) \to \mathbb{R}$ *such that (i) for any* $a_1, a_2 \in \mathcal{D}$, $a_1 \succsim a_2$ *if and only if* $U(a_1) \geq U(a_2)$; *and (ii) for any* $b = \{b_1, \ldots, b_n\} \in D$,

$$U(b) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(U(b_i))\right). \tag{3}$$

Due to equation (3), U and f are not independent. However, if we restrict the domain of U to the set of lotteries D_0, U (defined on D_0) does not depend on f; that is, equation (3) uniquely extends the utility of lotteries to the utility of all finite decision trees. To see how the representation works, for example, suppose again that $a = \{p, q, r\}$ and $b = \{s, a\} = \{s, \{p, q, r\}\}$. Knowing the utility of lotteries p, q, r, s, we can apply (3) to derive the utility of a,

$$U(a) = f^{-1}\left(\frac{1}{3}f(U(p)) + \frac{1}{3}f(U(q)) + \frac{1}{3}f(U(r))\right),$$

which in turn is used to derive the utility of b,

$$\begin{aligned} U(b) &= f^{-1}\left(\frac{1}{2}f(U(s)) + \frac{1}{2}f(U(a))\right) \qquad (4) \\ &= f^{-1}\left(\frac{1}{2}f(U(s)) + \frac{1}{2}f\left(f^{-1}\left(\frac{1}{3}f(U(p)) + \frac{1}{3}f(U(q)) + \frac{1}{3}f(U(r))\right)\right)\right) \\ &= f^{-1}\left(\frac{1}{2}f(U(s)) + \frac{1}{6}f(U(p)) + \frac{1}{6}f(U(q)) + \frac{1}{6}f(U(r))\right). \end{aligned}$$

Theorem 1 *The preference* $\succsim$ *has a* 0*-BRBI representation if and only if* $\succsim$ *satisfies Axioms 1–7. In the* 0*-BRBI representation of a nontrivial* $\succsim$*, the expected utility function* U *is unique up to a positive affine transformation, and fixing any* U*, the function* f *is unique up to a positive affine transformation.*

We offer two interpretations of the aggregator (3). The first interpretation is below, and the second will be introduced in Section 4. In the first interpretation, note that the aggregator in fully rational backward induction is the maximum function; that is, $U(b) = \max_i U(b_i)$ for any decision tree $b = \{b_1, \ldots, b_n\} \in D$. In our model, instead of the maximum function, the aggregator (3) is a general notion of *average*.[9] The decision maker evaluates subtrees *as if* she applies (3) recursively. The average aggregator (3) captures the idea that if the decision maker does not perform backward induction, she evaluates subtrees according to some "foggy" overall impression. As f gets arbitrarily convex (concave), (3) converges to the maximum (minimum) function. For example, suppose $U(D_0) = \mathbb{R}_{++}$. As $\gamma \to \infty$, $f(u) = u^\gamma$ ($\gamma > 0$) yields the maximum function and $f(u) = -u^{-\gamma}$ ($\gamma > 0$) yields the minimum function.

3.1 Nonrecursive Aggregation

Although equation (3) and the previous example of evaluating $b = \{s, a\} = \{s, \{p, q, r\}\}$ suggest that the aggregator works recursively, there is an equivalent but nonrecursive way to apply the aggregator. Nonrecursive aggregation may provide a better interpretation of the representation. In recursive aggregation, it is natural that the decision maker adopts some average aggregator—but perhaps less so that the decision maker does this recursively. In the equivalent nonrecursive aggregation introduced below, the decision maker applies the average aggregator only once to evaluate the decision tree, but the aggregator's weights over (terminal) lotteries will depend on the structure of the tree.

Consider $b = \{s, a\} = \{s, \{p, q, r\}\}$ again. As shown by the third equality of equation (4), rather than first evaluating a and then b recursively, an alternative way to evaluate b is to aggregate the utility b's lotteries s, p, q, r directly:

$$U(b) = f^{-1}\left(\frac{1}{2}f(U(s)) + \frac{1}{6}f(U(p)) + \frac{1}{6}f(U(q)) + \frac{1}{6}f(U(r))\right).$$

Note that instead of using the uniform weights ($1/n$) as in equation (3), each lottery's weight now depends on the structure of the decision tree. In this example, s has weight $1/2$ and the other lotteries have weight $1/6$.

Intuitively, we can think of the weight of a lottery as the share of attention that the

[9] In mathematics, Kolmogorov (1930) proposes an equation that is similar to (3) to define average.

lottery receives. The decision maker's attention is equally split at every node of a decision tree. Therefore, s will receive half of the attention and p, q, r will each receive 1/6 of the attention. This observation can be easily generalized.

3.2 Sketch of the Proof

The construction of the function f is similar to how one calibrates an expected utility function from the data on certainty equivalents for 50-50 gambles (see Machina (1987)). Recall that $D_0 = \Delta(X)$ is the set of Borel probability measures on a compact set X. The axiom *vNM* states that $\succsim$ has an expected utility representation on D_0. Let $U : D_0 \to \mathbb{R}$ be an expected utility representation of $\succsim$ on D_0. Since U is continuous and X is compact, find one of the best lotteries p_h and one of the worst lotteries p_l. Consider the nontrivial case in which $U(p_h) = 1$ and $U(p_l) = 0$.

From *Disjoint Set Betweenness*, one can show recursively that for any subtree c, $p_h \succsim c \succsim p_l$. To see this, for example, consider a depth-1 decision tree $a = \{p, q\} \in D_1$ and suppose $p \succsim q$. By *Disjoint Set Betweenness*, $p_h \succsim p \succsim a \succsim q \succsim p_l$. Next, consider some depth-2 decision tree $b = \{r, a\}$. As explained in Lemma 1, *Monotonicity* implies the recursivity of the representation; that is, if for some lottery s, $a \sim s$, then *Monotonicity* implies $b \sim \{s, r\}$. Since we already know that $p_h \succsim a \succsim p_l$, we know that s exists. Then, we can apply *Disjoint Set Betweenness* again to show that $p_h \succsim b \succsim p_l$. By induction, we can show for every decision tree c, $p_h \succsim c \succsim p_l$, which implies that we can find some α_c such that $\alpha_c p_h + (1 - \alpha_c)p_l \sim c$. Let $U(c) = \alpha_c$. Now, we have extended the domain of U from D_0 to $\mathcal{D}$. The question is whether the function U is consistent with equation (3).

We construct f as follows. Let $f(0) = f(U(p_l)) = 0$ and $f(1) = f(U(p_h)) = 1$. Define

$$f(U(\{p_h, p_l\})) := \frac{1}{2}f(U(p_h)) + \frac{1}{2}f(V(p_l)) = \frac{1}{2};$$

that is, if, for example, $U(\{p_h, p_l\}) = 1/5$, the equation above defines $f(1/5)$ to be $1/2$. To see why this construction of f is similar to the calibration of an expected utility function, think of $U(p_h)$ and $U(p_l)$ as x dollars and y dollars; $U(\{p_h, p_l\})$ as the certainty equivalent of the 50-50 gamble between x and y; and f as the expected utility function. Then, the equation above is similar to stating that the utility of the certainty equivalent is equal to the expected utility expression on the right-hand side.

Next, consider $\{p_h, \{p_h, p_l\}\}$ and define

$$f(U(\{p_h, \{p_h, p_l\}\})) := \frac{1}{2} f(U(\{p_h, p_l\})) + \frac{1}{2} f(U(p_h)) = \frac{3}{4}.$$

Similarly, consider $\{p_l, \{p_h, p_l\}\}$ and set $f(U(\{p_l, \{p_h, p_l\}\})) = \frac{1}{2} f(U(\{p_h, p_l\})) + \frac{1}{2} f(U(p_l)) = \frac{1}{4}$. We can continue in this fashion and define f on some subset of $[0, 1]$. Denote all the binary decision trees of this kind (whose terminal lotteries are either p_h or p_l) by $\hat{D}$. The subset of $[0, 1]$ on which f has been defined is $U\left(\hat{D}\right)$.

We want to verify that equation (3) holds on $\hat{D}$, and a key consequence of equation (3) to be verified is a bisymmetry property. Consider two binary decision trees, $\{\{a, b\}, \{c, d\}\}$ and $\{\{a, c\}, \{b, d\}\}$, that belong to $\hat{D}$. For equation (3) to hold on $\hat{D}$, it must be true that

$$\{\{a, b\}, \{c, d\}\} \sim \{\{a, c\}, \{b, d\}\}, \tag{5}$$

because

$$U(\{\{a, b\}, \{c, d\}\}) = U(\{\{a, c\}, \{b, d\}\}) = f^{-1}\left(\frac{1}{4} f(U(a)) + \cdots + \frac{1}{4} f(U(d))\right).$$

Preference for Accentuating Swaps ensures that (5) holds. Consider $\{\{a, b\}, \{c, d\}\}$ and suppose $b \succsim c$. First, find lotteries $p_1, \ldots, p_4$ such that $p_1 \sim a, \ldots, p_4 \sim d$. Thus, $p_2 \succsim p_3$ By *Monotonicity*, $\{\{a, b\}, \{c, d\}\} \sim \{\{p_1, p_2\}, \{p_3, p_4\}\}$ and $\{\{a, c\}, \{b, d\}\} \sim \{\{p_1, p_3\}, \{p_2, p_4\}\}$. According to *Preference for Accentuating Swaps*, since $|\{p_3, p_4\}| \geq |\{p_1, p_2\}|$, an accentuating swap of p_2 for p_3 should be weakly preferred to $\{\{p_1, p_2\}, \{p_3, p_4\}\}$. Therefore,

$\{\{p_1,p_3\},\{p_2,p_4\}\} \succsim \{\{p_1,p_2\},\{p_3,p_4\}\}$. However, we can swap p_2 back for p_3, and apply *Preference for Accentuating Swaps* again to conclude that $\{\{p_1,p_2\},\{p_3,p_4\}\} \succsim \{\{p_1,p_3\},\{p_2,p_4\}\}$. Thus, we have (5).

