Two-stage Distributionally Robust Optimal Power Flow with Flexible Loads

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***Abstract*—In this paper, we formulate a two-stage distributionally robust (DR) model for the optimal power flow (OPF) problem in the presence of uncertainties from wind power generation and load-based reserves. Assuming ambiguous distributions of the random variables, we minimize the costs of generation, reserves, and the worst-case expected value of the penalty cost of violating constraints. We consider a lifted support and a distributional ambiguity set parameterized by empirical means and absolute deviations of the random variables. We adopt an enhanced linear decision rule (ELDR) to derive a quadratic programming reformulation of the DR-OPF model, and compare its performance to that of a DR chance-constrained OPF model. We study the optimal solution patterns of the two approaches, compare their performance in out-of-sample simulations, and also numerically justify the use of the ELDR.**

***Index Terms*—load control, distributionally robust optimization, enhanced linear decision rule**

I. Introduction

Aggregations of electric loads can provide reserves to power systems to help manage uncertain power injections from renewables and loads [1]. However, because load usage patterns and ambient conditions are uncertain, we do not usually have perfect distributional information about the capacities of load-based reserves when we solve the optimal power flow (OPF) problem [2].

To manage uncertainty from renewables, loads, and/or load-based reserves researchers have applied robust and stochastic optimization approaches to the OPF problem, e.g., [3]–[5]. A common approach is to use chance constraints in which the probability of constraint violation is restricted [3], [6]–[11]. To handle cases in which the uncertainty distributions are unknown, recent work has developed distributionally robust (DR) OPF approaches, e.g., [12]–[16], which ensure that constraints are satisfied for all distributions within an ambiguity set. A two-stage DR-OPF problem is posed in [16] for joint energy and reserve dispatch with renewable generation.

Reference [17] develops a modular and tractable DR adaptive optimization framework by assuming the recourse decision is an affine function of the uncertainties (i.e., a linear decision rule (LDR)), which adapts decisions to uncertainties while evaluating the worst-case expected value of the second-stage cost over a distributional ambiguity set that incorporates the support and moment information. They propose an enhanced LDR (ELDR), which lifts the uncertainty to a higher dimensional space with auxiliary variables to address infeasibility issues of the LDR in complete recourse problems. This method has been applied to a unit commitment problem with uncertain wind generation [18].

In this paper, we develop a two-stage DR-OPF model to optimize energy and reserve dispatch under uncertainty in renewable generation, load consumption, and load-based reserve capacities. We use the method of [17], including the ELDR, to reformulate and solve the DR-OPF model. We compare the performance of the DR-OPF to that of a DR chance constrained OPF (DR CC-OPF) model and show that, in some circumstances, the DR-OPF produces solutions that are lower cost than and comparably reliable to the DR CC-OPF. However, in general, the performance of the DR-OPF and the DR CC-OPF is similar. We also numerically justify the applicability of the ELDR.

The remainder of this paper is organized as follows. In Section II we formulate the two-stage DR OPF problem and derive a quadratic programming (QP) reformulation. Section III describes the computational setup and Section IV compares the DR-OPF to the DR CC-OPF by showing their optimal solution patterns and out-of-sample performance based on the IEEE 9-bus system. In Section V, we conclude the paper and present future research directions.

II. Formulation

In Section II-A, we propose a two-stage DR-OPF variant of the single-stage DR CC-OPF model in [15], which assumes the DC power flow model. The DR-OPF optimizes energy dispatch in the first stage before realizing uncertain wind generation, load consumption, and load-based reserve capacities. Instead of measuring the uncertain outcomes using chance constraints, we define second-stage auxiliary variables as violations of constraints due to uncertainty, and penalize the amounts of constraint violation. The uncertainties have unknown distributions but we assume that we have historical data from which we know the empirical mean values and average absolute deviations, and we use them to construct an ambiguity set of possible distributions (see Section II-B). In

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Section II-C, we present a QP reformulation of the DR-OPF by applying the ELDR from [17].

A. Two-stage DR-OPF Formulation

We use the same notation as in [15] and $\langle\cdot\rangle$ to denote a stacked column vector. The generation, generator-based (G) and load-based (L) reserve capacities, and distribution vectors (which provide a policy for allocating real-time supply/demand mismatch to reserve providers (see [9], [19])) are defined as "here-and-now" (first-stage) decision variables, i.e., $x = \langle P_G, \overline{R}_G, \underline{R}_G, \overline{R}_L, \underline{R}_L, d_G, d_L\rangle$, which is decided before realizing the uncertainty.

