Distributionally Robust Chance Constrained Optimal Power Flow Assuming Log-Concave Distributions

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***Abstract*—Optimization formulations with chance constraints have been widely proposed to operate the power system under various uncertainties, such as renewable production and load consumption. Constraints like the system's physical limits are required to be satisfied at high confidence levels. Conventional solving methodologies either make assumptions on the underlying uncertainty distributions or give overly-conservative results. We develop a new distributionally robust (DR) chance constrained optimal power flow formulation in which the chance constraints are satisfied over a family of distributions with known first-order moments, ellipsoidal support, and an assumption that the probability distributions are log-concave. Since most practical uncertainties have log-concave probability distributions, including this assumption in the formulation reduces the objective costs as compared to traditional DR approaches without sacrificing reliability. We derive second-order cone approximations of the DR chance constraints, resulting in a tractable formulation that can be solved with commercial solvers. We evaluate the performance of our approach using a modified IEEE 9-bus system with uncertain wind power production and compare it to standard approaches. We find that our approach produces solutions that are sufficiently reliable and less costly than traditional DR approaches.**

***Index Terms*—Optimal power flow, chance constraint, log-concave distribution, distributionally robust optimization, uncertainty**

I. Introduction

Uncertainties resulting from, e.g., renewable generation or load consumption, complicate the optimal operation of power systems. Optimal power flow (OPF) formulations using chance constraints [1]–[5] have been proposed to limit the chance of violating physical constraints, such as generation and line limits. The key difficulty in solving chance constrained problems is that we usually do not know the true underlying probability distributions of the uncertain variables. Most existing work either assumes that the uncertainties follow known, empirical distributions [6], [7] or use randomized techniques that require constraint satisfaction for a large number of scenarios [8], [9]. The former often fail to guarantee reliability (unless the distributions are modeled perfectly) while the latter often give overly-conservative results. Recently, distributionally robust chance constrained (DRCC) OPF formulations have been developed, e.g., [10]–[16]. This new approach requires the chance constraints to be satisfied for all possible distributions with known statistical parameters (e.g., first and second-order moments [10], [12], [16] or likelihood to a data-based distribution [15]) producing highly reliable solutions generally at a lower cost than those obtained with randomized techniques. In addition to moment information, common structural properties of practical uncertainties, like unimodality [13], [14] and symmetry [11], can be enforced to further lower the costs. To solve DRCC OPF problems, the constraints are either exactly reformulated or approximated resulting in second-order cone programs (SOCPs) [17] or semidefinite programs (SDPs) [18], [19], which can both be directly solved using commercial solvers.

In this paper, we propose a new DRCC OPF formulation in which chance constraints are satisfied over a family of distributions (i.e., an ambiguity set) with known first-order moments, an ellipsoidal support, and an assumption that the probability distributions are log-concave [20], [21]. Most practical uncertainties follow log-concave distributions; Gaussian, Beta and Weibull distributions are all log-concave. For example, wind forecast errors are generally modeled as specific distributions in the log-concave family [22]–[24]. Including this assumption limits the distributions over which the chance constraints should be satisfied, reducing the conservatism and the objective cost of the solution. Meanwhile, assuming the real uncertainty distributions are log-concave, the solutions will be sufficiently reliable. We benchmark our approach against a DRCC approach that uses an ambiguity set with only moment and support requirements and a chance constrained approach that assumes all uncertainty follows multivariate normal distributions, which is a special type of log-concave distribution.

The contributions of the paper are as follows. 1) We derive a projection property to simplify the multi-dimensional ambiguity set into an equivalent single dimensional one. 2) We derive second-order cone (SOC) relaxing and conservative approximations (i.e., a sandwich approximation) of the distributionally robust chance constraints under our ambiguity set. Using the DC power flow equations, the resulting DRCC OPF is an

This work is supported by the U.S. National Science Foundation Awards CCF-1442495 and CMMI-1662774.

SOCP. We also derive exact SOC constraints for the simpler ambiguity set. 3) We apply the theoretical results to solve a DC OPF problem on a modified IEEE 9-bus system with wind uncertainty. 4) We compare our results to those produced by the two benchmark approaches described above and report the objective costs and reliability of all approaches. To the best of our knowledge, this is the first work to include the log-concavity structure in distributionally robust optimization.

The remainder of the paper is organized as follows. The DR chance constraints and ambiguity set are introduced in Section II. In Section III, we derive the projection property. In Section IV, we give our main theoretical results. In Section V, we present the case studies and their results. Finally, Section VI summarizes the paper and proposes some potential future research directions.

II. Distributionally Robust Chance Constraints

A. Formulation

Assume that ξ represents an n-dimensional random vector defined on a probability space $(\mathbb{R}^n, \mathcal{B}, \mathbb{P}_\xi)$ with Borel σ-algebra $\mathcal{B}$ and probability distribution $\mathbb{P}_\xi$. We define $x \in \mathbb{R}^m$ as the design variable vector and $1-\epsilon$ as the confidence level. A chance constraint seeks to respect the physical constraint with the pre-defined confidence level as follows:

$$\mathbb{P}_\xi\left(f(x,\xi) \le 0\right) \ge 1-\epsilon. \tag{1}$$

In practice, $\mathbb{P}_\xi$ is usually unknown and empirical estimation can be difficult and unreliable. To reduce the risk associated with modeling uncertainty, we consider a distributional ambiguity set $\mathcal{D}_\xi$ that incorporates plausible candidates of the true distribution $\mathbb{P}_\xi$. Then, we require that (1) is satisfied for all $\mathbb{P}_\xi \in \mathcal{D}_\xi$, which leads to the following DR chance constraint:

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi\left(f(x,\xi) \le 0\right) \ge 1-\epsilon. \tag{2}$$

