Distributionally Robust Chance Constrained Optimal Power Flow Assuming Log-Concave Distributions

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Background: Chance Constrained Optimization

- Widely used in different problems with uncertainty
- But hard to accurately estimate uncertainty distributions with historical data
- **Example**: Physical constraints should be satisfied for most of the possible realizations of renewable/load uncertainty in power systems planning problems
- Solving methodologies
  - Scenario approach: large number of scenarios/constraints
  - Probabilistically robust method: overconservativeness
  - Analytical reformulation: case dependent reliability
- General formulation

$$\mathbb{P}(L(x,\xi) \leq 0) \geq 1 - \epsilon$$

Background: Distributionally Robust Optimization [1]

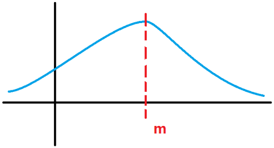
- Key properties
  - A probabilistic constraint is satisfied within an ambiguity set of distributions
  - The ambiguity set can be calibrated using the following information
    - Moment-based information: mean, covariance, higher-order moments
    - Density-based information: likelihood of a known probability density
    - Distribution structure: unimodality, support, smoothness, symmetry
  - Good trade-off between performance and computational tractability
- Example: distributionally robust chance constraint (DRCC)

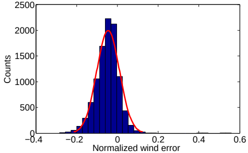
$$\inf_{\mathbb{P}\in\mathcal{P}^U} \mathbb{P}(L(x,\xi) \le 0) \ge 1-\epsilon$$

$$\mathcal{P}^U = \left\{\text{Unimodal},\ \mathbb{E}_{\mathbb{P}}(\xi)=\mu,\ \mathbb{E}_{\mathbb{P}}(\xi\xi^T)=S\right\}$$

[1] E. Delage and Y. Ye, "Distributionally Robust Optimization under Moment Uncertainty with Application to Data-driven problems," *Operations Research*, vol. 58, no. 3, pp. 595-612, 2010

- Univariate unimodal distribution
  - For an univariate random variable, the probability density/mass function (pdf/pmf) $f(\xi)$ has its maximum value at the location of mode, m
  - The pdf/pmf is non-decreasing when $\xi \leq m$ and non-increasing when $\xi > m$
- α-unimodality [2]
  - A generalised multivariate unimodality whose distribution structure is regulated by parameter α
  - With a random vector of size k, we have the following special features
    - When $\alpha = k = 1$, α-unimodality coincides with univariate unimodality
    - When $\alpha = k$, α-unimodality coincides with star unimodality such that the pdf/pmf is non-increasing along the ray from the mode to any other point in the space





[2] S.W. Dharmadhikari and K. Joat-Dev, "Unimodality, Convexity and Application," *Probability and Mathematical Statistics*, vol. 27. Academic Press, 1988

Conditional Value-at-Risk (CVaR)

Definition: CVaR

Given a random function $L(x, \xi)$ *and a constant* $\epsilon \in (0, 1)$*, the CVaR of* $L(x, \xi)$ *with confidence level* $(1 - \epsilon)$*, denoted* $\mathrm{CVaR}_\epsilon(L(x, \xi))$*, is defined as*

$$\mathrm{CVaR}_\epsilon(L(x, \xi)) = \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left[L(x, \xi) - \beta \right]^+ \right\}, \quad (1)$$

where $\mathbb{E}_{\mathbb{P}}[\cdot]^+ = \mathbb{E}_{\mathbb{P}}[\max(\cdot, 0)]$.

- CVaR [3, 4] is a popular tool used to measure the risk level of randomness
- CVaR evaluates the conditional expectation of $L(x, \xi)$ on the tail of its distribution
- Risk constraints (RC) imply chance constraints (CC) as below

$$\mathrm{CVaR}_\epsilon(L(x, \xi)) \leq 0 \;\Rightarrow\; \mathbb{P}\{L(x, \xi) \leq 0\} \geq 1 - \epsilon.$$

[3] P. Artzner, F. Delbaen, J.-M. Eber and D. Heath, "Coherent Measures of Risk," *Mathematical Finance*, 9: 203228, 1999

[4] R.T. Rockfellar and S. Uryasev, "Optimization of conditional value-at-risk," J.Risk 2, 21-41, 2002

Distributionally Robust Risk Constraints (DRRC)

Definition: DRRC

With the definition of RC and the ambiguity set $\mathcal{P}$, DRRC is defined as

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathrm{CVaR}_{\epsilon}(L(x,\xi)) \leq 0, \tag{2}$$

- Similarly, DRRC imply distributionally robust chance constraints (DRCC) as below

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathrm{CVaR}_{\epsilon}(L(x,\xi)) \leq 0 \;\Rightarrow\; \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(L(x,\xi) \leq 0) \geq 1-\epsilon$$