DR-OPF:

$$\min_{x\in X}\left\{c_x^{\mathsf{T}}\langle 1, P_G, P_G^2, \overline{R}_G, \underline{R}_G, \overline{R}_L, \underline{R}_L\rangle + \sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[Q(x,\widetilde{z})\right]\right\},$$

where $c_x = \langle c_0, c_1, c_2, \overline{c}_G, \underline{c}_G, \overline{c}_L, \underline{c}_L\rangle$ is the cost vector and $\widetilde{z}$ includes all uncertain variables including wind power production, load consumption, and load-based reserve capacities (upper and lower), i.e., $\widetilde{z} = \langle \widetilde{P}_W, \widetilde{P}_L, \widetilde{\overline{P}}_L, \widetilde{\underline{P}}_L\rangle$. The feasible region X consists of all constraints that do not involve uncertainty, i.e.,

$$\sum_{i=1}^{N_G} P_{G,i} = \sum_{i=1}^{N_L} P_{L,i}^f - \sum_{i=1}^{N_W} P_{W,i}^f, \tag{1}$$

$$\sum_{i=1}^{N_G} d_{G,i} + \sum_{i=1}^{N_L} d_{L,i} = 1, \tag{2}$$

$$x \geq \mathbf{0}, \tag{3}$$

where (1) enforces power balance and (2) normalizes the distribution vectors. The second term in the objective function is the worst-case (maximum) expected cost incurred in the second stage over an ambiguity set $\mathcal{F}$ of an unknown probability measure $\mathbb{P}$ of $\widetilde{z}$, which we will discuss in detail in Section II-B. Let R_G and R_L be the actual reserves provided by generators and loads, respectively, i.e., $R_G = -d_G P_m$ and $R_L = d_L P_m$, where $P_m = \sum_{i=1}^{N_W}(\widetilde{P}_{W,i} - P_{W,i}^f) - \sum_{i=1}^{N_L}(\widetilde{P}_{L,i} - P_{L,i}^f)$ is the real-time supply/demand mismatch. Then,

$$Q(x,\widetilde{z}) = \min_{y\in Y} c_y^{\mathsf{T}} y, \tag{4}$$

where the feasible region Y contains the following constraints

$$-P_{\text{line}} \leq B_{\text{flow}}\begin{bmatrix} 0 \\ B_{\text{bus}}^{-1}\hat{P}_{\text{inj}}\end{bmatrix} + y_{\text{line,l}},$$

$$B_{\text{flow}}\begin{bmatrix} 0 \\ B_{\text{bus}}^{-1}\hat{P}_{\text{inj}}\end{bmatrix} - y_{\text{line,u}} \leq P_{\text{line}}, \tag{5}$$

$$\underline{P}_G \leq P_G + R_G + y_{\text{Gl}},$$

$$P_G + R_G - y_{\text{Gu}} \leq \overline{P}_G, \tag{6}$$

$$\widetilde{\underline{P}}_L \leq \widetilde{P}_L + R_L + y_{\text{Ll}},$$

$$\widetilde{P}_L + R_L - y_{\text{Lu}} \leq \widetilde{\overline{P}}_L, \tag{7}$$

$$-\underline{R}_G \leq R_G + y_{\text{RGl}},$$

$$R_G - y_{\text{RGu}} \leq \overline{R}_G, \tag{8}$$

$$-\underline{R}_L \leq R_L + y_{\text{RLl}},$$

$$R_L - y_{\text{RLu}} \leq \overline{R}_L, \tag{9}$$

$$y \geq \mathbf{0}, \tag{10}$$

where $y = \langle y_{\text{Gl}}, y_{\text{Gu}}, y_{\text{Ll}}, y_{\text{Lu}}, y_{\text{RGl}}, y_{\text{RGu}}, y_{\text{RLl}}, y_{\text{RLu}}, y_{\text{line,l}}, y_{\text{line,u}}\rangle$ consists of the "wait-and-see" (second-stage) recourse variables, i.e., violations of the second-stage constraints by decision x after realizing $\widetilde{z}$. The cost vector $c_y = \langle c_{\text{Gl}}, c_{\text{Gu}}, c_{\text{Ll}}, c_{\text{Lu}}, c_{\text{RGl}}, c_{\text{RGu}}, c_{\text{RLl}}, c_{\text{RLu}}, c_{\text{line,l}}, c_{\text{line,u}}\rangle$ corresponds to the variable y. Constraints (5)–(9) limit line flows, generation, load, generator-based reserves, and load-based reserves, respectively.