In the rest of the paper, we further specify that the constraint function $f(x,\xi)$ is a bilinear function, i.e., $f(x,\xi) = a(x)^T\xi - b(x)$, where both $a(x): \mathbb{R}^m \to \mathbb{R}^n$ and $b(x): \mathbb{R}^m \to \mathbb{R}$ are affine functions of x. This form of bilinear function applies to many physical constraints in practice and is standard in existing DR chance constraint research. The DR chance constraint in the final form is then

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi\left(a(x)^T\xi \le b(x)\right) \ge 1-\epsilon. \tag{3}$$

B. Ambiguity Set

We assume $\mathcal{D}_\xi$ consists of all probability distributions $\mathbb{P}_\xi$ that live in an ellipsoid around its mean value μ and satisfies structural properties specified in set $\mathcal{P}^{\mathrm{s}}$, i.e.,

$$\begin{aligned}\mathcal{D}_\xi := \{\mathbb{P}_\xi \in \mathcal{P}^{\mathrm{s}}: \quad & \mathbb{E}_{\mathbb{P}_\xi}[\xi] = \mu,\\ & \|\Sigma^{-\frac{1}{2}}(\xi-\mu)\|_2 \le r \text{ almost surely}\},\end{aligned} \tag{4}$$

where matrix $\Sigma \succ 0$ defines the ellipsoid and r represents its radius. For example, we could choose Σ to be the empirical covariance matrix obtained from the data corresponding to ξ. In this paper, we consider the following two options for $\mathcal{D}_\xi$:

- Case 1: $\mathcal{P}^{\mathrm{s}}_1 = \{\mathbb{P}_\xi \text{ is log-concave}\}$,
- Case 2: $\mathcal{P}^{\mathrm{s}}_2 = \{\mathbb{P}_\xi \text{ is any probability distribution}\}$,

where Case 2 is our DRCC benchmark. We denote the ambiguity set corresponding to Case 1 as $\mathcal{D}^1_\xi$ and corresponding to Case 2 as $\mathcal{D}^2_\xi$, where $\mathcal{D}^1_\xi \subseteq \mathcal{D}^2_\xi$. We formally define log-concavity [20] as follows.

Definition 2.1: A probability distribution $\mathbb{P}$ is log-concave if and only if for all non-empty sets $A, B \in \mathcal{B}$ and for all $\theta \in (0,1)$, we have

$$\mathbb{P}\left(\theta A + (1-\theta)B\right) \ge [\mathbb{P}(A)]^\theta [\mathbb{P}(B)]^{1-\theta}. \tag{5}$$

A large family of probability distributions are log-concave, including Gaussian, Beta, and Weibull distributions and log-concavity is commonly assumed for many practical uncertainty distributions.

III. Projection Property

In this section, we derive a projection property that transforms the ambiguity set $\mathcal{D}^i_\xi$ of a random vector $\xi \in \mathbb{R}^n$ into an equivalent ambiguity set $\mathcal{D}^i_\zeta$ of a random variable $\zeta \in \mathbb{R}$, for $i = 1, 2$.

Lemma 3.1: For $i = 1, 2$, the following equality holds:

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}^i_\xi} \mathbb{P}_\xi\left(a(x)^T\xi \le b(x)\right) = \inf_{\mathbb{P}_\zeta \in \mathcal{D}^i_\zeta} \mathbb{P}_\zeta\left(\zeta \le g(x)\right), \tag{6}$$

where

$$g(x) = b(x) - a(x)^T\mu + r\|\Sigma^{\frac{1}{2}}a(x)\|_2 \tag{7}$$

and

$$\begin{aligned}\mathcal{D}^i_\zeta := \{\mathbb{P}_\zeta \in & \mathcal{P}^{\mathrm{s}}_i: \ \mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = r\|\Sigma^{\frac{1}{2}}a(x)\|_2,\\ & 0 \le \zeta \le 2r\|\Sigma^{\frac{1}{2}}a(x)\|_2 \text{ almost surely}\}.\end{aligned} \tag{8}$$

Proof: We provide the proof for $i=1$; the proof for $i=2$ is similar and so omitted.

On the one hand, we can pick any ξ with $\mathbb{P}_\xi \in \mathcal{D}^1_\xi$. Define $\zeta = a(x)^T(\xi-\mu) + r\|\Sigma^{\frac{1}{2}}a(x)\|_2$, we have $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = r\|\Sigma^{\frac{1}{2}}a(x)\|_2$ and $0 \le \zeta \le 2r\|\Sigma^{\frac{1}{2}}a(x)\|_2$ almost surely, where the bounds of ζ are valid because

$$\begin{aligned}\zeta &\le r\|\Sigma^{\frac{1}{2}}a(x)\|_2 + \max_{\xi:\|\Sigma^{-\frac{1}{2}}(\xi-\mu)\|_2 \le r} a(x)^T(\xi-\mu)\\ &= r\|\Sigma^{\frac{1}{2}}a(x)\|_2 + \max_{y:\|y\|_2 \le r} (\Sigma^{\frac{1}{2}}a(x))^T y\\ &= r\|\Sigma^{\frac{1}{2}}a(x)\|_2 + r\|\Sigma^{\frac{1}{2}}a(x)\|_2 = 2r\|\Sigma^{\frac{1}{2}}a(x)\|_2\end{aligned}$$

and

$$\begin{aligned}\zeta &\ge r\|\Sigma^{\frac{1}{2}}a(x)\|_2 + \min_{\xi:\|\Sigma^{-\frac{1}{2}}(\xi-\mu)\|_2 \le r} a(x)^T(\xi-\mu)\\ &= r\|\Sigma^{\frac{1}{2}}a(x)\|_2 - r\|\Sigma^{\frac{1}{2}}a(x)\|_2 = 0.\end{aligned}$$

Furthermore, from Lemma 2.1 of [20], we know that $\mathbb{P}_\zeta$ is log-concave. Hence, $\mathbb{P}_\zeta \in \mathcal{D}^1_\zeta$ and

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}^1_\xi} \mathbb{P}_\xi\left(a(x)^T\xi \le b(x)\right) \ge \inf_{\mathbb{P}_\zeta \in \mathcal{D}^1_\zeta} \mathbb{P}_\zeta\left(\zeta \le g(x)\right).$$

