Distributionally Robust Chance-Constrained Optimal Power Flow With Uncertain Renewables and Uncertain Reserves Provided by Loads

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***Abstract*—Aggregations of electric loads can provide reserves to power systems, but their available reserve capacities are time-varying and not perfectly known when the system operator computes the optimal generation and reserve schedule. In this paper, we formulate a chance constrained optimal power flow problem to procure minimum cost energy, generator reserves, and load reserves given uncertainty in renewable energy production, load consumption, and load reserve capacities. Assuming that uncertainty distributions are not perfectly known, we solve the problem with distributionally robust optimization, which ensures that chance constraints are satisfied for any distribution in an ambiguity set built upon the first two moments. We use two ambiguity sets to reformulate the model as a semidefinite program and a second-order cone program and run computational experiments on the IEEE 9-bus, 39-bus, and 118-bus systems. We compare the solutions to those given by two benchmark reformulations; the first assumes normally distributed uncertainty and the second uses large numbers of uncertainty samples. We find that the use of load reserves, even when load reserve capacities are uncertain, reduces operational costs. Also, the approach is able to meet reliability requirements, unlike the first benchmark approach and with lower computation times than the second benchmark approach.**

***Index Terms*—Chance-constrained optimal power flow (CC-OPF), convex optimization, load control, moment-based ambiguity set, uncertain reserves.**

NOMENCLATURE

A. Numbers

N_{line}	Number of transmission lines
N_B	Number of buses
N_G	Number of conventional generators
N_W	Number of wind power plants
N_L	Number of loads
m	Number of chance constraints
n	Number of decision variables

B. Parameters

$P_W^f \in \mathbb{R}^{N_W}$	Forecasted wind production
$P_L^f \in \mathbb{R}^{N_L}$	Forecasted load consumption
$\underline{P}_G, \overline{P}_G \in \mathbb{R}^{N_G}$	Min/max generator production
$P_{\text{line}} \in \mathbb{R}^{N_{\text{line}}}$	Line flow limit
c	Vector of energy/reserve costs
$B_{\text{bus}} \in \mathbb{R}^{(N_B-1)\times(N_B-1)}$	Bus susceptance matrix
$B_{\text{flow}} \in \mathbb{R}^{N_{\text{line}}\times N_B}$	Flow susceptance matrix
ϵ_i	Violation probability of chance constraint i

C. Random Variables

$\widetilde{P}_W \in \mathbb{R}^{N_W}$	Actual wind production
$\widetilde{P}_L \in \mathbb{R}^{N_L}$	Actual load consumption
$\underline{\widetilde{P}}_L, \overline{\widetilde{P}}_L \in \mathbb{R}^{N_L}$	Actual min/max possible load consumption

D. Auxiliary Random Variables

$P_{\text{mis}} \in \mathbb{R}$	Real-time supply/demand mismatch
$R_G \in \mathbb{R}^{N_G}$	Actual generator reserve dispatch
$R_L \in \mathbb{R}^{N_L}$	Actual load reserve dispatch
$P_{\text{inj}} \in \mathbb{R}^{N_B}$	Net power injections at each bus

E. Decision Variables

$P_G \in \mathbb{R}^{N_G}$	Generator production
$\overline{R}_G, \underline{R}_G \in \mathbb{R}^{N_G}$	Generator up/down-reserve capacities
$\overline{R}_L, \underline{R}_L \in \mathbb{R}^{N_L}$	Load up/down-reserve capacities
$\overline{d}_G, \underline{d}_G \in \mathbb{R}^{N_G}$	Generator up/down distribution vectors
$\overline{d}_L, \underline{d}_L \in \mathbb{R}^{N_L}$	Load up/down distribution vectors

I. INTRODUCTION

FLEXIBLE loads, such as heating and cooling systems, coordinated via load control algorithms are capable of providing reserves to power systems [1]–[6]. However, the capacity (in MW) of reserves that loads could provide is generally time-varying [7], [8]. For example, the reserve capacity of an aggregation of air conditioners is a function of weather and load usage patterns [9], which are uncertain. Therefore, when the power system operator computes the optimal generation and reserve schedule, the future reserve capacity that will be available from loads is not perfectly known. A conservative option would be to forecast the expected reserve capacity and schedule only a portion of it. Instead, we solve a chance constrained optimal power flow (CC-OPF) problem that explicitly considers uncertain reserves from loads, enabling us to more-effectively use this resource.

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CC-OPF problems to manage renewable energy and load uncertainty have been posed in [10]–[15], though none of this work considers load reserves or their uncertainty. Ref. [16] formulates a multi-period CC-OPF with uncertain load reserves. It is solved with a robust reformulation [17] of the scenario approach [18] that makes no assumption on the uncertainty distributions but requires large numbers of uncertainty samples. Ref. [19] reformulates the problem assuming Gaussian uncertainty. In practice, it may be difficult to obtain large numbers of uncertainty samples and we would expect the uncertainty to be non-Gaussian.

In this paper, which is an extension of our preliminary work [20], we formulate a CC-OPF with uncertain load reserves and reformulate it using distributionally robust (DR) optimization [21], which makes no assumption on uncertainty distributions and does not require large numbers of uncertainty samples. For simplicity of exposition, we use single-period formulation, rather than the multi-period formulation of [16], [19]. Using the empirical mean and covariance (i.e., the first two moments) of a small number of uncertainty samples, we construct two types of convex ambiguity sets for the unknown distribution, yielding a semidefinite programming (SDP) reformulation [22], and a tractable second-order conic programming (SOCP) reformulation [23]. We compare the two DR models with two benchmark approaches, the first of which is an SOCP model obtained by Gaussian approximation, and the second of which is a large-scale quadratic programming (QP) model obtained by scenario approximation.

There has been significant recent work on robust OPF, CC-OPF, and DR optimization applied to power system problems. For example, [12] formulates a robust OPF problem in which generator participation factors are chosen to ensure a feasible solution for all possible renewable energy injections. The formulation is extended to include chance constraints that manage normally-distributed load forecast error. Ref. [24] further extends the method to a multi-period formulation with energy storage. The method assumes the uncertainty set is known. CC-OPFs are posed by [10], [11], [14] and solved with scenario-based methods [11] and analytical reformulation assuming normally-distributed uncertainty [10], [13], [14]. DR optimization has been applied to the dynamic line rating problem [25] and to quantifying the probability of infeasible dispatch given uncertain wind power injections [26]. The latter uses data-determined uncertainty moments to derive an SDP formulation. Ref. [15] proposes DR analytical reformulations of a security constrained CC-OPF. Specifically, the mean, covariance, and structured properties (e.g., symmetry or unimodality) of the uncertainty are used to derive an upper bound for the inverse cumulative distribution function, resulting in a linear programming reformulation. Using upper bounds rather than moments, as in [26] and our approach, results in conservative solutions. Ref. [27] poses a robust CC-OPF problem assuming the uncertainty is normally-distributed but the parameters of the normal distributions are unknown. The paper develops a cutting-plane approach that scales to large systems, and demonstrates the computational efficiency by testing a network with 2209 buses. Note that none of the papers cited in this paragraph consider load reserves or their uncertainty.