B. Distributional Ambiguity Set

We use $[I]$ to denote the set of indices: $\{1,\ldots,I\}$. Consider the random vector $\widetilde{z} = [\widetilde{z}_i,\ i\in[I_1]]^{\mathsf{T}} \in \mathbb{R}^{I_1}$ of uncertainties with $I_1 = N_W + 3N_L$, where N_W is the number of wind power plants and N_L is the number of loads (and we assume all loads can provide reserves). Recall $\mathbb{P}$ as the true distribution measure of $\widetilde{z}$, and let $\mathcal{P}_0(\mathbb{R}^{I_1})$ be the set of all possible probability measures of $\widetilde{z}$ on $\mathbb{R}^{I_1}$. The distributional ambiguity set is

$$\mathcal{F} = \left\{\mathbb{P}\in\mathcal{P}_0\left(\mathbb{R}^{I_1}\right): \begin{array}{l} \mathbb{E}_{\mathbb{P}}[\widetilde{z}] = \mu \\ \mathbb{E}_{\mathbb{P}}[|\widetilde{z}_i - \mu_i|] \leq \sigma_i,\ \forall i\in[I_1] \\ \mathbb{P}(\widetilde{z}\in\mathcal{V}) = 1 \end{array}\right\}, \tag{11}$$

where $\mathcal{V}$ is the support of $\widetilde{z}$ specified as

$$\mathcal{V} = \left\{\widetilde{z}\in\mathbb{R}^{I_1} : \underline{z}_i \leq \widetilde{z}_i \leq \overline{z}_i,\ \forall i\in[I_1]\right\}. \tag{12}$$

The three constraints in (11) require that i) the mean of the random parameter matches a given vector μ (e.g., an empirical mean), ii) the average absolute deviation is bounded by an empirical value σ_i for each $i\in[I_1]$, and iii) all realizations of $\widetilde{z}$ are within the support $\mathcal{V}$.

Following [17] and [20], we define a *lifted ambiguity set* to encompass the primary random variable $\widetilde{z}$ and an auxiliary (or lifted) random variable $\widetilde{u}$ as

$$\mathcal{G} = \left\{\mathbb{Q}\in\mathcal{P}_0\left(\mathbb{R}^{I_1}\times\mathbb{R}^{I_2}\right): \begin{array}{l} \mathbb{E}_{\mathbb{Q}}[\widetilde{z}_i] = \mu_i,\ \forall i\in[I_1] \\ \mathbb{E}_{\mathbb{Q}}[\widetilde{u}_i] \leq \sigma_i,\ \forall i\in[I_2] \\ \mathbb{Q}((\widetilde{z},\widetilde{u})\in\bar{\mathcal{V}}) = 1 \end{array}\right\}, \tag{13}$$

where $\bar{\mathcal{V}}$ is the *lifted support set* defined as

$$\bar{\mathcal{V}} = \left\{(\widetilde{z}, \widetilde{u}) \in \mathbb{R}^{I_1} \times \mathbb{R}^{I_2}:\ \widetilde{z} \in \mathcal{V},\ |\widetilde{z}_i - \mu_i| \le \widetilde{u}_i,\ i \in [I_2]\right\}. \tag{14}$$

Here I_2 is the dimension of $\widetilde{u}$, which equals the dimension of I_1 in this paper since we associate one element of $\widetilde{u}$ with each element of $\widetilde{z}$. Since functions $|\widetilde{z}_i - \mu_i|$ are piecewise linear, we can express the lifted support set $\bar{\mathcal{V}}$ in a general linear form

$$\bar{\mathcal{V}} = \left\{(\widetilde{z}, \widetilde{u}) \in \mathbb{R}^{I_1} \times \mathbb{R}^{I_2}:\ C\widetilde{z} + D\widetilde{u} \le h\right\}, \tag{15}$$

where $C \in \mathbb{R}^{M_0 \times I_1}$, $D \in \mathbb{R}^{M_0 \times I_2}$, and $h \in \mathbb{R}^{M_0}$, with M_0 as the number of constraints in the lifted support set $\bar{\mathcal{V}}$.

The ambiguity set $\mathcal{F}$ is equivalent to the set of marginal distributions of $\widetilde{z}$ under probability $\mathbb{Q}$, for $\mathbb{Q} \in \mathcal{G}$ in (13). Although sets $\mathcal{F}$ and $\mathcal{G}$ are essentially the same with respect to $\widetilde{z}$, the introduction of the auxiliary variable $\widetilde{u}$ increases the flexibility of implementing the LDR (see [17] for more discussions about this ambiguity set variant).

C. Reformulation using Enhanced Linear Decision Rule

The DR-OPF in Section II-A is intractable because y is an arbitrary function of the uncertainty (see [5], [21]). To overcome this intractability, following [17], we apply the ELDR to restrict y to a class of affine functions of $\widetilde{z}$ and $\widetilde{u}$, leading to a QP reformulation of the DR-OPF.

