Seshadri Constants and Fujita's Conjecture via Positive Characteristic Methods

by

Takumi Murayama

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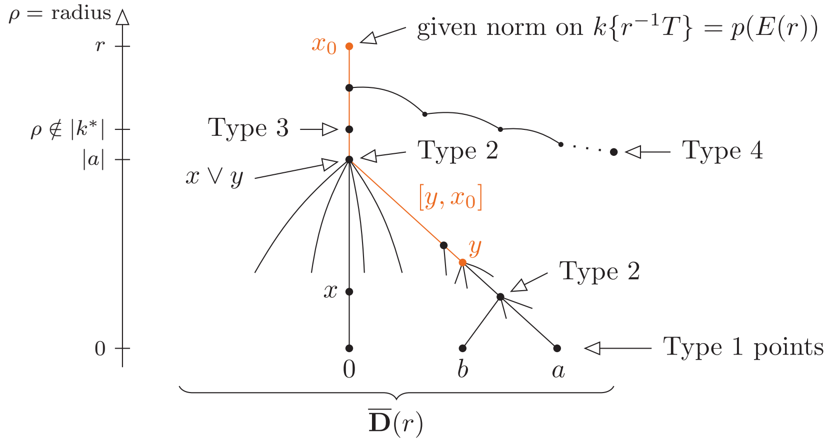
Doctoral Committee:

Professor Mircea Mustaţă, Chair
Professor Ratindranath Akhoury
Professor Melvin Hochster
Professor Mattias Jonsson
Professor Karen E. Smith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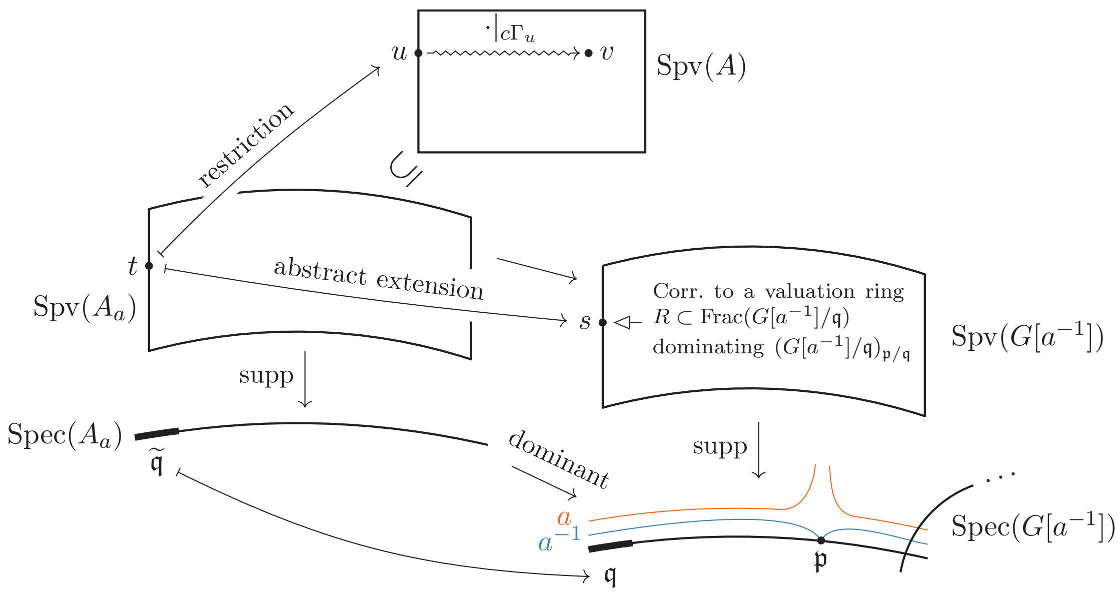
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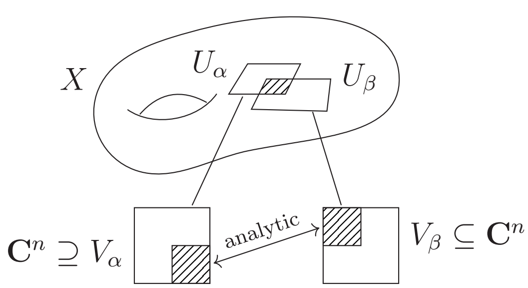
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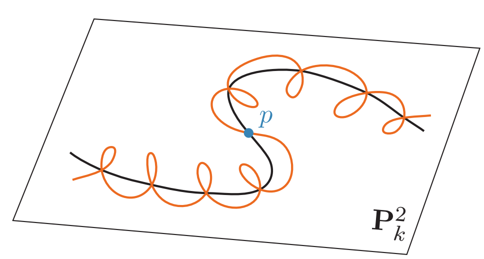
p
$\mathbf{P}^2_k$

Takumi Murayama
takumim@umich.edu
ORCID iD: 0000-0002-8404-8540

To my family

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List of Symbols

Symbols are grouped into three groups, depending on whether they start with punctuation, Greek letters, or Latin letters.

Symbol	Description
$(-)^!$	exceptional pullback of Grothendieck duality, 52
$(-)^*_{(-)}$	tight closure, 88, 155
$(-)^{[p^e]}$	eth Frobenius power of an ideal or module, 71, 88
$(-)^\circ$	complement of the union of minimal primes in a ring, 62
$(-)^\vee$	dual of a sheaf, 51
$\equiv_{\mathbf{k}}$	$\mathbf{k}$-numerical equivalence
$\sim_{\mathbf{k}}$	$\mathbf{k}$-linear equivalence
$(L^{\dim V} \cdot V)$	intersection product, 10
$(X, \Delta, \mathfrak{a}_\bullet^\lambda)$	log triple, 62
$\lceil D \rceil$	round-up of a divisor, 50
$\lfloor D \rfloor$	round-down of a divisor, 50
$\lvert D \rvert$	complete linear system, 54
$\lvert V \rvert$	linear system, 54
$\varepsilon(D; x)$	Seshadri constant, 17
$\varepsilon(\lVert D \rVert; x)$	moving Seshadri constant, 111
$\varepsilon_{\mathrm{jet}}(\lVert D \rVert; x)$	jet separation description for $\varepsilon(\lVert D \rVert; x)$, 117
$\varepsilon_N(\lVert D \rVert; x)$	Nakamaye's description for $\varepsilon(\lVert D \rVert; x)$, 114
$\varepsilon_F^\ell(D; x)$	ℓth Frobenius–Seshadri constant, 132
$\varepsilon_{F\text{-sig}}^\Delta(D; x)$	F-signature Seshadri constant, 144
$\tau(X, \Delta, \mathfrak{a}^t)$	test ideal of a triple, 82, 84
$\tau(X, \Delta, \mathfrak{a}_\bullet^\lambda)$	asymptotic test ideal of a triple, 87
$\tau(X, \Delta, t \cdot \lvert D \rvert)$	test ideal of a Cartier divisor, 87
$\tau(X, \Delta, \lambda \cdot \lVert D \rVert)$	asymptotic test ideal of a $\mathbf{Q}$-Cartier divisor, 88
Ω_X	cotangent bundle
ω_X	canonical bundle or sheaf, 53
$\omega_X^\bullet$	(normalized) dualizing complex, 53
$a(E, X, \Delta, \mathfrak{a}^t)$	discrepancy of E with respect to a triple, 63

Symbol	Description
$\mathfrak{a}_\bullet$	graded family of ideals, 54
$\mathfrak{a}_\bullet(D)$	graded family of ideals associated to a divisor, 54
$\operatorname{Ann}_R M$	annihilator of a module
$\mathfrak{b}(\lvert V\rvert)$	base ideal of a linear system, 54
$\mathbf{B}(D)$	stable base locus of a divisor, 55
$\mathbf{B}_+(D)$	augmented base locus of a divisor, 56
$\operatorname{Big}^{\{x\}}_{\mathbf{R}}(X)$	subcone of big cone consisting of ξ such that $x \notin \mathbf{B}_+(\xi)$, 112
$\operatorname{Bs}(\lvert V\rvert)$	base scheme of a linear system, 54
$\operatorname{Bs}(\lvert V\rvert)_{\mathrm{red}}$	base locus of a linear system, 54
$\mathbf{C}$	complex numbers
$\operatorname{Cart}_{\mathbf{k}}(X)$	group of $\mathbf{k}$-Cartier divisors, 49
$\deg f$	degree of generically finite map, 147
$\deg L$	degree of a line bundle or divisor on a curve
$\mathbf{D}_{\mathrm{qc}}(X)$	derived category of quasi-coherent sheaves, 52
$\mathbf{D}^+_{\mathrm{qc}}(X)$	bounded-below derived category of quasi-coherent sheaves, 52
$e(R)$	Hilbert–Samuel multiplicity of a local ring
$E_R(M)$	injective hull of a module
F^e	eth iterate of the (absolute) Frobenius morphism, 70
F^e_*R	the ring R with R-algebra structure given by F^e, 71
$\operatorname{fpt}_x((X,\Delta);\mathfrak{a})$	F-pure threshold of a pair with respect to an ideal, 80
$\mathbf{F}_q$	finite field with q elements
$H^i_{\mathrm{sing}}(X,\mathbf{Z})$	singular cohomology with coefficients in $\mathbf{Z}$
$H^i(X,\mathscr{F})$	sheaf cohomology
$h^i(X,\mathscr{F})$	$\dim_k(H^i(X,\mathscr{F}))$ when X is complete over a field k
$\widehat{h}^i(X,D)$	asymptotic cohomological function, 59
$H^0(X\vert V,L)$	image of $H^0(X,L) \to H^0(V,L\vert_V)$, 61
$h^0(X\vert V,L)$	$\dim_k H^0(X\vert V,L)$, 61
$\mathbf{h}^i(\mathscr{F})$	cohomology sheaf of a complex of sheaves
$I^{\Delta}_e(\mathfrak{m})$	eth Frobenius–degeneracy ideal, 144
$\mathcal{I}_W$	ideal defining a closed subscheme, 147
$\mathcal{J}(X,\Delta,\mathfrak{a}^t)$	multiplier ideal of a triple, 66
$\mathcal{J}(X,\Delta,\mathfrak{a}_\bullet^\lambda)$	asymptotic multiplier ideal of a triple, 67
$\mathcal{J}(X,\Delta,t\cdot\lvert D\rvert)$	multiplier ideal of a Cartier divisor, 68
$\mathcal{J}(X,\Delta,\lambda\cdot\lVert D\rVert)$	asymptotic multiplier ideal of a $\mathbf{Q}$-Cartier divisor, 68
K_X	canonical divisor, 53
$\mathscr{K}_X$	sheaf of total quotient rings, 49
$\operatorname{lct}_x((X,\Delta);\mathfrak{a})$	log canonical threshold of a pair with respect to an ideal, 65
$\operatorname{mult}_W D$	multiplicity of divisor along a subscheme, 147

Symbol	Description
$\mathfrak{m}_x$	ideal defining a closed point, 18
$\mathbf{N}$	natural numbers $\{0, 1, 2, \ldots\}$
$N^1_{\mathbf{R}}(X)$	Néron–Severi space, 99
$\mathrm{Nklt}(X, \Delta)$	non-klt locus of a pair, 150
$\mathcal{O}_{\mathbf{P}^n_k}(1)$	twisting sheaf of Serre
ord_E	divisorial valuation defined by a prime divisor, 63
$\mathcal{O}_X$	structure sheaf
$\mathcal{O}_X(D)$	sheaf associated to a Cartier divisor
$\mathbf{P}^n_k$	n-dimensional projective space over a field k, 1
$\mathbf{P}(E)$	projective bundle of one-dimensional quotients
$\mathscr{P}^\ell(L)$	bundle of principal parts, 41
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
$\mathbf{R}f_*$	derived pushforward
$s(D; x)$	largest ℓ such that $\mathcal{O}_X(D)$ separates ℓ-jets at x, 18
$s(\mathscr{F}; x)$	largest ℓ such that $\mathscr{F}$ separates ℓ-jets at x, 18
$\mathbf{S_e}$	category of noetherian schemes whose morphisms are separated and essentially of finite type, 52
$\operatorname{Spec} R$	spectrum of a ring
$\mathbf{Spec}_X \mathscr{A}$	relative spectrum of a sheaf of $\mathcal{O}_X$-algebras
$\mathrm{totaldiscrep}(X, \Delta, \mathfrak{a}^t)$	total discrepancy of a triple, 63
$\mathrm{Tr}^e_{X,D}$	the trace of Frobenius, 25, 77
T_X	tangent bundle
$V_\bullet$	graded linear system, 146
$\mathrm{vol}_X(D)$	volume of a line bundle or divisor, 60
$\mathrm{vol}_X(V_\bullet)$	volume of a graded linear system, 146
$\mathrm{vol}_{X\vert V}(D)$	restricted volume of a line bundle or divisor, 61
$\mathrm{WDiv}_{\mathbf{k}}(X)$	group of $\mathbf{k}$-Weil divisors, 49
$\mathbf{Z}$	integers

Abstract

In 1988, Fujita conjectured that there is an effective and uniform way to turn an ample line bundle on a smooth projective variety into a globally generated or very ample line bundle. We study Fujita's conjecture using Seshadri constants, which were first introduced by Demailly in 1992 with the hope that they could be used to prove cases of Fujita's conjecture. While examples of Miranda seemed to indicate that Seshadri constants could not be used to prove Fujita's conjecture, we present a new approach to Fujita's conjecture using Seshadri constants and positive characteristic methods. Our technique recovers some known results toward Fujita's conjecture over the complex numbers, without the use of vanishing theorems, and proves new results for complex varieties with singularities. Instead of vanishing theorems, we use positive characteristic techniques related to the Frobenius–Seshadri constants introduced by Mustaţă–Schwede and the author. As an application of our results, we give a characterization of projective space using Seshadri constants in positive characteristic, which was proved in characteristic zero by Bauer and Szemberg.

Chapter 1

Introduction

Algebraic geometry is the study of *algebraic varieties*, which are geometric spaces defined by polynomial equations. Some varieties are particularly simple, and the simplest algebraic varieties are perhaps the n-dimensional projective spaces $\mathbf{P}_k^n$. Recall that if k is a field (e.g. the complex numbers $\mathbf{C}$), then the *projective space* of dimension n over k is

$$\mathbf{P}_k^n := \frac{k^{n+1} \smallsetminus \{0\}}{k^*}.$$

A *projective variety* over k is an algebraic variety that is isomorphic to a subset of $\mathbf{P}_k^n$ defined as the zero set of homogeneous polynomials.

Projective spaces are very well understood. The most relevant property of projective space for us is its intersection theory. Since at least the Renaissance, artists have used the intersection theory of $\mathbf{P}_k^2$ to paint perspective: in Raphael's *School of Athens* (see Figure 1.1), every pair of lines not parallel to the plane of vision appear to intersect between the two central figures, Plato and Aristotle. Mathematically, a concise way to describe this feature is that the singular cohomology ring of $\mathbf{P}_{\mathbf{C}}^n$ can be described as

$$H_{\mathrm{sing}}^*\bigl(\mathbf{P}_{\mathbf{C}}^n, \mathbf{Z}\bigr) \simeq \frac{\mathbf{Z}[h]}{(h^{n+1})}, \tag{1.1}$$

where $h \in H^2(\mathbf{P}_{\mathbf{C}}^n, \mathbf{Z})$ is the cohomology class associated to a hyperplane.

In addition to its intersection theory, we understand many more things about projective spaces, in particular the values of various cohomological invariants associated to algebraic varieties that come from sheaf cohomology. It is therefore useful to know when a variety

Figure 1.1: Raphael's *School of Athens* (1509–1511)
Public domain, `https://commons.wikimedia.org/w/index.php?curid=2194482`

is projective space, prompting the following:

Question 1.1. *How can we identify when a given projective variety is projective space?*

Of course, not every projective variety is a projective space. For example, the hyperboloid

$$\mathbf{P}^1_k \times_k \mathbf{P}^1_k \simeq \left\{x^2 + y^2 - z^2 = w^2\right\} \subseteq \mathbf{P}^3_k$$

is an example of a ruled surface, and cannot be isomorphic to $\mathbf{P}^2_k$ since two lines in it may not intersect. See Figure 1.2 for a real-world example of this phenomenon: adjacent steel trusses that run vertically along the Kobe port tower are straight, and do not intersect. We therefore also ask:

Question 1.2. *Given a projective variety X, how can we find an embedding $X \hookrightarrow \mathbf{P}^N_k$, or even just a morphism $X \to \mathbf{P}^N_k$?*

We now state our first result, which gives one answer to Question 1.1. In the statement below, we recall that a smooth projective variety is *Fano* if the anti-canonical bundle $\omega_X^{-1} \coloneqq \bigwedge^{\dim X} T_X$ is ample, where a line bundle L on a variety X over a field k is *ample* if one of the following equivalent conditions hold (see Definition 2.1.1 and Theorem 2.1.2):

Figure 1.2: Kobe Port Tower in the Kobe harbor (2006)
By `663highland`, CC BY 2.5, `https://commons.wikimedia.org/w/index.php?curid=1389137`

(1) There exists an integer $\ell > 0$ such that $L^{\otimes \ell}$ is *very ample*, i.e., such that there exists an embedding $X \hookrightarrow \mathbf{P}^N_k$ for some N for which $L^{\otimes \ell} \simeq \mathcal{O}_{\mathbf{P}^N_k}(1)|_X$.

(2) For every coherent sheaf $\mathscr{F}$ on X, there exists an integer $\ell_0 \geq 0$ such that the sheaf $\mathscr{F} \otimes L^{\otimes \ell}$ is globally generated for all $\ell \geq \ell_0$.

Additionally, $e(\mathcal{O}_{C,x})$ denotes the *Hilbert–Samuel multiplicity* of C at x.

Theorem A. *Let X be a Fano variety of dimension n over an algebraically closed field k of positive characteristic. If there exists a closed point $x \in X$ with*

$$\deg\bigl(\omega_X^{-1}|_C\bigr) \geq e(\mathcal{O}_{C,x}) \cdot (n+1)$$

for every integral curve $C \subseteq X$ passing through x, then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n_k$.

An interesting feature of this theorem is that it only requires a positivity condition on ω_X^{-1} at *one* point $x \in X$. Bauer and Szemberg showed the analogous statement in characteristic zero. There have been some recent generalizations of both Bauer and

Szemberg's result and of Theorem A due to Liu and Zhuang; see Remark 3.2.3. There is also an interesting connection between Theorem A and the Mori–Mukai conjecture (see Conjecture 3.1.5), which states that if X is a Fano variety of dimension n such that the anti-canonical bundle ω_X^{-1} satisfies $\deg(\omega_X^{-1}|_C) \geq n+1$ for all rational curves $C \subseteq X$, then X is isomorphic to $\mathbf{P}^n_k$. Theorem A strengthens the positivity assumption on ω_X^{-1} to incorporate the multiplicity of the curves passing through x, but has the advantage of not having to impose any generality conditions on the point x. See §3.1 for further discussion.

Our next result is the main ingredient in proving Theorem A, and gives a partial answer to Question 1.2. We motivate this result by first stating Fujita's conjecture, a proof of which would answer Question 1.2. Below, $\omega_X \coloneqq \bigwedge^{\dim X} \Omega_X$ is the *canonical bundle* on X.

Conjecture 1.3 [Fuj87, Conj.; Fuj88, nº 1]**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field k, and let L be an ample line bundle on X. We then have the following:*

(*i*) (Fujita's freeness conjecture) $\omega_X \otimes L^{\otimes \ell}$ *is globally generated for all* $\ell \geq n+1$.

(*ii*) (Fujita's very ampleness conjecture) $\omega_X \otimes L^{\otimes \ell}$ *is very ample for all* $\ell \geq n+2$.

The essence of Fujita's conjecture is that an ample line bundle L can effectively and uniformly be turned into a globally generated or very ample line bundle. Over the complex numbers, Fujita's freeness conjecture holds in dimensions ≤ 5 [Rei88; EL93a; Kaw97; YZ], and Fujita's very ampleness conjecture holds in dimensions ≤ 2 [Rei88]. On the other hand, in arbitrary characteristic, much less is known. While the same proof as over the complex numbers works for curves, only partial results are known for surfaces [SB91; Ter99; DCF15], and in higher dimensions, we only know that Fujita's conjecture 1.3 holds when L is additionally assumed to be globally generated [Smi97]. See §2.1 and especially Table 2.1 for a summary of existing results.

We now describe our approach to Fujita's conjecture 1.3, and state our second main result. In 1992, Demailly introduced Seshadri constants to measure the local positivity of line bundles with the hope that they could be used to prove cases of Fujita's conjecture [Dem92, §6]. These constants are defined as follows. Let L be an ample line bundle on a projective variety X over an algebraically closed field, and consider a closed point $x \in X$. The *Seshadri constant* of L at x is

$$\varepsilon(L;x) \coloneqq \sup\bigl\{t \in \mathbf{R}_{\geq 0} \bigm| \mu^* L(-tE) \text{ is ample}\bigr\}, \tag{1.2}$$

where $\mu\colon \widetilde{X} \to X$ is the blowup of X at x with exceptional divisor E. The connection between Seshadri constants and Fujita's conjecture 1.3 is given by the following result, which says that if the Seshadri constant $\varepsilon(L;x)$ is sufficiently large, then $\omega_X \otimes L$ has many global sections. This is the main ingredient in the proof of Theorem A.

Theorem B. *Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic $p > 0$, and let L be an ample line bundle on X. Let $x \in X$ be a closed point, and consider an integer $\ell \geq 0$. If $\varepsilon(L;x) > n + \ell$, then $\omega_X \otimes L$ separates ℓ-jets at x, i.e., the restriction morphism*

$$H^0(X, \omega_X \otimes L) \longrightarrow H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1})$$

is surjective, where $\mathfrak{m}_x \subseteq \mathcal{O}_X$ is the ideal defining x.

In particular, then, to show Fujita's freeness conjecture 1.3(i), it would suffice to show that $\varepsilon(L;x) > \frac{n}{n+1}$ for every point $x \in X$, where $n = \dim X$. Theorem B was proved over the complex numbers by Demailly; see Proposition 2.2.6. In positive characteristic, the special case when $\ell = 0$ is due to Mustaţă and Schwede [MS14, Thm. 3.1]. Our contribution is that the same result holds for all $\ell \geq 0$ in positive characteristic.

Remark 1.4. Theorem B holds more generally for line bundles that are not necessarily ample, and for certain singular varieties over arbitrary fields; see Theorem 7.3.1. This version of Theorem B for singular varieties is new even over the complex numbers, and we do not know of a proof of this more general result that does not reduce to the positive characteristic case. Moreover, by combining Theorem 7.3.1 with lower bounds on Seshadri constants due to Ein, Küchle, and Lazarsfeld (Theorem 2.2.11), we obtain generic results toward Fujita's conjecture 1.3 for singular varieties; see Corollary 2.2.13 and Remark 2.2.14.

The main difficulty in proving Theorem B is that Kodaira-type vanishing theorems can fail in positive characteristic. Recall that if X is a smooth projective variety over the complex numbers, and L is an ample line bundle on X, then the Kodaira vanishing theorem states that

$$H^i(X, \omega_X \otimes L) = 0$$

for every $i > 0$. This vanishing theorem was a critical ingredient in Demailly's proof of Theorem B over the complex numbers. In positive characteristic, however, the Kodaira vanishing theorem is often false, as was first discovered by Raynaud [Ray78] (see Example 2.4.4). We note that the strategy behind known cases of Fujita's conjecture 1.3 is to construct global sections of $\omega_X \otimes L^{\otimes \ell}$ inductively by using versions of the Kodaira vanishing theorem to lift sections from smaller dimensional subvarieties. It has therefore been thought that the failure of vanishing theorems may be the greatest obstacle to making progress on Fujita's conjecture 1.3 in positive characteristic.

In order to replace vanishing theorems, we build on the theory of so-called "Frobenius techniques." A key insight in positive characteristic algebraic geometry is that while vanishing theorems are false, there is one major advantage to working in positive characteristic: every variety X has an interesting endomorphism, called the *Frobenius morphism*. This endomorphism $F \colon X \to X$ is defined as the identity map on points, and the p-power map

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow F_*\mathcal{O}_X(U) \\ f &\longmapsto f^p \end{aligned}$$

on functions over every open set $U \subseteq X$, where p is the characteristic of the ground field k. Even if one is only interested in algebraic geometry over the complex numbers, some results necessitate reducing to the case when the ground field is of positive characteristic and then using the Frobenius morphism. For example, this "reduction modulo p" technique is used in one proof of the Ax–Grothendieck theorem, which says that an injective polynomial endomorphism $\mathbf{C}^n \to \mathbf{C}^n$ is bijective [Ax68, Thm. C; EGAIV$_3$, Prop. 10.4.11], and in Mori's bend and break technique, which is used to find rational curves on varieties [Mor79, §2]. The latter in particular is a fundamental technique in modern birational geometry, but there is no known direct proof of Mori's theorems over the complex numbers.

In its current form, Frobenius techniques were developed simultaneously in commutative algebra (see, e.g., [HR76; HH90]) and in representation theory (see, e.g., [MR85; RR85]). Particularly important is the theory of tight closure developed by Hochster and Huneke, which was used by Smith to show special cases of Fujita's conjecture [Smi97; Smi00a].

The Frobenius techniques used in proving Theorem B can be used to give progress toward Fujita's conjecture 1.3. As mentioned above, Theorem B implies that to show Fujita's freeness conjecture 1.3(i), it would suffice to show that $\varepsilon(L;x) > \frac{n}{n+1}$ for every point $x \in X$, where $n = \dim X$. Unfortunately, Miranda showed that the Seshadri constant $\varepsilon(L;x)$ can get arbitrarily small at special points $x \in X$; see Example 2.2.9. Nevertheless, we show that the dimension n in the statement of Theorem B can be replaced by a smaller number, called the *log canonical threshold,* over which one has more control. See Definition 4.8.6 for a precise definition of the log canonical threshold. This invariant is associated to the data of the variety X together with a formal $\mathbf{Q}$-linear combination Δ of codimension one subvarieties of X, and measures how bad the singularities of X and Δ are. We also mention that $\varepsilon(\lVert H\rVert;x)$ below denotes the *moving Seshadri constant* of H at x, which is a version of the Seshadri constant defined above in (1.2) for line bundles that are not necessarily ample; see Definition 7.1.1.

Theorem C. *Let (X,Δ) be an effective log pair such that X is a projective normal variety over a field k of characteristic zero, and such that $K_X+\Delta$ is $\mathbf{Q}$-Cartier. Consider a k-rational point $x \in X$ such that (X,Δ) is klt, and suppose that D is a Cartier divisor on X such that $H = D - (K_X+\Delta)$ satisfies*

$$\varepsilon\bigl(\lVert H\rVert;x\bigr) > \mathrm{lct}_x\bigl((X,\Delta);\mathfrak{m}_x\bigr).$$

Then, $\mathcal{O}_X(D)$ has a global section not vanishing at x.

While we have stated Theorem C over a field of characteristic zero, our proof uses reduction modulo p and Frobenius techniques to reduce to a similar result in positive characteristic (Theorem 8.1.1).

Using Theorem C, we then show the following version of a theorem of Angehrn and Siu [AS95, Thm. 0.1]. Our statement is modeled after that in [Kol97, Thm. 5.8]. Below, $\mathrm{vol}_{X|Z}(H)$ denotes the *restricted volume,* which measures how many global sections $\mathcal{O}_Z(mH|_Z)$ has on Z that are restrictions of global sections of $\mathcal{O}_X(mH)$ on X as $m \to \infty$; see Definition 4.6.13.

Theorem D. *Let (X,Δ) be an effective log pair, where X is a normal projective variety over an algebraically closed field k of characteristic zero, Δ is a $\mathbf{Q}$-Weil divisor, and*

$K_X + \Delta$ is $\mathbf{Q}$-Cartier. Let $x \in X$ be a closed point such that (X, Δ) is klt at x, and let D be a Cartier divisor on X such that setting $H \coloneqq D - (K_X + \Delta)$, there exist positive numbers $c(m)$ with the following properties:

(i) For every positive dimensional variety $Z \subseteq X$ containing x, we have

$$\mathrm{vol}_{X|Z}(H) > c(\dim Z)^{\dim Z}.$$

(ii) The numbers $c(m)$ satisfy the inequality

$$\sum_{m=1}^{\dim X} \frac{m}{c(m)} \leq 1.$$

Then, $\mathcal{O}_X(D)$ has a global section not vanishing at x.

A version of this result for smooth complex projective varieties appears in [ELM+09, Thm. 2.20]. As a consequence, we recover the following result, which gives positive evidence toward Fujita's freeness conjecture 1.3(i).

Corollary 1.5 (cf. [AS95, Cor. 0.2])**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic zero, and let L be an ample line bundle on X. Then, the line bundle $\omega_X \otimes L^{\otimes \ell}$ is globally generated for all $\ell \geq \frac{1}{2}n(n+1)+1$.*

This corollary is obtained from Theorem D by setting $c(m) = \binom{n+1}{2}$ for every m. Since we prove Corollary 1.5 without the use of Kodaira-type vanishing theorems, Theorem D and Corollary 1.5 support the validity of the following:

Principle 1.6. *The failure of Kodaira-type vanishing theorems is* not *the main obstacle to proving Fujita's conjecture 1.3 over fields of positive characteristic.*

Instead, the difficulty is in constructing certain boundary divisors that are very singular at a point, but have mild singularities elsewhere; cf. Theorem 8.2.1.

Finally, we mention one intermediate result used in the proofs of Theorems C and D, which is of independent interest. This statement characterizes ampleness in terms of asymptotic growth of higher cohomology groups. It is well known that if X is a projective variety of dimension $n > 0$, then $h^i(X, \mathcal{O}_X(mL)) \coloneqq \dim_k H^i(X, \mathcal{O}_X(mL)) = O(m^n)$ for every Cartier divisor L; see [Laz04a, Ex. 1.2.20]. It is therefore natural to ask when cohomology groups have submaximal growth. The following result says that ample Cartier divisors L are characterized by having submaximal growth of higher cohomology groups for small perturbations of L.

Theorem E. *Let X be a projective variety of dimension $n > 0$ over a field k. Let L be an $\mathbf{R}$-Cartier divisor on X. Then, L is ample if and only if there exists a very ample Cartier divisor A on X and a real number $\varepsilon > 0$ such that*

$$\widehat{h}^i(X, L - tA) := \limsup_{m\to\infty} \frac{h^i\bigl(X, \mathcal{O}_X\bigl(\lceil m(L - tA)\rceil\bigr)\bigr)}{m^n/n!} = 0$$

for all $i > 0$ and for all $t \in [0, \varepsilon)$.

Here, the $\widehat{h}^i(X, -)$ are the *asymptotic higher cohomological functions* introduced by Küronya [Kür06]; see §4.6.3. Theorem E was first proved by de Fernex, Küronya, and Lazarsfeld over the complex numbers [dFKL07, Thm. 4.1]. We note that one can have $\widehat{h}^i(X, L) = 0$ for all $i > 0$ without L being ample, or even pseudoeffective; see Example 6.1.1.

1.1. Outline

This thesis is divided into two parts, followed by two appendices. The first part consists of Chapters 2 and 3, and is more introductory in nature. In Chapter 2, we give more motivation and many examples illustrating the questions we are studying in this thesis. After highlighting some difficulties in positive characteristic, we prove Theorem B. We then devote Chapter 3 to proving our characterization of projective space (Theorem A).

The second part of this thesis consists of the remaining chapters. In Chapters 4 and 5, we review some preliminary material that will be used in the rest of the thesis. Since almost all of this material is not new, we recommend the reader to skip ahead to the results they are interested in, and to refer back to these preliminary chapters as necessary. We then focus on proving Theorem 7.3.1, which is a generalization of Theorem B for singular varieties, and on proving Theorems C and D. To do so, we prove Theorem E in Chapter 6, which is used when we study moving Seshadri constants in Chapter 7. This latter chapter is also where we prove Theorem 7.3.1. Finally, we prove Theorems C and D in Chapter 8.

The two appendices are devoted to some technical aspects of the theory of F-singularities for rings and schemes whose Frobenius endomorphisms are not necessarily finite. Appendix A reviews the definitions of and relationships between different classes of F-singularities, and Appendix B develops a scheme-theoretic version of the gamma construction of Hochster–Huneke, which we use throughout the thesis to reduce to the case when the ground field k satisfies $[k : k^p] < \infty$, where $\operatorname{char} k = p > 0$.

1.2. Notation and conventions

We mostly follow the notation and conventions of [Har77] for generalities in algebraic geometry, of [Laz04a; Laz04b] for positivity of divisors, line bundles, and vector bundles, and of [Har66] for Grothendieck duality theory. See also the List of Symbols. A notable exception is that we do not assume anything a priori about the ground field that we work over, and in particular, the ground field may not be algebraically closed or even perfect.

All rings are commutative with identity. A *variety* is a reduced and irreducible scheme that is separated and of finite type over a field k. A *complete scheme* is a scheme that is proper over a field k. Intersection products $(L^{\dim V} \cdot V)$ are defined using Euler characteristics, following Kleiman; see [Kle05, App. B].

Chapter 2

Motivation and examples

In this chapter, we motivate the questions posed in the introduction with some more background and examples. The new material is a slight modification of Kollár's example 2.1.6 to work in arbitrary characteristic, and the proof of Theorem B; see §2.4.1. A different proof of Theorem B originally appeared in [Mur18, §3].

2.1. Fujita's conjecture

To motivate Fujita's conjectural answer to Question 1.2, we give some background. First, we recall the following definition.

Definition 2.1.1 (see [Har77, Def. on p. 120 and Thm. II.7.6])**.** Let X be a scheme over a field k, and let L be a line bundle on X. We say that L is *very ample* if there exists an embedding $X \hookrightarrow \mathbf{P}^N_k$ for some N for which $L \simeq \mathcal{O}_{\mathbf{P}^N_k}(1)|_X$. We say that L is *ample* if $L^{\otimes \ell}$ is very ample for some integer $\ell > 0$.

Ample line bundles can be characterized in the following manner.

Theorem 2.1.2 (Cartan–Serre–Grothendieck; see [Har77, Def. on p. 153 and Thm. II.7.6])**.** *Let X be a scheme of finite type over a field k, and let L be a line bundle on X. Then, L is ample if and only if for every coherent sheaf $\mathscr{F}$ on X, there exists an integer $\ell_0 \geq 0$ such that the sheaf $\mathscr{F} \otimes L^{\otimes \ell}$ is globally generated for all $\ell \geq \ell_0$.*

Because of the defining property in Definition 2.1.1 and the characterization in Theorem 2.1.2, we can ask the following mathematically precise version of Question 1.2.

Question 2.1.3. *Let L be an ample line bundle on a projective variety X. What power of L is very ample or globally generated?*

The best thing we could hope for is that the power needed in Question 2.1.3 depends on some invariants of X. For curves, we can give a very explicit answer to Question 2.1.3. We use the language of divisors instead of line bundles below to simplify notation.

Example 2.1.4 (Curves I; see [Har77, Cor. IV.3.2]). Let X be a smooth curve over an algebraically closed field k, i.e., a projective variety of dimension 1 over k. Let D be a divisor on X. We claim that the complete linear system $|D|$ is basepoint-free if $\deg D \geq 2g$, and is very ample if $\deg D \geq 2g+1$, where g is the genus of X. Recall that by [Har77, Prop. IV.3.1], the complete linear system $|D|$ is basepoint-free if and only if

$$h^0\bigl(X, \mathcal{O}_X(D-P)\bigr) = h^0\bigl(X, \mathcal{O}_X(D)\bigr) - 1$$

for every closed point $P \in X$, and is very ample if and only if

$$h^0\bigl(X, \mathcal{O}_X(D-P-Q)\bigr) = h^0\bigl(X, \mathcal{O}_X(D)\bigr) - 2$$

for every pair of closed points $P, Q \in X$. We will verify these properties below.

Suppose $\deg D \geq 2g$ (resp. $\deg D \geq 2g+1$). By Serre duality, we have $h^1(X, \mathcal{O}_X(D)) = 0$, and $h^1(X, \mathcal{O}_X(D-P-Q)) = 0$ for every closed point $P \in X$ (resp. $h^1(X, \mathcal{O}_X(D-P-Q)) = 0$ for every two closed points $P, Q \in X$). We therefore have

$$\begin{aligned} h^0\bigl(X, \mathcal{O}_X(D-P)\bigr) &= \deg(D-P) + 1 - g \\ &= \deg D - 1 + 1 - g = h^0\bigl(X, \mathcal{O}_X(D)\bigr) - 1 \\ h^0\bigl(X, \mathcal{O}_X(D-P-Q)\bigr) &= \deg(D-P-Q) + 1 - g \\ &= \deg D - 2 + 1 - g = h^0\bigl(X, \mathcal{O}_X(D)\bigr) - 2 \end{aligned}$$

in each case by the Riemann–Roch theorem [Har77, Thm. IV.1.3]. As a result, we see that if L is an ample divisor on X, the complete linear system $|\ell L|$ is basepoint-free for all $\ell \geq 2g$, and is very ample for all $\ell \geq 2g+1$, where g is the genus of X.

We can answer Question 2.1.3 for abelian varieties as well.

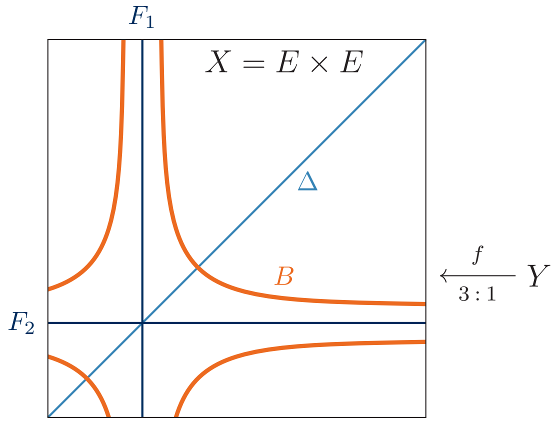

Figure 2.1: Kollár's example (Example 2.1.6)

Example 2.1.5 (Abelian varieties)**.** If L is an ample line bundle on an abelian variety A, then $L^{\otimes \ell}$ is globally generated for $\ell \geq 2$ and is very ample for $\ell \geq 3$ by a theorem of Lefschetz. See [Mum08, App. 1 on p. 57 and Thm. on p. 152].

On the other hand, the following example essentially due to Kollár shows that one cannot hope for such a simple answer on surfaces: different ample line bundles on the same surface may need to be raised to different powers to become very ample. Note that we have modified Kollár's example to work in arbitrary characteristic.

Example 2.1.6 (Kollár [EL93b, Ex. 3.7])**.** Let E be an elliptic curve over an algebraically closed field k. Let $X = E \times_k E$, let F_i be the divisors associated to the fibers of the projection morphisms $\mathrm{pr}_i \colon X \to E$ for $i \in \{1, 2\}$, and let Δ be the divisor associated to the diagonal in X. Set $R = F_1 + F_2$. Since $3R$ is very ample by Example 2.1.4, we can choose a smooth divisor $B \in |3R|$ by Bertini's theorem [Har77, Thm. II.8.18]; see Figure 2.1. For each integer $m \geq 2$, consider the divisor

$$A_m \coloneqq mF_1 + (m^2 - m + 1)F_2 - (m - 1)\Delta$$

on X. We can compute that $(A_m^2) = 2$ and $(A_m \cdot R) = m^2 - 2m + 3 > 0$, hence A_m is ample: these intersection conditions imply A_m is big by [Har77, Cor. V.1.8], and the fact that X is a homogeneous space implies A_m is ample by the Nakai–Moishezon criterion [Laz04a, Thm. 1.2.23] (see [Laz04a, Lem. 1.5.4]).

Now consider the triple cover $f \colon Y \to X$ branched over B, as constructed in [Laz04a, Prop. 4.1.6]. For every $m \geq 2$, the divisors $D_m \coloneqq f^*A_m$ are ample by [Laz04a, Prop. 1.2.13], but we claim that mD_m is not ample. It suffices to show that the pullback homomorphism

$$f^*\colon H^0\bigl(X, \mathcal{O}_X(mA_m)\bigr) \longrightarrow H^0\bigl(Y, \mathcal{O}_Y(mD_m)\bigr) \tag{2.1}$$

is an isomorphism, since if this were the case, then the morphism

$$Y \xrightarrow{|mD_m|} \mathbf{P}\bigl(H^0\bigl(Y, \mathcal{O}_Y(mD_m)\bigr)\bigr)$$

would factor through the $3:1$ morphism f. To show that (2.1) is an isomorphism, we first note that

$$\begin{aligned} f_*\bigl(\mathcal{O}_Y(mD_m)\bigr) &\simeq f_*\mathcal{O}_Y \otimes \mathcal{O}_X(mA_m) \\ &\simeq \mathcal{O}_X(mA_m) \oplus \mathcal{O}_X(mA_m - R) \oplus \mathcal{O}_X(mA_m - 2R) \end{aligned} \tag{2.2}$$

by the projection formula and by the construction of Y (see [Laz04a, Rem. 4.1.7]). On global sections, the inclusion $H^0(X, \mathcal{O}_X(mA_m)) \hookrightarrow H^0(X, f_*(\mathcal{O}_Y(mD_m)))$ induced by the isomorphism (2.2) can be identified with the pullback homomorphism (2.1) by the construction of Y. On the other hand, since $(mA_m - R)^2 < 0$ and $(mA_m - 2R)^2 < 0$, we have that $H^0(X, \mathcal{O}_X(mA_m - R)) = H^0(X, \mathcal{O}_X(mA_m - 2R)) = 0$ by [Laz04a, Lem. 1.5.4]. Thus, (2.1) is an isomorphism.

To get bounds only in terms of the dimension of X, Mukai suggested that the correct bundles to look at are *adjoint line bundles*, i.e., line bundles of the form $\omega_X \otimes L$, where ω_X is the canonical bundle on X. In this direction, Fujita conjectured the following:

Conjecture 1.3 [Fuj87, Conj.; Fuj88, n° 1]**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field, and let L be an ample line bundle on X. We then have the following:*

(*i*) (Fujita's freeness conjecture) $\omega_X \otimes L^{\otimes \ell}$ *is globally generated for all* $\ell \geq n+1$.

(*ii*) (Fujita's very ampleness conjecture) $\omega_X \otimes L^{\otimes \ell}$ *is very ample for all* $\ell \geq n+2$.

Note that both properties hold for some ℓ: (*i*) holds for some ℓ by Theorem 2.1.2, and for (*ii*), it suffices to note that if $\omega_X \otimes L^{\otimes \ell_1}$ is globally generated and $L^{\otimes \ell_2}$ is very ample, then their tensor product $\omega_X \otimes L^{\otimes(\ell_1+\ell_2)}$ is very ample [EGAII, Prop. 4.4.8]. The essence of Fujita's conjecture, then, is that the ℓ required can be bounded effectively in terms of only the dimension of X.

Fujita's conjecture 1.3 is known for some special classes of varieties.

Example 2.1.7 (Projective spaces and toric varieties)**.** If $X = \mathbf{P}^n_k$ for a field k and $L = \mathscr{O}_{\mathbf{P}^n_k}(1)$, then $\omega_X = \mathscr{O}_{\mathbf{P}^n_k}(-n-1)$ [Har77, Ex. II.8.20.2]. Thus, the bounds in Fujita's conjecture 1.3 are in some sense optimal.

Fujita's conjecture also holds for toric varieties. In the smooth case, this follows from Mori's cone theorem (see, e.g., [Laz04a, Rem. 10.4.6] and see [Mus02, Thm. 0.3] for a stronger statement), and in the singular case, see [Fuj03, Cor. 0.2; Pay06, Thm. 1].

Example 2.1.8 (Curves II)**.** Let X be a smooth curve over an algebraically closed field as in Example 2.1.4, and let L be an ample line bundle on X. By Example 2.1.4, since $\deg \omega_X = 2g - 2$ where g is the genus of X [Har77, Ex. IV.1.3.3], the line bundle $\omega_X \otimes L^{\otimes \ell}$ is globally generated if $\ell \geq 2$, and is very ample if $\ell \geq 3$.

Example 2.1.9 (Abelian varieties)**.** Since the canonical bundle ω_A is isomorphic to the structure sheaf $\mathscr{O}_A$ on an abelian variety A, Example 2.1.5 already shows that Fujita's conjecture holds for abelian varieties.

Example 2.1.10 (Ample and globally generated line bundles; see [Laz04a, Ex. 1.8.23])**.** Fujita's conjecture 1.3 holds when L is moreover assumed to be globally generated. In characteristic zero, this can be seen as follows. By Castelnuovo–Mumford regularity [Laz04a, Thm. 1.8.5], a coherent sheaf $\mathscr{F}$ on X is globally generated if $H^i(X, \mathscr{F} \otimes L^{\otimes -i}) = 0$ for all $i > 0$. Thus, the sheaf $\mathscr{F} = \omega_X \otimes L^{\otimes \ell}$ is globally generated for $\ell \geq n + 1$ since $H^i(X, \omega_X \otimes L^{\otimes(\ell - i)}) = 0$ by the Kodaira vanishing theorem [Laz04a, Thm. 4.2.1], proving (i). (ii) then follows from [Laz04a, Ex. 1.8.22].

We also mention generalizations of this example. In characteristic zero, the argument above works when X is only assumed to have rational singularities by [Laz04a, Ex. 4.3.13], and in positive characteristic, Smith used tight closure methods to recover an analogous result when X has F-rational singularities [Smi97, Thm. 3.2]. Keeler gave a proof of Smith's result using Castelnuovo–Mumford regularity, and also showed (ii) for smooth varieties when L is globally generated [Kee08, Thm. 1.1]. Note that Keeler's argument for (i) also applies to varieties with F-injective singularities in positive characteristic; see [Sch14, Thm. 3.4(i)].

Dimension	Result (over $\mathbf{C}$)	Method
1	Classical [Har77, Cor. IV.3.2(a)]	Riemann–Roch
2	Reider [Rei88, Thm. 1(i)]	Bogomolov instability
3	Ein–Lazarsfeld [EL93a, Cor. 2*]	Cohomological method of Kawamata–Reid–Shokurov
4	Kawamata [Kaw97, Thm. 4.1]	
5	Ye–Zhu [YZ, Main Thm.]	

Table 2.1: Known cases of Fujita's freeness conjecture over the complex numbers

For general smooth complex projective varieties, Fujita's freeness conjecture 1.3(i) holds in dimensions $n \leq 5$ (see Table 2.1) while Fujita's very ampleness conjecture 1.3(ii) is only known in dimensions $n \leq 2$ [Rei88, Thm. 1(ii)]. In positive characteristic, the usual statement of Fujita's conjecture holds for surfaces that are neither quasi-elliptic nor of general type [SB91, Cor. 8], and weaker bounds are known for quasi-elliptic and general type surfaces [Ter99, Thm.; DCF15, Thm. 1.4].

In arbitrary dimension, one of the best results toward Fujita's conjecture so far is the following result due to Angehrn and Siu, which they proved using analytic methods.

Theorem 2.1.11 [AS95, Cor. 0.2]**.** *Let X be a smooth complex projective variety of dimension n, and let L be an ample line bundle on X. Then, the line bundle $\omega_X \otimes L^{\otimes \ell}$ is globally generated for all $\ell \geq \frac{1}{2}n(n+1)+1$.*

Kollár later gave an algebraic proof of Theorem 2.1.11, which also applies to klt pairs [Kol97, Thm. 5.8]. Improved lower bounds for ℓ have also been obtained by Helmke [Hel97, Thm. 1.3; Hel99, Thm. 4.4] and Heier [Hei02, Thm. 1.4]. Note that Theorem 2.1.11 is a special case of Theorem D, which we will prove later in this thesis, since we can set $c(m) = \binom{n+1}{2}$ for all m; see Corollary 1.5.

2.2. Seshadri constants

To study Fujita's conjecture 1.3, Demailly introduced Seshadri constants, which measure the local positivity of nef divisors. Recall that an $\mathbf{R}$-Cartier divisor D is *nef* if $(D \cdot C) \geq 0$ for every curve $C \subseteq X$. See Definition 4.2.1 for the definition of an $\mathbf{R}$-Cartier divisor.

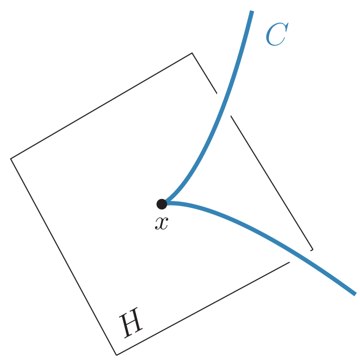

Figure 2.2: Computing the Seshadri constant of the hyperplane class on $\mathbf{P}^n_k$

Definition 2.2.1 (see [Laz04a, Def. 5.1.1])**.** Let X be a complete scheme over a field k, and let D be a nef $\mathbf{R}$-Cartier divisor on X. Let $x \in X$ be a k-rational point, and let $\mu\colon \widetilde{X} \to X$ be the blowup of X at x with exceptional divisor E. The *Seshadri constant* of D at x is

$$\varepsilon(D;x) \coloneqq \sup\bigl\{t \in \mathbf{R}_{\ge 0} \bigm| \mu^*D - tE \text{ is nef}\bigr\}.$$

We use the same notation for line bundles.

We will see later that this definition matches the definition in (1.2) (Lemma 2.4.1). This definition was motivated by Seshadri's criterion for ampleness, which says that when k is algebraically closed, an $\mathbf{R}$-Cartier divisor D is ample if and only if $\inf_{x\in X} \varepsilon(D;x) > 0$ [Laz04a, Thm. 1.4.13]. While originally defined in the context of Fujita's conjecture, Seshadri constants have also attracted attention as interesting geometric invariants in their own right; see [Laz04a, Ch. 5; BDRH$^+$09].

Before describing the connection between Seshadri constants and Fujita's conjecture 1.3, we compute a simple example. Note that Seshadri constants are very difficult to compute in general. We will use the fact from [Laz04a, Prop. 5.1.5] that

$$\varepsilon(D;x) = \inf_{C \ni x} \left\{ \frac{(D \cdot C)}{e(\mathcal{O}_{C,x})} \right\}, \tag{2.3}$$

where the infimum runs over all integral curves $C \subseteq X$ containing x, and $e(\mathcal{O}_{C,x})$ is the Hilbert–Samuel multiplicity of C at x.

Example 2.2.2 (Projective spaces; see Figure 2.2)**.** Consider $\mathbf{P}^n_k$ for an algebraically closed field k, and let $D = H$ be the hyperplane class. We claim that $\varepsilon(H;x) = 1$

for every closed point $x \in \mathbf{P}^n_k$. By Bézout's theorem [Har77, Thm. I.7.7], we have $(H \cdot C) \geq e(\mathcal{O}_{C,x})$ for every such curve $C \ni x$, hence $\varepsilon(H; x) \geq 1$ by (2.3). The inequality $\varepsilon(H; x) \leq 1$ also holds by considering the case when C is a line containing x.

Example 2.2.2 can be generalized as follows.

Example 2.2.3 (Ample and globally generated line bundles; see [Laz04a, Ex. 5.1.18])**.** We claim that if D is an ample and free Cartier divisor, then $\varepsilon(D; x) \geq 1$. Let $C \ni x$ be a curve; it suffices to show that $(D \cdot C) \geq e(\mathcal{O}_{C,x})$. Since the complete linear system $|D|$ is basepoint-free, there exists a divisor $H \in |D|$ such that H does not contain C. We then see that

$$(D \cdot C) = \deg(D|_C) \geq \ell(\mathcal{O}_{D|_C,x}) \geq e(\mathcal{O}_{C,x}),$$

where the first inequality follows from definition (see [GW10, Def. 15.29]) and the second inequality is a consequence of [Mat89, Thm. 14.10].

Demailly's original motivation for defining Seshadri constants seems to have been its potential application to Fujita's conjecture 1.3. Before we state the result realizing this connection, we make the following definition.

Definition 2.2.4. Let X be a scheme, and let $\mathscr{F}$ be a coherent sheaf on X. Fix a closed point $x \in X$, and denote by $\mathfrak{m}_x \subseteq \mathcal{O}_X$ the ideal sheaf defining x. For every integer $\ell \geq -1$, we say that $\mathscr{F}$ *separates ℓ-jets* at x if the restriction morphism

$$H^0(X, \mathscr{F}) \longrightarrow H^0(X, \mathscr{F}/\mathfrak{m}_x^{\ell+1}\mathscr{F}) \tag{2.4}$$

is surjective. We denote by $s(\mathscr{F}; x)$ the largest integer $\ell \geq -1$ such that $\mathscr{F}$ separates ℓ-jets at x. If $\mathscr{F} = \mathcal{O}_X(D)$ for a Cartier divisor D, then we denote $s(D; x) \coloneqq s(\mathcal{O}_X(D); x)$.

Remark 2.2.5. The convention that $s(\mathscr{F}; x) = -1$ if $\mathscr{F}$ does not separate ℓ-jets for every $\ell \geq 0$ is from [FMa, Def. 6.1]. This differs from the convention $s(\mathscr{F}; x) = -\infty$, which is used in [Dem92, p. 96] and [Mur18, Def. 2.1], and the convention $s(\mathscr{F}; x) = 0$, which is used in [ELM$^+$09, p. 646]. Our convention is chosen to make a variant of the Seshadri constant defined using jet separation (Definition 7.2.4) detect augmented base loci (Lemma 7.2.6), while distinguishing whether or not $\mathscr{F}$ has any non-vanishing global sections.

We now prove the following result due to Demailly, which connects Seshadri constants to separation of jets.

Proposition 2.2.6 [Dem92, Prop. 6.8(a)]**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic zero, and let L be a big and nef divisor on X. Let $x \in X$ be a closed point, and consider an integer $\ell \geq 0$. If $\varepsilon(L;x) > n+\ell$, then $\omega_X \otimes \mathcal{O}_X(L)$ separates ℓ-jets at x.*

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{m}_x^{\ell+1} \cdot \omega_X \otimes \mathcal{O}_X(L) \longrightarrow \omega_X \otimes \mathcal{O}_X(L) \longrightarrow \omega_X \otimes \mathcal{O}_X(L) \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1} \longrightarrow 0.$$

By the associated long exact sequence on sheaf cohomology, to show the surjectivity of the restriction morphism (2.4), it suffices to show that

$$\begin{aligned} H^1\bigl(X, \mathfrak{m}_x^{\ell+1} \cdot \omega_X \otimes \mathcal{O}_X(L)\bigr) &\simeq H^1\bigl(\widetilde{X}, \mu^*\bigl(\omega_X \otimes \mathcal{O}_X(L)\bigr)\bigl(-(\ell+1)E\bigr)\bigr) \\ &\simeq H^1\bigl(\widetilde{X}, \omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}\bigl(\mu^*L - (n+\ell)E\bigr)\bigr) = 0, \end{aligned}$$

where $\mu\colon \widetilde{X} \to X$ is the blowup of X at x. Here, the first isomorphism follows from the Leray spectral sequence and the quasi-isomorphism $\mathfrak{m}_x^{\ell+1} \simeq \mathbf{R}\mu_*\mathcal{O}_{\widetilde{X}}(-(\ell+1)E)$ [Laz04a, Lem. 4.3.16], and the second isomorphism follows from how the canonical bundle transforms under a blowup with a smooth center [Har77, Exer. II.8.5(b)]. The vanishing of the last group follows from the Kawamata–Viehweg vanishing theorem [Laz04a, Thm. 4.3.1] since $\mu^*L - (n+\ell)E$ is nef by the assumption $\varepsilon(L;x) > n+\ell$, and is big because by [Laz04a, Prop. 5.1.9], we have

$$\bigl(\mu^*L - (n+\ell)E\bigr)^n = (L^n) - (n+\ell)^n \geq \bigl(\varepsilon(L;x)\bigr)^n - (n+\ell)^n > 0. \qquad \square$$

Demailly showed that a similar technique can be used to deduce separation of points from the existence of lower bounds on Seshadri constants, and in particular, that if $\inf_{x\in X} \varepsilon(L;x) > 2n$ where $n = \dim X$, then $\omega_X \otimes \mathcal{O}_X(L)$ is very ample [Dem92, Prop. 6.8(b)]. Because of these results, Demailly asked:

Question 2.2.7 [Dem92, Quest. 6.9]**.** *Given a smooth projective variety X over an* *algebraically closed field and an ample divisor L on X, does there exist a lower bound for*

$$\varepsilon(L) \coloneqq \inf_{x \in X} \varepsilon(L;x)\,?$$

If such a lower bound were to exist, could we compute this lower bound explicitly in terms of geometric invariants of X?

Remark 2.2.8. We note that the divisors constructed in Kollár's example 2.1.6 do not give a counterexample to Question 2.2.7. In the notation of Example 2.1.6, the divisor $2A_m$ is free on X by Example 2.1.5. The pullback $2D_m = f^*(2A_m)$ is therefore ample and free, hence $\varepsilon(2D_m;x) \geq 1$ for every point $x \in Y$. By the homogeneity of Seshadri constants [Laz04a, Ex. 5.1.4], we have $\varepsilon(D_m;x) \geq 1/2$.

A very optimistic answer to Question 2.2.7 would be that $\varepsilon(L) > \frac{n}{n+1}$ where $n = \dim X$, since if this were the case, Proposition 2.2.6 would then imply Fujita's freeness conjecture 1.3(i). The following example of Miranda, however, shows that $\varepsilon(L)$ can become arbitrarily small, even on smooth surfaces.

Example 2.2.9 (Miranda [EL93b, Ex. 3.1])**.** Let $\delta > 0$ be arbitrary. We will construct a smooth projective surface X over an algebraically closed field k such that $\varepsilon(L;x) < \delta$ for an ample divisor L on X and a closed point $x \in X$.

Choose an integer $m \geq 1$ such that $\frac{1}{m} < \delta$, and let $\Gamma \subseteq \mathbf{P}^2_k$ be an integral curve of degree $d \geq 3$ and multiplicity m at a closed point $\xi \in \mathbf{P}^2_k$. Let $\Gamma' \subseteq \mathbf{P}^2_k$ be a general curve of degree d, which by generality we may assume is integral and intersects Γ in d^2 reduced points. We moreover claim that for general Γ', every curve in the pencil $|W|$ spanned by Γ and Γ' is irreducible. Note that such a pencil is a one-dimensional linear system, while the codimension of the space of reducible curves in $|dH|$ is

$$\begin{aligned}\binom{d+2}{2} - \max_{1 \leq i \leq d-1}&\left\{\binom{i+2}{2} + \binom{d-i+2}{2}\right\} + 1\\ &\geq \frac{(d+1)(d+2)}{2} - \left(\frac{d}{2}+1\right)\left(\frac{d}{2}+2\right) + 1 = \frac{d^2}{4} \geq 2,\end{aligned}$$

by the assumption $d \geq 3$. Thus, for general Γ', the pencil $|W|$ does not contain any reducible curves.