The contributions of this study are to i) formulate a single-period CC-OPF with uncertain load reserves; ii) reformulate the CC-OPF, using DR optimization with two types of ambiguity sets, as an SDP and an SOCP; and iii) compare the cost, reliability, computational time, and optimal decisions of the approach to that of two benchmark approaches. Beyond our preliminary work in [20], in this paper we have i) corrected the formulation, ii) added a more complete description of the SDP reformulation and derived an SOCP reformulation by using a strengthened ambiguity set, iii) generated results for the IEEE 39-bus and 118-bus systems (in [20] we considered only the IEEE 9-bus system), iv) computed solutions to a variety of additional cases to put the objective costs in context, and v) included comparisons of the optimal solutions, which give significant additional insights into the performance of various approaches. Our results show that the DR approach provides a good tradeoff between cost, reliability, and computational tractability as compared to the other approaches.

The paper is organized as follows. In Section II, we present the CC-OPF and the DR reformulations. Section III presents the two benchmark approaches. Sections IV and V present the computational results of the three approaches for the IEEE 9-bus, 39-bus, and 118-bus systems, respectively. Section VI summarizes the paper and discusses future research.

II. Modeling

We formulate a CC-OPF which minimizes the costs of producing energy and providing reserves while ensuring that stochastic constraints are satisfied with certain probabilities. We consider a power system in which a fraction of the load at each bus is flexible and can provide reserves by increasing/decreasing its consumption from its baseline consumption. However, we assume that each load's minimum and maximum possible consumption are uncertain. For example, the minimum/maximum possible consumption of an aggregation of air conditioners or heat pumps is a function of the number of air conditioners that are operating, which is a function of outdoor temperature, which is uncertain. Details on the underlying flexible load model used within our formulation are given in Section V-B.

As in [11], [13], [14], we use the dc (i.e., linearized) power flow approximation. Also, for ease of exposition and results interpretation, we consider single period (e.g., 1 h), unlike [16] which considers a multi-period OPF. We first formulate the CC-OPF and then its DR variant.

A. Chance-Constrained Optimal Power Flow

The linear inequalities that involve random variables are

$$\begin{aligned}\widetilde{A}x \geq \widetilde{b} = \Big\{ & \underline{P}_G \leq P_G + R_G \leq \overline{P}_G, \\ & \underline{\widetilde{P}}_L \leq \widetilde{P}_L + R_L \leq \overline{\widetilde{P}}_L, \\ & -\underline{R}_G \leq R_G \leq \overline{R}_G, \\ & -\underline{R}_L \leq R_L \leq \overline{R}_L, \\ & -P_{\text{line}} \leq B_{\text{flow}} \begin{bmatrix} 0 \\ B_{\text{bus}}^{-1}\hat{P}_{\text{inj}} \end{bmatrix} \leq P_{\text{line}} \Big\}, \end{aligned} \tag{1}$$

which limit generation, load, generator reserves, load reserves, and line flows, respectively. The notation is defined in Section I, with $\widetilde{\cdot}$ used to denote random variables. The vector of net power injections at each bus is $P_{\text{inj}} \in \mathbb{R}^{N_B}$ is $C_G(P_G + R_G) + C_W\widetilde{P}_W - C_L(\widetilde{P}_L + R_L)$, where C_G, C_W, and C_L are matrices mapping generators, wind power plants, and loads to buses; $\hat{P}_{\text{inj}} \in \mathbb{R}^{N_B-1}$ contains the last $N_B - 1$ rows of P_{inj}; B_{bus} is the bus susceptance matrix (including the susceptances between each bus except the slack bus, which is assumed to be Bus 1) and so the quantity in square brackets is the vector of voltage angles; and B_{flow} is the flow susceptance matrix (which is used to compute the line flows by multiplying each line susceptance by the difference in voltage angle across the line). The full optimization problem is

[CC-OPF]:

$$\min c^{\mathsf{T}}\langle 1, P_G, P_G^2, \overline{R}_G, \underline{R}_G, \overline{R}_L, \underline{R}_L\rangle \tag{2}$$

$$\text{s.t. } P_{\text{mis}} = \sum_{i=1}^{N_W}(\widetilde{P}_{W,i} - P_{W,i}^f) - \sum_{i=1}^{N_L}(\widetilde{P}_{L,i} - P_{L,i}^f) \tag{3}$$

$$\sum_{i=1}^{N_G} P_{G,i} = \sum_{i=1}^{N_L} P_{L,i}^f - \sum_{i=1}^{N_W} P_{W,i}^f \tag{4}$$

$$\sum_{i=1}^{N_G} \underline{d}_{G,i} + \sum_{i=1}^{N_L} \overline{d}_{L,i} = 1 \tag{5}$$

$$\sum_{i=1}^{N_G} \overline{d}_{G,i} + \sum_{i=1}^{N_L} \underline{d}_{L,i} = 1 \tag{6}$$

$$R_G = \overline{d}_G \max\{-P_{\text{mis}}, 0\} - \underline{d}_G \max\{P_{\text{mis}}, 0\} \tag{7}$$

$$R_L = \overline{d}_L \max\{P_{\text{mis}}, 0\} - \underline{d}_L \max\{-P_{\text{mis}}, 0\} \tag{8}$$

$$\mathbb{P}\left(\widetilde{A}_i x \geq \widetilde{b}_i\right) \geq 1 - \epsilon_i \quad \forall i = 1, \ldots, m \tag{9}$$

$$x = \langle P_G, \overline{R}_G, \underline{R}_G, \overline{R}_L, \underline{R}_L, \underline{d}_G, \overline{d}_G, \underline{d}_L, \overline{d}_L\rangle \geq \mathbf{0}, \tag{10}$$

where we use $\langle\cdot\rangle$ to denote a stacked column vector. The cost vector $c = \langle c_0, c_1, c_2, \overline{c}_G, \underline{c}_G, \overline{c}_L, \underline{c}_L\rangle$ corresponds to "here-and-now" decisions made before realizing the uncertainty; (3) calculates the real-time supply/demand mismatch; (4) enforces power balance; (5)–(6) normalize the distribution vectors, which provide a policy for allocating P_{mis} to generators and loads, as in [11], [14]; and (7)–(8) compute the actual reserves provided by generators and loads, respectively. We assume that there are m constraints in (1), and use $\widetilde{A}_i$ to represent the ith row of matrix $\widetilde{A}$ and $\widetilde{b}_i$ to represent the ith entry of vector $\widetilde{b}$. The number of rows in matrix $\widetilde{A}$ and entries in vector $\widetilde{b}$ is $(2N_{\text{line}} + 4N_G + 4N_L)$. Each chance constraint i in (9) should be satisfied with probability $1 - \epsilon_i$. The decision variables, listed in (10), are the generator production, generator reserve capacities, load reserve capacities, and distribution vectors. Note that the total reserve requirement (i.e., the sum of all generator and load reserves capacities) is determined endogenously, and is a function of the uncertainty.

We assume symmetric reserve dispatch, i.e., $\overline{d}_G = \underline{d}_G$ and $\overline{d}_L = \underline{d}_L$, since (7)–(8) become linear enabling DR reformulation (the Gaussian approximation approach also requires this assumption, while the scenario approximation approach does not). However, we assume unsymmetrical reserve procurement (as in CAISO and ERCOT [28]), i.e., $\underline{R}_L$ is not enforced to be equal to $\overline{R}_L$ and $\underline{R}_G$ is not enforced to be equal to $\overline{R}_G$. Modeling symmetrical reserve procurement (as in MISO, PJM, NYISO, and NE-ISO [28]) reduces the number of decision variables, does not change the form of the resulting optimization problems, and increases the objective cost.