Using *Monotonicity* and *Disjoint Set Betweenness*, it can be shown that the subset must be dense. By *Continuity*, we can extend the domain of f to the entire set $[0,1]$. The construction so far only deals with binary decision trees $\hat{D}$. In the last step, we show that (3) holds not only for $\hat{D}$, but also for all finite decision trees under the same f function.

4 κ-Stage Perfect Foresight

To accommodate violations of fully rational backward induction, many economists have considered the idea that the decision maker can perform fully rational backward induction only for a few stages, and then she uses some other aggregator/value function to evaluate subtrees beyond her perfect foresight (see footnote 2). Below, we characterize the model in which the decision maker has perfect foresight for κ stages, and uses the aggregator (3) to evaluate subtrees beyond κ stages.

To state the axioms, we first introduce some terminology and notations. For each decision tree $a \in D$, we use $\pi(a)$ to denote the set of lotteries that can be reached by making a series of choices in a. Since a is finite, $\pi(a)$ is finite. Because the preference $\succsim$ is complete and transitive, we can find the best lotteries of $\pi(a)$. Denote one of the best lotteries among $\pi(a)$ by $\bar{\pi}(a)$. For a lottery $p \in D_0$, let $\pi(p) = \{\bar{\pi}(p)\} = \{p\}$.

Confronting $a \in \mathcal{D}$, we say that a subtree b is a *j-stage subtree* of a if

$$b \in a^{(j-1)} \in \cdots \in a^{(1)} \in a.$$

We say that a is a 0-stage subtree of a. Next, suppose b is a j-stage subtree of a. We use $\phi_b^c(a) \in \mathcal{D}$ to denote the tree that differs from a only by replacing b with a subtree c (see

Figure 10).[10] By definition, c is a j-stage subtree of $\phi_b^c(a)$.

We first introduce a simple axiom that allows us to identify κ.

Axiom 8 *(Limited Perfect Foresight) For some $j \in \mathbb{N}$, if $a \sim \bar{\pi}(a)$ for all $a \in D_j$, then $b \in D_k$ and $k \leq j$ imply $b \sim \bar{\pi}(b)$.*

The axiom says that if a decision maker can solve all depth-j decision trees perfectly, she can also solve any depth-k decision tree perfectly as long as $k \leq j$. This axiom enables us to define κ and a preference $\succsim^*$ for subtrees beyond the decision maker's perfect foresight.

Definition 4 *Let κ be the number such that (i) for any $a \in D_\kappa$, $\bar{\pi}(a) \sim a$; and (ii) there exists some $b \in D_{\kappa+1}$ such that $\bar{\pi}(b) \not\sim b$. When $\kappa < +\infty$, we write $b \succsim^* c$ if for any $j \geq \kappa$ and $a \in \mathcal{D}$ such that b is a j-stage subtree of a, $a \succsim \phi_b^c(a)$. When $\kappa = +\infty$, $b \succsim^* c$ for any $b, c \in \mathcal{D}$.*

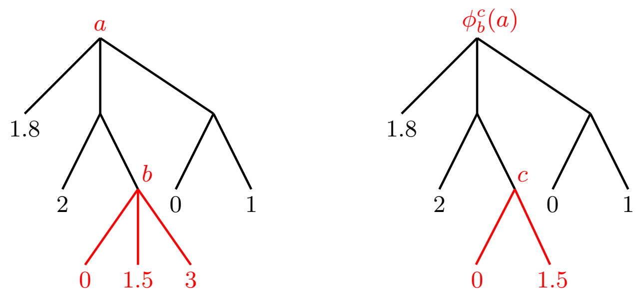

Figure 10: By replacing the 2-stage subtree $b = \{0, 1.5, 3\}$ of a with $c = \{0, 1.5\}$, we obtain a new decision tree $\phi_b^c(a)$. Suppose $\kappa \leq 2$. Then, $b \succsim^* c$ implies that $a \succsim \phi_b^c(a)$.

Since $p \sim \bar{\pi}(p)$ for any lottery p, we know that $\kappa \geq 0$. If $\kappa = +\infty$, the decision maker performs fully rational backward induction. The definition of $\succsim^*$ implies that if $b \succsim^* c$, replacing a subtree b beyond perfect foresight with another subtree c always makes the original decision tree worse. The next axiom requires that $\succsim^*$ be well-behaved.

[10] More precisely, $\phi_b^c(a)$ should also keep track of the choice path in a that leads to b; that is, $\phi_b^c(a)$ should also depend on $a^{(j-1)}, \ldots, a^{(1)}$ if we are considering subtree $b \in a^{(j-1)} \in \cdots \in a^{(1)} \in a$. This is because in general, subtree b may appear multiple times in a decision tree a. For simplicity, we omit the choice path that leads to b in the notation.

Axiom 9 *(κ-Consistent Weak Order) $\succsim^*$ is complete, transitive, and for any $p, q \in D_0$, $p \succsim q$ if and only if $p \succsim^* q$.*

This axiom implies that if replacing a j-stage ($j \geq \kappa$) subtree b of a with another subtree c makes a better, then whenever b is beyond perfect foresight in *any* decision tree, replacing b with c improves the tree. Moreover, $\succsim^*$ coincides with $\succsim$ on D_0; that is, the risk attitude stays constant no matter where the lotteries are located in decision trees.

Lastly, we impose axioms from Section 3 to $\succsim^*$.

Axiom 10 *(κ-Aggregator) $\succsim^*$ satisfies Axioms 3–7.*

Axioms 1 and 2 together with the axioms in this section lead to the following representation of the decision maker's preference.

Definition 5 *The preference $\succsim$ has a κ-Stage Boundedly Rational Backward Induction (κ-BRBI) representation if there exists a constant $\kappa \in \mathbb{N} \cup \{+\infty\}$; expected utility functions $U^* : \mathcal{D} \to \mathbb{R}$ and $U_j : \mathcal{D} \to \mathbb{R}$ for each $j \leq \kappa$; and a continuously strictly increasing function $f : U^*(\mathcal{D}) \to \mathbb{R}$ such that*

(i) for any $a, a' \in \mathcal{D}$, $a \succsim a'$ if and only if $U_0(a) \geq U_0(a')$;

(ii) $U_j(p) = U^*(p)$ for any $j \leq \kappa$ and $p \in D_0$;

(iii) if $b = \{b_1, \ldots, b_n\} \in D$ is a j-stage subtree of a decision tree c,

$$\begin{cases} U_j(b) = \max_i U_{j+1}(b_i) & \text{if } j < \kappa, \\ U_j(b) = U^*(b) = f^{-1}\left(\frac{1}{n}\sum_i f(U^*(b_i))\right) & \text{if } j = \kappa, \\ U^*(b) = f^{-1}\left(\frac{1}{n}\sum_i f(U^*(b_i))\right) & \text{if } j > \kappa. \end{cases} \tag{6}$$

Part (i) of the definition says that U_0 is the utility function that represents the preference. Part (ii) implies that no matter where a lottery is located in a decision tree, its expected utility does not change. Hence, let us define an expected utility function for lotteries; that is, for any κ-BRBI representation, we define $U : D_0 \to R$ to be the function such that for any $p \in D_0$,

$$U(p) := U_0(p) = \cdots = U_\kappa(p) = U^*(p). \tag{7}$$

What the structure of a decision tree affects is how the decision maker aggregates the utility of lotteries, which is reflected in part (iii). The first case in part (iii) is the aggregator within the decision maker's perfect foresight, in which the utility of a j-stage subtree b depends on j. To see why, suppose $b = \{s, \{p, q, r\}\}$ and $\kappa = 2$. If b is a 0-stage subtree, the utility of b should be equal to its best lottery's utility. However, if b is a 1-stage subtree, b's subtree $\{p, q, r\}$ is beyond the decision maker's perfect foresight. Hence, the utility of b may differ from its best lottery's utility. The second case in part (iii) is the transition stage, in which b is at the boundary of perfect foresight, and b's subtrees are beyond perfect foresight. Beyond perfect foresight, the utility function U^* is used instead, which is no longer aggregated through the maximum function. Lastly, note that although part (iii) works recursively, our discussion in Section 3.1 again applies.

To see how the representation works, consider again, for example, $a = \{p, q, r\}$ and $b = \{s, a\} = \{s, \{p, q, r\}\}$. We derive the utility of b. The case of $\kappa = 0$ is identical to the example in the previous section, except for some notational differences.[11] Suppose $\kappa = 1$. Since a is a 1-stage subtree of b, the second case of the definition's part (iii) implies that

$$U_1(a) = U^*(a) = f^{-1}\left(\frac{1}{3}f(U(p)) + \frac{1}{3}f(U(q)) + \frac{1}{3}f(U(r))\right).$$

However, for the 0-stage subtree b (of b), we will apply the first case of the definition's part (iii):

$$U_0(b) = \max\{U(s), U_1(a)\}.$$

The theorem below establishes the equivalence between the κ-BRBI representation and

[11] Suppose $\kappa = 0$. First, because b is a 0-stage subtree of b, the second case of the definition's part (iii) implies that $U_0(b) = U^*(b) = f^{-1}\left(\frac{1}{2}f(U^*(s)) + \frac{1}{2}f(U^*(a))\right)$. Since s is a lottery, part (ii) of the definition and equation (7) imply that $U^*(s) = U(s)$. The same applies to lotteries p, q, r. Next, a is a 1-stage subtree of b. The third case of the definition's part (iii) applies: $U^*(a) = f^{-1}\left(\frac{1}{3}f(U^*(p)) + \frac{1}{3}f(U^*(q)) + \frac{1}{3}f(U^*(r))\right) = f^{-1}\left(\frac{1}{3}f(U(p)) + \frac{1}{3}f(U(q)) + \frac{1}{3}f(U(r))\right)$, which can be plugged into the equation for $U_0(b)$ above.

the axioms.