Let M_1 be the number of constraints in Y, N_1 be the dimension of x, and N_2 be the dimension of vector y. We formulate $Q(x, \widetilde{z})$ in (4) as

$$Q(x, \widetilde{z}) = \min_{y} \left\{c_y^\mathsf{T} y:\ A(\widetilde{z})x + By \ge b(\widetilde{z})\right\}, \tag{16}$$

where $A \in \mathbb{R}^{M_1 \times N_1}$, $B \in \mathbb{R}^{M_1 \times N_2}$, and $b \in \mathbb{R}^{M_1}$. Functions A and b are affinely dependent on $\widetilde{z}$, i.e., $A(\widetilde{z}) = A^0 + \sum_{i=1}^{I_1} A^i \widetilde{z}_i$, $b(\widetilde{z}) = b^0 + \sum_{i=1}^{I_1} b^i \widetilde{z}_i$, where $A^0, A^1, \ldots, A^{I_1} \in \mathbb{R}^{M_1 \times N_1}$, and $b^0, b^1, \ldots, b^{I_1} \in \mathbb{R}^{M_1}$. We apply the ELDR

$$y(\widetilde{z}, \widetilde{u}) = y^0 + \sum_{i \in W} y_i^1 \widetilde{z}_i + \sum_{j \in U} y_j^2 \widetilde{u}_j, \tag{17}$$

where $y^0 \in \mathbb{R}^{N_2}$, $y_i^1 \in \mathbb{R}^{N_2}$ and $y_j^2 \in \mathbb{R}^{N_2}$ are coefficients of the uncertainties. The sets $W \subseteq [I_1]$ and $U \subseteq [I_2]$ reflect the information dependency of adaptive decision $y(\widetilde{z}, \widetilde{u})$, which depends on a subset of the uncertainties. By replacing y in (16) with (17), we obtain the upper bound for $\beta(x) = \sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}}[Q(x, \widetilde{z})]$, given in [17],

$$\overline{\beta}(x) = \min_{y^0, y^1, y^2} \sup_{\mathbb{Q} \in \mathcal{G}} \mathbb{E}_{\mathbb{Q}}\left[c_y^\mathsf{T} y(\widetilde{z}, \widetilde{u})\right] \tag{18a}$$

$$\text{s.t.} \quad A(\widetilde{z})x + By(\widetilde{z}, \widetilde{u}) \ge b(\widetilde{z}),\ \forall (\widetilde{z}, \widetilde{u}) \in \bar{\mathcal{V}}. \tag{18b}$$

After taking the dual of the inner maximization problem, (18) is equivalent to the monolithic minimization problem

$$\overline{\beta}(x) = \min_{r, s, t, y^0, y^1, y^2} r + s^\mathsf{T}\mu + t^\mathsf{T}\sigma \tag{19a}$$

$$\text{s.t.} \quad r + s^\mathsf{T}\widetilde{z} + t^\mathsf{T}\widetilde{u} \ge c_y^\mathsf{T} y(\widetilde{z}, \widetilde{u})\ \forall (\widetilde{z}, \widetilde{u}) \in \bar{\mathcal{V}} \tag{19b}$$

$$\text{(18b)}$$

$$t \ge 0, \tag{19c}$$

where $s \in \mathbb{R}^{I_1}, t \in \mathbb{R}^{I_2}$, and $r \in \mathbb{R}$ are dual variables associated with the three constraints in the ambiguity set $\mathcal{G}$. Note that (19) is a robust optimization model with uncertainty set $\bar{\mathcal{V}}$ of $(\widetilde{z}, \widetilde{u})$. By moving every term of each constraint in (18b) and (19b) to the left side, we consider optimization problems with the left sides as objective functions subject to constraints in the lifted support $\bar{\mathcal{V}}$. Associating dual variables with the constraints of the support $\bar{\mathcal{V}}$, we then reformulate (19) as a QP problem

$$\min_{r, s, t, y^0, y^1, y^2, \pi} r + s^\mathsf{T}\mu + t^\mathsf{T}\sigma \tag{20a}$$

$$\text{s.t.} \quad r - c_y^\mathsf{T} y^0 \ge \pi_0^\mathsf{T} h \tag{20b}$$

$$\pi_0^\mathsf{T} C_i = c_y^\mathsf{T} y_i^1 - s_i,\ i \in W \tag{20c}$$

$$\pi_0^\mathsf{T} C_i = -s_i,\ i \in [I_1]/W \tag{20d}$$

$$\pi_0^\mathsf{T} D_j = c_y^\mathsf{T} y_j^2 - t_j,\ j \in U \tag{20e}$$

$$\pi_0^\mathsf{T} D_j = -t_j,\ j \in [I_2]/U \tag{20f}$$

$$A_l^0 x + B_l y^0 - b_l \ge \pi_l^\mathsf{T} h,\ l \in [M_1] \tag{20g}$$

$$\pi_l^\mathsf{T} C_i = b_l^i - A_l^i x - B_l y_i^1,\ l \in [M_1],\ i \in W \tag{20h}$$

