Distributionally Robust Chance Constrained Optimal Power Flow Assuming Log-Concave Distributions

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Increasing Uncertainty

- Renewable energy production forecast error, load consumption forecast error, component failures
- How can we operate the power system in a way that (almost always) ensures feasibility?
- **Solution**:
  - More reserves
  - Stochastic optimal power flow to schedule reserve capacities in a way that balances system cost and reliability

Chance Constrained Optimization

- Used to obtain the lowest cost solution that satisfies constraints at certain (high) probabilities

$$\mathbb{P}_{\xi}(f(x,\xi) \leq 0) \geq 1 - \epsilon$$

- **Example**: In optimal power flow problems, physical constraints (e.g., line limits) should be satisfied for most of the possible realizations of renewable/load uncertainty [Zhang and Li 2011; Vrakopoulou et al. 2013; Roald et al. 2013; Bienstock et al. 2014, and many more]
- Hard to obtain accurate uncertainty distributions
- Solving methodologies
  - Assume a distribution, analytically reformulate, and solve [Roald et al. 2013; Bienstock et al. 2014; Li and Mathieu 2015]
  - Scenario approaches [Campi et al. 2009; Vrakopoulou et al. 2013]
  - Probabilistically robust methods [Margellos et al. 2014; Vrakopoulou et al. 2013]

Distributionally Robust Optimization [Delage and Ye 2010]

- Distributionally Robust Chance Constraint

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi(f(x,\xi) \leq 0) \geq 1 - \epsilon$$

- Chance constraints satisfied for all distributions within an ambiguity set defined using
  - Moment-based information: mean, covariance, higher-order moments
  - Density-based information: likelihood of a known probability density
  - Distribution structure: support, unimodality, symmetry, log-concavity
- Distributionally robust OPF [Roald et al. 2015; Zhang et al. 2017; Xie and Ahmed 2017; Summers et al. 2015; Lubin et al. 2016; Li et al. 2017]

Log-Concave Distributions [Dharmadhikari and Joat-Dev 1988]

- Our ambiguity set

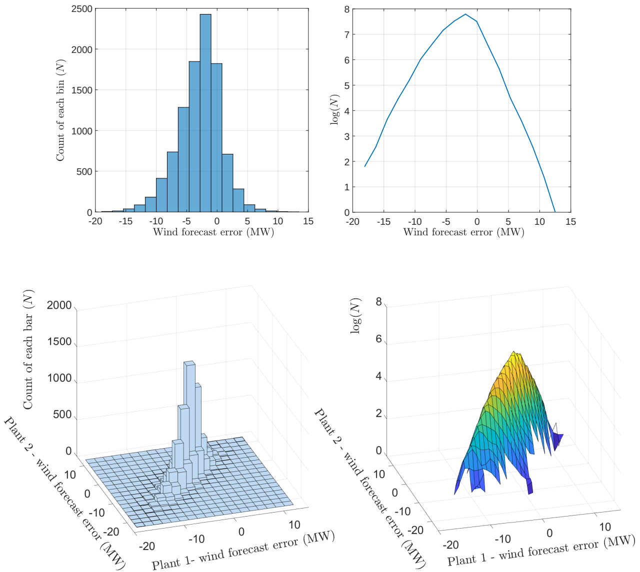
$$\mathcal{D}_\xi = \left\{ \mathbb{P}_\xi \text{ is log-concave}, \ \mathbb{E}_{\mathbb{P}_\xi}[\xi] = \mu, \ \|\Sigma^{-\frac{1}{2}}(\xi - \mu)\| \leq r \ \right\}$$

where μ is the mean and $\Sigma \succ 0$, r define an ellipsoidal support set.

- What is a log-concave distribution?
  - The logarithm of the probability density function (PDF) is concave.
    Note: log-concave PDF $\rightarrow$ log-concave CDF, unimodal PDF

- Why assume a log-concave distribution?
  - We want to make the most aggressive assumptions we can without assuming away any plausible distributions. Otherwise our solutions will be unnecessarily conservative, and costly!
  - A large family of well-known probability distributions (e.g., Uniform, Gaussian, Beta, and Weibull) are log-concave.
  - These distributions are generally good matches for empirical wind and load forecast uncertainty data [Pappala et al. 2009; Doherty and O'Malley 2005; Bludszuweit et al. 2008].