Theorem 2 *The preference $\succsim$ has a κ-BRBI representation if and only if $\succsim$ satisfies Axioms 1, 2, and 8–10. In the κ-BRBI representation of a nontrivial $\succsim$, κ is unique; the expected utility functions U_j's ($j \leq \kappa$) and U^* are unique up to a positive affine transformation; and fixing U_j's and U^*, the function f is unique up to a positive affine transformation.*

Now, we introduce the second interpretation of the κ-BRBI representation. In this interpretation of (6), it is *as if* the decision maker knows the structure of a decision tree but is unable to predict or does not trust her future selves' choices beyond κ stages. Conceptually, knowing the structure of a decision tree does not imply that she can identify the optimal path and keep track of it. In particular, when evaluating a decision tree b, she envisions that beyond κ stages, she will choose uniformly randomly among $a_1, \ldots, a_n$ for each j-stage subtree $a = \{a_1, \ldots, a_n\}$ of b ($j \geq \kappa$). As discussed previously, such a belief is called *Laplacian*; it captures the idea that the uniform prior should be applied to unknown events based on "the principle of insufficient reason" (Laplace (1824)). The Laplacian belief is biased/unsophisticated, because the decision maker does not actually randomize at future stages. The decision maker's attitude toward the Laplacian belief is captured by $f \circ U^*$, because (3) implies that

$$f \circ U^*(b) = \sum_{i=1}^{n} \frac{1}{n} \cdot f \circ U^*(b_i).$$

Therefore, f is a *second-order expected utility function*, and captures the difference between how the decision maker treats objective risk (captured by U^*) and the subjective uncertainty perceived in the Laplacian belief. As will be shown, the concavity of f describes the decision maker's *unpredictability aversion*, the same way the concavity of the expected utility function describes the decision maker's risk aversion.

Note that by definition, $U_j(p) = U^*(p)$ for any lottery p and $j \leq \kappa$. Therefore, U_j's and U^* are jointly unique up to a positive affine transformation. Similar to the case with $\kappa = 0$ in the previous section, κ and f uniquely extend the utility of lotteries to the utility of decision trees. The three parameters—the expected utility function for lotteries U as defined in equation (7), κ, and f—are independent. When the decision maker's preference can be represented by the κ-BRBI representation, as in Definition 5, we say that (U, κ, f) represents $\succsim$.

4.1 A Comparative Measure of Unpredictability Aversion

In the second interpretation of the κ-BRBI representation, the decision maker is unable to predict her choices beyond κ stages. However, if what lies beyond κ stages is a degenerate subtree, the decision maker does not need to make any prediction. Suppose there are two decision makers, labeled 1 and 2, who only look forward 0 stages perfectly for simplicity; that is, they are unable to predict their future choices in any nondegenerate decision tree. Confronting the same lottery and decision tree, if compared to decision maker 1, decision maker 2 is always more inclined to choose the lottery over the decision tree, decision maker 2 reveals that she is more averse to the situation in which she cannot make predictions. This idea can be extended to the case in which both decision makers can look forward more stages perfectly.

Formally, suppose $\succsim_i$ is decision maker i's preference. Recall that we write $b \succsim_i^* c$ if decision maker i always prefers to replace subtree c with b whenever c is beyond her perfect foresight.

Definition 6 *$\succsim_2$ is more unpredictability-averse than $\succsim_1$ if for any $p \in D_0$, $a \in \mathcal{D}$, $a \succsim_2^* p$ implies $a \succsim_1^* p$.*

We say that a function f_2 is more concave than f_1 if $f_2 = g \circ f_1$ for some strictly increasing and concave function g. The following theorem characterizes the comparative measure of unpredictability aversion.

Theorem 3 *Suppose $\left(\hat{U}_i, \kappa_i, \hat{f}_i\right)$ represents the nontrivial preference $\succsim_i$ and $\kappa_i < +\infty$. Then, $\succsim_2$ is more unpredictability-averse than $\succsim_1$ if and only if there exist (U, κ_1, f_1) and*

(U, κ_2, f_2) *that represent* $\succsim_1$ *and* $\succsim_2$*, respectively, such that* f_2 *is more concave than* f_1*.*

Therefore, the concavity of f characterizes a decision maker's attitude toward unpredictability, the same way that the concavity of an expected utility function characterizes the decision maker's attitude toward objective risk. Similar to the measure of absolute risk aversion, when f_1 and f_2 are twice differentiable, $-f''/f'$ can be used as the comparative measure of unpredictability aversion.

4.2 Choices in Decision Trees and Iterative Revisions of Plans

Confronting a decision tree $a = \{a_1, \dots, a_n\}$, the most preferred subtree a_i should be chosen. If a_i is not a lottery, it is natural to assume that the decision maker will continue to choose her most preferred subtree from $a_i = \{b_1, \dots, b_m\}$, and so on. Thus, we can apply the preference iteratively to describe the decision maker's choices (see Figure 4 for example). By doing so, we have assumed *history independence*: The decision maker's preference over subtrees does not depend on her past choices.

Under history independence, the decision maker's perfect foresight moves forward as she makes choices. She can see the first κ stages of a perfectly when choosing from $a = \{a_1, \dots, a_n\}$. After choosing $a_i \in a$, she will be able to see the $(\kappa + 1)^{\text{th}}$ stage of a perfectly. Therefore, the decision maker may iteratively revise her plan. In Figure 4, initially, the decision maker may believe that she will choose 2 at the next stage, but at the next stage, she realizes that 2 is not optimal.

Intuitively, at every stage, the decision maker has an "optimal" plan for the next κ choices following the current one, and the current one constitutes the first step of the plan. Our revealed preference theory, together with history independence, thus characterizes a decision maker who does not realize that her actual future choices may differ from her plan. The decision maker mistakenly believes that she is able to control her future choices, and as she makes choices, she may even revise what she believes she will do in the future. Therefore, our approach is similar to an approach to modeling choices under imperfect foresight discussed in Section 4 of Chapter 7 in Rubinstein (1998).

This approach is in contrast with an alternative approach discussed in Rubinstein (1998), the "multiselves" approach first suggested by Strotz (1955). The multiselves approach assumes that the decision maker at each stage chooses the utility-maximizing subtree *taking her next κ selves' strategies as given.* The decision maker becomes a set of "selves" whose strategies form some equilibrium. The equilibrium describes how the decision maker chooses in the decision tree. For example, in Jéhiel (1995), at each stage, a player of an infinite-horizon alternate-move game forms a finite forecast of the moves of the player's and her opponent's future selves. Although the forecast is limited, the equilibrium requires that the forecast be correct.

Both approaches have pros and cons. In addition to Rubinstein's (1998) discussion of why the multiselves approach may be inappropriate, it is also not clear whether the decision maker's first self should know her second self's strategy, because the second self's strategy is determined based on the $(\kappa+1)^{\text{th}}$ self's strategy, which the first self should not know.[12] For example, when an amateur plays a Rubik's Cube, she often has an initial plan for the next few moves, but as the configuration of the Rubik's Cube changes, the plan may no longer be appealing. This seems consistent with our approach. On the other hand, Rubinstein (1998) points out a difficulty in our approach that will arise in games: Since the decision maker does not know her own future choices, it is not clear whether we should assume that in equilibrium the decision maker knows the strategies of opponents who move after her.

Iterative revisions of plans lead to *time inconsistency.* Many other models generate time inconsistency too, such as models of changing tastes (see Strotz (1955)). There are three main differences between models of changing tastes and ours. First, in our model, the utility function over lotteries never changes. Therefore, our model has a clear welfare criterion. In contrast, the utility function over lotteries in models of changing tastes may change over

[12]However, see Jéhiel (1998) for a learning foundation for this approach.

time, and it is less clear how to choose the welfare criterion. Second, in our model, dominated lotteries affect the decision maker's evaluation and choices, because all lotteries beyond the perfect foresight are aggregated with equal weights. In models of changing tastes, if a lottery is always dominated regardless of tastes (for example, a degenerate lottery of losing a billion dollars in the presence of a degenerate lottery of losing zero dollars), it should not matter. Third, our decision maker applies the Laplacian belief to her future choices beyond the perfect foresight. In models of changing tastes, the decision maker can have a deterministic and complete plan for her future choices, even though the plan may not match her actual future choices. Lastly, sometimes decision trees can be viewed as frames of decision problems that unfold in a short period of time. In this situation, models of changing tastes may not be suitable.

Models of temptations (see Gul and Pesendorfer (2001), Dekel, Lipman, and Rustichini (2009), and Stovall (2010)) may also generate time inconsistency.[13] First, compared to our model, models of temptations usually implicitly assume that the decision maker is correct about her second-stage (tempted) preferences, while in our model, the decision maker can be wrong about her future choices. Second, the second point in the previous discussion continues to apply to models of temptations. Lastly, it can be shown that our model violates the independence axiom that is often imposed in models of temptations. However, it should be noted that our model satisfies the temptation-related axioms in, for example, Dekel et al. and Stovall. Those axioms are rather weak. Hence, they have temptation-based interpretations, and may also have imperfect-foresight-based interpretations.

5 κ and f in Decision Trees

We use a few examples below to illustrate that first, as κ increases or f becomes more convex, it is not necessarily the case that the decision maker will reach better lotteries in decision

[13]See Dekel and Lipman (2012) for the relation between models of random changing tastes and models of random temptations.

trees; second, a more convex f may be more helpful for the decision maker to reach better lotteries in decision trees with higher depth, and a higher κ, in contrast, may be more helpful in decision trees with lower depth.