We could also reformulate the CC-OPF, using sample average approximation, as a mixed-integer quadratic program. However, in [20], via extensive computational studies, we demonstrated that the formulation is computationally intractable, and leads to solutions that are generally worse than solutions obtained by the approaches in this paper, especially when the true distribution of the underlying uncertainty is unknown. Therefore, we do not include it here.

B. DR Reformulation

We first introduce the DR approach for reformulating the CC-OPF, which builds an ambiguity set to bound the probability density function (pdf) of the underlying uncertainty. Our approach follows that of [22], which we summarize here for completeness. To the best of our knowledge, this is the first application of this approach to a CC-OPF problem.

Consider each chance constraint i in (9). Suppose that $(\widetilde{A}_i, \widetilde{b}_i) = (\widetilde{A}_i(\xi^i), \widetilde{b}_i(\xi^i))$, where ξ^i includes the random variables affecting constraint i. Specifically, for all chance constraints except, $\underline{\widetilde{P}}_L \leq \widetilde{P}_L + R_L \leq \overline{\widetilde{P}}_L$, $\xi^i = \langle \widetilde{P}_W, \widetilde{P}_L\rangle$. For $\underline{\widetilde{P}}_L \leq \widetilde{P}_L + R_L$, $\xi^i = \langle \widetilde{P}_W, \widetilde{P}_L, \underline{\widetilde{P}}_L\rangle$ and for $\widetilde{P}_L + R_L \leq \overline{\widetilde{P}}_L$, $\xi^i = \langle \widetilde{P}_W, \widetilde{P}_L, \overline{\widetilde{P}}_L\rangle$. Then, the DR variants of the chance constraints are

$$\inf_{f(\xi^i)\in\mathcal{D}} \mathbb{P}_{\xi^i}(\widetilde{A}_i(\xi^i)x \geq \widetilde{b}_i(\xi^i)) \geq 1 - \epsilon_i \quad \forall i = 1, \ldots, m. \tag{11}$$

Without loss of generality, we can define $\widetilde{A}_i(\xi^i)$ and $\widetilde{b}_i(\xi^i)$ as affine functions of ξ^i (see [29]), i.e.,

$$\widetilde{A}_i(\xi^i) = A_{i0} + \sum_{k=1}^{K_i} A_{ik}\xi_k^i, \quad \widetilde{b}_i(\xi^i) = b_{i0} + \sum_{k=1}^{K_i} b_{ik}\xi_k^i,$$

where K_i is the dimension of ξ^i. Terms A_{i0} and b_{i0} are the deterministic parts of $\widetilde{A}_i(\xi^i)$ and $\widetilde{b}_i(\xi^i)$ and terms A_{ik} and b_{ik} are the affine coefficients of ξ_k^i $\forall k = 1, \ldots, K_i$. As a result, we can reformulate the inner constraint of (11) as $(\bar{A}_i^x)^{\mathsf{T}}\xi^i \leq \bar{b}_i^x$, where $\bar{A}_i^x = \langle b_{i1} - A_{i1}x, \ldots, b_{iK} - A_{iK}x\rangle$ and $\bar{b}_i^x = A_{i0}x - b_{i0}$.

For simplicity, we drop the index i of ξ^i in the following. Given data samples $\{\xi^\ell\}_{\ell=1}^N$ of ξ, we calculate the empirical mean vector $\mu_0 = \frac{1}{N}\sum_{\ell=1}^N \xi^\ell$ and covariance matrix $\Sigma_0 = \frac{1}{N}\sum_{\ell=1}^N(\xi^\ell - \mu_0)(\xi^\ell - \mu_0)^{\mathsf{T}}$, and build an ambiguity set [21]

$$\mathcal{D} = \left\{ f(\xi) : \begin{array}{c} \int_{\xi\in\mathbb{R}^K} f(\xi)d\xi = 1 \\ (\mathbb{E}[\xi] - \mu_0)^{\mathsf{T}}\Sigma_0^{-1}(\mathbb{E}[\xi] - \mu_0) \leq \gamma_1 \\ \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^{\mathsf{T}}] \preceq \gamma_2\Sigma_0 \end{array} \right\},$$

where $\mathbb{R}^K$ is the support of ξ. The ambiguity set $\mathcal{D}$ is determined by μ_0 and Σ_0, and by parameters γ_1 and γ_2. The three constraints in $\mathcal{D}$ ensure that (i) the integral of pdf $f(\xi)$ is one; (ii) the true mean of ξ lies in a μ_0-centered ellipsoid bounded by γ_1; and (iii) the true covariance matrix lies in a positive semi-definite cone bounded by $\gamma_2\Sigma_0$. Ref. [21] describes how the values of γ_1 and γ_2 can be chosen based on the data sample size, risk parameter, and desired confidence. In practice, the values of γ_1 and γ_2 represent a decision maker's risk preference and can be used to change solution conservatism. In general, larger values of γ_1 and γ_2 will lead to more conservative (robust) solutions. We follow [21] and set $\gamma_1 = 0$ and $\gamma_2 = 1$.

We solve a minimization problem over the ambiguity set $\mathcal{D}$ of $f(\xi)$, specifically, [21], [22]

$$z_{\mathcal{D}} = \min_{f(\xi)} \int_{\mathbb{R}^K} \mathbb{I}_{\mathcal{A}}(\xi) f(\xi) \mathrm{d}\xi \tag{12}$$

$$\text{s.t.} \int_{\mathbb{R}^K} f(\xi) d\xi = 1 \tag{13}$$

$$\int_{\mathbb{R}^K} \begin{bmatrix} \Sigma_0 & \xi - \mu_0 \\ (\xi - \mu_0)^\mathsf{T} & \gamma_1 \end{bmatrix} f(\xi) d\xi \succeq \mathbf{0} \tag{14}$$

$$\int_{\mathbb{R}^K} (\xi - \mu_0)(\xi - \mu_0)^\mathsf{T} f(\xi) d\xi \preceq \gamma_2 \Sigma_0, \tag{15}$$

where $\mathbb{I}_{\mathcal{A}}(\xi)$ is an indicator function which equals 1 if $\xi \in \mathcal{A} = \{\xi : (\bar{A}_i^x)^\mathsf{T}\xi \le \bar{b}_i^x\}$ and 0 otherwise, and (13)–(15) are the constraints in set $\mathcal{D}$ in integral form. The generalized inequality for symmetric matrices, $X \succeq Y$, means that $X - Y$ is a positive semidefinite matrix; similarly, $X \preceq Y$, means that $Y - X$ is a positive semidefinite matrix. A DR chance constraint i is satisfied when $z_{\mathcal{D}} \ge 1 - \epsilon_i$.

Note that the set $\mathcal{D}$ could be very conservative as the worst-case distributions consist of few discrete points [30]. Recent literature investigated inclusion of structural properties (e.g., unimodality) to exclude discrete distributions from the ambiguity set (see, e.g., [31], [32]). Others use higher order moments and strengthened supports (e.g., [33]) to reduce the conservatism of the DR approach. However, the construction of set $\mathcal{D}$ also needs to ensure tractability of the reformulation. In the next section, we propose an alternative ambiguity set that matches the exact mean and covariance, which leads to a tractable SOCP reformulation.