On the other hand, we can pick any ζ with $\mathbb{P}_\zeta \in \mathcal{D}^1_\zeta$ and define $\xi = \mu + (\zeta - r\|\Sigma^{\frac{1}{2}}a(x)\|_2)\frac{\Sigma a(x)}{a(x)^T\Sigma a(x)}$. We have $\mathbb{E}_{\mathbb{P}_\xi}[\xi] = \mu$ and $\|\Sigma^{-\frac{1}{2}}(\xi-\mu)\|_2 \leq r$ almost surely, which follows from

$$\|\Sigma^{-\frac{1}{2}}(\xi-\mu)\|_2 = \frac{\|\Sigma^{\frac{1}{2}}a(x)\|_2}{a(x)^T\Sigma a(x)}\left|\zeta - r\|\Sigma^{\frac{1}{2}}a(x)\|_2\right| \leq r.$$

Furthermore, from Lemma 2.1 of [20], we know that $\mathbb{P}_\xi$ is log-concave. Hence, $\mathbb{P}_\xi \in \mathcal{D}^1_\xi$ and

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}^1_\xi} \mathbb{P}_\xi\left(a(x)^T\xi \leq b(x)\right) \leq \inf_{\mathbb{P}_\zeta \in \mathcal{D}^1_\zeta} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right).$$

■

IV. Main Results

We next present an SOC sandwich approximation of (3) under ambiguity set $\mathcal{D}^1_\xi$ and an SOC exact reformulation under ambiguity set $\mathcal{D}^2_\xi$.

A. Sandwich Approximation for $\mathcal{D}^1_\xi$

First, we derive a conservative approximation for (3) under $\mathcal{D}^1_\xi$ by relaxing $\mathcal{P}^{\mathrm{S}}_1$ to a set consisting of all $\mathbb{P}_\xi$ with a log-concave cumulative distribution functions (CDFs). Letting $\mathcal{P}^{\mathrm{L}}$ represent the set of all $\mathbb{P}_\xi$ with a log-concave CDFs, we define

$$\begin{aligned}\mathcal{D}^{\mathrm{L}}_\zeta := \{\mathbb{P}_\zeta \in \mathcal{P}^{\mathrm{L}}:\ & \mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = r\|\Sigma^{\frac{1}{2}}a(x)\|_2,\\ & 0 \leq \zeta \leq 2r\|\Sigma^{\frac{1}{2}}a(x)\|_2 \text{ almost surely}\}.\end{aligned} \quad (9)$$

From Theorem 1 of [21], we have $\mathcal{D}^1_\zeta \subseteq \mathcal{D}^{\mathrm{L}}_\zeta$ and so

$$\inf_{\mathbb{P}_\zeta \in \mathcal{D}^{\mathrm{L}}_\zeta} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right) \leq \inf_{\mathbb{P}_\zeta \in \mathcal{D}^1_\zeta} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right). \quad (10)$$

Theorem 4.1: If $\epsilon \leq \frac{1}{4}$, then (3) under $\mathcal{D}^1_\xi$ is implied by the following SOC constraint:

$$a(x)^T\mu + \left[1 - \frac{2\log(1-\epsilon)}{d^*}\right] r\|\Sigma^{\frac{1}{2}}a(x)\|_2 \leq b(x), \quad (11)$$

where d^* is the unique root of function $\exp\{d\} - d/2 = 1$ on the interval $(-\infty, 0)$ and log represents the natural logarithm.

Proof: From the above discussion, it is clear that (3) under $\mathcal{D}^1_\xi$ is implied by

$$\inf_{\mathbb{P}_\zeta \in \mathcal{D}^{\mathrm{L}}_\zeta} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right) \geq 1-\epsilon. \quad (12)$$

Define $\pi = r\|\Sigma^{\frac{1}{2}}a(x)\|_2$ and the CDF of ζ as $F_\zeta(z) = \mathbb{P}_\zeta(\zeta \leq z)$. Then, $F_\zeta(z)$ is log-concave in z for any $\mathbb{P}_\zeta \in \mathcal{D}^{\mathrm{L}}_\zeta$.

We claim that, without loss of optimality, we can focus on those $\mathbb{P}_\zeta$ with $\log(F_\zeta(z))$ being the minimum of an affine function of z and the constant-zero function when computing $\inf_{\mathbb{P}_\zeta \in \mathcal{D}^{\mathrm{L}}_\zeta} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right)$. To see this, we pick any ζ with $\mathbb{P}_\zeta \in \mathcal{D}^{\mathrm{L}}_\zeta$. Then $\log(F_\zeta(z))$ is concave and nondecreasing. Consider a tangent of $\log(F_\zeta(z))$ at $z^* = g(x)$, i.e., $G(z) = k(z-z^*) + \log(F_\zeta(z^*))$ with $k \in \partial\log(F_\zeta(z^*))$. As $\log(F_\zeta(z))$ is nondecreasing and concave, we have $k \geq 0$ and $G(z) \geq \log(F_\zeta(z))$ for all $z \in [0, 2\pi]$. Define $\hat{F}_\zeta(z) = \min\{1, \exp(G(z))\}$, which satisfies the property of a log-concave CDF and yields a probability measure $\hat{\mathbb{P}}_\zeta$. In addition, we have $\hat{\mathbb{P}}_\zeta\left(\zeta \leq z^*\right) = \mathbb{P}_\zeta\left(\zeta \leq z^*\right)$ and $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] \geq \mathbb{E}_{\hat{\mathbb{P}}_\zeta}[\zeta]$. Then, we manipulate $G(z)$ via the following two steps.

1) Increase the horizontal intercept to 2π with vertical intercept fixed, and
2) decrease the vertical intercept towards $-\infty$ with the horizontal intercept fixed.