As κ increases, the decision maker looks forward more stages perfectly. As f becomes more convex, the decision maker is less unpredictability-averse; that is, her aggregator beyond the perfect foresight is closer to the fully rational decision maker's. Whether a higher κ or a more convex f implies that the decision maker will reach better lotteries, however, depends on the decision tree.[14]

Suppose (U, κ, f) represents the decision maker's preference, $U(x) = x$ whenever $x \in X$ is a real number, $\kappa = 0$, and f is the identity function. Consider the following decision tree,

$$a = \{\{\{3, 1\}, 4\}, \{2, 4 + \varepsilon\}\},$$

in which ε is a small positive number. Since $\{\{3, 1\}, 4\}$ is indifferent to $\{2, 4\}$, the decision maker will choose $\{2, 4+\varepsilon\}$ and end up with $4+\varepsilon$. Now, if f becomes convex, it can be shown that as long as ε is sufficiently small, the decision maker will choose $\{\{3, 1\}, 4\}$ over $\{2, 4+\varepsilon\}$ and end up with 4. This is because when f is convex, the subtree $\{3, 1\}$ from $\{\{3, 1\}, 4\}$ appears better than 2 from $\{2, 4 + \varepsilon\}$. The decision maker with a "better" aggregator gets distracted by 3 and misses $4 + \varepsilon$.

Increasing κ may also cause the decision maker to turn away from the best lottery. Consider another decision tree

$$b = \{\{3, \{2, 4 + \varepsilon\}\}, \{2, 4\}\}.$$

The value of $\{3, \{2, 4 + \varepsilon\}\}$ is $3 + \varepsilon/4$, which is higher than the value of $\{2, 4\}$. Therefore, the decision maker will reach $4 + \varepsilon$. Now, suppose κ becomes 1. The value of $\{3, \{2, 4 + \varepsilon\}\}$ becomes $3 + \varepsilon/2$, while the value of $\{2, 4\}$ becomes 4. The decision maker ends up with 4.

[14] Jéhiel (1995) also finds that decision makers who can see fewer stages may perform better.

The decision maker who sees more stages perfectly gets distracted by 4 and misses $4 + \varepsilon$.

Of course, fixing any $\varepsilon > 0$, if κ becomes sufficiently high or the aggregator becomes close enough to the maximum function, the decision maker should not be worse off. In general, however, the κ-BRBI representation is flexible enough to interact with the decision tree and generate interesting behavioral predictions.

The next example illustrates how κ's role may differ from f's. Suppose the decision maker needs to go to a building. She can either take a taxi or drive. If she takes a taxi, the utility is t. If she drives, there is a series of exits along the way. For simplicity, assume that only one exit leads to the building and the others are equally bad. The utility of the correct exit is 2 and the utility of other exits is 0. By driving and choosing the correct exit, the decision maker obtains the highest utility.

Suppose the following decision tree

$$c = \{t, \underbrace{\{0, \{0, \ldots, \{0,}_{n-1}\{2, \underbrace{\{0, \{0, \ldots, \{0, 0\}\}\}}_{m}\}\}\}\}\}$$

describes this situation.[15] It can be verified that regardless of what κ and f are, once the decision maker chooses to drive herself, she will reach the correct exit. However, the decision maker may not understand at early stages that she is able to do that if $\kappa < n$.

Suppose there are two decision makers. Decision maker 1 has $\kappa_1 = \kappa$ $(0 < \kappa \leq n)$ and $f_1(u) = u$. Decision maker 2 has $\kappa_2 = 0$ and $f_2(u) = u^\gamma$ $(\gamma > 1)$. At the first stage, for decision maker 1, the utility of driving is $2^\kappa \cdot 2^{1-n}$, and for decision maker 2, the utility of driving is $\left(\frac{1}{2^n} 2^\gamma\right)^{1/\gamma} = 2^{1-n/\gamma}$. Let us vary n. The smallest n is $n = \kappa$, in which case

$$2^\kappa \cdot 2^{1-n} = 2 > 2^{1-n/\gamma}.$$

Therefore, there exists some $t \in (0, 2)$ such that decision maker 1 will choose to drive, but

[15] Although eventually m does not play any role, we include the variable m so that this example better fits the description of the problem; that is, it is not necessarily the case that the final exit is the correct one.

decision maker 2 will take a taxi. The situation will change as n increases. Whenever n is larger than $\frac{\kappa}{1-1/\gamma}$, $2^{1-n/\gamma}$ will be higher than $2^{\kappa} \cdot 2^{1-n}$; that is, there exists some $t \in (0,2)$ such that decision maker 2 will drive but decision maker 1 will take a taxi. Hence, intuitively, if n is large, the difference between κ_1 and κ_2 is less important and the decision maker with a more convex f is likely to do better, and vice versa.

References

Aczél, János (1966), *Lectures on Functional Equations and Their Applications*, volume 19. Academic Press.

Ahn, David S. (2008), "Ambiguity Without a State Space." *Review of Economic Studies*, 75, 3–28.

Asheim, Geir B. and Martin Dufwenberg (2003), "Deductive Reasoning in Extensive Games." *Economic Journal*, 113, 305–325.

Binmore, Ken (1999), "Why Experiment in Economics?" *Economic Journal*, 109, F16–F24.

Binmore, Ken, John McCarthy, Giovanni Ponti, Larry Samuelson, and Avner Shaked (2002), "A Backward Induction Experiment." *Journal of Economic Theory*, 104, 48–88.

Bolker, Ethan D. (1966), "Functions Resembling Quotients of Measures." *Transactions of the American Mathematical Society*, 124, 292–312.

Camerer, Colin F., Eric J. Johnson, Talia Rymon, and Sankar Sen (1993), "Cognition and Framing in Sequential Bargaining for Gains and Losses." *Frontiers of Game Theory*, 104, 27–47.

Dekel, Eddie and Barton L. Lipman (2012), "Costly Self-Control and Random Self-Indulgence." *Econometrica*, 80, 1271–1302.

Dekel, Eddie, Barton L. Lipman, and Aldo Rustichini (2009), "Temptation-Driven Preferences." *Review of Economic Studies*, 76, 937–971.

Fudenberg, Drew and Tomasz Strzalecki (2015), "Dynamic Logit with Choice Aversion." *Econometrica*, 83, 651–691.

Gabaix, Xavier, David Laibson, Guillermo Moloche, and Stephen Weinberg (2006), "Costly Information Acquisition: Experimental Analysis of a Boundedly Rational Model." *American Economic Review*, 96, 1043–1068.

Ghirardato, Paolo, Fabio Maccheroni, and Massimo Marinacci (2004), "Differentiating Ambiguity and Ambiguity Attitude." *Journal of Economic Theory*, 118, 133–173.

Gul, Faruk and Wolfgang Pesendorfer (2001), "Temptation and Self-Control." *Econometrica*, 82, 1873–1912.

Güth, Werner and Reinhard Tietz (1990), "Ultimatum Bargaining Behavior: A Survey and Comparison of Experimental Results." *Journal of Economic Psychology*, 11, 417–449.

Jéhiel, Philippe (1995), "Limited Horizon Forecast in Repeated Alternate Games." *Journal of Economic Theory*, 67, 497–519.

Jéhiel, Philippe (1998), "Learning to Play Limited Forecast Equilibria." *Games and Economic Behavior*, 22, 274–298.

Jéhiel, Philippe (2001), "Limited Foresight May Force Cooperation." *Review of Economic Studies*, 68, 369–391.

Ke, Shaowei (2018), "Rational Expectation of Mistakes and a Measure of Error-Proneness." *Theoretical Economics*, 13, 527–552.

Kolmogorov, Andrey N. (1930), *On the Notion of Mean*, 144–146. Kluwer.

Laplace, Pierre-Simon (1824), *Essai Philosophique sur les Probabilités*. Dover (English Translation).

Levitt, Steven D., John A. List, and Sally E. Sadoff (2011), "Checkmate: Exploring Backward Induction among Chess Players." *American Economic Review*, 101, 975–990.

Machina, Mark J. (1987), "Choice under Uncertainty: Problems Solved and Unsolved." *Journal of Economic Perspectives*, 50, 121–154.

Mantovani, Marco (2014), "Limited Backward Induction: Foresight and Behavior in Sequential Games." *mimeo*.

McKelvey, Richard D. and Thomas R. Palfrey (1992), "An Experimental Study of the Centipede Game." *Econometrica*, 60, 803–836.

Palacios-Huerta, Ignacio and Oscar Volij (2009), "Field Centipedes." *American Economic Review*, 99, 1619–1635.

Rampal, Jeevant (2018), "Limited Foresight Equilibrium." *mimeo*.

Rubinstein, Ariel (1990), "New Directions in Economic Theory - Bounded Rationality." In *Foerder Institute for Economic Research, Tel-Aviv University*.

Rubinstein, Ariel (1998), *Modeling Bounded Rationality*. MIT Press.

Stovall, John E. (2010), "Multiple Temptations." *Econometrica*, 78, 349–376.

Strotz, Robert H. (1955), "Myopia and Inconsistency in Dynamic Utility Maximization." *Review of Economic Studies*, 23, 165–180.

Zermelo, Ernst (1913), "Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels." *Proceedings of the Fifth International Congress of Mathematicians*, 2, 501–504.

A Appendix

We first prove Lemma 1, which will be used to establish Theorem 1. Theorem 1, in turn, will be used to establish Theorem 2.

Proof of Lemma 1: We only show that the axioms imply the representation. According to *vNM*, let $U : D_0 \to R$ be a continuous function such that

$$U(p) = \int_X U \; dp$$

represents $\succsim$ on D_0. Since X is compact, $U(D_0)$ is compact and there exists a best lottery and a worst lottery in X. Let us use x^h and x^l to denote the best and the worst lottery, respectively. From the expected utility function U, we construct a utility representation of $\succsim$. To do this, we first prove two lemmas.