- In this work, we use the ambiguity set $\mathcal{P} := \{\mathbb{P} \in \mathcal{P}^{\alpha} : \mathbb{E}_{\mathbb{P}}[\xi] = \mu, \mathbb{E}_{\mathbb{P}}[\xi\xi^T] = S\}$ where μ and S can be estimated based on the historical data of ξ
- $\mathcal{P}^{\alpha}$ represents all of the α-unimodal distributions with mode at the origin

Objective and Contributions

- Objective
  - Previous work on distributionally robust optimization (DRO) either does not consider unimodality in the ambiguity set or assumes fixed variables
  - We want to include multivariate unimodality into the ambiguity set of DRO problems and give tractable reformulations
- Contribution of this work
  - We use risk constraints (CVaR) to control both the probability and magnitude of constraint violations
  - We incorporate multivariate unimodality into the distributionally robust risk-constrained optimization (DRRCO) problem with adjustable design variables and obtain two tractable approximate semidefinite programming (SDP) reformulations
  - We study the impact of the mode location and the value of the unimodality parameter α on the optimal solution

Previous Results

Lemma (adapted from [5])

Let $\widetilde{\mu} := (\frac{\alpha+1}{\alpha})\mu$ *and* $\widetilde{S} := (\frac{\alpha+2}{\alpha})S$. *For* $\mathcal{P} := \{\mathbb{P} \in \mathcal{P}^\alpha : \mathbb{E}_\mathbb{P}[\xi] = \mu, \mathbb{E}_\mathbb{P}[\xi\xi^T] = S\}$ *as assumed, we have*

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}\left[L(x,\xi) - \beta\right]^+ = \sup_{\mathbb{P}\in\mathcal{P}(\widetilde{\mu},\widetilde{S})} \mathbb{E}_\mathbb{P}\left[\widetilde{f}(x,\xi)\right], \tag{3}$$

where $\mathcal{P}(\widetilde{\mu},\widetilde{S}) := \{\mathbb{P} : \mathbb{E}_\mathbb{P}[\xi] = \widetilde{\mu}, \mathbb{E}_\mathbb{P}[\xi\xi^T] = \widetilde{S}\}$ *and* β *as a constant*

Assumption 1

We assume $L(x,\xi) = y_0(x) + y(x)^T\xi$ *where where* $y_0(x)$ *and* $y(x)$ *represent two affine functions of* x

[5] B. Van Parys, P. Goulart and M. Morari, "Distributionally Robust Expectation Inequalities for Structured distributions," *Optimization Online* 2015

Previous Results

- The transformation in the lemma above does not include the unimodality assumption in the ambiguity set
- If we define $a := y(x)^T \xi$ and $b := \beta - y_0(x)$, we have $\widetilde{f}(x, \xi)$ as below
- Evaluating the expectation of $\widetilde{f}(\xi)$ is computationally challenging

Lemma (cont'd)

$$\textit{if} \quad b \leq 0 \qquad \widetilde{f}(\xi) := \begin{cases} -\left(\frac{b}{\alpha+1}\right)\left(\frac{b}{a}\right)^{\alpha} & \textit{if } a \leq b \\ \left(\frac{\alpha}{\alpha+1}\right) a - b, & \textit{otherwise} \end{cases} \tag{4}$$

$$\textit{if} \quad b > 0 \qquad \widetilde{f}(\xi) := \begin{cases} 0 & \textit{if } a \leq b \\ \left(\frac{\alpha}{\alpha+1}\right) a - b + \left(\frac{b}{\alpha+1}\right)\left(\frac{b}{a}\right)^{\alpha}, & \textit{otherwise} \end{cases} . \tag{5}$$

Sandwich Approximation

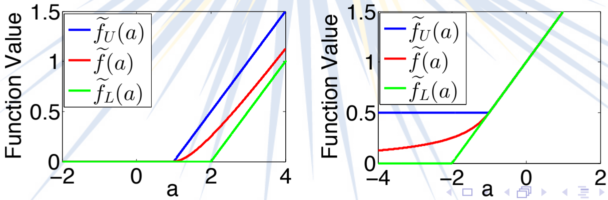
- To reduce the computational challenge, we replace $\widetilde{f}(x,\xi)$ with a sandwich approximation of $\widetilde{f}_U(x,\xi)$ and $\widetilde{f}_L(x,\xi)$
- Both approximations are the tightest 2-piecewise linear approximations of $\widetilde{f}(x,\xi)$

Theorem: Approximations

$$\widetilde{f}_U(x,\xi) = \left(\frac{\alpha}{\alpha+1}\right)[y(x)^T\xi + y_0(x) - \beta]^+ + \left(\frac{1}{\alpha+1}\right)[y_0(x) - \beta]^+ \quad (6)$$

$$\widetilde{f}_L(x,\xi) = \left[\left(\frac{\alpha}{\alpha+1}\right)y(x)^T\xi + y_0(x) - \beta\right]^+ \quad (7)$$

For given x, $\quad \widetilde{f}_L(x,\xi) \le \widetilde{f}(x,\xi) \le \widetilde{f}_U(x,\xi) \quad \forall\, \xi \in \mathbb{R}^k \quad (8)$