Both actions will decrease $\hat{\mathbb{P}}_\zeta\left(\zeta \leq z^*\right)$ and increase $\mathbb{E}_{\hat{\mathbb{P}}_\zeta}[\zeta]$, and so we could stop as soon as $\hat{\mathbb{P}}_\zeta\left(\zeta \leq z^*\right) \leq \mathbb{P}_\zeta\left(\zeta \leq z^*\right)$ and $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \mathbb{E}_{\hat{\mathbb{P}}_\zeta}[\zeta]$. This proves the claim.

Based on the claim, we recast the left side of (12) as follows.

$$\min_{c,d} \quad \min\{1, \exp\{cz^* + d\}\} \quad (13)$$

$$\text{s.t.} \quad \int_0^{2\pi} \left[1 - \min\{1, \exp\{cz+d\}\}\right] dz = \pi \quad (14)$$

$$\exp\{c(2\pi) + d\} \geq 1 \quad (15)$$

$$c > 0, \quad d < 0, \quad (16)$$

where (14) enforces $\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \pi$, (15) enforces $0 \leq \zeta \leq 2\pi$ almost surely, and (16) makes sure that CDF $\min\{1, \exp\{cz+d\}\}$ is nondecreasing and nontrivial. Note that (15) is equivalent to $-d/c \leq 2\pi$. Then, it follows that (14) is equivalent to

$$\begin{aligned}\pi &= \int_0^{2\pi} \left[1 - \min\{1, \exp\{cz+d\}\}\right] dz\\ &= \int_0^{-d/c} \left[1 - \exp\{cz+d\}\right] dz\\ &= \frac{1}{c}\left[\exp\{d\} - d - 1\right],\end{aligned}$$

and so $c = (1/\pi)[\exp\{d\} - d - 1]$. Thus, (15) is equivalent to $2\pi \geq -d/c = -d\pi/[\exp\{d\} - d - 1]$, or $\exp\{d\} \geq 1 + d/2$ because $\exp\{d\} - d - 1 > 0$ when $d < 0$. It follows that the optimal objective value of problem (13)–(16) equals that of the following problem:

$$\min_{d<0} \quad \min\left\{1, \exp\left\{\frac{\exp\{d\} - d - 1}{\pi} z^* + d\right\}\right\} \quad (17)$$

$$\text{s.t.} \quad \exp\{d\} \geq \frac{d}{2} + 1. \quad (18)$$

We analyze the objective function in (17) by considering the following two scenarios.

- Scenario 1. If $z^* \geq 2\pi$, then

$$\begin{aligned}&(\exp\{d\} - d - 1)\left(\frac{z^*}{\pi}\right) + d\\ \geq &(\exp\{d\} - d - 1)\left(\frac{2\pi}{\pi}\right) + d\\ = &2\left(\exp\{d\} - \frac{d}{2} - 1\right) \geq 0\end{aligned}$$

  for all $d < 0$ and $\exp\{d\} \geq d/2 + 1$. Then, the optimal objective value of problem in (13)–(16) equals 1. Note that this makes sense because when $z^* \geq 2\pi$, we always have $\zeta \leq z^*$ for any $\zeta \in [0, 2\pi]$.

- Scenario 2. If $z^* < 2\pi$, then by the definition of d^*

$$
\begin{aligned}
&(\exp\{d^*\} - d^* - 1)\left(\frac{z^*}{\pi}\right) + d^* \\
<&(\exp\{d^*\} - d^* - 1)\left(\frac{2\pi}{\pi}\right) + d^* \\
=&2\left(\exp\{d^*\} - \frac{d^*}{2} - 1\right) = 0
\end{aligned}
$$

Hence, there exists a $d < 0$ with $\exp\{d\} \geq d/2 + 1$ such that $(\exp\{d\} - d - 1)(z^*/\pi) + d < 0$. It follows that the objective function in (17) is equivalent to

$$\min_{d<0} \exp\left\{(\exp\{d\} - d - 1)\left(\frac{z^*}{\pi}\right) + d\right\}.$$

Finally, we recast (12) by discussing the following two scenarios.

- Scenario 1. If $z^* \geq 2\pi$, then (12) always holds.
- Scenario 2. If $z^* < 2\pi$, then (12) holds if and only if the optimal objective value of problem (17)–(18) is greater than or equal to $1 - \epsilon$, or equivalently,

$$z^* \geq \max_{d<0;\ \exp\{d\}\geq d/2+1} \left\{\frac{\log(1-\epsilon) - d}{\exp\{d\} - d - 1}\right\}\pi.$$

Note that $d < 0$ and $\exp\{d\} \geq d/2 + 1$ is equivalent to $d \leq d^*$ because $h(d) = \exp\{d\} - d/2 - 1$ is convex in d with roots $d = 0$ and $d = d^* \approx -1.59$. In addition, $H(d) = \frac{\log(1-\epsilon)-d}{\exp\{d\}-d-1}$ is nondecreasing in d because $\epsilon \leq 1/4$. It follows that (12) is equivalent to

$$z^* \geq \left\{\frac{\log(1-\epsilon) - d^*}{\exp\{d^*\} - d^* - 1}\right\}\pi = \left\{2 - \frac{2\log(1-\epsilon)}{d^*}\right\}\pi.$$

Note that $2 - \frac{2\log(1-\epsilon)}{d^*} < 2$ as $d^* < 0$ and $1 - \epsilon \in (0, 1)$.

Summarizing the above two scenarios, (12) is equivalent to

$$z^* \geq \left\{2 - \frac{2\log(1-\epsilon)}{d^*}\right\}\pi.$$

The proof is complete given the definition of z^*. ■

Second, we derive a relaxing approximation for (3) by focusing on a particular distribution in $\mathcal{D}_\zeta^1$. More specifically, let $\mathbb{P}_\zeta^{\mathrm{U}}$ represent the uniform distribution on the interval $[0, 2r\|\Sigma^{1/2}a(x)\|_2]$, then (3) implies

$$\mathbb{P}_\zeta^{\mathrm{U}}\{\zeta \leq g(x)\} \geq 1 - \epsilon. \tag{19}$$

which can be recast as

$$a(x)^T\mu + r(1 - 2\epsilon)\|\Sigma^{\frac{1}{2}}a(x)\|_2 \leq b(x). \tag{20}$$

The conservative approximation and this relaxing approximation are used as the sandwich approximation.