We now consider the blowup $X \to \mathbf{P}^2_k$ along $\Gamma \cap \Gamma'$ (see Figure 2.3). Since we have

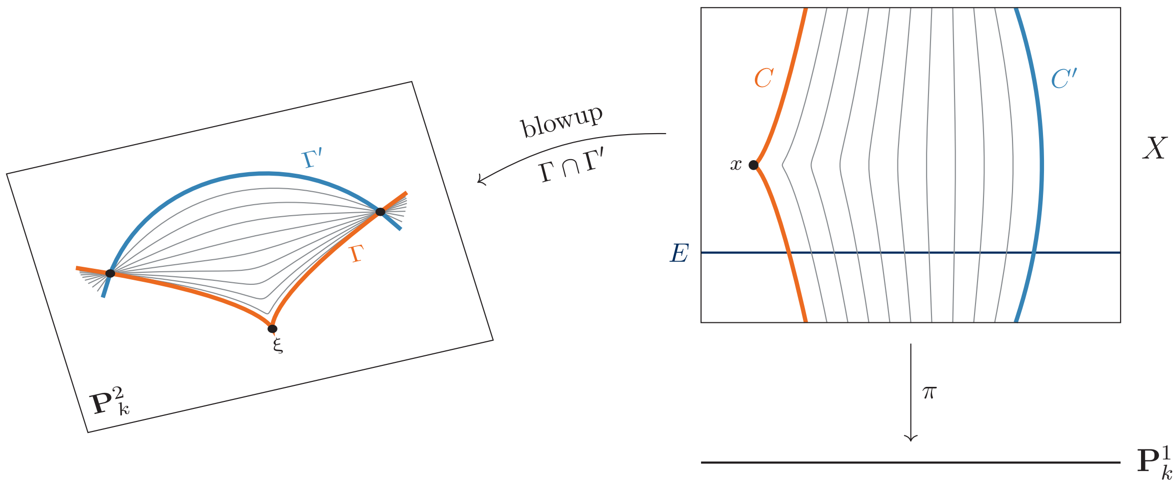

Figure 2.3: Miranda's example (Example 2.2.9)
Illustration inspired by [Laz04a, Fig. 5.1]

blown up the base locus of $|W|$, there is an induced morphism $\pi\colon X \to \mathbf{P}^1_k$ whose fibers correspond to curves in the pencil $|W|$. Let C and C' be the strict transforms of Γ and Γ' in X, respectively, let $x \in C$ be the strict transform of $\xi \in \Gamma$, and let E be an exceptional divisor of the blowup $X \to \mathbf{P}^2_k$. We claim that the divisor $L = aC + E$ on X is ample for $a \geq 2$. First, note that since $(C \cdot E) = 1$, we have $(L^2) = 2a - 1$ and $(L \cdot E) = a - 1$. If Z is a curve on X different from E, we then have

$$(L \cdot Z) = (C \cdot Z) + (E \cdot Z) \geq 0 \tag{2.5}$$

since C is basepoint-free and $(E \cdot Z) \geq 0$. By the Nakai–Moishezon criterion [Laz04a, Thm. 1.2.23], to show that L is ample, it suffices to show that equality cannot hold in (2.5). If equality holds, then $(C \cdot Z) = 0$, in which case $\pi(Z)$ is a point. On the other hand, since every curve in the pencil $|W|$ is irreducible, this implies Z is a fiber of π, in which case $(E \cdot Z) > 0$, a contradiction. Thus, L is ample. Finally, we note that

$$\varepsilon(L; x) \leq \frac{(L \cdot C)}{m} = \frac{1}{m} < \delta.$$

Remark 2.2.10. As noted by Viehweg, Miranda's example can be used to construct varieties of any dimension with arbitrarily small Seshadri constants [EL93b, Ex. 3.2]. Letting X be as constructed in Miranda's example 2.2.9, for every $n \geq 2$, the n-

dimensional smooth projective variety $X \times_k \mathbf{P}^{n-2}_k$ satisfies

$$\varepsilon\bigl(p_1^*L \otimes p_2^*\mathcal{O}(1); (x, z)\bigr) \le \varepsilon(L; x)$$

for every $z \in \mathbf{P}^{n-2}_k$ by considering the curve $C \times_k \{z\}$, where p_1, p_2 are the first and second projection morphisms, respectively.

Bauer has also shown that Miranda's example is not as exceptional as it might appear: suitable blowups of any surface with Picard number one have arbitrarily small Seshadri constants [Bau99, Prop. 3.3].

Despite Miranda's example, Ein, Küchle, and Lazarsfeld were able to prove that at very general points on complex projective varieties, lower bounds for $\varepsilon(L; x)$ do exist.

Theorem 2.2.11 [EKL95, Thm. 1]**.** *Let X be a complex projective variety of dimension n, and let L be a big and nef divisor on X. Then, for all closed points $x \in X$ outside of a countable union of proper closed subvarieties in X, we have*

$$\varepsilon(L; x) \ge \frac{1}{n}.$$

Moreover, for every $\delta > 0$, the locus

$$\left\{ x \in X \;\middle|\; \varepsilon(L; x) > \frac{1}{n + \delta} \right\}$$

contains a Zariski-open dense set in $X(\mathbf{C})$.

When L is ample and X is smooth of dimension $n \le 3$, the lower bound in Theorem 2.2.11 can be improved to $\varepsilon(L; x) \ge 1/(n-1)$ [EL93b, Thm.; CN14, Thm. 1.2]. The case $n = 2$ supports the following strengthening of Theorem 2.2.11.

Conjecture 2.2.12 [EKL95, p. 194]**.** *Let X be a projective variety over an algebraically closed field, and let L be a big and nef divisor on X. Then, for all closed points $x \in X$ outside of a countable union of proper closed subvarieties of X, we have $\varepsilon(L; x) \ge 1$.*

By combining Proposition 2.2.6 and Theorem 2.2.11, we obtain the following:

Corollary 2.2.13. *Let X be a smooth complex projective variety of dimension n, and let L be a big and nef divisor on X. Then, the bundle $\omega_X \otimes L^{\otimes m}$ separates ℓ-jets at all* *general points $x \in X$ for all $m \geq n(n+\ell)+1$. In particular, we have*

$$h^0(X, \omega_X \otimes L^{\otimes m}) \geq \binom{n+\ell}{n}$$

for all $m \geq n(n+\ell)+1$.

Remark 2.2.14. By replacing Proposition 2.2.6 with Theorem 7.3.1, we see that Corollary 2.2.13 holds for X with singularities of at worst dense F-injective type. See Definition 5.6.7 for the definition of this class of singularities. In particular, Corollary 2.2.13 holds for X with at worst rational singularities by Figure 5.1.

2.3. A relative Fujita-type conjecture

We also mention the following relative version of Fujita's conjecture. Inspired by Kollár and Viehweg's work on weak positivity, which partially answers an analogue of Question 1.2 for *families* of varieties, Popa and Schnell proposed the following:

Conjecture 2.3.1 [PS14, Conj. 1.3]**.** *Let $f\colon Y \to X$ be a morphism of smooth complex projective varieties, where X is of dimension n, and let L be an ample line bundle on X. Then, for every $k \geq 1$, the sheaf $f_*\omega_Y^{\otimes k} \otimes L^{\otimes m}$ is globally generated for all $m \geq k(n+1)$.*

Note that if f is the identity morphism $X \to X$, then Conjecture 2.3.1 is identical to Fujita's freeness conjecture 1.3(i). Popa and Schnell proved Conjecture 2.3.1 when $\dim X = 1$ [PS14, Prop. 2.11], or when L is additionally assumed to be globally generated [PS14, Thm. 1.4]. This latter result was shown using Castelnuovo–Mumford regularity in a similar fashion to Example 2.1.10, with Ambro and Fujino's Kollár-type vanishing theorem replacing the Kodaira vanishing theorem in the proof.

In joint work with Yajnaseni Dutta, we proved the following effective global generation result in the spirit of Conjecture 2.3.1, which we later extended to higher-order jets in joint work with Mihai Fulger. Note that the case when (Y, Δ) is klt and $k = 1$ is due to de Cataldo [dC98, Thm. 2.2].

Theorem 2.3.2 [DM19, Thm. A; FMa, Cor. 8.2]**.** *Let $f\colon Y \to X$ be a surjective morphism of complex projective varieties, where X is of dimension n. Let (Y, Δ) be a log canonical pair and let L be a big and nef line bundle on X. Consider a Cartier divisor*

P on Y such that $P \sim_{\mathbf{R}} k(K_Y + \Delta)$ for some integer $k \geq 1$. Then, the sheaf

$$f_*\mathcal{O}_Y(P) \otimes L^{\otimes m}$$

separates ℓ-jets at all general points $x \in X$ for all $m \geq k(n(n+\ell)+1)$.

The proof of Theorem 2.3.2 is a relativization of the argument in Proposition 2.2.6, and uses the lower bound on Seshadri constants in Theorem 2.2.11. A generic global generation result in this direction was first obtained by Dutta for klt $\mathbf{Q}$-pairs (Y, Δ) [Dut, Thm. A]. Using analytic techniques, Deng and Iwai later obtained improvements of Dutta's original result for klt pairs with better lower bounds, under the additional assumption that X is smooth and L is ample [Den, Thm. C; Iwa, Thm. 1.5]. In [DM19, Thm. B], we proved algebraic versions of Deng's and Iwai's results as a consequence of a new weak positivity result for pairs [DM19, Thms. E and F]. Note, however, that only our methods in [DM19; FMa] apply to log canonical pairs.

Remark 2.3.3. In positive characteristic, there is an example of a curve fibration over $\mathbf{P}^1_k$ which gives a counterexample both to Popa and Schnell's relative Fujita-type conjecture 2.3.1, and to the analogue of Theorem 2.3.2 in positive characteristic. The example is based on a construction due to Moret-Bailly [MB81]; see [SZ, Prop. 4.11].

2.4. Difficulties in positive characteristic

While most of the questions, conjectures, and examples seen so far have been stated over fields of arbitrary characteristic, the majority of the results stated, in particular on Fujita's conjecture (Table 2.1 and Theorem 2.1.11) and lower bounds on Seshadri constants (Theorem 2.2.11), are only known over fields of characteristic zero. The most problematic situation is when the ground field k is an imperfect field of characteristic $p > 0$, in which case there are at least three major difficulties. First, since k is of characteristic $p > 0$,

(I) Resolutions of singularities are not known to exist (see [Hau10]), and

(II) Kodaira-type vanishing theorems are false [Ray78] (see §2.4.2).

A common workaround for the lack of resolutions is to use de Jong's theory of alterations [dJ96]. The lack of vanishing theorems is harder to circumvent, however, since over the complex numbers, vanishing theorems are a fundamental ingredient used to construct global sections of line bundles. A useful workaround is to exploit the Frobenius morphism $F\colon X \to X$ and its Grothendieck trace $F_*\omega_X^\bullet \to \omega_X^\bullet$; see [PST17; Pat18]. For imperfect fields, however, this approach runs into another problem:

(III) Applications of Frobenius techniques in algebraic geometry usually require the ground field k to be *F-finite*, i.e., satisfy $[k : k^p] < \infty$.

The last issue arises since Grothendieck duality cannot be applied to the Frobenius if it is not finite. Working around this last issue is the focus of Appendix B.

2.4.1. Proof of Theorem B

To illustrate how Frobenius techniques can be used in practice, we prove the following positive characteristic version of Proposition 2.2.6.

Theorem B. *Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic $p > 0$, and let L be an ample line bundle on X. Let $x \in X$ be a closed point, and consider an integer $\ell \geq 0$. If $\varepsilon(L;x) > n+\ell$, then $\omega_X \otimes L$ separates ℓ-jets at x.*

The case $\ell = 0$ is due to Mustaţă and Schwede [MS14, Thm. 3.1]. The case for arbitrary $\ell \geq 0$ first appeared in [Mur18, Thm. A]. These proofs used a positive-characteristic version of Seshadri constants called *Frobenius–Seshadri constants* $\varepsilon_F^\ell(L;x)$; see Remark 7.3.4. Note that we will later prove a generalization of Theorem B; see Theorem 7.3.1.

We give a new proof of Theorem B, which is an adaptation of the proof in [PST17, Exer. 6.3], which proves the case when $\ell = 0$. As in the proof of [MS14, Thm. 3.1], the main ingredient in the proof is the *Grothendieck trace*

$$\mathrm{Tr}_X\colon F_*\omega_X \longrightarrow \omega_X$$

associated to the (absolute) Frobenius morphism $F\colon X \to X$. Recall that the Frobenius morphism is defined as the identity map on points, and the p-power map

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow F_*\mathcal{O}_X(U)\\ f &\longmapsto f^p \end{aligned}$$

on functions over every open set $U \subseteq X$. This map Tr_X is a morphism of $\mathcal{O}_X$-modules, which can be obtained by applying Grothendieck duality for finite flat morphisms to the (absolute) Frobenius morphism $F\colon X \to X$; see §4.4. Note that F is finite since k is F-finite (see Example 5.3.2), and is flat by Kunz's theorem [Kun69, Thm. 2.1] since X is smooth. By [BK05, Lem. 1.3.6], we can also describe the trace map locally by

$$\prod_{i=1}^{n} x_i^{a_i}\, dx \longmapsto \prod_{i=1}^{n} x_i^{\frac{a_i-p+1}{p}}\, dx, \tag{2.6}$$

where $x_1, x_2, \ldots, x_n \in \mathcal{O}_X(U)$ is a choice of local coordinates on an affine open subset $U \subseteq X$, and $dx \coloneqq dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. By convention, the expression on the right-hand side of (2.6) is zero unless all exponents are integers. See [BK05, §1.3] for the definition and basic properties of the morphism Tr_X from this point of view, where it is also called the *Cartier operator*.

The trace map Tr_X satisfies the following key properties needed for our proof:

(a) Since X is smooth, the trace map Tr_X and its eth iterates $\mathrm{Tr}_X^e\colon F_*^e\omega_X \to \omega_X$ are surjective for every $e \geq 0$ [BK05, Thm. 1.3.4].

(b) If $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent ideal sheaf, then for every $e \geq 0$, the map Tr_X^e satisfies

$$\mathrm{Tr}_X^e\bigl(F_*^e(\mathfrak{a}^{[p^e]} \cdot \omega_X)\bigr) = \mathfrak{a} \cdot \mathrm{Tr}_X^e(F_*^e\omega_X) = \mathfrak{a} \cdot \omega_X. \tag{2.7}$$

Here, $\mathfrak{a}^{[p^e]}$ is the eth *Frobenius power* of $\mathfrak{a}$, which is the ideal sheaf locally generated by p^eth powers of local generators of $\mathfrak{a}$. Note that (2.7) follows from (a) by considering the $\mathcal{O}_X$-module structure on $F_*^e\omega_X$.

We need one more general result about Seshadri constants of ample divisors. Note that this result shows that the definition of the Seshadri constant in (1.2) matches that in Definition 2.2.1.

Lemma 2.4.1. *Let X be a projective scheme over a field k, and let D be an ample $\mathbf{R}$-Cartier divisor on X. Consider a k-rational point $x \in X$, and let $\mu\colon \widetilde{X} \to X$ be the blowup of X at x with exceptional divisor E. For every $\delta \in (0, \varepsilon(D;x))$, the $\mathbf{R}$-Cartier divisor $\mu^*D - \delta E$ is ample.*

Proof. Let $V \subseteq \widetilde{X}$ be a subvariety. If $V \not\subseteq E$, then V is the strict transform of a subvariety $V_0 \subseteq X$, and

$$\Bigl(\bigl(\mu^* D - \delta E\bigr)^{\dim V} \cdot V\Bigr) = \bigl(D^{\dim V} \cdot V_0\bigr) - \delta\, e(\mathcal{O}_{V_0,x}) > 0$$

by the assumption $\varepsilon(D;x) > \delta$ and [Laz04a, Prop. 5.1.9]. Otherwise, if $V \subseteq E$, then

$$\Bigl(\bigl(\mu^* D - \delta E\bigr)^{\dim V} \cdot V\Bigr) = \Bigl(\bigl(-\delta E|_E\bigr)^{\dim V} \cdot V\Bigr) > 0$$

since $\mathcal{O}_E(-E|_E) \simeq \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ is very ample. Thus, the divisor $\mu^* D - \delta E$ is ample by the Nakai–Moishezon criterion [Laz04a, Thm. 1.2.23]. □

We can now prove Theorem B.

Proof of Theorem B. First, we claim that it suffices to show that the restriction morphism

$$\varphi_e \colon H^0(X, \omega_X \otimes L^{\otimes p^e}) \longrightarrow H^0\bigl(X, \omega_X \otimes L^{\otimes p^e} \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell p^e + n(p^e-1)+1}\bigr)$$

is surjective for some $e \geq 0$. By (2.7), the map Tr_X^e induces a morphism

$$F^e_*\bigl((\mathfrak{m}_x^{\ell+1})^{[p^e]} \cdot \omega_X\bigr) \longrightarrow \mathfrak{m}_x^{\ell+1} \cdot \omega_X.$$

Twisting this morphism by L and applying the projection formula yields a morphism

$$F^e_*\bigl((\mathfrak{m}_x^{\ell+1})^{[p^e]} \cdot \omega_X \otimes L^{\otimes p^e}\bigr) \longrightarrow \mathfrak{m}_x^{\ell+1} \cdot \omega_X \otimes L. \tag{2.8}$$

Here, we use the fact that $F^* L \simeq L^{\otimes p}$ since pulling back by the Frobenius morphism raises the transition functions defining L to the pth power. Since the Frobenius morphism F is affine, the pushforward functor F^e_* is exact, hence we obtain the exactness of the left column in the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
F^e_*\big((\mathfrak{m}_x^{\ell+1})^{[p^e]} \cdot \omega_X \otimes L^{\otimes p^e}\big) \arrow[r] \arrow[d] & \mathfrak{m}_x^{\ell+1} \cdot \omega_X \otimes L \arrow[d] \\
F^e_*\big(\omega_X \otimes L^{\otimes p^e}\big) \arrow[r, two heads] \arrow[d] & \omega_X \otimes L \arrow[d] \\
F^e_*\big(\omega_X \otimes L^{\otimes p^e} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\big) \arrow[r, two heads] \arrow[d] & \omega_X \otimes L \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1} \arrow[d] \\
0 & 0
\end{tikzcd}
\tag{2.9}
$$

The top horizontal arrow is the map in (2.8); the middle horizontal arrow is obtained from Tr^e_X in a similar fashion by twisting by L and by applying the projection formula, hence is surjective by (a). The surjectivity of the middle horizontal arrow also implies the bottom horizontal arrow is surjective by the snake lemma. Now by the pigeonhole principle (see [HH02, Lem. 2.4(a)] or Lemma 5.2.1), we have the inclusion $\mathfrak{m}_x^{\ell p^e+n(p^e-1)+1} \subseteq (\mathfrak{m}_x^{\ell+1})^{[p^e]}$ for every $e \geq 0$, which yields the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathfrak{m}_x^{\ell p^e+n(p^e-1)+1} \cdot \omega_X \otimes L^{\otimes p^e} \arrow[r, hook] \arrow[d] & (\mathfrak{m}_x^{\ell+1})^{[p^e]} \cdot \omega_X \otimes L^{\otimes p^e} \arrow[d] \\
\omega_X \otimes L^{\otimes p^e} \arrow[r, equal] \arrow[d] & \omega_X \otimes L^{\otimes p^e} \arrow[d] \\
\omega_X \otimes L^{\otimes p^e} \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell p^e+n(p^e-1)+1} \arrow[r, two heads] \arrow[d] & \omega_X \otimes L^{\otimes p^e} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]} \arrow[d] \\
0 & 0
\end{tikzcd}
\tag{2.10}
$$

By applying $F^e_*(-)$ to (2.10), combining it with (2.9), and taking global sections in the bottom half of both diagrams, we obtain the following commutative square:

$$\begin{tikzcd}
H^0(X,\omega_X\otimes L^{\otimes p^e}) \arrow[r] \arrow[d, "\varphi_e"'] & H^0(X,\omega_X\otimes L) \arrow[d, "\rho"] \\
H^0\bigl(X,\omega_X\otimes L^{\otimes p^e}\otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell p^e+n(p^e-1)+1}\bigr) \arrow[r, two heads, "\psi"] & H^0\bigl(X,\omega_X\otimes L\otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1}\bigr)
\end{tikzcd}$$

Note that ψ is surjective because the kernel of the corresponding morphism of sheaves is a skyscraper sheaf supported at x. Now assuming that φ_e is surjective, we see that the composition from the top left corner to the bottom right corner is surjective, hence the restriction morphism ρ is necessarily surjective as well.

We now show that φ_e is surjective for some e. By the long exact sequence on sheaf cohomology, it suffices to show that

$$\begin{aligned}
&H^1\bigl(X,\mathfrak{m}_x^{\ell p^e+n(p^e-1)+1}\cdot\omega_X\otimes L^{\otimes p^e}\bigr)\\
&\qquad\simeq H^1\Bigl(\widetilde{X},\mu^*\bigl(\omega_X\otimes L^{\otimes p^e}\bigr)\bigl(-\bigl(\ell p^e+n(p^e-1)+1\bigr)E\bigr)\Bigr)\\
&\qquad\simeq H^1\Bigl(\widetilde{X},\omega_{\widetilde{X}}\otimes\bigl(\mu^*L\bigl(-(n+\ell)E\bigr)\bigr)^{\otimes p^e}\Bigr)=0,
\end{aligned}$$

where $\mu\colon\widetilde{X}\to X$ is the blowup of X at x. The first isomorphism follows from the Leray spectral sequence and the quasi-isomorphism

$$\mathfrak{m}_x^{\ell p^e+n(p^e-1)+1}\simeq \mathbf{R}\mu_*\mathcal{O}_{\widetilde{X}}\bigl(-\bigl(\ell p^e+n(p^e-1)+1\bigr)E\bigr)$$

from [Laz04a, Lem. 4.3.16], and the second isomorphism follows from how the canonical bundle transforms under a blowup with a smooth center [Har77, Exer. II.8.5(b)]. The vanishing of the last group follows from Serre vanishing for e sufficiently large [Har77, Prop. III.5.3] since $\mu^*L-(n+\ell)E$ is ample by Lemma 2.4.1. □

Remark 2.4.2. The proof of Theorem B works under the weaker assumption that X is regular and k is F-finite. Moreover, by using the gamma construction (Theorem B.1.1) to reduce to the case when k is F-finite, the proof of Theorem B yields a statement over arbitrary fields of characteristic $p>0$. Since this more general version of Theorem B follows from Theorem 7.3.1, we have chosen to prove this weaker result for simplicity.

Remark 2.4.3. If $\dim X=2$, then it suffices for L to be big and nef instead of ample in

Theorem B. To prove this, it suffices to replace Serre vanishing with a vanishing theorem of Szpiro [Szp79, Prop. 2.1] and Lewin-Ménégaux [LM81, Prop. 2], which asserts that for a big and nef divisor L on a smooth projective surface X, we have

$$H^1\bigl(X, \mathcal{O}_X(-mL)\bigr) = 0$$

for m sufficiently large. Fujita has shown that a similar vanishing theorem also holds for higher-dimensional projective varieties that are only assumed to be normal [Fuj83, Thm. 7.5], although the positivity condition on L is stronger. Fujita's theorem cannot be used to prove Theorem B in higher dimensions for big and nef divisors L, however, since the required vanishing $H^{n-1}(X, \mathcal{O}_X(-mL)) = 0$ does not hold in general, even as $m \to \infty$; see Example 2.4.5.

2.4.2. Raynaud's counterexample to Kodaira vanishing

To illustrate what goes wrong in positive characteristic, we give a version of Raynaud's original example showing that Kodaira vanishing is false in positive characteristic, with some changes in presentation following Mukai [Muk79; Muk13]. See also [Tak10; Zhe17]. Note that Mukai also constructs versions of Raynaud's example in higher dimensions.

Example 2.4.4 (Raynaud [Ray78; Muk79; Muk13])**.** Let k be an algebraically closed field of characteristic $p > 0$. The construction proceeds in four steps.

Step 1. *Construction of a smooth projective curve C over k and a Cartier divisor D on C such that the morphism*

$$F^*\colon H^1\bigl(C, \mathcal{O}_C(-D)\bigr) \longrightarrow H^1\bigl(C, \mathcal{O}_C(-pD)\bigr) \tag{2.11}$$

induced by the Frobenius morphism is not injective.

Let $h > 0$ be an integer, let P be a polynomial of degree h in one variable over k, and consider the plane curve

$$C = \overline{\bigl\{P(x^p) - x = y^{ph-1}\bigr\}} \subseteq \mathbf{P}^2_k$$

of degree ph, where $\mathbf{P}^2_k$ has variables x, y, z, and $P(x^p) - x = y^{ph-1}$ is the equation defining C on the open set $\{z \neq 0\}$. Note that C has exactly one point ∞ along

$\{z = 0\}$. By the Jacobian criterion [Har77, Exer. I.5.8], since the homogeneous Jacobian $(-z^{ph-1},\ y^{ph-2}z,\ xz^{ph-2} - y^{ph-1})$ associated to C has full rank along C, we see that C is smooth.

We claim that the morphism (2.11) is not injective for the divisor $D = h(ph-3) \cdot \infty$. By [Tan72, Lem. 12], since the kernel of the morphism in (2.11) can be described by

$$\ker(F^*) \simeq \bigl\{ df \in \Omega_{K(C)} \bigm| f \in K(C) \text{ such that } (df) \geq pD \bigr\},$$

it suffices to construct a rational function $f \in K(C)$ satisfying $(df) \geq pD$. Here, (df) is the divisor of zeroes and poles of the differential form df. Consider the rational function $y \in K(C)$. By the relation $-dx = -y^{ph-2}dy$ on $C \smallsetminus \{\infty\}$, we see that Ω_C is generated by dy over $C \smallsetminus \{\infty\}$, hence dy has no poles or zeroes away from ∞. Since by [Har77, Ex. V.1.5.1], we have

$$\deg \Omega_C = 2g(C) - 2 = ph(ph-3), \tag{2.12}$$

we obtain $(dy) = ph(ph-3) \cdot \infty = pD$, as desired. We note that C is an example of a *Tango curve*.

Step 2. *Construction of a projective bundle* $\pi \colon \mathbf{P}(E) \to C$ *with two distinguished divisors* F *and* G *arising from sections of* $\mathbf{P}(E)$ *and of* $\mathbf{P}(E^{(p)})$.

By identifying the sheaf cohomology groups in (2.11) with Ext^1 groups [Har77, Prop. III.6.3], we obtain a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C(D) \longrightarrow 0 \tag{2.13}$$

such that after pulling back via the Frobenius morphism $F \colon C \to C$ on C, the resulting short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E^{(p)} \longrightarrow \mathcal{O}_C(pD) \longrightarrow 0 \tag{2.14}$$

splits. The projective bundles of one-dimensional quotients $\mathbf{P}(E)$ and $\mathbf{P}(E^{(p)})$ associated

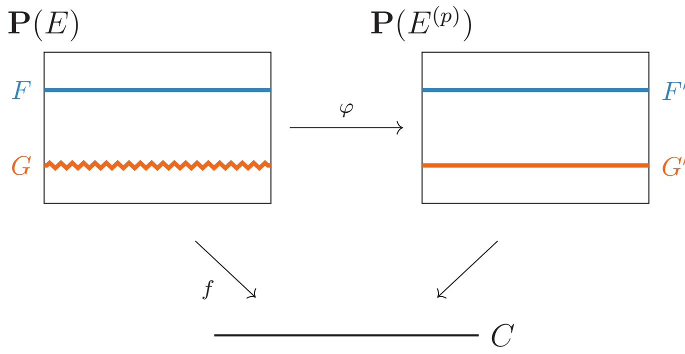

Figure 2.4: Raynaud's example (Example 2.4.4)
Illustration from [Muk79, Fig. on p. 18]

to E and $E^{(p)}$ fit into the pullback diagram

$$\begin{tikzcd}
\mathbf{P}(E) \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "f"'] & & \\
& \mathbf{P}(E^{(p)}) \arrow[r] \arrow[d] & \mathbf{P}(E) \arrow[d, "f"] \\
& C \arrow[r, "F"] & C
\end{tikzcd}$$

where $\varphi\colon \mathbf{P}(E) \to \mathbf{P}(E^{(p)})$ is the relative Frobenius morphism for $\mathbf{P}(E)$ over C.

We now note that $f\colon \mathbf{P}(E) \to C$ has a section $C \to \mathbf{P}(E)$ with image $F \simeq C$, which corresponds to the surjection in (2.13). The image $F' = \varphi(F)$ of F gives the section of $\mathbf{P}(E^{(p)}) \to C$ corresponding to the surjection in (2.14), and the fact that (2.14) splits implies $\mathbf{P}(E^{(p)})$ also has another section $C \to \mathbf{P}(E^{(p)})$ with image $G' \simeq C$ such that $F' \cap G' = \emptyset$ [Har77, Exer. V.2.2]. We denote by $G \coloneqq \varphi^{-1}(G')$ the scheme-theoretic inverse of G', which is a smooth variety by [Muk13, Prop. 1.7].[1] Note that $F \cap G = \emptyset$ since $F' \cap G' = \emptyset$; see Figure 2.4. By [Har77, Prop. V.2.6], we have the linear equivalences $0 \sim \xi - F$ and $f^*(pD) \sim p\xi - G$ on $\mathbf{P}(E)$, where ξ is the divisor class associated to $\mathscr{O}_{\mathbf{P}(E)}(1)$, hence

$$G - pF \sim -f^*(pD). \tag{2.15}$$

[1] The fact that (2.13) does not split but (2.14) does split is used here. We mention that [Muk13, Prop. 1.7] is proved for a higher-dimensional generalization of our example. See [Tak10, Thm. 3] for a simpler statement that suffices for our purposes.

Step 3. *Construction of a cyclic cover $\pi\colon X \to \mathbf{P}(E)$ where X is a smooth surface and a suitable ample divisor $\widetilde{D}$ on X.*

Let $r \geq 2$ be an integer such that $r \mid p+1$ and $r \mid h(ph-3)$ for some choice of integer $h > 0$ in Step 1. For example, if $p \neq 2$ then we can set $r = 2$ for arbitrary $h > 0$; if $p = 2$, then we can set $r = 3$ for $h > 0$ such that $3 \mid h$. By adding $(p+1)F$ to (2.15), we have $G + F \sim (p+1)F - f^*(pD) \sim rM$, where

$$M \coloneqq \frac{p+1}{r}F - f^*\biggl(\frac{ph(ph-3)}{r} \cdot \infty\biggr),$$

since $D = h(ph-3) \cdot \infty$. We can therefore construct a degree r cyclic cover

$$X \coloneqq \mathbf{Spec}_{\mathbf{P}(E)}\Bigl(\mathcal{O}_{\mathbf{P}(E)} \oplus \mathcal{O}_{\mathbf{P}(E)}(-M) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}(E)}\bigl(-(r-1)M\bigr)\Bigr) \xrightarrow{\pi} \mathbf{P}(E)$$

branched along $G + F$, which is smooth by the fact that both $\mathbf{P}(E)$ and $G + F$ are smooth [Laz04a, Prop. 4.1.6]. We then set

$$\widetilde{D} = \widetilde{F} + (f \circ \pi)^*\biggl(\frac{h(ph-3)}{r} \cdot \infty\biggr),$$

where $\widetilde{F} = \pi^{-1}(F)_{\mathrm{red}}$ is the inverse image of F with reduced scheme structure. To show that $\widetilde{D}$ is ample, we first note that $F + f^*(h(ph-3)\cdot\infty)$ is ample by the Nakai–Moishezon criterion [Laz04a, Thm. 1.2.23] since it intersects both the section F and the fibers of the ruled surface $f\colon \mathbf{P}(E) \to C$ positively. The pullback $r\widetilde{D} \sim \pi^*(rF) + \pi^*f^*(h(ph-3)\cdot\infty)$ is therefore also ample by [Laz04a, Prop. 4.1.6], hence $\widetilde{D}$ is ample. We note that X is an example of a *Raynaud surface*.

Step 4. *Proof that $H^1(X, \mathcal{O}_X(-\widetilde{D})) \neq 0$.*

First, we note that

$$\begin{aligned} H^1\bigl(X, \mathcal{O}_X(-\widetilde{D})\bigr) &\simeq H^1\bigl(\mathbf{P}(E), \pi_*\mathcal{O}_X(-\widetilde{D})\bigr) \\ &\simeq H^1\Biggl(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(-F) \oplus \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbf{P}(E)}(-iM)\Biggr). \end{aligned}$$

The first isomorphism holds by the fact that π is finite. The second isomorphism holds by properties of cyclic covers; see [Zhe16, Prop. 6.3.4] for a proof using local coordinates, or see [Zhe17, Prop. 3.3] for a shorter proof. Now consider the Leray spectral sequence

$$\begin{aligned} E_2^{p,q} = H^p\Big(C, R^q f_*\Big(\mathcal{O}_{\mathbf{P}(E)}(-F) \oplus \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbf{P}(E)}(-iM)\Big)\Big) \\ \Rightarrow H^{p+q}\Big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(-F) \oplus \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbf{P}(E)}(-iM)\Big) \end{aligned} \tag{2.16}$$

which already degenerates on the E_2 page. We have that $E_2^{1,0} = 0$, since the pushforward

$$\begin{aligned} &f_*\Big(\mathcal{O}_{\mathbf{P}(E)}(-F) \oplus \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbf{P}(E)}(-iM)\Big) \\ &\qquad \simeq f_*\mathcal{O}_{\mathbf{P}(E)}(-1) \oplus \bigoplus_{i=1}^{r-1} f_*\mathcal{O}_{\mathbf{P}(E)}\Big(-\frac{i(p+1)}{r}\Big) \otimes \mathcal{O}_C\Big(\frac{iph(ph-3)}{r} \cdot \infty\Big) \end{aligned}$$

is zero by the fact that $f_*\mathcal{O}_{\mathbf{P}(E)}(-n) = 0$ for $n > 0$. Thus, the Leray spectral sequence (2.16) implies that

$$H^1\big(X, \mathcal{O}_X(-\widetilde{D})\big) \simeq H^0\Big(C, R^1 f_*\Big(\mathcal{O}_{\mathbf{P}(E)}(-F) \oplus \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbf{P}(E)}(-iM)\Big)\Big). \tag{2.17}$$

Since $R^1 f_*(\mathcal{O}_{\mathbf{P}(E)}(-F)) \simeq R^1 f_*(\mathcal{O}_{\mathbf{P}(E)}(-1)) = 0$, we will consider the summands containing $\mathcal{O}_{\mathbf{P}(E)}(-iM)$. By [Har77, Exer. II.8.4(c)], we have

$$R^1 f_*\Big(\mathcal{O}_{\mathbf{P}(E)}\Big(-\frac{i(p+1)}{r}\Big)\Big) \simeq \Big(f_*\mathcal{O}_{\mathbf{P}(E)}\Big(\frac{i(p+1)-2r}{r}\Big)\Big)^\vee \simeq \big(\mathrm{Sym}^{\frac{i(p+1)-2r}{r}} E\big)^\vee,$$

hence the projection formula implies

$$R^1 f_*\big(\mathcal{O}_{\mathbf{P}(E)}(-iM)\big) \simeq \big(\mathrm{Sym}^{\frac{i(p+1)-2r}{r}} E\big)^\vee \otimes \mathcal{O}_C\Big(\frac{iph(ph-3)}{r} \cdot \infty\Big).$$

The short exact sequence (2.13) implies that there is a surjection $\mathrm{Sym}^{\frac{i(p+1)-2r}{r}} E \twoheadrightarrow \mathcal{O}_C\big(\frac{i(p+1)-2r}{r} D\big)$, hence there is an injection

$$\mathcal{O}_C\Big(\frac{(2r-i)h(ph-3)}{r} \cdot \infty\Big) \hookrightarrow R^1 f_*\big(\mathcal{O}_{\mathbf{P}(E)}(-iM)\big).$$

We therefore have

$$H^0\left(C, \mathcal{O}_C\left(\frac{(2r-i)h(ph-3)}{2} \cdot \infty\right)\right) \hookrightarrow H^0\bigl(C, R^1 f_*\bigl(\mathcal{O}_{\mathbf{P}(E)}(-iM)\bigr)\bigr),$$

where the left-hand side is nonzero as long as $2r - i \geq 0$. By the assumption $r \geq 2$, the left-hand side is nonzero for $i = 1$, hence (2.17) implies $H^1(X, \mathcal{O}_X(-\widetilde{D})) \neq 0$.

As noted by Fujita, Raynaud's example 2.4.4 also gives counterexamples to the vanishing theorem in Remark 2.4.3 for smooth projective varieties of dimension 3.

Example 2.4.5 [Fuj83, (7.10)]**.** Let X and $\widetilde{D}$ be as constructed in Example 2.4.4, and consider the $\mathbf{P}^1$-bundle

$$Y \coloneqq \mathbf{P}\bigl(\mathcal{O}_X(\widetilde{D}) \oplus \mathcal{O}_X\bigr) \xrightarrow{\pi} X$$

over X. Note that $\mathcal{O}_Y(1)$ is big and nef by [Laz04a, Lems. 2.3.2(iii) and 2.3.2(iv)]. We claim that $H^2(Y, \mathcal{O}_Y(-m)) \neq 0$ for all $m \geq 2$. We have

$$\begin{aligned} H^2\bigl(Y, \mathcal{O}_Y(-m)\bigr)^\vee &\simeq H^1\bigl(Y, \omega_Y \otimes \mathcal{O}_Y(m)\bigr) \\ &\simeq H^1\Bigl(Y, \mathcal{O}_Y(m-2) \otimes \pi^*\bigl(\omega_X \otimes \mathcal{O}_X(\widetilde{D})\bigr)\Bigr) \end{aligned}$$

by Serre duality [Har77, Cor. 7.7] and by [Har77, Exer. II.8.4(b)], respectively. By the projection formula, we therefore have

$$H^2\bigl(Y, \mathcal{O}_Y(-m)\bigr)^\vee \simeq H^1\bigl(X, \mathrm{Sym}^{m-2}\bigl(\mathcal{O}_X(\widetilde{D}) \oplus \mathcal{O}_X\bigr) \otimes \omega_X \otimes \mathcal{O}_X(\widetilde{D})\bigr).$$

The right-hand side contains $H^1(X, \omega_X \otimes \mathcal{O}_X(\widetilde{D})) \simeq H^1(X, \mathcal{O}_X(-\widetilde{D}))^\vee$ as a direct summand for all $m \geq 2$. Since $H^1(X, \mathcal{O}_X(-\widetilde{D})) \neq 0$ by the construction in Example 2.4.4, we see that $H^2(Y, \mathcal{O}_Y(-m)) \neq 0$ for all $m \geq 2$.

Chapter 3

Characterizations of projective space

In this chapter, we describe how Seshadri constants can be used to study the following:

Question 1.1. *How can we identify when a given projective variety is projective space?*

Recall that a smooth projective variety X of dimension n is *Fano* if its anti-canonical bundle $\omega_X^{-1} := \bigwedge^n T_X$ is ample. In this chapter, we prove the following characterization of projective space amongst Fano varieties using Seshadri constants. Note that the lower bound $\deg(\omega_X^{-1}|_C) \geq e(\mathcal{O}_{C,x}) \cdot (n+1)$ below is equivalent to $\varepsilon(\omega_X^{-1}; x) \geq n+1$; see the statement of Theorem A*.

Theorem A. *Let X be a Fano variety of dimension n over an algebraically closed field k of positive characteristic. If there exists a closed point $x \in X$ with*

$$\deg\bigl(\omega_X^{-1}|_C\bigr) \geq e(\mathcal{O}_{C,x}) \cdot (n+1)$$

for every integral curve $C \subseteq X$ passing through x, then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n_k$.

This result is originally due to Bauer and Szemberg in characteristic zero [BS09, Thm. 1.7]. The material in this chapter is from [Mur18, §4].

3.1. Background

We start by motivating the statement of Theorem A. Our story begins with the following observation about projective space.

Principle 3.1.1. *The n-dimensional projective space $\mathbf{P}^n_k$ over a field k has a "positive" tangent bundle T_X. For example, we have the following:*

(1) *T_X is an ample vector bundle* [Laz04b, Prop. 6.3.1(i)].

(2) *There exists an ample line bundle H on $\mathbf{P}^n_k$ such that $\bigwedge^n T_X \simeq H^{\otimes(n+1)}$.*

(3) *$\bigwedge^n T_X = \omega_X^{-1}$ is ample (i.e., X is Fano) and $\deg(\omega_X^{-1}|_C) \geq n+1$ for all integral curves $C \subseteq X$.*

Note that (2) and (3) hold since $\omega_{\mathbf{P}^n_k}^{-1} = \mathcal{O}_{\mathbf{P}^n_k}(n+1)$ [Har77, Ex. II.8.20.2]. We recall that a vector bundle E on X is *ample* if $\mathcal{O}_{\mathbf{P}(E)}(1)$ is ample [Laz04b, Def. 6.1.1].

These properties seem very special, and lead us to ask the following more specific version of Question 1.1.

Question 3.1.2. *Let X be a smooth projective variety of dimension n over an algebraically closed field k. If X satisfies one of* (1)–(3)*, is X isomorphic to $\mathbf{P}^n_k$?*

Many results in this direction are known. The first result, due to Mori, is in some sense the birthplace of modern birational geometry and the minimal model program.

Theorem 3.1.3 [Mor79, Thm. 8]**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field k. If* (1) *holds, then X is isomorphic to $\mathbf{P}^n_k$.*

The sufficiency of (1) was first conjectured by Frankel [Fra61, Conj.] in the analytic context, and by Hartshorne [Har70, Prob. III.2.3] in the algebraic context. The idea of Mori's proof is to produce many copies of $\mathbf{P}^1_k$ inside X passing through a point $x_0 \in X$ using *bend and break* techniques. Letting $\mu\colon \widetilde{X} \to X$ be the blowup of X at x_0, Mori shows that $\widetilde{X}$ has the structure of a $\mathbf{P}^1_k$-bundle over $\mathbf{P}^{n-1}_k$; see Figure 3.1 for an illustration. This $\mathbf{P}^1_k$-bundle structure for $\widetilde{X}$ forces $X \simeq \mathbf{P}^n_k$ [Kol96, Lem. V.3.7.8]. An interesting feature of Mori's bend and break techniques is that in characteristic zero, Mori's techniques require reducing modulo p and utilizing the Frobenius morphism. It is unknown whether one can prove Theorem 3.1.3 directly, without reducing modulo p.

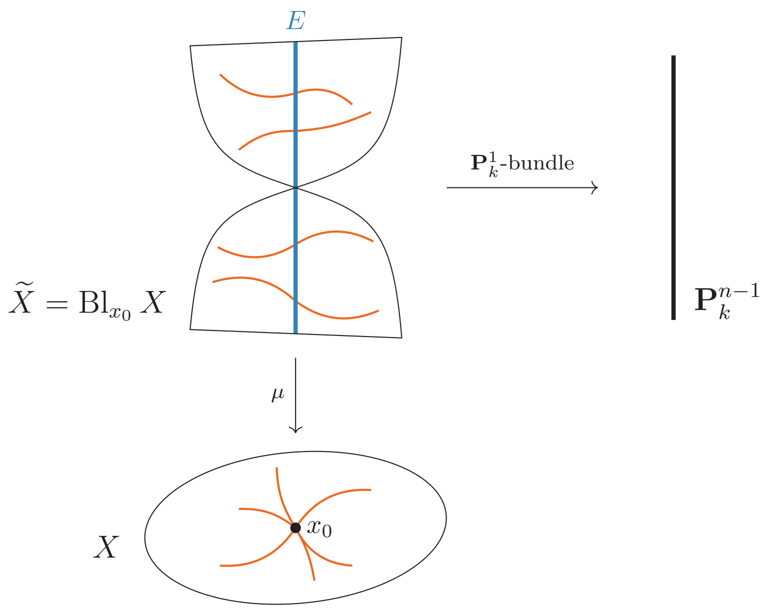

Figure 3.1: Mori's characterization of $\mathbf{P}^n_k$

The next result was actually known before Mori's theorem 3.1.3. The analogous result in positive characteristic, however, took much longer.

Theorem 3.1.4 [KO73, Cor. to Thm. 1.1; KK00, Cor. 2]**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field k. If* (2) *holds, then X is isomorphic to $\mathbf{P}^n_k$.*

This result is due to Kobayashi and Ochiai in characteristic zero [KO73], and to Kachi and Kollár in positive characteristic [KK00]. The methods of [KO73] are topological and complex analytic in nature, while [KK00] uses Mori's bend and break techniques. Theorem 3.1.4 illustrates the general philosophy that methods from topology and complex analysis can often be replaced by Frobenius techniques in positive characteristic.

Finally, we consider the following:

Conjecture 3.1.5 (Mori–Mukai [Kol96, Conj. V.1.7])**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field k. If* (3) *holds, then X is isomorphic to $\mathbf{P}^n_k$.*

By using results of Kebekus [Keb02] on families of singular rational curves, Cho, Miyaoka, and Shepherd-Barron proved this conjecture in characteristic zero [CMSB02].

More precisely, they showed the following statement, which is stronger than the Mori–Mukai conjecture 3.1.5 since Fano varieties are uniruled [Kol96, Cor. IV.1.15].

Theorem 3.1.6 [CMSB02, Cor. 0.4(11)]**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic zero. If X is uniruled, and the inequality*

$$\deg\bigl(\omega_X^{-1}|_C\bigr) \geq n+1$$

holds for every rational curve $C \subseteq X$ passing through a general closed point $x_0 \in X$, then X is isomorphic to $\mathbf{P}^n_k$.

Because of the assumption on the characteristic, we ask the following:

Question 3.1.7. *Is the Mori–Mukai conjecture 3.1.5 true in positive characteristic?*

In arbitrary characteristic, as far as we know the only result in this direction is the following result due to Kachi and Kollár, which we state using the language of divisors.

Theorem 3.1.8 [KK00, Cor. 3]**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field k. Suppose K_X is not nef. If*

(a) $(-K_X \cdot C) \geq n+1$ *for every rational curve $C \subseteq X$; and*

(b) $(-K_X)^n \geq (n+1)^n$,

then X is isomorphic to $\mathbf{P}^n_k$.

The major issue in trying to mimic the proof in [CMSB02] is the use of deformation theory and, more specifically, the use of generic smoothness in studying deformations of curves. Note that generic smoothness is false in positive characteristic, since the absolute Frobenius morphism $F\colon X \to X$ for a smooth variety X is nowhere smooth [Har77, Ex. III.10.5.1]. One way to interpret Theorem A is that we avoid issues with generic smoothness and deformation theory by building singularities into the statement of the Mori–Mukai conjecture 3.1.5. The advantage of this modification is that Theorem A can be interpreted in terms of Seshadri constants in the following manner.

Theorem A*. *Let X be a Fano variety of dimension n over an algebraically closed field k of positive characteristic. If there exists a closed point $x \in X$ with $\varepsilon(\omega_X^{-1}; x) \geq (n+1)$, then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n_k$.*

Note that the conditions in Theorem A and in Theorem A* are equivalent by (2.3).

Using this reinterpretation, we can show that Theorem A is a consequence of Theorem 3.1.6 in characteristic zero, and a version of Theorem A assuming a slightly stronger lower bound on $\varepsilon(\omega_X^{-1}; x)$ at *all* points $x \in X$ is a consequence of Theorem 3.1.8. The statement in characteristic zero gives a different proof of [BS09, Thm. 1.7].

Proposition 3.1.9. *Let X be a Fano variety of dimension n over an algebraically closed field k. Suppose one of the following is satisfied:*

(i) *We have* $\operatorname{char} k = 0$ *and* $\varepsilon(\omega_X^{-1}; x) > n$ *holds for a single closed point* $x \in X$*; or*

(ii) *We have* $\operatorname{char} k = p > 0$ *and* $\varepsilon(\omega_X^{-1}; x) \geq n + 1$ *holds for* all *closed points* $x \in X$.

Then, X is isomorphic to $\mathbf{P}^n_k$.

Proof. For (i), we use Theorem 3.1.6. It suffices to verify the condition $\deg(\omega_X^{-1}|_C) > n$. First, the locus $\{x \in X \mid \varepsilon(\omega_X^{-1}; x) > n\}$ contains a Zariski open set [EKL95, Lem. 1.4], hence we have $\varepsilon(\omega_X^{-1}; x_0) > n$ at a general point $x_0 \in X$. By the alternative characterization of Seshadri constants in terms of curves in (2.3), we have the chain of inequalities

$$n < \varepsilon(\omega_X^{-1}; x_0) \leq \frac{\deg(\omega_X^{-1}|_C)}{e(\mathcal{O}_{C,x_0})} \leq \deg(\omega_X^{-1}|_C)$$

for any rational curve C containing x_0.

For (ii), we use Theorem 3.1.8. The verification of condition (a) proceeds as in (i) by applying (2.3) to a closed point $x \in C$ contained in a given rational curve $C \subseteq X$. For condition (b), we use the inequality $\varepsilon(\omega_X^{-1}; x) \leq \sqrt[n]{(-K_X)^n}$, which is [Laz04a, eq. 5.2]. The inequality $\varepsilon(\omega_X^{-1}; x) \geq n + 1$ then implies condition (b). □

Given the similarity between the Mori–Mukai conjecture 3.1.5 and Theorem A, we ask the following:

Question 3.1.10. *Let X be a Fano variety of dimension n over an algebraically closed field k. If the inequality*

$$\deg\bigl(\omega_X^{-1}|_C\bigr) \geq n + 1$$

holds for every curve $C \subseteq X$, then does there exist a closed point $x \in X$ with

$$\deg(\omega_X^{-1}|_C) \geq e(\mathcal{O}_{C,x}) \cdot (n + 1)$$

for every curve $C \subseteq X$ containing x?

The answer to this question is "yes" in characteristic zero by using Theorem 3.1.6, since Theorem 3.1.6 implies $X \simeq \mathbf{P}^n_k$, and therefore the required positivity property on ω_X^{-1} holds by Example 2.2.2. If one could answer Question 3.1.10 affirmatively independently of Theorem 3.1.6, then [BS09, Thm. 1.7] would give an alternative proof of the Mori–Mukai conjecture 3.1.5 in characteristic zero, and Theorem A would resolve their conjecture in positive characteristic.

Finally, we mention another conjectural characterization of projective space that ties in with our discussion in Chapter 1.

Remark 3.1.11. In Chapter 1, we noted that the cohomology ring of $\mathbf{P}^n_{\mathbf{C}}$ is $\mathbf{Z}[h]/(h^{n+1})$; see (1.1). Fujita conjectured that a smooth complex projective variety of dimension n with this singular cohomology ring is isomorphic to $\mathbf{P}^n_{\mathbf{C}}$ [Fuj80, Conj. $\mathrm{C_n}$]. Fujita himself proved this conjecture in dimensions $n \le 5$ [Fuj80, Thm. 1] (under the additional assumption that X is Fano), and Libgober and Wood proved this conjecture in dimensions $n \le 6$ [LW90, Thm. 1]. See also [Deb, Thm. 2]. It is unclear what the right formulation of this conjecture would be in positive characteristic.

3.2. Proof of Theorem A

We now turn to the proof of Theorem A. The main technical tool is the notion of bundles of principal parts, which are also known as jet bundles in the literature. See [LT95, §4] or [EGAIV$_4$, §16] for a detailed discussion.

Definition 3.2.1. Let X be a variety over an algebraically closed field k. Denote by p_1 and p_2 the projections

$$\begin{tikzcd} & X \times_k X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

Let $\mathscr{I} \subset \mathcal{O}_{X\times X}$ be the ideal defining the diagonal, and let L be a line bundle on X. For each integer $\ell \ge 0$, the *ℓth bundle of principal parts* associated to L is the sheaf

$$\mathscr{P}^\ell(L) := p_{1*}(p_2^*L \otimes \mathcal{O}_{X\times X}/\mathscr{I}^{\ell+1}).$$

Note that $\mathscr{P}^0(L) \simeq L$, since the diagonal in $X \times X$ is isomorphic to X.

We will use the following facts about these sheaves from [LT95, §4], under the assumption that X is a smooth variety over an algebraically closed field.

(*a*) There exists a short exact sequence [LT95, n° 4.2]

$$0 \longrightarrow \mathrm{Sym}^\ell(\Omega_X) \otimes L \longrightarrow \mathscr{P}^\ell(L) \longrightarrow \mathscr{P}^{\ell-1}(L) \longrightarrow 0, \tag{3.1}$$

where Ω_X denotes the cotangent bundle on X. By using induction and this short exact sequence, we see that the sheaf $\mathscr{P}^\ell(L)$ is a vector bundle for all integers $\ell \geq 0$.

(*b*) There exists an identification $\mathscr{P}^\ell(L) \simeq p_{2*}(p_2^*L \otimes \mathcal{O}_{X\times X}/\mathscr{I}^{\ell+1})$, and by applying adjunction to the map $p_2^*L \to p_2^*L \otimes \mathcal{O}_{X\times X}/\mathscr{I}^{\ell+1}$, there is a morphism $d^\ell \colon L \to \mathscr{P}^\ell(L)$ of sheaves [LT95, n° 4.1], such that the diagram

$$\begin{tikzcd} H^0(X,L) \arrow[r, "H^0(d^\ell)"] \arrow[d] & H^0\big(X, \mathscr{P}^\ell(L)\big) \arrow[d] \\ H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1}) & H^0\big(X, \mathscr{P}^\ell(L) \otimes \mathcal{O}_X/\mathfrak{m}_x\big) \arrow[l, "\sim"'] \end{tikzcd}$$

commutes for all closed points $x \in X$ [LT95, Lem. 4.5(1)], where the vertical arrows are the restriction maps. Thus, if L separates ℓ-jets at x, then $\mathscr{P}^\ell(L)$ is globally generated at x.

We will also use the following description of the determinant of the ℓth bundle of principal parts. This description is stated in [DRS01, p. 1660].

Lemma 3.2.2. *Let X be a smooth variety of dimension n over an algebraically closed field, and let L be a line bundle on X. Then, for each $\ell \geq 0$, we have an isomorphism*

$$\det(\mathscr{P}^\ell(L)) \simeq \big(\omega_X^\ell \otimes L^{\otimes(n+1)}\big)^{\frac{1}{n+1}\binom{n+\ell}{n}}.$$

Proof. We proceed by induction on $\ell \geq 0$. If $\ell = 0$, then $\mathscr{P}^0(L) \simeq L$, so we are done.

Now suppose $\ell > 0$. Since X is smooth, the cotangent bundle Ω_X has rank n, and we have isomorphisms

$$\det\bigl(\operatorname{Sym}^\ell(\Omega_X)\otimes L\bigr) \simeq \det\bigl(\operatorname{Sym}^\ell(\Omega_X)\bigr)\otimes L^{\otimes\binom{n+\ell-1}{n-1}} \simeq \omega_X^{\otimes\binom{n+\ell-1}{n}}\otimes L^{\otimes\binom{n+\ell-1}{n-1}}.$$

By induction and taking top exterior powers in the short exact sequence (3.1), we obtain

$$\begin{aligned}\det(\mathscr{P}^\ell(L)) &\simeq \omega_X^{\otimes\binom{n+\ell-1}{n}}\otimes L^{\otimes\binom{n+\ell-1}{n-1}}\otimes\det(\mathscr{P}^{\ell-1}(L))\\ &\simeq \omega_X^{\otimes\binom{n+\ell-1}{n}}\otimes L^{\otimes\binom{n+\ell-1}{n-1}}\otimes\bigl(\omega_X^{\otimes(\ell-1)}\otimes L^{\otimes(n+1)}\bigr)^{\otimes\frac{1}{n+1}\binom{n+\ell-1}{n}}\\ &\simeq \bigl(\omega_X^{\otimes\ell}\otimes L^{\otimes(n+1)}\bigr)^{\otimes\frac{1}{n+1}\binom{n+\ell}{n}}.\end{aligned}$$

Note that the last isomorphism holds because of the identities

$$\binom{n+\ell-1}{n}+\frac{\ell-1}{n+1}\binom{n+\ell-1}{n} = \frac{n+\ell}{n+1}\binom{n+\ell-1}{n} = \frac{\ell}{n+1}\binom{n+\ell}{n},$$
$$\binom{n+\ell-1}{n-1}+\binom{n+\ell-1}{n} = \binom{n+\ell}{n}$$

involving binomial coefficients. $\square$

We now prove Theorem A. We actually show the equivalent formulation in Theorem A*. To prove Theorem A*, we mostly follow the proof of [BS09, Thm. 1.7], although we must be more careful with tensor operations in positive characteristic.