$$\pi_l^\mathsf{T} C_i = b_l^i - A_l^i x,\ l \in [M_1],\ i \in [I_1]/W \tag{20i}$$

$$\pi_l^\mathsf{T} D_j = -B_l y_j^2,\ l \in [M_1],\ j \in U \tag{20j}$$

$$\pi_l^\mathsf{T} D_j = 0,\ l \in [M_1],\ j \in [I_2]/U \tag{20k}$$

$$\pi_0 \ge 0,\ \pi_l \ge 0,\ l \in [M_1] \tag{20l}$$

$$t \ge 0, \tag{20m}$$

where $\pi_l \in \mathbb{R}^{M_0}$ for $l = 0, 1, \ldots, M_1$ are dual variables of constraints in the lifted support set $\bar{\mathcal{V}}$. We transform the semi-infinite constraints (18b) and (19b) into a finite number of dual linear constraints. The vector C_i is the i^{th} column of matrix C, D_j is the j^{th} column of matrix D (both appearing in the linear form (15) of the lifted support set $\bar{\mathcal{V}}$), and b_l^i is the l^{th} element of vector b^i. The vectors A_l^i and B_l represent the l^{th} row of matrices A^i and B, respectively. The number of constraints in (20) grows polynomially in the size of sets W and U.

III. Computational Setup

We solve the QP approximation of the DR-OPF (20) on the IEEE 9-bus system from MATPOWER [22]. As in [15], we add a single wind power plant to Bus 6 (75 MW rated capacity, 50 MW forecasted output) and assume that 30% of the load at Buses 5, 7, and 9 ($P_L^f = (90, 100, 125)$ MW) is uncertain but can provide reserves. We assume the remaining 70% is perfectly forecastable. We assume that all generator-based reserves are higher-cost than load-based reserves, specifically, $\overline{c}_G = \underline{c}_G = 10 \times \mathbf{1}$ and $\overline{c}_L = \underline{c}_L = 5 \times \mathbf{1}$, where $\mathbf{1}$ is a vector of ones of appropriate size. We use the same procedure as in [15] to generate 10,000 i.i.d. samples of $\widetilde{P}_W, \widetilde{P}_L, \widetilde{\overline{P}}_L, \widetilde{\underline{P}}_L$, comprising the sample set Ω. As in [15], we assume that a decision maker has partial knowledge of Ω, i.e., 20 data points out of Ω. With these sample points, we calculate the empirical mean values and average absolute deviations, used as μ and σ in the lifted ambiguity set $\mathcal{G}$. The upper bound and the lower bound of $\widetilde{z}$ in the lifted support $\bar{\mathcal{V}}$ are set as the the maximum and minimum values in the 20 sample set. Except the line flow constraints (5), the penalty costs for violations on lower bounds $c_{\text{Gl}}, c_{\text{Ll}}, c_{\text{RGl}}, c_{\text{RLl}}$ are 500 and for upper bounds $c_{\text{Gu}}, c_{\text{Lu}}, c_{\text{RGu}}, c_{\text{RLu}}$ are 1000. We vary $c_{\text{line,u}}$ and $c_{\text{line,l}}$ between 250 and 400 to penalize $y_{\text{line,l}}$ and $y_{\text{line,u}}$, respectively, in Sections IV-A and IV-B. All computations are performed on a Windows 10 machine with Intel(R) Core(TM) i5-5200U CPU 2.20 GHz and 4GB memory. All optimization problems are solved by CVX implemented in MATLAB with Mosek 7.1.0.12 as the solver.

IV. Computational Results

We compare the solution patterns of the DR-OPF and the DR CC-OPF in Section IV-A and out-of-sample performance in Section IV-B. In Section IV-C, we demonstrate the linear relationship between constraint violation amounts and uncertain quantities to justify the use of the ELDR. In the DR CC-OPF, instead of penalizing the violation of constraints (5)–(9) in the objective function, individual DR chance constraints are employed to ensure that the probability of violating these constraints is less than $\epsilon \in [0, 1]$.