Semidefinite Programming (SDP) Reformulation

- With the sandwich approximation, we obtain two tractable SDP reformulations using $\widetilde{f}_{L,U}(x,\xi)$ respectively
- Both approximations will provide upper and lower bounds on real objective value
- The idea can be extended using multiple piecewise linear approximations

Example: SDP reformulation using $\widetilde{f}_L(x,\xi)$

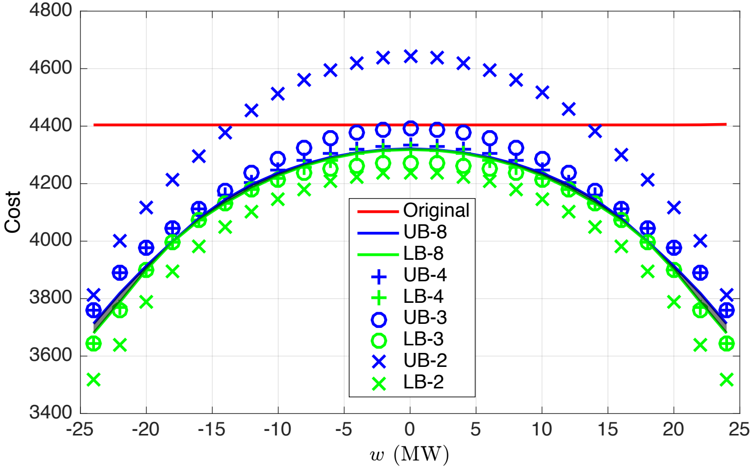
$$\beta + \frac{1}{\epsilon}\operatorname{Tr}(\widetilde{\Omega}\cdot M) \leq 0, \quad M \in \mathbb{S}^+_{k+1}$$

$$M - \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)y(x) \\ \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)y(x)^T & y_0(x) - \beta \end{bmatrix} \succeq 0, \quad \widetilde{\Omega} = \begin{bmatrix} \widetilde{S} & \widetilde{\mu} \\ \widetilde{\mu}^T & 1 \end{bmatrix} \qquad (9)$$

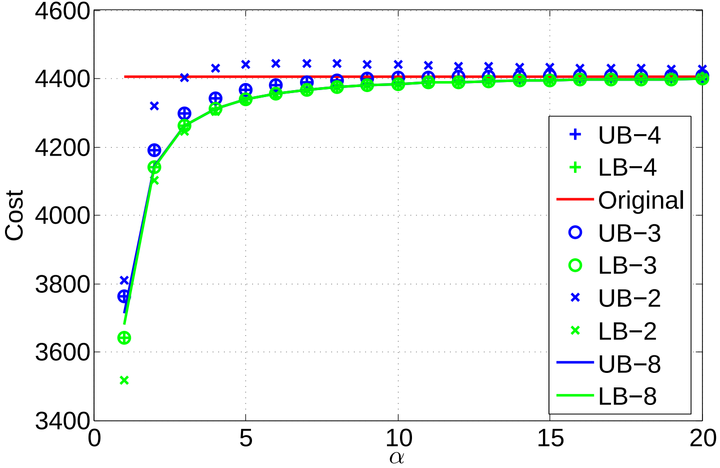
Simulation Setup

- **Objective**: Evaluate the performance of the approximate SDP reformulations of the DRRCO problem and compare to that without the assumption of unimodality
- **Problem**: Optimal power flow to schedule generation and reserve capacities under wind power production uncertainty
  - Uncongested modified IEEE 9-bus power system
  - Assume two independent wind farms
  - Confidence level 95%
  - Assume the mode of wind production uncertainty is at the origin
- Case studies
  - Case study 1: Fixing the covariance matrix of the random vector, analyze the optimal solution with different mean vectors
  - Case study 2: Fixing both mean and covariance, analyze the influence of unimodality parameter α
  - For the above two tests, compare the performance for different number of approximation pieces

Simulation Results: Case Study 1



Simulation Results: Case Study 2



Conclusion

- Including unimodality assumption within DRRCO problems leads to better objective values as our ambiguity set is more practical and better defined
- When mode moves further from mean, the skewness effect on the distribution will result in less conservative results
- When α increases, the impact of unimodality gets weaker and the solution will converge to that without assuming unimodality
- Multi-cut approximations provide tighter bounds on objective value as cut number increases
- Future work
    - Joint risk/chance constraints; Other structural features; More realistic case studies

Summary of All Completed Extensions

- Distributionally Robust Chance-Constrained Optimization
  - Sandwich approximation with a second-order cone programming (SOCP) formulation
  - Exact reformulation with an SOCP formulation and an efficient algorithm
- Distributionally Robust Risk-Constrained Optimization
  - Sandwich approximation with an SDP formulation
  - Exact reformulation with an SDP formulation,but no efficient algorithm
  - Exact reformulation with an SOCP formulation and an efficient algorithm
- Case Studies
  - Stochastic optimal power flow problem with both wind uncertainty and line congestion

Questions

Thank you!

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