The DR chance constraints (11) are equivalent to the following SDP model [22]:

$$\gamma_2\Sigma_0 \cdot G_i + 1 - r_i + \Sigma_0 \cdot H_i + \gamma_1 q_i \le \epsilon_i y_i \tag{16}$$

$$\begin{bmatrix} G_i & -p_i \\ -p_i^\mathsf{T} & 1 - r_i \end{bmatrix} \succeq \begin{bmatrix} \mathbf{0} & \frac{1}{2}\bar{A}_i^x \\ \frac{1}{2}(\bar{A}_i^x)^\mathsf{T} & y_i + (\bar{A}_i^x)^\mathsf{T}\mu_0 - \bar{b}_i^x \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} G_i & -p_i \\ -p_i^\mathsf{T} & 1 - r_i \end{bmatrix} \succeq \mathbf{0} \tag{18}$$

$$\begin{bmatrix} H_i & p_i \\ p_i^\mathsf{T} & q_i \end{bmatrix} \succeq \mathbf{0} \tag{19}$$

$$y_i \ge 0, \tag{20}$$

for $i = 1, \dots, m$, where we use $X \cdot Y$ to denote the Frobenius inner product of X and Y (i.e., $X \cdot Y = \mathrm{tr}(X^\mathsf{T}Y)$) and $y_i \in \mathbb{R}$, $r_i \in \mathbb{R}$, $q_i \in \mathbb{R}$, $p_i \in \mathbb{R}^K$, $H_i \in \mathbb{R}^{K\times K}$, and $G_i \in \mathbb{R}^{K\times K}$. The proof, given in [22], uses conic duality, and is summarized as follows. Denote $\mathbb{S}_+^K$ as the set of symmetric positive semidefinite $K \times K$ matrices. Define dual variables r_i for (13), G_i for (14), and

$$\begin{bmatrix} H_i & p_i \\ p_i^\mathsf{T} & q_i \end{bmatrix}$$

for (15). The conic dual of (12)–(15) is [22]

$$z_{\mathcal{D}} = \max_{G_i, H_i, p_i, q_i, r_i} -\gamma_2\Sigma_0 \cdot G_i + r_i - \Sigma_0 \cdot H_i - \gamma_1 q_i \tag{21}$$

$$\text{s.t.} \quad (\xi - \mu_0)^\mathsf{T}(-G_i)(\xi - \mu_0) + 2p_i^\mathsf{T}(\xi - \mu_0) + r_i \le \mathbb{I}_{\mathcal{A}}(\xi), \ \forall \xi \in \mathbb{R}^K \tag{22}$$

$$G_i \in \mathbb{S}_+^{K\times K} \tag{23}$$

$$\begin{bmatrix} H_i & p_i \\ p_i^\mathsf{T} & q_i \end{bmatrix} \in \mathbb{S}_+^{(K+1)\times(K+1)}. \tag{24}$$

As strong duality holds for the primal and dual problems, the existence of feasible solutions to (11) is equivalent to having $z_{\mathcal{D}} \ge 1 - \epsilon_i, \ \forall i = 1, \dots, m$. After applying Lemma 1 in [22], the SDP formulation (16)–(20) directly follows from the dual formulation (21)–(24) by replacing the semi-infinite constraints (22) by finite SDP constraints.

Importantly, note that the above approach for bounding the unknown pdf $f(\xi)$ is general and allows the uncertainty ξ to be time-varying and correlated. The complete DR CC-OPF model is

$$\min_{x,y,r,H,G,p,q} \{(2) : (3) - (8), (10), (16) - (20) \, \forall i = 1, \dots m\}.$$

C. An Alternative Ambiguity Set and DR Reformulation

In this section, we provide a DR reformulation based on an alternative ambiguity set

$$\mathcal{D}' = \left\{ f(\xi) : \begin{array}{c} \int_{\xi \in \mathbb{R}^K} f(\xi) d\xi = 1 \\ \mathbb{E}[\xi] - \mu_0 = 0 \\ \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^\mathsf{T}] = \Sigma_0 \end{array} \right\},$$

which requires that the true mean and covariance matrix of ξ, given by any distribution in set $\mathcal{D}'$, be exactly the empirical mean μ_0 and covariance Σ_0. When using $\gamma_1 = 0$ and $\gamma_2 = 1$ in the set $\mathcal{D}$, we have $\mathcal{D}' \subset \mathcal{D}$, and thus this set produces less conservative solutions by placing more trust in μ_0 and Σ_0.

With this ambiguity set, the problem can be reformulated as an SOCP, which is more efficient to compute than the SDP reformation using set $\mathcal{D}$. Specifically, we rewrite the DR constraints (11) as

$$\sup_{f(\xi) \in \mathcal{D}'} \mathbb{P}_\xi \left((\bar{A}_i^x)^\mathsf{T}\xi \ge \bar{b}_i^x \right) \le \epsilon_i \quad \forall i = 1, \dots, m, \tag{25}$$

which are equivalent to

$$\sqrt{(\bar{A}_i^x)^\top \Sigma_0 \bar{A}_i^x} \le \sqrt{\frac{\epsilon_i}{1-\epsilon_i}} (\bar{b}_i^x - (\mu_0)^\top \bar{A}_i^x) \quad \forall i = 1, \ldots, m. \tag{26}$$

The derivation follows a variant of the Chebyshev inequality and was given in [23] for general DR individual chance constraints. We demonstrate the procedure and result for the DR CC-OPF as follows. Consider a variant of the Chebyshev inequality for random variable X with mean μ and variance σ^2 as $\mathbb{P}[X \ge (1+\delta)\mu] \le \frac{\sigma^2}{\sigma^2 + \mu^2\delta^2}$ for any constant $0 \le \delta \le 1$. According to [34], there exists a distribution in $\mathcal{D}'$ to make the inequality tight. Then, the equivalence between (25) and (26) can be established by letting

$$\delta = -1 + \frac{\bar{b}_i^x}{\mathbb{E}[(\bar{A}_i^x)^\top \xi]} = -1 + \frac{\bar{b}_i^x}{(\bar{A}_i^x)^\top \mathbb{E}[\xi]}.$$

Therefore, by using a new ambiguity set $\mathcal{D}'$, we can solve the alternative DR CC-OPF model

$$\min_{x,y,r,H,G,p,q} \{(2) : (3)\text{–}(8), (10), (26)\}.$$

III. Benchmark Approaches

We compare the DR approach to two benchmark approaches used in the literature to solve CC-OPF problems. Both use statistical information and derive convex approximations for the exact CC-OPF.

A. Reformulation via Gaussian Approximation

Assuming Gaussian uncertainty, the CC-OPF can be reformulated as a convex program [13], [14]. Prior research has not applied this approach to a CC-OPF with uncertain load reserves, with the exception of [19], which used the multi-period CC-OPF formulation from [16]. We briefly describe the derivation of the convex program. First, consider constraints (9) in an equivalent form

$$\mathbb{P}\left(\widetilde{A}_i' \bar{x} \le b_i'\right) \ge 1 - \epsilon_i \quad i = 1, \ldots, m, \tag{27}$$

where only the constraint vector $\widetilde{A}_i'$ is uncertain and the right-hand side scalar b_i' is deterministic. This is because for each constraint i of the form (9) where both $\widetilde{A}_i$ and $\widetilde{b}_i$ are random, we can always define an artificial variable $x_b \in \mathbb{R}$ and rewrite $\widetilde{A}_i x \ge \widetilde{b}_i$ as

$$-\widetilde{A}_i x + \widetilde{b}_i x_b \le 0 \;\Leftrightarrow\; \langle -\widetilde{A}_i, \widetilde{b}_i \rangle^\top \langle x, x_b \rangle \le 0,$$

for which we enforce $x_b = 1$. Consequently, we have $\widetilde{A}_i' = \langle -\widetilde{A}_i, \widetilde{b}_i \rangle$, $\bar{x} = \langle x, x_b \rangle$, and $b_i' = 0, \forall i = 1, \ldots, m$ in (27).