Third, we consider another relaxing approximation of (3) by restricting $\mathbb{P}_\xi$ to be normally-distributed with mean μ and covariance matrix Σ. Then, based on existing results (e.g., [6]), (3) implies the following SOC constraint:

$$a(x)^T\mu + \Phi_N^{-1}(1 - \epsilon)\|\Sigma^{\frac{1}{2}}a(x)\|_2 \leq b(x). \tag{21}$$

where Φ_N is the inverse CDF of the standard normal distribution. This relaxing approximation is used as a benchmark.

B. Exact reformulation for $\mathcal{D}_\xi^2$

We next derive an exact reformulation for (3) under $\mathcal{D}_\xi^2$, which we use as the DRCC benchmark.

Theorem 4.2: If $\epsilon < \frac{1}{2}$, then (3) under $\mathcal{D}_\xi^2$ is equivalent to the following SOC constraint:

$$a(x)^T\mu + r\|\Sigma^{\frac{1}{2}}a(x)\|_2 \leq b(x). \tag{22}$$

Proof: We prove by contradiction.

First, if $g(x) < r\|\Sigma^{\frac{1}{2}}a(x)\|_2$, then we consider $\hat{\mathcal{P}}_\zeta \in \mathcal{D}_\zeta^2$ such that $\zeta = r\|\Sigma^{1/2}a(x)\|_2$ almost surely. It follows that $\hat{\mathcal{P}}_\zeta(\zeta \leq g(x)) = 0$ and so $\inf_{\mathbb{P}_\zeta\in\mathcal{D}_\zeta^1} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right) = 0 < 1-\epsilon$.

Second, if $r\|\Sigma^{\frac{1}{2}}a(x)\|_2 \leq g(x) < 2r\|\Sigma^{\frac{1}{2}}a(x)\|_2$, then there exists a point $\tilde{\zeta}$ such that $g(x) < \tilde{\zeta} < 2r\|\Sigma^{\frac{1}{2}}a(x)\|_2$. Consider a specific distribution $\hat{\mathbb{P}}_\zeta \in \mathcal{D}_\zeta^2$ that puts equal weight at $\tilde{\zeta}$ and $2r\|\Sigma^{\frac{1}{2}}a(x)\|_2 - \tilde{\zeta}$. We have

$$\inf_{\mathbb{P}_\zeta\in\mathcal{D}_\zeta^2} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right) \leq \hat{\mathbb{P}}_\zeta\left(\zeta \leq g(x)\right) = \frac{1}{2} < 1 - \epsilon.$$

Hence, (3) is valid if and only if $g(x) \geq 2r\|\Sigma^{1/2}a(x)\|_2$. ■

Note that (3) under $\mathcal{D}_\xi^2$ is conservative because it is invariant for any $\epsilon \in (0, 1/2)$.

V. Case Study

A. Problem Formulation and Setup

We consider the DCOPF problem from [13], which is similar to that of [3], [6]. Assuming two wind power plants, the random variables are the wind forecast error $\widetilde{W} = [W_1, W_2]^T$. Assuming the system has N_G generators and N_B buses, the design variables are generation $P_G \in \mathbb{R}^{N_G}$, up and down reserve capacities $R^{up} \in \mathbb{R}^{N_G}, R^{dn} \in \mathbb{R}^{N_G}$, and a "distribution vector" $d \in \mathbb{R}^{N_G}$, which parametrizes an affine reserve dispatch policy that allocates real-time supply/demand mismatch to generators providing reserves. The problem is

$$
\begin{aligned}
\min\ & P_G^T[C_1]P_G + C_2^TP_G + C_R^T(R^{up} + R^{dn}) && (23)\\
\text{s.t.}\ & -P_l \leq A_sP_{\text{inj}} \leq P_l && (24)\\
& R = -d(W_1 + W_2) && (25)\\
& P_{\text{inj}} = C_G(P_G + R) + C_W(P_W^f + \widetilde{W}) - C_LP_L && (26)\\
& \underline{P}_G \leq P_G + R \leq \overline{P}_G && (27)\\
& -R^{dn} \leq R \leq R^{up} && (28)\\
& \mathbf{1}_{1\times N_G}d = 1 && (29)\\
& \mathbf{1}_{1\times N_B}(C_GP_G + C_WP_W^f - C_LP_L) = 0 && (30)\\
& P_G \geq \mathbf{0}_{N_G\times 1},\ d \geq \mathbf{0}_{N_G\times 1}, && (31)\\
& R^{up} \geq \mathbf{0}_{N_G\times 1},\ R^{dn} \geq \mathbf{0}_{N_G\times 1} && (32)
\end{aligned}
$$

where $[C_1] \in \mathbb{R}^{N_G\times N_G}$, $C_2 \in \mathbb{R}^{N_G}$, and $C_R \in \mathbb{R}^{N_G}$ are cost parameters. Constraint (24) uses the DC power flow approximation to relate the power injections P_{inj} to the line flows using parameter matrix A_s, which includes the network impedances, and constrains the flows to below to the line limits P_l. Constraint (25) is the affine reserve dispatch policy that computes the real-time reserve actions R; (26) defines the

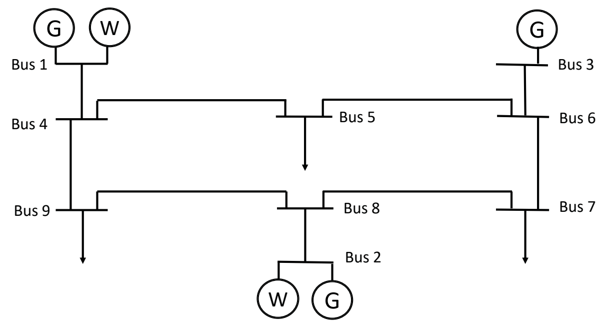

Figure 1. Modified IEEE 9-bus system.