Wind Forecast Uncertainty Data



Contributions of Our Work

- We derive a projection property to simplify the multi-dimensional ambiguity set into an equivalent single dimensional one.
- We incorporate the log-concavity into the distributionally robust chance-constrained optimization problem and obtain second-order cone conservative and relaxing approximations, i.e., a sandwich approximation.
- We apply our results to a DCOPF on a modified IEEE 9-bus system with wind uncertainty and compare to
  - distributionally robust optimization without the assumption of log-concavity
  - analytical reformulation assuming Gaussian distributions

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Projection Property

- Our ambiguity set:

$$\mathcal{D}_\xi = \left\{ \mathbb{P}_\xi \text{ is log-concave}, \ \mathbb{E}_{\mathbb{P}_\xi}[\xi] = \mu, \ \|\Sigma^{-\frac{1}{2}}(\xi - \mu)\|_2 \le r \right\}$$

- We derive a projection property that transforms the ambiguity set $\mathcal{D}_\xi$ of a random vector $\xi \in \mathbb{R}^n$ into an equivalent set $\mathcal{D}_\zeta$ of a random variable $\zeta \in \mathbb{R}$.

Lemma: Projection Property

The following equality holds:

$$\inf_{\mathbb{P}_\xi \in \mathcal{D}_\xi} \mathbb{P}_\xi \left(a(x)^T \xi \le b(x) \right) = \inf_{\mathbb{P}_\zeta \in \mathcal{D}_\zeta} \mathbb{P}_\zeta \left(\zeta \le g(x) \right),$$

where

$$g(x) = b(x) - a(x)^T \mu + \underbrace{r\|\Sigma^{\frac{1}{2}} a(x)\|_2}_{\pi},$$

and

$$\mathcal{D}_\zeta := \{\mathbb{P}_\zeta \text{ is log-concave}, \ \mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \pi, \ 0 \le \zeta \le 2\pi\}.$$

Conservative Approximation

- We have been unable to derive an exact reformulation of

$$\inf_{\mathbb{P}_\zeta \in \mathcal{D}_\zeta} \mathbb{P}_\zeta\left(\zeta \leq g(x)\right) \geq 1 - \epsilon$$

- Instead, we relax $\mathcal{D}_\zeta$ to $\mathcal{D}_\zeta^C \supseteq \mathcal{D}_\zeta$ where

$$\mathcal{D}_\zeta^C := \{\text{CDF of } \mathbb{P}_\zeta \text{ is log-concave},\ \mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \pi,\ 0 \leq \zeta \leq 2\pi\}$$

Theorem: Conservative Approximation

If $\epsilon \leq \frac{1}{4}$, then the distributionally robust chance constraint is implied by the second order cone constraint

$$a(x)^T \mu + \left[1 - \frac{2\log(1-\epsilon)}{d^*}\right] \pi \leq b(x),$$

where d^* is the unique root of function $\exp\{d\} - d/2 = 1$ on the interval $(-\infty, 0)$ and $\log$ represents the natural logarithm.

Relaxing Approximation, DR Benchmark, and Gaussian

- **Relaxing Approximation:** Assume ζ follows a uniform distribution, which is a particular distribution in $\mathcal{D}_\zeta$. Then the distributionally robust chance constraint is the second order cone constraint

$$a(x)^T\mu + (1-2\epsilon)\pi \le b(x)$$

- **DR Benchmark:** Our distributionally robust ambiguity set without the log-concavity assumption, i.e., $\mathcal{D}_\zeta := \{\mathbb{E}_{\mathbb{P}_\zeta}[\zeta] = \pi,\ 0 \le \zeta \le 2\pi\}$. If $\epsilon < \frac{1}{2}$, then the distributionally robust chance constraint is equivalent to the second order cone constraint

$$a(x)^T\mu + \pi \le b(x)$$

- **Gaussian Assumption:** Assume the distribution is multivariate Gaussian. Then the chance constraint is equivalent to the second order cone constraint

$$a(x)^T\mu + \Phi_N^{-1}(1-\epsilon)\|\Sigma^{\frac{1}{2}}a(x)\|_2 \le b(x)$$

where Φ_N is the CDF.