A. Solution Patterns

1) Uncongested system: We set $c_{\text{line,u}} = c_{\text{line,l}}$ and vary their values among $\{250, 280, 310, 340, 370, 400\}$ for the DR-OPF. For the DR CC-OPF, we vary the desired reliability $1 - \epsilon$ among $\{95\%, 85\%, 75\%, 65\%\}$. Figure 1 shows the solution patterns under different costs/reliabilities when the system is uncongested. The bars corresponding to P_{G1}, P_{G2}, P_{G3} are the generation amounts at Generators 1–3. The bars corresponding to $R_G^* = \overline{R}_G + \underline{R}_G$ and $R_L^* = \overline{R}_L + \underline{R}_L$ are the reserve capacities at generators and loads, respectively. The bars corresponding to R^* are the total reserve capacities. The solutions to P_G are identical for both models. As shown in Fig. 1a, for all values of $c_{\text{line,u}} = c_{\text{line,l}}$ the total reserve capacity is the same (i.e., 53.83 MW). This is because the total reserve capacity is decided by the uncertainty support size and we keep the same support size for all penalty cost choices. Moreover, the second-stage expected penalty costs are all zero, because the constraints (5)-(10) are easily satisfied in the uncongested system. As shown in Fig. 1b, reserve capacities increase as $1 - \epsilon$ increases, since more reserves can better manage the uncertainties and thus ensure higher reliability.

2) Congested system: We decrease the line flow limit between Bus 5 and Bus 6 from 150 MW to 40 MW to produce congestion. As shown in Fig. 2a, no reserve capacity is assigned to any generator for $c_{\text{line}} = 250$ and 280, but for $c_{\text{line}} \geq 310$, the generator at Bus 3 takes over for the load at Bus 5. In contrast, in Fig. 2, the load at Bus 7 provides a larger share of the reserves as $1 - \epsilon$ increases, and the generators do not need to provide reserves. For both models, the total reserve capacity is the same as in the uncongested system.

B. Out-of-Sample Test Results

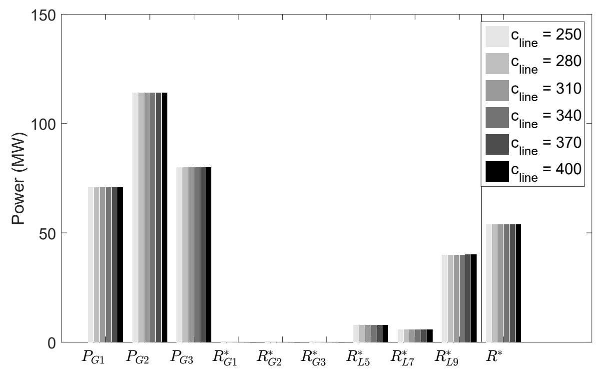
1) Result comparison with similar costs: Using the congested system, we consider cases with different congested lines. In Table I, the column "Line" shows which line is congested. For example, "1-4" means that we lower the flow limit on the transmission line from Bus 1 to Bus 4 until it is congested. For all the cases except "8-9", we decrease the flow limit to 60 MW. For "8-9", we decrease the limit to 80 MW. For the DR-OPF, $c_{\text{line}} = 400$; for the DR CC-OPF, $1 - \epsilon = 95\%$. Columns "Cost", "Reliability", and "R^*" show the total cost, reliability, and reserve capacities. We calculate the "Reliability" as the percentage of the 10,000 samples in the set Ω in which a solution does not violate the limit of the congested line. The costs in Table I only include the first-stage cost in the DR-OPF model, which is the whole objective cost in the DR CC-OPF. When a model produces a lower costs and higher reliability we have bolded the results. In each bolded cases, the DR-OPF is slightly cheaper than the counterpart DR CC-OPF, while the reliability is the same or better than the counterpart DR CC-OPF. The reserve capacities are the same for each instance of each model since each instance of the DR-OPF uses the same support size and since each instance of the DR CC-OPF uses the same $1 - \epsilon$, empirical mean, and covariance for constructing its ambiguity set.

TABLE I: Comparison of cost, reliability, and total reserve capacity for the DR-OPF and DR CC-OPF