Lemma 2 *For any decision tree $a = \{a_1, \ldots, a_n\} \in D$, and distinct lotteries $p_1, \ldots, p_n$. The following statements are true:*

1. *If $p_i \succsim a_i$ for each i, $\{p_1, \ldots, p_n\} \succsim a$;*
2. *If $p_i \succsim a_i$ for each i and for some j, $p_j \succ a_j$, then $\{p_1, \ldots, p_n\} \succ a$;*
3. *If $a_i \succsim p_i$ for each i, $a \succsim \{p_1, \ldots, p_n\}$;*
4. *If $a_i \succsim p_i$ for each i and for some j, $a_j \succ p_j$, then $a \succ \{p_1, \ldots, p_n\}$;*
5. *If $p_i \sim a_i$ for each i, $\{p_1, \ldots, p_n\} \sim a$;*

Proof. We only show the first statement; the rest are similar. Applying *Monotonicity* repeatedly, we have $\{p_1, \ldots, p_n\} \succsim \{a_1, p_2, \ldots, p_n\} \succsim \{a_1, a_2, p_3, \ldots, p_n\} \succsim \cdots \succsim \{a_1, \ldots, a_n\}$. ■

For any decision tree $a = \{a_1, \ldots, a_n\}$, we use $\pi^h(a)$ to denote the best subtree among $a_1, \ldots, a_n$, and use $\pi^l(a)$ to denote the worst subtree among $a_1, \ldots, a_n$.

Lemma 3 *For any decision tree a, $x^h \succsim \pi^h(a) \succsim a \succsim \pi^l(a) \succsim x^l$.*

Proof. We show that for any $a \in D$,

$$x^h \succsim \pi^h(a) \succsim a \succsim \pi^l(a) \succsim x^l \tag{8}$$

by two inductions. First, suppose $a \in D_1$. Clearly, (8) holds if $|a| = 1$. If for any $a \in D_1$ such that $|a| < n$, (8) holds. Take any $b \in D_1$ and $|b| = n$. Let $c = b \setminus \{\pi^h(b)\}$. By defintion, $b = \{\pi^h(b)\} \cup c$ and $|c| < n$. By definition, $\pi^h(b) \succsim \pi^h(c)$. Since $|c| < n$, $\pi^h(c) \succsim c$. It must be true that $\pi^h(b) \succsim c$. By *Indifference to Trivial Extensions*, $\pi^h(b) \sim \{\pi^h(b)\}$. By *Disjoint Set Betweenness*, $\pi^h(b) \sim \{\pi^h(b)\} \succsim b \succsim c$. We know that $x^h \succsim \pi^h(b)$ and $\pi^h(c) \succsim c \succsim \pi^l(c) \succsim \pi^l(b) \succsim x^l$. Therefore, (8) holds for b. By induction, we know that for any $a \in D_1$, (8) holds.

Next, we show that for any $a \in D$, (8) holds. Suppose that for some $1 < k \in \mathbb{N}$, we have shown that for any $b \in D_j$ such that $j < k$, (8) holds. We want to show that for any $a = \{a_1, \ldots, a_n\} \in D_k$, (8) holds. By definition, each $a_i \in a$ must belong to one and only one D_{k_i} such that $k_i < k$. Therefore, $x^h \succsim a_i \succsim x^l$. Clearly, there exists some $\alpha_i \in [0, 1]$ such that

$$a_i \sim \alpha_i x^h + (1 - \alpha_i)x^l =: \hat{p}_i. \tag{9}$$

We want to apply Lemma 2 to show that a is indifferent to some depth-1 decision tree. At this point, it is possible that there are some $a_i, a_j \in a$ such that $a_i \sim a_j$, which means that $\hat{p}_i$ and $\hat{p}_j$ are identical and prevents us from applying Lemma 2. However, by richness of X, there are countably infinitely many elements of X that are indifferent to x^h and x^l, respectively. Therefore, we can always find different $x_i^h, x_i^l \in X$ such that $x_i^h \sim x^h$ and $x_i^l \sim x^l$ to construct the mixture in (9) for each a_i. Denote the lottery $\alpha_i x_i^h + (1 - \alpha_i)x_i^l$ by p_i. We can ensure that $p_1, \ldots, p_n$ are distinct. By Lemma 2,

$$a \sim \{p_1, \ldots, p_n\} =: b.$$

Note that $b \in D_1$. Therefore, $x^h \succsim \pi^h(b) \succsim b \succsim \pi^l(b) \succsim x^l$. Since $\pi^h(b) \sim \pi^h(a)$ and $\pi^l(b) \sim \pi^l(a)$, we know that (8) holds for any $a \in D_k$. By induction, for any $a \in D$, (8) holds. ■

Therefore, for each decision tree $a \in D$, we can find a unique $\alpha \in [0,1]$ such that $a \sim \alpha x^h + (1-\alpha)x^l$. Define $U(a) := \alpha U(x^h) + (1-\alpha)U(x^l)$. Then, we have extended U's domain from D_0 to $\mathcal{D}$. It is straightforward to verify that U represents $\succsim$ on $\mathcal{D}$.

Next, we want to show that there exist a sequence of continuously strictly increasing symmetric functions $g_n : U(\mathcal{D})^n \to \mathbb{R}$ for $n \in \mathbb{N}$ such that for any $b = \{b_1, \ldots, b_n\} \in D$,

$$U(b) = g_n(U(b_1), \ldots, U(b_n)). \tag{10}$$

If we can show (10), it is automatically true that for any $u_1, \ldots, u_n \in U(\mathcal{D})$,

$$\min u_i \leq g_n(u_1, \ldots, u_n) \leq \max u_i,$$

because by Lemma 3, for any $b = \{b_1, \ldots, b_n\} \in D$, $\min U(b_i) \leq U(b) \leq \max U(b_i)$.

Lemma 2 ensures that g_n is well defined, because for any $a, b \in D$ such that $a \neq b$ and $U(a_i) = U(b_i)$, we must have $U(a) = U(b)$. Lemma 2 also implies that g_n is increasing. Since decision trees are defined using sets, the function g_n's are clearly symmetric. *Continuity* implies that g_n is continuous. To see this, suppose that in $U(D_0)^n$, a sequence of n-tuples $\left(\left(u_1^{(j)}, \ldots, u_n^{(j)}\right)\right)_{j=1}^{\infty}$ converges to some $(u_1, \ldots, u_n)$. For each $\left(u_1^{(j)}, \ldots, u_n^{(j)}\right)$ and $(u_1, \ldots, u_n)$, we can find an n-tuple of lotteries giving the desired n-tuple of utility such that each lottery's support is $\{x^h, x^l\}$. It can be verified that these lotteries converge in d^n. Then, standard arguments will follow.

■

Proof of Theorem 1: We first show that the axioms imply the representation. First, we can apply Lemma 1 and know that there exist a sequence of continuously strictly increasing functions $g_n : U(\mathcal{D})^n \to \mathbb{R}$ for $n \in \mathbb{N}$ and an expected utility function $U : \mathcal{D} \to \mathbb{R}$ such that (i) U represents $\succsim$, (ii) for any $b = \{b_1, \ldots, b_n\} \in D$, $U(b) = g_n(U(b_1), \ldots, U(b_n))$, and (iii) $\min u_i \leq g_n(u_1, \ldots, u_n) \leq \max u_i$ for any $u_i \in U(\mathcal{D})$. Note that part (iii) implies that $g_1(u) = u$ and $g_n(u, \ldots, u) = u$ for any $u \in U(\mathcal{D})$.

Let us first focus on g_2. We use *Preference for Accentuating Swaps* to prove a property of g_2 in the following lemma. Next, we will apply a result from Aczél (1966), which shows that there exists a continuously strictly increasing function $f : U(\mathcal{D}) \to \mathbb{R}$ such that

$$g_2(u_1, u_2) = f^{-1}\left(\frac{1}{2}f(u_1) + \frac{1}{2}f(u_2)\right). \tag{11}$$

The main steps of Aczél's proof are described in Section 3.2.

Lemma 4 *For any $u_1, \ldots, u_4 \in U(\mathcal{D})$, $g_2(g_2(u_1, u_2), g_2(u_3, u_4)) = g_2(g_2(u_1, u_3), g_2(u_2, u_4))$.*

Proof. For any $u_1, \ldots, u_4 \in U(\mathcal{D})$, by the richness assumption of X, we can find four distinct lotteries $p_1, \ldots, p_4$ such that $U(p_i) = u_i$. Consider the depth-1 decision trees $a = \{\{p_1, p_2\}, \{p_3, p_4\}\}$ and $b = \{\{p_1, p_3\}, \{p_2, p_4\}\}$. Without loss of generality, assume that $u_2 \geq u_3$; that is, $p_2 \succsim p_3$. Since $|\{p_1, p_2\}| \geq |\{p_3, p_4\}|$, $b = \Delta_{p_3}^{p_2}(a)$. Therefore, by *Preference for Accentuating Swaps*, $b \succsim a$. Now, also note that $|\{p_2, p_4\}| \geq |\{p_1, p_3\}|$, and hence $a = \Delta_{p_3}^{p_2}(b)$. We apply *Preference for Accentuating Swaps* again, and find that $a \succsim b$. Since $a \sim b$,

$$\begin{aligned} U(a) &= g_2(U(\{p_1, p_2\}), U(\{p_3, p_4\}) \\ &= g_2(g_2(u_1, u_2), g_2(u_3, u_4)) \\ &= U(b) = g_2(U(\{p_1, p_3\}), U(\{p_2, p_4\}) \\ &= g_2(g_2(u_1, u_3), g_2(u_2, u_4)). \end{aligned}$$

■

Therefore, g_2 is symmetric, strictly increasing, and continuous, and satisfies $g_2(u, u) = u$

and $g_2(g_2(u_1,u_2),g_2(u_3,u_4)) = g_2(g_2(u_1,u_3),g_2(u_2,u_4))$. If $\succsim$ is trivial, that is, $U(\mathcal{D})$ consists of only one number, then equation (11) is trivially true for any f. Otherwise, according to Aczél (1966), we know that there exists a continuously strictly increasing function $f : U(\mathcal{D}) \to \mathbb{R}$ such that (11) holds. Thus, for any decision tree $a = \{a_1, a_2\} \in D$,

$$U(a) = g_2(U(a_1), U(a_2)) = f^{-1}\left(\frac{1}{2}f(U(a_1)) + \frac{1}{2}f(U(a_2))\right).$$

Since $g_1(u) = u$, for any $b = \{b'\} \in D$, it is trivially true that $U(b) = f^{-1}(f(U(b')))$. Hence,

$$g_n(u_1,\ldots,u_n) = f^{-1}\left(\frac{1}{n}\sum_i f(u_i)\right) \tag{12}$$

holds for $n = 1, 2$.