If $\widetilde{A}_i'$ follows a multivariate normal distribution $\mathcal{N}(\mu_i, \Sigma_i)$ with μ_i and Σ_i being the mean and covariance of $\widetilde{A}_i'$, respectively, then $\widetilde{A}_i' \bar{x} - b_i'$ follows a multivariate normal distribution $\mathcal{N}(\mu_i^\top \bar{x} - b_i', \bar{x}^\top \Sigma_i \bar{x})$. As a result,

$$\mathbb{P}(\widetilde{A}_i' \bar{x} \le b_i') = \Phi\left(\frac{b_i' - \mu_i^\top \bar{x}}{\bar{x}^\top \Sigma_i \bar{x}}\right) \quad i = 1, \ldots, m, \tag{28}$$

following which, constraints (9) are equivalent to

$$b_i' - \mu_i^\top \bar{x} + \Phi^{-1}(\epsilon_i)\sqrt{\bar{x}^\top \Sigma_i \bar{x}} \ge 0 \quad i = 1, \ldots, m, \tag{29}$$

where $\Phi^{-1}(\epsilon_i)$ denotes the ϵ_i-quantile of the standard normal distribution. We rewrite (29) as

$$b_i' - \mu_i^\top \bar{x} \ge \Phi^{-1}(1-\epsilon_i)\sqrt{\bar{x}^\top \Sigma_i \bar{x}} \quad i = 1, \ldots, m, \tag{30}$$

which are SOCP constraints if $\Phi^{-1}(1-\epsilon_i) \ge 0$, i.e., $1 - \epsilon_i \ge 0.5$. This is a mild assumption since the chance constraints must be satisfied with probabilities much higher than 0.5. The first benchmark approach solves an SOCP model:

$$\min_x \{(2) : (3)\text{–}(8), (10), (30)\}.$$

If the uncertainties are not Gaussian, then the above model is a convex approximation for the exact CC-OPF.

Given the definitions of $\bar{A}_i^x$, $\bar{b}_i^x$, μ_0, Σ_0 in (26) and $\widetilde{A}_i'$, b_i', μ_i, Σ_i, $\bar{x}$ in (30), we can verify that $\bar{b}_i^x - (\mu_0)^\top \bar{A}_i^x$ is equivalent to $b_i' - \mu_i^\top \bar{x}$, and $(\bar{A}_i^x)^\top \Sigma_0 \bar{A}_i^x$ is equivalent to $\bar{x}^\top \Sigma_i \bar{x}$, for all $i = 1, \ldots, m$. For a particular ϵ_i, the coefficient $\sqrt{(1-\epsilon_i)/\epsilon_i}$ (which would appear on the left side of (26) after dividing both sides by $\sqrt{\epsilon_i/(1-\epsilon_i)}$) is greater than the corresponding coefficient, $\Phi^{-1}(1-\epsilon_i)$, in (30). Thus, the DR CC-OPF model with constraint (26) will result in more conservative solutions than the Gaussian approximation approach.

B. Reformulation via Scenario Approximation

Ref. [18] proposes a scenario approximation method for CC optimization problems by enforcing $A_i^s x \ge b_i^s$, $\forall i = 1, \ldots, m$ in all samples s in an approximate sample set Ω_{ap}. The samples in Ω_{ap} could be (i) data observations or (ii) generated from a known distribution by using Monte Carlo sampling. It usually requires a large sample size $|\Omega_{\text{ap}}|$ to guarantee reliability $1 - \epsilon_i$ with high confidence. Each chance constraint i in (9) is replaced with

$$A_i^s x \ge b_i^s \;\forall s \in \Omega_{\text{ap}}. \tag{31}$$

It is shown in [18] that $|\Omega_{\text{ap}}| \ge \frac{2}{\epsilon}(\ln\frac{1}{\beta} + n)$ where $1 - \beta$ is the confidence level and n is the dimension of the decision vector x. Therefore, the second benchmark approach solves a QP model:

$$\min_x \{(2) : (3)\text{–}(8), (10), (31)\}.$$

A variant of this approach was applied in [11], [16], which use a robust reformulation [17] that reduces the required number of samples and the computational time. Moreover, one could employ scenario reduction procedures from the sample average approximation literature to further improve the solution time of the above model.

IV. Studies on the IEEE 9-Bus System

We first compute solutions for the IEEE 9-bus system using the two benchmark approaches and the DR approach, assuming unknown distributions of the uncertainties. We refer to the approaches as:

1) A1: Gaussian approximation approach (SOCP)
2) A2: Scenario approximation approach (QP)

3) A3: DR approach with ambiguity set $\mathcal{D}$ (SDP)
4) A4: DR approach with ambiguity set $\mathcal{D}'$ (SOCP)

All computational tests are performed on a Windows 7 machine with Intel(R) Core(TM) i7-2600 CPU 3.40 GHz and 8GB memory. All models are solved by CVX implemented in MATLAB with MOSEK as the optimization solver [35].

A. Test System

We obtained parameters for the IEEE 9-bus system from MATPOWER [36], and assume that generator reserves are more expensive than load reserves, specifically, $\overline{c}_G = \underline{c}_G = 10 \times \mathbf{1}$ and $\overline{c}_L = \underline{c}_L = 9.8 \times \mathbf{1}$, where $\mathbf{1}$ is a vector of ones of appropriate size. If generator reserves are less expensive than load reserves, the optimal solution is not to use load reserves as they are uncertain and so less valuable to the system. If the cost of generator and load reserves are set equal, the optimal solution is generally to use a combination of both types of reserves. This is because use of generator reserves can increase generator production costs as generators are dispatched sub-optimally to accommodate reserve provision, and this makes use of some (less valuable) load reserves cost effective.

We add one wind power plant at Bus 6 with rated capacity 75 MW and forecasted output 50 MW. We assume 70% of the load at each bus is perfectly forecastable and not flexible. The remaining load at each bus is flexible but uncertain (in terms of both consumption and min/max possible consumption). The load forecasts are set equal to the test case loads.

B. Flexible Load Modeling

We assume that the flexible loads are aggregations of electric heat pumps, and each load aggregation can be modeled as a thermal energy storage unit [7], [8] with power capacity P_c, energy capacity E_c, and baseline power consumption P_L, which all vary with (uncertain) outdoor air temperature $\widetilde{\theta}$. We use Fig. 1 of [16] as a look-up table to determine $P_c(\widetilde{\theta})$, $E_c(\widetilde{\theta})$, and $\widetilde{P}_L = P_L(\widetilde{\theta})$.

If the power system operator does not provide a method to manage the energy states of energy-constrained reserve resources, the real-time reserve signal will determine the energy states of the load aggregations at the end of each scheduling period—a zero-mean signal will result in final energy states equivalent to the initial energy states, while all other signals will result in final energy states that are different than initial energy states. At the beginning of each scheduling period, we can compute the capacity (in MW) of reserves that each load aggregation can provide as a function of its forecasted P_c, E_c, P_L, and its current energy state assuming, in the worst case, that the real-time reserve signal will be equivalent to its minimum or maximum over the entire scheduling period (e.g., 1 hour). To mitigate the conservatism that results from computing reserve capacities in this way, several power system operators in the U.S. are now managing the energy states of energy-constrained reserve resources. For example, the California ISO uses the 5-minute energy market to ensure that regulation signals are zero-mean over 15-minute periods [37].