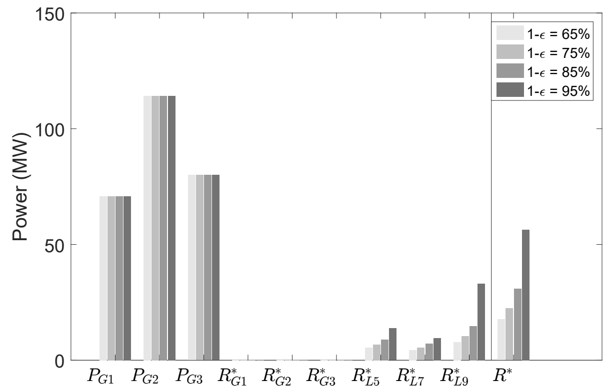
Line	Model	Cost	Reliability	R^*
1-4	DR-OPF	**4388.16**	100.00%	53.83
	DR CC-OPF	4401.08	100.00%	56.42
4-5	DR-OPF	**4369.12**	100.00%	53.83
	DR CC-OPF	4382.05	100.00%	56.42
5-6	DR-OPF	**4651.04**	**100.00%**	53.83
	DR CC-OPF	4655.00	99.95%	56.42
3-6	DR-OPF	**4437.25**	100.00%	53.83
	DR CC-OPF	4450.17	100.00%	56.42
6-7	DR-OPF	**4388.73**	**99.89%**	53.83
	DR CC-OPF	4396.97	99.78%	56.42
7-8	DR-OPF	**4369.12**	100.00%	53.83
	DR CC-OPF	4382.05	100.00%	56.42
8-2	DR-OPF	**4787.64**	100.00%	53.83
	DR CC-OPF	4800.56	100.00%	56.42
8-9	DR-OPF	**4375.74**	**99.99%**	53.83
	DR CC-OPF	4382.06	98.45%	56.42
9-4	DR-OPF	**4371.73**	**99.55%**	53.83
	DR CC-OPF	4382.07	98.45%	56.42

In Table II, we show results for nearly identical cost solutions of the DR-OPF and the DR CC-OPF when line 1-4 is congested. In all cases, the desired reliability is lower than that used to generate the results in Table I. In the first two cases, the DR CC-OPF achieves higher reliabilities than the DR OPF for the same cost solution, while in the last case the DR OPF achieves a higher reliability than the DR CC-OPF. These results indicate that as the desired reliability increases the DR OPF performs (slightly) better than the DR CC-OPF in terms of cost and reliability.

2) Comparison of violation amount: In addition to reliabilities, in Fig. 3 we report histograms of the magnitudes of line flow constraint violations corresponding to the results in Table II. The lighter colored bars correspond to the DR CC-OPF and the darker colored bars correspond to the DR-OPF. Since the DR-OPF penalizes the the magnitude of constraint violations

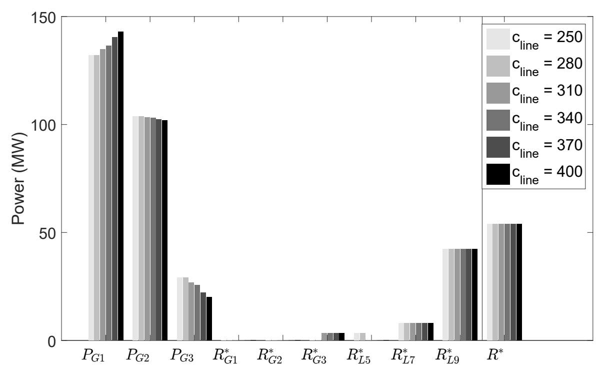


(a) Solutions given different c_{line} penalty in DR-OPF

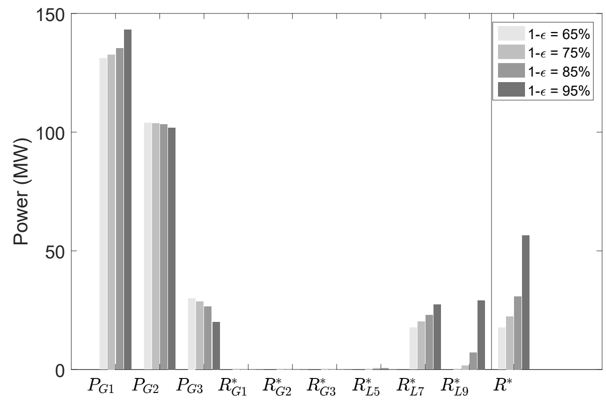


(b) Solutions given different $1-\epsilon$ in DR CC-OPF

Fig. 1: Solution patterns in the uncongested 9-bus system



(a) Solutions given different c_{line} penalty in DR-OPF



(b) Solutions given different $1-\epsilon$ in DR CC-OPF

Fig. 2: Solution patterns in the congested 9-bus system

TABLE II: Comparison of cost, reliability, and total reserve capacity under equivalent costs

$c_{\text{line}}/1-\epsilon$	Model	Cost	Reliability	R^*
320	DR-OPF	5007.29	92.31%	30.74
79.9%	DR CC-OPF	5007.51	**94.20%**	25.80
338	DR-OPF	5048.54	95.44%	30.74
83.4%	DR CC-OPF	5048.60	**95.93%**	29.01
350	DR-OPF	5098.47	**97.72%**	30.74
86.6%	DR CC-OPF	5098.50	97.66%	32.90

we would expect its violations to be of a smaller magnitude than those of the DR CC-OPF; however, we find that that the results are similar.