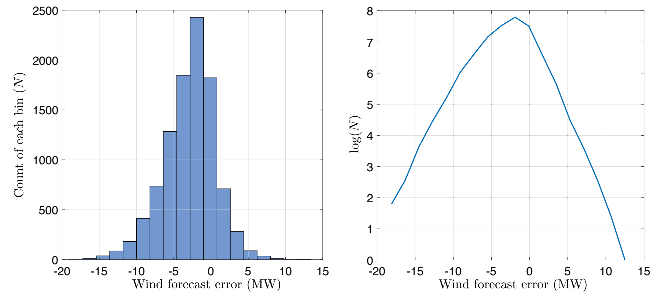

Figure 2. Histogram of univariate wind forecast errors and its logarithmic profile.

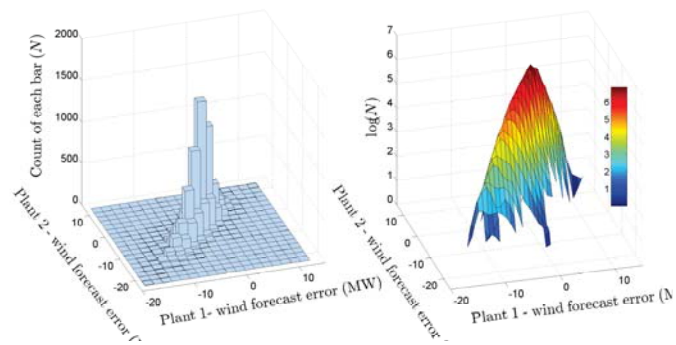

Figure 3. Histogram of bivariate wind forecast errors and its logarithmic profile.

real-time power injections, where P_W^f is the wind forecast and C_G, C_W, and C_L are matrices that map generators, wind power plants, and loads to their buses; (27) constrains generator production to within its limits $[\underline{P}_G, \overline{P}_G]$; (28) requires the real-time reserve actions be within the reserve capacity; (29) normalizes the distribution vector; (30) enforces power balance for the wind forecast; and (31), (32) ensures all decision variables are non-negative. Real-time power balance is enforced by (25) and (29) which require reserve actions to exactly compensate for the wind forecast error. Constraints (24), (27), and (28) are reformulated as chance constraints. Each constraint is enforced individually, not jointly.

We test our approach on the modified IEEE 9-bus system shown in Fig. 1. We use the network parameters and generation costs from [25], and set the reserve cost $C_R = 10C_2$. We set the wind power forecasts to $P_W^f = [66.8, 68.1]$ MW. We use the same wind power forecast uncertainty data as in [5], which was generated using a Markov Chain Monte Carlo mechanism [26] applied to real data from Germany. The 24-hour forecasts are made day-ahead but we only use those associated with the first hour since we solve a single period problem. We congest the system by increasing each load by 50% and reducing the line limit of the line connecting buses 2 and 8 from 250 MW to 200 MW. We pick this transmission line because it connects a wind power plant to the system and its forecasted power flow is close to its limit. All optimization problems are solved using CVX with the Mosek solver [27], [28].

B. Empirical Wind Forecast Error Distributions

We verify the log-concavity of the wind forecast error distributions using the full data set (10,000 scenarios) with the exception of statistical outliers (total probability$< 0.1\%$). Note that the outliers are also not used when empirically estimating the first-order moment μ, covariance Σ, and radius r, but they are used when evaluating the reliability of the solution.

In Fig. 2, we show the histogram of univariate wind forecast errors (i.e., all errors analyzed together) and its logarithmic profile, which appears to be concave. Figure 3 shows the histogram of the bivariate wind forecast error (i.e., errors were used to generate paired forecasts for the two wind power plants) and its logarithmic profile, which also appears to be concave. Figures 2 and 3 empirically justify our assumption that $\mathbb{P}_\xi$ in the ambiguity set $\mathcal{D}_\xi^1$ is log-concave.

C. Simulation Results

1) Objective Cost: We first compare the optimal objective cost across the following four DRCC OPF formulations:

- Gaussian: relaxing approximation in which $\mathbb{P}_\xi$ is normally-distributed, given in (21).
- RA: relaxing approximation of $\mathcal{D}_\xi^1$, given in (20).
- CA: conservative approximation of $\mathcal{D}_\xi^1$, given in (11).
- BM: benchmark based on $\mathcal{D}_\xi^2$, given in (22).

For each formulation, we test two violation levels $\epsilon = 5\%, 10\%$ and four data sizes $N = 500$, 1000, 2000, and 5000. We replicate the test associated with each (ϵ, N) combination 20 times by re-drawing N error scenarios. Table I displays the minimum, average, and maximum objective costs for each formulation over all replicates. BM yields the same optimal objective costs at different confidence levels as it is invariant under any $\epsilon \in (0, 1/2)$ (see Theorem 4.2). We also observe that, for all formulations, the range of the optimal objective cost becomes narrower as the data size grows because of better estimation of μ, Σ, and r. In addition, we observe that the optimal objective cost of a purely moment-based formulation (i.e., Gaussian) depends less on the data size as compared with the formulations using both moment and support information. The reason is that the support is more sensitive to outliers than the moments are. We observe that the optimal objective cost increases in the same order as the list above due to the increasing size of the corresponding ambiguity set.