Assuming a one hour scheduling period and that a load aggregation providing reserves must be able to operate at full capacity for 15-minutes, which is consistent with [37], we compute

$$\widetilde{\underline{P}}_L = \max\left(P_L(\widetilde{\theta}) - 4(E_c(\widetilde{\theta})/2), 0\right) \tag{32}$$

$$\widetilde{\overline{P}}_L = \min\left(P_L(\widetilde{\theta}) + 4(E_c(\widetilde{\theta})/2), P_c(\widetilde{\theta})\right) \tag{33}$$

where the first values within the min/max operators ensure that the energy constraints are not violated and the second values ensure that the power constraints are not violated. Specifically, we assume that initially the heat pumps are operating normally, and so their initial energy state is half their energy capacity. Then, they can increase or decrease their aggregate power consumption by four times $E_c(\widetilde{\theta})/2$ (since there are four 15-minute intervals in one hour) unless the resulting power consumption would violate their power constraints, i.e., the aggregate power must be between 0 and $P_c(\widetilde{\theta})$.

C. Random Sample Generation and Selection

We use forecasted and actual outdoor air temperatures from eleven weather stations in Switzerland to compute temperature errors, and then add these errors to assumed temperature forecasts at each load bus $\theta^f = [13, 10, 14]^\top$ °C to create temperature samples, with which we compute samples of $\widetilde{P}_L$, $\widetilde{\underline{P}}_L$, and $\widetilde{\overline{P}}_L$. We scale the samples to be consistent with our load forecast assumptions. Additionally, we use scaled versions of the wind power samples used in [16], which were computed by applying the Markov chain mechanism developed in [38] to forecasted and actual hourly wind power data from Germany from 2006-2011.

We generate 10,000 i.i.d. samples of $\widetilde{P}_W$, $\widetilde{P}_L$, $\widetilde{\underline{P}}_L$, $\widetilde{\overline{P}}_L$, which comprise the support set Ω. For A1, A3, and A4, we assume a decision maker only has limited knowledge of Ω, and so we randomly select 20 samples from Ω. With these samples, we construct the ambiguity set of the unknown pdf by computing their empirical mean and covariance, which we use to build the SOCP constraints (30) in A1, the SDP constraints (16)–(20) in A3, and the SOCP constraints (26) in A4. For A2, we use the bound in [18] to choose the number of samples for the approximate sample set Ω_{ap} in (31). Specifically, for $1 - \epsilon_i = 95\%$ and a confidence parameter $\beta = 0.05$, we select 900 random samples since the bound $|\Omega_{\text{ap}}| \geq \frac{2}{\epsilon}(\ln\frac{1}{\beta} + n) = 932$ where $n = 21$ is the dimension of decision variable vector for the 9-bus system. For $1 - \epsilon_i = 90\%$ and $\beta = 0.05$, we select 500 random samples.

After solving the CC-OPF with A1–A4, we fix the optimal solutions x and test their performance on all 10,000 samples in the support set Ω. For each approach, we re-solve the optimization problem and test the solution ten times, and report the average, maximum, and minimum objective cost, solution reliability (the percent of samples for which the constraints are satisfied when x is fixed), and CPU time.

TABLE I
COST, RELIABILITY, AND CPU TIME OF A1–A4 FOR THE IEEE 9-BUS SYSTEM WITH NO CONGESTION

		A1: Gaussian		A2: Scenario		A3: DR (SDP)		A4: DR (SOCP)	
	$1-\epsilon_i =$	95%	90%	95%	90%	95%	90%	95%	90%
Objective cost	avg	4392.63	4330.41	4758.32	4738.73	4875.35	4633.57	4875.41	4633.61
	max	4478.08	4394.57	4895.40	4812.65	5102.61	4789.59	5102.65	4789.62
	min	4308.60	4262.52	4678.17	4649.48	4652.84	4480.48	4652.92	4480.59
Reliability (%)	avg	84.47	75.63	99.65	99.57	99.43	97.45	99.65	97.95
	max	94.07	86.69	99.87	99.83	99.83	99.56	99.83	99.74
	min	65.40	61.98	99.36	99.26	97.60	90.99	98.80	91.94
CPU Time (s)	avg	0.03	0.03	15.21	3.51	0.47	0.46	0.44	0.37
	max	0.05	0.06	15.41	3.85	0.55	0.53	0.36	0.41
	min	0.03	0.02	14.88	3.34	0.34	0.41	0.39	0.34

D. Results and Solution Patterns

Table I shows the results for the 9-bus system with no congestion (which occurs when we use the test case line flow limits). We observe that A1 solves the fastest and A2 the slowest. The CPU time taken by A2 also depends on the probability of chance constraint violation as $1-\epsilon_i = 95\%$ requires many more samples than $1-\epsilon_i = 90\%$. In contrast, the solution time of A1, A3, and A4 are independent of the probability of chance constraint violation, and only depend upon the problem size. Approaches A2, A3, and A4 yield higher objective costs than A1. However, A1 results in much lower reliability since the underlying uncertainty is not Gaussian, while A2, A3, and A4 result in reliabilities above the requirements. Note that A3 and A4 perform similarly in terms of cost, reliability, and CPU time. Since, given our choice of γ_1 and γ_2, we have $\mathcal{D}' \subseteq \mathcal{D}$, A4 should result in objective cost values that are less than or equal to those of A3 in each instance. However, in some instances, due to solver limitations for handling SDP models, some results produced with A3 are not fully optimized (i.e., CVX reported "solved/inaccurate"). In contrast, all of the results produced with A4 are fully optimized. Thus, A3 sometimes produces lower costs than A4. When $1-\epsilon_i = 90\%$, A1 performs substantially worse than when $1-\epsilon_i = 95\%$, while A2, A3, and A4 yield similar reliability results as they are less sensitive to the change in $1-\epsilon_i$. In the remainder of the paper, we use $1-\epsilon_i = 95\%$, which is more reasonable for power systems.

To put the objective costs in context, we computed the solution to three additional cases. Case 1 assumes no uncertainty, and so we solve a deterministic problem that results in no reserve procurement. Case 2 assumes wind and load uncertainty, but that loads cannot provide reserves, which is consistent with power system operation today. Case 3 assumes that loads can provide reserves, and they are certain (in terms of both consumption and min/max possible consumption) giving us an upper-bound on the cost reductions possible with load reserves. Each is solved with A3. The results for the uncongested system are shown in Table II along with the comparable results of the complete formulation (referred to as Case 4), i.e., the results of A3 for $1-\epsilon_i = 95\%$ from Table I. Case 1 has the lowest cost, while Case 2 has the highest. Comparing Case 2 and Case 3 gives

TABLE II
OBJECTIVE COSTS OF VARIOUS UNCERTAINTY CASES FOR THE IEEE 9-BUS SYSTEM WITH NO CONGESTION ($1-\epsilon_i = 95\% \quad \forall i$).

		Case 1	Case 2	Case 3	Case 4
Objective cost	avg	4099.97	4891.11	4352.64	4875.35
	max	-	5122.41	4402.08	5102.61
	min	-	4664.12	4292.27	4652.84

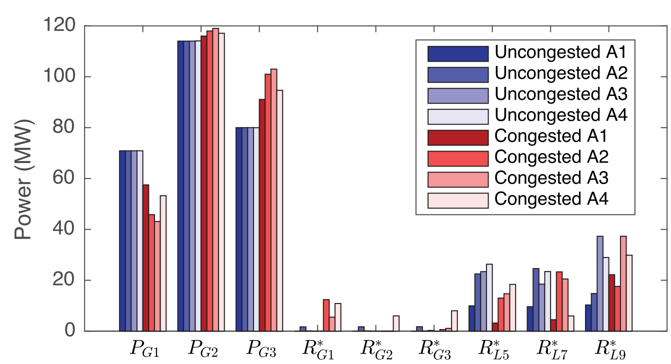

Fig. 1. Optimal generator production and reserve capacities for each generator/load in the IEEE 9-bus system.

the maximum possible value of load reserves, while comparing Case 3 and Case 4 gives the cost of load uncertainty.