Case Study: IEEE 9-bus system with two wind farms

Problem: Schedule generation P_G and reserve capacities R^{up}, R^{dn} under wind forecast error W_1, W_2 using the DC power flow equations. From [Li et al. 2016], similar to the formulations in [Vrakopoulou et al. 2013, Bienstock et al. 2014].

$$
\begin{aligned}
\min\ & P_G^T[C_1]P_G + C_2^T P_G + C_R^T(R^{up} + R^{dn}) \\
\text{s.t.} & -P_l \le A_s P_{\text{inj}} \le P_l \\
& P_{\text{inj}} = C_G(P_G + R) + C_W(P_W^f + \widetilde{W}) - C_L P_L \\
& R = -d(W_1 + W_2) \\
& \underline{P}_G \le P_G + R \le \overline{P}_G \\
& -R^{dn} \le R \le R^{up} \\
& \mathbf{1}_{1\times N_G} d = 1 \\
& \mathbf{1}_{1\times N_B}(C_G P_G + C_W P_W^f - C_L P_L) = 0 \\
& P_G \ge \mathbf{0}_{N_G\times 1},\ d \ge \mathbf{0}_{N_G\times 1} \\
& R^{up} \ge \mathbf{0}_{N_G\times 1},\ R^{dn} \ge \mathbf{0}_{N_G\times 1}
\end{aligned}
$$

where $\widetilde{W} = [W_1, W_2]^T$.

Case Study: Objectives

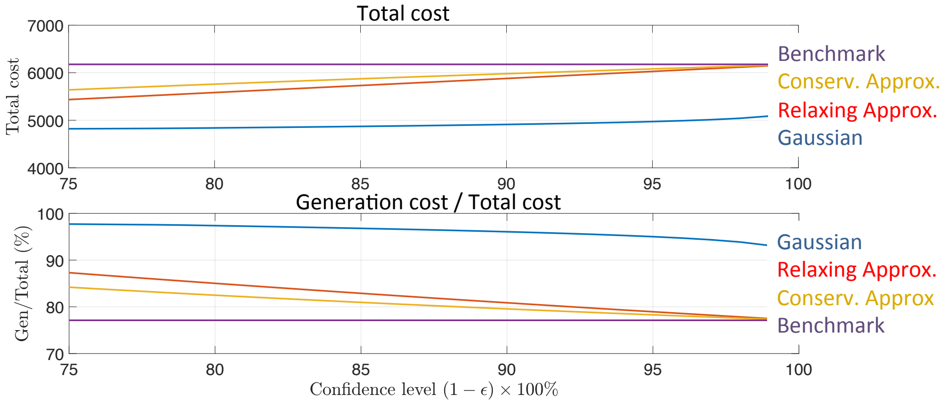
- Is it worth it?
- How tight is the sandwich approximation?
- How do the results change with confidence level $1 - \epsilon$ and the uncertainty data size for generating the ambiguity sets?

Results: Objective Cost Comparison

Results are the average of 20 runs in which we re-draw the uncertainty data used to generate the ambiguity sets (μ, Σ, r).

Data size	500		1000		2000		5000	
1-ϵ	95%	90%	95%	90%	95%	90%	95%	90%
Gaussian	4903	4858	4907	4862	4907	4861	4906	4861
Relax. Approx.	5468	5377	5918	5772	6001	5845	6046	5885
Conserv. Approx.	5501	5439	5970	5870	6057	5950	6104	5994
Benchmark	5561	5561	6066	6066	6159	6159	6211	6211

Results: Costs vs. Confidence Level



Simulation Results: Reliability

- Data size $N = 5000$:

$1-\epsilon$		Gaussian	Relax.	Conserv.	Benchmark
95%	min	87.5	99.7	99.8	99.8
	avg	88.7	99.9	99.9	100
	max	90.2	100	100	100
75%	min	53.2	99.5	99.5	99.8
	avg	55.2	99.8	99.8	100
	max	57.5	100	100	100

- Data size $N = 500$:

$1-\epsilon$		Gaussian	Relax.	Conserv.	Benchmark
95%	min	85.4	98.3	98.5	98.9
	avg	86.9	99.1	99.3	99.5
	max	89.2	99.6	99.8	99.8
75%	min	51.6	88.7	94.7	98.9
	avg	53.4	90	95.9	99.5
	max	55.4	92.1	96.7	99.8

Conclusions

- The objective costs of the sandwich approximation depend on the accuracy of the moment and support information.
- As the confidence level increases, the gap between the conservative and relaxing approximations reduce and the effect of the log-concavity assumption becomes weaker.
- The sandwich approximation provides a better trade-off between optimal objective cost and reliability than both the Gaussian approach and the Benchmark (without log-concavity).
- The approximations work well even if a small number of data points are used to estimate the statistical parameters.
- Future work: exact reformulation, other statistical information, joint chance constraint, more realistic case studies

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