TABLE I

OPTIMAL OBJECTIVE COST OF DRCC OPF UNDER VARIOUS FORMULATIONS, DATA SIZES, AND CONFIDENCE LEVELS

	Data size	500		1000		2000		5000	
	1-ϵ	95%	90%	95%	90%	95%	90%	95%	90%
Gaussian	min	4891	4849	4894	4851	4895	4852	4903	4858
	avg	4903	4858	4907	4862	4907	4861	4906	4861
	max	4927	4877	4925	4876	4915	4868	4913	4866
Relaxing Approximation (RA)	min	5078	5035	5192	5134	5748	5623	5985	5830
	avg	5468	5377	5918	5772	6001	5845	6046	5885
	max	6069	5904	6119	5948	6110	5940	6121	5950
Conservative Approximation (CA)	min	5094	5064	5212	5173	5793	5708	6040	5934
	avg	5501	5439	5970	5870	6057	5950	6104	5994
	max	6127	6015	6180	6063	6171	6054	6183	6065
Benchmark (BM)	min	5122	5122	5250	5250	5875	5875	6141	6141
	avg	5561	5561	6066	6066	6159	6159	6211	6211
	max	6235	6235	6292	6292	6282	6282	6295	6295

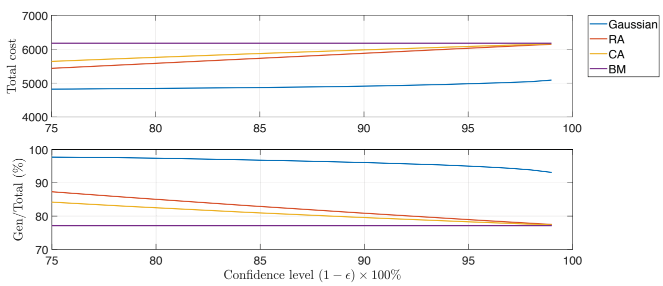

Figure 4. Optimal objective costs and generation cost percentage at various confidence levels $(1-\epsilon)$.

We next employ the full data set to construct the ambiguity sets. Then, we solve all formulations with varying confidence levels from 75% to 99% (i.e., $\epsilon \in [1, 25]\%$). Figure 4 displays the optimal objective costs and the generation cost as a percent of the total cost for each formulation. We observe that the optimal objective costs of RA and CA increase at a faster rate than that of Gaussian. Consistent with Table I, BM yields a constant optimal objective cost independent of ϵ. At high confidence levels ($\epsilon \leq 5\%$), the objective costs of RA and CA become closer and they both converge to that of BM as ϵ decreases. This indicates that the sandwich approximations become tighter as ϵ decreases. Meanwhile, the impact of the log-concavity assumption on the ambiguity sets becomes weaker. The optimal objective cost increases in the same order as in Table I (i.e., Gaussian, RA, CA, and BM). We also observe that the generation cost percentage of all formulations except BM decreases as the confidence level increases because we need to procure more reserve capacity to balance wind forecast error at a higher confidence levels. More reserve capacity is required by more conservative approaches (i.e., reserve capacity increases and generation cost percentage decreases in the following order: Gaussian, RA, CA, and BM). We also observe a similar convergence of the generation cost percentages associated with RA, CA, and BM as the confidence level increases.

2) Reliability: We evaluate the empirical reliability of all formulations via an out-of-sample Monte Carlo analysis for data sizes $N = 500,\ 5000$ (used to generate the statistical information needed in the formulation) and confidence levels $1-\epsilon = 95\%, 75\%$. For each formulation, we select the solution associated with the minimum optimal objective value reported in Table I and test it on 20 randomly-generated groups of 1000 out-of-sample wind forecast errors. We define the empirical reliability as the percentage of errors for which all chance constraints are satisfied by the selected optimal solution. Tables II and III show the results. With a data size of 5000, we get an accurate estimate of the statistical information and hence RA, CA, and BM all achieve high overall reliability, while Gaussian does not meet the reliability requirement. With a data size of 500, the reliability drops since the estimated statistical information is less accurate, but RA and CA still give sufficiently high reliability and are less conservative than BM. This demonstrates that the sandwich approximation (RA and CA) provides a good trade-off between cost and reliability. Additionally, the results in Table II demonstrate that, for large data size, we may be able to heuristically decrease the confidence level (e.g., to 75%) and still obtain solutions that achieve high reliability (i.e., $> 99\%$) at lower cost.

TABLE II

OVERALL RELIABILITY (%) WITH DATA SIZE 5000

$1-\epsilon$		Gaussian	RA	CA	BM
95%	min	87.5	99.7	99.8	99.8
	avg	88.7	99.9	99.9	100
	max	90.2	100	100	100
75%	min	53.2	99.5	99.5	99.8
	avg	55.2	99.8	99.8	100
	max	57.5	100	100	100

TABLE III

OVERALL RELIABILITY (%) WITH DATA SIZE 500

$1-\epsilon$		Gaussian	RA	CA	BM
95%	min	85.4	98.3	98.5	98.9
	avg	86.9	99.1	99.3	99.5
	max	89.2	99.6	99.8	99.8
75%	min	51.6	88.7	94.7	98.9
	avg	53.4	90	95.9	99.5
	max	55.4	92.1	96.7	99.8

VI. Conclusions and Future Work

In this paper, we derived distributionally robust chance constraints corresponding to log-concave uncertainty distributions with known first-order moments and ellipsoidal support. We derived a projection property and a tractable sandwich approximation (i.e., relaxing and conservative approximations) of the distributionally robust chance constraints as second-order cone constraints. We compared the approximations to a benchmark using only moment and support information and another that assumed normally-distributed uncertainty. The optimal objective costs of the sandwich approximation depend on the accuracy of the moment and support information. As the confidence level increases, the gap between the conservative and relaxing approximation reduce and the effect of the log-concavity assumption becomes weaker. The sandwich approximation provides a better trade-off between optimal objective cost and reliability than either benchmark, where the benchmark assuming normal distributions is cheap but not sufficiently reliable and the benchmark assuming known first-order moments and support (without log-concavity) is expensive. Our approach works well even if only a small number of data points are available to estimate the statistical parameters.

Future research directions include deriving an exact reformulation of the distributionally robust chance constraints assuming log-concave distributions and incorporating more statistical information like higher-order moments (i.e., uncertainty correlations) and skewed bounded supports. Other directions include analyzing joint chance constraint, testing on more realistic systems with higher-dimensional uncertainties, and development of distributionally robust methods that incorporate the AC power flow equations.