To prove (12) for the case of $n > 2$, we need the following two lemmas. To state the first lemma, let us extend our definition of swaps. For any $a = \{a_1, a_2, \ldots, a_n\} \in D$ such that $a_1, a_2 \in D$, $|a_1| \geq |a_2|$, $b \in a_1 \backslash a_2$, $c \in a_2 \backslash a_1$, $a_1' := a_1 \backslash \{b\} \cup \{c\} \notin \{a_3, \ldots, a_n\}$, and $a_2' := a_2 \backslash \{c\} \cup \{b\} \notin \{a_3, \ldots, a_n\}$, a swap of b for c is

$$\Delta_c^b(a) := a \backslash \{a_1, a_2\} \cup \{a_1', a_2'\}.$$

When we write $\Delta_c^b(a)$ to denote the swap of b for c, implicitly we have imposed the assumptions in the definition. The difference between this definition of swaps and our original definition in Section 3 is that in Section 3, we require swaps to be defined only for some depth-2 decision trees.

Lemma 5 *For any decision tree $a = \{a_1, \ldots, a_n\}$ such that $b \in a_1 \backslash a_2$, $c \in a_2 \backslash a_1$, and $|a_1| \geq |a_2|$, if $b \succsim c$, then $\Delta_c^b(a) \succsim a$. Moreover, if $|a_1| = |a_2|$, then $\Delta_c^b(a) \sim a$.*

Proof. Let us relabel subtrees b and c by b_1 and c_1, respectively. Suppose $a = \{a_1, \ldots, a_n\}$, $a_1 = \{b_1, \ldots, b_{m_1}\}$ and $a_2 = \{c_1, \ldots, c_{m_2}\}$. We know that $m_1 \geq m_2$ and $b_1 \succsim c_1$. By richness of X, we find distinct lotteries $p_3, \ldots, p_n$ such that $p_i \sim a_i$ for $i = 3, \ldots, n$. Then, we find distinct lotteries $q_1, \ldots, q_{m_1}$ and $r_1, \ldots, r_{m_2}$ such that $q_j \sim b_j$ for $j = 1, \ldots, m_1$ and $r_k \sim c_k$ for $k = 1, \ldots, m_2$. By Lemma 2, $a_1 \sim \{q_1, \ldots, q_{m_1}\} =: \hat{a}_1$, and $a_2 \sim \{r_1, \ldots, r_{m_2}\} =: \hat{a}_2$. Again, by Lemma 2, we know that $a \sim \{\hat{a}_1, \hat{a}_2, p_3, \ldots, p_n\} =: \hat{a}$. Clearly, $\hat{a} \in D_2$. Since $b_1 \succsim c_1$, $q_1 \succsim r_1$. Then, we can apply *Preference for Accentuating Swaps* and find that $\Delta^{q_1}_{r_1}(\hat{a}) \succsim \hat{a} \sim a$. Lastly, showing that $\Delta^{q_1}_{r_1}(\hat{a}) \sim \Delta^b_c(a)$ is similar to how we show that $\hat{a} \sim a$.

Next, suppose $|a_1| = |a_2|$. We still have $\Delta^b_c(a) \succsim a$, as shown in the previous paragraph. Moreover, in $\Delta^b_c(a) = \{a'_1, a'_2, a_3, \ldots, a_n\}$, we know that $|a'_1| = |a'_2|$, $b \in a'_2 \backslash a'_1$, $c \in a'_1 \backslash a'_2$. Then, we can apply the argument from the previous paragraph again, and show that $\Delta^b_c(\Delta^b_c(a)) \succsim \Delta^b_c(a)$. Obviously, $\Delta^b_c(\Delta^b_c(a)) = a$. Therefore, $\Delta^b_c(a) \sim a$. ■

Lemma 6 *Suppose $a = \{a_1, \ldots, a_n\}$ is a decision tree such that (i) for some $m \geq 1$, $a_i = \{a_{i,1}, \ldots, a_{i,m}\}$ for every $i = 1, \ldots, n$, and (ii) $a_i \cap a_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$. Then, $a \sim \bigcup_{i=1}^n a_i$.*

Proof. First, we find for each $a_{i,k}$, $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, $n \times m$ distinct copies of lotteries that are indifferent to $a_{i,k}$; that is, by richness of X, we find $\{p^\tau_{i,k}\}_{\tau=1}^{n \times m} \subset \Delta(X)$ for each $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ such that (i) $p^\tau_{i,k} \sim a_{i,k}$ for all i, k, τ, and (ii) $p^\tau_{i,k}$'s are distinct for all i, k, τ. Define $a^\tau_i := \{p^\tau_{i,1}, \ldots, p^\tau_{i,m}\}$ for each i, τ, and define $a^\tau := \{a^\tau_1, \ldots, a^\tau_n\}$ for each τ. According to Lemma 2,

$$a_i \sim a^\tau_i \tag{13}$$

for each i, τ, and $a \sim a^\tau$ for each τ. Also according to Lemma 2, $\{a^1, \ldots, a^{mn}\} \sim \{a\}$. By *Indifference to Trivial Extensions*, $a \sim \{a\}$. Therefore, $a \sim \{a^1, \ldots, a^{mn}\} =: b$.

Note that by construction, every a^σ_i is a 2-stage subtree of b, since $a^\sigma_i \in a^\sigma \in b$. Consider any a^σ and a^τ, $\sigma, \tau \in \{1, \ldots, mn\}$. Note that $|a^\sigma| = |a^\tau| = n$. Therefore, according to the second part of Lemma 5, $\Delta^{a^\sigma_i}_{a^\tau_j}(b) \sim b$ for any $i, j \in \{1, \ldots, n\}$; that is, if we swap any 2-stage subtree a^σ_i of b for another 2-stage subtree a^τ_j of b, the swapped decision tree $\Delta^{a^\sigma_i}_{a^\tau_j}(b)$ is indifferent to the original decision tree b. Moreover, every 1-stage subtree of the swapped decision tree $\Delta^{a^\sigma_i}_{a^\tau_j}(b)$ has the same cardinality. Therefore, we can continue to swap any two

2-stage subtrees of $\Delta_{a_j^\tau}^{a_i^\sigma}(b)$, and end up with a new swapped decision tree that is indifferent to $\Delta_{a_j^\tau}^{a_i^\sigma}(b)$ and, hence, to b. We can keep swapping 2-stage subtrees of b until we obtain the following decision tree: $c = \{c_1, \ldots, c_{mn}\}$ in which $c_{(i-1)m+1} = \{a_i^1, \ldots, a_i^n\}$, $c_{(i-1)m+2} = \{a_i^{n+1}, \ldots, a_i^{2n}\}$, $\ldots$, $c_{(i-1)m+k} = \left\{a_i^{(k-1)n+1}, \ldots, a_i^{kn}\right\}$, $\ldots$, $c_{im} = \left\{a_i^{(m-1)n+1}, \ldots, a_i^{mn}\right\}$, for each $i = 1, \ldots, n$. Moreover, $b \sim c$.

It can be verified that we can indeed perform swaps to convert b into c. First, decision tree c has as many (mn) 1-stage subtrees as b. Second, each 1-stage subtree of c has as many (n) subtrees as each 1-stage subtree of b. Third, each 2-stage subtree of c has as many (m) subtrees as each 2-stage subtree of b. Lastly, every a_i^σ shows up once in some subtree of c and only once.

Because of (13), subtrees of $c_{(i-1)m+k} = \left\{a_i^{(k-1)n+1}, \ldots, a_i^{kn}\right\}$ are indifferent to each other for any $i = 1, \ldots, n$ and $k = 1, \ldots, m$. According to Lemma 3, $c_{(i-1)m+k} \sim a_i \sim a_i^k$ for any $i = 1, \ldots, n$ and $k = 1, \ldots, m$. Therefore, by Lemma 2,

$$c \sim \{a_1^1, \ldots, a_1^m, a_2^1, \ldots, a_2^m, \ldots, a_n^1, \ldots, a_n^m\} =: d.$$

Lastly, we want to apply Lemma 5 to d. Since $d = \{\{a_{1,1}^1, \ldots, a_{1,m}^1\}, \{a_{1,1}^2, \ldots, a_{1,m}^2\}, \ldots, \{a_{1,1}^m, \ldots, a_{1,m}^m\}, \{a_{2,1}^1, \ldots, a_{2,m}^1\}, \{a_{2,1}^2, \ldots, a_{2,m}^2\}, \ldots, \{a_{2,1}^m, \ldots, a_{2,m}^m\}, \ldots, \{a_{n,1}^1, \ldots, a_{n,m}^1\}, \{a_{n,1}^2, \ldots, a_{n,m}^2\}, \ldots, \{a_{n,1}^m, \ldots, a_{n,m}^m\}\}$, each subtree $a_i^k = \{a_{i,1}^k, \ldots, a_{i,m}^k\}$ of d is of size m. The second half of Lemma 5 implies that swapping any two 2-stage subtrees of d will yield a new decision tree that is indifferent to the original decision tree. Again, we can swap 2-stage subtrees of d many times, and obtain

$$d' := \{\{a_{1,1}^1, \ldots, a_{1,1}^m\}, \{a_{1,2}^1, \ldots, a_{1,2}^m\}, \ldots, \{a_{n,m}^1, \ldots, a_{n,m}^m\}\} \sim d.$$