Proof of Theorem A.* We first show that $\mathscr{P}^{n+1}(\omega_X^{-1})$ is a trivial bundle. First, $\omega_X^{-1} \simeq \omega_X\otimes(\omega_X^{-1})^{\otimes 2}$ separates $(n+1)$-jets by Theorem 7.3.1 (or the special cases in Proposition 2.2.6 and Theorem B) since $\varepsilon(\omega_X^{-1};x)\geq n+1$. By property (b) of bundles of principal parts, we therefore have that $\mathscr{P}^{n+1}(\omega_X^{-1})$ is globally generated at x. On the other hand, by Lemma 3.2.2 applied to $L=\omega_X^{-1}$, we have an isomorphism $\det(\mathscr{P}^{n+1}(\omega_X^{-1}))\simeq\mathcal{O}_X$. Now to show that $\mathscr{P}^{n+1}(\omega_X^{-1})$ is a trivial bundle, consider the following diagram:

$$\begin{tikzcd}\det\bigl(\mathscr{P}^{n+1}(\omega_X^{-1})\bigr) \arrow[r, "\sim"] \arrow[d, two heads] & \mathcal{O}_X \arrow[d, two heads]\\ \det\bigl(\mathscr{P}^{n+1}(\omega_X^{-1})\otimes\mathcal{O}_X/\mathfrak{m}_x\bigr) \arrow[r, "\sim"] & \mathcal{O}_X/\mathfrak{m}_x\end{tikzcd}$$

Suppose the isomorphism in the top row is given by a non-vanishing global section

$$s \in H^0\big(X, \det\big(\mathscr{P}^{n+1}(\omega_X^{-1})\big)\big).$$

Let $s_{1,x} \wedge s_{2,x} \wedge \cdots \wedge s_{r,x}$ be the image of s in $\det\big(\mathscr{P}^{n+1}(\omega_X^{-1}) \otimes \mathcal{O}_X/\mathfrak{m}_x\big)$, which gives the isomorphism in the bottom row. Then, since $\mathscr{P}^{n+1}(\omega_X^{-1})$ is globally generated at x, each $s_{i,x}$ can be lifted to a global section $\widetilde{s}_i \in H^0\big(X, \mathscr{P}^{n+1}(\omega_X^{-1})\big)$. Because the exterior product $\widetilde{s}_1 \wedge \widetilde{s}_2 \wedge \cdots \wedge \widetilde{s}_r$ does not vanish at x, this exterior product does not vanish anywhere, since $H^0(X, \mathcal{O}_X) = k$ [Har77, Thm. I.3.4(a)]. Thus, the global sections $\widetilde{s}_i$ give a frame for $\mathscr{P}^{n+1}(\omega_X^{-1})$, and therefore $\mathscr{P}^{n+1}(\omega_X^{-1})$ is a trivial bundle.

To show $X \simeq \mathbf{P}^n_k$, we use a generalization of Mori's characterization of projective space [Kol96, Thm. V.3.2]. It suffices to show that for every non-constant morphism $f\colon \mathbf{P}^1_k \to X$, the pull back f^*T_X is a sum of line bundles of positive degree. Since every vector bundle on $\mathbf{P}^1_k$ splits [Har77, Exer. V.2.6], we may write

$$f^*(T_X) \simeq \bigoplus_{i=1}^{n} \mathcal{O}(a_i) \qquad \text{and} \qquad f^*(\omega_X^{-1}) \simeq \mathcal{O}(b),$$

where b is positive since ω_X^{-1} is ample. We want to show that each a_i is positive. We have

$$f^*(\Omega_X) \simeq f^*(T_X)^\vee \simeq \bigoplus_{i=1}^{n} \mathcal{O}(-a_i).$$

Dualizing the short exact sequence (3.1) for $\ell = n+1$, we have the short exact sequence

$$0 \longrightarrow \mathscr{P}^n(\omega_X^{-1})^\vee \longrightarrow \mathscr{P}^{n+1}(\omega_X^{-1})^\vee \longrightarrow (\operatorname{Sym}^{n+1}\Omega_X)^\vee \otimes \omega_X \longrightarrow 0.$$

The quotient on the right is globally generated because it is a quotient of the trivial bundle $\mathscr{P}^{n+1}(\omega_X^{-1})^\vee$. We have isomorphisms

$$\begin{aligned} f^*\big((\operatorname{Sym}^{n+1}\Omega_X)^\vee \otimes \omega_X\big) &\simeq \big(\operatorname{Sym}^{n+1} f^*(\Omega_X)\big)^\vee \otimes f^*(\omega_X) \\ &\simeq \left(\operatorname{Sym}^{n+1} \bigoplus_{i=1}^{n} \mathcal{O}(-a_i)\right)^\vee \otimes \mathcal{O}(-b), \end{aligned}$$

and this bundle is globally generated since it is the pullback of a globally generated bundle. By expanding out the symmetric power on the right-hand side, we have a surjection

$$f^*\bigl((\operatorname{Sym}^{n+1}\Omega_X)^\vee \otimes \omega_X\bigr) \twoheadrightarrow \bigoplus_{i=1}^{n} \mathcal{O}\bigl((n+1)a_i - b\bigr),$$

hence the direct sum on the right-hand side is also globally generated. Finally, this implies $(n+1)a_i - b \geq 0$, and therefore since $b > 0$, we have that $a_i > 0$ as required. $\square$

Remark 3.2.3. Liu and Zhuang's characteristic zero statement in [LZ18, Thm. 2] is stronger than Theorem A: it only assumes that X is $\mathbf{Q}$-Fano, and in particular that X is not necessarily smooth. While a version of Theorem B holds for a large class of singular varieties (see Theorem 7.3.1) the rest of our approach does not generalize to the non-smooth setting, since Mori's characterization of projective space uses bend and break techniques. Zhuang has since proved [LZ18, Thm. 2] in positive characteristic by studying the global F-singularities of the blowup of X at x [Zhu, Thm. 3]. Zhuang has also shown a version of Theorem A using lower bounds on the moving Seshadri constant $\varepsilon(\|{-K_X}\|; x)$ without the assumption that X is Fano, but only in characteristic zero [Zhu18, Thm. 1.7].

Chapter 4

Preliminaries in arbitrary characteristic

In this chapter, we collect some background material that will be used throughout the rest of this thesis. The only new result is Proposition 4.6.7, which describes how sufficiently large twists of a coherent sheaf by a big $\mathbf{Q}$-Cartier divisor D are globally generated away from the augmented base locus of D.

4.1. Morphisms essentially of finite type

Recall that a ring homomorphism $A \to B$ is *essentially of finite type* if B is isomorphic (as A-algebras) to a localization of an A-algebra of finite type [EGAIV$_1$, (1.3.8)]. The corresponding scheme-theoretic notion is the following:

Definition 4.1.1 [Nay09, Def. 2.1(a)]**.** Let $f\colon X \to Y$ be a morphism of schemes.

(a) We say that f is *locally essentially of finite type* if there is an affine open covering

$$Y = \bigcup_i \operatorname{Spec} A_i$$

such that for every i, there is an affine open covering

$$f^{-1}(\operatorname{Spec} A_i) = \bigcup_j \operatorname{Spec} B_{ij}$$

for which the corresponding ring homomorphisms $A_i \to B_{ij}$ are essentially of finite type.

(*b*) We say that f is *essentially of finite type* if it is locally essentially of finite type and quasi-compact.

We will also use the following alternative characterization of these morphisms.

Lemma 4.1.2. *A morphism* $f\colon X \to Y$ *of schemes is locally essentially of finite type (resp. essentially of finite type) if and only if for every affine open subset* $\operatorname{Spec} A \subseteq Y$*, there is an affine open covering (resp. finite affine open covering)*

$$f^{-1}(\operatorname{Spec} A) = \bigcup_i \operatorname{Spec} B_i$$

for which the corresponding ring homomorphisms $A \to B_i$ *are essentially of finite type.*

Proof. It suffices to show the statement for morphisms locally essentially of finite type since a similar statement holds for quasi-compactness [$\mathrm{EGAI}_{\mathrm{new}}$, p. 290]. Moreover, the direction $\Leftarrow$ is clear, hence it remains to prove the direction $\Rightarrow$.

Fix coverings for f as in Definition 4.1.1(a), and let $\operatorname{Spec} A \subseteq Y$ be an arbitrary open affine subset. By [GW10, Lem. 3.3], there exist $g_k \in A$ such that $\operatorname{Spec} A = \bigcup_k \operatorname{Spec} A_{g_k}$ and such that $\operatorname{Spec} A_{g_k} = \operatorname{Spec}(A_i)_{h_k}$ as open subsets in Y for some $h_k \in A_i$. The preimage of $\operatorname{Spec}(A_i)_{h_k}$ is covered by the $\operatorname{Spec}(B_{ij})_{h_k}$, and the compositions

$$A \longrightarrow A_{g_k} \xrightarrow{\sim} (A_i)_{h_k} \longrightarrow (B_{ij})_{h_k}$$

are essentially of finite type since the class of ring homomorphisms essentially of finite type is stable under composition and base change [EGAIV_1, Prop. 1.3.9]. We therefore use the affine open covering

$$f^{-1}(\operatorname{Spec} A) = \bigcup_{i,j,k} \operatorname{Spec}(B_{ij})_{h_k}. \qquad \square$$

Using this characterization, we can show the following:

Lemma 4.1.3 [Nay09, (2.2)]**.** *The class of morphisms (locally) essentially of finite type is closed under composition and base change.*

Proof. It suffices to show the statement for morphisms locally essentially of finite type since the corresponding statement holds for quasi-compactness [EGAI_{new}, Prop. 6.1.5].

For composition, let $f\colon X \to Y$ and $g\colon Y \to Z$ be locally essentially of finite type. Let $\operatorname{Spec} A \subseteq Z$ be an arbitrary affine open set. By Lemma 4.1.2, there exists an affine open covering

$$g^{-1}(\operatorname{Spec} A) = \bigcup_i \operatorname{Spec} B_i$$

where the corresponding ring homomorphisms $A \to B_i$ are essentially of finite type. Applying f^{-1} and using Lemma 4.1.2 again, there exists an affine open covering

$$(g \circ f)^{-1}(\operatorname{Spec} A) = \bigcup_i f^{-1}(\operatorname{Spec} B_i) = \bigcup_{i,j} \operatorname{Spec} C_{ij},$$

where the corresponding ring homomorphisms $B_i \to C_{ij}$ are essentially of finite type. The compositions $A \to B_i \to C_{ij}$ are essentially of finite type by [EGAIV_1, Prop. 1.3.9(i)], hence $g \circ f$ is locally essentially of finite type.

For base change, let $f\colon X \to Y$ be locally essentially of finite type and fix coverings for f as in Definition 4.1.1(a). Let $g\colon Y' \to Y$ be an arbitrary morphism of schemes, and denote the base change of f by $f'\colon X' \to Y'$. Choose an affine open covering

$$g^{-1}(\operatorname{Spec} A_i) = \bigcup_k \operatorname{Spec} C_{ik},$$

in which case $Y' = \bigcup_{i,k} \operatorname{Spec} C_{ik}$. Then, the affine open covering

$$f'^{-1}(\operatorname{Spec} C_{ik}) = \bigcup_j \operatorname{Spec}(B_{ij} \otimes_{A_i} C_{ik})$$

is such that the corresponding ring homomorphisms $C_{ik} \to B_{ij} \otimes_{A_i} C_{ik}$ are essentially of finite type by base change [EGAIV_1, Prop. 1.3.9(ii)]. □

Remark 4.1.4. This notion of morphisms essentially of finite type is somewhat subtle. For example, even if $\operatorname{Spec} B \to \operatorname{Spec} A$ is essentially of finite type, it is not known whether the corresponding ring homomorphism $A \to B$ is essentially of finite type [Nay09, (2.3)].

4.2. Cartier and Weil divisors

We will work often with **Q**- or **R**-coefficients for both Cartier and Weil divisors.

Definition 4.2.1 (see [EGAIV$_4$, Def. 21.1.2; Laz04a, §1.3])**.** Let X be a locally noetherian scheme. A *Cartier divisor* on X is an element of the abelian group

$$\operatorname{Cart}(X) \coloneqq H^0\bigl(X, \mathscr{K}_X^*/\mathcal{O}_X^*\bigr),$$

where $\mathscr{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ [Kle79, p. 204], and $\mathscr{K}_X^*$ (resp. $\mathcal{O}_X^*$) is the subsheaf of $\mathscr{K}_X$ (resp. $\mathcal{O}_X$) consisting of invertible sections. Concretely, a Cartier divisor is represented by the data $\{(U_i, f_i)\}_i$, where $f_i \in \mathscr{K}_X^*(U_i)$ are local sections, and $X = \bigcup_i U_i$. A Cartier divisor D is *effective* if the functions $f_i \in \mathscr{K}^*(U_i)$ are regular on U_i, i.e., if $f_i \in \mathcal{O}_X(U_i)$.

A $\mathbf{Q}$-*Cartier divisor* (resp. $\mathbf{R}$-*Cartier divisor*) is an element of the group $\operatorname{Cart}_{\mathbf{Q}}(X) \coloneqq \operatorname{Cart}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ (resp. $\operatorname{Cart}_{\mathbf{R}}(X) \coloneqq \operatorname{Cart}(X) \otimes_{\mathbf{Z}} \mathbf{R}$). A $\mathbf{Q}$-Cartier divisor (resp. $\mathbf{R}$-Cartier divisor) is *effective* if it is a $\mathbf{Q}_{\geq 0}$-linear combination (resp. $\mathbf{R}_{\geq 0}$-linear combination) of effective Cartier divisors. A $\mathbf{Q}$-Cartier divisor (resp. $\mathbf{R}$-*Cartier divisor*) *is a Cartier divisor* if it is in the image of the map $\operatorname{Cart}(X) \to \operatorname{Cart}_{\mathbf{Q}}(X)$ (resp. $\operatorname{Cart}(X) \to \operatorname{Cart}_{\mathbf{R}}(X)$).

Definition 4.2.2 (see [EGAIV$_4$, §21.6; Laz04a, §1.3])**.** Let X be a locally noetherian scheme. A *Weil divisor* on X is a formal $\mathbf{Z}$-linear combination of codimension 1 cycles on X. These form an abelian group, which we denote by $\operatorname{WDiv}(X)$. A Weil divisor D on X is *effective* if D is a formal $\mathbf{Z}_{\geq 0}$-linear combination of codimension 1 cycles on X.

A $\mathbf{Q}$-*Weil divisor* (resp. $\mathbf{R}$-*Weil divisor*) is an element of the group $\operatorname{WDiv}_{\mathbf{Q}}(X) \coloneqq \operatorname{WDiv}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ (resp. $\operatorname{WDiv}_{\mathbf{R}}(X) \coloneqq \operatorname{WDiv}(X) \otimes_{\mathbf{Z}} \mathbf{R}$). A $\mathbf{Q}$-Weil divisor (resp. $\mathbf{R}$-Weil divisor) is *effective* if it is a $\mathbf{Q}_{\geq 0}$-linear combination (resp. $\mathbf{R}_{\geq 0}$-linear combination) of effective Weil divisors. A $\mathbf{Q}$-Weil divisor (resp. $\mathbf{R}$-*Weil divisor*) *is a Weil divisor* if it is in the image of the map $\operatorname{WDiv}(X) \to \operatorname{WDiv}_{\mathbf{Q}}(X)$ (resp. $\operatorname{WDiv}(X) \to \operatorname{WDiv}_{\mathbf{R}}(X)$).

If X is a locally noetherian scheme, then there is a *cycle map*

$$\operatorname{cyc}\colon \operatorname{Cart}(X) \longrightarrow \operatorname{WDiv}(X)$$

sending a Cartier divisor to its associated Weil divisor; see [EGAIV$_4$, §21.6]. If X is locally factorial, then the cycle map cyc is bijective [EGAIV$_4$, Thm. 21.6.9(*ii*)], hence we can identify Cartier and Weil divisors, as well as their corresponding versions with **Q**- or **R**-coefficients.

Even if X is not locally factorial, as long as X is normal, we can pass from Cartier divisors to Weil divisors:

Definition 4.2.3. Let X be a locally noetherian scheme. If X is a normal, then the cycle map cyc is injective [EGAIV$_4$, Thm. 21.6.9(i)]. We then say that a Weil divisor (resp. **Q**-Weil divisor, **R**-Weil divisor) *is Cartier* (resp. **Q**-*Cartier*, **R**-*Cartier*) if it is in the image of cyc (resp. $\mathrm{cyc} \otimes_{\mathbf{Z}} \mathbf{Q}$, $\mathrm{cyc} \otimes_{\mathbf{Z}} \mathbf{R}$).

Finally, we will use the following conventions for rounding up and down.

Definition 4.2.4 (see [BGGJ$^+$, Def. 3.4.1; Laz04b, Def. 9.1.2])**.** Let X be a locally noetherian scheme, and let $D \in \mathrm{Cart}_{\mathbf{R}}(X)$. A *decomposition* $\mathcal{D}$ of D is an expression

$$D = \sum_{i=1}^{r} a_i D_i$$

for some $a_i \in \mathbf{R}$ and Cartier divisors D_i. We note that such a decomposition is not unique, since $\mathrm{Cart}(X)$ may have torsion. The *round-up* and *round-down* of D with respect to $\mathcal{D}$ are the Cartier divisors

$$\lceil D \rceil_{\mathcal{D}} := \sum_{i=1}^{r} \lceil a_i \rceil D_i \qquad \text{and} \qquad \lfloor D \rfloor_{\mathcal{D}} := \sum_{i=1}^{r} \lfloor a_i \rfloor D_i,$$

respectively. Note that the round-up and round-down depend on the decomposition $\mathcal{D}$.

Now let $D \in \mathrm{WDiv}_{\mathbf{R}}(X)$. By definition, we have

$$D = \sum_{i=1}^{r} a_i D_i.$$

Then, the *round-up* and *round-down* of D are the Weil divisors

$$\lceil D \rceil := \sum_{i=1}^{r} \lceil a_i \rceil D_i \qquad \text{and} \qquad \lfloor D \rfloor := \sum_{i=1}^{r} \lfloor a_i \rfloor D_i,$$

respectively.

4.3. Reflexive sheaves

We will need some basic results on reflexive sheaves, which we collect here. Our main reference is [Har94, §1].

Definition 4.3.1. Let $\mathscr{F}$ be a coherent sheaf on a scheme X. The *dual* of $\mathscr{F}$ is $\mathscr{F}^\vee := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. We say that $\mathscr{F}$ is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism. We say that $\mathscr{F}$ is *normal* if for every open subset $U \subseteq X$ and every subset $Y \subseteq U$ of codimension ≥ 2, the restriction map $\mathscr{F}(U) \to \mathscr{F}(U \smallsetminus Y)$ is bijective.

We note that all locally free sheaves are reflexive [Har77, Exer. II.5.1(a)].

By the following result, all reflexive sheaves on reasonably nice schemes are normal. Below, we recall that a noetherian scheme X *satisfies* G_i for an integer $i \geq 0$ if for every point $x \in X$ such that $\dim \mathcal{O}_{X,x} \leq i$, the local ring $\mathcal{O}_{X,x}$ is Gorenstein.

Proposition 4.3.2 [Har94, Prop. 1.11]**.** *Let X be a noetherian scheme satisfying G_1 and S_2. Then, every reflexive sheaf $\mathscr{F}$ is normal.*

We will also need the following:

Lemma 4.3.3. *Let X be a locally noetherian scheme satisfying G_0 and S_1. Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X. If $\mathscr{G}$ is reflexive, then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is also reflexive.*

Proof. Since $\mathscr{G}$ is reflexive, we have

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \simeq \mathscr{H}om_{\mathcal{O}_X}\bigl(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{G}^\vee, \mathcal{O}_X)\bigr) \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}^\vee, \mathcal{O}_X)$$

where the second isomorphism is by tensor–hom adjunction. Since the dual of any coherent sheaf is reflexive [Har94, Cor. 1.8], we are done. $\square$

We will often use this fact to extend morphisms from the complement of codimension at least two.

Corollary 4.3.4. *Let X be a locally noetherian scheme satisfying G_1 and S_2, and let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X such that $\mathscr{F}$ is reflexive. If $U \subseteq X$ is an open subset such that* $\operatorname{codim}(X \smallsetminus U) \geq 2$, *then every morphism $\varphi\colon \mathscr{G}|_U \to \mathscr{F}|_U$ extends uniquely to a morphism $\widetilde{\varphi}\colon \mathscr{G} \to \mathscr{F}$.*

Proof. The morphism φ corresponds to a section of the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{G},\mathscr{F})$ over U. The sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{G},\mathscr{F})$ is reflexive by Lemma 4.3.3, hence the section φ extends uniquely to a section $\widetilde{\varphi}$ of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{G},\mathscr{F})$ over X by Proposition 4.3.2. □

The following result says that on noetherian schemes satisfying G_1 and S_2, reflexive sheaves are determined by their codimension one behavior.

Theorem 4.3.5 [Har94, Thm. 1.12]**.** *Let X be a noetherian scheme satisfying G_1 and S_2, and let $Y \subseteq X$ be a closed subset of codimension at least 2. Then, the restriction functor induces an equivalence of categories*

$$\begin{aligned} \left\{\begin{matrix}\textit{reflexive coherent}\\ \mathcal{O}_X\textit{-modules}\end{matrix}\right\} &\longrightarrow \left\{\begin{matrix}\textit{reflexive coherent}\\ \mathcal{O}_{X\smallsetminus Y}\textit{-modules}\end{matrix}\right\}\\ \mathscr{F} &\longmapsto \mathscr{F}|_{X\smallsetminus Y}\end{aligned}$$

4.4. Dualizing complexes and Grothendieck duality

The main references for this section are [Har66; Con00], although we need the extension of the theory to separated morphisms that are essentially of finite type, following [Nay09]. In the statement below, recall that for a noetherian scheme X, $\mathbf{D}_{\mathrm{qc}}(X)$ denotes the derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology, and $\mathbf{D}^{+}_{\mathrm{qc}}(X)$ is the full subcategory of $\mathbf{D}_{\mathrm{qc}}(X)$ whose objects are complexes $\mathscr{F}$ such that $\mathbf{h}^i\mathscr{F} = 0$ for all $i \ll 0$.

Theorem 4.4.1 ([Nay09, Thm. 5.3]; cf. [Har66, Cor. V.3.4])**.** *Let $\mathbf{S_e}$ denote the subcategory of the category of schemes whose objects are noetherian schemes, and whose morphisms are separated and essentially of finite type morphisms of schemes. Then, there exists a contravariant $\mathbf{D}^{+}_{\mathrm{qc}}$-valued pseudofunctor $(-)^!$ on $\mathbf{S_e}$ such that*

(*i*) *For proper morphisms, $(-)^!$ is pseudofunctorially isomorphic to the right adjoint of the right-derived direct image pseudofunctor $\mathbf{R}f_*$;*

(*ii*) *For essentially étale morphisms, $(-)^!$ equals the inverse image pseudofunctor $(-)^*$;*

(*iii*) *For every cartesian diagram*

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, "i"] & Y \end{tikzcd}$$

of noetherian schemes, where f is proper and i is flat, there is a flat base change isomorphism $j^*f^! \overset{\sim}{\to} g^!i^*$.

We note that a morphism is *essentially étale* (resp. *essentially smooth*) if it is separated, formally étale (resp. formally smooth), and essentially of finite type [Nay09, (5.1) and (5.4)]. Note that for certain classes of morphisms, the pseudofunctor $(-)^!$ has concrete descriptions; see [Har66, Prop. III.6.5 and Thm. III.6.7] for finite morphisms, and see [Har66, III.2; Nay09, (5.4)] for essentially smooth morphisms.

Theorem 4.4.1 allows us to define the following:

Definition 4.4.2. Let $h\colon X \to \operatorname{Spec} k$ be an equidimensional scheme that is separated and essentially of finite type over a field k. The *normalized dualizing complex* for X is $\omega_X^\bullet \coloneqq h^!k$, where $h^!$ is the functor in Theorem 4.4.1. The *canonical sheaf* on X is the coherent sheaf

$$\omega_X \coloneqq \mathbf{h}^{-\dim X}\omega_X^\bullet.$$

Note that the canonical sheaf is reflexive if X satisfies G_1 and S_2, since it is S_2 by [Har07, Lem. 1.3], hence reflexive [Har94, Thm. 1.9]. If X is normal, we can therefore define a *canonical divisor* K_X as a choice of Weil divisor whose associated sheaf $\mathcal{O}_X(K_X)$ is isomorphic to ω_X. Note that K_X is only well-defined up to linear equivalence.

When X is essentially smooth, the canonical sheaf ω_X is isomorphic to the invertible sheaf of top differential forms $\bigwedge^{\dim X}\Omega_X$ [Har66, III.2; Nay09, (5.4)]. When X is Gorenstein, the canonical sheaf ω_X is invertible [Har66, Exer. V.9.7; Nay09, (5.10)].

4.5. Base ideals and base loci

We define the classical notions of base ideals and base loci for Cartier divisors.

Definition 4.5.1 (see [Laz04a, §1.1.B])**.** Let X be a scheme over a field k, and let L be a line bundle on X. If $V \subseteq H^0(X, L)$ is a finite-dimensional k-vector space, then the associated projective space $|V| \coloneqq \mathbf{P}(V^\vee)$ of one-dimensional subspaces of V is called a *linear system*. If $V = H^0(X, L)$, then $|V|$ is the *complete linear system* associated to L. The *base ideal* of $|V|$ is

$$\mathfrak{b}\bigl(|V|\bigr) \coloneqq \operatorname{im}\bigl(V \otimes_k L^{-1} \xrightarrow{\text{eval}} \mathcal{O}_X\bigr). \tag{4.1}$$

The *base scheme* of $|V|$ is the closed subscheme $\operatorname{Bs}(|V|)$ of X defined by $\mathfrak{b}(|V|)$, and the *base locus* of $|V|$ is the underlying closed subset $\operatorname{Bs}(|V|)_{\mathrm{red}}$ of $\operatorname{Bs}(|V|)$.

If the line bundle L is of the form $\mathcal{O}_X(D)$ for a Cartier divisor D, then the complete linear system associated to $\mathcal{O}_X(D)$ is denoted by $|D|$.

Note that if X is either projective over a field or reduced, then every line bundle L on X is of the form $\mathcal{O}_X(D)$ for a Cartier divisor D [Laz04a, Ex. 1.1.5].

We will need the following description for how base ideals transform under birational morphisms.

Lemma 4.5.2. *Let $f\colon X' \to X$ be a birational morphism between complete varieties, where X is normal. Then, for every Cartier divisor D on X, we have*

$$f^{-1}\mathfrak{b}\bigl(|D|\bigr) \cdot \mathcal{O}_{X'} = \mathfrak{b}\bigl(|f^*D|\bigr).$$

Proof. Since X is normal, we have $f_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ [Har77, Proof of Cor. III.11.4]. By the projection formula, we then have $H^0(X, \mathcal{O}_X(D)) = H^0(X', \mathcal{O}_{X'}(f^*D))$, and the lemma then follows by pulling back the evaluation map (4.1). $\square$

We define the notion of a graded family of ideals, of which base ideals will be an important example.

Definition 4.5.3 (see [Laz04b, Def. 2.4.14])**.** Let X be a locally noetherian scheme. A *graded family of ideals* $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \mathbf{N}}$ on X is a collection of coherent ideal sheaves $\mathfrak{a}_m \subseteq \mathcal{O}_X$ such that $\mathfrak{a}_0 = \mathcal{O}_X$, and such that for all $m, n \geq 0$, we have $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$.

We now describe how base ideals can form a graded family of ideals.

Example 4.5.4 (see [Laz04a, Ex. 1.1.9])**.** Let X be a complete scheme over a field k, and let D be a $\mathbf{Q}$-Cartier divisor on X. We define a graded family of ideals $\mathfrak{a}_\bullet(D)$ by setting

$$\mathfrak{a}_m(D) = \begin{cases} \mathfrak{b}\bigl(|mD|\bigr) & \text{if } mD \text{ is integral} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathfrak{b}(|mD|)$ is the base ideal of the complete linear series $|mD|$ (Definition 4.5.1). Note that $\mathfrak{a}_\bullet(D)$ is a graded family since the multiplication map $H^0(X, mD) \otimes_k H^0(X, nD) \to H^0(X, (m+n)D)$ induces an inclusion $\mathfrak{b}(|mD|) \cdot \mathfrak{b}(|nD|) \subseteq \mathfrak{b}(|(m+n)D|)$.

4.6. Asymptotic invariants of line bundles

In this section, we review some aspects of the theory of asymptotic invariants of Cartier divisors and their base loci. We have taken care to work over arbitrary fields; see [ELM+05] for an overview on the theory of asymptotic invariants for smooth complex varieties.

4.6.1. Stable base loci

We start by defining a stable "asymptotic" version of the base locus due to Fujita.

Definition 4.6.1 [Fuj83, Def. 1.17]**.** Let X be a complete scheme over a field, and let D be a Cartier divisor on X. The *stable base locus* of D is the closed subset

$$\mathbf{B}(D) \coloneqq \bigcap_m \mathrm{Bs}\bigl(|mD|\bigr)_{\mathrm{red}} \tag{4.2}$$

of X, where the intersection runs over every integer $m > 0$. The noetherian property implies $\mathbf{B}(D) = \mathbf{B}(nD)$ for every integer $n > 0$ [Laz04a, Ex. 2.1.23], hence the formula (4.2) can be used for $\mathbf{Q}$-Cartier divisors D by taking the intersection over every integer $m > 0$ such that mD is a Cartier divisor.

The stable base locus is not a numerical invariant of D, as we can see in the following example.

Example 4.6.2 (cf. [Laz04b, Ex. 10.3.3])**.** Let C be an elliptic curve, and let P_1 and P_2 be degree zero divisors on C that are torsion and non-torsion, respectively. Then, P_1 is semiample, hence $\mathbf{B}(P_1) = \emptyset$. On the other hand, we have $\mathbf{B}(P_2) = C$ since no multiple of P_2 has global sections.

Using this observation, we also construct an example where the stable base locus is not a numerical invariant, even for big and nef divisors. The construction below is due to Cutkosky; see [Laz04a, §2.3.B]. Let A be divisor on C of degree 1, and for every degree zero divisor P on C, consider the projective space bundle

$$X_P \coloneqq \mathbf{P}\bigl(\mathcal{O}_C(P) \oplus \mathcal{O}_C(A+P)\bigr) \longrightarrow C.$$

Denote by ξ_P the divisor on X_P corresponding to the tautological line bundle $\mathcal{O}_{X_P}(1)$, which is big and nef by [Laz04a, Lems. 2.3.2(iii) and 2.3.2(iv)]. Since the bundles defining X_P only differ by a twist by the divisor P, the varieties X_P are all naturally isomorphic to X_0, and under this identification, the divisors ξ_P are numerically equivalent divisors on X_0. Now let P_1 and P_2 be as in the previous paragraph. Then, since P_1 is semiample, ξ_{P_1} is also semiample by [Laz04a, Lem. 2.3.2(v)], hence $\mathbf{B}(\xi_{P_1}) = \emptyset$. On the other hand, $\mathbf{B}(\xi_{P_2})$ contains the section $C \simeq \mathbf{P}(\mathcal{O}_C(P_2)) \subseteq X_{P_2}$ corresponding to the first projection $\mathcal{O}_C(P_2) \oplus \mathcal{O}_C(A+P_2) \to \mathcal{O}_C(P_2)$, since $\mathbf{B}(P_2) = C$.

We will see in the next subsection how one can define a numerically invariant approximation of $\mathbf{B}(D)$.

4.6.2. Augmented base loci

We define a numerically invariant upper approximation of the stable base locus, which was first introduced by Nakamaye [Nak00].

Definition 4.6.3 (see [ELM$^+$06, Def. 1.2])**.** Let X be a projective scheme over a field, and let D be an $\mathbf{R}$-Cartier divisor on X. The *augmented base locus* of D is the closed subset

$$\mathbf{B}_+(D) \coloneqq \bigcap_A \mathbf{B}(D-A)$$

of X, where the intersection runs over all ample $\mathbf{R}$-Cartier divisors A such that $D-A$ is a $\mathbf{Q}$-Cartier divisor. If X is a variety, then by [ELM$^+$06, Rem. 1.3], we have

$$\mathbf{B}_+(D) = \bigcap_{D \equiv_{\mathbf{R}} A+E} \operatorname{Supp} E$$

where the intersections runs over all $\mathbf{R}$-numerical equivalences $D \equiv_{\mathbf{R}} A + E$ where A is an ample $\mathbf{Q}$-Cartier divisor and E is an effective $\mathbf{R}$-Cartier divisor.

We note that D is ample if and only if $\mathbf{B}_+(D) = \emptyset$, and if X is a variety, then D is big if and only if $\mathbf{B}_+(D) \neq X$; see [ELM+06, Ex. 1.7].

We will need the following birational transformation rule for augmented base loci.

Proposition 4.6.4 (cf. [BBP13, Prop. 2.3])**.** *Let $f\colon X' \to X$ be a birational morphism between normal projective varieties. If D is an $\mathbf{R}$-Cartier divisor on X, then we have*

$$\mathbf{B}_+(f^*D) = f^{-1}\big(\mathbf{B}_+(D)\big) \cup \operatorname{Exc}(f). \tag{4.3}$$

The proof of [BBP13, Prop. 2.3] applies in this setting after setting $F = 0$ in their notation, since this makes the application of the negativity lemma unnecessary.

Remark 4.6.5. If one works over an algebraically closed field, then the augmented base locus on the left-hand side of (4.3) can be replaced by $\mathbf{B}_+(f^*D + F)$, where F is any f-exceptional $\mathbf{R}$-Cartier divisor on X'. The proof of this follows [BBP13, Prop. 2.3], after proving the negativity lemma in arbitrary characteristic (see [Bir16, (2.3)]).

We also need the following description for the augmented base locus for nef Cartier divisors, which is originally due to Nakamaye for smooth projective varieties over algebraically closed fields of characteristic zero.

Theorem 4.6.6 ([Bir17, Thm. 1.4]; cf. [Nak00, Thm. 0.3])**.** *Let X be a projective scheme over a field, and suppose D is a nef $\mathbf{R}$-Cartier divisor. Then, we have*

$$\mathbf{B}_+(D) = \bigcup_{(L^{\dim V} \cdot V) = 0} V,$$

where V runs over all positive-dimensional subvarieties $V \subseteq X$ such that $(L^{\dim V} \cdot V) = 0$.

We will also need the following result, which describes how $\mathbf{B}_+(D)$ is the locus where D is ample. Regularity in the proof below is in the sense of Castelnuovo and Mumford; see [Laz04a, Def. 1.8.4] for the definition.

Proposition 4.6.7 (cf. [Kür13, Prop. 2.7; FMa, Lem. 7.12])**.** *Let X be a projective scheme over a field, and let D be a $\mathbf{Q}$-Cartier divisor on X with a decomposition $\mathcal{D}$.*

Then, $\mathbf{B}_+(D)$ *is the smallest closed subset of* X *such that for every coherent sheaf* $\mathscr{F}$ *on* X *and for every* $\mathbf{R}$*-Cartier divisor* E *with decomposition* $\mathcal{E}$*, there exists an integer* n_0 *such that the sheaves*

$$\mathscr{F} \otimes \mathcal{O}_X\big(\lfloor E + nD \rfloor + P\big) \qquad \text{and} \qquad \mathscr{F} \otimes \mathcal{O}_X\big(\lceil E + nD \rceil + P\big)$$

are globally generated on $X \smallsetminus \mathbf{B}_+(D)$ *for every integer* $n \geq n_0$ *and every nef Cartier divisor* P*, where the rounding is done with respect to* $\mathcal{D}$ *and* $\mathcal{E}$*.*

If X *is normal, then the same conclusion holds for* $\mathbf{Q}$*-Cartier* $\mathbf{Q}$*-Weil divisors* D *and* $\mathbf{R}$*-Weil divisors* E*, where the rounding is done in the sense of* $\mathbf{R}$*-Weil divisors.*

Proof. We first show that $\mathbf{B}_+(D)$ satisfies the condition in the proposition. If $\mathbf{B}_+(D) = X$, then the condition trivially holds. We therefore assume that $\mathbf{B}_+(D) \neq X$.

Let A be an ample and free Cartier divisor on X. By [ELM+06, Prop. 1.5] and [Laz04a, Prop. 2.1.21], there exist positive integers q and r such that qrD is a Cartier divisor and

$$\mathbf{B}_+(D) = \mathbf{B}(rD - A) = \operatorname{Bs}\big(\lvert q(rD - A)\rvert\big)_{\mathrm{red}}. \tag{4.4}$$

After possibly replacing A and r by qr and qA, respectively, we can assume that r is an integer such that rD is Cartier, and $\mathbf{B}_+(D) = \operatorname{Bs}(rD - A)_{\mathrm{red}}$.

Now we claim that there exists an integer m_0 such that $\mathscr{F} \otimes \mathcal{O}_X(mA + \lfloor E + jD \rfloor + P)$ (resp. $\mathscr{F} \otimes \mathcal{O}_X(mA + \lceil E + jD \rceil + P)$) is globally generated for every $m \geq m_0$, every $1 \leq j < r$, and every nef Cartier divisor P, where $\lfloor E + jD \rfloor$ (resp. $\lceil E + jD \rceil$) should either be interpreted in the sense of $\mathbf{R}$-Cartier divisors with respect to the decomposition $\mathcal{D}$ and $\mathcal{E}$, or interpreted in the sense of $\mathbf{R}$-Cartier $\mathbf{R}$-Weil divisors in the situation when X is normal. By Fujita's vanishing theorem [Fuj83, Thm. 5.1], there exists an integer m_1 such that for all integers $m \geq m_1$ and all $i > 0$, we have

$$\begin{aligned} H^i\big(X, \mathscr{F} \otimes \mathcal{O}_X\big(mA + \lfloor E + jD \rfloor + P\big)\big) &= 0 \\ H^i\big(X, \mathscr{F} \otimes \mathcal{O}_X\big(mA + \lceil E + jD \rceil + P\big)\big) &= 0 \end{aligned}$$

for all $0 \leq j < r$, and every nef Cartier divisor P. Thus, if $m \geq m_1 + \dim X$, then the coherent sheaf $\mathscr{F} \otimes \mathcal{O}_X(mA + \lfloor E + jD \rfloor + P)$ (resp. $\mathscr{F} \otimes \mathcal{O}_X(mA + \lceil E + jD \rceil + P)$) is 0-regular with respect to A, hence is globally generated by [Laz04a, Thm. 1.8.5(i)]. It therefore suffices to set $m_0 = m_1 + \dim X$.

To prove that $\mathbf{B}_+(D)$ satisfies the condition in the proposition, we note that by the above, the sheaves

$$\begin{aligned}&\mathscr{F} \otimes \mathcal{O}_X\bigl(mA + \lfloor E + jD \rfloor + P\bigr) \otimes \mathcal{O}_X\bigl(q(rD - A)\bigr)\\&\mathscr{F} \otimes \mathcal{O}_X\bigl(mA + \lceil E + jD \rceil + P\bigr) \otimes \mathcal{O}_X\bigl(q(rD - A)\bigr)\end{aligned}$$

are globally generated away from $\mathbf{B}_+(D)$ for all $m \geq m_0$, all $q \geq 1$, all $0 \leq j < r$, and every nef Cartier divisor P. Setting $q = m$, we see that the sheaves

$$\begin{aligned}\mathscr{F} \otimes \mathcal{O}_X\bigl(mrD + \lfloor E + jD \rfloor + P\bigr) &\simeq \mathscr{F} \otimes \mathcal{O}_X\bigl(\lfloor E + (mr + j)D \rfloor + P\bigr)\\\mathscr{F} \otimes \mathcal{O}_X\bigl(mrD + \lceil E + jD \rceil + P\bigr) &\simeq \mathscr{F} \otimes \mathcal{O}_X\bigl(\lceil E + (mr + j)D \rceil + P\bigr)\end{aligned}$$

are globally generated away from $\mathbf{B}_+(D)$ for all $m \geq m_0$, all $0 \leq j < r$, and every nef Cartier divisor P. It therefore suffices to set $n_0 = m_0 r$.

Finally, we show $\mathbf{B}_+(D)$ is the smallest closed subset satisfying the condition in the proposition. Let $x \in \mathbf{B}_+(D)$; it suffices to show that for $\mathscr{F} = \mathcal{O}_X(-A)$ where A is ample, the sheaf $\mathscr{F} \otimes \mathcal{O}_X(mD) = \mathcal{O}_X(nD - A)$ is not globally generated at x for all $n \geq 0$ such that nD is a Cartier divisor. This follows from [ELM$^+$06, Prop. 1.5] since $x \in \mathbf{B}_+(D)$. $\square$

4.6.3. Asymptotic cohomological functions

We now review Küronya's asymptotic cohomological functions with suitable modifications to work over arbitrary fields, following [Kür06, §2] and [BGGJ$^+$, §3]. Asymptotic cohomological functions are defined as follows.

Definition 4.6.8 [BGGJ$^+$, Def. 3.4.6]**.** Let X be a projective scheme of dimension n over a field. For every integer $i \geq 0$, the *ith asymptotic cohomological function* $\widehat{h}^i(X, -)$ on X is the function defined by setting

$$\widehat{h}^i(X, D) \coloneqq \limsup_{m \to \infty} \frac{h^i\bigl(X, \mathcal{O}_X\bigl(\lceil mD \rceil_{\mathcal{D}}\bigr)\bigr)}{m^n/n!}$$

for an $\mathbf{R}$-Cartier divisor D, where $\mathcal{D}$ is a decomposition of D (see Definition 4.2.4).

The numbers $\widehat{h}^i(X,D)$ only depend on the $\mathbf{R}$-linear equivalence class of D and are independent of the decomposition $\mathcal{D}$ by [BGGJ$^+$, Rem. 3.4.5], hence $\widehat{h}^i(X,-)$ gives rise to a well-defined function $\mathrm{Cart}_{\mathbf{R}}(X) \to \mathbf{R}$ and $\mathrm{Cart}_{\mathbf{R}}(X)/{\sim_{\mathbf{R}}} \to \mathbf{R}$.

A key property of asymptotic cohomological functions is the following:

Proposition 4.6.9 [BGGJ$^+$, Prop. 3.4.8]**.** *Let X be a projective scheme of dimension n over a field. For every $i \geq 0$, the function $\widehat{h}^i(X,-)$ on $\mathrm{Cart}_{\mathbf{R}}(X)$ is homogeneous of degree n, and is continuous on every finite-dimensional $\mathbf{R}$-subspace of $\mathrm{Cart}_{\mathbf{R}}(X)$ with respect to every norm.*

Proposition 4.6.9 shows that Definition 4.6.8 is equivalent to Küronya's original definition in [Kür06, Def. 2.1], and that when $i = 0$, the asymptotic cohomological function $\widehat{h}^i(X,D)$ matches the *volume function* $\mathrm{vol}_X(D)$ from [Laz04a, §2.2]. Proposition 4.6.9 also allows us to prove that asymptotic cohomological functions behave well with respect to generically finite morphisms.

Proposition 4.6.10 (cf. [Kür06, Prop. 2.9(1)])**.** *Let $f\colon Y \to X$ be a surjective morphism of projective varieties, and consider an $\mathbf{R}$-Cartier divisor D on X. Suppose f is generically finite of degree d. Then, for every i, we have*

$$\widehat{h}^i(Y, f^*D) = d \cdot \widehat{h}^i(X,D).$$

Proof. The proof of [Kür06, Prop. 2.9(1)] works in our setting with the additional hypothesis that D is a Cartier divisor. It therefore suffices to reduce to this case. If the statement holds for Cartier divisors D, then it also holds for $D \in \mathrm{Cart}_{\mathbf{Q}}(X)$ by homogeneity of $\widehat{h}^i$ (Proposition 4.6.9). Moreover, the subspace of $\mathrm{Cart}_{\mathbf{R}}(X)$ spanned by the Cartier divisors appearing in a decomposition of D is finite-dimensional, hence by approximating each coefficient in D by rational numbers, Proposition 4.6.9 implies the statement for $D \in \mathrm{Cart}_{\mathbf{R}}(X)$ by continuity. □

Remark 4.6.11. We will repeatedly use the same steps as in the proof of Proposition 4.6.10 to prove statements about $\widehat{h}^i(X,D)$ for arbitrary $\mathbf{R}$-Cartier divisors by reducing to the case when D is a Cartier divisor. If D is an $\mathbf{R}$-Cartier divisor, we can write D as the limit of $\mathbf{Q}$-Cartier divisors by approximating each coefficient in a decomposition of D by rational numbers, and continuity of asymptotic cohomological functions (Proposition 4.6.9) then allows us to reduce to the case when D is a $\mathbf{Q}$-Cartier divisor. By homogeneity of asymptotic cohomology functions (Proposition 4.6.9), one can then reduce to the case when D is a Cartier divisor.

We also need the following:

Proposition 4.6.12 (Asymptotic Serre duality; cf. [Kür06, Cor. 2.11])**.** *Let X be a projective variety of dimension n, and let D be an $\mathbf{R}$-Cartier divisor on X. Then, for every $0 \le i \le n$, we have*
$$\widehat{h}^i(X, D) = \widehat{h}^{n-i}(X, -D).$$

Proof. As in Remark 4.6.11, it suffices to consider the case when D is a Cartier divisor. Let $f\colon Y \to X$ be a regular alteration of degree d [dJ96, Thm. 4.1]. We then have

$$\widehat{h}^i(Y, f^*D) = \limsup_{m\to\infty} \frac{h^{n-i}\bigl(Y, \mathcal{O}_Y\bigl(K_Y - f^*(mD)\bigr)\bigr)}{m^n/n!} = \widehat{h}^{n-i}(Y, -f^*D)$$

by Serre duality and [BGGJ+, Lem. 3.2.1], respectively. By Proposition 4.6.10, the left-hand side is equal to $d \cdot \widehat{h}^i(X, D)$ and the right-hand side is equal to $d \cdot \widehat{h}^{n-i}(X, -D)$, hence the statement follows after dividing by d. $\square$

4.6.4. Restricted volumes

We will also need the following variant of the volume function $\mathrm{vol}_X(D) = \widehat{h}^0(X, D)$.

Definition 4.6.13 [ELM+09, Def. 2.1]**.** Let X be a projective variety of dimension n over a field k, and let $V \subseteq X$ be a subvariety of dimension $d \ge 1$. Consider a $\mathbf{Q}$-Cartier divisor D on X. The *restricted volume* of D along V is

$$\mathrm{vol}_{X|V}(D) \coloneqq \limsup_{m\to\infty} \frac{h^0(X|V, \mathcal{O}_X(mD))}{m^d/d!},$$

where

$$H^0\bigl(X|V, \mathcal{O}_X(mD)\bigr) \coloneqq \mathrm{im}\Bigl(H^0\bigl(X, \mathcal{O}_X(mD)\bigr) \to H^0\bigl(V, \mathcal{O}_V(mD|_V)\bigr)\Bigr),$$

and $h^0(X|V, \mathcal{O}_X(mD)) \coloneqq \dim_k H^0(X|V, \mathcal{O}_X(mD))$.

4.7. Log pairs and log triples

To simplify notation, we will use the following conventions for log pairs and log triples. Recall that if R is a ring, then R° is the complement of the union of the minimal primes of R.

Definition 4.7.1. A *log triple* $(X, \Delta, \mathfrak{a}_\bullet^\lambda)$ consists of

(i) an excellent reduced noetherian scheme X;

(ii) an $\mathbf{R}$-Weil divisor Δ on X; and

(iii) a symbol $\mathfrak{a}_\bullet^\lambda$ where $\mathfrak{a}_\bullet$ is a graded family of ideals on X such that for every open affine subset $U = \operatorname{Spec} R \subseteq X$, we have $\mathfrak{a}_m(U) \cap R^\circ \neq \emptyset$ for some $m > 0$, and λ is a real number;

where we assume that X is normal and integral if $\Delta \neq 0$. We say that $(X, \Delta, \mathfrak{a}_\bullet^\lambda)$ is *effective* if $\Delta \geq 0$ and $\lambda \geq 0$. We drop λ from our notation if $\lambda = 1$. If $\mathfrak{a}_\bullet = \{\mathfrak{a}^m\}_{m=0}^\infty$ for some fixed ideal sheaf $\mathfrak{a}$, then we denote the log triple by $(X, \Delta, \mathfrak{a}^t)$ where $t = \lambda$. If $X = \operatorname{Spec} R$ for a ring R, then we denote the log triple by $(R, \Delta, \mathfrak{a}_\bullet^\lambda)$, and denote by $R(\lfloor \Delta \rfloor)$ (resp. $R(\lceil \Delta \rceil)$) the ring of global sections of $\mathcal{O}_{\operatorname{Spec} R}(\lfloor \Delta \rfloor)$ (resp. $\mathcal{O}_{\operatorname{Spec} R}(\lceil \Delta \rceil)$). A *log pair* (X, Δ) (resp. $(X, \mathfrak{a}_\bullet^\lambda)$) is a log triple such that $\mathfrak{a}_m = \mathcal{O}_X$ for all m (resp. $\Delta = 0$).

We will often call log triples (resp. log pairs) *triples* (resp. *pairs*) when there is no risk of confusion.

4.8. Singularities of pairs and triples

We will need the notion of singularities of log pairs and log triples. We mostly follow the conventions of [Kol97, §3], with some adaptations to work with log triples as well.

Definition 4.8.1 (Discrepancies; cf. [Kol97, Defs. 3.3 and 3.4])**.** Let $(X, \Delta, \mathfrak{a}^t)$ be a log triple, where X is a normal variety and $K_X + \Delta$ is $\mathbf{R}$-Cartier. Write $\Delta = \sum d_i D_i$. Suppose $f \colon Y \to X$ is a birational morphism from a normal variety Y, and choose canonical divisors K_Y and K_X such that $f_* K_Y = K_X$. In this case, we may write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta) E, \tag{4.5}$$

where the E are distinct prime Weil divisors over X. The right-hand side is not unique since we allow non-exceptional divisors to appear on the right-hand side. To make the sum on the right-hand side unique, we adopt the convention that a non-exceptional divisor E appears on the right-hand side of (4.5) if and only if $E = f_*^{-1} D_i$ for some i, in which case we set $a(E, X, \Delta) = -d_i$.

For each E, the real number $a(E, X, \Delta)$ is called the *discrepancy of E* with respect to (X, Δ). Note that if $f' \colon Y' \to X$ is another birational morphism and $E' \subseteq Y'$ is the birational transform of E, then $a(E, X, \Delta) = a(E', X, \Delta)$, hence the discrepancy of E only depends on E and not on Y.

The *discrepancy of E* with respect to $(X, \Delta, \mathfrak{a}^t)$ is

$$a(E, X, \Delta, \mathfrak{a}^t) \coloneqq a(E, X, \Delta) - t \cdot \operatorname{ord}_E(\mathfrak{a})$$

where ord_E is the divisorial valuation on the function field of X defined by E.

The *total discrepancy* of $(X, \Delta, \mathfrak{a}^t)$ is

$$\operatorname{totaldiscrep}(X, \Delta, \mathfrak{a}^t) \coloneqq \inf_{f \colon Y \to X} \bigl\{ a(E, X, \Delta, \mathfrak{a}^t) \bigm| E \text{ is a Weil divisor on } Y \bigr\}$$

where the infimum runs over all birational morphisms $f \colon Y \to X$ as above.

Definition 4.8.2 (Singularities of pairs and triples; cf. [Kol97, Def. 3.5])**.** Let $(X, \Delta, \mathfrak{a}^t)$ be a log triple, where X is a normal variety and $K_X + \Delta$ is $\mathbf{R}$-Cartier. We say that $(X, \Delta, \mathfrak{a}^t)$ is *sub-klt* if $\operatorname{totaldiscrep}(E, X, \Delta, \mathfrak{a}^t) > -1$, and is *sub-log canonical* if $\operatorname{totaldiscrep}(E, X, \Delta, \mathfrak{a}^t) \geq -1$. A sub-klt (resp. sub-log canonical) log triple $(X, \Delta, \mathfrak{a}^t)$ is *klt* (resp. *log canonical*) if $(X, \Delta, \mathfrak{a}^t)$ is effective.

When we say that $(X, \Delta, \mathfrak{a}^t)$ is sub-klt (resp. sub-log canonical, klt, log canonical) at a point $x \in X$, we mean that there exists an open neighborhood $U \subseteq X$ of x such that $(U, \Delta|_U, \mathfrak{a}|_U^t)$ is sub-klt (resp. sub-log canonical, klt, log canonical).

We note that *klt* is short for *Kawamata log terminal*.

Next, we recall the following:

Definition 4.8.3. A *log resolution* of a log triple $(X, \Delta, \mathfrak{a}^t)$ is a projective, birational morphism $f \colon Y \to X$, with Y regular, such that

(*i*) We have $f^{-1}\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for an effective Cartier divisor F;

(*ii*) If $\Delta = \sum_i d_i D_i$, and $\widetilde{D}_i$ is the strict transform of D_i, then the divisor $\operatorname{ExcDiv}(f) + F + \sum_i \widetilde{D}_i$ has simple normal crossing support, where $\operatorname{ExcDiv}(f)$ is the sum of exceptional divisors of f.

Note that log resolutions exist for varieties over a field of characteristic zero [Hir64a; Hir64b], and even for reduced noetherian quasi-excellent $\mathbf{Q}$-schemes [Tem18].

Remark 4.8.4. Since the existence of log resolutions is not stated explicitly in [Tem18], we describe how this follows from results therein. First, apply the principalization result in [Tem18, Thm. 1.1.11] to the closed subscheme $\operatorname{Supp}\Delta \cup Z(\mathfrak{a})$ to obtain a resolution $g\colon X' \to X$ such that $g^{-1}(\operatorname{Supp}\Delta \cup Z(\mathfrak{a}))$ is a divisor with simple normal crossing support. Then, one can apply [Tem18, Thm. 1.1.9] to the subscheme $g^{-1}(\operatorname{Supp}\Delta \cup Z(\mathfrak{a})) \cup \operatorname{ExcDiv}(g)$ to ensure that the simple normal crossing condition in (*ii*) holds.

The result below says that to check what singularities a given log triple has, it suffices to check on a log resolution.

Lemma 4.8.5. *Let $(X, \Delta, \mathfrak{a}^t)$ be a log triple, and consider a log resolution $f\colon Y \to X$ for $(X, \Delta, \mathfrak{a}^t)$. Choose canonical divisors K_Y and K_X such that $f_*K_Y = K_X$, and write*

$$K_Y - f^*(K_X + \Delta) - tF = \sum_E a(E, X, \Delta)E$$

using our conventions in Definition 4.8.1 for the right-hand side, where F is the effective Cartier divisor defined by $f^{-1}\mathfrak{a} \cdot \mathcal{O}_Y$. Then, we have that $(X, \Delta, \mathfrak{a}^t)$ is sub-klt (resp. sub-log canonical) if and only if $\min_E\{a(E, X, \Delta)\} > -1$ (resp. ≥ -1), where E runs over all prime divisors on Y.

Proof. The statement for log pairs is [Kol97, Cor. 3.13]. The statement for log triples then follows, since $(X, \Delta, \mathfrak{a}^t)$ is sub-klt (resp. sub-log canonical) if and only $(Y, \Delta_Y + tF)$ is sub-klt (resp. sub-log canonical), where Δ_Y is defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$. $\square$

4.8.1. Log canonical thresholds

We also define the following:

Definition 4.8.6 (Log canonical threshold; cf. [Kol97, Def. 8.1])**.** Let $(X, \Delta, \mathfrak{a})$ be a triple and let $x \in X$ be a closed point. The *log canonical threshold* of (X, Δ) at x with respect to $\mathfrak{a}$ is

$$\operatorname{lct}_x\bigl((X,\Delta);\mathfrak{a}\bigr) \coloneqq \sup\bigl\{c \in \mathbf{R}_{\geq 0} \bigm| (X,\Delta,\mathfrak{a}^c) \text{ is sub-log canonical at } x\bigr\},$$

where if (X,Δ) is not sub-log canonical, then we set $\operatorname{lct}_x((X,\Delta);\mathfrak{a}) = -\infty$. If $\mathfrak{a} = \mathcal{O}_X(-D)$ for a Cartier divisor D, then we denote

$$\operatorname{lct}_x\bigl((X,\Delta);D\bigr) \coloneqq \operatorname{lct}_x\bigl((X,\Delta);\mathcal{O}_X(-D)\bigr).$$

We also drop Δ from our notation if $\Delta = 0$.

Log canonical thresholds can be computed on a log resolution:

Proposition 4.8.7 (cf. [Kol97, Prop. 8.5])**.** *Let $(X,\Delta,\mathfrak{a})$ be a log triple such that (X,Δ) is sub-log canonical, and let $x \in X$ be a closed point. Consider a log resolution $f\colon Y \to X$ for $(X,\Delta,\mathfrak{a})$. Using our conventions in Definition 4.8.1, write*

$$K_Y - f^*(K_X+\Delta) = \sum_j a_j E_j \qquad \text{and} \qquad F = \sum_j b_j E_j,$$

where F is the effective Cartier divisor defined by $f^{-1}\mathfrak{a}\cdot\mathcal{O}_Y$. Then,

$$\operatorname{lct}_x\bigl((X,\Delta);\mathfrak{a}\bigr) = \min_{\{j \mid f(E_j)=\{x\}\}}\biggl\{\frac{a_j+1}{b_j}\biggr\}.$$

Proof. This follows from Lemma 4.8.5, since $(X,\Delta,\mathfrak{a}^c)$ is sub-log canonical if and only if $a_j - cb_j \geq -1$ for all j. □

We compute one example of a log canonical threshold.

Example 4.8.8 (Cuspidal cubic)**.** Let k be an algebraically closed field, and consider the cuspidal cubic $C = \{x^2+y^3=0\} \subseteq \mathbf{A}^2_k$. We would like to compute the log canonical threshold $\operatorname{lct}_0(\mathbf{A}^2_k;C)$, where $0 \in \mathbf{A}^2_k$ is the origin. First, there is a log resolution $\pi\colon W \to \mathbf{A}^2_k$ as in Figure 4.1, which is constructed as a sequence of blowups at the intersection of the divisors shown, where

$$\pi^*C = 2E_1 + 3E_2 + 6E_3 + \widetilde{C} \qquad \text{and} \qquad K_W - \pi^*K_{\mathbf{A}^2_k} = E_1 + 2E_2 + 4E_3.$$

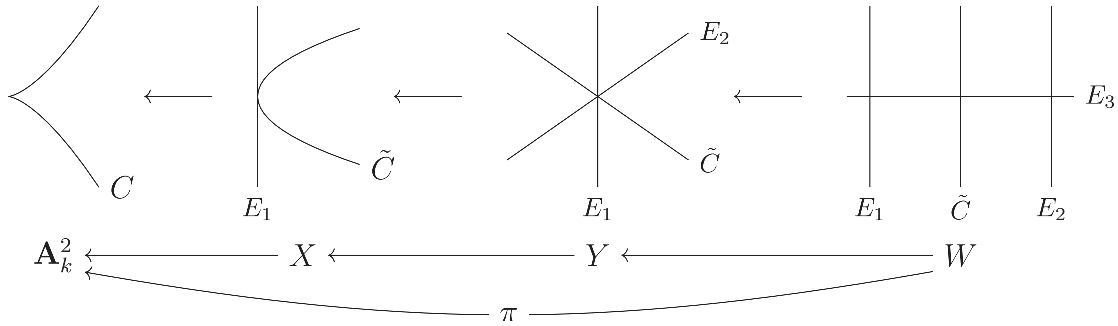

Figure 4.1: Log resolution of a cuspidal cubic

By Proposition 4.8.7, we then have

$$\operatorname{lct}_0(\mathbf{A}^2_k; C) = \min\left\{\frac{1+1}{2}, \frac{2+1}{3}, \frac{4+1}{6}\right\} = \frac{5}{6}.$$

4.9. Multiplier ideals

We briefly review the theory of multiplier ideals. Multiplier ideals were first defined by Nadel in the analytic setting [Nad90, Def. 2.5]. We recommend [Laz04b, Pt. 3] for an overview on this topic. We work in the more general setting of excellent $\mathbf{Q}$-schemes, following [dFM09, §2] and [JM12, App. A].