We also explore the performance of the approaches under congestion. Specifically, we decrease the line flow limit of the transmission line between Buses 5 and 6 from 150 to 40 MW, resulting in congestion on multiple lines. The results are shown in Table III. All three approaches yield higher costs as compared to the UC cases in Table I. A1's reliability is poor; its optimal solution can only satisfy the chance constraints in 78.69% of the samples within Ω. In contrast, A2, A3, and A4 achieve higher reliability than required. In each case, the CPU time increases when the system is congested (C); however, the increase for A4 is small, especially as compare to the increase for A3.

In Fig. 1, we plot the optimal generator production and reserve capacities for each generator/load in each approach for $1-\epsilon_i = 95\%$ for both the UC and C cases. The values plotted are the average over the 10 runs and R^* denotes the sum of

TABLE III
COST, RELIABILITY, AND CPU TIME OF A1–A4 FOR THE IEEE 9-BUS SYSTEM WITH CONGESTION ($1 - \epsilon_i = 95\% \quad \forall i$).

		A1	A2	A3	A4
Objective cost	avg	4428.72	4884.12	5030.31	5036.35
	max	4538.47	5036.97	5328.71	5330.85
	min	4345.12	4789.28	4767.10	4767.12
Reliability (%)	avg	78.69	99.44	99.27	99.53
	max	91.94	99.79	99.81	99.85
	min	63.52	99.12	97.93	98.43
CPU Time (s)	avg	0.05	48.93	6.40	0.47
	max	0.19	49.44	6.77	0.52
	min	0.02	48.22	6.08	0.42

TABLE IV
RESERVES PROCURED BY A1–A4 IN THE IEEE 9-BUS SYSTEM

		Generator Reserves	Load Reserves	Total Reserves
UC	A1	4.7×10^{-5}	29.86	29.86
	A2	5.19	61.89	67.07
	A3	0.34	78.77	79.11
	A4	0.42	78.70	79.12
C	A1	1.2×10^{-4}	29.86	29.86
	A2	13.06	54.01	67.07
	A3	6.63	72.47	79.10
	A4	24.91	54.21	79.12

the up and down reserve capacities, i.e., $\overline{R} + \underline{R}$. In Table IV we show the optimal reserve procurement given by the three approaches for $1 - \epsilon_i = 95\%$ for both the UC and C cases. In the column "Generator Reserves" we list the sum of all generator reserves, i.e., $\sum_{j=1}^{3} R^*_{G,j}$ and "Load Reserves" are defined similarly. The last column "Total Reserves" is the sum of the generator reserves and load reserves. We can make several observations from Fig. 1 and Table IV. First, when the system is not congested, the generator production is the same for each approach; however, with congestion the generator production is different for each approach and generators provide more of the total reserves. A1 procures the fewest reserves (corresponding to the lowest cost solution) and A3/A4 the most (corresponding to the highest cost solution). For each approach, the total reserves procured in the UC and C cases is approximately the same.

V. STUDIES ON THE IEEE 39-BUS AND 118-BUS SYSTEMS

A. *Test System and Sample Generation/Selection*

We also tested all approaches on the IEEE 39-bus system and both A1 and A4 on the IEEE 118-bus system. We were unable test A2 or A3 on the 118-bus system as the required CPU times were extremely large. However, it is worth noting that SDP solvers are not yet mature and it is expected that it will become feasible to solve large SDP problems in the near term, which would make A3 more useful.

We obtained test system parameters from MATPOWER [36], and again assume $\overline{c}_G = \underline{c}_G = 10 \times \mathbf{1}$, and $\overline{c}_L = \underline{c}_L = 9.8 \times \mathbf{1}$.

TABLE V
COST, RELIABILITY, AND CPU TIME OF A1–A4 FOR THE IEEE 39-BUS SYSTEM WITH NO CONGESTION ($1 - \epsilon_i = 95\%$).

		A1	A2	A3	A4
Objective cost	avg	39232.81	40822.56	40214.75	40223.52
	max	39351.18	40952.34	40489.98	40541.92
	min	39060.95	40685.37	39772.05	39764.59
Reliability (%)	avg	85.13	99.89	93.89	99.65
	max	92.28	99.92	97.97	99.82
	min	71.61	99.84	86.34	99.11
CPU Time (s)	avg	0.10	998.11	427.05	0.76
	max	0.12	1036.17	745.54	0.95
	min	0.09	971.84	286.75	0.67

TABLE VI
COST, RELIABILITY, AND CPU TIME OF A1– A4 FOR THE IEEE 39-BUS SYSTEM WITH CONGESTION ($1 - \epsilon_i = 95\%$).

		A1	A2	A3	A4
Objective cost	avg	42350.49	45489.17	44765.94	44787.62
	max	42788.23	45871.69	45676.41	45744.09
	min	41888.16	45187.54	43996.11	44070.00
Reliability (%)	avg	72.38	99.79	93.25	99.10
	max	86.90	99.87	98.23	99.77
	min	60.47	99.70	85.74	96.60
CPU Time (s)	avg	0.10	1003.45	502.69	0.72
	max	0.12	1040.64	666.80	0.78
	min	0.09	978.80	395.95	0.63

TABLE VII
OBJECTIVE COSTS OF VARIOUS UNCERTAINTY CASES FOR THE IEEE 39-BUS SYSTEM WITH NO CONGESTION ($1 - \epsilon_i = 95\%$, $\forall i$).

		Case 1	Case 2	Case 3	Case 4
Objective cost	avg	38629.05	40259.96	39423.98	40214.75
	max	-	40581.04	39566.85	40489.98
	min	-	39796.61	39217.81	39772.05

For the 39-bus system, we add one wind power plant at Bus 6 with rated capacity 300 MW and forecasted output 200 MW. For the 118-bus system, we add three wind power plants at Buses 6, 8, and 15, each with rated capacity 300 MW and forecasted output 200 MW. We treat each wind power injection as an uncorrelated random variable. For both systems, we assume 95% of the load at each bus is perfectly forecastable and not flexible, and the remaining load is flexible but uncertain, with the load forecasts set equal to the test case loads.

The temperature for each load bus is randomly selected between 10 and 15°C, and we use the same procedures as used for the 9-bus system to generate 10,000 i.i.d. samples of $\widetilde{P}_W$, $\widetilde{P}_L$, $\widetilde{\underline{P}}_L$, $\widetilde{\overline{P}}_L$, which comprise the support set Ω. For A1, A3, and A4, we still use 20 samples randomly picked from the set Ω. For the 39-bus system, A2 now requires 4000 samples within Ω_{ap} since the number of decision variables increases to $n = 103$. Moreover, the number of chance constraints increases to $m = 216$ as compared to $m = 42$ in the 9-bus system.