Since $a_{i,k}^\sigma \sim a_{i,k}^\tau$ for any $\sigma, \tau \in \{1, \ldots, m\}$, we can apply Lemma 3 and show that

$$\{a_{i,j}^1, \ldots, a_{i,j}^m\} \sim \{a_{i,j}\} \sim a_{i,j}$$

for each i, j. Hence, d' is indifferent to $\{a_{1,1}, a_{1,2}, \ldots, a_{n,m}\} = \bigcup_{i=1}^n a_i$. Since we have $a \sim b \sim c \sim d \sim d'$, we know that $a \sim \bigcup_{i=1}^n a_i$. ■

Now suppose that (12) works for all $m \leq n$ for some $n > 1$. Take any $a = \{a_1, \ldots, a_{n+1}\}$. By richness of X, let us find distinct lotteries $p_1, \ldots, p_{n-1}$ such that none of them belongs to a and $p_i \sim a$ for $i = 1, \ldots, n-1$. By Lemma 2, $\{p_1, \ldots, p_{n-1}\} \sim p_i \sim a$. Thus, by *Disjoint Set Betweenness*, $a \sim a \cup \{p_1, \ldots, p_{n-1}\}$. Now, consider a decision tree $b := \{\{a_1, \ldots, a_n\}, \{a_{n+1}, p_1, \ldots, p_{n-1}\}\}$. According to Lemma 6, $b \sim \{a_1, \ldots, a_n\} \cup \{a_{n+1}, p_1, \ldots, p_{n-1}\} = a \cup \{p_1, \ldots, p_{n-1}\} \sim a$. Define $b_1 := \{a_1, \ldots, a_n\}$, and $b_2 := \{a_{n+1}, p_1, \ldots, p_{n-1}\}$. Since $|b| = 2$, $|b_i| = n$, and $U(p_i) = U(a) = U(b)$, we know that

$$\begin{aligned}
f(U(a)) &= f(U(b)) = \frac{1}{2} f(U(b_1)) + \frac{1}{2} f(U(b_2)) \\
&= \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n} f(U(a_i)) + \frac{1}{n} f(U(a_{n+1})) + \frac{1}{n}\sum_{j=1}^{n-1} f(U(p_j))\right) \\
&= \frac{1}{2n}\left(\sum_{i=1}^{n+1} f(U(a_i)) + \sum_{j=1}^{n-1} f(U(p_j))\right) \\
&= \frac{1}{2n}\sum_{i=1}^{n+1} f(U(a_i)) + \frac{n-1}{2n} f(U(a)).
\end{aligned}$$

Thus,

$$U(a) = f^{-1}\left(\frac{1}{n+1}\sum_{i=1}^{n+1} f(U(a_i))\right).$$

This shows that (12) works for all n using the same f function. The idea behind this step is illustrated in the figure below.

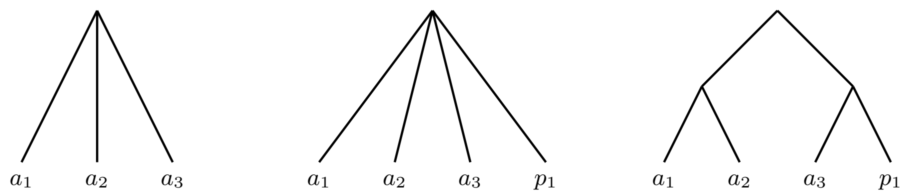

Figure 11: Since equation (12) holds for $n = 2$, we can compute the utility of the right-hand decision tree by applying g_2 for the second stage and then for the first stage. We use it to show that (12) holds for $n = 3$. *Disjoint Set Betweenness* is used to show that the left-hand decision tree is indifferent to the decision tree in the middle, because $p_1 \sim a$. Lemma 6 shows that the decision tree in the middle is indifferent to the right-hand decision tree. By letting the utility of the left-hand decision tree be equal to the utility of the right-hand decision tree, we show that (12) holds for $n = 3$.

Next, we show that the representation implies the axioms. We only show that *Disjoint Set Betweenness* and *Preference for Accentuating Swaps* hold. Consider any $a, b \in D$ such that $a \cap b = \emptyset$, say $a = \{a_1, \ldots, a_m\}$ and $b = \{b_1, \ldots, b_n\}$. If $a \succsim b$, then $U(a) \geq U(b)$. Since $f(U(a)) = \frac{1}{m}\sum_{i=1}^{m} f(U(a_i))$ and $f(U(b)) = \frac{1}{n}\sum_{i=1}^{n} f(U(b_i))$,

$$\begin{aligned} f(U(a \cup b)) &= \frac{1}{m+n}\left(\sum_{i=1}^{m} f(U(a_i)) + \sum_{i=1}^{n} f(U(b_i))\right) \\ &= \frac{m}{m+n} f(U(a)) + \frac{n}{m+n} f(U(b)). \end{aligned}$$

Thus, $U(a) \geq U(a \cup b) \geq U(b)$, and hence *Disjoint Set Betweenness* is satisfied.

For $a = \{a_1, \ldots, a_n\}$ such that $a_1, a_2 \in D_1$, $|a_1| \geq |a_2|$, $p \in a_1 \backslash a_2$, $q \in a_2 \backslash a_1$, $a_1' := a_1 \backslash \{p\} \cup \{q\} \notin \{a_3, \ldots, a_n\}$, and $a_2' := a_2 \backslash \{q\} \cup \{p\} \notin \{a_3, \ldots, a_n\}$, we have

$$\begin{aligned} |a| \times (f(U(\Delta_q^p(a))) - f(U(a))) &= f(U(a_1')) + f(U(a_2')) - f(U(a_1)) - f(U(a_2)) \\ &= (f(U(p)) - f(U(q)))\left(\frac{1}{|a_2|} - \frac{1}{|a_1|}\right) \geq 0. \end{aligned}$$

Therefore, *Preference for Accentuating Swaps* is satisfied.

Lastly, we show the uniqueness of the representation. Since $\succsim$ is nontrivial and X is compact, we know that $U(\mathcal{D}) = [u^l, u^h]$ for some $u^l < u^h$. Since U is an expected utility function on D_0, we know that U is unique up to a positive affine transformation. Fix the expected utility function U. To show f's uniqueness, suppose that f and g both satisfy equation (3). Consider $p, q \in D_0$. Define $u_1 := U(p)$, $u_2 := U(q)$, and $u_3 := U(\{p, q\})$. We have

$$f^{-1}\left(\frac{1}{2}f(u_1) + \frac{1}{2}f(u_2)\right) = g^{-1}\left(\frac{1}{2}g(u_1) + \frac{1}{2}g(u_2)\right). \tag{14}$$

Define $t_1 := f(u_1)$ and $t_2 := f(u_2)$. Equation (14) becomes

$$g \circ f^{-1}\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) = \frac{1}{2}g \circ f^{-1}(t_1) + \frac{1}{2}g \circ f^{-1}(t_2).$$

Since u_1 and u_2 are arbitrarily chosen from some nontrivial interval $[u^l, u^h]$, by Jensen's inequality, it must be true that

$$g \circ f^{-1}(t) = \alpha t + \beta,$$

and hence $g(u) = \alpha f(u) + \beta$. Since both f and g are strictly increasing, $\alpha > 0$.

■

Proof of Theorem 2: We only show that the axioms imply the representation. First, let $U : D_0 \to \mathbb{R}$ be an expected utility representation of $\succsim$ on D_0 implied by *vNM*. Under *Limited Perfect Foresight*, κ is uniquely determined from the preference. If $\kappa = +\infty$, then for any a, $a \sim \bar{\pi}(a)$; that is, the decision maker evaluates a decision tree using fully rational backward induction. Define a function $U_0 : \mathcal{D} \to \mathbb{R}$ such that for a lottery $p \in D_0$, $U_0(p) = U(p)$, and for a decision tree $b = \{b_1, \ldots, b_n\}$,

$$U_0(b) = \max_i U_0(b_i). \tag{15}$$

This definition is recursive.[16] It can easily be seen that U_0 is the recursive value function used by fully rational backward induction, and U_0 represents $\succsim$. To fit the representation definition, for each $j \in \mathbb{N} \cup \{+\infty\}$, we define $U_j : \mathcal{D} \to \mathbb{R}$ to be the function such that $U_j(a) = U_0(a)$ for each subtree $a \in \mathcal{D}$; that is, the other U_j's play no role when $\kappa = +\infty$.

Suppose $\kappa < +\infty$. Then, $\succsim^*$ is not trivially defined. First, the axiom *κ-Aggregator* implies that the preference $\succsim^*$ on $\mathcal{D}$ satisfies *Monotonicity*, *Disjoint Set Betweenness*, *Indifference to Trivial Extensions*, *Continuity*, and *Preference for Accentuating Swaps*. The axiom *κ-Consistent Weak Order* implies that $\succsim^*$ satisfies *Weak Order* and *vNM*, because $\succsim$ on D_0 satisfies *vNM* and $\succsim^*$ coincides with $\succsim$ on D_0. Therefore, according to Theorem 1, we can find an expected utility function $U^* : \mathcal{D} \to \mathbb{R}$ and a continuously strictly increasing function $f : U^*(\mathcal{D}) \to \mathbb{R}$ such that (i) for any $a_1, a_2 \in \mathcal{D}$, $a_1 \succsim^* a_2$ if and only if $U^*(a_1) \geq U^*(a_2)$, and (ii) for any $b = \{b_1, \ldots, b_n\} \in D$,

$$U^*(b) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(U^*(b_i))\right).$$

According to *κ-Consistent Weak Order* and the uniqueness of U^*, we can, without loss of generality, assume that U^* and U coincide on D_0. As we have seen in the proof of Theorem 1, for every decision tree a, we can find a lottery p^a such that $a \sim^* p^a$.

If $\kappa = 0$, according to the definition of $\succsim^*$, $\succsim$ becomes identical to $\succsim^*$. In this case, Theorem 2 reduces to Theorem 1: Define $U_0 : \mathcal{D} \to \mathbb{R}$ such that $U_0 = U^*$, and we know that U_0 represents $\succsim$.