C. Justification of the ELDR

In this section, we show that the inclusion of the auxiliary variable $\widetilde{u}$ in (17) provides a better fit for the second-stage decision variables than only including $\widetilde{z}$ i.e., $y(\widetilde{z}) = \hat{y}^0 + \sum_{i\in W} \hat{y}^1_i \widetilde{z}_i$, where $\hat{y}^0 \in \mathbb{R}^{N_2}$ is the constant and $\hat{y}^1_i \in \mathbb{R}^{N_2}$ are the coefficients of the QP approximation (20) using all the samples in set Ω to obtain $\widetilde{y}$ and $\widetilde{u} = |\widetilde{z} - \mu|$ for each realization of $\widetilde{z}$. To do this, we derive least square regressions of $\widetilde{y}$ versus $\widetilde{z}$ (referred to as Regression 1) and versus $\langle \widetilde{z}, \widetilde{u} \rangle$ (referred to as Regression 2), respectively. We compute their p-values and adjusted R-squared values (see [23]). If the p-value is sufficiently small, i.e., ≤ 0.05, there is a significant relationship between the variables in the linear regression model. The adjusted R-squared value is the percentage of the response variable variation explained by a linear model [23].

We lower the cost parameter c_y, which increases the number of violations, enabling regression with more data points. In Table III, we show the adjusted R-squared values of the DR-OPF for $0.010c_y$ and $0.028c_y$ for constraints (8) and (9), each of which contain six inequalities. For both regressions, the p-value results are less than 0.05. For all constraints, the adjusted R-squared values of Regression 2 are higher than those of Regression 1, which shows the explanatory power added by the auxiliary variable $\widetilde{u}$.

V. Conclusion

In this paper, we compared the performance of multi-stage distributionally robust OPF and distributionally robust

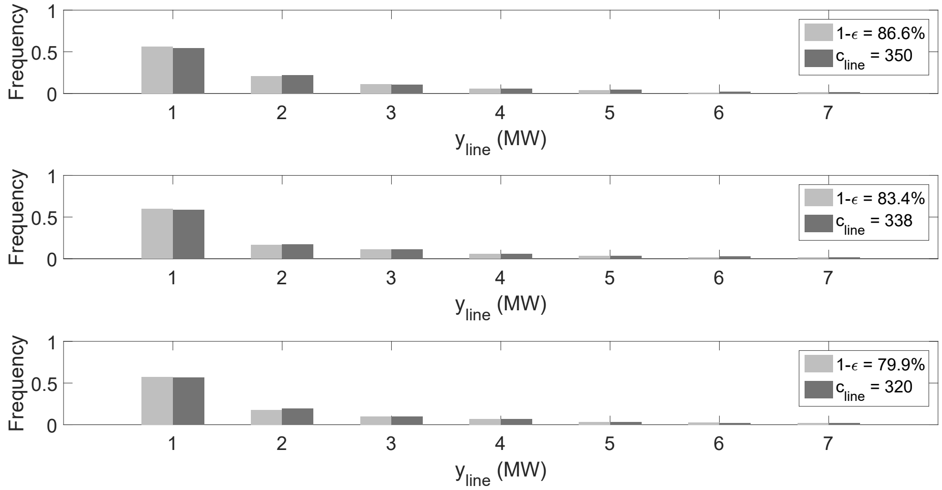


Fig. 3: Histogram of constraint violations under equivalent costs

TABLE III: Adjusted R-squared values of linear regressions

Penalty	Constraint (8)	1	2	3	4	5	6
$0.010c_y$	Regression 1	0.86	0.86	0.86	0.54	0.54	0.54
	Regression 2	0.91	0.91	0.91	0.70	0.70	0.70
Penalty	Constraint (9)	1	2	3	4	5	6
$0.028c_y$	Regression 1	0.54	0.54	0.54	0.42	0.42	0.42
	Regression 2	0.70	0.70	0.70	0.59	0.59	0.59

chance constrained OPF models including renewable, load, and load-based reserve uncertainty. Out of sample tests showed that the DR-OPF model yielded solutions with slightly better reliability performance and lower cost than DR CC-OPF when both models need to achieve high reliability (close to 100%). When the reliability requirement is relatively low, DR-OPF could perform worse than DR CC-OPF. We also provided justification for the enhanced linear decision rule (used to formulate the problem as a quadratic program) to show that the inclusion of auxiliary variables strengthened its explanatory power. For future research, a natural extension is to apply the ELDR to a multi-stage DR-OPF problem.

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