Definition 4.9.1 [Laz04b, Def. 9.3.60]**.** Let $(X, \Delta, \mathfrak{a}^t)$ be an effective log triple such that X is a $\mathbf{Q}$-scheme and such that $K_X + \Delta$ is $\mathbf{R}$-Cartier. Fix a log resolution $f\colon Y \to X$ of $(X, \Delta, \mathfrak{a}^t)$ so that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_{\widetilde{X}}(-D)$ for an effective divisor D. Note that such a log resolution exists by Remark 4.8.4. The *multiplier ideal* is

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) := f_*\mathcal{O}_Y\bigl(K_Y - \lfloor f^*(K_X + \Delta) + tD \rfloor\bigr).$$

This definition does not depend on the choice of log resolution; see [Laz04b, Thm. 9.2.18; dFM09, Prop. 2.2].

If $(X, \Delta, \mathfrak{b}^s)$ is another effective log triple on X, then we can analogously define the multiplier ideal

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s) := f_*\mathcal{O}_Y\bigl(K_Y - \lfloor f^*(K_X + \Delta) + tD + sF \rfloor\bigr).$$

where f is a simultaneous log resolution for $(X, \Delta, \mathfrak{a})$ and $(X, \Delta, \mathfrak{b})$ such that $\mathfrak{b} \cdot \mathcal{O}_Y = \mathcal{O}_{\widetilde{X}}(-F)$ for an effective divisor F.

Note that if $(X, \Delta, \mathfrak{a}^t)$ is an effective log triple such that X is a normal variety and $K_X + \Delta$ is $\mathbf{R}$-Cartier, then $(X, \Delta, \mathfrak{a}^t)$ is klt if and only if $\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \mathcal{O}_X$ [Laz04a, p. 165]. See [Laz04b, Ex. 9.2.30, Prop. 9.2.32, and p. 185] for some other basic properties of multiplier ideals, which carry over to the setting of excellent $\mathbf{Q}$-schemes. Some more subtle properties in our context are checked in [dFM09, Prop. 2.3; JM12, App. A]. In particular, the subadditivity theorem of Demailly–Ein–Lazarsfeld [DEL00, Thm. on p. 137] holds for regular excellent $\mathbf{Q}$-schemes; see [JM12, Thm. A.2].

We will also need an asymptotic version of multiplier ideals for graded families of ideals. Note that an asymptotic version of multiplier ideals first appeared in the work of Siu [Siu98, pp. 668–669] in the analytic setting.

Definition 4.9.2 (see [Laz04a, Def. 11.1.15])**.** Let $(X, \Delta, \mathfrak{a}_\bullet^\lambda)$ be an effective log triple such that X is a $\mathbf{Q}$-scheme and such that $K_X + \Delta$ is $\mathbf{R}$-Cartier. If m and r are positive integers, then

$$\mathcal{J}(X, \Delta, \mathfrak{a}_m^{\lambda/m}) = \mathcal{J}\big(X, \Delta, (\mathfrak{a}_m^r)^{\lambda/(mr)}\big) \subseteq \mathcal{J}\big(X, \Delta, \mathfrak{a}_{mr}^{\lambda/(mr)}\big),$$

by [Laz04b, Rem. 9.2.4 and Prop. 9.2.32(*iii*)], and by the graded property $\mathfrak{a}_m^r \subseteq \mathfrak{a}_{mr}$. Thus, the set of ideals

$$\big\{\mathcal{J}(X, \Delta, \mathfrak{a}_m^{\lambda/m})\big\}_{m=1}^{\infty} \tag{4.6}$$

is partially ordered, and has a unique maximal element by the noetherian property that coincides with $\mathcal{J}(X, \Delta, \mathfrak{a}_m^{\lambda/m})$ for m sufficiently large and divisible. The *asymptotic multiplier ideal*

$$\mathcal{J}(X, \Delta, \mathfrak{a}_\bullet^\lambda) \subseteq \mathcal{O}_X$$

is the maximal element of the partially ordered set (4.6).

If $(X, \Delta, \mathfrak{b}_\bullet^\lambda)$ is another effective log triple on X, then we can analogously define the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}_\bullet^\lambda \cdot \mathfrak{b}_\bullet^\mu)$.

The following examples of multiplier ideals will be the most useful in our applications.

Example 4.9.3 (see [Laz04b, Defs. 9.2.10 and 11.1.2])**.** Suppose (X, Δ) is an effective log pair where X is a complete scheme over a field of characteristic zero. If D is a Cartier divisor such that $H^0(X, \mathcal{O}_X(mD)) \neq 0$ for some positive integer m, then for every real number $t \geq 0$, we set

$$\mathcal{J}\bigl(X, \Delta, t \cdot |D|\bigr) \coloneqq \mathcal{J}\bigl(X, \Delta, \mathfrak{b}\bigl(|D|\bigr)^t\bigr).$$

If D is a $\mathbf{Q}$-Cartier divisor such that $H^0(X, \mathcal{O}_X(mD)) \neq 0$ for some sufficiently divisible $m > 0$, then for every real number $\lambda \geq 0$, we set

$$\mathcal{J}\bigl(X, \Delta, \lambda \cdot \|D\|\bigr) \coloneqq \mathcal{J}\bigl(X, \Delta, \mathfrak{a}_\bullet(D)^\lambda\bigr),$$

where $\mathfrak{a}_\bullet(D)$ is the graded family of ideals defined in Example 4.5.4.

Finally, we have the following uniform global generation result for the (asymptotic) multiplier ideals defined in Example 4.9.3.

Theorem 4.9.4 (cf. [Laz04b, Prop. 9.4.26 and Cor. 11.2.13])**.** *Let (X, Δ) be an effective log pair where X is a normal projective variety over a field k of characteristic zero and Δ is an effective $\mathbf{Q}$-Weil divisor such that $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Let D, L, and H be Cartier divisors on X such that H is ample and free. If λ is a non-negative real number such that $L - (K_X + \Delta + \lambda \cdot D)$ is ample, then the sheaves*

$$\mathcal{J}\bigl(X, \Delta, \lambda \cdot |D|\bigr) \otimes \mathcal{O}_X(L + dH) \quad \textit{and} \quad \mathcal{J}\bigl(X, \Delta, \lambda \cdot \|D\|\bigr) \otimes \mathcal{O}_X(L + dH)$$

are globally generated for every integer $d > \dim X$.

Proof. By choosing n sufficiently divisible such that

$$\mathcal{J}\bigl(X, \Delta, \lambda \cdot \|D\|\bigr) = \mathcal{J}\bigl(X, \Delta, (\lambda/n) \cdot |nD|\bigr),$$

it suffices to consider the case for the usual multiplier ideals. Global generation follows from Nadel vanishing when k is algebraically closed [Laz04b, Prop. 9.4.26 and Rem. 9.4.27], hence it suffices to reduce to this case.

Let $\pi\colon \overline{X} \coloneqq X \times_k \overline{k} \to X$ denote the base extension to the algebraic closure $\overline{k}$ of k. Since the extension $k \subseteq \overline{k}$ is faithfully flat, the sheaf $\mathcal{J}(X, \Delta, \lambda \cdot |D|) \otimes \mathcal{O}_X(L + dH)$ is globally generated if its pullback to $\overline{X}$ is globally generated. The pullback to $\overline{X}$ is isomorphic to

$$\mathcal{J}\bigl(\overline{X}, \pi^*\Delta, \lambda \cdot |\pi^*D|\bigr) \otimes \mathcal{O}_{\overline{X}}(\pi^*L + d\pi^*H) \tag{4.7}$$

since the formation of multiplier ideals commutes with faithfully flat base change [JM12, Prop. 1.9]. Moreover, the **R**-Cartier divisor

$$\pi^*\bigl(L - (K_X + \Delta + \lambda \cdot D)\bigr) = \pi^* L - (K_{\overline{X}} + \pi^*\Delta + \lambda \cdot \pi^* D)$$

is ample and $\pi^* H$ is ample and free by faithfully flat base change [EGAIV$_2$, Cor. 2.7.2]. To apply the special case when k is algebraically closed, we note that while $\overline{X}$ may not be irreducible, it is still the disjoint union of normal varieties [Mat89, Rem. on pp. 64–65]. Thus, by applying [Laz04b, Prop. 9.4.26 and Rem. 9.4.27] to each connected component of $\overline{X}$ individually, we see that the sheaf in (4.7) is globally generated. □

Chapter 5

Preliminaries in positive characteristic

In this chapter, we review some preliminaries on commutative algebra and algebraic geometry in positive characteristic. See [ST12] and [TW18] for overviews on the topic. See also [PST17] and [Pat18] for more geometric applications.

The only new material is a new, short proof of the subadditivity theorem for test ideals (Theorem 5.5.8), and some material on F-pure triples in §5.4.

5.1. Conventions on the Frobenius morphism

We start by establishing our conventions for the Frobenius morphism.

Definition 5.1.1. Let X be a scheme of characteristic $p > 0$. The *(absolute) Frobenius morphism* is the morphism $F\colon X \to X$ of schemes given by the identity on points and the p-power map

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow F_*\mathcal{O}_X(U)\\ f &\longmapsto f^p \end{aligned}$$

on structure sheaves for every open subset $U \subseteq X$. If R is a ring of characteristic $p > 0$, we denote the corresponding ring homomorphism by $F\colon R \to F_*R$. For every integer $e \geq 0$, the eth iterate of the Frobenius morphism is denoted by $F^e\colon X \to X$ and $F^e\colon R \to F^e_*R$. If $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent ideal sheaf, we define the *eth Frobenius* *power* $\mathfrak{a}^{[p^e]}$ to be the inverse image of $\mathfrak{a}$ under the eth iterate of the Frobenius morphism. Locally, if $\mathfrak{a}$ is generated by $(h_i)_{i\in I}$, then $\mathfrak{a}^{[p^e]}$ is generated by $(h_i^{p^e})_{i\in I}$.

We note that the notation F^e_*R is used to remind us that the R-algebra structure on F^e_*R is given by the ring homomorphism F^e.

5.2. The pigeonhole principle

A surprisingly important fact in this thesis is the following combinatorial result based on the pigeonhole principle.

Lemma 5.2.1 (cf. [HH02, Lem. 2.4(a)])**.** *Let R be a commutative ring of characteristic $p > 0$. Then, for any ideal $\mathfrak{a}$ generated by n elements and for all non-negative integers e and ℓ, we have the sequence of inclusions*

$$\mathfrak{a}^{\ell p^e+n(p^e-1)+1} \subseteq (\mathfrak{a}^{\ell+1})^{[p^e]} \subseteq \mathfrak{a}^{(\ell+1)p^e}. \tag{5.1}$$

Moreover, if R is a regular local ring of dimension n and $\mathfrak{m}$ is the maximal ideal of R, then

$$\mathfrak{m}^{\ell p^e+n(p^e-1)} \not\subseteq (\mathfrak{m}^{\ell+1})^{[p^e]}.$$

Proof. The second inclusion in (5.1) is clear by the definition of Frobenius powers. We want to show the first inclusion. Let $y_1, y_2, \ldots, y_n$ be a set of generators for $\mathfrak{a}$. The ideal $\mathfrak{a}^{\ell p^e+n(p^e-1)+1}$ is generated by all elements of the form

$$\prod_{i=1}^n y_i^{a_i} \quad \text{such that} \quad \sum_{i=1}^n a_i = \ell p^e + n(p^e-1)+1, \tag{5.2}$$

and the ideal $(\mathfrak{a}^{\ell+1})^{[p^e]}$ is generated by all elements of the form

$$\prod_{i=1}^n y_i^{p^e b_i} \quad \text{such that} \quad \sum_{i=1}^n b_i = \ell+1. \tag{5.3}$$

We want to show that the elements (5.2) are divisible by some elements of the form (5.3). By the division algorithm, we may write $a_i = a_{i,0} + p^e a_i'$ for some non-negative integers

$a_{i,0}$ and a'_i such that $0 \leq a_{i,0} \leq p^e - 1$. We then have

$$\prod_{i=1}^{n} y_i^{a_i} = \prod_{i=1}^{n} y_i^{a_{i,0}} \cdot \prod_{i=1}^{n} y_i^{p^e a'_i},$$

and since $a_{i,0} \leq p^e - 1$, we have that $\sum_{i=1}^{n} a_{i,0} \leq n(p^e - 1)$. Thus, we have the inequality

$$\ell p^e + n(p^e - 1) + 1 = \sum_{i=1}^{n} a_i \leq n(p^e - 1) + \sum_{i=1}^{n} p^e a'_i,$$

which implies $\ell + p^{-e} \leq \sum_{i=1}^{n} a'_i$. Since the right-hand side of this inequality is an integer, we have that $\ell + 1 \leq \sum_{i=1}^{n} a'_i$, i.e., the element $\prod_{i=1}^{n} y_i^{p^e a'_i}$ is divisible by one of the form (5.3). Thus, each element of the form in (5.2) is divisible by one of the form in (5.3).

Now suppose R is a regular local ring of dimension n, and $\mathfrak{m}$ is the maximal ideal of R. Let $y_1, y_2, \ldots, y_n$ be a regular system of parameters. Then, we have

$$y_{i_0}^{\ell p^e} \cdot \prod_{i=1}^{n} y_i^{p^e - 1} \in \mathfrak{m}^{\ell p^e + n(p^e - 1)}$$

for all $i_0 \in \{1, 2, \ldots, n\}$. This monomial does not lie in $(\mathfrak{m}^{\ell+1})^{[p^e]}$ since its image is not in the extension of $(\mathfrak{m}^{\ell+1})^{[p^e]}$ in the completion of R at $\mathfrak{m}$, which is isomorphic to a formal power series ring with variables $y_1, y_2, \ldots, y_n$ by the Cohen structure theorem. $\square$

We moreover show that asymptotically, the number n of elements generating $\mathfrak{a}$ can be replaced by the analytic spread of $\mathfrak{a}$. See [HS06, Def. 5.1.5] for the definition of analytic spread.

Lemma 5.2.2. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of characteristic $p > 0$. Then, for every ideal $\mathfrak{a}$ of analytic spread h, there exists an integer $t \geq 0$ such that for all non-negative integers e and ℓ, we have the sequence of inclusions*

$$\mathfrak{a}^{\ell p^e + h(p^e - 1) + 1 + t} \subseteq (\mathfrak{a}^{\ell+1})^{[p^e]} \subseteq \mathfrak{a}^{(\ell+1)p^e}. \tag{5.4}$$

In particular, if $\mathfrak{a} = \mathfrak{m}$, then (5.4) holds for $h = \dim R$.

Proof. The right inclusion in (5.4) is clear as in Lemma 5.2.1. It therefore suffices to prove the left inclusion in (5.4). We first reduce to the case when k is infinite. Consider the ring $S = R[x]_{\mathfrak{m}R[x]}$ as in [HS06, §8.4]. Then, S is a noetherian local ring of characteristic $p > 0$ such that $R \subseteq S$ is faithfully flat and $S/\mathfrak{m}S \simeq k(x)$ is infinite. Since we can check the inclusions in (5.4) after a faithfully flat extension [Mat89, Thm. 7.5(ii)], and since analytic spread does not change after passing to S [HS06, Lem. 8.4.2(4)], we can replace R with S to assume that k is infinite.

We now prove the left inclusion in (5.4) under the assumption that k is infinite. Recall that since k is infinite, there exists an ideal $\mathfrak{q} \subseteq \mathfrak{a}$ called a *minimal reduction ideal* and an integer $t > 0$ such that $\mathfrak{q}$ is generated by h elements, and $\mathfrak{a}^{s+t} = \mathfrak{q}^s \cdot \mathfrak{a}^t$ for every integer $s \geq 0$; see [HS06, Def. 1.2.1 and Prop. 8.3.7]. Setting $s = \ell p^e + h(p^e - 1) + 1$, we have

$$\mathfrak{a}^{\ell p^e + h(p^e-1)+1+t} = \mathfrak{q}^{\ell p^e + h(p^e-1)+1} \cdot \mathfrak{a}^t \subseteq (\mathfrak{q}^{\ell+1})^{[p^e]} \cdot \mathfrak{a}^t \subseteq (\mathfrak{a}^{\ell+1})^{[p^e]}$$

for all non-negative integers e and ℓ, where the first inclusion holds by Lemma 5.2.1. The special case for $\mathfrak{a} = \mathfrak{m}$ follows from [HS06, Cor. 8.3.9]. □

5.3. *F*-finite schemes

As mentioned in §2.4, in positive characteristic, one often needs to restrict or reduce to the case when the Frobenius morphism is finite. We isolate this class of schemes.

Definition 5.3.1. Let X be a scheme of characteristic $p > 0$. We say that X is *F-finite* if the (absolute) Frobenius morphism $F\colon X \to X$ is finite. We say that a ring R of characteristic $p > 0$ is *F-finite* if $\operatorname{Spec} R$ is F-finite, or equivalently if $F\colon R \to F_*R$ is module-finite.

Note that a field k is F-finite if and only if $[k : k^p] < \infty$. F-finite schemes are ubiquitous in geometric contexts because of the following:

Example 5.3.2 (see [Kun76, p. 999; BMS08, Ex. 2.1])**.** If X is a scheme that is locally essentially of finite type over an F-finite scheme of characteristic $p > 0$, then X is F-finite. In particular, schemes essentially of finite type over perfect or F-finite fields are F-finite.

If a scheme X of characteristic $p > 0$ is F-finite, then Grothendieck duality (Theorem 4.4.1) can be applied to the Frobenius morphism since it is finite. The F-finiteness condition implies other desirable conditions as well.

Theorem 5.3.3 [Kun76, Thm. 2.5; Gab04, Rem. 13.6]**.** *Let R be a noetherian F-finite ring of characteristic $p > 0$. Then, R is excellent and is isomorphic to a quotient of a regular ring of finite Krull dimension. In particular, R admits a dualizing complex.*

See [Har66, Def. on p. 258] for the definition of a dualizing complex.

5.4. $\boldsymbol{F}$-singularities of pairs and triples

We now define F-singularities for log triples in the sense of Definition 4.7.1. These are common generalizations of the notions for log pairs (X, Δ) and $(X, \mathfrak{a}^t)$ due to Hara–Watanabe [HW02] and Takagi [Tak04b], respectively. While an equivalent definition of strong F-regular triples has appeared before (see Example 5.4.5), the definition of F-pure triples appears to be new.

We note that we assume F-finiteness throughout. See Appendix A for an overview on F-singularities for rings, where we work without F-finiteness assumptions.

Definition 5.4.1 (cf. [HW02, Def. 2.1; Tak04b, Def. 3.1])**.** Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is an F-finite local ring of characteristic $p > 0$.

(*a*) The triple $(R, \Delta, \mathfrak{a}^t)$ is *F-pure* if there exists an integer $e' > 0$ such that for all $e \geq e'$, there exists an element $d \in \mathfrak{a}^{\lfloor (p^e-1)t \rfloor}$ for which the composition

$$R \xrightarrow{F^e} F^e_* R \hookrightarrow F^e_* R\big(\lfloor (p^e-1)\Delta \rfloor\big) \xrightarrow{F^e_*(-\cdot d)} F^e_* R\big(\lfloor (p^e-1)\Delta \rfloor\big) \tag{5.5}$$

splits as an R-module homomorphism.

(*b*) The triple $(R, \Delta, \mathfrak{a}^t)$ is *sharply F-pure* if there exists an integer $e > 0$ and an element $d \in \mathfrak{a}^{\lceil (p^e-1)t \rceil}$ for which the composition

$$R \xrightarrow{F^e} F^e_* R \hookrightarrow F^e_* R\big(\lceil (p^e-1)\Delta \rceil\big) \xrightarrow{F^e_*(-\cdot d)} F^e_* R\big(\lceil (p^e-1)\Delta \rceil\big) \tag{5.6}$$

splits as an R-module homomorphism.

(*c*) The triple $(R, \Delta, \mathfrak{a}^t)$ is *strongly F-regular* if for all $c \in R^\circ$, there exists an integer $e > 0$ and an element $d \in \mathfrak{a}^{\lfloor (p^e-1)t \rfloor}$ for which the composition

$$R \xrightarrow{F^e} F^e_* R \hookrightarrow F^e_* R\big(\lfloor (p^e-1)\Delta \rfloor\big) \xrightarrow{F^e_*(-\cdot cd)} F^e_* R\big(\lfloor (p^e-1)\Delta \rfloor\big) \tag{5.7}$$

splits as an R-module homomorphism.

Now suppose that $(X, \Delta, \mathfrak{a}^t)$ is an effective log triple such that X is an F-finite scheme of characteristic $p > 0$, and let $x \in X$ be a point. The triple $(X, \Delta, \mathfrak{a}^t)$ is *F-pure* (resp. *sharply F-pure*, *strongly F-regular*) at x if the localized triple $(\mathcal{O}_{X,x}, \Delta|_{\operatorname{Spec}\mathcal{O}_{X,x}}, \mathfrak{a}_x^t)$ is F-pure (resp. sharply F-pure, strongly F-regular). The triple $(X, \Delta, \mathfrak{a}^t)$ is *F-pure* (resp. *sharply F-pure*, *strongly F-regular*) if it is F-pure (resp. strongly F-regular) at every point $x \in X$.

Remark 5.4.2. A triple $(R, 0, R^1)$ as in Definition 5.4.1 is F-pure if and only if R is F-pure in the sense of Hochster–Roberts (since F-purity and F-splitting coincide F-finite rings; see Figure A.1), and is strongly F-regular if and only if R is strongly F-regular in the sense of Hochster–Huneke (Definition A.7(a)).

We collect some basic properties of F-singularities for triples.

Proposition 5.4.3 (cf. [HW02, Prop. 2.2; Tak04b, Prop. 3.3])**.** *Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is an F-finite local ring of characteristic $p > 0$.*

(i) *If $(R, \Delta, \mathfrak{a}^t)$ is F-pure (resp. sharply F-pure, strongly F-regular), then so is $(R, \Delta', \mathfrak{b}^s)$ for every triple such that $\Delta' \leq \Delta$, $\mathfrak{b} \supseteq \mathfrak{a}$, and $s \in [0, t]$.*

(ii) *If $(R, \Delta, \mathfrak{a}^t)$ is F-pure, then $\lceil \Delta \rceil$ is reduced, i.e., the nonzero coefficients of $\lceil \Delta \rceil$ are equal to* 1.

(iii) *$(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular if and only if for all $c \in R^\circ$, there exists an integer $e' > 0$ such that for all $e \geq e'$, there exists $d \in \mathfrak{a}^{\lceil p^e t \rceil}$ for which the composition*

$$R \xrightarrow{F^e} F^e_* R \hookrightarrow F^e_* R\bigl(\lceil p^e \Delta \rceil\bigr) \xrightarrow{F^e_*(-\cdot cd)} F^e_* R\bigl(\lceil p^e \Delta \rceil\bigr) \tag{5.8}$$

splits as an R-module homomorphism.

(iv) *We have the implications*

$$\begin{array}{ccc} \textit{strongly } F\textit{-regular} & \Longrightarrow & \textit{sharply } F\textit{-pure} \\ & \searrow\!\!\!\searrow \quad \swarrow\!\!\!\swarrow & \\ & F\textit{-pure} & \end{array}$$

where the dashed implication holds when Δ is Cartier and $\mathfrak{a}$ is locally principal.

Proof. (i) follows since the splitting conditions for $(R, \Delta', \mathfrak{b}^s)$ are weaker than those for $(R, \Delta, \mathfrak{a}^t)$. For (ii), we note that (R, Δ) is F-pure by (i), hence (ii) follows from [HW02, Prop. 2.2(4)].

For (iii), we note that $\Leftarrow$ is clear. For $\Rightarrow$, the case when $\Delta = 0$ is shown in [Tak04b, Prop. 3.3(3)], hence it suffices to consider when $\Delta \neq 0$, in which case R is a normal domain by our conventions in Definition 4.7.1. Let $c' \in R^\circ$ be arbitrary; we want to show that for $c = c'$, the composition (5.8) splits for some $d \in \mathfrak{a}^{\lceil p^e t \rceil}$ for all $e \gg 0$. Choose nonzero elements $a \in R(-2\lceil \Delta \rceil)$ and $b \in \mathfrak{a}^{2\lceil t \rceil}$, in which case

$$\begin{gathered} a \cdot R\bigl(\lceil p^e \Delta \rceil\bigr) \subseteq R\bigl(\lceil p^e \Delta \rceil - 2\lceil \Delta \rceil\bigr) \subseteq R\bigl(\lfloor (p^e - 1)\Delta \rfloor\bigr) \\ b \cdot \mathfrak{a}^{\lfloor (p^e-1)t \rfloor} \subseteq \mathfrak{a}^{2\lceil t \rceil + \lfloor (p^e-1)t \rfloor} \subseteq \mathfrak{a}^{\lceil p^e t \rceil} \end{gathered}$$

for every integer $e > 0$. By the assumption that $(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular, there exist $e' > 0$ and $d' \in \mathfrak{a}^{\lfloor (p^{e'}-1)t \rfloor}$ such that the composition (5.7) splits for $e = e'$ and with $c = abc'$. This composition factors as

$$\begin{aligned} R \xrightarrow{F^{e'}} F^{e'}_* R \lhook\joinrel\longrightarrow F^{e'}_* R\bigl(\lceil p^{e'} \Delta \rceil\bigr) \\ \xrightarrow{F^{e'}_*(-\cdot bc'd')} F^{e'}_* R\bigl(\lceil p^{e'} \Delta \rceil\bigr) \xrightarrow{F^{e'}_*(-\cdot a)} F^{e'}_* R\bigl(\lfloor (p^{e'} - 1)\Delta \rfloor\bigr), \end{aligned}$$

hence the composition of the first three homomorphisms splits. Now since R is F-pure by (i), the homomorphism

$$F^{e'}_* R\bigl(\lceil p^{e'} \Delta \rceil\bigr) \xrightarrow{F^{e'}_*(F^{e-e'}(\lceil p^{e'} \Delta \rceil))} F^e_* R\bigl(p^{e-e'} \lceil p^{e'} \Delta \rceil\bigr),$$

which is obtained by twisting the $(e - e')$th iterate of the Frobenius homomorphism by $\lceil p^{e'} \Delta \rceil$ and applying $F^{e'}_*$, also splits for every $e \geq e'$. The composition

$$\begin{aligned} R \xrightarrow{F^{e'}} F^{e'}_* R \lhook\joinrel\longrightarrow F^{e'}_* R\bigl(\lceil p^{e'} \Delta \rceil\bigr) \xrightarrow{F^{e'}_*(-\cdot bc'd')} F^{e'}_* R\bigl(\lceil p^{e'} \Delta \rceil\bigr) \\ \xrightarrow{F^{e'}_*(F^{e-e'}(\lceil p^{e'} \Delta \rceil))} F^e_* R\bigl(p^{e-e'} \lceil p^{e'} \Delta \rceil\bigr) \end{aligned}$$

therefore splits for $e \geq e'$. Finally, this composition factors as

$$R \xrightarrow{F^e} F^e_* R \hookrightarrow F^e_* R\big(\lceil p^e \Delta \rceil\big) \xrightarrow{F^e_*(- \cdot (bc'd')^{p^{e-e'}})} F^e_* R\big(\lceil p^e \Delta \rceil\big) \hookrightarrow F^e_* R\big(p^{e-e'} \lceil p^{e'} \Delta \rceil\big),$$

hence the composition (5.8) splits for $c = c'$ and

$$d = b^{p^{e-e'}} (c')^{p^{e-e'}-1} (d')^{p^{e-e'}} \in \big(\mathfrak{a}^{2\lceil t \rceil + \lfloor (p^{e'}-1)t \rfloor}\big)^{p^{e-e'}} \subseteq \big(\mathfrak{a}^{\lceil p^{e'} t \rceil}\big)^{p^{e-e'}} \subseteq \mathfrak{a}^{\lceil p^e t \rceil}.$$

Finally, for (iv), we note that strong F-regularity implies sharp F-purity by (iii), and the dashed implication holds by [Sch08, Prop. 3.5]. $\square$

Remark 5.4.4. It seems to be unknown whether sharp F-purity implies F-purity in general [Sch08, Ques. 3.8].

Example 5.4.5. While Proposition 5.4.3(iii) shows that the rounding "$\lfloor (p^e - 1) - \rfloor$" in Definition 5.4.1$(c)$ can be replaced by "$\lceil (p^e - 1) - \rceil$" (this is the convention in [Sch10b, Def. 2.11; Sch10a, Def. 3.2]), this is not the case for F-purity. For example, the pair

$$\big(\mathbf{F}_2[\![x, y, z]\!], (x^2 + y^5 + z^5)^{1/2}\big)$$

from [MY09, Ex. 4.3] is F-pure but not sharply F-pure by [Her12, Thm. 4.1].

5.4.1. The trace of Frobenius

We now describe variants of the Grothendieck trace map associated to the Frobenius morphism, and its relationship to F-singularities. This material is essentially contained in [Sch09a], although we use some of the notation of [Tan15, §2] and [CTX15, §2.3].

Proposition 5.4.6 [CTX15, Def.-Prop. 2.5]**.** *Let X be a normal scheme essentially of finite type over an F-finite field of characteristic $p > 0$, let D be an effective Weil divisor on X, and let e be a positive integer. Then, there exists a homomorphism*

$$\mathrm{Tr}^e_{X,D} \colon F^e_*\big(\mathcal{O}_X\big((1-p^e)K_X - D\big)\big) \longrightarrow \mathcal{O}_X$$

of $\mathcal{O}_X$-modules that fits into a commutative diagram

$$\begin{tikzcd}
F^e_*\big(\mathcal{O}_X\big((1-p^e)K_X - D\big)\big) \arrow[rr, "\mathrm{Tr}^e_{X,D}"] \arrow[d, "\theta"', "\wr"] & & \mathcal{O}_X \arrow[d, "\wr"] \\
\mathscr{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(D)\big), \mathcal{O}_X\big) \arrow[r, "(F^e_D)^*"] & \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)
\end{tikzcd} \tag{5.9}$$

of $\mathcal{O}_X$-modules, where the left vertical arrow is an isomorphism of $(F^e_*\mathcal{O}_X, \mathcal{O}_X)$-bimodules and the right vertical arrow is an isomorphism of $\mathcal{O}_X$-modules.

Proof. Consider the composition map

$$\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D),$$

which we denote by F^e_D. Applying the contravariant functor $\mathscr{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$, we have the top arrow in the commutative diagram

$$\begin{tikzcd}
\mathscr{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(D)\big), \mathcal{O}_X\big) \arrow[r, "(F^e_D)^*"] & \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \\
\mathscr{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(p^eK_X + D)\big), \mathcal{O}_X(K_X)\big) \arrow[r] \arrow[u, "\wr"] & \mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X(K_X), \mathcal{O}_X(K_X)\big) \arrow[u, "\wr"]
\end{tikzcd}$$

where the vertical arrows are isomorphisms by restricting to the regular locus of X, by the fact that $\mathcal{O}_X(K_X)$ is a reflexive sheaf, and by Lemma 4.3.3 and Theorem 4.3.5. By Grothendieck duality for finite morphisms (see Theorem 4.4.1), the sheaf in the bottom left corner satisfies

$$\begin{aligned}
\mathscr{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(p^eK_X + D)\big), \mathcal{O}_X(K_X)\big) &\simeq F^e_*\,\mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X(p^eK_X + D), F^{e!}\mathcal{O}_X(K_X)\big) \\
&\simeq F^e_*\,\mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X(p^eK_X + D), \mathcal{O}_X(K_X)\big) \\
&\simeq F^e_*\mathcal{O}_X\big((1-p^e)K_X - D\big)
\end{aligned}$$

where the second isomorphism follows from the fact that $F^{e!}\omega_X \simeq \omega_X$ by Definition 4.4.2 and Theorem 4.4.1, and the last isomorphism follows from restricting to the regular locus of X and using the reflexivity of the sheaves involved [Har94, Prop. 2.7]. We can therefore define θ to be the composition of isomorphisms

$$\begin{aligned} F^e_* \mathcal{O}_X\big((1-p^e)K_X - D\big) &\simeq \mathscr{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(p^e K_X + D)\big), \mathcal{O}_X(K_X)\big) \\ &\simeq \mathscr{H}om_{\mathcal{O}_X}\big(F^e_*\big(\mathcal{O}_X(D)\big), \mathcal{O}_X\big). \end{aligned}$$

Note that θ is an isomorphism of left-$F^e_*\mathcal{O}_X$-modules by tracing through these isomorphisms, where the left-$F^e_*\mathcal{O}_X$-module structure comes from precomposition by multiplication by an element in $F^e_*\mathcal{O}_X$. $\square$

We then use Proposition 5.4.6 to prove the following characterization of F-singularities.

Corollary 5.4.7. *Let $(X, \Delta, \mathfrak{a}^t)$ be an effective log triple such that X is F-finite and of characteristic $p > 0$.*

(i) The triple $(X, \Delta, \mathfrak{a}^t)$ is F-pure if and only if there exists an integer $e' > 0$ such that for all $e \geq e'$, the morphism

$$\begin{aligned} &F^e_*\big(\mathfrak{a}^{\lfloor (p^e-1)t \rfloor} \cdot \mathcal{O}_X\big((1-p^e)K_X - \lfloor (p^e-1)\Delta \rfloor\big)\big) \\ &\qquad \hookrightarrow F^e_*\big(\mathcal{O}_X\big((1-p^e)K_X - \lfloor (p^e-1)\Delta \rfloor\big)\big) \xrightarrow{\mathrm{Tr}^e_{X, \lfloor (p^e-1)\Delta \rfloor}} \mathcal{O}_X \end{aligned}$$

is surjective.

(ii) The triple $(X, \Delta, \mathfrak{a}^t)$ is sharply F-pure if and only if there exists an integer $e > 0$ such that the morphism

$$\begin{aligned} &F^e_*\big(\mathfrak{a}^{\lceil (p^e-1)t \rceil} \cdot \mathcal{O}_X\big((1-p^e)K_X - \lceil (p^e-1)\Delta \rceil\big)\big) \\ &\qquad \hookrightarrow F^e_*\big(\mathcal{O}_X\big((1-p^e)K_X - \lceil (p^e-1)\Delta \rceil\big)\big) \xrightarrow{\mathrm{Tr}^e_{X, \lceil (p^e-1)\Delta \rceil}} \mathcal{O}_X \end{aligned}$$

is surjective.

(iii) The triple $(X, \Delta, \mathfrak{a}^t)$ is strongly F-regular if and only if for every Cartier divisor E on X, there there exists an integer $e > 0$ such that the morphism

$$\begin{aligned} &F^e_*\big(\mathfrak{a}^{\lfloor (p^e-1)t \rfloor} \cdot \mathcal{O}_X\big((1-p^e)K_X - \lfloor (p^e-1)\Delta \rfloor - E\big)\big) \\ &\qquad \hookrightarrow F^e_*\big(\mathcal{O}_X\big((1-p^e)K_X - \lfloor (p^e-1)\Delta \rfloor - E\big)\big) \xrightarrow{\mathrm{Tr}^e_{X, \lfloor (p^e-1)\Delta \rfloor + E}} \mathcal{O}_X \end{aligned}$$

is surjective.

Proof. We first consider the case of a pair (X, Δ). Let D stand for one of $\lfloor (p^e - 1)\Delta \rfloor$, $\lceil (p^e - 1)\Delta \rceil$, or $\lfloor (p^e - 1)\Delta \rfloor + E$. By Proposition 5.4.6, we see that $\mathrm{Tr}^e_{X,D}$ is surjective if and only if

$$(F^e_D)^* \colon \mathscr{H}om_{\mathcal{O}_X}\bigl(F^e_*\bigl(\mathcal{O}_X(D)\bigr), \mathcal{O}_X\bigr) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

is surjective. Since X is F-finite, these morphisms are surjective if and only if for every $x \in X$, the morphism

$$(F^e_D)^* \colon \mathrm{Hom}_{\mathcal{O}_{X,x}}\bigl(F^e_*\bigl(\mathcal{O}_{X,x}(D)\bigr), \mathcal{O}_{X,x}\bigr) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x})$$

is surjective. Finally, this condition is equivalent to the splitting of the map

$$\mathcal{O}_{X,x} \longrightarrow F^e_*\mathcal{O}_{X,x} \hookrightarrow F^e_*\bigl(\mathcal{O}_{X,x}(D)\bigr),$$

hence all three statements follow by comparing this condition to Definition 5.4.1.

Finally, the case for a triple $(X, \Delta, \mathfrak{a}^t)$ follows from the fact that under the isomorphism θ in Proposition 5.4.6, the multiplication $F^e_*(- \cdot d)$ or $F^e_*(- \cdot cd)$ in Definition 5.4.1 corresponds to precomposition of the trace $\mathrm{Tr}^e_{X,D}$ by multiplication by an element in $\mathfrak{a}^{\lfloor (p^e-1)t \rfloor}$ or $\mathfrak{a}^{\lceil (p^e-1)t \rceil}$. $\square$

5.4.2. *F*-pure thresholds

We define the F-pure threshold, which is the positive characteristic analogue of the log canonical threshold.

Definition 5.4.8 [TW04, Def. 2.1]**.** Let $(X, \Delta, \mathfrak{a})$ be an effective log triple such that X is F-finite of characteristic $p > 0$. The *F-pure threshold* of the pair (X, Δ) with respect to $\mathfrak{a}$ at a point $x \in X$ is

$$\mathrm{fpt}_x\bigl((X, \Delta); \mathfrak{a}\bigr) \coloneqq \sup\bigl\{c \in \mathbf{R}_{\geq 0} \bigm| \text{the triple } (X, \Delta, \mathfrak{a}^c) \text{ is } F\text{-pure at } x\bigr\},$$

where if (X, Δ) is not F-pure at x, then we set $\mathrm{fpt}_x((X, \Delta); \mathfrak{a}) = -\infty$.

F-pure thresholds can be very different from log canonical thresholds, even for the same defining equation.

Example 5.4.9 [MTW05, Ex. 4.3]**.** Let $R = k[\![x, y]\!]$ with maximal ideal $\mathfrak{m}$, where k is an F-finite field of characteristic $p > 0$, and let $f = x^2 + y^3$. The F-pure threshold then depends on the characteristic of k:

$$\operatorname{fpt}_{\mathfrak{m}}\bigl(\operatorname{Spec} R; x^2 + y^3\bigr) = \begin{cases} \dfrac{1}{2} & \text{if } p = 2 \\ \dfrac{2}{3} & \text{if } p = 3 \\ \dfrac{5}{6} & \text{if } p \equiv 1 \bmod 3 \\ \dfrac{5}{6} - \dfrac{1}{6p} & \text{if } p \equiv 2 \bmod 3 \text{ and } p \neq 2 \end{cases}$$

We see that as $p \to \infty$, the F-pure threshold approaches the log canonical threshold as computed in Example 4.8.8, as predicted by Theorem 5.6.8. For more examples of similar phenomena, see [TW04, Exs. 2.4 and 2.5; MTW05, §4; CHSW16].

5.5. Test ideals

We review the theory of test ideals, which are the positive characteristic analogues of multiplier ideals. Test ideals for rings were originally defined by Hochster and Huneke [HH90, Def. 8.22] using tight closure, and versions for pairs and triples were first defined by Hara–Yoshida [HY03, Def.-Thm. 6.5] and Takagi [Tak04a, Def. 2.6; Tak08, Def. 2.2] using generalized versions of tight closure; see Remark 5.5.14. While test ideals can be defined in this way without F-finiteness assumptions, we will assume F-finiteness throughout and define test ideals using the notion of F-compatibility, following Schwede [Sch10b]. We recommend [ST12, §6; TW18, §5] for surveys on this topic.

We start with the following definition.

Definition 5.5.1 [Sch10b, Def. 3.1]**.** Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is an F-finite ring of characteristic $p > 0$. An ideal $J \subseteq R$ is *uniformly* $(\Delta, \mathfrak{a}^t, F)$-*compatible* if for every integer $e > 0$ and every $\varphi \in \operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$, we have

$$\varphi\Big(F^e_*\big(J \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil}\big)\Big) \subseteq J. \tag{5.10}$$

We drop Δ or $\mathfrak{a}^t$ from our notation when working with pairs or the ring itself.

If $(R, \Delta, \mathfrak{b}^s)$ is another effective log triple on R, then we can analogously define uniform $(\Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s, F)$-compatibility by using the ideal $\mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathfrak{b}^{\lceil s(p^e-1)\rceil}$ in (5.10).

We can now define test ideals.

Definition 5.5.2 [Sch10b, Def. 3.1 and Thm. 6.3]**.** Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is an F-finite ring of characteristic $p > 0$. The *test ideal*

$$\tau(R, \Delta, \mathfrak{a}^t) \subseteq R$$

is the smallest ideal which is uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible and whose intersection with R° is nonempty. We drop Δ or $\mathfrak{a}^t$ from our notation when working with pairs or the ring itself. We also often drop the ring R from our notation if it is clear from context.

If $(R, \Delta, \mathfrak{b}^s)$ is another effective log triple on R, then we can analogously define the test ideal $\tau(R, \Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s)$ as the smallest uniformly $(\Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s, F)$-compatible ideal that intersects R°.

The test ideal as defined in Definition 5.5.2 exists since it matches the earlier notion (see Remark 5.5.14) defined using tight closure [Sch10b, Thm. 6.3]. We briefly describe a direct proof of existence, following [Sch11]. The key ingredient is the following:

Definition 5.5.3 (cf. [Sch11, Def. 3.19])**.** Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is an F-finite ring of characteristic $p > 0$. An element $c \in R^\circ$ is a *big sharp test element* for $(R, \Delta, \mathfrak{a}^t)$ if, for every $d \in R^\circ$, there exists $\varphi \in \operatorname{Hom}_R(F^e_*R(\lceil (p^e-1)\Delta\rceil), R)$ for some integer $e > 0$ such that

$$c \in \varphi\Big(F^e_*\big(d \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil}\big)\Big). \tag{5.11}$$

If c is a big sharp test element, then $c'c$ is also for all $c' \in R^\circ$ by considering the composition $(c' \cdot -) \circ \varphi$ for φ as in (5.11).

If $(R, \Delta, \mathfrak{b}^s)$ is another effective log triple on R, then we can analogously define the big sharp test elements for $(R, \Delta, \mathfrak{a}^t)$ and $(R, \Delta, \mathfrak{b}^s)$ by using the ideal $\mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathfrak{b}^{\lceil s(p^e-1)\rceil}$ in (5.11).

Various versions of test elements were shown to exist in the context of tight closure; see [HH90, §6; HY03, Thm. 6.4; Tak04a, Thm. 2.5(2); Hoc07, Thm. on p. 90]. Big sharp test elements as defined in Definition 5.5.3 exist by [Sch10b, Lem. 2.17; Sch11, Prop. 3.21]. Assuming this fact, we can show that test ideals exist. Note that the description (5.12) is originally due to Hara and Takagi [HT04, Lem. 2.1].

Theorem 5.5.4 [Sch11, Thm. 3.18]**.** *Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is an F-finite ring of characteristic $p > 0$. Then, for every choice of big sharp test element $c \in R^\circ$ for the triple $(R, \Delta, \mathfrak{a}^t)$, we have*

$$\tau(R, \Delta, \mathfrak{a}^t) = \sum_{e=0}^{\infty} \sum_{\varphi_e} \varphi_e\Big(F^e_*\big(c \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil}\big)\Big), \tag{5.12}$$

where φ_e ranges over all elements in $\operatorname{Hom}_R(F^e_* R(\lceil (p^e-1)\Delta\rceil), R)$. In particular, the test ideal $\tau(R, \Delta, \mathfrak{a}^t)$ exists. If $(R, \Delta, \mathfrak{b}^s)$ is another effective log triple, then $\tau(R, \Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s)$ exists by replacing c with a big sharp test element for $(R, \Delta, \mathfrak{a}^t)$ and $(R, \Delta, \mathfrak{b}^s)$ and by using the ideal $\mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathfrak{b}^{\lceil s(p^e-1)\rceil}$ in (5.12).

Proof. By definition of a big sharp test element (Definition 5.5.3), we have $c \in J$ for every $(\Delta, \mathfrak{a}^t, F)$-compatible ideal $J \subseteq R$. On the other hand, the ideal on the right-hand side of (5.12) is the smallest $(\Delta, \mathfrak{a}^t, F)$-compatible ideal containing c, hence must coincide with $\tau(R, \Delta, \mathfrak{a}^t)$. The proof for $\tau(R, \Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s)$ is similar. □

To define test ideals on schemes, we use the following consequence of the proof of the existence of big sharp test elements.

Proposition 5.5.5 [Sch11, Prop. 3.23(*ii*)]**.** *Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is an F-finite ring of characteristic $p > 0$. For every multiplicative set $W \subseteq R$, we have*

$$W^{-1}\tau(R, \Delta, \mathfrak{a}^t) = \tau\big(W^{-1}R, \Delta|_{\operatorname{Spec} W^{-1}R}, (W^{-1}\mathfrak{a})^t\big),$$

and similarly for $\tau(R, \Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s)$.

We can now define test ideals on schemes.

Definition 5.5.6. Let $(X, \Delta, \mathfrak{a}^t)$ be an effective log triple such that X is an F-finite scheme of characteristic $p > 0$. By Proposition 5.5.5, we can define the *test ideal*

$\tau(X, \Delta, \mathfrak{a}^t) \subseteq \mathcal{O}_X$ locally on every open affine subset $U = \operatorname{Spec} R \subseteq X$ by

$$\tau(X, \Delta, \mathfrak{a}^t)(U) = \tau\bigl(R, \Delta|_U, \mathfrak{a}(U)^t\bigr).$$

We drop Δ or $\mathfrak{a}^t$ from our notation when working with pairs or the scheme itself. We also often drop the scheme X from our notation if it is clear from context.

If $(X, \Delta, \mathfrak{b}^s)$ is another effective log triple on X, then we can analogously define the test ideal $\tau(X, \Delta, \mathfrak{a}^t \cdot \mathfrak{b}^s)$.

We now state some properties of test ideals that we will use often, which are reminiscent of those for multiplier ideals in [Laz04b, §9.2].

Proposition 5.5.7 (see [TW18, Prop. 5.6])**.** *Let $(X, \Delta, \mathfrak{a}^t)$ be an effective log triple such that X is an F-finite scheme of characteristic $p > 0$.*

(i) If $(X, \mathfrak{b})$ is a log pair on X, then $\tau(\Delta, \mathfrak{a}^t) \cdot \mathfrak{b} \subseteq \tau(\Delta, \mathfrak{a}^t \cdot \mathfrak{b})$.

(ii) Let $(X, \Delta', \mathfrak{b}^s)$ be another effective log triple on X. If $\Delta \geq \Delta'$ and $\mathfrak{a}^{\lceil t(p^e-1)\rceil} \subseteq \mathfrak{b}^{\lceil s(p^e-1)\rceil}$ for every integer $e > 0$, then $\tau(\Delta, \mathfrak{a}^t) \subseteq \tau(\Delta', \mathfrak{b}^s)$.

(iii) For every non-negative real number s, we have $\tau(\Delta, \mathfrak{a}^s \cdot \mathfrak{a}^t) = \tau(\Delta, \mathfrak{a}^{s+t})$.

(iv) For every non-negative integer m, we have $\tau(\Delta, (\mathfrak{a}^m)^t) = \tau(\Delta, \mathfrak{a}^{mt})$.

(v) There exists $\varepsilon > 0$ such that for all $s \in [t, t+\varepsilon]$, we have $\tau(\Delta, \mathfrak{a}^t) = \tau(\Delta, \mathfrak{a}^s)$.

(vi) Suppose that X is normal. For every effective Cartier divisor D on X, there exists $\varepsilon > 0$ such that for all $\delta \in [0, \varepsilon]$, we have $\tau(\Delta, \mathfrak{a}^t) = \tau(\Delta + \delta D, \mathfrak{a}^t)$.

(vii) The triple $(X, \Delta, \mathfrak{a}^t)$ is strongly F-regular if and only if $\tau(\Delta, \mathfrak{a}^t) = \mathcal{O}_X$.

We will define strong F-regularity in Definition 5.4.1(c).

Proof. Since test ideals are defined locally, it suffices to consider the case when $X = \operatorname{Spec} R$. Fix a big sharp test element $c \in R^\circ$. We will freely use the description of the test ideal in Theorem 5.5.4.

To show (i), we note that

$$\begin{aligned}
\tau(\Delta, \mathfrak{a}^t) \cdot \mathfrak{b} &= \sum_{e=0}^{\infty} \sum_{\varphi_e} \varphi_e\Big(F_*^e\big(c \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil}\big)\Big) \cdot \mathfrak{b} \\
&= \sum_{e=0}^{\infty} \sum_{\varphi_e} \varphi_e\Big(F_*^e\big(c \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathfrak{b}^{[p^e]}\big)\Big) \\
&\subseteq \sum_{e=0}^{\infty} \sum_{\varphi_e} \varphi_e\Big(F_*^e\big(c \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathfrak{b}^{p^e-1}\big)\Big) = \tau(\Delta, \mathfrak{a}^t \cdot \mathfrak{b}).
\end{aligned}$$

To show (ii), it suffices to note that if an ideal $J \subseteq R$ is $(\Delta', \mathfrak{b}^s, F)$-compatible, then J is $(\Delta, \mathfrak{a}^t, F)$-compatible, since

$$\varphi\Big(F_*^e\big(J \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil}\big)\Big) \subseteq \varphi\Big(F_*^e\big(J \cdot \mathfrak{b}^{\lceil s(p^e-1)\rceil}\big)\Big) \subseteq J$$

for all

$$\varphi \in \operatorname{Hom}_R\big(F_*^e R\big(\lceil (p^e-1)\Delta\rceil\big), R\big) \subseteq \operatorname{Hom}_R\big(F_*^e R\big(\lceil (p^e-1)\Delta'\rceil\big), R\big).$$

To show (iii), we first note that the inclusion $\subseteq$ holds by (ii) since

$$\mathfrak{a}^{\lceil s(p^e-1)\rceil} \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil} \subseteq \mathfrak{a}^{\lceil (s+t)(p^e-1)\rceil}$$

for every integer $e > 0$. To show the reverse inclusion $\supseteq$, note that

$$\begin{aligned}
\emptyset \neq \mathfrak{a}^{\lceil s(p^e-1)\rceil + \lceil t(p^e-1)\rceil - \lceil (s+t)(p^e-1)\rceil} &\cap R^\circ \\
&\subseteq \Big(\big(\mathfrak{a}^{\lceil s(p^e-1)\rceil} \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil}\big) : \mathfrak{a}^{\lceil (s+t)(p^e-1)\rceil}\Big) \cap R^\circ,
\end{aligned}$$

hence we can choose an element c' in the set on the right-hand side. Then, the product cc' is a big sharp test element, hence

$$\begin{aligned}
\tau(\Delta, \mathfrak{a}^{s+t}) &= \sum_{e=0}^{\infty} \sum_{\varphi_e} \varphi_e\Big(F_*^e\big(cc' \cdot \mathfrak{a}^{\lceil (s+t)(p^e-1)\rceil}\big)\Big) \\
&\subseteq \sum_{e=0}^{\infty} \sum_{\varphi_e} \varphi_e\Big(F_*^e\big(c \cdot \mathfrak{a}^{\lceil s(p^e-1)\rceil} \cdot \mathfrak{a}^{\lceil t(p^e-1)\rceil}\big)\Big) = \tau(\Delta, \mathfrak{a}^s \cdot \mathfrak{a}^t).
\end{aligned}$$

(iv) then follows from applying (iii) m times.

See [ST14, Lem. 6.1] and [Sat18, Prop. 2.14(2)] for proofs of (v) and (vi), respectively. Note that in the proof of [Sat18, Prop. 2.14(2)], one should follow the proof of [ST14, Lem. 6.1] to reduce to the case when $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$.

See [Sch11, Prop. 3.23(iii)] for a proof of (vii). □

We give a new proof of the following very important property of test ideals.

Theorem 5.5.8 (Subadditivity [HY03, Thm. 6.10(2)])**.** *Let $(X, \mathfrak{a}^t)$ and $(X, \mathfrak{b}^s)$ be two effective log pairs where X is an F-finite regular scheme of characteristic $p > 0$. Then, we have*

$$\tau(\mathfrak{a}^t \cdot \mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^s).$$

Proof. By Proposition 5.5.5, it suffices to consider the case when $X = \operatorname{Spec} R$ for a regular local ring R. By [Sch10b, Prop. 3.11], for a regular ring R, an ideal $J \subseteq R$ is uniformly $(\mathfrak{a}^t \cdot \mathfrak{b}^s, F)$-compatible if and only if for every integer $e \geq 0$, we have

$$\mathfrak{a}^{\lceil t(p^e-1) \rceil} \cdot \mathfrak{b}^{\lceil s(p^e-1) \rceil} \subseteq (J^{[p^e]} : J).$$

It therefore suffices to show the chain of inclusions

$$\begin{aligned} \mathfrak{a}^{\lceil t(p^e-1) \rceil} \cdot \mathfrak{b}^{\lceil s(p^e-1) \rceil} &\subseteq \bigl(\tau(\mathfrak{a}^t)^{[p^e]} : \tau(\mathfrak{a}^t)\bigr) \cdot \bigl(\tau(\mathfrak{b}^s)^{[p^e]} : \tau(\mathfrak{b}^s)\bigr) \\ &\subseteq \Bigl(\bigl(\tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^s)\bigr)^{[p^e]} : \bigl(\tau(\mathfrak{a}^t)^{[p^e]} \cdot \tau(\mathfrak{b}^s)^{[p^e]}\bigr)\Bigr) \end{aligned}$$

since $\tau(\mathfrak{a}^t \cdot \mathfrak{b}^s)$ is the smallest $(\mathfrak{a}^t \cdot \mathfrak{b}^s, F)$-compatible ideal by definition. The first inclusion follows from the fact that $\tau(\mathfrak{a}^t)$ and $\tau(\mathfrak{b}^s)$ are uniformly $(\mathfrak{a}^t, F)$- and $(\mathfrak{b}^s, F)$-compatible, respectively. The second inclusion follows from the fact that in general, $(I_1 : J_1) \cdot (I_2 : J_2) \subseteq (I_1 I_2 : J_1 J_2)$. □

Remark 5.5.9. Subadditivity (Theorem 5.5.8) was originally proved by Hara and Yoshida [HY03, Thm. 6.10(2)] using tight closure. We have included a proof purely in the language of F-compatibility to be consistent with our choice of definition (Definition 5.5.2); see [BMS08, Prop. 2.11(iv)] for another approach. Our proof can also be used to show a more general form of subadditivity: if $(X, \mathfrak{a}_\bullet)$ and $(X, \mathfrak{b}_\bullet)$ are two pairs as in [Sch10b, Def. 2.3], then

$$\tau(\mathfrak{a}_\bullet \cdot \mathfrak{b}_\bullet) \subseteq \tau(\mathfrak{a}_\bullet) \cdot \tau(\mathfrak{b}_\bullet).$$

Here, the test ideal is as described in [Sch11, Thm. 6.3]. We have avoided this notation since it clashes with that of asymptotic test ideals below.

Because of the formal properties of test ideals in Proposition 5.5.7, we can define the following asymptotic version of test ideals.

Definition 5.5.10 [Sat18, Prop.-Def. 2.16]**.** Let $(X, \Delta, \mathfrak{a}_\bullet^\lambda)$ be an effective log triple such that X is F-finite and of characteristic $p > 0$. If m and r are positive integers, then

$$\tau(X, \Delta, \mathfrak{a}_m^{\lambda/m}) = \tau\bigl(X, \Delta, (\mathfrak{a}_m^r)^{\lambda/(mr)}\bigr) \subseteq \tau\bigl(X, \Delta, \mathfrak{a}_{mr}^{\lambda/(mr)}\bigr),$$

by Propositions 5.5.7(iv) and 5.5.7(ii), and by the graded property $\mathfrak{a}_m^r \subseteq \mathfrak{a}_{mr}$. Thus, the set of ideals

$$\bigl\{\tau(X, \Delta, \mathfrak{a}_m^{\lambda/m})\bigr\}_{m=1}^{\infty} \tag{5.13}$$

is partially ordered, and has a unique maximal element by the noetherian property that coincides with $\tau(X, \Delta, \mathfrak{a}_m^{\lambda/m})$ for m sufficiently large and divisible. The *asymptotic test ideal*

$$\tau(X, \Delta, \mathfrak{a}_\bullet^\lambda) \subseteq \mathcal{O}_X$$

is the maximal element of the partially ordered set (5.13).

If $(X, \Delta, \mathfrak{b}_\bullet^\mu)$ is another effective log triple on X, then we can analogously define the test ideal $\tau(X, \Delta, \mathfrak{a}_\bullet^\lambda \cdot \mathfrak{b}_\bullet^\mu)$.

Asymptotic test ideals satisfy properties analogous to those in Proposition 5.5.7 and Theorem 5.5.8.

Remark 5.5.11. Definition 5.5.10 is due to Mustaţă when X is regular and $\Delta = 0$ [Mus13, pp. 540–541]. An asymptotic version of the test ideal was first defined by Hara [Har05, Prop.-Def. 2.9], although this ideal differs from that in Definition 5.5.10 in general; see [TY08, Rem. 1.4].

The following examples of test ideals will be the most useful in our applications.