Next, suppose that $0 < \kappa < +\infty$. For any $\hat{a}, \hat{b} \in D$ such that $\hat{a}$ is a j-stage subtree of $\hat{b}$ and $j \geq \kappa$, according to the definition of $\succsim^*$, we know that

$$\phi_{\hat{a}}^{p^{\hat{a}}}\left(\hat{b}\right) \sim \hat{b}, \tag{16}$$

[16]The utility of lotteries is given by U. Then, equation (15) defines the utility of all depth-1 decision trees; this, in turn, defines the utility of depth-2 decision trees, because depth-2 decision trees only consist of lotteries and depth-1 decision trees, and so on.

in which $p^{\hat{a}} \sim^* \hat{a}$. Take any depth-$k$ decision tree b such that $k > \kappa$. By definition, b has at least one κ-stage subtree that is not a lottery. Denote all its κ-stage subtrees that are not lotteries by $a_1, \ldots, a_n$. We can replace a_i's with lotteries p^{a_i}'s one after another; that is, we construct decision trees $c_1 := \phi_{a_1}^{p^{a_1}}(b)$, $c_2 := \phi_{a_2}^{p^{a_2}}(c_1)$, $\ldots$, $c_{n-1} := \phi_{a_{n-1}}^{p^{a_{n-1}}}(c_{n-2})$, and $b^\kappa := \phi_{a_n}^{p^{a_n}}(c_{n-1})$. Due to (16), $b \sim c_1 \sim c_2 \sim \cdots \sim c_{n-1} \sim b^\kappa$. By construction, b^κ is a depth-κ decision tree. Hence, according to *Limited Perfect Foresight* and the definition of κ,

$$\bar{\pi}(b^\kappa) \sim b^\kappa \sim b. \tag{17}$$

Define functions $U_j : \mathcal{D} \to \mathbb{R}$ for each integer $0 \leq j \leq \kappa$ such that (i) $U_j(p) = U^*(p)$ for any $p \in D_0$ and any j, and (ii) if $b = \{b_1, \ldots, b_n\} \in D$ is a j-stage subtree of a decision tree c, then

$$\begin{cases} U_j(b) = \max_i U_{j+1}(b_i) & \text{if } j < \kappa, \\ U_j(b) = U^*(b) = f^{-1}\left(\frac{1}{n}\sum_i f(U^*(b_i))\right) & \text{if } j = \kappa, \\ U^*(b) = f^{-1}\left(\frac{1}{n}\sum_i f(U^*(b_i))\right) & \text{if } j > \kappa.^{17} \end{cases} \tag{18}$$

We want to show that U_0 represents $\succsim$. Recall that for any decision tree b, $\pi(b)$ is the set of lotteries that can be possibly reached in b. Let $\pi^\kappa(b)$ denote the lotteries that can possibly be reached in b within κ stages. First, take any depth-k ($k > \kappa$) decision tree b in which $a_1, \ldots, a_n$ are all its κ-stage subtrees that are not lotteries. Clearly, $\pi^\kappa(b) \cap \pi(a_i) = \emptyset$ for all i, and

$$\pi^\kappa(b) \cup \pi(a_1) \cup \cdots \cup \pi(a_n) = \pi(b).$$

Since U^* represents $\succsim^*$, $U^*(a_i) = U^*(p^{a_i})$, $i = 1, \ldots, n$. Then, the second line of (18) implies that $U_\kappa(a_i) = U^*(a_i) = U^*(p^{a_i})$. Therefore, it can easily be verified that (18) implies that

$$U_0(b) = U_0(b^\kappa) = \max_{p \in \pi^\kappa(b) \cup \{p^{a_i}\}_{i=1}^n} U^*(p) = \max_{p \in \pi(b^\kappa)} U^*(p) = U^*(\bar{\pi}(b^\kappa)). \tag{19}$$

Lastly, for any depth-k' decision tree c ($k' \leq \kappa$), we know that $c \sim \bar{\pi}(c)$. Equation (18) implies that $U_0(c) = U^*(\bar{\pi}(c))$.

Now, we need to consider three cases. First, suppose $a_1, a_2 \in \bigcup_{i=0}^{\kappa} D_i$. Then, $a_1 \sim \bar{\pi}(a_1) \in D_0$ and $a_2 \sim \bar{\pi}(a_2) \in D_0$. Note that U^* represents $\succsim$ on D_0. Therefore, $\bar{\pi}(a_1) \succsim \bar{\pi}(a_2)$ if and only if $U^*(\bar{\pi}(a_1)) = U_0(a_1) \geq U^*(\bar{\pi}(a_2)) = U_0(a_2)$ implies that $a_1 \succsim a_2$ if and only if $U_0(a_1) \geq U_0(a_2)$. Second, suppose $a_1, a_2 \in \bigcup_{i=\kappa+1}^{+\infty} D_i$. Then, $a_1 \sim \bar{\pi}(a_1^{\kappa}) \in D_0$ and $a_2 \sim \bar{\pi}(a_2^{\kappa}) \in D_0$. According to (19), since $\bar{\pi}(a_1^{\kappa}) \succsim \bar{\pi}(a_2^{\kappa})$ if and only if $U^*(\bar{\pi}(a_1^{\kappa})) = U_0(a_1) \geq U^*(\bar{\pi}(a_2^{\kappa})) = U_0(a_2)$, we have $a_1 \succsim a_2$ if and only if $U_0(a_1) \geq U_0(a_2)$. The last case, in which one subtree is in $\bigcup_{i=0}^{\kappa} D_i$ and the other is in $\bigcup_{i=\kappa+1}^{+\infty} D_i$, follows from similar arguments. Lastly, the uniqueness of the representation follows from the uniqueness result in Theorem 1.

■

Proof of Theorem 3: First, we prove sufficiency. Suppose that $\succsim_1$ and $\succsim_2$ can be represented by (U, κ_1, f_1) and (U, κ_2, f_2), respectively, and κ_j's are finite. Then, $\succsim_1^*$ must coincide with $\succsim_2^*$ on D_0. Take any $p \in D_0$ and $a = \{q_1, \ldots, q_n\} \in D_1$. Let U_j^* be the utility function that decision maker j uses for subtrees beyond κ stages, and let $u_i := U_j^*(q_i) = U(q_i)$. Since $f_2 = g \circ f_1$,

$$\begin{aligned} f_2(U_2^*(a)) &= \frac{1}{n}\sum_{i=1}^{n} f_2(u_i) \\ g \circ f_1(U_2^*(a)) &= \frac{1}{n}\sum_{i=1}^{n} g \circ f_1(u_i). \end{aligned}$$

On the other hand, $f_1(U_1^*(a)) = \frac{1}{n}\sum_i f_1(u_i)$. By Jensen's inequality,

$$\begin{aligned} f_2(U_2^*(a)) &= \frac{1}{n}\sum g \circ f_1(u_i) \leq g\left(\frac{1}{n}\sum f_1(u_i)\right) \\ &= g \circ f_1(U_1^*(a)) = f_2(U_1^*(a)). \end{aligned}$$

Therefore, $U_1^*(a) \geq U_2^*(a)$, and hence $a \succsim_2^* p$ implies $a \succsim_1^* p$. Next, suppose we have shown that for some m, $a \succsim_2^* p$ implies $a \succsim_1^* p$ for any $p \in D_0$ and $a \in \bigcup_{k=0}^{m} D_k$. Consider

$b = \{b_1, \ldots, b_n\} \in D_{m+1}$. By the induction hypothesis, we have $U_1^*(b_i) \geq U_2^*(b_i)$ for each i, and thus

$$\begin{aligned} U_1^*(b) &= f_1^{-1}\left(\frac{1}{n}\sum f_1(U_1^*(b_i))\right) \geq f_1^{-1}\left(\frac{1}{n}\sum f_1(U_2^*(b_i))\right) \\ &\geq f_2^{-1}\left(\frac{1}{n}\sum f_2(U_2^*(b_i))\right) = U_2^*(b). \end{aligned}$$

The second inequality is similar to what we previously derive for the case of $a \in D_1$. Therefore, again we know that $b \succsim_2^* p$ implies $b \succsim_1^* p$.

Next, we prove necessity. The definition of unpredictability aversion implies that for any two lotteries $p, q \in D_0$,

$$q \succsim_2^* p \Rightarrow q \succsim_1^* p. \tag{20}$$

Since $\succsim_j^*$ on D_0 satisfies *vNM*, it is well known that (20) implies that $\succsim_1^*$ and $\succsim_2^*$ must coincide on D_0 (see, for example, Ghirardato, Maccheroni, and Marinacci (2004)). Suppose U represents $\succsim_1^*$ on D_0 and $\succsim_2^*$ on D_0. We know from the uniqueness of the κ-BRBI representation that there exist (U, κ_1, f_1) and (U, κ_2, f_2) that represent $\succsim_1^*$ and $\succsim_2^*$, respectively.

Define $g := f_2 \circ f_1^{-1}$. The function g is strictly increasing. We know that for any $p \in D_0$ and $a = \{q_1, \ldots, q_n\} \in D_1$, $a \succsim_2^* p$ implies $a \succsim_1^* p$. Again, let $u_i := U_j^*(q_i) = U(q_i)$. The proof of Theorem 1 shows that there exists a lottery r such that $a \sim_2^* r$; that is, $U_2^*(a) = f_2^{-1}(\frac{1}{n}\sum f_2(u_i)) = U(r)$. We know that $U_1^*(a) \geq U(r)$, which implies that

$$\begin{aligned} f_1^{-1}\left(\frac{1}{n}\sum f_1(u_i)\right) &\geq f_2^{-1}\left(\frac{1}{n}\sum f_2(u_i)\right) \\ g\left(\frac{1}{n}\sum f_1(u_i)\right) &\geq \frac{1}{n}\sum f_2(u_i). \end{aligned}$$

Define $t_i := f_1(u_i)$. The inequality above becomes $\frac{1}{n}\sum g(t_i) \leq g(\frac{1}{n}\sum t_i)$, which implies that g is concave.

■