References

[1] H. Zhang and P. Li, "Chance constrained programming for optimal power flow under uncertainty," *IEEE Transactions on Power Systems*, vol. 26, no. 4, pp. 2417–2424, 2011.

[2] R. A. Jabr, "Adjustable robust OPF with renewable energy sources," *IEEE Transactions on Power Systems*, vol. 28, no. 4, pp. 4742–4751, 2013.

[3] M. Vrakopoulou, K. Margellos, J. Lygeros, and G. Andersson, "A probabilistic framework for reserve scheduling and N-1 security assessment of systems with high wind power penetration," *IEEE Transactions on Power Systems*, vol. 28, no. 4, pp. 3885–3896, 2013.

[4] L. Roald, F. Oldewurtel, T. Krause, and G. Andersson, "Analytical reformulation of security constrained optimal power flow with probabilistic constraints," in *Proceedings of IEEE PowerTech Conference*, Grenoble, France, 2013.

[5] M. Vrakopoulou, J. L. Mathieu, and G. Andersson, "Stochastic optimal power flow with uncertain reserves from demand response," in *Proceedings of Hawaii International Conference on System Science*, 2014.

[6] D. Bienstock, M. Chertkov, and S. Harnett, "Chance-constrained optimal power flow: Risk-aware network control under uncertainty," *SIAM Review*, vol. 56, no. 3, pp. 461–495, 2014.

[7] B. Li and J. L. Mathieu, "Analytical reformulation of chance-constrained optimal power flow with uncertain load control," in *Proceedings of IEEE PowerTech Conference*, Eindhoven, Netherlands, 2015.

[8] M. C. Campi, S. Garatti, and M. Prandini, "The scenario approach for systems and control design," *Annual Reviews in Control*, vol. 33, no. 2, pp. 149 – 157, 2009.

[9] K. Margellos, P. Goulart, and J. Lygeros, "On the road between robust optimization and the scenario approach for chance constrained optimization problems," *IEEE Transactions on Automatic Control*, vol. 59, no. 8, pp. 2258–2263, 2014.

[10] Y. Zhang, S. Shen, and J. L. Mathieu, "Distributionally robust chance-constrained optimal power flow with uncertain renewables and uncertain reserves provided by loads," *IEEE Transactions on Power Systems*, vol. 32, no. 2, pp. 1378–1388, 2017.

[11] L. Roald, F. Oldewurtel, B. Van Parys, and G. Andersson, "Security constrained optimal power flow with distributionally robust chance constraints," *arXiv preprint arXiv:1508.06061*, 2015.

[12] W. Xie and S. Ahmed, "Distributionally robust chance constrained optimal power flow with renewables: A conic reformulation," *IEEE Transactions on Power Systems (in press)*, 2017.

[13] B. Li, R. Jiang, and J. L. Mathieu, "Distributionally robust risk-constrained optimal power flow using moment and unimodality information," in *Proceedings of IEEE Conference on Decision and Control*, Las Vegas, NV, 2016.

[14] T. Summers, J. Warrington, M. Morari, and J. Lygeros, "Stochastic optimal power flow based on conditional value at risk and distributional robustness," *International Journal of Electrical Power & Energy Systems*, vol. 72, pp. 116 – 125, 2015.

[15] Y. Guo, K. Baker, E. Dall'Anese, Z. Hu, and T. Summers, "Stochastic optimal power flow based on data-driven distributionally robust optimization," *arXiv preprint arXiv:1706.04267*, 2017.

[16] M. Lubin, Y. Dvorkin, and S. Backhaus, "A robust approach to chance constrained optimal power flow with renewable generation," *IEEE Transactions on Power Systems*, vol. 31, no. 5, pp. 3840–3849, 2016.

[17] B. Li, R. Jiang, and J. Mathieu, "Ambiguous risk constraints with moment and unimodality information," *Mathematical Programming (Accepted)*, 2017.

[18] R. Jiang and Y. Guan, "Data-driven chance constrained stochastic program," *Mathematical Programming*, vol. 158, no. 1, pp. 291–327, 2016.

[19] S. Zymler, D. Kuhn, and B. Rustem, "Distributionally robust joint chance constraints with second-order moment information," *Mathematical Programming*, vol. 137, no. 1, pp. 167–198, 2013.

[20] S. W. Dharmadhikari and K. Joag-Dev, *Unimodality, convexity, and applications*. Academic Press, 1988.

[21] M. Bagnoli and T. Bergstrom, "Log-concave probability and its applications," *Economic Theory*, vol. 26, no. 2, pp. 445–469, 2005.

[22] V. S. Pappala, I. Erlich, K. Rohrig, and J. Dobschinski, "A stochastic model for the optimal operation of a wind-thermal power system," *IEEE Transactions on Power Systems*, vol. 24, no. 2, pp. 940–950, 2009.

[23] R. Doherty and M. O'Malley, "A new approach to quantify reserve demand in systems with significant installed wind capacity," *IEEE Transactions on Power Systems*, vol. 20, no. 2, pp. 587–595, 2005.

[24] H. Bludszuweit, J. A. Dominguez-Navarro, and A. Llombart, "Statistical analysis of wind power forecast error," *IEEE Transactions on Power Systems*, vol. 23, no. 3, pp. 983–991, 2008.

[25] R. D. Zimmerman, C. E. Murillo-Sanchez, R. J. Thomas, and L. Fellow, "MATPOWER: Steady-state operations, planning and analysis tools for power systems research and education," *IEEE Transactions on Power Systems*, vol. 26, no. 1, pp. 12–19, 2011.

[26] G. Papaefthymiou and B. Klockl, "MCMC for wind power simulation," *IEEE Transactions on Energy Conversion*, vol. 23, no. 1, pp. 234–240, 2008.

[27] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," http://cvxr.com/cvx, 2014.

[28] ——, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, 2008.