Example 5.5.12 (see [Sat18, Def. 2.36])**.** Suppose (X, Δ) is an effective log pair where X is a complete scheme over an F-finite field of characteristic $p > 0$. If D is a Cartier divisor such that $H^0(X, \mathcal{O}_X(D)) \neq 0$, then for every real number $t \geq 0$, we set

$$\tau\bigl(X, \Delta, t \cdot |D|\bigr) \coloneqq \tau\bigl(X, \Delta, \mathfrak{b}\bigl(|D|\bigr)^t\bigr).$$

If D is a $\mathbf{Q}$-Cartier divisor such that $H^0(X, \mathcal{O}_X(mD)) \neq 0$ for some $m > 0$ such that mD is Cartier, then for every real number $\lambda \geq 0$, we set

$$\tau\bigl(X, \Delta, \lambda \cdot \lVert D \rVert\bigr) \coloneqq \tau\bigl(X, \Delta, \mathfrak{a}_\bullet(D)^\lambda\bigr),$$

where $\mathfrak{a}_\bullet(D)$ is the graded family of ideals defined in Example 4.5.4.

Finally, we have the following uniform global generation result for (asymptotic) test ideals.

Theorem 5.5.13 ([Sat18, Prop. 4.1]; cf. [Sch14, Thm. 4.3; Mus13, Thm. 4.1])**.** *Let (X, Δ) be an effective log pair where X is a normal projective variety over an F-finite field of characteristic $p > 0$ and Δ is an effective $\mathbf{Q}$-Weil divisor such that $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Let D, L, and H be Cartier divisors on X such that H is ample and free. If λ is a non-negative real number such that $L - (K_X + \Delta + \lambda \cdot D)$ is ample, then the sheaves*

$$\tau\bigl(X, \Delta, \lambda \cdot |D|\bigr) \otimes \mathcal{O}_X(L + dH) \quad \textit{and} \quad \tau\bigl(X, \Delta, \lambda \cdot \lVert D \rVert\bigr) \otimes \mathcal{O}_X(L + dH)$$

are globally generated for every integer $d > \dim X$.

Remark 5.5.14 (Test ideals via tight closure)*.* We briefly recall an alternative definition for test ideals via tight closure, following [Tak08, §2] and [Sch10b, §2.2]. Let $(R, \Delta, \mathfrak{a}^t)$ be an effective log triple such that R is a ring of characteristic $p > 0$, and let $\iota \colon N \hookrightarrow M$ be an inclusion of R-modules. For every integer $e > 0$, let

$$N_M^{[p^e],\Delta} \coloneqq \operatorname{im}\Bigl(N \otimes_R F^e_* R \xrightarrow{\iota \otimes_R \mathrm{id}} M \otimes_R F^e_* R \longrightarrow M \otimes_R F^e_* R\bigl(\lfloor (p^e - 1)\Delta \rfloor\bigr)\Bigr).$$

The $(\Delta, \mathfrak{a}^t)$-*tight closure* of N in M is the R-module

$$N_M^{*(\Delta,\mathfrak{a}^t)} \coloneqq \left\{ z \in M \;\middle|\; \begin{array}{c} \text{there exists } c \in R^\circ \text{ such that} \\ z \otimes c\mathfrak{a}^{\lceil p^e t \rceil} \subseteq N_M^{[p^e],\Delta} \text{ for all } e \gg 0 \end{array} \right\}.$$

Now let $E \coloneqq \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum of the injective hulls of the residue fields $R/\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subseteq R$. The *test ideal* of $(R, \Delta, \mathfrak{a}^t)$ is

$$\tau(R, \Delta, \mathfrak{a}^t) \coloneqq \operatorname{Ann}_R\bigl(0_E^{*(\Delta,\mathfrak{a}^t)}\bigr) \subseteq R.$$

By [Sch10b, Thm. 6.3], this test ideal is equal to the test ideal defined in Definition 5.5.2 as long as R is F-finite.

Remark 5.5.15 (Big vs. finitistic test ideals). Our test ideals correspond to the (*non-finitistic* or *big*) test ideal defined by Lyubeznik and Smith [LS01, §7] when $\Delta = 0$, $\mathfrak{a} = R$, and $t = 1$, instead of the original (*finitistic*) test ideal defined by Hochster and Huneke [HH90, Def. 8.22]. Note that the definitions for test ideals of pairs in [HY03, Def. 1.1; Tak04a, Def. 2.1] specialize to the finitistic test ideal. The corresponding non-finitistic notion first appears in [HT04, Def. 1.4].

5.6. Reduction modulo $\mathfrak{p}$

Finally, we review the theory of reduction modulo $\mathfrak{p}$, and the relationship between singularities in characteristic zero and characteristic $p > 0$. What follows is a small part of the general discussion in [EGAIV$_3$, §8].

Setup 5.6.1 [EGAIV$_3$, (8.2.2), (8.5.1), and (8.8.1)]**.** We will denote by $\{(S_\lambda, u_{\lambda\mu})\}_{\lambda\in\Lambda}$ a filtered inverse system of schemes with affine transition morphisms $u_{\lambda\mu}\colon S_\mu \to S_\lambda$ for $\lambda \le \mu$, where Λ has a unique minimal element 0. We then set $S \coloneqq \varprojlim_{\lambda\in\Lambda} S_\lambda$ with projection morphisms $u_\lambda\colon S \to S_\lambda$.

Now suppose an element $\alpha \in \Lambda$ and schemes X_α and Y_α over S_α are given. We then denote by

$$\bigl\{(X_\lambda, v_{\lambda\mu})\bigr\}_{\lambda\in\Lambda} \qquad \text{and} \qquad \bigl\{(Y_\lambda, w_{\lambda\mu})\bigr\}_{\lambda\in\Lambda}$$

the inverse systems induced by $\{(S_\lambda, u_{\lambda\mu})\}$, where

$$\begin{aligned} X_\lambda &\coloneqq X_\alpha \times_{S_\alpha} S_\lambda \\ v_{\lambda\mu} &\coloneqq \mathrm{id}_{X_\alpha} \times u_{\lambda\mu} \end{aligned} \qquad \text{and} \qquad \begin{aligned} Y_\lambda &\coloneqq Y_\alpha \times_{S_\alpha} S_\lambda \\ w_{\lambda\mu} &\coloneqq \mathrm{id}_{Y_\alpha} \times u_{\lambda\mu} \end{aligned}$$

for $\alpha \le \lambda \le \mu$. The inverse limits of these inverse systems are $X = X_\alpha \times_{S_\alpha} S$ and $Y = Y_\alpha \times_{S_\alpha} S$, respectively, with projection morphisms $v_\lambda\colon X \to X_\lambda$ and $w_\lambda\colon Y \to Y_\lambda$. We then have the following canonical map of sets:

$$\varinjlim_{\lambda\in\Lambda} \mathrm{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \longrightarrow \mathrm{Hom}_S(X, Y). \tag{5.14}$$

Similarly, suppose an element $\alpha \in \Lambda$, a scheme X_α, and $\mathcal{O}_{X_\alpha}$-modules $\mathscr{F}_\alpha$ and $\mathscr{G}_\alpha$ are given. We then denote by

$$\{\mathscr{F}_\lambda\}_{\lambda\in\Lambda} \qquad \text{and} \qquad \{\mathscr{G}_\lambda\}_{\lambda\in\Lambda}$$

the inverse systems induced by $\{(X_\lambda, v_{\lambda\mu})\}$ and $\{(Y_\lambda, w_{\lambda\mu})\}$, where

$$\mathscr{F}_\lambda \coloneqq v_{\alpha\lambda}^*(\mathscr{F}_\alpha) \qquad \text{and} \qquad \mathscr{G}_\lambda \coloneqq w_{\alpha\lambda}^*(\mathscr{G}_\alpha)$$

for $\alpha \leq \lambda \leq \mu$. Note that these families verify the conditions $\mathscr{F}_\mu = v_{\lambda\mu}^*(\mathscr{F}_\lambda)$ and $\mathscr{G}_\mu = w_{\lambda\mu}^*(\mathscr{G}_\lambda)$. The $\mathcal{O}_X$-modules $\mathscr{F} = v_\alpha^*(\mathscr{F}_\alpha)$ and $\mathscr{G} = w_\alpha^*(\mathscr{G}_\alpha)$ then satisfy $\mathscr{F} = v_\lambda^*(\mathscr{F}_\lambda)$ and $\mathscr{G} = w_\lambda^*(\mathscr{G}_\lambda)$ for every $\lambda \geq \alpha$, and we have the following canonical map of abelian groups:

$$\varinjlim_{\lambda\in\Lambda} \operatorname{Hom}_{\mathcal{O}_{X_\lambda}}(\mathscr{F}_\lambda, \mathscr{G}_\lambda) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}). \tag{5.15}$$

Theorem 5.6.2 (Spreading out; see [EGAIV$_3$, Thms. 8.8.2 and 8.5.2])**.** *Fix notation as in Setup 5.6.1.*

(i) *Suppose S_0 is quasi-compact and quasi-separated. For every scheme X of finite presentation over S, there exists $\lambda \in \Lambda$, a scheme X_λ of finite presentation over S_λ, and an S-isomorphism $X \xrightarrow{\sim} X_\lambda \times_{S_\lambda} S$.*

(ii) *Suppose X_α is quasi-compact (resp. quasi-compact and quasi-separated) over S_α, and Y_α is locally of finite type (resp. locally of finite presentation) over S_α for some $\alpha \in \Lambda$. Then, the map* (5.14) *is injective (resp. bijective).*

(iii) *Suppose X_α is quasi-compact and quasi-separated over S_α, and that S_α is quasi-compact and quasi-separated. For every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ of finite presentation, there exists $\lambda \in \Lambda$ and a quasi-coherent $\mathcal{O}_{X_\lambda}$-module $\mathscr{F}_\lambda$ of finite presentation such that $\mathscr{F}$ is isomorphic to $u_\lambda^*(\mathscr{F}_\lambda)$.*

(iv) *Suppose X_α is quasi-compact (resp. quasi-compact and quasi-separated) and that $\mathscr{F}_\lambda$ is quasi-coherent of finite type (resp. of finite presentation) and $\mathscr{G}_\lambda$ is quasi-coherent for some $\alpha \in \Lambda$. Then, the map* (5.15) *is injective (resp. bijective).*

We give the resulting objects in Theorem 5.6.2 a name.

Property of morphism of schemes	Proof
closed immersion	[EGAIV$_3$, Thm. 8.10.5(iv)]
flat	[EGAIV$_3$, Thm. 11.2.6(ii)]
projective	[EGAIV$_3$, Thm. 8.10.5($xiii$)]
proper	[EGAIV$_3$, Thm. 8.10.5(xii)]
separated	[EGAIV$_3$, Thm. 8.10.5(v)]
smooth	[EGAIV$_4$, Thm. 17.7.8(ii)]
Property of sheaf	**Proof**
flat	[EGAIV$_3$, Thm. 11.2.6(ii)]
locally free of rank n	[EGAIV$_3$, Prop. 8.5.5]

Table 5.1: Some properties preserved under spreading out
We assume that S_0 is quasi-compact and quasi-separated
and that X_α and Y_α are of finite presentation over S_α.

Definition 5.6.3. Fix notation as in Setup 5.6.1. We say that X_λ (resp. $\mathscr{F}_\lambda$) is a *model* of X (resp. $\mathscr{F}$) over S_λ in the situation of Theorem 5.6.2(i) (resp. 5.6.2(iii)). If in the situation of Theorem 5.6.2(ii) (resp. 5.6.2(iv)), the map in (5.14) (resp. (5.15)) is bijective, and f_λ (resp. φ_λ) is a lift of $f \in \operatorname{Hom}_S(X,Y)$ (resp. $\varphi \in \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$) under this map, then we also say that f_λ (resp. φ_λ) is a *model* of f (resp. φ) over S_λ.

Now let $\mathcal{P}$ be a property of schemes (resp. morphisms of schemes, modules, morphisms of modules). If a model X_λ (resp. f_λ, $\mathscr{F}_\lambda$, φ_λ) can always be chosen such that X (resp. f, $\mathscr{F}$, φ) has $\mathcal{P}$ if and only if X_λ (resp. f_λ, $\mathscr{F}_\lambda$, φ_λ) has $\mathcal{P}$, then we say that $\mathcal{P}$ is *preserved under spreading out*.

We record in Table 5.1 some properties of schemes, morphisms, sheaves, and morphisms of sheaves that can be descended to a model that we will use. See the properties labeled (IND) in [GW10, App. C] and the properties in the "spreading out" column in [Poo17, App. C.1, Table 1] for more exhaustive lists.

We now specialize to the case where $S = \operatorname{Spec} k$ for a field k of characteristic zero.

Definition 5.6.4. Let k be a field of characteristic zero, and write $k = \varinjlim_{\lambda\in\Lambda} A_\lambda$, where the rings A_λ are finite type extensions of $\mathbf{Z}$ in k. Let $S_\lambda = \operatorname{Spec} A_\lambda$ in Setup 5.6.1. Given models over S_λ as in Definition 5.6.3, for every closed point $\mathfrak{p} \in \operatorname{Spec} A_\lambda$, we say that $X_\mathfrak{p} \coloneqq X_\lambda \times_{A_\lambda} \kappa(\mathfrak{p})$ (resp. $\mathscr{F}_\mathfrak{p} \coloneqq \mathscr{F}|_{X_\mathfrak{p}}$, $f_\mathfrak{p} \coloneqq f_\lambda|_{X_\mathfrak{p}} \colon X_\mathfrak{p} \to Y_\mathfrak{p}$, $\varphi_\mathfrak{p} \coloneqq \varphi|_{\mathscr{F}_\mathfrak{p}} \colon \mathscr{F}_\mathfrak{p} \to \mathscr{G}_\mathfrak{p}$) is the *reduction modulo* $\mathfrak{p}$ of X (resp. $\mathscr{F}$, f, φ).

Now let $\mathcal{P}$ be a property of schemes (resp. morphisms of schemes, modules, morphisms of modules). If a model X_λ (resp. f_λ, $\mathscr{F}_\lambda$, φ_λ) can always be chosen such that X (resp. f, $\mathscr{F}$, φ) has $\mathcal{P}$ if and only if $X_\mathfrak{p}$ (resp. $f_\mathfrak{p}$, $\mathscr{F}_\mathfrak{p}$, $\varphi_\mathfrak{p}$) has $\mathcal{P}$ for every $\mathfrak{p} \in \operatorname{Spec} A_\lambda$, then we say that $\mathcal{P}$ is *preserved under reduction modulo* $\mathfrak{p}$.

An important feature of reduction modulo $\mathfrak{p}$ is the following:

Lemma 5.6.5. *With notation as in Setup 5.6.1 and Definition 5.6.4, for every $\lambda \in \Lambda$, the residue fields $\kappa(\mathfrak{p})$ of A_λ are finite fields for every $\mathfrak{p} \in \operatorname{Spec} A_\lambda$. Moreover, the set $\{\operatorname{char} \kappa(\mathfrak{p})\}_{\mathfrak{p} \in \operatorname{Spec} A_\lambda} \subseteq \mathbf{N}$ is unbounded for every $\lambda \in \Lambda$.*

Proof. The first statement is [EGAIV$_3$, Lem. 10.4.11.1]. For the second, consider the morphism $u_\lambda \colon \operatorname{Spec} A_\lambda \to \operatorname{Spec} \mathbf{Z}$, which is of finite type. By Chevalley's theorem [EGAIV$_1$, Thm. 1.8.4], the image of u_λ is constructible. Moreover, since $\mathbf{Z} \to A_\lambda$ is injective, the morphism u_λ is dominant, and in particular the image contains $(0) \in \operatorname{Spec} \mathbf{Z}$. Thus, the image of u_λ is open, and therefore contains points $\mathfrak{p} \in \operatorname{Spec} A_\lambda$ with residue fields of unbounded characteristic. $\square$

We record in Table 5.2 some properties of schemes, morphisms, sheaves, and morphisms of sheaves that are preserved under reduction modulo $\mathfrak{p}$. Note that these properties are constructible on $\operatorname{Spec} A_\lambda$, hence for arbitrary models, as long as the original object over k satisfied the property listed, these properties will hold when $\operatorname{char} \kappa(\mathfrak{p})$ is sufficiently large.

We will also need to spread out more than what we have discussed above. We discuss these operations below.

Remark 5.6.6 (Spreading out and reduction modulo $\mathfrak{p}$ for other objects). Fix notation as in Definition 5.6.4. We will freely use the properties in Tables 5.1 and 5.2.

(*a*) (Ideal sheaves) Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf. We can then spread out $\mathfrak{a}$ and the inclusion into $\mathcal{O}_X$ to a model $\mathfrak{a}_\lambda \to \mathcal{O}_{X_\lambda}$. We can further assume that $\mathfrak{a}_\mathfrak{p} \to \mathcal{O}_{X_\mathfrak{p}}$ is injective for all $\mathfrak{p} \in \operatorname{Spec} A_\lambda$.

(*b*) (Cartier divisors) Let D be an effective Cartier divisor on X. We can then spread out the ideal sheaf $\mathcal{O}_X(-D)$ to a model $\mathcal{O}_{X_\lambda}(-D_\lambda)$ on X_λ, which remains invertible. Thus, $D_\mathfrak{p}$ is an effective Cartier divisor for all $\mathfrak{p} \in \operatorname{Spec} A_\lambda$, since $\mathcal{O}_{X_\mathfrak{p}}(-D_\mathfrak{p}) \to \mathcal{O}_{X_\mathfrak{p}}$

Property of scheme	Proof
dimension n (when X irreducible)	[EGAIV$_3$, Cor. 9.5.6]
geometrically irreducible	[EGAIV$_3$, Thm. 9.7.7(i)]
geometrically normal	[EGAIV$_3$, Prop. 9.9.4(iii)]
geometrically reduced	[EGAIV$_3$, Thm. 9.7.7(iii)]
Property of sheaf	**Proof**
(very) ample over k (when X/k proper)	[EGAIV$_3$, Prop. 9.6.3]
Property of morphism of sheaves	**Proof**
bijective	[EGAIV$_3$, Cor. 9.4.5]
injective	[EGAIV$_3$, Cor. 9.4.5]
surjective	[EGAIV$_3$, Cor. 9.4.5]

Table 5.2: Some properties preserved under reduction modulo $\mathfrak{p}$

is injective for all $\mathfrak{p} \in \operatorname{Spec} A_\lambda$. This can be extended to arbitrary Cartier divisors and to **Q**- and **R**-coefficients by linearity.

(c) (Weil divisors) Suppose X is irreducible, and suppose D is a prime Weil divisor on X. Then, one can find $\lambda \in \Lambda$ such that X and D have models X_λ and D_λ over S_λ such that every $D_\mathfrak{p}$ is a prime Weil divisor (by preserving dimension, integrality, and the fact that $D \hookrightarrow X$ is a closed immersion) on $X_\mathfrak{p}$ (by preserving irreducibility and dimension of X). This can be extended to arbitrary Weil divisors and to **Q**- and **R**-coefficients by linearity.

If X is normal, and D is a Cartier divisor (resp. **Q**-Cartier divisor, **R**-Cartier divisor) on X, then by simultaneously choosing models for a Cartier divisor (resp. **Q**-Cartier divisor, **R**-Cartier divisor) and the Weil divisor (resp. **Q**-Weil divisor, **R**-Weil divisor) associated to it, we can preserve the property of being a Cartier divisor (resp. **Q**-Cartier divisor, **R**-Cartier divisor) under reduction modulo $\mathfrak{p}$.

5.6.1. Singularities vs. *F*-singularities

We can now define the following notions in characteristic zero obtained via reduction modulo $\mathfrak{p}$. See Definition A.8 for the definition of F-injective singularities in positive characteristic.

$$
\begin{tikzcd}
\text{klt} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[rr, Leftrightarrow, bend left] & \text{rational} \arrow[d, Rightarrow] \arrow[rr, Leftrightarrow, bend left] & \text{strongly } F\text{-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-rational} \arrow[d, Rightarrow] \\
\text{log canonical} \arrow[r, Rightarrow] & \text{Du Bois} & F\text{-pure} \arrow[r, Rightarrow] \arrow[ll, Rightarrow, bend left] & F\text{-injective} \arrow[ll, Rightarrow, bend left]
\end{tikzcd}
$$

Figure 5.1: Singularities vs. *F*-singularities
The left- and right-hand sides of the diagram are connected via reduction modulo $\mathfrak{p}$.
This is a simplified version of [ST14, Fig. on p. 86]. See [ST14, p. 86] for references for each implication.

Definition 5.6.7. Fix notation as in Setup 5.6.1, Definition 5.6.4, and Remark 5.6.6. Let X be a scheme of finite type over a field k of characteristic zero. We say that X is of *F-injective type* (resp. *dense F-injective type*) if there exists a model X_λ over A_λ such that $X_\mathfrak{p}$ is F-injective for an open dense (resp. dense) set of closed points $\mathfrak{p} \in \operatorname{Spec} A_\lambda$.

Now let $(X, \Delta, \mathfrak{a})$ be an effective log triple such that X is normal and of finite type over a field k of characteristic zero. Fix models X_λ, Δ_λ, and $\mathfrak{a}_\lambda$ over $\operatorname{Spec} A_\lambda$. We say that $(X, \Delta, \mathfrak{a})$ is of *F-pure type* (resp. *dense F-pure type*) if $(X_\mathfrak{p}, \Delta_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}^t)$ is F-pure for an open dense (resp. dense) set of closed points $\mathfrak{p} \in \operatorname{Spec} A_\lambda$. We say that $(X, \Delta, \mathfrak{a})$ is of *strongly F-regular type* (resp. *dense strongly F-regular type*) if $(X_\mathfrak{p}, \Delta_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}^t)$ is strongly F-regular for an open dense (resp. dense) set of closed points $\mathfrak{p} \in \operatorname{Spec} A_\lambda$.

One can define similar notions for all F-singularities of rings and of pairs and triples. The notions defined above are those that appear in the sequel.

We will need the following result connecting singularities of pairs and F-singularities of pairs, which relates multiplier ideals and test ideals under reduction modulo $\mathfrak{p}$. For rings, this result is due to Smith [Smi00b, Thm. 3.1] and Hara [Har01, Thm. 5.9], and for pairs, this result is due to Takagi [Tak04a, Thm. 3.2] and Hara–Yoshida [HY03, Thm. 6.8]. There are many more results describing how singularities and F-singularities are related, which we will not state explicitly; see Figure 5.1 for a summary of what is known.

Theorem 5.6.8 (see [Tak08, Thm. 2.5])**.** *Let $(X, \Delta, \mathfrak{a})$ be an effective log triple such that X is normal and finite type over a field k of characteristic zero, and such that $K_X + \Delta$*

is $\mathbf{R}$*-Cartier. With notation as in Setup 5.6.1, Definition 5.6.4, and Remark 5.6.6, fix models* X_λ*,* Δ_λ*, and* $\mathfrak{a}_\lambda$ *over* $\operatorname{Spec} A_\lambda$*. Then, for all* $t \geq 0$*, we have*

$$\mathcal{J}\big((X,\Delta);\mathfrak{a}^t\big)_{\mathfrak{p}} = \tau\big((X_{\mathfrak{p}},\Delta_{\mathfrak{p}});\mathfrak{a}_{\mathfrak{p}}^t\big) \tag{5.16}$$

when $\operatorname{char}\kappa(\mathfrak{p})$ *is sufficiently large. In particular,* $(X,\Delta,\mathfrak{a}^t)$ *is klt if and only if* $(X,\Delta,\mathfrak{a}^t)$ *is of strongly* F*-regular type. Moreover, for every sequence of closed points* $\mathfrak{p} \in \operatorname{Spec} A_\lambda$ *such that the characteristic of* $\kappa(\mathfrak{p})$ *goes to infinity, we have that the limit of the* F*-pure thresholds* $\operatorname{fpt}_x((X_{\mathfrak{p}},\Delta_{\mathfrak{p}});\mathfrak{a}_{\mathfrak{p}})$ *is the log canonical threshold* $\operatorname{lct}_x((X,\Delta);\mathfrak{a})$*.*

Note that implicit in the statement of Theorem 5.6.8 is that both objects in (5.16) make sense. For the left-hand side, this requires choosing a model of a log resolution as well, from which one obtains a model of $\mathcal{J}((X,\Delta);\mathfrak{a}^t)$. We also note that the characteristic of $\kappa(\mathfrak{p})$ is unbounded by Lemma 5.6.5.

Proof. All but the last part of the statement of Theorem 5.6.8 is proved in [Tak08, Thm. 2.5]. To prove this last statement, let $\{\mathfrak{p}_i\}_{i\in\mathbf{N}}$ be a sequence of closed points in $\operatorname{Spec} A_\lambda$ such that $\operatorname{char}\kappa(\mathfrak{p}_i) \to \infty$ as $i \to \infty$. We claim that for every $s \geq 0$, we have

$$\operatorname{lct}_x\big((X,\Delta);\mathfrak{a}\big) \geq \operatorname{fpt}_x\big((X_{\mathfrak{p}_i},\Delta_{\mathfrak{p}_i});\mathfrak{a}_{\mathfrak{p}_i}\big) > s \tag{5.17}$$

for $i \gg 0$. The first inequality automatically holds since the inclusion $\supseteq$ in (5.16) holds for every $\mathfrak{p} \in \operatorname{Spec} A_\lambda$; see [Sch10b, Thm. 6.7]. The second inequality holds for $i \gg 0$ since (5.16) holds for $s = t$ when $\operatorname{char}\kappa(\mathfrak{p}_i)$ is sufficiently large. $\square$

Chapter 6

The ampleness criterion of de Fernex–Küronya–Lazarsfeld

In this chapter, we prove a criterion for ampleness using asymptotic cohomological functions (Theorem E), which is originally due to de Fernex, Küronya, and Lazarsfeld over the complex numbers [dFKL07, Thm. 4.1]. A key ingredient is a lemma asserting that the base ideals associated to multiples of a non-nef divisor grow at least like powers of an ideal defining a curve (Proposition 6.2.1). This material is mostly from [Mur], with some modifications in the proof of Proposition 6.2.1 using ideas from [MPST, Lems. 4.3 and 4.4].

We briefly describe the main difficulties in adapting the proof of [dFKL07, Thm. 4.1] to positive characteristic. First, the proof of [dFKL07, Prop. 3.1] requires resolutions of singularities, and because of this, we need to adapt the proof to use alterations instead. Second, we need to replace asymptotic multiplier ideals with asymptotic test ideals in the same proof, which requires reducing to the case when the ground field is F-finite by using the gamma construction (Theorem B.1.1). Finally, [dFKL07] uses the assumption that the ground field is uncountable to choose countably many very general divisors that facilitate an inductive argument. Our version of [dFKL07, Thm. 4.1] therefore needs to reduce to this case.

6.1. Motivation and statement

We start by motivating the statement of our ampleness criterion. Let X be a projective variety of dimension $n > 0$. For every Cartier divisor L on X, we have

$$h^i\bigl(X, \mathcal{O}_X(mL)\bigr) = O(m^n)$$

for every i; see [Laz04a, Ex. 1.2.20]. In [dFKL07, Thm. 4.1], de Fernex, Küronya, and Lazarsfeld asked when the higher cohomology groups have submaximal growth, i.e., when

$$h^i\bigl(X, \mathcal{O}_X(mL)\bigr) = o(m^n).$$

They proved that over the complex numbers, ample Cartier divisors L are characterized by having submaximal growth of higher cohomology groups for small perturbations of L.

We prove the following version of their result, which is valid over arbitrary fields, and in particular, is valid over possibly imperfect fields of positive characteristic.

Theorem E. *Let X be a projective variety of dimension $n > 0$ over a field k. Let L be an $\mathbf{R}$-Cartier divisor on X, and consider the following property:*

($\star$) *There exists a very ample Cartier divisor A on X and a real number $\varepsilon > 0$ such that*
$$\widehat{h}^i(X, L - tA) \coloneqq \limsup_{m\to\infty} \frac{h^i\bigl(X, \mathcal{O}_X\bigl(\lceil m(L - tA)\rceil\bigr)\bigr)}{m^n/n!} = 0$$
for all $i > 0$ and for all $t \in [0, \varepsilon)$.

Then, L is ample if and only if L satisfies ($\star$) for some pair (A, ε).

We note that one can have $\widehat{h}^i(X, L) = 0$ for all $i > 0$ without L being ample, or even pseudoeffective, as seen in the following example.

Example 6.1.1 [Kür06, Ex. 3.3]**.** Let A be a abelian variety of dimension g over an algebraically closed field k, and let L be a line bundle on A. We recall that $K(L) \subseteq A$ is defined to be the maximal closed subscheme of A such that the *Mumford bundle*

$$\Lambda(L) \coloneqq m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}$$

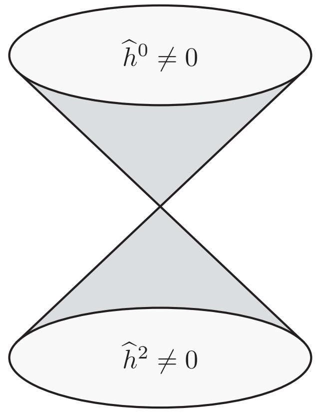

Figure 6.1: Asymptotic cohomological functions on an abelian surface
Illustration from [ELM$^+$05, Fig. 4]

is trivial on $A \times_k A$ [Mum08, p. 115], where m is the multiplication map and p_1, p_2 are the first and second projections, respectively. We also recall that L is *non-degenerate* if $K(L)$ is finite [Mum08, p. 145n]. By Mumford's index theorem [Mum08, Thm. on p. 140], we have

$$h^i(A, L) = \begin{cases} (-1)^{i(L)} \cdot (L^g) & \text{if } i = i(L) \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

for non-degenerate line bundles L, where $i(L)$ is the index of L [Mum08, p. 145]. In particular, this holds for ample line bundles L on A by [Mum08, App. 1 on p. 57], in which case $i(L) = 0$ by the proof of [Mum08, Thm. on p. 140].

Now let ξ be a nef $\mathbf{R}$-Cartier divisor on A. Then, ξ can be written as the limit of ample $\mathbf{Q}$-Cartier divisors on A. Thus, by using (6.1) and the homogeneity and continuity of asymptotic cohomological functions (see Remark 4.6.11), we have

$$\widehat{h}^i(A, \xi) = \begin{cases} (\xi^g) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and we note that $(\xi^g) = 0$ if ξ is nef but not ample [Laz04a, Cor. 1.5.18]. By asymptotic Serre duality (Proposition 4.6.12), we therefore see that for nef but not ample $\mathbf{R}$-Cartier divisors ξ, we have $\widehat{h}^i(A, -\xi) = 0$ for all i, even though $-\xi$ is not ample, or even pseudoeffective.

We now illustrate this phenomenon in a more concrete situation. Recall that if X is a complete scheme over a field, then the *Néron–Severi space* is the $\mathbf{R}$-vector space

$$N^1_{\mathbf{R}}(X) := \mathrm{Cart}_{\mathbf{R}}(X)/{\equiv_{\mathbf{R}}}, \tag{6.2}$$

where $\equiv_{\mathbf{R}}$ denotes $\mathbf{R}$-linear equivalence. This vector space is finite-dimensional by [Cut15, Prop. 2.3]. Now if A is an abelian surface, then the ample cone in $N^1_{\mathbf{R}}(X)$ is $\{\xi \in N^1_{\mathbf{R}}(X) \mid \widehat{h}^0(\xi) \neq 0\}$. By [Laz04a, Lem. 1.5.4], the classes $-\xi$ considered above for nef but not ample $\mathbf{R}$-Cartier divisors ξ correspond to classes in the boundary of the cone $\{\xi \in N^1_{\mathbf{R}}(X) \mid \widehat{h}^2(\xi) \neq 0\}$. See Figure 6.1 for an illustration of the case when the Picard rank $\rho(A)$ of A is 3. We note that if $A = E \times_k E$ for a sufficiently general elliptic curve E, then $\rho(A) = 3$. This follows from the fact that $\mathrm{End}_k(E) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathbf{Q}$ for sufficiently general E by a theorem of Deuring [Mum08, Thm. on p. 201], hence $\rho(A) = 3$ by a lemma of Murty [Laf, Prop. 2.3].

6.2. A lemma on base loci

A key ingredient in our proof of Theorem E is the following result on base loci, which is the analogue of [dFKL07, Prop. 3.1] over arbitrary fields. The lemma says that base ideals associated to multiples of non-nef divisors grow like powers of an ideal defining a curve.

Proposition 6.2.1. *Let V be a normal projective variety of dimension at least two over a field k. Let D be a Cartier divisor on V, and suppose there exists an integral curve $Z \subseteq V$ such that $(D \cdot Z) < 0$. Denoting by $\mathfrak{a} \subseteq \mathcal{O}_V$ the ideal sheaf defining Z, there exist positive integers q and c such that for every integer $m \geq c$, we have*

$$\mathfrak{b}\bigl(|mqD|\bigr) \subseteq \mathfrak{a}^{m-c}.$$

Here, $\mathfrak{b}(|D|)$ denotes the base ideal of the Cartier divisor D; see Definition 4.5.1.

To use Bertini theorems, we need to reduce to the case when the ground field k is infinite. Moreover, in positive characteristic, we use asymptotic test ideals instead of asymptotic multiplier ideals, which requires also reducing to the case where the ground field is F-finite.

Lemma 6.2.2. *To prove Proposition 6.2.1, we may assume that the ground field k is infinite, and in positive characteristic, we may also assume that k is F-finite.*

Proof. We first construct a sequence $k \subseteq k' \subseteq K$ of two field extensions such that $V \times_k K$ is integral and normal, where k' is infinite and K is F-finite in positive characteristic. If k is already infinite, then let $k' = k$. Otherwise, consider the purely transcendental extension $k \subseteq k(x)$. To show that $V \times_k k'$ is integral and normal, let $\bigcup_j U_j$ be an affine open covering of V. Then, $V \times_k k'$ is covered by affine open subsets that are localizations of the normal varieties $U_j \times_k \operatorname{Spec} k[x]$, which pairwise intersect, hence $V \times_k k'$ is integral and normal. The same argument shows that $Z \times_k k'$ is an integral curve. We set $K = k'$ in characteristic zero, and in positive characteristic, the gamma construction (Theorem B.1.6) shows that there is a field extension $k' \subseteq K$ such that K is F-finite, $V \times_k K$ is integral and normal, and $Z \times_k K$ is integral. Note that K is infinite since it contains the infinite field k'.

We now show that the special case when k is infinite and F-finite implies the general case. Let $\pi\colon V \times_k k' \to V$ be the first projection morphism, which we note is faithfully flat by base change. Since $(\pi^*D \cdot \pi^*Z) = (D \cdot Z) < 0$ by [Kle05, Prop. B.17], the special case of Proposition 6.2.1 implies

$$\mathfrak{b}\big(|mq\,\pi^*D|\big) \subseteq (\pi^{-1}\mathfrak{a} \cdot \mathcal{O}_{V \times_k k'})^{m-c}.$$

Then, since π is faithfully flat and since $\mathfrak{b}(|mq\,\pi^*D|) = \pi^{-1}\mathfrak{b}(|mqD|) \cdot \mathcal{O}_{V \times_k k'}$ by flat base change, we have $\mathfrak{b}(|mqD|) \subseteq \mathfrak{a}^{m-c}$ by [Mat89, Thm. 7.5(ii)]. □

Remark 6.2.3. When k is F-finite of characteristic $p > 0$, then one can set K to be $k(x^{1/p^\infty})$ in the proof of Lemma 6.2.2, since integrality and normality are preserved under limits of schemes with affine and flat transition morphisms [EGAIV$_2$, Cor. 5.13.4].

We now focus on proving Proposition 6.2.1 in positive characteristic; see Remark 6.2.4 for the characteristic zero case. We have incorporated some ideas from [MPST, Lems. 4.3 and 4.4]. In the proof below, we will use the fact [Kle05, Lem. B.12] that if W is a one-dimensional subscheme of a complete scheme X over a field, and if D is a Cartier divisor on X, then

$$(D \cdot W) = \sum_{\alpha} \operatorname{length}_{\mathcal{O}_{X,\eta_\alpha}}\bigl(\mathcal{O}_{W_\alpha,\eta_\alpha}\bigr) \cdot (D \cdot W_\alpha), \tag{6.3}$$

where the W_α are the one-dimensional components of W with generic points $\eta_\alpha \in W_\alpha$.

Proof of Proposition 6.2.1 in positive characteristic. By Lemma 6.2.2, it suffices to consider the case when the ground field k is infinite and F-finite. The statement is trivial if $H^0(V, \mathcal{O}_V(mD)) = 0$ for every integer $m > 0$, since in this case $\mathfrak{b}(|mqD|) = 0$ for all positive integers m, q. We therefore assume $H^0(V, \mathcal{O}_V(mD)) \neq 0$ for some integer $m > 0$.

We first set some notation. Let $\eta\colon V_1 \to V$ be the normalized blowup of $Z \subseteq V$, and denote $E \coloneqq \eta^{-1}(Z)$. Consider a regular alteration $\varphi\colon V' \to V_1$ for (V_1, E) as in [dJ96, Thm. 4.1], and set $D' \coloneqq (\eta \circ \varphi)^* D$. Note that in this case, $E' \coloneqq \varphi^* E = (\eta \circ \varphi)^{-1}(Z)$ is a Cartier divisor with simple normal crossing support. The proof proceeds in four steps.

Step 1. *It suffices to show that there exists a positive integer a such that for every integer $m > 0$, we have*

$$\mathfrak{b}\bigl(|maD'|\bigr) \subseteq \mathcal{O}_{V'}(-mE'_{\mathrm{red}}). \tag{6.4}$$

Consider the commutative diagram

$$\begin{tikzcd} V' \arrow[r, "\varphi_2"] \arrow[dr, "\varphi"'] & V_2 \arrow[d, "\varphi_1"] \\ & V_1 \arrow[r, "\eta"] & V \end{tikzcd}$$

where the triangle is the Stein factorization for φ [Har77, Cor. III.11.5]. Note that by construction of the Stein factorization, the scheme V_2 is a normal projective variety. Now by setting b to be the largest coefficient appearing in E', we see that $\mathcal{O}_{V'}(-bE_{\mathrm{red}}) \subseteq \mathcal{O}_{V'}(-E)$. Thus, we have

$$\varphi_2^{-1}\mathfrak{b}\bigl(|mab\,(\eta \circ \varphi_1)^* D|\bigr) \cdot \mathcal{O}_{V'} = \mathfrak{b}\bigl(|mabD'|\bigr) \subseteq \mathcal{O}_{V'}(-mbE'_{\mathrm{red}}) \subseteq \mathcal{O}_{V'}(-mE') \tag{6.5}$$

by (6.4), where the first equality holds by Lemma 4.5.2 since φ_2 is birational. Setting

$q = ab$ and pushing forward by φ_2, we have

$$\begin{aligned}\mathfrak{b}\big(|mq\,(\eta\circ\varphi_1)^*D|\big) &= \mathfrak{b}\big(|mq\,(\eta\circ\varphi_1)^*D|\big)\cdot\varphi_{2*}\mathcal{O}_{V'}\\ &= \varphi_{2*}\Big(\varphi_2^{-1}\mathfrak{b}\big(|mq\,(\eta\circ\varphi_1)^*D|\big)\cdot\mathcal{O}_{V'}\Big)\subseteq\mathcal{O}_{V_2}(-m\varphi_1^*E)\end{aligned}$$

where the first equality and last inclusion hold by the fact that V_2 is normal, hence $\varphi_{2*}\mathcal{O}_{V'} = \mathcal{O}_{V_2}$ [Har77, Proof of Cor. III.11.4], and the second equality holds by definition of restriction of scalars. Next, we push forward by φ_1 and intersect with the subsheaf $\mathcal{O}_{V_1}\subseteq\varphi_{1*}\mathcal{O}_{V_2}$ to obtain the chain of inclusions

$$\begin{aligned}\mathfrak{b}\big(|mq\,\eta^*D|\big) &\subseteq \varphi_{1*}\Big(\mathfrak{b}\big(|mq\,(\eta\circ\varphi_1)^*D|\big)\Big)\cap\mathcal{O}_{V_1}\\ &\subseteq\varphi_{1*}\big(\mathcal{O}_{V_2}(-m\,\varphi_1^*E)\big)\cap\mathcal{O}_{V_1} = \mathcal{O}_{V_1}(-mE),\end{aligned}$$

where the last equality holds by the fact that φ_1 is finite, hence integral, and then by properties of integral closure [HS06, Props. 1.5.2 and 1.6.1]. Finally, we push forward by η to obtain

$$\mathfrak{b}\big(|mqD|\big)\subseteq\eta_*\mathfrak{b}\big(|mq\,\eta^*D|\big)\subseteq\overline{\mathfrak{a}^m},$$

where $\overline{\mathfrak{a}^m}$ is the integral closure of $\mathfrak{a}^m$ [Laz04b, Rem. 9.6.4]. By [HS06, Cor. 1.2.5], there exists an integer c such that $\overline{\mathfrak{a}^{\ell+1}} = \mathfrak{a}\cdot\overline{\mathfrak{a}^\ell}$ for all $\ell\geq c$ [HS06, Cor. 1.2.5]. We therefore have $\overline{\mathfrak{a}^m}\subseteq\mathfrak{a}^{m-c}$ for all $m\geq c$, concluding Step 1.

In the rest of the proof, we consider another Stein factorization [Har77, Cor. III.11.5]

$$\begin{tikzcd} V' \arrow[r, "\mu"] \arrow[dr, "\psi"'] & \widetilde{V} \arrow[d, "\nu"] \\ & V \end{tikzcd} \tag{6.6}$$

this time for the morphism $\psi = \eta\circ\varphi$, in which case $\widetilde{V}$ is a normal projective variety. Let $\widetilde{Z} \coloneqq \nu^{-1}(Z)$ be the scheme-theoretic inverse image of Z under ν, and write

$$\widetilde{Z} = \bigcup_\alpha \widetilde{Z}_\alpha$$

where $\widetilde{Z}_\alpha$ are the irreducible components of $\widetilde{Z}$. Since ν is finite, every $\widetilde{Z}_\alpha$ is one-dimensional and dominates Z, hence the projection formula and (6.3) imply $\nu^*D\cdot\widetilde{Z}_\alpha < 0$. We also note that $E' = \mu^{-1}(\widetilde{Z})$ is a Cartier divisor with simple normal crossing support by the factorization (6.6).

We also fix the following notation. Fix a very ample Cartier divisor H on V', and set $A = K_{V'} + (\dim V' + 1)H$. For every subvariety $W \subseteq V'$, a *complete intersection curve* is a curve formed by taking the intersection of $\dim W - 1$ hyperplane sections in $\bigl|H|_W\bigr|$, and a *general complete intersection curve* is one formed by taking these hyperplane sections to be general in $\bigl|H|_W\bigr|$. For each positive integer q, we will consider the asymptotic test ideal

$$\tau\bigl(V', \|qD'\|\bigr) = \tau\bigl(\|qD'\|\bigr) \subseteq \mathcal{O}_{V'}.$$

By uniform global generation for test ideals (Theorem 5.5.13), the sheaf

$$\tau\bigl(\|qD'\|\bigr) \otimes \mathcal{O}_{V'}(qD' + A) \tag{6.7}$$

is globally generated for every integer $q > 0$.

Step 2. *There exists an integer $\ell_0 > 0$ such that for every integer $\ell > \ell_0$ and for every irreducible component F of E'_{red} that dominates $(Z_\alpha)_{\mathrm{red}}$ for some α, we have*

$$\tau\bigl(\|\ell D'\|\bigr) \subseteq \mathcal{O}_{V'}(-F).$$

Let $C \subseteq F$ be a general complete intersection curve; note that C is integral by Bertini's theorem [FOV99, Thm. 3.4.10 and Cor. 3.4.14] and dominates $(Z_\alpha)_{\mathrm{red}}$ for some α, hence $(D' \cdot C) < 0$ by the projection formula and (6.3). If for some integer $q > 0$, the curve C is not contained in the zero locus of $\tau(\|qD'\|)$, then the fact that the sheaf (6.7) is globally generated implies

$$\bigl((qD' + A) \cdot C\bigr) \geq 0.$$

Letting $\ell_{0F} = -(A\cdot C)/(D'\cdot C)$, we see that the ideal $\tau(\|\ell D'\|)$ vanishes everywhere along C for every integer $\ell > \ell_{0F}$. By varying C, the ideal $\tau(\|\ell D'\|)$ must vanish everywhere along F for every integer $\ell > \ell_{0F}$, hence we can set $\ell_0 = \max_F\{\ell_{0F}\}$.

Step 3. *Let E'_i be an irreducible component of E'_{red} not dominating Z_α for every α. Suppose E'_j is another irreducible component of E'_{red} such that $E'_i \cap E'_j \neq \emptyset$ and for which* *there exists an integer ℓ_j such that for every integer $\ell > \ell_j$, we have*

$$\tau\big(\|\ell D'\|\big) \subseteq \mathcal{O}_{V'}(-E'_j).$$

Then, there is an integer $\ell_i \geq \ell_j$ such that for every integer $\ell > \ell_i$, we have

$$\tau\big(\|\ell D'\|\big) \subseteq \mathcal{O}_{V'}(-E'_i).$$

Let $C \subseteq E'_i$ be a complete intersection curve. By the assumption that E' has simple normal crossing support, there exists at least one closed point $P \in C \cap E'_j$. For every $\ell > \ell_j$ and every $m > 0$, we have the sequence of inclusions

$$\begin{aligned}\Big(\tau\big(\|m\ell D'\|\big) \otimes \mathcal{O}_{V'}(m\ell D' + A)\Big) \cdot \mathcal{O}_C &\subseteq \Big(\tau\big(\|\ell D'\|\big)^m \otimes \mathcal{O}_{V'}(m\ell D' + A)\Big) \cdot \mathcal{O}_C \\ &\subseteq \Big(\mathcal{O}_{V'}(-mE'_j) \otimes \mathcal{O}_{V'}(m\ell D' + A)\Big) \cdot \mathcal{O}_C \subseteq \mathcal{O}_C(A|_C - mP)\end{aligned} \tag{6.8}$$

where the first two inclusions follow from subadditivity (Theorem 5.5.8) and by assumption, respectively. The last inclusion holds since C maps to a closed point in V, hence $\mathcal{O}_C(D') = \mathcal{O}_C$. By the global generation of the sheaf in (6.7) for $q = m\ell$, the inclusion (6.8) implies that for every integer $\ell > \ell_j$, if $\tau(\|m\ell D'\|)$ does not vanish everywhere along C, then $(A \cdot C) \geq m$. Choosing $\ell_i = \ell_j \cdot ((A \cdot C) + 1)$, we see that $\tau(\|\ell D'\|)$ vanishes everywhere along C for every integer $\ell > \ell_i$. By varying C, we have $\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E'_i)$ for every integer $\ell > \ell_i$.

Step 4. *There exists a positive integer a such that for every integer $m > 0$, we have $\mathfrak{b}(|maD'|) \subseteq \mathcal{O}_{V'}(-mE'_{\mathrm{red}})$.*

Write

$$E'_{\mathrm{red}} = \bigcup_j \bigcup_{i \in I_j} E'_{ij},$$

where the E'_{ij} are the irreducible components of E'_{red}, and the $\bigcup_{i \in I_j} E'_{ij}$ are the connected components of E'_{red}. Since V is normal, each preimage $\mu^{-1}(Z_\alpha)$ is connected by Zariski's main theorem [Har77, Cor. III.11.4], hence each connected component $\bigcup_{i \in I_j} E'_{ij}$ of E'_{red} contains an irreducible component $E'_{i_0 j}$ that dominates $(Z_\alpha)_{\mathrm{red}}$ for some α. By Step 2, there exists an integer ℓ_0 such that for every j, we have $\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E'_{i_0 j})$ for every integer $\ell > \ell_0$. For each j, by applying Step 3 $(|I_j| - 1)$ times to the jth connected component $\bigcup_{i\in I_j} E'_{ij}$ of E', we can find ℓ_j such that $\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E'_{ij})$ for every $i \in I_j$ and for every integer $\ell > \ell_j$. Setting $a = \max_j\{\ell_j\} + 1$, we have $\tau(\|aD'\|) \subseteq \mathcal{O}_{V'}(-E'_{\mathrm{red}})$. Thus, for every integer $m > 0$, we have

$$\mathfrak{b}\bigl(|maD'|\bigr) \subseteq \tau\bigl(|maD'|\bigr) \subseteq \tau\bigl(\|maD'\|\bigr) \subseteq \tau\bigl(\|aD'\|\bigr)^m \subseteq \mathcal{O}_{V'}(-mE'_{\mathrm{red}}),$$

where the first inclusion follows by the fact that V' is regular hence strongly F-regular (Propositions 5.5.7(i) and 5.5.7(vii)), the second inclusion is by definition of the asymptotic test ideal, and the third inclusion is by subadditivity (Theorem 5.5.8). This concludes the proof of Step 4, hence also of Proposition 6.2.1 by Step 1. $\square$

Remark 6.2.4. When $\operatorname{char} k = 0$, it suffices to replace asymptotic test ideals in the proof above with asymptotic multipliers ideals $\mathcal{J}(\|D\|)$ as defined in Definition 4.9.2 by replacing Proposition 5.5.7, Theorem 5.5.8, and Theorem 5.5.13 with [dFM09, Prop. 2.3], [JM12, Thm. A.2], and Theorem 4.9.4, respectively.

6.3. Proof of Theorem E

We now prove Theorem E. We first note that the direction $\Rightarrow$ in Theorem E follows from existing results.

Proof of $\Rightarrow$ in Theorem E. Let A be a very ample Cartier divisor. Then, for all t such that $L - tA$ is ample, we have $\widehat{h}^i(X, L - tA) = 0$ by Serre vanishing and by homogeneity and continuity (see Remark 4.6.11). $\square$

For the direction $\Leftarrow$, it suffices to show Theorem E for Cartier divisors L by continuity and homogeneity (see Remark 4.6.11). We also make the following two reductions. Recall that an $\mathbf{R}$-Cartier divisor L on X *satisfies* $(\star)$ for a pair (A, ε) consisting of a very ample Cartier divisor A on X and a real number $\varepsilon > 0$ if $\widehat{h}^i(X, L - tA) = 0$ for all $i > 0$ and all $t \in [0, \varepsilon)$.

Lemma 6.3.1. *To prove the direction $\Leftarrow$ in Theorem E, we may assume that the ground field k is uncountable.*

Proof. Consider the purely transcendental extension

$$k' := k(x_\alpha)_{\alpha \in A}$$

where $\{x_\alpha\}_{\alpha \in A}$ is an uncountable set of indeterminates; note that k' is uncountable by construction. We claim that $X \times_k k'$ is integral. Let $\bigcup_j U_j$ be an affine open covering of X. Then, $X \times_k k'$ is covered by affine open subsets that are localizations of the integral varieties $U_j \times_k \operatorname{Spec} k[x_\alpha]_{\alpha \in A}$, which pairwise intersect, hence $X \times_k k'$ is integral.

Now suppose X is a projective variety over k, and let L be an Cartier divisor satisfying $(\star)$ for some pair (A, ε). Let

$$\pi \colon X \times_k k' \longrightarrow X$$

be the first projection map, which we note is faithfully flat by base change. Then, the pullback $\pi^* A$ of A is very ample, and to show that L is ample, it suffices to show that $\pi^* L$ is ample by flat base change and Serre's criterion for ampleness. By the special case of Theorem E over the ground field k', it therefore suffices to show that $\pi^* L$ satisfies $(\star)$ for the pair $(\pi^* A, \varepsilon)$.

We want to show that for every $i > 0$ and for all $t \in [0, \varepsilon)$, we have

$$\widehat{h}^i(X, L - tA) = \widehat{h}^i\bigl(X \times_k K, \pi^*(L - tA)\bigr) = 0. \tag{6.9}$$

For every $D \in \operatorname{Cart}(X)$ and every $i \geq 0$, the number $h^i(X, \mathcal{O}_X(D))$ is invariant under ground field extensions by flat base change, hence $\widehat{h}^i(X, D)$ is also. By homogeneity and continuity (see Remark 4.6.11), the number $\widehat{h}^i(X, D)$ is also invariant under ground field extensions for $D \in \operatorname{Cart}_{\mathbf{R}}(X)$, hence (6.9) holds. $\square$

Lemma 6.3.2. *To prove the direction $\Leftarrow$ in Theorem E, it suffices to show that every Cartier divisor satisfying $(\star)$ is nef.*

Proof. Suppose L is a Cartier divisor satisfying $(\star)$ for a pair (A, ε). Choose $\delta \in (0, \varepsilon) \cap \mathbf{Q}$ and let m be a positive integer such that $m\delta$ is an integer. Then, the Cartier divisor $m(L - \delta A)$ is nef since

$$\begin{aligned}
\widehat{h}^i\bigl(X, m(L - \delta A) - tA\bigr) &= \widehat{h}^i\bigl(X, mL - (t + m\delta)A\bigr) \\
&= m \cdot \widehat{h}^i\Bigl(X, L - \Bigl(\frac{t}{m} + \delta\Bigr)A\Bigr) = 0
\end{aligned}$$

for all $t \in [0, m\varepsilon - \delta)$ by homogeneity (Proposition 4.6.9). Thus, the Cartier divisor $L = (L - \delta A) + \delta A$ is ample by [Laz04a, Cor. 1.4.10]. $\square$

We will also need the following result to allow for an inductive proof. Note that the proof in [dFKL07] works in our setting.

Lemma 6.3.3 [dFKL07, Lem. 4.3]**.** *Let X be a projective variety of dimension $n > 0$ over an uncountable field, and let L be a Cartier divisor on X. Suppose L satisfies $(\star)$ for a pair (A, ε), and let $E \in |A|$ be a very general divisor. Then, the restriction $L|_E$ satisfies $(\star)$ for the pair $(A|_E, \varepsilon)$.*

We can now show the direction $\Leftarrow$ in Theorem E; by Lemma 6.3.2, we need to show that every Cartier divisor satisfying $(\star)$ is nef. Recall that by Lemma 6.3.1, we may assume that the ground field k is uncountable. Our proof follows that in [dFKL07, pp. 450–454] after reducing to a setting where Proposition 6.2.1 applies, although we have to be more careful in positive characteristic.

Proof of $\Leftarrow$ in Theorem E. We proceed by induction on $\dim X$. Suppose $\dim X = 1$; we will show the contrapositive. If L is not nef, then $\deg L < 0$ and $-L$ is ample. Thus, by asymptotic Serre duality (Proposition 4.6.12), we have $\widehat{h}^1(X, L) = \widehat{h}^0(X, -L) \neq 0$, hence $(\star)$ does not hold for every choice of (A, ε).

We now assume $\dim X \geq 2$. Suppose by way of contradiction that there is a non-nef Cartier divisor L satisfying $(\star)$, and let $Z \subseteq X$ be an integral curve such that $(L \cdot Z) < 0$. Our goal is to show that

$$\widehat{h}^1(X, L - \delta A) \neq 0 \tag{6.10}$$

for $0 < \delta \ll 1$, contradicting $(\star)$. Let $F \in |A|$ be a very general divisor. By Bertini's theorem [FOV99, Thm. 3.4.10 and Cor. 3.4.14], we may assume that F is a subvariety of X, in which case by inductive hypothesis and Lemma 6.3.3, we have that $L|_F$ is ample. Since ampleness is an open condition in families [EGAIV$_3$, Cor. 9.6.4], there exists an integer $b > 0$ such that bL is very ample along the generic divisor $F_\eta \in |A|$. By possibly replacing b with a multiple, we may also assume that $mbL|_{F_\eta}$ has vanishing higher cohomology for every integer $m > 0$. Since the ground field k is uncountable, we can then choose a sequence of very general Cartier divisors $\{E_\beta\}_{\beta=1}^{\infty} \subseteq |A|$ such that the following properties hold:

(a) E_β is a subvariety of X for all β (by Bertini's theorem [FOV99, Thm. 3.4.10 and Cor. 3.4.14]);

(b) For all β, $bL|_{E_\beta}$ is very ample and $mbL|_{E_\beta}$ has vanishing higher cohomology for every integer $m > 0$ (by the constructibility of very ampleness in families [EGAIV$_3$, Prop. 9.6.3] and by semicontinuity); and

(c) For every positive integer r, the k-dimension of cohomology groups of the form

$$H^j\bigl(E_{\beta_1} \cap E_{\beta_2} \cap \cdots \cap E_{\beta_r}, \mathcal{O}_{E_{\beta_1} \cap E_{\beta_2} \cap \cdots \cap E_{\beta_r}}(mL)\bigr) \tag{6.11}$$

for non-negative integers j and m is independent of the r-tuple $(\beta_1, \beta_2, \ldots, \beta_r)$ (by semicontinuity; see [Kür06, Prop. 5.5]).

We will denote by $h^j(\mathcal{O}_{E_1 \cap E_2 \cap \cdots \cap E_r}(mL))$ the dimensions of the cohomology groups (6.11). By homogeneity (Proposition 4.6.9), we can replace L by bL so that $L|_{E_\beta}$ is very ample with vanishing higher cohomology for all β.