We run cases without and with congestion. To produce congestion in the 39-bus system, we modify the test case line flow

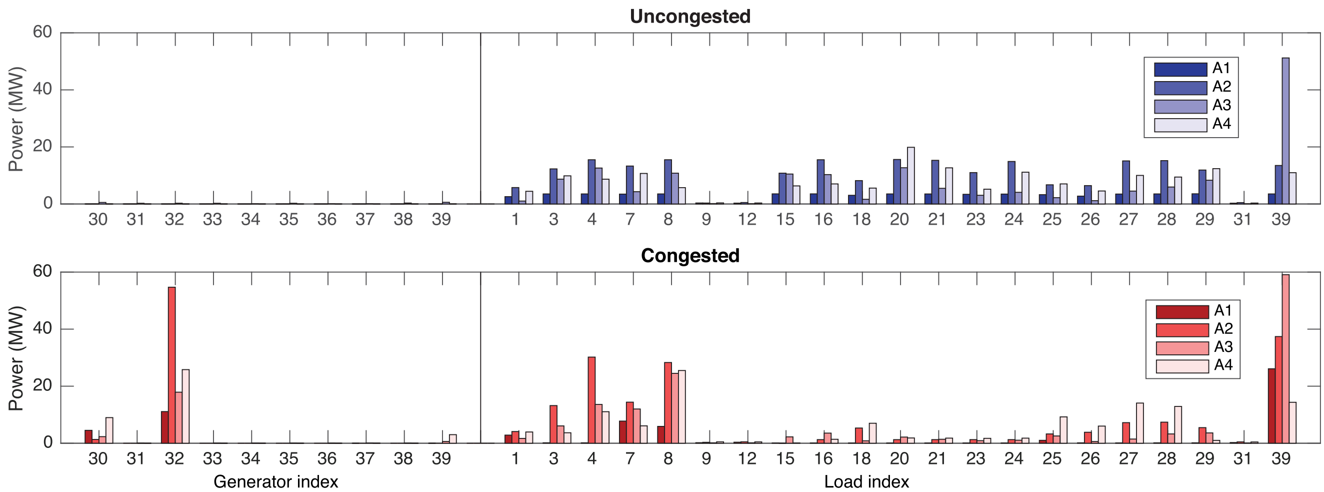

Fig. 2. Optimal reserve capacities R^* for each generator/load in the IEEE 39-bus system.

TABLE VIII
RESERVES PROCURED BY A1–A4 IN THE IEEE 39-BUS SYSTEM

		Generator Reserves	Load Reserves	Total Reserves
UC	A1	0.01	61.60	61.61
	A2	0.07	223.76	223.83
	A3	2.64	159.03	161.67
	A4	0.04	162.66	162.70
C	A1	15.59	46.02	61.61
	A2	56.01	167.82	223.83
	A3	21.11	141.04	162.15
	A4	37.83	124.89	162.72

limits by decreasing the flow limit of the line between Buses 2 and 3 from 500 MW to 350 MW and between Buses 13 and 14 from 600 to 200 MW. To produce congestion in the 118-bus system, we set all line flow limits to 180 MW.

B. Results and Solution Patterns

For the 39-bus system, we report the objective cost, reliability, and CPU time of approaches A1–A4 for the uncongested system in Table V and the congested system in Table VI. For comparison, the objective costs of Cases 1–4 (as defined in Section IV-D) for the uncongested system solved with A3 are in Table VII. Fig. 2 shows the optimal reserve procurement (averaged over the 10 runs) in each approach in the uncongested and congested systems and Table VIII summarizes the generator, load, and total reserves procured by each approach in the UC and C cases.

In the 39-bus system, A3 and A4 are *less costly* than A2, since they procure less reserves. Again, A1 is the least expensive, with the best CPU time and the worst reliability. Comparing A3 and A4, A4 achieves higher reliability with significantly lower CPU time and only a slight increase in cost. A2's reliability is slightly better than A4's (both are well-above the reliability requirement), but its CPU time is three orders of magnitude larger. It is also worth noting that A2 requires 5583.30 seconds on average to read in data and construct the models (this time is not included in "CPU seconds") while the average time for A3 and A4 are 298.13 and 21.05 seconds, respectively.

TABLE IX
COST, RELIABILITY, AND CPU TIME OF A1 AND A4 FOR THE IEEE 118-BUS SYSTEM ($1 - \epsilon_i = 95\%$).

		Uncongested		Congested	
		A1	A4	A1	A4
Objective cost	avg	105 060	107 530	105 480	108 120
	max	105 430	108 790	105 860	109 390
	min	104 710	106 360	105 110	106 920
Reliability (%)	avg	29.68	96.57	24.23	95.30
	max	44.62	98.71	40.90	98.63
	min	13.89	92.58	10.11	89.49
CPU Time (s)	avg	7.90	10.71	7.69	11.78
	max	8.69	13.05	8.41	12.76
	min	7.50	9.72	7.36	10.95

TABLE X
RESERVES PROCURED BY A1 AND A4 IN THE IEEE 118-BUS SYSTEM

		Generator Reserves	Load Reserves	Total Reserves
UC	A1	49.62	64.22	113.84
	A4	94.85	206.23	301.07
C	A1	51.65	62.19	113.84
	A4	104.96	196.10	301.06

Without congestion, the generator reserves are barely used (they are not clearly visible in the plot, but the totals are listed in Table VIII). However, with congestion generator reserves, especially from generators 30 and 32 increase. Again, for each approach, the total reserves procured by the system are approximately the same in the UC and C cases.

For the 118-bus system, we report the objective cost, reliability, and CPU time of A1 and A4 for the uncongested and congested systems in Table IX. Table X summarizes the generator, load, and total reserves procured by both approaches in the UC and C cases. In both the UC and C cases, A4 achieves reliabilities above the requirement with modest CPU time (around 10 seconds) demonstrating that the approach should scale to realistically-sized systems. In contrast, A1's reliabilities are well-below the requirement (25-30%, rather than the required 95%). However, the cost of A4's solution is higher, since more reserves are procured, as seen in Table X.

VI. Conclusion

In this paper, we posed a single-period CC-OPF with uncertain reserves from loads. We reformulated the problem using DR optimization and two different ambiguity sets resulting in an SDP problem and an SOCP problem, and compared them to two other reformulations. We conducted a number of computational experiments on the uncongested and congested IEEE 9-bus, 39-bus, and 118-bus systems, and compared the results of the four approaches in terms of objective cost, reliability, CPU, and optimal solution. We find that use of load reserves, even when their reserve capacities are uncertain, decreases system operational costs. We also find that, in contrast to the Gaussian approximation approach, the DR approach yields solutions with reliabilities close to the specified requirements. Additionally, both DR approaches require less computation time than the scenario approximation approach, which requires large numbers of uncertainty samples (900 for the 9-bus system and 4000 for the 39-bus system). Furthermore, the DR reformulation that uses SOCP produces solutions with reliabilities above the requirements and requires only modest CPU time (approximately 10 seconds for the IEEE 118-bus system with multiple wind power plants and congestion). In summary, the DR approach, which relies on moments calculated from small uncertainty sample sets (here we use only 20) but makes no assumption on the underlying uncertainty distributions, provides a good tradeoff between performance and computational tractability.

Future research includes quantifying the relationship between result quality and the amount of data used to construct the moment-based ambiguity set. We will apply the DR approach to multi-period, larger-scale problems that better capture the complex and correlated uncertainties associated with load reserves and renewable energy. Additionally, we will explore methods to include structural properties (e.g., unimodality), higher-order moments, and/or strengthened supports of the random variable ξ to reduce the conservatism of the solutions. The main challenge will be deriving tractable reformulations or approximations of the DR CC-OPF model.

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