To show (6.10), we now follow the proof in [dFKL07, pp. 453–454] with appropriate modifications. Given positive integers m and r, consider the complex

$$\begin{aligned} K^\bullet_{m,r} &\coloneqq \Biggl(\bigotimes_{\beta=1}^{r} (\mathcal{O}_X \longrightarrow \mathcal{O}_{E_\beta})\Biggr) \otimes \mathcal{O}_X(mL) \\ &= \Biggl\{\mathcal{O}_X(mL) \longrightarrow \bigoplus_{\beta=1}^{r} \mathcal{O}_{E_\beta}(mL) \longrightarrow \bigoplus_{1 \le \beta_1 < \beta_2 \le r} \mathcal{O}_{E_{\beta_1} \cap E_{\beta_2}}(mL) \longrightarrow \cdots\Biggr\}. \end{aligned}$$

By [Kür06, Cor. 4.2], this complex is acyclic away from $\mathcal{O}_X(mL)$, hence is a resolution for $\mathcal{O}_X(mL - rA)$. In particular, we have

$$H^j\bigl(X, \mathcal{O}_X(mL - rA)\bigr) = \mathbf{H}^j(X, K^\bullet_{m,r}).$$

The right-hand side is computed by an E_1-spectral sequence whose first page is shown in Figure 6.2. This spectral sequence yields a natural inclusion

$$\frac{\ker(u_{m,r})}{\operatorname{im}(v_{m,r})} \subseteq H^1\bigl(X, \mathcal{O}_X(mL - rA)\bigr). \tag{6.12}$$

We want to bound the left-hand side of (6.12) from below. First, there exists a constant

$$
\begin{array}{c|cccccc}
E_1 & & & & & & \\
\vdots & \vdots & & & & & \\
2 & H^2\big(\mathcal{O}_X(mL)\big) & & & 0 & & \\
1 & H^1\big(\mathcal{O}_X(mL)\big) & & & & & \\
0 & H^0\big(\mathcal{O}_X(mL)\big) & \xrightarrow{v_{m,r}} & \bigoplus_{\beta=1}^{r} H^0\big(\mathcal{O}_{E_\beta}(mL)\big) & \xrightarrow{u_{m,r}} & \bigoplus_{1\le\beta_1<\beta_2\le r} H^0\big(\mathcal{O}_{E_{\beta_1}\cap E_{\beta_2}}(mL)\big) & \cdots \\
\hline
& 0 & & 1 & & 2 & \cdots
\end{array}
$$

Figure 6.2: Hypercohomology spectral sequence computing $H^j(X, \mathcal{O}_X(mL - rA))$

$C_1 > 0$ such that $h^0(\mathcal{O}_{E_1\cap E_2}(mL)) \le C_1 \cdot m^{n-2}$ for all $m \gg 0$ [Laz04a, Ex. 1.2.20]. Thus, we have

$$\operatorname{codim}\Big(\ker(u_{m,r}) \subseteq \bigoplus_{\beta=1}^{r} H^0\big(E_\beta, \mathcal{O}_{E_\beta}(mL)\big)\Big) \le C_2 \cdot r^2 m^{n-2}$$

for some C_2 and for all $m \gg 0$. Now by Proposition 6.2.1, there are positive integers q and c such that $\mathfrak{b}(|mqL|) \subseteq \mathfrak{a}^{m-c}$ for all $m > c$, where $\mathfrak{a}$ is the ideal sheaf of Z. By replacing L by qL, we can assume that this inclusion holds for $q = 1$. The morphism $v_{m,r}$ therefore fits into the following commutative diagram:

$$
\begin{tikzcd}
H^0\big(X, \mathcal{O}_X(mL) \otimes \mathfrak{a}^{m-c}\big) \arrow[r, "v'_{m,r}"] \arrow[d, equal] & \bigoplus_{\beta=1}^{r} H^0\big(E_\beta, \mathcal{O}_{E_\beta}(m \otimes \mathfrak{a}^{m-c}\big) \arrow[d, hook] \\
H^0\big(X, \mathcal{O}_X(mL)\big) \arrow[r, "v_{m,r}"] & \bigoplus_{\beta=1}^{r} H^0\big(E_\beta, \mathcal{O}_{E_\beta}(mL)\big)
\end{tikzcd}
$$

We claim that there exists a constant $C_3 > 0$ such that for all $m \gg 0$,

$$\operatorname{codim}\Big(H^0\big(E_\beta, \mathcal{O}_{E_\beta}(mL) \otimes \mathfrak{a}^{m-c}\big) \subseteq H^0\big(E_\beta, \mathcal{O}_{E_\beta}(mL)\big)\Big) \ge C_3 \cdot m^{n-1}. \qquad (6.13)$$

Granted this, we have

$$\dim\left(\frac{\ker(u_{m,r})}{\operatorname{im}(v_{m,r})}\right) \ge C_4 \cdot \big(rm^{n-1} - r^2 m^{n-2}\big)$$

for some constant $C_4 > 0$ and for all $m \gg 0$. Fixing a rational number $0 < \delta \ll 1$ and setting $r = m\delta$ for an integer $m > 0$ such that $m\delta$ is an integer, we then see that there exists a constant $C_5 > 0$ such that

$$h^1\bigl(X, \mathcal{O}_X\bigl(m(L - \delta A)\bigr)\bigr) \geq C_5 \cdot \delta m^n$$

for all $m \gg 0$, contradicting $(\star)$.

It remains to show (6.13). Since the vanishing locus of $\mathfrak{a}$ may have no k-rational points, we will pass to the algebraic closure of k to bound the codimension on the left-hand side of (6.13) from below. Let $\overline{E}_\beta \coloneqq E_\beta \times_k \overline{k}$, and denote by $\pi\colon \overline{E}_\beta \to E_\beta$ the projection morphism. Note that

$$\begin{aligned}&\operatorname{codim}\Bigl(H^0\bigl(E_\beta, \mathcal{O}_{E_\beta}(mL) \otimes \mathfrak{a}^{m-c}\bigr) \subseteq H^0\bigl(E_\beta, \mathcal{O}_{E_\beta}(mL)\bigr)\Bigr)\\ &\qquad = \operatorname{codim}\Bigl(H^0\bigl(\overline{E}_\beta, \mathcal{O}_{\overline{E}_\beta}(m\,\pi^*L) \otimes \pi^{-1}\mathfrak{a}^{m-c} \cdot \mathcal{O}_{\overline{E}_\beta}\bigr) \subseteq H^0\bigl(\overline{E}_\beta, \mathcal{O}_{\overline{E}_\beta}(m\,\pi^*L)\bigr)\Bigr)\end{aligned}$$

by the flatness of $k \subseteq \overline{k}$. Since $\mathcal{O}_{\overline{E}_\beta}(\pi^*L)$ is very ample by base change, we can choose a closed point $x \in Z(\pi^{-1}\mathfrak{a} \cdot \mathcal{O}_{\overline{E}_\beta}) \cap \overline{E}_\beta$, in which case $\mathcal{O}_{\overline{E}_\beta}(m\,\pi^*L)$ separates $(m-c)$-jets at x by [Ito13, Proof of Lem. 3.7] (see also Lemma 7.2.5). Finally, the dimension of the space of $(m-c)$-jets at x is at least that for a regular point of a variety of dimension n, hence

$$\begin{aligned}&\operatorname{codim}\Bigl(H^0\bigl(\overline{E}_\beta, \mathcal{O}_{\overline{E}_\beta}(m\,\pi^*L) \otimes \pi^{-1}\mathfrak{a}^{m-c} \cdot \mathcal{O}_{\overline{E}_\beta}\bigr) \subseteq H^0\bigl(\overline{E}_\beta, \mathcal{O}_{\overline{E}_\beta}(m\,\pi^*L)\bigr)\Bigr)\\ &\qquad \geq \operatorname{codim}\Bigl(H^0\bigl(\overline{E}_\beta, \mathcal{O}_{\overline{E}_\beta}(m\,\pi^*L) \otimes \mathfrak{m}_x^{m-c} \cdot \mathcal{O}_{\overline{E}_\beta}\bigr) \subseteq H^0\bigl(\overline{E}_\beta, \mathcal{O}_{\overline{E}_\beta}(m\,\pi^*L)\bigr)\Bigr)\\ &\qquad \geq \binom{m-c+n}{n-1} \geq C_3 \cdot m^{n-1}\end{aligned}$$

for some constant $C_3 > 0$ and all $m \gg 0$, as required. $\square$

Chapter 7

Moving Seshadri constants

Moving Seshadri constants were defined by Nakamaye [Nak03] as a generalization of the Seshadri constant introduced in §2.2 to arbitrary **R**-Cartier divisors. In this chapter, we extend basic results on moving Seshadri constants from [Nak03; ELM$^+$09, §6] to the setting of possibly singular varieties over arbitrary fields. These results are new even for complex projective varieties that are not smooth. Some of this material will appear in joint work with Mihai Fulger [FMb].

7.1. Definition and basic properties

Following [ELM$^+$09], we define the moving Seshadri constant as follows:

Definition 7.1.1 (cf. [ELM$^+$09, Def. 6.1])**.** Let X be a normal projective variety over a field k and let D be an $\mathbf{R}$-Cartier divisor on X. Consider a k-rational point $x \in X$. If $x \notin \mathbf{B}_+(D)$, then the *moving Seshadri constant* of D at x is

$$\varepsilon\bigl(\lVert D\rVert;x\bigr) \coloneqq \sup_{f^*D\equiv_{\mathbf{R}}A+E} \varepsilon(A;x) \tag{7.1}$$

where the supremum runs over all birational morphisms $f\colon X' \to X$ from normal projective varieties X' that are isomorphisms over a neighborhood of x, and $\mathbf{R}$-numerical equivalences $f^*D \equiv_{\mathbf{R}} A + E$ where A is an ample $\mathbf{R}$-Cartier divisor and E is an effective $\mathbf{R}$-Cartier divisor such that $x \notin f(\operatorname{Supp}(E))$. If $x \in \mathbf{B}_+(D)$, then we set $\varepsilon(\lVert D\rVert;x) = 0$. By definition, $x \in \mathbf{B}_+(D)$ if and only if $\varepsilon(\lVert D\rVert;x) = 0$.

Note that $\mathbf{R}$-numerical equivalences of the form in (7.1) exist since $f^{-1}(x) \notin \mathbf{B}_+(f^*D)$ by Proposition 4.6.4.

We collect some elementary properties of moving Seshadri constants. Recall that if $x \in X$ is a closed point, then $\operatorname{Big}^{\{x\}}_{\mathbf{R}}(X)$ denotes the open convex subcone of the big cone consisting of big $\mathbf{R}$-Cartier divisor classes $D \in N^1_{\mathbf{R}}(X)$ such that $x \notin \mathbf{B}_+(D)$ [ELM+09, Def. 5.1]. Here $N^1_{\mathbf{R}}(X)$ is the *Néron–Severi space* associated to X defined in (6.2).

Proposition 7.1.2 (cf. [ELM+09, Prop. 6.3 and Rem. 6.5])**.** *Let X be a normal projective variety over a field k and let $x \in X$ be a k-rational point. Then, the function*

$$\begin{aligned} \operatorname{Big}^{\{x\}}_{\mathbf{R}}(X) &\longrightarrow \mathbf{R}_{>0} \\ D &\longmapsto \varepsilon\bigl(\lVert D\rVert; x\bigr) \end{aligned}$$

is continuous. Moreover, if D is an $\mathbf{R}$-Cartier divisor, then we have the following:

(*i*) $\varepsilon(\lVert D\rVert; x) \le (\mathrm{vol}_X(D)/e(\mathcal{O}_{X,x}))^{1/\dim X}$.

(*ii*) *If D and E are numerically equivalent $\mathbf{R}$-Cartier divisors, then* $\varepsilon(\lVert D\rVert; x) = \varepsilon(\lVert E\rVert; x)$.

(*iii*) $\varepsilon(\lVert \lambda D\rVert; x) = \lambda \cdot \varepsilon(\lVert D\rVert; x)$ *for every positive real number λ.*

(*iv*) *If D' is another $\mathbf{R}$-Cartier divisor such that $x \notin \mathbf{B}_+(D) \cup \mathbf{B}_+(D')$, then*

$$\varepsilon\bigl(\lVert D + D'\rVert; x\bigr) \ge \varepsilon\bigl(\lVert D\rVert; x\bigr) + \varepsilon\bigl(\lVert D'\rVert; x\bigr).$$

(*v*) *If D is a nef $\mathbf{R}$-Cartier divisor, then* $\varepsilon(\lVert D\rVert; x) = \varepsilon(D; x)$.

Proof. (*i*)–(*iv*) follow by definition and from the analogous properties for usual Seshadri constants; for (*i*), the analogous property is [Laz04a, Prop. 5.1.9]. The continuity property follows from (*iii*) and (*iv*) by [ELM+09, Rem. 5.4].

We now prove (*v*), following [ELM+09, Rem. 6.5]. If $x \in \mathbf{B}_+(D)$, then $\varepsilon(\lVert D\rVert; x) = 0$ by definition, while $\varepsilon(D; x) = 0$ by combining Theorem 4.6.6 and [Laz04a, Prop. 5.1.9]. It therefore suffices to consider the case when $x \notin \mathbf{B}_+(D)$. As in Definition 7.1.1, choose a birational morphism $f \colon X' \to X$ with a decomposition $f^*D \equiv_{\mathbf{R}} A + E$. We have

$$\varepsilon(D; x) = \varepsilon\bigl(f^*D; f^{-1}(x)\bigr) \ge \varepsilon\bigl(A; f^{-1}(x)\bigr)$$

where the first equality holds since D is nef and f is an isomorphism at x. The second inequality holds by combining (2.3) and the fact that $x \notin f(\mathrm{Supp}(E))$, hence $E \cdot C \geq 0$ for every integral curve $C \subseteq X'$ passing through $f^{-1}(x)$. Taking the supremum over all f as in Definition 7.1.1, we have the inequality $\varepsilon(D;x) \geq \varepsilon(\|D\|;x)$. For the opposite inequality, write $D \equiv_{\mathbf{R}} A + E$ with A an ample $\mathbf{R}$-Cartier divisor and E an effective $\mathbf{R}$-Cartier divisor not containing x in its support. For every integer $n \geq 1$, we can write

$$D \equiv_{\mathbf{R}} \frac{1}{n}A + \frac{n-1}{n}D + \frac{1}{n}E,$$

hence setting $A_n \coloneqq \frac{1}{n}A + \frac{n-1}{n}D$, we have $D \equiv_{\mathbf{R}} A_n + \frac{1}{n}E$ for an ample $\mathbf{R}$-Cartier divisor A_n and a fixed effective $\mathbf{R}$-Cartier divisor E. We therefore have

$$\varepsilon\bigl(\|D\|;x\bigr) \geq \varepsilon(A_n;x) \tag{7.2}$$

for all n. Now we note that using the characterization in (2.3), we have

$$\varepsilon(A_n;x) = \inf_{C \ni x}\left\{\frac{(A_n \cdot C)}{e(\mathcal{O}_{C,x})}\right\} \geq \inf_{C \ni x}\left\{\frac{(D \cdot C)}{e(\mathcal{O}_{C,x})}\right\} + \frac{1}{n}\inf_{C \ni x}\left\{\frac{(-E \cdot C)}{e(\mathcal{O}_{C,x})}\right\}. \tag{7.3}$$

Taking the limit $n \to \infty$ in (7.2), we obtain the inequality $\varepsilon(\|D\|;x) \geq \varepsilon(D;x)$. □

Remark 7.1.3. We note that the continuity statement in Proposition 7.1.2 is *not* the analogue of [ELM$^+$09, Thm. 6.2], which states that if X is a smooth complex projective variety and $x \in X$ is a closed point, then the function $D \mapsto \varepsilon(\|D\|;x)$ is continuous on the entire Néron–Severi space $N^1_{\mathbf{R}}(X)$. A proof of this statement would require extending the main results about restricted volume functions in [ELM$^+$09] to our setting.

7.2. Alternative descriptions

We now give alternative characterizations of the moving Seshadri constant defined in Definition 7.1.1.

7.2.1. Nakamaye's description

Moving Seshadri constants were first defined by Nakamaye by decomposing the complete linear system $|D|$ into its moving and fixed parts on a birational model of X. The following is a version of his definition that works over arbitrary fields.

Definition 7.2.1 (cf. [Nak03, Def. 0.4]). Let X be a normal projective variety over a field k and let D be a $\mathbf{Q}$-Cartier divisor on X. Let $x \notin \mathbf{B}(D)$ be a k-rational point. For every integer $m \geq 1$ such that mD is Cartier and $x \notin \operatorname{Bs}(|mD|)$, let $\pi_m \colon X_m \to X$ be a morphism from a normal projective variety X_m that is an isomorphism over a neighborhood of x, and such that $\pi_m^{-1}\mathfrak{b}(|mD|) \cdot \mathcal{O}_{X_m} = \mathcal{O}_{X_m}(-F_m)$ for an effective Cartier divisor F_m. We can then write

$$\left|\pi_m^*(mD)\right| = |M_m| + F_m,$$

where $|M_m|$ is the moving part and F_m is the fixed part of the linear system $|\pi_m^*(mD)|$, respectively. We then set

$$\varepsilon_N\bigl(\|D\|; x\bigr) \coloneqq \limsup_{m\to\infty} \frac{\varepsilon\bigl(M_m; \pi_m^{-1}(x)\bigr)}{m}$$

where the limit supremum is taken over all m such that mD is integral.

To make sure that $\varepsilon_N(\|D\|; x)$ is well-defined, we show that $\varepsilon(M_m; \pi_m^{-1}(x))$ does not depend on the choice of morphism π_m. First, any two morphisms $\pi_m \colon X_m \to X$ and $\pi'_m \colon X'_m \to X$ as above can be dominated by a morphism $\pi''_m \colon X''_m \to X$ satisfying the same properties, and the normality of the varieties X, X_m, X'_m, X''_m imply that the moving parts on X_m and X'_m pullback to the moving part on X''_m. Since π_m, π'_m, π''_m are all isomorphisms in a neighborhood of x, we see that the Seshadri constants of M_m, M'_m, M''_m are equal, hence $\varepsilon(M_m; \pi_m^{-1}(x))$ does not depend on the choice of π_m.

We now show that the limit supremum used to define $\varepsilon_N(\|D\|; x)$ is equal to a limit.

Lemma 7.2.2. *With notation as in Definition 7.2.1, we have*

$$\varepsilon_N\bigl(\|D\|; x\bigr) = \lim_{m\to\infty} \frac{\varepsilon\bigl(M_m; \pi_m^{-1}(x)\bigr)}{m} = \sup_m \frac{\varepsilon\bigl(M_m; \pi_m^{-1}(x)\bigr)}{m}.$$

Proof. Let m and n be positive integers such that mD and nD are Cartier divisors. Choose $\pi\colon X' \to X$ that satisfies the properties in Definition 7.2.1 for $|mD|$, $|nD|$, and $|(m+n)D|$, for example by blowup up the base loci for all three linear systems, and then taking a normalization. Since we have $M_{m+n} = M_m + M_n + E$ for some effective divisor E with $\pi^{-1}(x) \notin \mathrm{Supp}(E)$, we deduce that

$$\varepsilon\bigl(M_{m+n};\pi^{-1}(x)\bigr) \geq \varepsilon\bigl(M_m;\pi^{-1}(x)\bigr) + \varepsilon\bigl(M_n;\pi^{-1}(x)\bigr).$$

Thus, the sequence $\{\varepsilon(M_m;\pi_m^{-1}(x))\}_m$ is superadditive, and Fekete's lemma [PS98, Pt. I, n$^{\text{o}}$ 98] implies that the limit supremum is equal to the limit and the supremum. $\square$

Nakamaye's definition coincides with the one in Definition 7.1.1 for $x \notin \mathbf{B}(D)$.

Proposition 7.2.3 (cf. [ELM$^+$09, Prop. 6.4])**.** *Let X be a normal projective variety over a field k and let D be a $\mathbf{Q}$-Cartier divisor. If $x \notin \mathbf{B}(D)$ is a k-rational point, then $\varepsilon(\|D\|;x) = \varepsilon_N(\|D\|;x)$.*

Proof. By Proposition 7.1.2(iii) and Lemma 7.2.2, both invariants are homogeneous with respect to taking integer multiples of D. It therefore suffices to consider the case when D is integral, $\mathbf{B}(D) = \mathrm{Bs}(|D|)_{\mathrm{red}}$, and $|D|$ induces a rational map birational onto its image. Let m be a positive integer, and let $\pi_m\colon X_m \to X$ be as in Definition 7.2.1. Writing $|\pi_m^*(mD)| = |M_m| + F_m$, we note that by assumption on x, we have that $\pi_m^{-1}(x) \notin \mathrm{Supp}(F_m)$.

First, suppose that $x \in \mathbf{B}_+(D)$. By assumption, $x \notin \mathbf{B}(D) = \mathrm{Bs}(|D|)_{\mathrm{red}}$, hence $\pi_m^{-1}(x) \notin \mathrm{Bs}(|\pi_m^*D|)_{\mathrm{red}} = \mathrm{Supp}(F_m)$ by the normality of X. Thus, Proposition 4.6.4 implies $\pi_m^{-1}(x) \in \mathbf{B}_+(M_m)$, and Theorem 4.6.6 implies there exists a subvariety $V \subseteq X_m$ such that $\pi_m^{-1}(x) \in V$ and $(M_m^{\dim V} \cdot V) = 0$. We therefore see that $\varepsilon(M_m;\pi_m^{-1}(x)) = 0$ by [Laz04a, Prop. 5.1.9]. Since this is true for every m, we have that $\varepsilon_N(\|D\|;x) = 0$, hence $\varepsilon(\|D\|;x) = \varepsilon_N(\|D\|;x)$ when $x \in \mathbf{B}_+(D)$.

In the rest of the proof, we therefore assume that $x \notin \mathbf{B}_+(D)$. Since $|M_m|$ induces a map birational onto the image of X' by the assumption that $|D|$ induces a rational map birational onto its image, we may write $M_m \equiv_{\mathbf{Q}} A + E$, where A and E are ample and effective $\mathbf{Q}$-Cartier divisors, respectively, such that $\pi_m^{-1}(x) \notin \mathrm{Supp}(E)$ [Laz04a, Cor.

2.2.7]. For every integer $n \geq 1$, we can write

$$M_m \equiv_{\mathbf{Q}} \frac{1}{n}A + \frac{n-1}{n}M_m + \frac{1}{n}E,$$

hence setting $A_n \coloneqq \frac{1}{n}A + \frac{n-1}{n}M_m$, we have $M_m \equiv_{\mathbf{Q}} A_n + \frac{1}{n}E$ for an ample $\mathbf{Q}$-Cartier divisor A_n and a fixed effective $\mathbf{Q}$-Cartier divisor E. By Definition 7.1.1, we see that

$$\varepsilon\bigl(\|D\|; x\bigr) \geq \frac{\varepsilon\bigl(A_n; \pi_m^{-1}(x)\bigr)}{m}$$

for every n. Using (7.3) and taking the limit as $n \to \infty$, we see that

$$\varepsilon\bigl(\|D\|; x\bigr) \geq \frac{\varepsilon\bigl(M_m; \pi_m^{-1}(x)\bigr)}{m},$$

hence $\varepsilon(\|D\|; x) \geq \varepsilon_N(\|D\|; x)$.

We now show the reverse inequality. Let $f\colon X' \to X$ and $f^*D \equiv_{\mathbf{R}} A + E$ be as in Definition 7.1.1. Fix m such that mA is a very ample Cartier divisor. By taking a normalized blowup of the base locus of $f^*(mD)$, which by assumption is an isomorphism in a neighborhood of $f^{-1}(x)$, we can write

$$|f^*(mD)| = |M_m| + F_m$$

as in Definition 7.2.1. Since mA is free, we have $M_m \sim mA + F'_m$ for an effective Cartier divisor F'_m such that $F'_m \leq mE$, hence $f^{-1}(x) \notin \operatorname{Supp}(F'_m)$. We therefore have

$$\frac{\varepsilon\bigl(M_m; \pi_m^{-1}(x)\bigr)}{m} \geq \varepsilon\bigl(A; f^{-1}(x)\bigr),$$

hence $\varepsilon_N(\|D\|; x) \geq \varepsilon(\|D\|; x)$. $\square$

7.2.2. A description in terms of jet separation

We now show that just as for usual Seshadri constants of ample Cartier divisors at regular closed points on projective varieties [Laz04a, Thm. 5.1.17], moving Seshadri constants can be described using separation of jets. Note that this description (Proposition 7.2.10) is new even in characteristic zero for singular points.

Recall from Definition 2.2.4 that if X is a scheme, $x \in X$ is a closed point, and $\ell \geq -1$ is an integer, a coherent sheaf $\mathscr{F}$ *separates ℓ-jets* at x if the restriction morphism

$$H^0(X, \mathscr{F}) \longrightarrow H^0(X, \mathscr{F} \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1})$$

is surjective, and that we denote by $s(\mathscr{F}; x)$ the largest integer $\ell \geq -1$ such that $\mathscr{F}$ separates ℓ-jets at x. We can then define moving Seshadri constants using jet separation.

Definition 7.2.4. Let X be a projective variety over a field k and let D be a $\mathbf{Q}$-Cartier divisor on X. Consider a k-rational point $x \in X$ with defining ideal $\mathfrak{m}_x \subseteq \mathcal{O}_X$. We set

$$\varepsilon_{\mathrm{jet}}\bigl(\lVert D\rVert; x\bigr) \coloneqq \limsup_{m \to \infty} \frac{s(mD; x)}{m},$$

where the limit supremum runs over all integers $m \geq 1$ such that mD is integral.

We now prove that the limit supremum used to define $\varepsilon_{\mathrm{jet}}(\lVert D\rVert; x)$ is equal to a limit.

Lemma 7.2.5 ([FMa, Lem. 6.4]; cf. [Ito13, Proof of Lem. 3.7])**.** *Let X be a scheme, and let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on X. Then, for every closed point $x \in X$ such that $s(\mathscr{F}; x) \geq 0$ and $s(\mathscr{G}; x) \geq 0$, we have*

$$s(\mathscr{F}; x) + s(\mathscr{G}; x) \leq s(\mathscr{F} \otimes \mathscr{G}; x).$$

With notation in Definition 7.2.4, we therefore have

$$\varepsilon_{\mathrm{jet}}\bigl(\lVert D\rVert; x\bigr) = \lim_{m \to \infty} \frac{s(mD; x)}{m} = \sup_{m \geq 1} \frac{s(mD; x)}{m}.$$

Proof. We first show that a coherent sheaf $\mathscr{F}$ separates ℓ-jets if and only if

$$H^0(X, \mathfrak{m}_x^i \mathscr{F}) \longrightarrow H^0(X, \mathfrak{m}_x^i \mathscr{F}/\mathfrak{m}_x^{i+1}\mathscr{F}) \tag{7.4}$$

is surjective for every $i \in \{0, 1, \ldots, \ell\}$. We proceed by induction on ℓ. If $\ell = 0$, then there is nothing to show. Now suppose $\ell > 0$. By induction and the fact that a coherent sheaf separating ℓ-jets also separates all lower order jets, it suffices to show that if $\mathscr{F}$ separates $(\ell - 1)$-jets, then $\mathscr{F}$ separates ℓ-jets if and only if (7.4) is surjective for $i = \ell$.

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_x^\ell \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}/\mathfrak{m}_x^\ell \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{m}_x^\ell \mathscr{F}/\mathfrak{m}_x^{\ell+1}\mathscr{F} \arrow[r] & \mathscr{F}/\mathfrak{m}_x^{\ell+1}\mathscr{F} \arrow[r] & \mathscr{F}/\mathfrak{m}_x^\ell \mathscr{F} \arrow[r] & 0
\end{tikzcd}$$

Taking global sections, we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathfrak{m}_x^\ell \mathscr{F}) \arrow[r] \arrow[d] & H^0(X, \mathscr{F}) \arrow[r] \arrow[d] & H^0(X, \mathscr{F}/\mathfrak{m}_x^\ell \mathscr{F}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^0(X, \mathfrak{m}_x^\ell \mathscr{F}/\mathfrak{m}_x^{\ell+1}\mathscr{F}) \arrow[r] & H^0(X, \mathscr{F}/\mathfrak{m}_x^{\ell+1}\mathscr{F}) \arrow[r] & H^0(X, \mathscr{F}/\mathfrak{m}_x^\ell \mathscr{F})
\end{tikzcd}$$

where the top row remains exact by the assumption that $\mathscr{F}$ separates $(\ell-1)$-jets. By the snake lemma, we see that the left vertical arrow is surjective if and only if the middle vertical arrow is surjective, as desired.

We now prove the lemma. Suppose $\mathscr{F}$ separates i-jets and $\mathscr{G}$ separates j-jets. We then have the commutative diagram

$$\begin{tikzcd}
H^0(X, \mathfrak{m}_x^i \mathscr{F}) \otimes H^0(X, \mathfrak{m}_x^j \mathscr{G}) \arrow[r] \arrow[d] & H^0(X, \mathfrak{m}_x^i \mathscr{F}/\mathfrak{m}_x^{i+1}\mathscr{F} \otimes \mathfrak{m}_x^j \mathscr{G}/\mathfrak{m}_x^{j+1}\mathscr{G}) \arrow[d] \\
H^0\bigl(X, \mathfrak{m}_x^{i+j}(\mathscr{F} \otimes \mathscr{G})\bigr) \arrow[r] & H^0\bigl(X, \mathfrak{m}_x^{i+j}(\mathscr{F} \otimes \mathscr{G})/\mathfrak{m}_x^{i+j+1}(\mathscr{F} \otimes \mathscr{G})\bigr)
\end{tikzcd}$$

Since the top horizontal arrow is surjective by assumption, and the right vertical arrow is surjective, essentially by the surjectivity of

$$\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1} \otimes \mathfrak{m}_x^j/\mathfrak{m}_x^{j+1} \simeq (\mathfrak{m}_x^i \otimes \mathfrak{m}_x^j) \otimes \mathcal{O}_X/\mathfrak{m}_x \twoheadrightarrow \mathfrak{m}_x^{i+j}/\mathfrak{m}_x^{i+j+1},$$

we see that the composition from the top left corner to the bottom right corner is surjective, hence the bottom horizontal arrow is surjective. By running through all combinations of integers $i \le s(\mathscr{F}; x)$ and $j \le s(\mathscr{G}; x)$, we see that $s(\mathscr{F}; x) + s(\mathscr{G}; x) \le s(\mathscr{F} \otimes \mathscr{G}; x)$ by the argument in the previous paragraph.

The last statement about $\varepsilon_{\mathrm{jet}}(\|D\|; x)$ follows from Fekete's lemma [PS98, Pt. I, n° 98], since we have shown the superadditivity of the sequence $\{s(mD; x)\}_{m \ge 1}$ provided that

$x \notin \mathbf{B}(D)$. If $x \in \mathbf{B}(D)$, then $s(mD;x) = -1$ for all $m \geq 1$ such that mD is integral, hence the limit, limit supremum, and supremum are all equal to zero. $\square$

The constant $\varepsilon_{\mathrm{jet}}(\|D\|;x)$ detects $\mathbf{B}_+(D)$.

Lemma 7.2.6. *Let X be a normal projective variety over a field k and let D be a $\mathbf{Q}$-Cartier divisor on X. Consider a k-rational point $x \in X$. Then, $x \in \mathbf{B}_+(D)$ if and only if $\varepsilon_{\mathrm{jet}}(\|D\|;x) = 0$.*

Proof. For $\Leftarrow$, note that $\varepsilon_{\mathrm{jet}}(\|D\|;x) = 0$ implies $s(mD;x) \leq 0$ for all m such that mD is integral, since Lemma 7.2.5 implies $\varepsilon_{\mathrm{jet}}(\|D\|;x)$ is a supremum. If $x \in \mathbf{B}(D)$, then $x \in \mathbf{B}_+(D)$ as well, hence it suffices to consider the case when $x \notin \mathbf{B}(D)$. Suppose $x \notin \mathbf{B}_+(D)$, and let A be a very ample Cartier divisor on X. By [ELM+06, Prop. 1.5] and [Laz04a, Prop. 2.1.21], there exist positive integers q, r such that

$$\mathbf{B}_+(D) = \mathbf{B}(rD - A) = \mathrm{Bs}\bigl(\bigl|q(rD - A)\bigr|\bigr)_{\mathrm{red}}.$$

Since $x \notin \mathbf{B}_+(D)$, we see that $|q(rD-A)|$ is basepoint-free at x. Moreover, since $\mathcal{O}_X(qA)$ separates 1-jets at x by the very ampleness of A, we see that $\mathcal{O}_X(qrD)$ separates 1-jets at x by Lemma 7.2.5, a contradiction.

For $\Rightarrow$, we note that if $x \in \mathbf{B}(D)$, then $s(mD;x) = -1$ for all m such that mD is integral, hence it suffices to consider the case when $x \in \mathbf{B}_+(D) \smallsetminus \mathbf{B}(D)$. Suppose $s(mD;x) > 0$ for some integer $m > 0$. By possibly replacing m with a large and divisible enough multiple, we may assume that $\mathbf{B}(mD) = \mathrm{Bs}(|mD|)_{\mathrm{red}}$ by Lemma 7.2.5 and [Laz04a, Prop. 2.1.21]. Then, for every subvariety $V \subseteq X$ containing x, we have $V \nsubseteq \mathrm{Bs}(|mD|)_{\mathrm{red}}$. Moreover, since $\mathcal{O}_X(mD)$ separates tangent directions at x, there exists $E \in |mD|$ not containing V, in which case $(E^{\dim V} \cdot V) > 0$.

Now let $\pi_m \colon X_m \to X$ be the normalized blowup of $\mathfrak{b}(|mD|)$, and write $|\pi_m^*(mD)| = |M_m| + F_m$ where $\mathcal{O}_{X_m}(-F_m) = \mathfrak{b}(|mD|) \cdot \mathcal{O}_{X_m}$. By Proposition 4.6.4 and the definition of the augmented base locus, we have

$$\pi_m^{-1}(x) \in \mathbf{B}_+(\pi_m^* D) \subseteq \mathbf{B}_+(M_m) \cup \mathrm{Supp}(F_m).$$

The fact that $x \notin \mathbf{B}(D)$ implies $\pi_m^{-1}(x) \in \mathbf{B}_+(M_m)$. By Theorem 4.6.6, there therefore exists a subvariety $W \subseteq X_m$ such that $\pi_m^{-1}(x) \in W$ and $(M_m^{\dim W} \cdot W) = 0$. Now choose

$E \in |mD|$ as in the previous paragraph for $V = \pi_{m*}W$. Since $F_m = (\pi_m)^{-1}\operatorname{Bs}(|mD|)$, we have that $\pi_m^*E - F_m$ is an effective Cartier divisor that contains $\pi_m^{-1}(x)$ but does not contain W, hence

$$(M_m^{\dim W} \cdot W) = \bigl((\pi_m^*E - F_m)^{\dim W} \cdot W\bigr) > 0,$$

a contradiction. We therefore have $\varepsilon_{\mathrm{jet}}(\|D\|;x) = 0$ if $x \in \mathbf{B}_+(D)$. $\square$

This lemma has the following consequence:

Corollary 7.2.7. *Let X be a normal projective variety over an algebraically closed field k, and let D be a $\mathbf{Q}$-Cartier divisor on X. Then, D is ample if and only if $\varepsilon_{\mathrm{jet}}(\|D\|;x) > 0$ for every closed point $x \in X$.*

Proof. By Lemma 7.2.6, we have that $\varepsilon_{\mathrm{jet}}(\|D\|;x) > 0$ for every closed point $x \in X$ if and only if $\mathbf{B}_+(D) = \emptyset$. This condition is equivalent to the ampleness of D by [ELM$^+$06, Ex. 1.7]. $\square$

We now collect some properties of $\varepsilon_{\mathrm{jet}}(\|D\|;x)$ analogous to those in Proposition 7.1.2. Below, $\operatorname{vol}_{X|V}(D)$ denotes the restricted volume of D along a subvariety V, as defined in Definition 4.6.13.

Proposition 7.2.8. *Let X be a projective variety over a field k and let $x \in X$ be a k-rational point. Then, the function*

$$\begin{aligned} \operatorname{Big}^{\{x\}}_{\mathbf{Q}}(X) &\longrightarrow \mathbf{R}_{>0} \\ D &\longmapsto \varepsilon_{\mathrm{jet}}\bigl(\|D\|;x\bigr) \end{aligned} \tag{7.5}$$

is continuous, and extends to a continuous function $\operatorname{Big}^{\{x\}}_{\mathbf{R}}(X) \to \mathbf{R}_{>0}$. Moreover, if D is a $\mathbf{R}$-Cartier divisor, then we have the following:

(i) $\varepsilon_{\mathrm{jet}}(\|D\|;x) \le (\operatorname{vol}_{X|V}(D)/e(\mathcal{O}_{V,x}))^{1/\dim V}$ *for every positive-dimensional subvariety $V \subseteq X$ containing x;*

(ii) If D and E are numerically equivalent $\mathbf{R}$-Cartier divisors, then $\varepsilon_{\mathrm{jet}}(\|D\|;x) = \varepsilon_{\mathrm{jet}}(\|E\|;x)$;

(iii) $\varepsilon_{\text{jet}}(\|\lambda D\|;x) = \lambda \cdot \varepsilon_{\text{jet}}(\|D\|;x)$ *for every positive real number* λ*;*

(iv) If D' *is another* $\mathbf{R}$*-Cartier divisor such that* $x \notin \mathbf{B}_+(D) \cup \mathbf{B}_+(D')$*, then*

$$\varepsilon_{\text{jet}}\big(\|D + D'\|;x\big) \geq \varepsilon_{\text{jet}}\big(\|D\|;x\big) + \varepsilon_{\text{jet}}\big(\|D'\|;x\big).$$

Proof. We will prove (i)–(iv) for $\mathbf{Q}$-Cartier divisors D, D', E and $\lambda \in \mathbf{Q}_{>0}$. Then, (ii) will imply that the function (7.5) is well-defined, and the fact that it extends to a continuous function on $\mathrm{Big}^{\{x\}}_{\mathbf{R}}(X)$ follows from (iii) and (iv) by [ELM+09, Rem. 5.4], since $\varepsilon_{\text{jet}}(\|A\|;x) > 0$ for ample A (Corollary 7.2.7). Finally, the general case for (i)–(iv) will follow by continuity.

We first prove (iii) when D is a $\mathbf{Q}$-Cartier divisor and $\lambda \in \mathbf{Q}_{>0}$. We have

$$\begin{aligned}
\lambda \cdot \varepsilon_{\text{jet}}\big(\|D\|;x\big) = \lambda \cdot \lim_{m\to\infty} \frac{s(mD;x)}{m} &= \lim_{m\to\infty} \frac{s(mD;x)}{m/\lambda} \\
= \lim_{m\to\infty} \frac{s(m\lambda D;x)}{m} &= \varepsilon_{\text{jet}}\big(\|\lambda D\|;x\big)
\end{aligned}$$

where the third equality holds since both sides are equal to the limits running over all m sufficiently divisible. To show the remaining properties, then, it suffices to consider the case when D, D', E are Cartier divisors.

Next, we prove (i) when D is a Cartier divisor. Since the inequality trivially holds when $\varepsilon_{\text{jet}}(\|D\|;x) = 0$, we may assume that $\varepsilon_{\text{jet}}(\|D\|;x) > 0$. In this case, we have

$$\begin{aligned}
\frac{\mathrm{vol}_{X|V}(D)}{\mathrm{mult}_x V} &= \lim_{m\to\infty} \frac{h^0\big(X|V, \mathcal{O}_X(mD)\big)}{m^{\dim V}/(\dim V)!} \cdot \lim_{\ell\to\infty} \frac{(\ell+1)^{\dim V}/(\dim V)!}{h^0\big(V, \mathcal{O}_V(mD) \otimes \mathcal{O}_V/\mathfrak{m}_x^{\ell+1}\big)} \\
&= \lim_{m\to\infty} \frac{h^0\big(X|V, \mathcal{O}_X(mD)\big)}{h^0\big(V, \mathcal{O}_V(mD) \otimes \mathcal{O}_V/\mathfrak{m}_x^{s(mD;x)+1}\big)} \cdot \left(\frac{s(mD;x)+1}{m}\right)^{\dim X},
\end{aligned}$$

where the second equality follows from setting $\ell = s(mD;x)$, and then from the fact that $s(mD;x) \to \infty$ as $m \to \infty$. By definition of $s(mD;x)$ and the commutativity of the diagram

$$\begin{array}{ccc} H^0\big(X, \mathcal{O}_X(mD)\big) & \longrightarrow & H^0\big(X, \mathcal{O}_X(mD) \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1}\big) \\ \downarrow & & \downarrow\!\!\!\downarrow \\ H^0\big(V, \mathcal{O}_V(mD)\big) & \longrightarrow & H^0\big(V, \mathcal{O}_V(mD) \otimes \mathcal{O}_V/\mathfrak{m}_x^{\ell+1}\big) \end{array}$$

we have $h^0(X|V, \mathcal{O}_X(mD)) \geq h^0(V, \mathcal{O}_V(mD) \otimes \mathcal{O}_V/\mathfrak{m}_x^{s(mD;x)+1})$. Thus,

$$\frac{\operatorname{vol}_{X|V}(D)}{\operatorname{mult}_x V} \geq \lim_{m\to\infty} \left(\frac{s(mD;x)+1}{m}\right)^{\dim V} = \varepsilon_{\text{jet}}\big(\|D\|;x\big)^{\dim V}.$$

We now prove (ii). First, recall that $\mathbf{B}_+(D)$ only depends on the numerical class of D, and that if $x \in \mathbf{B}_+(D)$, then $\varepsilon_{\text{jet}}(\|D\|;x) = 0$ by Lemma 7.2.6. We can therefore assume that $x \notin \mathbf{B}_+(D)$. By assumption, there exists a numerically trivial Cartier divisor P such that $D \sim E + P$, and Proposition 4.6.7 implies that there exists a positive integer j such that $\mathcal{O}_X(jD + iP)$ is globally generated away from $\mathbf{B}_+(D)$ for all integers i. For every m, we therefore see that

$$s(mD;x) \leq s\big((m+j)D + (m+j)P;x\big) = s(iE;x)$$

by setting $i = m+j$, where the inequality follows from Lemma 7.2.5 since $\mathcal{O}_X(jD + (m+j)P)$ separates 0-jets at x. Dividing by m and taking limits, we see that $\varepsilon_{\text{jet}}(\|D\|;x) \leq \varepsilon_{\text{jet}}(\|E\|;x)$. Repeating the argument above after switching the roles of D and E, we have $\varepsilon_{\text{jet}}(\|D\|;x) = \varepsilon_{\text{jet}}(\|E\|;x)$.

Finally, (iv) follows from Lemma 7.2.5. □

Remark 7.2.9. In Proposition 7.2.8(i), one can ask whether

$$\varepsilon_{\text{jet}}\big(\|D\|;x\big) = \inf_{V \ni x} \left\{\frac{\operatorname{vol}_{X|V}(D)}{e(\mathcal{O}_{V,x})}\right\}^{1/\dim V},$$

where the infimum runs over all subvarieties $V \subseteq X$ containing x. This holds for smooth varieties over the complex numbers [ELM$^+$09, Prop. 6.7], or when D is nef [Laz04a, Prop. 5.1.9]. A proof of this statement in the generality of Proposition 7.2.8 would require extending the main results about restricted volume functions in [ELM$^+$09] to our setting.

We can now prove our comparison result. Note that we do not know of any examples where the equalities in the statement below do not hold without the additional assumptions on D and X.

Proposition 7.2.10 (cf. [ELM$^+$09, Prop. 6.6])**.** *Let X be a projective variety over a field k, and let D be a $\mathbf{Q}$-Cartier divisor. For every k-rational point $x \in X$, we have*

(*i*) $\varepsilon(D;x) = \varepsilon_{\mathrm{jet}}(\lVert D \rVert;x)$ *if D is nef and $x \notin \mathbf{B}_+(D)$, and*

(*ii*) $\varepsilon(\lVert D \rVert;x) = \varepsilon_{\mathrm{jet}}(\lVert D \rVert;x)$ *if X is normal.*

We first show that the case when D is nef implies the general case, under the assumption that X is normal.

Proof that (*i*) *implies* (*ii*)*.* Since both sides are zero if $x \in \mathbf{B}_+(D)$ (Definition 7.1.1 and Lemma 7.2.6), we may assume that $x \notin \mathbf{B}_+(D)$. By homogeneity (Propositions 7.1.2 and 7.2.8), it suffices to consider the case when D is a Cartier divisor and both sides are positive, in which case we still have $x \notin \mathbf{B}_+(D)$. Note that in particular, we have $x \notin \mathbf{B}(D)$.

For every integer $m \geq 1$ such that $x \notin \mathrm{Bs}(\lvert mD \rvert)$, let $\pi_m \colon X_m \to X$ be as in Definition 7.2.1, and write

$$\bigl\lvert \pi_m^*(mD) \bigr\rvert = \lvert M_m \rvert + F_m,$$

where $\lvert M_m \rvert$ is the moving part and F_m is the fixed part of the linear system $\lvert \pi_m^*(mD) \rvert$. Since X is normal, we have $\pi_{m*}\mathcal{O}_{X_m} \simeq \mathcal{O}_X$ [Har77, Proof of Cor. III.11.4]. Note that the base ideal of $\lvert \pi_m^*(mD) \rvert$ is $\mathcal{O}_{X_m}(-F_m)$, and in particular, we have $x_m \coloneqq \pi_m^{-1}(x) \notin \operatorname{Supp} F_m$. Thus, the inclusion

$$H^0\bigl(X_m, \mathcal{O}_{X_m}(M_m)\bigr) \subseteq H^0\bigl(X_m, \pi_m^*\mathcal{O}_X(mD)\bigr)$$

induced by multiplication by F_m is a bijection, and M_m is a free Cartier divisor. We then have the commutative diagram

$$\begin{tikzcd}
H^0\big(X_m, \mathcal{O}_{X_m}(nM_m)\big) \arrow[r] \arrow[d, hook, "\cdot nF_m"'] & H^0\big(X_m, \mathcal{O}_{X_m}(nM_m) \otimes \mathcal{O}_X/\mathfrak{m}_{x_m}^{\ell+1}\big) \arrow[d, "\cdot nF_m"', "\wr"] \\
H^0\big(X_m, \pi_m^*\mathcal{O}_X(mnD)\big) \arrow[r] & H^0\big(X_m, \pi_m^*\mathcal{O}_X(mnD) \otimes \mathcal{O}_X/\mathfrak{m}_{x_m}^{\ell+1}\big) \\
H^0\big(X, \mathcal{O}_X(mnD)\big) \arrow[r] \arrow[u, "\wr"'] & H^0\big(X, \mathcal{O}_X(mnD) \otimes \mathcal{O}_X/\mathfrak{m}_{x}^{\ell+1}\big) \arrow[u, "\wr"']
\end{tikzcd} \tag{7.6}$$

for all integers $n \geq 1$ and $\ell \geq -1$, where the top left vertical arrow is an isomorphism for $n = 1$ by the discussion above, the bottom left vertical arrow is an isomorphism by the fact that $\pi_{m*}\mathcal{O}_{X_m} \simeq \mathcal{O}_X$, and the right vertical arrows are isomorphisms by the fact that $x \notin \operatorname{Supp} F_m$ and π_m is an isomorphism in a neighborhood of x, respectively.

To show the inequality $\geq$ in (ii), let m be such that $\varepsilon(M_m; x_m) > 0$, in which case $x_m \notin \mathbf{B}_+(M_m)$ by Theorem 4.6.6 and [Laz04a, Prop. 5.1.9]. Note that this property holds for all sufficiently large m by Proposition 7.2.3. We then have the chain of inequalities

$$\varepsilon\big(\|D\|; x\big) \geq \frac{\varepsilon(M_m; x_m)}{m} \geq \frac{s(M_m; x_m)}{m} = \frac{s(mD; x)}{m}$$

for all such m, where the second inequality follows from (i), and the equality follows from the commutativity of the diagram (7.6). Taking the limit as $m \to \infty$, we have the inequality $\geq$ in (ii).

To show the inequality $\leq$ in (ii), let $\delta > 0$ be arbitrary. For $m \gg 0$ and $n \gg 0$, we have the following chain of inequalities:

$$\varepsilon\big(\|D\|; x\big) \leq \frac{\varepsilon(M_m; x_m)}{m} + \frac{\delta}{2} \leq \frac{s(nM_m; x_m)}{mn} + \delta \leq \frac{s(mnD; x_m)}{mn} + \delta.$$

For the middle inequality, we need m to be sufficiently large such that $\varepsilon(M_m; x_m) > 0$ as in the previous paragraph, in which case the inequality follows from (i) for $n \gg 0$. The last inequality follows from the commutativity of the diagram (7.6). Taking the limit as $m \to \infty$, and using the fact that δ was arbitrary, we have the inequality $\leq$ in (ii). $\square$

To prove (i), we need the following elementary lemma:

Lemma 7.2.11. *Let X be a projective variety of dimension n, and let $x \in X$ be a closed point with defining ideal $\mathfrak{m}_x \subseteq \mathcal{O}_X$. Let L be a Cartier divisor on X.*

(i) If L is ample, then for m sufficiently large, we have $H^i(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^a) = 0$ for all $i > 1$ and $a \geq 0$.

(ii) If $H^1(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^a) \neq 0$ for some $a, m > 0$, then $H^1(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^{a+1}) \neq 0$.

Proof. For (i), consider the exact sequence

$$H^{i-1}\bigl(X, \mathcal{O}_X(mL) \otimes \mathcal{O}_X/\mathfrak{m}_x^a\bigr) \longrightarrow H^i\bigl(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^a\bigr) \longrightarrow H^i\bigl(X, \mathcal{O}_X(mL)\bigr).$$

Note that the left-hand term vanishes if $i > 1$ since $\mathcal{O}_X/\mathfrak{m}_x^a$ has zero-dimensional support. We have that $H^i(X, \mathcal{O}_X(mL)) = 0$ for all m sufficiently large by Serre vanishing, hence the exact sequence implies $H^i(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^a) = 0$ as well.

For (ii), consider the exact sequence

$$H^1\bigl(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^{a+1}\bigr) \longrightarrow H^1\bigl(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^a\bigr) \longrightarrow H^1\bigl(X, \mathcal{O}_X(mL) \otimes \mathfrak{m}_x^a/\mathfrak{m}_x^{a+1}\bigr).$$

The sheaf $\mathfrak{m}_x^a/\mathfrak{m}_x^{a+1}$ has zero-dimensional support, hence the right-hand term vanishes, and the desired non-vanishing follows. □

We can now prove Proposition 7.2.10(i). Part of the proof below was suggested by Harold Blum, following the strategy in [Fuj18, Thm. 2.3].

Proof of Proposition 7.2.10(i). By continuity and homogeneity (Proposition 7.2.8), it suffices to consider the case when D is an ample Cartier divisor.

We prove the inequality $\geq$ in (i). Let $0 < \delta \ll 1$ be arbitrary, and fix positive integers p_0, q_0 such that

$$0 < \frac{p_0}{2q_0} < \varepsilon(D; x) < \frac{p_0}{q_0} < \varepsilon(D; x) + \delta.$$

Then, denoting by $\mu\colon \widetilde{X} \to X$ the blowup at x with exceptional divisor E, we have that $A \coloneqq 2q_0\mu^*D - p_0E$ is ample while $B \coloneqq q_0\mu^*D - p_0E$ is not ample by Lemma 2.4.1. By Theorem E and the homogeneity of asymptotic cohomological functions (Proposition 4.6.9), for some integer $r \gg 2$ and for some $i \geq 1$, we have that

$$H^i\bigl(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\bigl(mr\bigl(B - (1/r)A\bigr)\bigr)\bigr) \neq 0$$

for infinitely many m. Now

$$\begin{aligned} mr\bigl(B-(1/r)A\bigr) &= mrq_0\bigl(1-(2/r)\bigr)\mu^*D - mrp_0\bigl(1-(1/r)\bigr)E \\ &= m\bigl((r-2)q_0\mu^*D-(r-1)p_0E\bigr), \end{aligned}$$

and defining $q_1=(r-2)q_0$ and $p_1=(r-1)p_0$, the Leray spectral sequence applied to the blowup morphism μ [Laz04a, Lem. 5.4.24] implies

$$H^i\bigl(X,\mathcal{O}_X(mq_1D)\otimes\mathfrak{m}_x^{mp_1}\bigr)\simeq H^i\bigl(\widetilde{X},\mathcal{O}_{\widetilde{X}}\bigl(m(q_1\mu^*D-p_1E)\bigr)\bigr)\neq 0$$

for infinitely many m. By Lemma 7.2.11(i), we must have $i=1$. Since

$$H^1\bigl(X,\mathcal{O}_X(mq_1D)\bigr)=0$$

for all $m\gg 0$ by Serre vanishing, this implies that mq_1D does not separate (mp_1-1)-jets at x, hence $mp_1-1>s(mq_1D;x)$ for infinitely many m. Dividing the inequality by mq_1 and taking limits as $m\to\infty$, we have

$$\varepsilon(D;x)+\delta>\frac{p_0}{q_0}>\frac{(r-1)p_0}{(r-2)q_0}=\frac{p_1}{q_1}\geq\lim_{m\to\infty}\frac{s(mq_1D;x)}{mq_1}=\varepsilon_{\mathrm{jet}}\bigl(\|D\|;x\bigr),$$

where the limit runs over all m sufficiently large and divisible, and the last equality holds by the fact that $\varepsilon_{\mathrm{jet}}(\|D\|;x)$ is computed by a limit (Lemma 7.2.5). Finally, since δ was arbitrary, the inequality $\geq$ in (i) follows.

We now prove the inequality $\leq$ in (i). Let $0<\delta\ll 1$ be arbitrary, and fix positive integers p_0,q_0 such that

$$\varepsilon(D;x)-\delta<\frac{p_0}{q_0}<\varepsilon(D;x).$$

Then, denoting by $\mu\colon\widetilde{X}\to X$ the blowup at x with exceptional divisor E, we have that $q_0\mu^*D-p_0E$ is ample, hence by Fujita's vanishing theorem [Fuj83, Thm. 5.1] there exists a natural number n_0 such that

$$H^1\bigl(\widetilde{X},\mathcal{O}_{\widetilde{X}}\bigl(n(q_0\mu^*D-p_0E)+P\bigr)\bigr)=0$$

for every integer $n\geq n_0$ and all nef Cartier divisors P on $\widetilde{X}$. Now let $m\geq n_0q_0$ be an integer, and write $m = nq_0 + q_1$ with $0 \le q_1 < q_0$ and $n \ge n_0$. Applying the vanishing above for $P = q_1\mu^*D$, we have that

$$H^1\bigl(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(m\mu^*D - np_0E)\bigr) = 0.$$

By the Leray spectral sequence applied to the blowup morphism μ [Laz04a, Lem. 5.4.24], we have an isomorphism

$$H^1\Bigl(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\bigl(m\mu^*D - np_0E\bigr)\Bigr) \simeq H^1\Bigl(X, \mathcal{O}_X(mD) \otimes \mathfrak{m}_x^{np_0}\bigr)\Bigr) = 0$$

for $m \gg 0$ (which implies $n \gg 0$). Thus, we see that $\mathcal{O}_X(mD)$ separates $(np_0 - 1)$-jets at x. Now consider the following chain of inequalities:

$$\begin{aligned}\frac{s(mD;x)}{m} \ge \frac{np_0 - 1}{m} \ge \frac{np_0 - 1}{(n+1)q_0} &= \frac{n}{n+1} \cdot \frac{p_0}{q_0} - \frac{1}{(n+1)q_0} \\ &> \frac{n}{n+1}\bigl(\varepsilon(D;x) - \delta\bigr) - \frac{1}{(n+1)q_0}.\end{aligned}$$

Taking the limit as $m \to \infty$, we have that $n \to \infty$ as well, hence $\varepsilon_{\mathrm{jet}}(\|D\|;x) \ge \varepsilon(D;x) - \delta$. Finally, since δ was arbitrary, the inequality $\le$ in (i) follows. $\square$

Remark 7.2.12. We give an alternative proof of Proposition 7.2.10(i) in [FMa, Thm. 6.3].

7.3. A generalization of Theorem B

Our goal in this section is to prove the following generalization of Theorem B.

Theorem 7.3.1. *Let X be a projective variety of dimension n over a field k, and let L be a line bundle on X. Let $x \in X$ be a k-rational point such that either X has singularities of dense F-injective type at x in characteristic zero, or X has F-injective singularities at x in positive characteristic. Suppose that for some integer $\ell \ge 0$, one of the following holds:*

- (i) *L is nef and $\varepsilon(L;x) > n + \ell$; or*
- (ii) *X is normal and $\varepsilon(\|L\|;x) > n + \ell$.*

Then, the sheaf $\omega_X \otimes L$ separates ℓ-jets at x.

We will first show the statement in positive characteristic, from which we will deduce the characteristic zero case via reduction modulo p.

7.3.1. Proof in positive characteristic

We state the main technical result that will imply Theorem 7.3.1.

Theorem 7.3.2. *Let X be a projective variety of dimension n over a field k of characteristic $p > 0$, and let L be a Cartier divisor on X. Consider a k-rational point $x \in X \smallsetminus \mathbf{B}_+(L)$ and consider a coherent sheaf $\mathscr{F}$ on X together with a morphism $\tau\colon F^g_*\mathscr{F} \to \mathscr{F}$ that is surjective at x. If $\varepsilon_{\mathrm{jet}}(\|L\|; x) > n + \ell$ for an integer $\ell \geq 0$, then $\mathscr{F} \otimes \mathcal{O}_X(L)$ separates ℓ-jets at x.*

We note that a coherent sheaf $\mathscr{F}$ on X together with a morphism $\tau\colon F^g_*\mathscr{F} \to \mathscr{F}$ is an example of a *Cartier module* as defined in [BB11].

Proof. We proceed in a sequence of steps, following the outline of the proof of [MS14, Thm. 3.1] and [Mur18, Thm. C].

We first claim that for every integer $t > 0$, there exist a positive integer m_0 and a sequence $\{d_e\}$ such that $\mathcal{O}_X(m_0 d_e L)$ separates $(\ell p^{ge} + n(p^{ge} - 1) + t)$-jets at x for all $e > 0$, and such that $p^{ge} - m_0 d_e \to \infty$ as $e \to \infty$. Let $0 < \delta \ll 1$. By Lemma 7.2.5, there exists an m_0 such that

$$\frac{s(m_0 L; x)}{m_0} > (1 + \delta)(n + \ell).$$

Now for every integer $e > 0$, let

$$d_e = \left\lceil \frac{\ell p^{ge} + n(p^{ge} - 1) + t}{s(m_0 L; x)} \right\rceil.$$

By the superadditivity property (Lemma 7.2.5), we have

$$s(m_0 d_e L; x) \geq d_e \cdot s(m_0 L; x) \geq \ell p^{ge} + n(p^{ge} - 1) + t,$$

hence $\mathscr{O}_X(m_0 d_e L)$ separates $(\ell p^{ge} + n(p^{ge}-1)+t)$-jets at x. We now claim that $p^{ge} - m_0 d_e \to \infty$ as $e \to \infty$. Note that

$$\begin{aligned}
p^{ge} - m_0 d_e &= p^{ge} - m_0 \cdot \left\lceil \frac{\ell p^{ge} + n(p^{ge}-1)+t}{s(m_0 H; x)} \right\rceil \\
&\geq p^{ge} - \bigl(\ell p^{ge} + n(p^{ge}-1)+t\bigr) \cdot \frac{m_0}{s(m_0 H; x)} - m_0 \\
&\geq p^{ge} - \bigl(\ell p^{ge} + n(p^{ge}-1)+t\bigr) \cdot \frac{1}{(1+\delta)(n+\ell)} - m_0
\end{aligned}$$

and as $e \to \infty$, we have

$$\begin{aligned}
\lim_{e\to\infty} (p^{ge} - m_0 d_e) &\geq \lim_{e\to\infty} \left(p^{ge} - (\ell p^{ge} + n(p^{ge}-1)+t) \cdot \frac{1}{(1+\delta)(n+\ell)} - m_0 \right) \\
&= \lim_{e\to\infty} p^{ge} \left(1 - \frac{1}{1+\delta} \right) - m_0 = \infty.
\end{aligned}$$

We therefore see that $\mathscr{O}_X(m_0 d_e L)$ separates $(\ell p^{ge} + n(p^{ge}-1)+t)$-jets at x, and that $p^{ge} - m_0 d_e \to \infty$ as $e \to \infty$.

We now show that there exist a positive integer e such that the restriction morphism

$$H^0\bigl(X, \mathscr{F} \otimes \mathscr{O}_X(p^{ge}L)\bigr) \longrightarrow H^0\left(X, \frac{\mathscr{F} \otimes \mathscr{O}_X(p^{ge}L)}{(\mathfrak{m}_x^{\ell+1})^{[p^{ge}]}(\mathscr{F} \otimes \mathscr{O}_X(p^{ge}L))}\right) \tag{7.7}$$

is surjective. By Lemma 5.2.2, there exists an integer $t \geq 0$ such that

$$\mathfrak{m}_x^{\ell p^{ge} + n(p^{ge}-1)+1+t} \subseteq (\mathfrak{m}_x^{\ell+1})^{[p^{ge}]}, \tag{7.8}$$

for all $e > 0$. Now let m_0 and $\{d_e\}$ as in the previous paragraph. Since $x \notin \mathbf{B}_+(L)$ and since $p^{ge} - m_0 d_e \to \infty$, Proposition 4.6.7 implies $\mathscr{F} \otimes \mathscr{O}_X((p^{ge} - m_0 d_e)L)$ is globally generated at x for some $e \gg 0$. Since $\mathscr{O}_X(m_0 d_e L)$ separates $(\ell p^{ge} + n(p^{ge}-1)+t)$-jets at x, Lemma 7.2.5 implies

$$\mathscr{F} \otimes \mathscr{O}_X(p^{ge}L) \simeq \mathscr{F} \otimes \mathscr{O}_X\bigl((p^{ge} - m_0 d_e)L\bigr) \otimes \mathscr{O}_X(m_0 d_e L)$$

separates $(\ell p^{ge} + n(p^{ge}-1)+t)$-jets at x. The inclusion (7.8) then implies the surjectivity of (7.7).

We now use the eth iterate τ^e of the morphism τ defined as the composition

$$F_*^{ge}\mathscr{F} \xrightarrow{F_*^{g(e-1)}\tau} F_*^{g(e-1)}\mathscr{F} \xrightarrow{F_*^{g(e-2)}\tau} \cdots \xrightarrow{\tau} \mathscr{F}$$

to take out the factors of p^{ge}. Note that τ^e is surjective at x by assumption, since the Frobenius and its iterates are affine morphisms. Twisting τ^e by $\mathcal{O}_X(L)$, we have a morphism

$$F_*^{ge}(\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L)) \longrightarrow \mathscr{F} \otimes \mathcal{O}_X(L)$$

that is surjective at x, and by considering the $\mathcal{O}_X$-module structure on $F_*^{ge}(\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L))$, we obtain a morphism

$$F_*^{ge}\big((\mathfrak{m}_x^{\ell+1})^{[p^{ge}]}\big(\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L)\big)\big) \longrightarrow \mathfrak{m}_x^{\ell+1}\big(\mathscr{F} \otimes \mathcal{O}_X(L)\big).$$

We therefore have the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
F_*^{ge}\big((\mathfrak{m}_x^{\ell+1})^{[p^{ge}]}\big(\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L)\big)\big) \arrow[r] \arrow[d] & \mathfrak{m}_x^{\ell+1}\big(\mathscr{F} \otimes \mathcal{O}_X(L)\big) \arrow[d] \\
F_*^{ge}\big(\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L)\big) \arrow[r] \arrow[d] & \mathscr{F} \otimes \mathcal{O}_X(L) \arrow[d] \\
F_*^{ge}\left(\dfrac{\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L)}{(\mathfrak{m}_x^{\ell+1})^{[p^{ge}]}(\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L))}\right) \arrow[r] \arrow[d] & \dfrac{\mathscr{F} \otimes \mathcal{O}_X(L)}{\mathfrak{m}_x^{\ell+1}(\mathscr{F} \otimes \mathcal{O}_X(L))} \arrow[d] \\
0 & 0
\end{tikzcd}$$

where the horizontal arrows are induced by τ^e, and are therefore surjective at x. Note that the left column is exact since the Frobenius morphism F is affine. Taking global sections in the bottom square, we obtain the following commutative square:

$$\begin{tikzcd} H^0\bigl(X, \mathscr{F} \otimes \mathcal{O}_X(p^{ge}L)\bigr) \arrow[r] \arrow[d, two heads, "\varphi"'] & H^0\bigl(X, \mathscr{F} \otimes \mathcal{O}_X(L)\bigr) \arrow[d] \\ H^0\left(X, \dfrac{\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L)}{(\mathfrak{m}_x^{\ell+1})^{[p^{ge}]}(\mathscr{F} \otimes \mathcal{O}_X(p^{ge}L))}\right) \arrow[r, two heads, "\psi"] & H^0\left(X, \dfrac{\mathscr{F} \otimes \mathcal{O}_X(L)}{\mathfrak{m}_x^{\ell+1}(\mathscr{F} \otimes \mathcal{O}_X(L))}\right) \end{tikzcd}$$

where ψ is surjective since the corresponding morphism of sheaves is a surjective morphism of skyscraper sheaves supported at x. Since the restriction map φ is surjective by the previous paragraph, the right vertical map is necessarily surjective by the commutativity of the diagram. Thus, the sheaf $\mathscr{F} \otimes \mathcal{O}_X(L)$ indeed separates ℓ-jets at x. $\square$

To prove Theorem 7.3.1, we need the following elementary lemma.

Lemma 7.3.3. *Let X be a projective variety over a field k, let $\mathscr{F}$ be a coherent sheaf on X, and let $x \in X$ be a k-rational point. Consider a field extension $k \subseteq k'$ such that $X \times_k k'$ is a variety, and such that denoting by*

$$\pi\colon X \times_k k' \longrightarrow X$$

the first projection morphism, the inverse image $\pi^{-1}(x)$ of x consists of a single k'-rational point. Then, for every integer $\ell \geq 0$, the sheaf $\mathscr{F}$ separates ℓ-jets at x if and only if $\pi^*\mathscr{F}$ separates ℓ-jets at $\pi^{-1}(x)$. In particular, $\varepsilon_{\mathrm{jet}}(\lVert D\rVert; x) = \varepsilon_{\mathrm{jet}}(\lVert \pi^*D\rVert; \pi^{-1}(x))$ for every $\mathbf{Q}$-Cartier divisor D on X.

Proof. The first statement follows from faithfully flat base change, which also implies $\varepsilon_{\mathrm{jet}}(\lVert D\rVert; x) = \varepsilon_{\mathrm{jet}}(\lVert \pi^*D\rVert; \pi^{-1}(x))$ for Cartier divisors. We then obtain the same equality for $\mathbf{Q}$-Cartier divisors by homogeneity (Proposition 7.2.8(*iii*)). $\square$

We now prove Theorem 7.3.1.

Proof of Theorem 7.3.1 in positive characteristic. In either setting, we note that

$$\varepsilon_{\mathrm{jet}}\bigl(\lVert L\rVert; x\bigr) > n + \ell$$

by Proposition 7.2.10, where for (i), we note that $\varepsilon(L;x) > n + \ell$ implies $x \notin \mathbf{B}_+(L)$ by Theorem 4.6.6 and [Laz04a, Prop. 5.1.9]. We therefore need to check that the rest of the hypotheses in Theorem 7.3.2 can be satisfied.

We first claim that we can reduce to the case when k is F-finite. By Theorem B.1.6 applied simultaneously to X, $\operatorname{Spec}\mathcal{O}_{X,x}$, and $\{x\}$, there exists a field extension $k \subseteq k^\Gamma$ such that denoting the projection morphism by $\pi^\Gamma \colon X^\Gamma \to X$, the scheme X^Γ is a variety and $x^\Gamma := (\pi^\Gamma)^{-1}(x)$ is a closed k^Γ-rational point such that $\mathcal{O}_{X^\Gamma,x^\Gamma}$ is F-injective. The formation of ω_X is compatible with ground field extensions [Har66, Cor. V.3.4(a)] as is the nefness of L [Kle05, Prop. B.17], and $\varepsilon_{\mathrm{jet}}(\lVert L\rVert;x)$ is invariant under the ground field extension $k \subseteq k^\Gamma$ by Lemma 7.3.3. Since the condition that $\omega_X \otimes \mathcal{O}_X(L)$ separates ℓ-jets can also be checked after base change to k^Γ by Lemma 7.3.3, it therefore suffices to consider the case when k is F-finite. In this case, we can apply Theorem 7.3.2 for $\mathscr{F} = \omega_X$, since the trace morphism $\mathrm{Tr}_X \colon F_*\omega_X \to \omega_X$ is surjective at x by the F-injectivity of $\mathcal{O}_{X,x}$ (Lemma A.9). □

Remark 7.3.4. The original proof of Theorem B in [Mur18, Thm. A] inspired the proof of Theorem 7.3.2 given above. The idea is that the surjectivity of restriction maps of the form in (7.8) can be detected by *Frobenius–Seshadri constants*, which are a positive-characteristic version of Seshadri constants introduced in [MS14] and [Mur18]. These constants are defined as follows: Let L be a Cartier divisor on a complete variety X over a field k, and let $x \in X$ be a k-rational point. Denote by $s_F^\ell(mL;x)$ the largest integer $e \geq 0$ such that the restriction map

$$H^0\bigl(X, \mathcal{O}_X(mL)\bigr) \longrightarrow H^0\bigl(X, \mathcal{O}_X(mL) \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\bigr)$$

is surjective. Then, the *ℓth Frobenius–Seshadri constant* of L at x is

$$\varepsilon_F^\ell(L;x) := \limsup_{m\to\infty} \frac{p^{s_F^\ell(mL;x)} - 1}{m/(\ell+1)}.$$

See [MS14, §2; Mur18, §2] for basic properties of these constants. In particular, lower bounds of the form $\varepsilon_F^\ell(L;x) > \ell + 1$ imply $\omega_X \otimes L$ separates ℓ-jets at x [Mur18, Thm. C]. One can then deduce Theorem 7.3.1 since the pigeonhole principle (Lemma 5.2.2) implies

$$\frac{\ell+1}{\ell+n}\cdot\varepsilon_{\mathrm{jet}}\bigl(\|L\|;x\bigr)\le\varepsilon_F^{\ell}(L;x)\le\varepsilon_{\mathrm{jet}}\bigl(\|L\|;x\bigr),$$

where $n=\dim X$. See the proof of [Mur18, Prop. 2.9].

7.3.2. Proof in characteristic zero

To prove Theorem 7.3.1 in characteristic zero, we fix the following notation.

Setup 7.3.5. Let X be a projective variety over a field k of characteristic zero, and consider a k-rational point $x\in X$ with defining ideal $\mathfrak{m}_x\subseteq\mathcal{O}_X$. We then have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, hook, "i_x"] \arrow[dr, equal] & X \arrow[d, "\pi"] \\ & \operatorname{Spec} k \end{tikzcd}$$

where i_x is the closed embedding corresponding to point x. By spreading out the entire diagram as in Theorem 5.6.2, there exists a domain $A_\lambda\subseteq k$ that is of finite type over $\mathbf{Z}$ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_\lambda \arrow[r, hook, "i_{x,\lambda}"] \arrow[dr, equal] & X_\lambda \arrow[d, "\pi_\lambda"] \\ & \operatorname{Spec} A_\lambda \end{tikzcd}$$

that base changes to the commutative diagram above, where π_λ is of finite type. After possibly enlarging A_λ by inverting finitely many elements, and with notation as in Definition 5.6.4, we can assume the following properties by Tables 5.1 and 5.2:

(a) $i_{x,\lambda}$ is a closed embedding;

(b) π_λ is flat and projective; and

(c) $X_{\mathfrak{p}}$ is integral for every closed point $\mathfrak{p}\in\operatorname{Spec}A_\lambda$.

We will denote the ideal sheaf defining the image of $i_{x,\lambda}$ as $\mathfrak{m}_{x_\lambda}$, and the corresponding subscheme by x_λ. By Remark 5.6.6, we can also spread out coherent sheaves, Cartier divisors, and $\mathbf{Q}$-Cartier divisors from X to X_λ.

We will also need the following result, which describes how separation of jets and how $\varepsilon_{\mathrm{jet}}(\|D\|;x)$ behave under reduction modulo $\mathfrak{p}$. We note that in the description of different loci below, we allow $\mathfrak{q}$ to be non-closed points in $\operatorname{Spec} A_\lambda$.

Lemma 7.3.6. *Let X and $\mathfrak{m}_x$ be as in Setup 7.3.5, with models X_λ and $\mathfrak{m}_{x\lambda}$ over* $\operatorname{Spec} A_\lambda$*, respectively.*

(*i*) *Let $\mathscr{F}$ be a coherent sheaf on X together with a model $\mathscr{F}_\lambda$ over* $\operatorname{Spec} A_\lambda$. *Let $\ell \geq -1$ be an integer, and suppose that $\mathscr{F}_\lambda$ and $\mathscr{F}_\lambda/\mathfrak{m}_{x_\lambda}^{\ell+1}\mathscr{F}_\lambda$ are flat and cohomologically flat in degree zero over* $\operatorname{Spec} A_\lambda$. *Then, the locus*

$$\bigl\{\mathfrak{q} \in \operatorname{Spec} A_\lambda \bigm| \mathscr{F}_\mathfrak{q} \text{ separates } \ell\text{-jets at } x_\mathfrak{q}\bigr\}$$

is open in $\operatorname{Spec} A_\lambda$.

(*ii*) *Let D be a $\mathbf{Q}$-Cartier divisor on X together with a model D_λ over* $\operatorname{Spec} A_\lambda$. *Then, for every integer $m > 0$ such that mD_λ is Cartier, the locus*

$$\bigl\{\mathfrak{q} \in \operatorname{Spec} A_\lambda \bigm| mD_\mathfrak{q} \text{ separates } \ell\text{-jets at } x_\mathfrak{q}\bigr\}$$

contains an open set in $\operatorname{Spec} A_\lambda$ *for every integer $\ell \geq -1$.*

(*iii*) *Let D be a $\mathbf{Q}$-Cartier divisor on X together with a model D_λ over* $\operatorname{Spec} A_\lambda$, *and let $\delta > 0$ be a real number such that $\varepsilon_{\mathrm{jet}}(\|D\|;x) > \delta$. Then, the locus*

$$\bigl\{\mathfrak{q} \in \operatorname{Spec} A_\lambda \bigm| \varepsilon_{\mathrm{jet}}\bigl(\|D_\mathfrak{q}\|;x_\mathfrak{q}\bigr) > \delta\bigr\} \tag{7.9}$$

contains a non-empty open set in $\operatorname{Spec} A_\lambda$.

Proof. We first prove (*i*). By cohomology and base change [Ill05, Cor. 8.3.11], the locus where $\mathscr{F}_\mathfrak{q}$ does not separate ℓ-jets at $x_\mathfrak{q}$ is

$$\operatorname{Supp}\Bigl(\operatorname{coker}\bigl(\pi_{\lambda*}\mathscr{F}_\lambda \longrightarrow \pi_{\lambda*}(\mathscr{F}_\lambda/\mathfrak{m}_{x_\lambda}^{\ell+1}\mathscr{F}_\lambda)\bigr)\Bigr),$$

where $\pi_\lambda\colon X_\lambda \to \operatorname{Spec} A_\lambda$ is as in Setup 7.3.5. Since π_λ is proper, both direct image sheaves are coherent, and the cokernel above is also coherent. Thus, the support of this cokernel is closed in $\operatorname{Spec} A_\lambda$, which implies (*i*). (*ii*) then follows from (*i*) by setting

$\mathscr{F} = \mathcal{O}_X(mD)$, after possibly enlarging A_λ by inverting finitely many elements to assume that $\mathcal{O}_{X_\lambda}(mD_\lambda)$ and $\mathcal{O}_{X_\lambda}(mD_\lambda) \otimes \mathcal{O}_{X_\lambda}/\mathfrak{m}_{x_\lambda}^{\ell+1}$ are flat and cohomologically flat in degree zero over $\operatorname{Spec} A_\lambda$ by generic flatness [EGAIV$_2$, Thm. 6.9.1] and by [Ill05, Cor. 8.3.11].

We now show (*iii*). By Lemma 7.2.5, $\varepsilon_{\text{jet}}(\|D\|; x)$ is a limit, hence there exists an integer $m > 0$ such that mD is a Cartier divisor, and such that

$$\frac{s(mD; x)}{m} > \delta. \tag{7.10}$$

Since $s(mD; x)$ is an integer, this inequality is equivalent to $s(mD; x) \geq \lfloor m\delta \rfloor + 1$. By (*ii*), the locus

$$\bigl\{\mathfrak{q} \in \operatorname{Spec} A_\lambda \bigm| mD_{\mathfrak{q}} \text{ separates } \bigl(\lfloor m\delta \rfloor + 1\bigr)\text{-jets at } x_{\mathfrak{q}}\bigr\} \tag{7.11}$$

contains an open set, which is nonempty by (7.10) since the generic point of $\operatorname{Spec} A_\lambda$ is contained in this set by flat base change, and since $A_\lambda \subseteq k$ is flat. Now if $mD_{\mathfrak{q}}$ separates $(\lfloor m\delta \rfloor + 1)$-jets at $x_{\mathfrak{q}}$, then we have the inequality

$$\varepsilon\bigl(\|D_{\mathfrak{q}}\|; x_{\mathfrak{q}}\bigr) \geq \frac{s(mD_{\mathfrak{q}}; x)}{m} > \delta$$

by the fact that $s(mD_{\mathfrak{q}}; x)$ is an integer. The locus (7.11) is therefore contained in the locus (7.9), and (*iii*) follows. $\square$

We will also use the following:

Lemma 7.3.7. *Let X be as in Setup 7.3.5, with a model X_λ over* $\operatorname{Spec} A_\lambda$. *If $\omega^i_{X_\lambda/A_\lambda}$ is flat over A_λ for every i, then the base change isomorphism*

$$\omega^\bullet_{X_\lambda/A_\lambda}|^{\mathbf{L}}_{X_{\mathfrak{p}}} \xrightarrow{\sim} \omega^\bullet_{X_{\mathfrak{p}}}$$

from [Lip09, Cor. 4.4.3] *induces an isomorphism*

$$\omega_{X_\lambda/A_\lambda}|_{X_{\mathfrak{p}}} \xrightarrow{\sim} \omega_{X_{\mathfrak{p}}}.$$

Proof. This statement follows from the flatness of $\omega^i_{X_\lambda/A_\lambda}$ and a spectral sequence for sheaf $\mathscr{T}or$; see [EGAIII$_2$, Cor. 6.5.9]. $\square$

We can now prove Theorem 7.3.1 in characteristic zero via reduction modulo $\mathfrak{p}$.

Proof of Theorem 7.3.1 in characteristic zero. As in the proof in positive characteristic, we note that

$$\varepsilon_{\mathrm{jet}}\bigl(\|L\|;x\bigr) > n+\ell$$

by Proposition 7.2.10, where for (i), we note that $\varepsilon(L;x) > n+\ell$ implies $x \notin \mathbf{B}_+(L)$ by Theorem 4.6.6 and [Laz04a, Prop. 5.1.9]. We use the notation in Setup 7.3.5. After possibly further enlarging A_λ by inverting finitely many elements, we may assume that

(a) the Cartier divisor L spreads out to a Cartier divisor L_λ on X_λ such that $\mathcal{O}_{X_\lambda}(L_\lambda)$ is flat and cohomologically flat in degree zero over A_λ;

(b) the sheaves $\omega^i_{X_\lambda/A_\lambda}$ and $\omega_{X_\lambda/A_\lambda}$ are flat and cohomologically flat in degree zero over A_λ for every i; and

(c) the sheaf $\omega_{X_\lambda/A_\lambda} \otimes \mathcal{O}_{X_\lambda}(L_\lambda) \otimes \mathcal{O}_{X_\lambda}/\mathfrak{m}^{\ell+1}_{x_\lambda}$ is flat and cohomologically flat in degree zero over A_λ.

Here, we have used Remark 5.6.6, generic flatness [EGAIV$_2$, Thm. 6.9.1], and cohomology and base change [Ill05, Cor. 8.3.11].

We can now prove Theorem 7.3.1 in characteristic zero. First, we have a base change isomorphism

$$\omega_{X_\lambda/A_\lambda}|_{X_\mathfrak{p}} \xrightarrow{\sim} \omega_{X_\mathfrak{p}}$$

by Lemma 7.3.7 and the assumption (b). Since $\mathcal{O}_X(L_\lambda)$ is also flat and cohomologically flat in degree zero over $\operatorname{Spec} A_\lambda$, the sheaf $\omega_{X_\lambda/A_\lambda} \otimes \mathcal{O}_{X_\lambda}(L_\lambda)$ is a model of $\omega_X \otimes \mathcal{O}_X(L)$ over $\operatorname{Spec} A_\lambda$, and is flat and cohomologically flat in degree zero over $\operatorname{Spec} A_\lambda$.

We now claim that $\omega_{X_\mathfrak{p}} \otimes \mathcal{O}_{X_\mathfrak{p}}(L_\mathfrak{p})$ separates ℓ-jets at $x_\mathfrak{p}$ for some closed point $\mathfrak{p} \in \operatorname{Spec} A_\lambda$. Note that $\varepsilon_{\mathrm{jet}}(\|L_\mathfrak{p}\|;x_\mathfrak{p}) > n+\ell$ holds for all $\mathfrak{p}$ in an open dense subset of $\operatorname{Spec} A_\lambda$ by Lemma 7.3.6(iii). We now claim we can apply Theorem 7.3.2 to show that $\omega_{X_\mathfrak{p}} \otimes \mathcal{O}_{X_\mathfrak{p}}(L_\mathfrak{p})$ separates ℓ-jets at $x_\mathfrak{p}$ for all $\mathfrak{p}$ such that $\varepsilon_{\mathrm{jet}}(\|L_\mathfrak{p}\|;x_\mathfrak{p}) > n+\ell$ and such that $X_\mathfrak{p}$ is F-injective. First, we note that $\kappa(\mathfrak{p})$ is F-finite for all $\mathfrak{p} \in \operatorname{Spec} A_\lambda$, since it is a finite field by Lemma 5.6.5. Thus, Lemma A.9 implies that the Frobenius trace $\mathrm{Tr}_{X_\mathfrak{p}}\colon F_*\omega_{X_\mathfrak{p}} \to \omega_{X_\mathfrak{p}}$ on $X_\mathfrak{p}$ exists and is surjective. We can therefore apply Theorem 7.3.2 to show that $\omega_{X_\mathfrak{p}} \otimes \mathcal{O}_{X_\mathfrak{p}}(L_\mathfrak{p})$ separates ℓ-jets at $x_\mathfrak{p}$.

Finally, we show that $\omega_X \otimes \mathcal{O}_X(L)$ separates ℓ-jets at x. Since the extension $\operatorname{Frac} A_\lambda \subseteq k$ is flat, it suffices by flat base change to show that $\omega_{X_\eta} \otimes \mathcal{O}_{X_\eta}(L_\eta)$ separates ℓ-jets at x_η, where $\eta \in \operatorname{Spec} A_\lambda$ is the generic point. But this follows from the previous paragraph from Lemma 7.3.6(i), since $\omega_{X_\mathfrak{p}} \otimes \mathcal{O}_{X_\mathfrak{p}}(L_\mathfrak{p})$ separates ℓ-jets at $x_\mathfrak{p}$ for some closed point $\mathfrak{p} \in \operatorname{Spec} A_\lambda$. $\square$

Chapter 8

The Angehrn–Siu theorem

The goal of this chapter is to prove the following version of the Angehrn–Siu theorem [AS95, Thm. 0.1].

Theorem D. *Let (X, Δ) be an effective log pair, where X is a normal projective variety over an algebraically closed field k of characteristic zero, Δ is a $\mathbf{Q}$-Weil divisor, and $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Let $x \in X$ be a closed point such that (X, Δ) is klt at x, and let D be a Cartier divisor on X such that setting $H \coloneqq D - (K_X + \Delta)$, there exist positive numbers $c(m)$ with the following properties:*

(i) For every positive dimensional variety $Z \subseteq X$ containing x, we have

$$\operatorname{vol}_{X|Z}(H) > c(\dim Z)^{\dim Z}.$$

(ii) The numbers $c(m)$ satisfy the inequality

$$\sum_{m=1}^{\dim X} \frac{m}{c(m)} \leq 1.$$

Then, $\mathcal{O}_X(D)$ has a global section not vanishing at x.

Before we prove this statement, we will need to prove a replacement for the Nadel vanishing theorem [Laz04b, Thm. 9.4.17].

8.1. The lifting theorem

A major obstacle in proving the Angehrn–Siu theorem in positive characteristic is that Kodaira-type vanishing theorems are false; see Example 2.4.4. While the result below is not yet strong enough to prove their theorem in positive characteristic, it does give a replacement for the Nadel vanishing theorem in characteristic zero, after reduction modulo $\mathfrak{p}$.

We start by stating the characteristic zero version of the result.

Theorem C. *Let (X, Δ) be an effective log pair such that X is a projective normal variety over a field k of characteristic zero, and such that $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Consider a k-rational point $x \in X$ such that (X, Δ) is of dense F-pure type at x. Suppose that D is a Cartier divisor on X such that $H = D - (K_X + \Delta)$ satisfies*

$$\varepsilon\bigl(\lVert H\rVert; x\bigr) > \operatorname{lct}_x\bigl((X, \Delta); \mathfrak{m}_x\bigr).$$

Then, $\mathcal{O}_X(D)$ has a global section not vanishing at x.

Note that in Chapter 1, we stated Theorem C with "dense F-pure type" replaced by "klt." The formulation in Chapter 1 follows from this one since klt pairs are of dense strongly F-regular type by Theorem 5.6.8, hence of dense F-pure type by Proposition 5.4.3(iv).

Theorem C follows from the following result via reduction modulo $\mathfrak{p}$.

Theorem 8.1.1. *Let (X, Δ) be an effective log pair such that X is a projective normal variety over an F-finite field k of characteristic $p > 0$, and such that $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Consider a k-rational point $x \in X$ such that (X, Δ) is F-pure at x. Suppose that D is a Cartier divisor on X such that $H = D - (K_X + \Delta)$ satisfies*

$$\varepsilon_{\mathrm{jet}}\bigl(\lVert H\rVert; x\bigr) > \operatorname{fpt}_x\bigl((X, \Delta); \mathfrak{m}_x\bigr).$$

Then, $\mathcal{O}_X(D)$ has a global section not vanishing at x.

We first prove Theorem C, assuming Theorem 8.1.1.

Proof of Theorem C. Let X_λ, Δ_λ, $\mathfrak{m}_{x\lambda}$, and D_λ be models of X, Δ, $\mathfrak{m}_x$, and D over a finitely generated $\mathbf{Z}$-algebra $A_\lambda \subseteq k$ as in Theorem 5.6.2 and Remark 5.6.6; cf. Setup 7.3.5.

After possibly enlarging A_λ by inverting finitely many elements, we may assume that $X_\mathfrak{p}$ is normal for every $\mathfrak{p} \in A_\lambda$ (by Table 5.2), and moreover, we may assume that $\omega_{X_\lambda/A_\lambda}$ is a model for $\omega_X \simeq \mathcal{O}_X(K_X)$ (by Lemma 7.3.7) that is flat and cohomologically flat over $\operatorname{Spec} A_\lambda$ (by generic flatness [EGAIV$_2$, Thm. 6.9.1] and cohomology and base change [Ill05, Cor. 8.3.11]). By assumption, $(X_\mathfrak{p}, \Delta_\mathfrak{p})$ is F-pure for a dense set of $\mathfrak{p} \in \operatorname{Spec} A_\lambda$, and we can also assume that for these $\mathfrak{p}$, the F-pure threshold of $(X_\mathfrak{p}, \Delta_\mathfrak{p})$ with respect to $\mathfrak{m}_{x_\mathfrak{p}}$ is strictly less than $\varepsilon_{\mathrm{jet}}(\|H\|; x)$ by Theorem 5.6.8. Note that here we have used Proposition 7.2.10 to say that $\varepsilon(\|H\|; x) = \varepsilon_{\mathrm{jet}}(\|H\|; x)$. By Lemma 7.3.6($iii$), we have

$$\varepsilon_{\mathrm{jet}}\bigl(\|H_\mathfrak{p}\|; x_\mathfrak{p}\bigr) > \operatorname{fpt}_x\bigl((X_\mathfrak{p}, \Delta_\mathfrak{p}); \mathfrak{m}_{x_\mathfrak{p}}\bigr)$$

for all but finitely many $\mathfrak{p} \in \operatorname{Spec} A_\lambda$, where $H_\mathfrak{p} = D_\mathfrak{p} - (K_{X_\mathfrak{p}} + \Delta_\mathfrak{p})$. Theorem 8.1.1 therefore implies that $\mathcal{O}_X(D_\mathfrak{p})$ has a global section not vanishing at $x_\mathfrak{p}$ after reduction modulo $\mathfrak{p}$ for infinitely many $\mathfrak{p} \in \operatorname{Spec} A$, where we note that $\kappa(\mathfrak{p})$ is F-finite for every $\mathfrak{p} \in \operatorname{Spec} A_\lambda$, since $\kappa(\mathfrak{p})$ is a finite field by Lemma 5.6.5. Thus, $\mathcal{O}_{X_\eta}(D_\eta)$ has a global section not vanishing at x_η by Lemma 7.3.6(i), where $\eta \in \operatorname{Spec} A_\lambda$ is the generic point. Finally, since $\operatorname{Frac} A_\lambda \subseteq k$ is flat, we see that $\mathcal{O}_X(D)$ also has a global section not vanishing at x by flat base change. □

We now prove Theorem 8.1.1.

Proof of Theorem 8.1.1. We proceed in a sequence of steps, following the outline of the proof of [MS14, Thm. 3.1] and [Mur18, Thm. C].

Denoting by c the F-pure threshold $\operatorname{fpt}_x((X, \Delta); \mathfrak{m}_x)$ of (X, Δ) with respect to $\mathfrak{m}_x$, fix $\delta > 0$ such that

$$\varepsilon_{\mathrm{jet}}(\|H\|; x) > (1 + 2\delta)c.$$

We first claim that there exists a positive integer m_0 and a sequence $\{d_e\}$ such that m_0H is Cartier, $\mathcal{O}_X(m_0 d_e H)$ separates $(\lfloor (p^e - 1)(1 + \delta)c \rfloor - 1)$-jets at x for all $e > 0$, and $p^e - m_0 d_e \to \infty$ as $e \to \infty$. By Lemma 7.2.5, there exists a positive integer m_0 such that m_0H is Cartier and

$$\frac{s(m_0H; x)}{m_0} > (1 + 2\delta)c.$$

Now for every integer $e > 0$, let

$$d_e = \left\lceil \frac{\lfloor (p^e-1)(1+\delta)c \rfloor - 1}{s(m_0H;x)} \right\rceil.$$

By the superadditivity property (Lemma 7.2.5), we have

$$s(m_0d_eH;x) \geq d_e \cdot s(m_0H;x) \geq \lfloor (p^e-1)(1+\delta)c \rfloor - 1,$$

hence $\mathcal{O}_X(m_0d_eH)$ separates $(\lfloor (p^e-1)(1+\delta)c \rfloor - 1)$-jets at x. We now claim that $p^e - m_0d_e \to \infty$ as $e \to \infty$. Note that

$$\begin{aligned}
p^e - m_0d_e &= p^e - m_0 \cdot \left\lceil \frac{\lfloor (p^e-1)(1+\delta)c \rfloor - 1}{s(m_0H;x)} \right\rceil \\
&\geq p^e - \bigl(\lfloor (p^e-1)(1+\delta)c \rfloor - 1\bigr) \cdot \frac{m_0}{s(m_0H;x)} - m_0 \\
&\geq p^e - \frac{\lfloor (p^e-1)(1+\delta)c \rfloor - 1}{(1+2\delta)c} - m_0
\end{aligned}$$

and as $e \to \infty$, we have

$$\begin{aligned}
\lim_{e\to\infty}(p^e - m_0d_e) &\geq \lim_{e\to\infty}\left(p^e - \frac{\lfloor (p^e-1)(1+\delta)c \rfloor - 1}{(1+2\delta)c} - m_0\right) \\
&= \lim_{e\to\infty} p^e\left(1 - \frac{1+\delta}{1+2\delta}\right) - m_0 = \infty.
\end{aligned}$$

We therefore see that $\mathcal{O}_X(m_0d_eH)$ separates $(\lfloor (p^e-1)(1+\delta)c \rfloor - 1)$-jets at x, and that $p^e - m_0d_e \to \infty$ as $e \to \infty$.

We now show that there exists a positive integer e such that $\mathcal{O}_X(\lceil K_X + \Delta + p^eH \rceil)$ separates $(\lfloor (p^e-1)(1+\delta)c \rfloor - 1)$-jets at x. Let m_0 and $\{d_e\}$ as in the previous paragraph. Since $x \notin \mathbf{B}_+(H)$ and since $p^e - m_0d_e \to \infty$, Proposition 4.6.7 implies that the sheaf

$$\mathcal{O}_X\bigl(\lceil K_X + \Delta + (p^e - m_0d_e)H \rceil\bigr)$$

is globally generated at x for all $e \gg 0$. Since $\mathcal{O}_X(m_0d_eH)$ separates $(\lfloor (p^e-1)(1+\delta)c \rfloor - 1)$-

jets at x, Lemma 7.2.5 implies

$$\mathcal{O}_X\bigl(\lceil K_X + \Delta + (p^e - m_0 d_e)H\rceil\bigr) \otimes \mathcal{O}_X(m_0 d_e H) \simeq \mathcal{O}_X\bigl(\lceil K_X + \Delta + p^e H\rceil\bigr)$$

separates $(\lfloor (p^e-1)(1+\delta)c\rfloor - 1)$-jets at x.

We now use the trace morphism $\mathrm{Tr}^e_{X,\lfloor (p^e-1)\Delta\rfloor}$ to take out the factors of p^e. Note that $\mathrm{Tr}^e_{X,\lfloor (p^e-1)\Delta\rfloor}$ is surjective at x by assumption, since (X,Δ) is F-pure at x (Corollary 5.4.7(i)). Twisting $\mathrm{Tr}^e_{X,\lfloor (p^e-1)\Delta\rfloor}$ by $\mathcal{O}_X(D)$, we have a morphism

$$F^e_*\bigl(\mathcal{O}_X\bigl((1-p^e)K_X - \lfloor (p^e-1)\Delta\rfloor + p^e D\bigr)\bigr) \xrightarrow{\mathrm{Tr}^e_{X,\lfloor (p^e-1)\Delta\rfloor}(D)} \mathcal{O}_X(D) \tag{8.1}$$

that is surjective at x, where the source can be identified with

$$F^e_*\bigl(\mathcal{O}_X\bigl(\lceil (1-p^e)(K_X+\Delta) + p^e D\rceil\bigr)\bigr) = F^e_*\bigl(\mathcal{O}_X\bigl(\lceil K_X + \Delta + p^e H\rceil\bigr)\bigr).$$

The triple $(X,\Delta,\mathfrak{m}_x^{(1+\delta)c})$ is not F-pure, since c is the F-pure threshold $\mathrm{fpt}_x((X,\Delta);\mathfrak{m}_x)$. Thus, the morphism

$$F^e_*\bigl(\mathfrak{m}_x^{\lfloor (p^e-1)(1+\delta)c\rfloor} \cdot \mathcal{O}_X\bigl(\lceil K_X + \Delta + p^e H\rceil\bigr)\bigr) \xrightarrow{\mathrm{Tr}^e_{X,\lfloor (p^e-1)\Delta\rfloor}(D)} \mathcal{O}_X(D) \tag{8.2}$$

induced by the trace morphism (8.1) is not surjective at x by Corollary 5.4.7. We therefore see that the morphism (8.2) induces a morphism

$$F^e_*\bigl(\mathfrak{m}_x^{\lfloor (p^e-1)(1+\delta)c\rfloor} \cdot \mathcal{O}_X\bigl(\lceil K_X + \Delta + p^e H\rceil\bigr)\bigr) \xrightarrow{\mathrm{Tr}^e_{X,\lfloor (p^e-1)\Delta\rfloor}(D)} \mathfrak{m}_x \cdot \mathcal{O}_X(D)$$

since the target is $\mathcal{O}_{X,y}(D)$ after localizing at every point $y \neq x$, and at x, the non-surjectivity of the localization of (8.2) at x is equivalent to having image in $\mathfrak{m}_x \cdot \mathcal{O}_{X,x}(D)$, by the fact that $\mathcal{O}_{X,x}$ is local. We therefore have the commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] \\
& F^e_*\big(\mathfrak{m}_x^{\lfloor (p^e-1)(1+\delta)c\rfloor} \cdot \mathcal{O}_X\big(\lceil K_X+\Delta+p^eH\rceil\big)\big) \arrow[r] \arrow[d] & \mathfrak{m}_x \cdot \mathcal{O}_X(D) \arrow[d] \\
& F^e_*\big(\mathcal{O}_X\big(\lceil K_X+\Delta+p^eH\rceil\big)\big) \arrow[r] \arrow[d] & \mathcal{O}_X(D) \arrow[d] \\
& F^e_*\left(\dfrac{\mathcal{O}_X\big(\lceil K_X+\Delta+p^eH\rceil\big)}{\mathfrak{m}_x^{\lfloor (p^e-1)(1+\delta)c\rfloor} \cdot \mathcal{O}_X\big(\lceil K_X+\Delta+p^eH\rceil\big)}\right) \arrow[r] \arrow[d] & \dfrac{\mathcal{O}_X(D)}{\mathfrak{m}_x \cdot \mathcal{O}_X(D)} \arrow[d] \\
& 0 & 0
\end{tikzcd}$$

where the bottom two horizontal arrows are induced by $\mathrm{Tr}^e_{X,\lfloor (p^e-1)\Delta\rfloor}$, and are therefore surjective at x. Note that the left column is exact since the Frobenius morphism F is affine. Taking global sections in the bottom square, we obtain the following commutative square:

$$\begin{tikzcd}
H^0\big(X, \mathcal{O}_X\big(\lceil K_X+\Delta+p^eH\rceil\big)\big) \arrow[r] \arrow[d, two heads, "\varphi"'] & H^0\big(X, \mathcal{O}_X(D)\big) \arrow[d] \\
H^0\left(X, \dfrac{\mathcal{O}_X\big(\lceil K_X+\Delta+p^eH\rceil\big)}{\mathfrak{m}_x^{\lfloor (p^e-1)(1+\delta)c\rfloor} \cdot \mathcal{O}_X\big(\lceil K_X+\Delta+p^eH\rceil\big)}\right) \arrow[r, two heads, "\psi"] & H^0\left(X, \dfrac{\mathcal{O}_X(D)}{\mathfrak{m}_x \cdot \mathcal{O}_X(D)}\right)
\end{tikzcd}$$

where ψ is surjective since the corresponding morphism of sheaves is a surjective morphism of skyscraper sheaves supported at x. Since the restriction map φ is surjective by the previous paragraph, the right vertical map is necessarily surjective by the commutativity of the diagram. Thus, the sheaf $\mathcal{O}_X(D)$ has a global section not vanishing at x. $\square$

Remark 8.1.2. One can also prove a weaker version of Theorem 8.1.1 using another variant of Frobenius–Seshadri constants (cf. Remark 7.3.4). The relevant version of the Seshadri constant is defined using the *Frobenius degeneracy ideals* first introduced by Yao [Yao06, Rem. 2.3(1)] and Aberbach–Enescu [AE05, Def. 3.1]. If (R, Δ) is a sharply

F-pure pair where R is an F-finite local ring of characteristic $p > 0$ with maximal ideal $\mathfrak{m} \subseteq R$, then following [Tuc12, Def. 4.3; BST12, Def. 3.3], the *eth Frobenius degeneracy ideal* is

$$I_e^{\Delta}(\mathfrak{m}) \coloneqq \Bigl\{ f \in R \Bigm| \varphi(f) \in \mathfrak{m} \text{ for all } \varphi \in \operatorname{Hom}_R\bigl(F_*^e R(\lceil (p^e - 1)\Delta \rceil), R\bigr) \Bigr\}.$$

Note that we have followed the terminology from [CRST, Def. 2.6]. Following [DSNB18, Lem. 3.9 and Prop. 3.10], one can show that

$$\mathfrak{m}^{\lfloor p^e \operatorname{fpt}((R,\Delta);\mathfrak{m}) \rfloor + 1} \subseteq I_e^{\Delta}(\mathfrak{m}). \tag{8.3}$$

Now let (X, Δ) be a sharply F-pure pair, where X is a complete variety over an F-finite field of characteristic $p > 0$. For a $\mathbf{Q}$-Cartier divisor H on X and for every integer $m \geq 1$ such that mH is a Cartier divisor, denote by $s_{F\text{-sig}}^{\Delta}(mH; x)$ the largest integer such that the restriction map

$$H^0\bigl(X, \mathcal{O}_X(mH)\bigr) \longrightarrow H^0\bigl(X, \mathcal{O}_X(mH) \otimes \mathcal{O}_X / I_e^{\Delta}(\mathfrak{m}_x)\bigr)$$

is surjective. Then, the *F-signature Seshadri constant* of H at x is

$$\varepsilon_{F\text{-sig}}^{\Delta}(H; x) \coloneqq \limsup_{m \to \infty} \frac{p^{s_{F\text{-sig}}^{\Delta}(mH;x)} - 1}{m},$$

where the limit supremum runs over all m such that mH is integral. The inclusion (8.3) then implies

$$\varepsilon_{\text{jet}}\bigl(\lVert H \rVert; x\bigr) \leq \operatorname{fpt}_x\bigl((X, \Delta); \mathfrak{m}_x\bigr) \cdot \varepsilon_{F\text{-sig}}^{\Delta}(H; x).$$

Using the strategy in Theorem 8.1.1, one can show that a lower bound of the form $\varepsilon_{F\text{-sig}}^{\Delta}(H; x) > 1$ implies the existence of global sections of $\mathcal{O}_X(D)$ as in Theorem 8.1.1. This version of the Frobenius–Seshadri constant is difficult to work with since we do not know if the analogues of [MS14, Lem. 2.5 and Prop. 2.6] or [Mur18, Lem. 2.4 and Prop. 2.5] hold. The core issue is that the sequence $\{I_e^{\Delta}(\mathfrak{m}_x)\}_{e \in \mathbf{N}}$ does not necessarily form a p-family of ideals in the sense of [HJ18, Def. 1.1].

8.2. Constructing singular divisors and proof of Theorem D

The goal in this section is to prove the following result, which will be the other crucial ingredient in proving Theorem D.

Theorem 8.2.1 (cf. [Kol97, Thm. 6.4])**.** *Let (X, Δ) be an effective log pair, where X is a normal projective variety over an algebraically closed field k of characteristic zero, Δ is a $\mathbf{Q}$-Weil divisor, and $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Let $x \in X$ be a closed point such that (X, Δ) is klt at x. Let D be a Cartier divisor on X such that setting $N \coloneqq D - (K_X + \Delta)$, there exist positive numbers $c(m)$ with the following properties:*

(i) For every positive dimensional variety $Z \subseteq X$ containing x, we have

$$\mathrm{vol}_{X|Z}(N) > c(\dim Z)^{\dim Z}.$$

(ii) The numbers $c(m)$ satisfy the inequality

$$\sum_{m=1}^{\dim X} \frac{m}{c(m)} \le 1.$$

Then, there exist an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-Weil divisor $E \sim_{\mathbf{Q}} bN$ for some $b \in (0,1)$, and an open neighborhood $X^0 \subseteq X$ of x such that $(X^0, \Delta + E)$ is log canonical, $(X^0, \Delta + E)$ is klt on $X^0 \smallsetminus \{x\}$, and $(X, \Delta + E)$ is not klt at x.

Assuming this, Theorem D is not difficult.

Proof of Theorem D. Fix a closed point $x \in X$. By Theorem 8.2.1, there exists a boundary divisor $E \sim_{\mathbf{Q}} bN$ for some $b \in (0,1)$ such that $(X, \Delta + E)$ is strictly log canonical at x, but is klt in a punctured neighborhood of x. Now let $f\colon Y \to X$ be a log resolution for $(X, \Delta + E, \mathfrak{m}_x)$. Then, there is divisor $F \subseteq Y$ over x such that $a(F, X, \Delta + E) = -1$ by Lemma 4.8.5. For $0 < \delta \ll 1$, we claim that $(X, \Delta + (1-\delta)E)$ is klt in a neighborhood of x, and that

$$\mathrm{lct}_x\bigl((X, \Delta + (1-\delta)E); \mathfrak{m}_x\bigr) < (1-b) \cdot \varepsilon\bigl(\|N\|; x\bigr).$$

Note that the right-hand side is positive since $b \in (0,1)$. The property that $(X, \Delta + (1-\delta)E)$ is klt in a neighborhood of x follows from Lemma 4.8.5, and the inequality above follows for $0 < \delta \ll 1$ from the computation of the log canonical threshold in Proposition 4.8.7, since the Cartier divisor defined by $f^{-1}\mathfrak{m}_x \cdot \mathcal{O}_Y$ contains F as a component. We therefore have

$$D - \bigl(K_X + \Delta + (1-\delta)E\bigr) \sim_{\mathbf{Q}} N - (1-\delta)bN = \bigl(1 - (1-\delta)b\bigr)N,$$

hence the conditions of Theorem C are satisfied for $H = D - (K_X + \Delta + (1-\delta)E)$, since

$$\begin{aligned}\varepsilon\bigl(\|H\|;x\bigr) = \bigl(1-(1-\delta)b\bigr)\cdot\varepsilon\bigl(\|N\|;x\bigr) &> \bigl(1-b\bigr)\cdot\varepsilon\bigl(\|N\|;x\bigr)\\ &> \operatorname{lct}_x\bigl((X, \Delta + (1-\delta)E); \mathfrak{m}_x\bigr). \qquad \square\end{aligned}$$

It therefore remains to show Theorem 8.2.1. The idea of the proof is to first produce a divisor that is highly singular at a point, and then cut down the dimension of the non-klt locus at the point until the non-klt locus is isolated. We mostly follow the proofs in [Kol97, §6] and [Liu, §3], with suitable changes to deal with the weaker positivity condition on N.

We start with the following result. Recall that if D is a $\mathbf{Q}$-Cartier divisor on a variety X over a field k, then a *graded linear system* $V_\bullet$ associated to D is a sequence of subspaces $V_m \subseteq H^0(X, \mathcal{O}_X(mD))$ for m such that mD is a Cartier divisor, which satisfies the property that the multiplication map $V_m \otimes V_n \to H^0(X, \mathcal{O}_X((m+n)D))$ has image in V_{m+n} [Laz04a, Def. 2.4.1]. The *volume* of $V_\bullet$ is

$$\operatorname{vol}_X(V_\bullet) \coloneqq \limsup_{m\to\infty} \frac{\dim_k V_m}{m^n/n!},$$

where $n = \dim X$ [Laz04a, Def. 2.4.12]. If $f\colon Y \to X$ is a morphism, and $V_\bullet$ is a graded linear system associated to a $\mathbf{Q}$-Cartier D on X, then the graded linear series $f^*V_\bullet$ is given by setting

$$f^*V_m \coloneqq \operatorname{im}\Bigl(V_m \subseteq H^0\bigl(X, \mathcal{O}_X(mD)\bigr) \longrightarrow H^0\bigl(Y, \mathcal{O}_Y(m\,f^*D)\bigr)\Bigr),$$

where the morphism is induced by the pullback morphism $\mathcal{O}_X(mD) \to f_*\mathcal{O}_Y(m\,f^*D)$. In particular, if f is an inclusion $Y \subseteq X$ of a closed subvariety, then we set $V_\bullet|_Y \coloneqq f^*V_\bullet$.

Lemma 8.2.2 (cf. [Kol97, Lem. 6.1; Fuj11, Lem. 12.2])**.** *Let $f\colon Y \to Z$ be a surjective projective morphism from a normal variety Y to an affine variety Z over an algebraically closed field k, and let W be a general closed subvariety of Y such that $f|_W\colon W \to Z$ is generically finite and generically regular. Consider a $\mathbf{Q}$-Cartier divisor M on Y, and let $V_\bullet$ be a graded linear system associated to M. Then, for every $\varepsilon > 0$, there exists a positive integer $t > 0$ such that tM is a Cartier divisor, and such that there exists an effective Cartier divisor $D_t = D_t(\varepsilon) \in |V_t|$ on Y such that setting $D := \frac{1}{t}D_t$, we have*

$$\operatorname{mult}_W D \geq \left(\frac{\operatorname{vol}_F(V_\bullet|_F)}{\deg(f|_W)}\right)^{1/n} - \varepsilon, \tag{8.4}$$

where F is a general fiber of f and $n = \dim F$.

Here, $\operatorname{mult}_W D$ is the maximum integer s such that D vanishes to order s everywhere along W, and $\deg(f|_W)$ is the degree of the generically finite morphism $f|_W$.

Proof. Let $t > 0$ be an integer such that tM is Cartier, and let $\mathcal{I}_W \subseteq \mathcal{O}_Y$ be the ideal sheaf defining W. Then, for every integer $s > 0$, we have the short exact sequence

$$0 \longrightarrow \mathcal{I}_W^s \otimes \mathcal{O}_Y(tM) \longrightarrow \mathcal{O}_Y(tM) \longrightarrow \mathcal{O}_Y(tM) \otimes \mathcal{O}_Y/\mathcal{I}_W \longrightarrow 0,$$

and pushing forward by f, we obtain the left-exact sequence

$$0 \longrightarrow f_*\bigl(\mathcal{I}_W^s \otimes \mathcal{O}_Y(tM)\bigr) \longrightarrow f_*\bigl(\mathcal{O}_Y(tM)\bigr) \longrightarrow f_*\bigl(\mathcal{O}_Y(tM) \otimes \mathcal{O}_Y/\mathcal{I}_W^s\bigr). \tag{8.5}$$

Now choose integers $s, t > 0$ such that

$$\left(\frac{\operatorname{vol}_F(V_\bullet|_F)}{\deg(f|_W)}\right)^{1/n} > \frac{s}{t} > \left(\frac{\operatorname{vol}_F(V_\bullet|_F)}{\deg(f|_W)}\right)^{1/n} - \varepsilon,$$

and recall that for every regular point $x \in F$, we have

$$h^0(F, \mathcal{O}_F/\mathfrak{m}_x^s) = \binom{n+s-1}{n} = \frac{s^n}{n!} + O(s^{n-1}).$$

Thus, after possibly replacing s and t by multiples, we may assume without loss of generality that tM is Cartier, and that

$$\begin{aligned}\dim_k V_t|_F - h^0\bigl(F, \mathcal{O}_F(tM|_F) \otimes \mathcal{O}_F/\mathcal{I}^s_{W\cap F}\bigr)\\ = \dim_k V_t|_F - \deg(f|_W) \cdot \binom{n+s-1}{n} > 0\end{aligned} \tag{8.6}$$

by the definition of volume, which implies that $V_t|_F$ has sections vanishing to order s everywhere along $W \cap F$. Now by generic flatness [EGAIV$_2$, Thm. 6.9.1] and by [Ill05, Cor. 8.3.11], the sheaf $\mathcal{O}_Y(tM) \otimes \mathcal{O}_Y/\mathcal{I}^s_W$ is generically flat and generically cohomologically flat in degree zero over Z. By cohomology and base change [Ill05, Cor. 8.3.11], the estimate (8.6) together with the exact sequence (8.5) therefore implies that the sheaf $f_*(\mathcal{O}_Y(tM) \otimes \mathcal{O}_Y/\mathcal{I}^s_W)$ is nonzero, hence has a global section by the fact that Z is affine. We then let $D_t = D_t(\varepsilon)$ be a Cartier divisor corresponding to a section in

$$H^0\bigl(Z, f_*\bigl(\mathcal{O}_Y(tM) \otimes \mathcal{O}_Y/\mathcal{I}^s_W\bigr)\bigr) = H^0\bigl(Y, \mathcal{O}_Y(tM) \otimes \mathcal{O}_Y/\mathcal{I}^s_W\bigr),$$

in which case (8.4) holds for $D \coloneqq \frac{1}{t}D_t$. $\square$

When $Z = \operatorname{Spec} k$ is a point and W is a closed point $x \in X$, we see that Lemma 8.2.2 gives a way to construct an effective $\mathbf{Q}$-Cartier divisor $D(\varepsilon)$ that is singular at x. However, this divisor may have very bad singularities in a neighborhood of x. The proof of Theorem 8.2.1 is devoted to ensuring that one can replace $D(\varepsilon)$ with a divisor with mild singularities in a neighborhood of x.

In the course of the proof, we will need the following:

Lemma 8.2.3 (cf. [Kol97, Cor. 7.8])**.** *Let (Y, Δ) be a klt pair over an algebraically closed field k of characteristic zero, and let $y \in Y$ be a closed point. Let C be a smooth affine curve,and let B be a $\mathbf{Q}$-Cartier divisor on $Y \times C$ such that $\{y\} \times C \subseteq \operatorname{Supp} B$. Let $0 \in C$ be a closed point such that $(Y \times \{c\}, \Delta \times \{c\} + B|_{Y\times\{c\}})$ is not log canonical at y for all closed points $c \in C$ in a punctured neighborhood of 0. Then, $(Y \times \{0\}, \Delta \times \{0\} + B|_{Y\times\{0\}})$ is not log canonical at y.*

Kollár's proof of [Kol97, Cor. 7.8] uses the Kollár–Shokurov connectedness theorem [Kol97, Thm. 7.4], among other results. The proof of this connectedness theorem uses vanishing theorems. We therefore give a proof of Lemma 8.2.3 that uses reduction modulo $\mathfrak{p}$ instead of vanishing theorems.

Proof. Set $X = Y \times C$, and suppose that $(Y \times \{0\}, \Delta \times \{0\} + B|_{Y \times \{0\}})$ is log canonical. Then, by inversion of adjunction for log canonical pairs [Tak04b, Thm. 4.2], we see that $(X, \Delta \times C + B + Y \times \{0\})$ is log canonical in a neighborhood $U \subseteq X$ of $(y, 0)$. Letting $W = U \cap (\{y\} \times C)$, we see that

$$\bigl(Y \times \{c\}, \Delta \times \{c\} + (B + Y \times \{0\})|_{Y \times \{c\}}\bigr) = \bigl(Y \times \{c\}, \Delta \times \{c\} + B|_{Y \times \{c\}}\bigr)$$

is log canonical at y for general closed points $c \in W$ by Reid's Bertini-type theorem [Kol97, Prop. 7.7], which is a contradiction. $\square$

We can now prove Theorem 8.2.1.

Proof of Theorem 8.2.1. We prove Theorem 8.2.1 in a sequence of steps.

Step 1. *Finding a singular divisor at x.*

Theorem 8.2.4 (cf. [Kol97, Thm. 6.7.1; Liu, Prop. 3.3])**.** *Let (X, Δ) be an effective log pair, where X is a normal projective variety over an algebraically closed field k of characteristic zero, Δ is a $\mathbf{Q}$-Weil divisor, and $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Consider a closed point $x \in X$ such that (X, Δ) is klt at x. Let H be a $\mathbf{Q}$-Cartier divisor on X, and let $V_\bullet$ be a graded linear system associated to H such that $\operatorname{vol}_X(V_\bullet) > n^n$. Then, there exists an effective $\mathbf{Q}$-divisor $B_x \sim_{\mathbf{Q}} H$ that is a multiple of a divisor in $|V_t|$ for some t such that $(X, \Delta + B_x)$ is not log canonical at x.*

Proof. If $x \in X$ is a regular point, then this immediately follows from Lemma 8.2.2 by setting $Z = \operatorname{Spec} k$ and $W = \{x\}$. Otherwise, consider the second projection morphism $X \times \mathbf{A}^1_k \to \mathbf{A}^1_k$, and let $C' \subseteq X \times \mathbf{A}^1_k$ be a general curve passing through $(x, 0)$ that is finite over $\mathbf{A}^1_k$. Let $\nu\colon C \to C'$ be the normalization of C'. We then have the commutative diagram

$$\begin{tikzcd}
C \arrow[dr, dashed, "\sigma"] \arrow[ddr, equal, bend right] \arrow[ddrr, bend left] \\
& X \times C \arrow[rr] \arrow[d, "p_2"'] & & X \times \mathbf{A}^1_k \arrow[d, "p_2"] \\
& C \arrow[r, "\nu"] & C' \arrow[r] \arrow[ur, hook] & \mathbf{A}^1_k
\end{tikzcd}$$

where the outer rectangle is cartesian. By the universal property of fiber products, this cartesian rectangle induces a section $\sigma\colon C \to X \times C$ of $p_2\colon X \times C \to C$, such that $\sigma(C)$

passes through $(x,0) \in X \times C$ for some closed point $0 \in C$. By applying Lemma 8.2.2 to the graded linear system $p_1^* V_\bullet$ on $X \times C$, the surjective morphism $p_2 \colon X \times C \to C$, and the subvariety $\sigma(C) \subseteq X \times C$, we see that for some $t > 0$, there exists an effective $\mathbf{Q}$-Cartier divisor $B \in p_1^* V_t$ such that $\frac{1}{t} B|_{X \times \{c\}}$ has multiplicity greater than n at (x,c) for every $c \in C$ in a punctured neighborhood of $0 \in C$. By taking the normalized blowup at (x,c), we see that the pair

$$\left(X \times \{c\}, \Delta \times \{c\} + \frac{1}{t} B|_{X \times \{c\}} \right)$$

is not log canonical at (x,c). We then take $B_x \coloneqq \frac{1}{t} B|_{X \times \{0\}}$, which we identify with its image in X under the isomorphism $X \times \{0\} \simeq X$. By Lemma 8.2.3, the pair

$$\big(X \times \{0\}, \Delta \times \{0\} + B_x \big) \simeq (X, \Delta + B_x)$$

is not log canonical at x. $\square$

Step 2. *Inductive step.*

The following result is the main part of the proof of Theorem 8.2.1. Below, $\operatorname{Nklt}(X,\Delta)$ is the *non-klt locus* of (X,Δ), which is the vanishing locus of the multiplier ideal $\mathcal{J}(X,\Delta)$.

Theorem 8.2.5 (cf. [Kol97, Thm. 6.8.1; Liu, Prop. 3.4])**.** *Let (X,Δ) be an effective log pair, where X is a normal projective variety over an algebraically closed field k of characteristic zero, Δ is a $\mathbf{Q}$-Weil divisor, and $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Consider a closed point $x \in X$ such that (X,Δ) is klt at x. Let D be an effective $\mathbf{Q}$-Cartier divisor such that $(X, \Delta + D)$ is log canonical on a neighborhood X^0 of x, and suppose that $\operatorname{Nklt}(X, \Delta + D) = Z \cup Z'$, where Z is irreducible, $x \in Z$, and $x \notin Z'$. Set $m = \dim Z$. Let H be a $\mathbf{Q}$-Cartier divisor such that $\operatorname{vol}_{X|Z}(H) > m^m$. Then, there exists an effective $\mathbf{Q}$-Cartier divisor $B \sim_{\mathbf{Q}} H$ and rational numbers $0 < \delta \ll 1$ and $0 < c < 1$ such that*

- *(i)* $(X, \Delta + (1-\delta)D + cB)$ *is log canonical in a neighborhood of x, and*
- *(ii)* $\operatorname{Nklt}(X, \Delta + (1-\delta)D + cB) = Z_1 \cup Z_1'$, *where $x \in Z_1$, $x \notin Z_1'$, and* $\dim Z_1 < \dim Z$.

Proof. By assumption, there is a proper birational morphism $f \colon Y \to X$ from a normal variety Y, and a divisor $E \subseteq Y$ such that $a(X, \Delta + D, E) = -1$ and $f(E) = Z$. Write

$$K_Y = f^*(K_X + \Delta + D) + \sum_i e_i E_i, \tag{8.7}$$

where $E = E_1$ and $e_1 = -1$. Let $Z^0 \subseteq Z$ be an open subset such that $f|_E \colon E \to Z$ is smooth over Z^0, and such that if $z \in Z^0$, then $(f|_E)^{-1}(z) \not\subseteq E_i$ for $i \neq 1$.

Now let $t \gg 0$ such that tH is Cartier, and such that $\mathcal{O}_X(tH) \otimes I_Z$ is globally generated away from $\mathbf{B}_+(H)$. We then make the following:

Claim 8.2.6 (cf. [Kol97, Clms. 6.8.3 and 6.8.4]). *We can construct a divisor $F_x \sim tH|_Z$ such that*

(*i*) $\operatorname{mult}_x F_x > tm$,

(*ii*) *F_x is the image of a Cartier divisor F_x^X on X under the restriction morphism*

$$H^0\big(X, \mathcal{O}_X(tH)\big) \longrightarrow H^0\big(Z, \mathcal{O}_Z(tH|_Z)\big), \tag{8.8}$$

(*iii*) *$(X, \Delta + D + \frac{1}{t}F_x^X)$ is klt on $X^0 \smallsetminus (Z \cup Z' \cup \mathbf{B}_+(D))$,*

(*iv*) *$(X, \Delta + D + \frac{1}{t}F_x^X)$ is log canonical at the generic point of Z, and*

(*v*) *$(X, \Delta + D + \frac{1}{t}F_x^X)$ is not log canonical at z.*

Proof. As in the proof of Theorem 8.2.4, we first construct a regular affine curve C such that the projection $p_2 \colon Z \times C \to C$ has a section $\sigma \colon C \to Z \times C$ for which a closed point $0 \in C$ maps to x. Now let $p_1^* V_\bullet$ be the graded linear system obtained by pulling back the graded linear system arising as the image of the restriction maps (8.8) via the first projection morphism $p_1 \colon Z \times C \to Z$. By Lemma 8.2.2, there exists an effective Cartier divisor $F \sim t\, p_1^* H|_Z$ on $Z \times C$ such that $\operatorname{mult}_{\sigma(C)} F|_{Z \times C} > tm$, and by construction, $F = F^X|_{Z \times C}$ for an effective Cartier divisor $F^X \in |t\, q_1^* H|$ on $X \times C$, where $q_1 \colon X \times C \to X$ is the first projection. The restriction $F_x \coloneqq F|_{Z \times \{\sigma(0)\}}$ then satisfies (*i*) and (*ii*). Note that (*iv*) also follows from construction, since F_x does not vanish everywhere along Z.

We now show that F_x satisfies (*iii*). First, we note that the sublinear system $|B| \subseteq |t\, q_1^* H|$ on $X \times C$ spanned by those effective Cartier divisors B' such that either $Z \times C \subseteq B'$

or $B'|_{Z\times C} = F$ is basepoint-free on $(X \smallsetminus (Z \cup \mathbf{B}_+(H))) \times C$ by the assumption that $\mathcal{O}_X(tH) \otimes I_Z$ is globally generated away from $\mathbf{B}_+(H)$. Thus, (iii) follows from the Kollár–Bertini theorem [Kol97, Thm. 4.8.2] by choosing F^X generally in $|B|$.

It remains to show (v). By Lemma 8.2.3, it suffices to show that for every $0 \neq c \in C$ such that $\sigma(c) \in Z^0$, the pair $(X, \Delta + D + \frac{1}{t}F^X_{\sigma(c)})$ is not log canonical at $\sigma(c)$, where $F^X_{\sigma(c)} \coloneqq F|_{X\times\{\sigma(c)\}}$. Let y be the generic point of $(f|_E)^{-1}(\sigma(c))$. Writing (8.7) as before, we also write

$$f^*F^X_{\sigma(c)} = F^Y_{\sigma(c)} + \sum_i tf_iE_i$$

where $F^Y_{\sigma(c)}$ is the strict transform $f^{-1}_*F^X_{\sigma(c)}$ of $F^X_{\sigma(c)}$. We then have

$$K_Y + \frac{1}{t}F^Y_{\sigma(c)} + \sum(f_i - e_i)E_i \sim_{\mathbf{Q}} f^*\Big(K_X + \Delta + D + \frac{1}{t}F^X_{\sigma(c)}\Big).$$

Now $(X, \Delta + D + \frac{1}{t}F^X_{\sigma(c)})$ is not log canonical at $\sigma(c)$ if $(Y, \frac{1}{t}F^Y_{\sigma(c)} + \sum(f_i - e_i)E_i)$ is not sub-log canonical at y. Since $Z \not\subseteq F^X_{\sigma(c)}$, we know that $f_1 = 0$. Thus, $\sum(f_i - e_i)E_i = E + \sum_{i\neq 1}(f_i - e_i)E_i$, and by assumption none of the E_i contain y when $i \neq 1$. Moreover, $(Y, \frac{1}{t}F^Y_{\sigma(c)} + \sum(f_i - e_i)E_i)$ is not sub-log canonical at y if and only if $(Y, \frac{1}{t}F^Y_{\sigma(c)} + E)$ is not log canonical at y. By inversion of adjunction [Tak04b, Thm. 4.2], the latter holds if and only if $(E, \frac{1}{t}f^*F^X_{\sigma(c)}|_E) = (E, \frac{1}{t}(f|_E)^*F_{\sigma(c)})$ is not log canonical at y. Now E is smooth at y, and y has codimension m in E and $\frac{1}{t}(f|_E)^*F_{\sigma(c)}$ has multiplicity $> m$. We then see that $(E, \frac{1}{t}(f|_E)^*F_{\sigma(c)})$ is not log canonical at y by taking the normalized blowup at y. This concludes the proof of Claim 8.2.6. □

To finish the proof of Theorem 8.2.5, we apply Claim 8.2.6 and set $B = \frac{1}{t}F^X_x$. Note that $(X, \Delta + (1-\delta)D)$ is klt at the generic point of Z for every $\delta > 0$ by the assumptions that (X, Δ) is klt, that $(X, \Delta + D)$ is log canonical in a neighborhood of x, and on the non-klt locus of $(X, \Delta + D)$. Now choose $0 < \delta \ll 1$ such that $(X, \Delta + (1-\delta)D + B)$ is not log canonical at x. Letting c be the log canonical threshold of $(X, \Delta + (1-\delta)D)$ with respect to B, we then see that $(X, \Delta + (1-\delta)D + cB)$ is log canonical but not klt at x, and that $\operatorname{Nklt}(X, \Delta + (1-\delta)D + cB) = Z_1 \cup Z_1'$, where $x \in Z_1$, $x \notin Z_1'$, and $\dim Z_1 < \dim Z$. □

We can almost show Theorem 8.2.1 using Theorem 8.2.5 and induction. However, the resulting pair in Theorem 8.2.5 may be such that $\operatorname{Nklt}(X, \Delta + (1-\delta)D + cB)$ has many irreducible components passing through x. We take care of this using the following:

Step 3. *Tie breaking.*

Lemma 8.2.7 (cf. [Kol97, Lem. 6.9.1])**.** *Let (X,Δ) be an effective log pair, where X is a normal projective variety over a field k of characteristic zero, Δ is a $\mathbf{Q}$-Weil divisor, and $K_X+\Delta$ is $\mathbf{Q}$-Cartier. Consider a k-rational point $x\in X$ such that (X,Δ) is klt at x. Let D be an effective $\mathbf{Q}$-Cartier divisor on X such that $(X,\Delta+D)$ is log canonical in a neighborhood of x. Let $\operatorname{Nklt}(X,\Delta+D)=\bigcup_i Z_i$ be the irreducible decomposition of $\operatorname{Nklt}(X,\Delta+D)$, where we label Z_1 such that $x\in Z_1$. Let H be a $\mathbf{Q}$-Cartier divisor on X such that $x\notin \mathbf{B}_+(H)$. Then, for every $0<\delta\ll 1$, there is an effective $\mathbf{Q}$-Cartier divisor $B\sim_{\mathbf{Q}} H$ and $0<c<1$ such that*

(*i*) *$(X,\Delta+(1-\delta)D+cB)$ is log canonical in a neighborhood of x, and*

(*ii*) *$\operatorname{Nklt}(X,\Delta+(1-\delta)D+cB)=W\cup W'$ where $x\in W$, $x\notin W'$, and $W\subseteq Z_1$.*

Proof. Let $t\gg 1$ such that tH is Cartier and such that $\mathcal{O}_X(tH)\otimes I_{Z_1}$ is globally generated away from $\mathbf{B}_+(H)$ (Proposition 4.6.7). Let B' correspond to a general section in $H^0(X,\mathcal{O}_X(tH)\otimes I_{Z_1})$. By the Kollár–Bertini theorem [Kol97, Thm. 4.8.2], we see that $(X,\Delta+(1-\delta)D+bB')$ is klt outside Z_1 in a neighborhood of x for $b<1$. However, it is not log canonical along Z_1 for $1>b\gg\delta>0$. Now choose $b=1/t$ and $1/t\gg\delta>0$. Then, by letting $c\in(0,1)$ be the log canonical threshold of $(X,\Delta+(1-\delta)D)$ with respect to $(1/t)B'$, we see that $(X,\Delta+(1-\delta)D+(c/t)B')$ is log canonical but not klt at x. We can then set $B=\frac{1}{t}B'$. □

Step 4. *Proof of Theorem 8.2.1.*

We prove the following theorem by induction on j.

Theorem 8.2.8. *With notation as in Theorem 8.2.1, let $j\in\{1,2,\dots,n\}$. Then, for every*

$$d_j\geq\sum_{m=n-j}^{n}\frac{m}{c(m)},\tag{8.9}$$

there exists an effective $\mathbf{Q}$-Cartier divisor $D_j\sim_{\mathbf{Q}} d_jN$ and an open neighborhood $X^0\subseteq X$ of x such that for some $b_j\in(0,1)$, we have that

(*i*) *$(X^0,\Delta+b_jD_j)$ is log canonical,* *(ii)* $\operatorname{codim}(\operatorname{Nklt}(X^0, \Delta + b_j D_j), X^0) \geq j$, *and*

(iii) $(X, \Delta + b_j D_j)$ *is not klt at* x.

Proof. Set $D_0 = \emptyset$. By induction, we will assume that D_j has already been constructed, and we are trying to construct D_{j+1}.

For $j + 1 = 1$, we construct D_1 by applying Theorem 8.2.4, and set b_1 to be the log canonical threshold $\operatorname{lct}_x((X, \Delta); D_1)$. Now consider the case when $j + 1 > 1$. First choose a positive real number $\varepsilon < (j + 1) \cdot c(j + 1)^{-1}$. By Lemma 8.2.7 and by inductive hypothesis, there exists a $\mathbf{Q}$-Cartier divisor $B_j \sim_{\mathbf{Q}} \varepsilon N$ such that for some $\delta > 0$, the $\mathbf{Q}$-Cartier divisor

$$D'_j \coloneqq (1 - \delta) b_j D_j + B_j \sim_{\mathbf{Q}} \bigl((1 - \delta) b_j d_j + \varepsilon\bigr) N$$

satisfies conditions (i)–(iii) for b_j replaced by 1, and in addition, either $Z \coloneqq \operatorname{Nklt}(X^0, \Delta + D'_j)$ is irreducible of codimension at least j at x, or it has codimension at least $j + 1$ at x. In the latter case, let M be a general member of $|tN|$ for $t \gg 1$. By assumption in (8.9), for all rational numbers $0 < \gamma \ll 1$, we have $d_{j+1} \geq (1 + \gamma)((1 - \delta) b_j d_j + \varepsilon)$. Thus, we can set

$$D_{j+1} \coloneqq (1 + \gamma) D'_j + \frac{1}{t}\bigl(d_{j+1} - (1 + \gamma)\bigl((1 - \delta) b_j d_j + \varepsilon\bigr)\bigr) M \sim_{\mathbf{Q}} d_{j+1} N,$$

and this $\mathbf{Q}$-divisor D_{j+1} satisfies conditions (i)–(iii) for $b_{j+1} = 1/(1 + \gamma)$.

It remains to consider the case when Z is irreducible of codimension at least j at x. Set $H = ((j + 1) \cdot c(j + 1)^{-1} - \varepsilon) N$. For $0 < \varepsilon \ll 1$, we have that $(H^j \cdot Z) > j^j$, hence we can apply Theorem 8.2.5 to obtain a $\mathbf{Q}$-Cartier divisor

$$D_{j+1} \sim_{\mathbf{Q}} \bigl((j + 1) \cdot c(j + 1)^{-1} - \varepsilon\bigr) N$$

satisfying conditions (i)–(iii) for b_{j+1} replaced by the rational number c in the statement of Theorem 8.2.5. $\square$

Finally, the case $j = \dim X$ in Theorem 8.2.8 is Theorem 8.2.1, concluding the proof of Theorem 8.2.1. $\square$

Appendix A

F-singularities for non-*F*-finite rings

In this appendix, we review classes of singularities defined using the Frobenius morphism, taking care to avoid F-finiteness assumptions. Most of this material is well-known, but some of the implications in Figure A.1 are new, at least for non-F-finite rings. We recommend [TW18] for a survey of F-singularities (mostly in the F-finite setting), and [DS16, §6] and [Has10a, §3] as references for the material on strong F-regularity in the non-F-finite setting. Some of this material appears in [Mur, Apps. A and B] and [DM].

To define different versions of F-rationality, we will need the following:

Definition A.1 [HH90, Def. 2.1]**.** Let R be a noetherian ring. A sequence of elements $x_1, x_2, \ldots, x_n \in R$ is a *sequence of parameters* if, for every prime ideal $\mathfrak{p}$ containing $(x_1, x_2, \ldots, x_n)$, the images of $x_1, x_2, \ldots, x_n$ in $R_\mathfrak{p}$ are part of a system of parameters in $R_\mathfrak{p}$. An ideal $I \subseteq R$ is a *parameter ideal* if I can be generated by a sequence of parameters in R.

We now begin defining different classes of singularities. We start with F-singularities defined using tight closure. Recall that if R is a ring, then R° is the complement of the union of the minimal primes of R.

Definition A.2 [HH90, Def. 8.2]**.** Let R be a ring of characteristic $p > 0$, and let $\iota\colon N \hookrightarrow M$ be an inclusion of R-modules. The *tight closure* of N in M is the R-module

$$N_M^* \coloneqq \left\{ x \in M \;\middle|\; \begin{array}{c} \text{there exists } c \in R^\circ \text{ such that for all } e \gg 0, \\ x \otimes c \in \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon N \otimes_R F_*^e R \to M \otimes_R F_*^e R\bigr) \end{array} \right\}.$$

We say that N is *tightly closed* in M if $N_M^* = N$.

Definition A.3 (F-singularities via tight closure)**.** Let R be a noetherian ring of characteristic $p > 0$. We say that

(a) R is *strongly F-regular* if $N_M^* = N$ for every inclusion $N \hookrightarrow M$ of R-modules [Hoc07, Def. on p. 166];

(b) R is *weakly F-regular* if $I_R^* = I$ for every ideal $I \subseteq R$ [HH90, Def. 4.5];

(c) R is *F-regular* if $R_{\mathfrak{p}}$ is weakly F-regular for every prime ideal $\mathfrak{p} \subseteq R$ [HH90, Def. 4.5];

(d) R is *F-rational* if $I_R^* = I$ for every parameter ideal $I \subseteq R$ [FW89, Def. 1.10].

The original definition of F-regularity asserted that localizations at every multiplicative set are weakly F-regular, but (c) is equivalent to this definition by [HH90, Cor. 4.15].

Remark A.4. Note that (a) is not the usual definition of strong F-regularity, although it coincides with the usual definition (Definition A.7(a)) for F-finite rings; see Figure A.1.

Next, we define F-singularities via purity of homomorphisms involving the Frobenius. We recall that a ring homomorphism $\varphi\colon R \to S$ is *pure* if the induced homomorphism $\mathrm{id}_M \otimes_R \varphi\colon M \otimes_R R \to M \otimes_R S$ is injective for every R-module M. To simplify notation, we fix the following:

Notation A.5. Let R be a ring of characteristic $p > 0$. For every $c \in R$ and every integer $e > 0$, we denote by λ_c^e the composition

$$R \xrightarrow{F^e} F_*^e R \xrightarrow{F_*^e(-\cdot c)} F_*^e R.$$

Definition A.6 (F-singularities via purity)**.** Let R be a noetherian ring of characteristic $p > 0$. For every $c \in R$, we say that R is *F-pure along c* if λ_c^e is pure for some $e > 0$. Moreover, we say that

(a) R is *F-pure regular* if R is F-pure along every $c \in R^\circ$ [HH94, Rem. 5.3];

(b) R is *F-pure* if R is F-pure along $1 \in R$ [HR76, p. 121];

(c) R is *strongly F-rational* if for every $c \in R^\circ$, there exists $e_0 > 0$ such that for all $e \geq e_0$, the induced homomorphism $\mathrm{id}_{R/I} \otimes_R \lambda_c^e$ is injective for every parameter ideal $I \subseteq R$ [Vél95, Def. 1.2].

The terminology *F-pure regular* is from [DS16, Def. 6.1.1] to distinguish (a) from Definition A.3(a). F-pure regular rings are also called *very strongly F-regular* [Has10a, Def. 3.4].

Next, we define F-singularities via splitting of homomorphisms involving the Frobenius. We use the same notation as for F-singularities defined using purity (Notation A.5).

Definition A.7 (F-singularities via splitting)**.** Let R be a noetherian ring of characteristic $p > 0$. For every $c \in R$, we say that R is *F-split along c* if λ_c^e splits as an R-module homomorphism for some $e > 0$. Moreover, we say that

(a) R is *split F-regular* if R is F-split along every $c \in R^\circ$ [HH94, Def. 5.1];

(b) R is *F-split* if R is F-split along $1 \in R$ [MR85, Def. 2].

The terminology *split F-regular* is from [DS16, Def. 6.6.1]. When R is F-finite, split F-regularity is usually known as strong F-regularity in the literature; see Remark A.4.

Finally, we define F-injective singularities.

Definition A.8 [Fed83, Def. on p. 473]**.** A noetherian local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ is *F-injective* if the R-module homomorphism $H^i_\mathfrak{m}(F)\colon H^i_\mathfrak{m}(R) \to H^i_\mathfrak{m}(F_*R)$ induced by Frobenius is injective for all i. An arbitrary noetherian ring R of characteristic $p > 0$ is *F-injective* if $R_\mathfrak{m}$ is F-injective for every maximal ideal $\mathfrak{m} \subseteq R$.

We characterize F-finite rings that are F-injective using Grothendieck duality. This characterization is already implicit in [Fed83, Rem. on p. 473] and the proof of [Sch09b, Prop. 4.3]. Note that if R is an F-finite ring, then the exceptional pullback $F^!$ from Grothendieck duality exists by Theorem 4.4.1, and R has a normalized dualizing complex $\omega_R^\bullet$ by Theorem 5.3.3.

Lemma A.9 (cf. [Fed83, Rem. on p. 473])**.** *Let R be an F-finite noetherian ring of characteristic $p > 0$. Then, R is F-injective if and only if the R-module homomorphisms*

$$\mathbf{h}^{-i}\mathrm{Tr}_F\colon \mathbf{h}^{-i}F_*F^!\omega_R^\bullet \longrightarrow \mathbf{h}^{-i}\omega_R^\bullet \tag{A.1}$$

induced by the Grothendieck trace of Frobenius are surjective for all i.

Lemma A.9 is most useful when R is essentially of finite type over an F-finite field, in which case $F^!\omega_R^\bullet \simeq \omega_R^\bullet$ in the derived category $\mathbf{D}^+_{\mathrm{qc}}(R)$ (Theorem 4.4.1), hence the homomorphisms in (A.1) can be written as $\mathbf{h}^{-i}F_*\omega_R^\bullet \to \mathbf{h}^{-i}\omega_R^\bullet$.

Proof. By Grothendieck local duality [Har66, Cor. V.6.3], R is F-injective if and only if

$$F^*\colon\ \operatorname{Ext}_R^{-i}(F_*R, \omega_R^\bullet) \longrightarrow \operatorname{Ext}_R^{-i}(R, \omega_R^\bullet)$$

is surjective for all i. By Grothendieck duality for finite morphisms (Theorem 4.4.1), this occurs if and only if

$$F_* \operatorname{Ext}_R^{-i}(R, F^!\omega_R^\bullet) \longrightarrow \operatorname{Ext}_R^{-i}(R, \omega_R^\bullet)$$

is surjective for all i. Since $\operatorname{Ext}_R^{-i}(R, -) = \mathbf{h}^{-i}(-)$ and by the description of the Grothendieck duality isomorphism [Har66, Thm. III.6.7], this is equivalent to the surjectivity of (A.1) for all i. $\square$

The relationship between these classes of singularities is summarized in Figure A.1. Most of the implications therein appear in the literature; see Table A.1. We now show the remaining implications in Figure A.1, for which we could not find a suitable reference.

Proofs of implications not appearing in Table A.1. Weakly F-regular + Gorenstein away from isolated points + Cohen–Macaulay $\Rightarrow$ strongly F-regular. Let R be a ring satisfying these properties. To show that R is strongly F-regular, it suffices to show that 0 is tightly closed in $E_\mathfrak{m} \coloneqq E_{R_\mathfrak{m}}(R/\mathfrak{m})$ for every maximal ideal $\mathfrak{m} \subseteq R$ [Has10a, Lem. 3.6]. Since $R_\mathfrak{m}$ is weakly R-regular [HH90, Cor. 4.15], every submodule of a finitely generated module is tightly closed [HH90, Prop. 8.7], hence the *finitistic tight closure* $0^{*\mathrm{fg}}_{E_\mathfrak{m}}$ as defined in [HH90, Def. 8.19] is zero. Since $0^{*\mathrm{fg}}_{E_\mathfrak{m}} = 0^*_{E_\mathfrak{m}}$ under the hypotheses on R [LS01, Thm. 8.8], we see that 0 is tightly closed in $E_\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subseteq R$, hence R is strongly F-regular.

Weakly F-regular + $\mathbf{N}$-graded $\Rightarrow$ split F-regular. We adapt the proof of [LS99, Cor. 4.4]. Let R be the $\mathbf{N}$-graded ring with irrelevant ideal $\mathfrak{m}$. By assumption (see [LS99, §3]), the ring R is finitely generated over a field $R_0 = k$ of characteristic $p > 0$. The localization $R_\mathfrak{m}$ of R is weakly F-regular by [HH90, Cor. 4.15]. Now let L be the perfect closure of k, and let $\mathfrak{m}'$ be the expansion of $\mathfrak{m}$ in $R \otimes_k L$; since R is graded, $\mathfrak{m}'$ is the irrelevant ideal in $R \otimes_k L$. The ring homomorphism $R_\mathfrak{m} \to R_\mathfrak{m} \otimes_k L \simeq (R \otimes_k L)_{\mathfrak{m}'}$ is purely inseparable and $\mathfrak{m}$ expands to $\mathfrak{m}'$, hence $(R \otimes_k L)_{\mathfrak{m}'}$ is weakly F-regular by [HH94, Thm. 6.17(b)]. By the proof of [LS99, Cor. 4.3], the ring $R \otimes_k L$ is split F-regular. Finally, R is a direct summand of $R \otimes_k L$ as an R-module, hence R is split F-regular as well [HH94, Thm. 5.5(e)].

F-rational + F-finite $\Rightarrow$ strongly F-rational. The hypotheses of [Vél95, Thm. 1.12] are satisfied when the ring is F-finite since an F-finite ring is excellent and is isomorphic to a quotient of a regular ring of finite Krull dimension by Theorem 5.3.3.

F-rational $\Rightarrow$ F-injective. We adapt the proof of [QS17, Prop. 6.9]. Let R be the F-rational ring, and consider a maximal ideal $\mathfrak{m} \subseteq R$. By [QS17, Thm. 3.7], it suffices to show that every ideal $I \subseteq R_\mathfrak{m}$ generated by a system of parameters in $R_\mathfrak{m}$ is *Frobenius closed* in the sense of [HH94, (10.2)]. Write $I = (a_1, a_2, \ldots, a_t)$, where t is the height of $\mathfrak{m}$ and $a_i \in R$ for every i. Note that $\mathfrak{m}$ is minimal over $(a_1, a_2, \ldots, a_t)$. Let J be the $\mathfrak{m}$-primary component of $(a_1, a_2, \ldots, a_t)$ in R. Then, we have $I = JR_\mathfrak{p}$, $\operatorname{ht} J = t$, and $\dim R/J \le d - t$, where $d = \dim R$. We claim there exist elements $b_1, b_2, \ldots, b_t \in J^2$ such that setting $x_i = a_i + b_i$, the sequence $x_1, x_2, \ldots, x_t$ is a sequence of parameters. For $i = 1$, we have

$$(a_1) + J^2 \not\subseteq \bigcup_{\substack{\mathfrak{p} \in \operatorname{Ass} R \\ \dim R/\mathfrak{p} = d}} \mathfrak{p}.$$

Thus, by a theorem of Davis [Kap74, Thm. 124], there exists $b_1 \in J^2$ such that

$$x_1 \coloneqq a_1 + b_1 \notin \bigcup_{\substack{\mathfrak{p} \in \operatorname{Ass} R \\ \dim R/\mathfrak{p} = d}} \mathfrak{p}.$$

For every $1 < i \le t$, the same method implies there exist $b_i \in J^2$ such that

$$x_i \coloneqq a_i + b_i \notin \bigcup_{\substack{\mathfrak{p} \in \operatorname{Ass}(R/(x_1, x_2, \ldots, x_{i-1})) \\ \dim R/\mathfrak{p} = d-i+1}} \mathfrak{p}.$$

We then see that $x_1, x_2, \ldots, x_t$ form a sequence of parameters in R, since they form a sequence of parameters after localizing to $R_\mathfrak{m}$, and are not all contained in any other prime ideal by construction. Now $(x_1, x_2, \dots, x_t)R_{\mathfrak{m}} \subseteq I$ and $I = (x_1, x_2, \dots, x_t)R_{\mathfrak{m}} + I^2$, hence Nakayama's lemma implies $I = (x_1, x_2, \dots, x_t)R_{\mathfrak{m}}$; see [Mat89, Cor. to Thm. 2.2]. By assumption, the ideal $(x_1, x_2, \dots, x_t)$ is tightly closed in R, hence Frobenius closed in R. Since Frobenius closure localizes [QS17, Lem. 3.3], we therefore see that $I = (x_1, x_2, \dots, x_t)R_{\mathfrak{m}}$ is Frobenius closed in $R_{\mathfrak{m}}$. $\square$

Remark A.10. The condition that R is the image of a Cohen–Macaulay ring is not too restrictive in practice. For instance, it suffices for R to be local and excellent [Kaw02, Cor. 1.2] or for R to have a dualizing complex [Kaw02, Cor. 1.4]. The latter property holds when R is F-finite; see Theorem 5.3.3.

Remark A.11. In the implication *Weakly F-regular + Gorenstein away from isolated points + Cohen–Macaulay ⇒ strongly F-regular,* MacCrimmon [Mac96, Thm. 3.3.2] showed that for F-finite rings, the Gorenstein condition can be weakened to the condition of being $\mathbf{Q}$-Gorenstein away from isolated points. The implication *weakly F-regular + F-finite ⇒ split F-regular* is a famous open problem, which was solved in dimensions at most three by Williams [Wil95, §4]. See [Abe02] for other situations in which this implication is known and for a proof of MacCrimmon's theorem (see [Abe02, (2.2.4)]).

Remark A.12. By using the gamma construction (see Appendix B), one can weaken the F-finiteness hypotheses appearing in Figure A.1. For strong F-regularity and F-purity, see Theorem B.2.3, and for F-rationality, see [Vél95, Thm. 3.8].

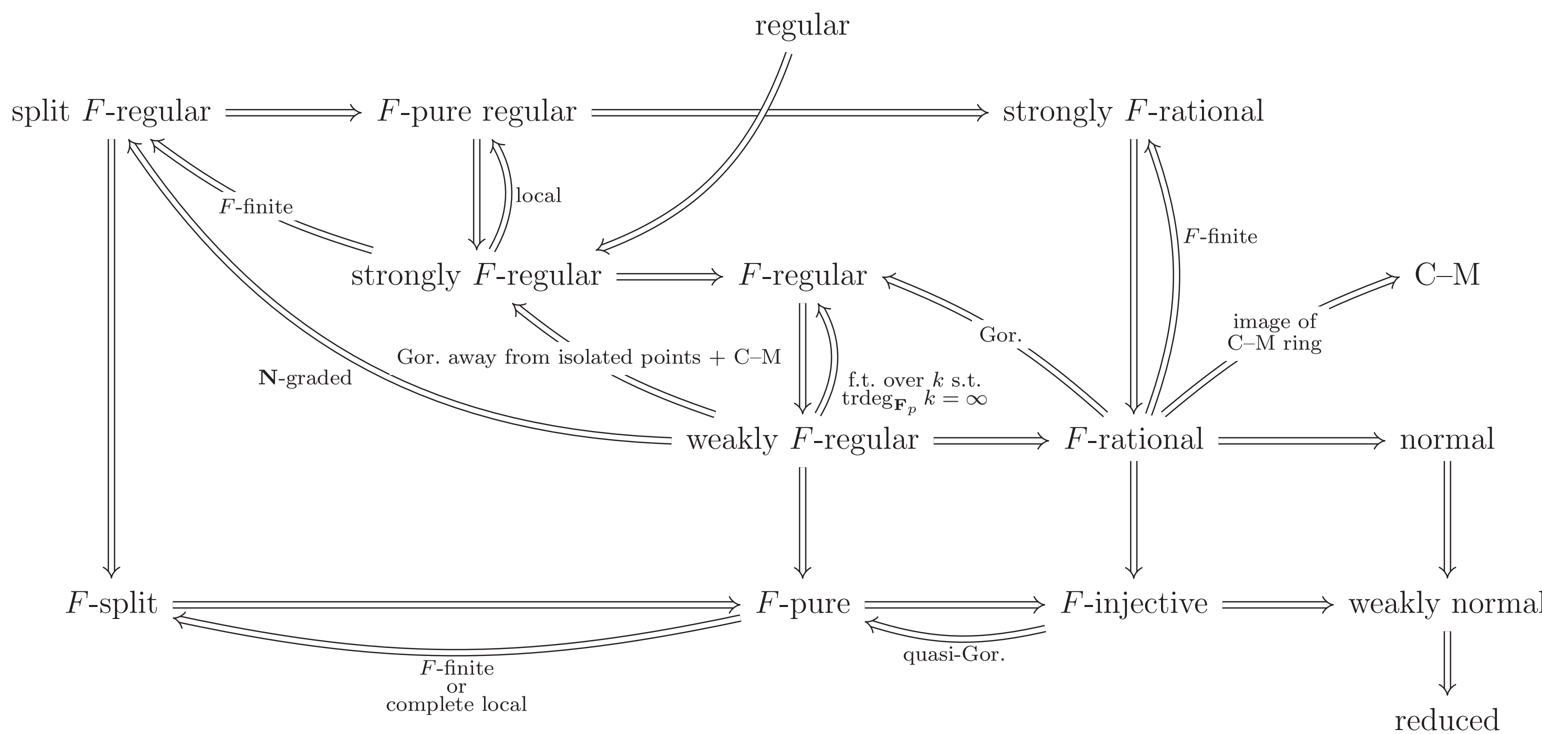

Figure A.1: Relationships between different classes of F-singularities
Here, "C–M" (resp. "Gor.") is an abbreviation for "Cohen–Macaulay" (resp. "Gorenstein"), and "f.t. over k s.t. $\operatorname{trdeg}_{\mathbf{F}_p} k = \infty$" means that the ring is of finite type over a field k that has infinite transcendence degree over its prime subfield.

Implication			Proof
split F-regular	$\Longrightarrow$	F-split	Definition
F-regular	$\Longrightarrow$	weakly F-regular	Definition
weakly F-regular	$\Longrightarrow$	F-rational	Definition
split F-regular	$\Longrightarrow$	F-pure regular	split maps are pure
F-split	$\Longrightarrow$	F-pure	split maps are pure
regular	$\Longrightarrow$	strongly F-regular	[DS16, Thm. 6.2.1]
F-pure regular	$\Longrightarrow$	strongly F-regular	[Has10a, Lem. 3.8]
F-pure regular	$\Longrightarrow$	strongly F-rational	[DS16, Rem. 6.1.5]
strongly F-regular	$\Longrightarrow$	F-regular	[Has10a, Cor. 3.7]
weakly F-regular	$\Longrightarrow$	F-pure	[FW89, Rem. 1.6]
F-pure	$\Longrightarrow$	F-injective	[HR74, Cor. 6.8]
strongly F-rational	$\Longrightarrow$	F-rational	[Vél95, Prop. 1.4]
F-rational	$\Longrightarrow$	normal	[HH94, Thm. 4.2(b)]
F-rational + image of C–M ring	$\Longrightarrow$	Cohen–Macaulay	[HH94, Thm. 4.2(c)]
F-injective	$\Longrightarrow$	weakly normal	[DM, Cor. 3.5]
strongly F-regular + F-finite	$\Longrightarrow$	split F-regular	[Has10a, Lem. 3.9]
strongly F-regular + local	$\Longrightarrow$	F-pure regular	[Has10a, Lem. 3.6]
weakly F-regular + f.t. over k s.t. $\operatorname{trdeg}_{\mathbf{F}_p} k = \infty$	$\Longrightarrow$	F-regular	[HH94, Thm. 8.1]
F-pure + F-finite	$\Longrightarrow$	F-split	[HR76, Cor. 5.3]
F-pure + complete local	$\Longrightarrow$	F-split	[Fed83, Lem. 1.2]
F-rational + Gor.	$\Longrightarrow$	F-regular	[HH94, Cor. 4.7(a)]
F-injective + quasi-Gor.	$\Longrightarrow$	F-pure	[EH08, Rem. 3.8]

Table A.1: Proofs of relationships between different classes of *F*-singularities

Appendix B

The gamma construction of Hochster–Huneke

We prove a scheme-theoretic version of the gamma construction of Hochster–Huneke [HH94], which we use to systematically reduce questions about varieties over an arbitrary imperfect field to the same questions over an F-finite field (that is still imperfect). In commutative algebra, the construction was first introduced in order to prove that test elements (in the sense of tight closure) exist for rings that are essentially of finite type over an excellent local ring of characteristic $p > 0$. The material below is from [Mur].

B.1. Construction and main result

The following is the main consequence of the gamma construction:

Theorem B.1.1. *Let X be a scheme essentially of finite type over a field k of characteristic $p > 0$, and let $\mathcal{Q}$ be a set of properties in the following list: local complete intersection, Gorenstein, Cohen–Macaulay, S_n, R_n, normal, weakly normal, reduced, strongly F-regular, F-pure, F-rational, F-injective. Then, there exists a purely inseparable field extension $k \subseteq k^\Gamma$ such that k^Γ is F-finite and such that the projection morphism*

$$\pi^\Gamma \colon X \times_k k^\Gamma \longrightarrow X$$

is a homeomorphism that identifies $\mathcal{P}$ loci for every $\mathcal{P} \in \mathcal{Q}$.

Here, we recall that for a scheme X and a property $\mathcal{P}$ of local rings on X, the $\mathcal{P}$ *locus* of X is

$$\mathcal{P}(X) \coloneqq \{x \in X \mid \mathcal{O}_{X,x} \text{ is } \mathcal{P}\}.$$

We will in fact show a more general result (Theorem B.1.6), which allows for k to be replaced by a complete local ring, and allows finitely many schemes instead of just one. Note that Theorem B.1.1 for weak normality, F-purity, and F-injectivity are new even in the affine setting.

Before describing the construction, we motivate the idea behind the construction with the following:

Example B.1.2. Let k be a non-F-finite field of characteristic $p > 2$, and let $a \in k \smallsetminus k^p$. For example, we can let $k = \mathbf{F}_p(x_i)_{i\in\mathbf{N}}$ and let $a = x_0$. Let $S = k[x,y]$ and $f = y^2 + x^p - a \in S$, and consider Chevalley's example [Zar47, Ex. 3]

$$R = S/(f) = \frac{k[x,y]}{y^2 + x^p - a}.$$

We claim that R is regular. Note that R is smooth everywhere except at the maximal ideal $(x^p - a, y)$, since the Jacobian for R is $(0,\ 2y)$. It therefore suffices to show that R is regular at $\mathfrak{m}_R \coloneqq (x^p - a, y)R \subseteq R$. To avoid confusion, we denote by $\mathfrak{m}_S$ the ideal $(x^p - a, y)S \subseteq S$. We have

$$\dim_{S/\mathfrak{m}_S}\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2}\right) = 2,$$

since S is regular. On the other hand, the defining equation $f = y^2 + x^p - a$ for R is nonzero modulo $\mathfrak{m}_S^2$, hence

$$\dim_{R/\mathfrak{m}_R}\left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}\right) = \dim_{S/\mathfrak{m}_S}\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2 + (f)}\right) = 1.$$

Thus, $R_{\mathfrak{m}_R}$ is regular, and R is regular everywhere.

We would now like to find a field extension $k \subseteq k'$ such that $R \otimes_k k'$ is F-finite and regular. First, we claim that setting $k' = k_{\mathrm{perf}}$ will result in an F-finite ring that is not regular. Set

$$R' \coloneqq R \otimes_k k(a^{1/p}) \simeq \frac{k(a^{1/p})[x,y]}{y^2 + x^p - a} \simeq \frac{k(a^{1/p})[x,y]}{y^2 + (x - a^{1/p})^p},$$

and denote $\mathfrak{m}_{R'} = (x - a^{1/p}, y)R'$. We have that

$$y^2 + x^p - a = y^2 + (x - a^{1/p})^2 \cdot (x - a^{1/p})^{2-p} \in (x - a^{1/p}, y)^2,$$

hence

$$\dim_{R'/\mathfrak{m}_{R'}}\left(\frac{\mathfrak{m}_{R'}}{\mathfrak{m}_{R'}^2}\right) = \dim_{S/\mathfrak{m}_S}\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2 + (f)}\right) = 2.$$

Thus, we see that R' is not regular at the maximal ideal $\mathfrak{m}_{R'}$. We therefore want to find a field extension $k \subseteq k'$ that avoids adjoining $a^{1/p}$, such that k' is still F-finite. The gamma construction (Theorem B.1.1) ensures the existence of such an extension, although we note that in the specific case where $k = \mathbf{F}_p(x_i)_{i \in \mathbf{N}}$ and $a = x_0$ above, we can set $k' = \mathbf{F}_p(x_0)(x_i)_{i \in \mathbf{N} \smallsetminus \{0\}}$.

We now give an account of Hochster and Huneke's construction.

Construction B.1.3 [HH94, (6.7) and (6.11)]**.** Let $(A, \mathfrak{m}, k)$ be a noetherian complete local ring of characteristic $p > 0$. By the Cohen structure theorem, we may identify k with a coefficient field $k \subseteq A$. Moreover, by Zorn's lemma (see [Mat89, p. 202]), we may choose a *p-basis* Λ for k, which is a subset $\Lambda \subseteq k$ such that $k = k^p(\Lambda)$, and such that for every finite subset $\Sigma \subseteq \Lambda$ with s elements, we have $[k^p(\Sigma) : k^p] = p^s$.

Now let $\Gamma \subseteq \Lambda$ be a cofinite subset, i.e., a subset Γ of Λ such that $\Lambda \smallsetminus \Gamma$ is a finite set. For each integer $e \geq 0$, consider the subfield $k_e^\Gamma = k[\lambda^{1/p^e}]_{\lambda \in \Gamma} \subseteq k_{\mathrm{perf}}$ of some perfect closure k_{perf} of k. These form an ascending chain, and we then set

$$A^\Gamma := \varinjlim_{e} k_e^\Gamma [\![A]\!],$$

where $k_e^\Gamma[\![A]\!]$ is the completion of $k_e^\Gamma \otimes_k A$ at the extended ideal $\mathfrak{m} \cdot (k_e^\Gamma \otimes_k A)$. Note that if $A = k$ is a field, then $A^\Gamma = k^\Gamma$ is a field by construction.

Finally, let X be a scheme essentially of finite type over A, and consider two cofinite subsets $\Gamma \subseteq \Lambda$ and $\Gamma' \subseteq \Lambda$ such that $\Gamma \subseteq \Gamma'$. We then have the following commutative diagram whose vertical faces are cartesian:

$$\begin{tikzcd}
X^{\Gamma'} \arrow[rr, "\pi^{\Gamma\Gamma'}"] \arrow[dr, "\pi^{\Gamma'}"] \arrow[dd] & & X^{\Gamma} \arrow[dl, "\pi^{\Gamma}"'] \arrow[dd] \\
& X \arrow[dd] & \\
\operatorname{Spec} A^{\Gamma'} \arrow[rr] \arrow[dr] & & \operatorname{Spec} A^{\Gamma} \arrow[dl] \\
& \operatorname{Spec} A &
\end{tikzcd}$$

We list some elementary properties of the gamma construction.

Lemma B.1.4. *Fix notation as in Construction B.1.3, and let $\Gamma \subseteq \Lambda$ be a cofinite subset.*

(*i*) *The ring A^Γ and the scheme X^Γ are noetherian and F-finite.*

(*ii*) *The morphism π^Γ is a faithfully flat universal homeomorphism with local complete intersection fibers.*

(*iii*) *Given a cofinite subset $\Gamma \subseteq \Gamma'$, the morphism $\pi^{\Gamma\Gamma'}$ is a faithfully flat universal homeomorphism.*

Proof. The ring A^Γ is noetherian and F-finite [HH94, (6.11)], hence X^Γ is also by Example 5.3.2 and the fact that morphisms essentially of finite type are preserved under base change (Lemma 4.1.3). The ring extensions $A \subseteq A^\Gamma$ and $A^\Gamma \subseteq A^{\Gamma'}$ are purely inseparable and faithfully flat [HH94, (6.11)], hence induce faithfully flat universal homeomorphisms on spectra [EGAIV$_2$, Prop. 2.4.5(*i*)]. Thus, the morphisms π^Γ and $\pi^{\Gamma\Gamma'}$ are faithfully flat universal homeomorphisms by base change. Finally, the ring extension $A \subseteq A^\Gamma$ is flat with local complete intersection fibers [Has10a, Lem. 3.19], hence π^Γ is also by base change [Avr75, Cor. 4]. □

Our goal now is to prove that if a local property of schemes satisfies certain conditions, then the property is preserved when passing from X to X^Γ for "small enough" Γ.

Proposition B.1.5. *Fix notation as in Construction B.1.3, and let $\mathcal{P}$ be a property of local rings of characteristic $p > 0$.*

(i) Suppose that for every flat local homomorphism $B \to C$ of noetherian local rings with local complete intersection fibers, if B is $\mathcal{P}$, then C is $\mathcal{P}$. Then, $\pi^\Gamma(\mathcal{P}(X^\Gamma)) = \mathcal{P}(X)$ for every cofinite subset $\Gamma \subseteq \Lambda$.

(ii) Consider the following conditions:

($\Gamma1$) *If B is a noetherian F-finite ring of characteristic $p > 0$, then $\mathcal{P}(\operatorname{Spec} B)$ is open.*

($\Gamma2$) *For every flat local homomorphism $B \to C$ of noetherian local rings of characteristic $p > 0$ with zero-dimensional fibers, if C is $\mathcal{P}$, then B is $\mathcal{P}$.*

($\Gamma3$) *For every local ring B essentially of finite type over A, if B is $\mathcal{P}$, then there exists a cofinite subset $\Gamma_1 \subseteq \Lambda$ such that B^Γ is $\mathcal{P}$ for every cofinite subset $\Gamma \subseteq \Gamma_1$.*

($\Gamma3'$) *For every flat local homomorphism $B \to C$ of noetherian local rings of characteristic $p > 0$ such that the closed fiber is a field, if B is $\mathcal{P}$, then C is $\mathcal{P}$.*

If $\mathcal{P}$ satisfies ($\Gamma1$), ($\Gamma2$), *and one of either* ($\Gamma3$) *or* ($\Gamma3'$), *then there exists a cofinite subset $\Gamma_0 \subseteq \Lambda$ such that $\pi^\Gamma(\mathcal{P}(X^\Gamma)) = \mathcal{P}(X)$ for every cofinite subset $\Gamma \subseteq \Gamma_0$.*

Proof. For (i), it suffices to note that π^Γ is faithfully flat with local complete intersection fibers by Lemma B.1.4(ii).

For (ii), we first note that ($\Gamma3'$) implies ($\Gamma3$), since there exists a cofinite subset $\Gamma_1 \subseteq \Lambda$ such that the closed fiber is a field for every cofinite subset $\Gamma \subseteq \Gamma_1$ by [HH94, Lem. 6.13(b)]. From now on, we therefore assume that $\mathcal{P}$ satisfies ($\Gamma1$), ($\Gamma2$), and ($\Gamma3$).

For every cofinite subset $\Gamma \subseteq \Lambda$, the set $\mathcal{P}(X^\Gamma)$ is open by ($\Gamma1$) since X^Γ is noetherian and F-finite by Lemma B.1.4(i). Moreover, the morphisms π^Γ and $\pi^{\Gamma\Gamma'}$ are faithfully flat universal homeomorphisms for every cofinite subset $\Gamma' \subseteq \Lambda$ such that $\Gamma \subseteq \Gamma'$ by Lemmas B.1.4(ii) and B.1.4(iii), hence by ($\Gamma2$), we have the inclusions

$$\mathcal{P}(X) \supseteq \pi^\Gamma\bigl(\mathcal{P}(X^\Gamma)\bigr) \supseteq \pi^{\Gamma'}\bigl(\mathcal{P}(X^{\Gamma'})\bigr) \tag{B.1}$$

in X, where $\pi^\Gamma(\mathcal{P}(X^\Gamma))$ and $\pi^{\Gamma'}(\mathcal{P}(X^{\Gamma'}))$ are open. Since X is noetherian, it satisfies the ascending chain condition on the open sets $\pi^\Gamma(\mathcal{P}(X^\Gamma))$, hence we can choose a cofinite subset $\Gamma_0 \subseteq \Lambda$ such that $\pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$ is maximal with respect to inclusion.

We claim that $\mathcal{P}(X) = \pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$ for every cofinite subset $\Gamma \subseteq \Gamma_0$. By (B.1), it suffices to show the inclusion $\subseteq$. Suppose there exists $x \in \mathcal{P}(X) \smallsetminus \pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$. By ($\Gamma$3), there exists a cofinite subset $\Gamma_1 \subseteq \Lambda$ such that $(\pi^{\Gamma})^{-1}(x) \in \mathcal{P}(X^{\Gamma})$ for every cofinite subset $\Gamma \subseteq \Gamma_1$. Choosing $\Gamma = \Gamma_0 \cap \Gamma_1$, we have $x \in \pi^{\Gamma}(\mathcal{P}(X^{\Gamma})) \smallsetminus \pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$, contradicting the maximality of $\pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$. $\square$

We now prove that the properties in Theorem B.1.1 are preserved when passing to X^{Γ}. Special cases of the following result appear in [HH94, Lem. 6.13], [Vél95, Thm. 2.2], [EH08, Lem. 2.9], [Has10a, Lems. 3.23 and 3.30], and [Ma14, Prop. 5.6].

Theorem B.1.6. *Fix notation as in Construction B.1.3.*

(i) *For every cofinite subset $\Gamma \subseteq \Lambda$, the map π^{Γ} identifies local complete intersection, Gorenstein, Cohen–Macaulay, and S_n loci.*

(ii) *There exists a cofinite subset $\Gamma_0 \subseteq \Lambda$ such that π^{Γ} identifies R_n (resp. normal, weakly normal, reduced, strongly F-regular, F-pure, F-rational, F-injective) loci for every cofinite subset $\Gamma \subseteq \Gamma_0$.*

Note that Theorem B.1.6 implies Theorem B.1.1 since if A is a field, then A^{Γ} is also by Construction B.1.3, and moreover if one wants to preserve more than one property at once, then it suffices to intersect the various Γ_0 for the different properties.

Proof. For (i), it suffices to note that these properties satisfy the condition in Proposition B.1.5(i) by [Avr75, Cor. 2] and [Mat89, Thm. 23.4, Cor. to Thm. 23.3, and Thm. 23.9(iii)], respectively.

We now prove (ii). We first note that (ii) holds for regularity since (Γ1) holds by the excellence of X^{Γ}, and (Γ2) and (Γ3′) hold by [Mat89, Thm. 23.7]. Since π^{Γ} preserves the dimension of local rings, we therefore see that (ii) holds for R_n. (ii) for normality and reducedness then follows from (i) since they are equivalent to $R_1 + S_2$ and $R_0 + S_1$, respectively.

To prove (ii) holds in the remaining cases, we check the conditions in Proposition B.1.5(ii). For weak normality, (Γ1) holds by [BF93, Thm. 7.1.3], and (Γ2) holds by [Man80, Cor. II.2]. To show that (Γ3) holds, recall by [Man80, Thm. I.6] that a reduced ring B is weakly normal if and only if

$$B \longrightarrow B^{\nu} \underset{b \mapsto 1 \otimes b}{\overset{b \mapsto b \otimes 1}{\rightrightarrows}} (B^{\nu} \otimes_B B^{\nu})_{\mathrm{red}} \tag{B.2}$$

is an equalizer diagram, where B^ν is the normalization of B. Now suppose B is weakly normal, and let $\Gamma_1 \subseteq \Lambda$ be a cofinite subset such that B^Γ is reduced, $(B^\nu)^\Gamma$ is normal, and $((B^\nu \otimes_B B^\nu)_{\mathrm{red}})^\Gamma$ is reduced for every cofinite subset $\Gamma \subseteq \Gamma_1$; such a Γ_1 exists by the previous paragraph. We claim that B^Γ is weakly normal for every $\Gamma \subseteq \Gamma_1$ cofinite in Λ. Since (B.2) is an equalizer diagram and $A \subseteq A^\Gamma$ is flat, the diagram

$$B^\Gamma \longrightarrow (B^\nu)^\Gamma \underset{b \mapsto 1 \otimes b}{\overset{b \mapsto b \otimes 1}{\rightrightarrows}} \big((B^\nu \otimes_B B^\nu)_{\mathrm{red}}\big)^\Gamma$$

is an equalizer diagram. Moreover, since $B^\Gamma \subseteq (B^\nu)^\Gamma$ is an integral extension of rings with the same total ring of fractions, and $(B^\nu)^\Gamma$ is normal, we see that $(B^\nu)^\Gamma = (B^\Gamma)^\nu$. Finally, $((B^\nu \otimes_B B^\nu)_{\mathrm{red}})^\Gamma$ is reduced, hence we have the natural isomorphism

$$\big((B^\nu \otimes_B B^\nu)_{\mathrm{red}}\big)^\Gamma \simeq \big((B^\Gamma)^\nu \otimes_{B^\Gamma} (B^\Gamma)^\nu\big)_{\mathrm{red}}.$$

Thus, since the analogue of (B.2) with B replaced by B^Γ is an equalizer diagram, we see that B^Γ is weakly normal for every $\Gamma \subseteq \Gamma_1$ cofinite in Λ, hence (Γ3) holds for weak normality.

We now prove (*ii*) for strong F-regularity, F-purity, and F-rationality. First, (Γ1) holds for strong F-regularity by [Has10a, Lem. 3.29], and the same argument shows that (Γ1) holds for F-purity since the F-pure and F-split loci coincide for F-finite rings [HR76, Cor. 5.3]. Next, (Γ1) for F-rationality holds by [Vél95, Thm. 1.11] since the reduced locus is open and reduced F-finite rings are admissible in the sense of [Vél95, Def. 1.5] by Theorem 5.3.3. It then suffices to note that (Γ2) holds by [Has10a, Lem. 3.17], [HR76, Prop. 5.13], and [Vél95, (6) on p. 440], respectively, and (Γ3) holds by [Has10a, Cor. 3.31], [Ma14, Prop. 5.4], and [Vél95, Lem. 2.3], respectively.

Finally, we prove (*ii*) for F-injectivity. First, (Γ1) and (Γ2) hold by [Mur, Lem. A.2] and [Mur, Lem. A.3], respectively. The proof of [EH08, Lem. 2.9(*b*)] implies (Γ3), since the residue field of B is a finite extension of k, hence socles of artinian B-modules are finite-dimensional k-vector spaces. $\square$

B.2. Applications

We now give some applications of the gamma construction (Theorem B.1.6). See also [Mur, §3.2] for applications to the minimal model program over imperfect fields.

B.2.1. Openness of F-singularities

We have the following consequence of Theorem B.1.6, which was first attributed to Hoshi in [Has10b, Thm. 3.2]. Note that the analogous statements for strong F-regularity and F-rationality appear in [Has10a, Prop. 3.33] and [Vél95, Thm. 3.5], respectively.

Corollary B.2.1. *Let X be a scheme essentially of finite type over a local G-ring $(A, \mathfrak{m})$ of characteristic $p > 0$. Then, the F-pure locus is open in X.*

Recall that a noetherian ring R is a *G-ring* if, for every prime ideal $\mathfrak{p} \subseteq R$, the completion homomorphisms $R_\mathfrak{p} \to \widehat{R_\mathfrak{p}}$ are regular in the sense of [EGAIV$_2$, Def. 6.8.1].

Proof. Let $A \to \widehat{A}$ be the completion of A at $\mathfrak{m}$, and let Λ be a p-basis for $\widehat{A}/\mathfrak{m}\widehat{A}$ as in Construction B.1.3. For every cofinite subset $\Gamma \subseteq \Lambda$, consider the commutative diagram

$$\begin{tikzcd}
X \times_A \widehat{A}^\Gamma \arrow[r, "\pi^\Gamma"] \arrow[d] & X \times_A \widehat{A} \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \widehat{A}^\Gamma \arrow[r] & \operatorname{Spec} \widehat{A} \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

where the squares are cartesian. By Theorem B.1.6, there exists a cofinite subset $\Gamma \subseteq \Lambda$ such that π^Γ is a homeomorphism identifying F-pure loci. Since $X \times_A \widehat{A}^\Gamma$ is F-finite, the F-pure locus in $X \times_A \widehat{A}$ is therefore open by the fact that (Γ1) holds for F-purity; see the proof of Theorem B.1.6(ii).

Now let $x \in X \times_A \widehat{A}$. Since $A \to \widehat{A}$ is a regular homomorphism, the morphism π is also regular by base change [EGAIV$_2$, Prop. 6.8.3(iii)]. Thus, $\mathcal{O}_{X \times_A \widehat{A}, x}$ is F-pure if and only if $\mathcal{O}_{X, \pi(x)}$ is F-pure by [HR76, Prop. 5.13] and [Has10a, Props. 2.4(4) and 2.4(6)]. Denoting the F-pure locus in X by W, we see that $\pi^{-1}(W)$ is the F-pure locus in $X \times_A \widehat{A}$. Since $\pi^{-1}(W)$ is open and π is quasi-compact and faithfully flat by base change, the F-pure locus $W \subseteq X$ is open by [EGAIV$_2$, Cor. 2.3.12]. $\square$

Remark B.2.2. Although [Mur, Lem. A.2] shows that the F-injective locus is open under F-finiteness hypotheses, and the gamma construction (Theorem B.1.6) implies that the F-injective locus is open for schemes essentially of finite type over *complete* local rings, the fact that the F-injective locus is open under the hypotheses of Corollary B.2.1 is a recent result due to Rankeya Datta and the author [DM, Thm. B].

B.2.2. *F*-singularities for rings essentially of finite type

We finally show that for rings to which the gamma construction applies, the notions of strong F-regularity and split F-regularity coincide, as do the notions of F-purity and F-splitting. This result is unpublished work of Rankeya Datta and the author.

Theorem B.2.3. *Let R be a ring essentially of finite type over a noetherian complete local ring $(A, \mathfrak{m}, k)$ of characteristic $p > 0$. If R is strongly F-regular (resp. F-pure), then R is split F-regular (resp. F-split).*

We first show the following preliminary result, which was communicated to Hochster by Auslander (although it may be older).

Lemma B.2.4 (cf. [Fed83, Lem. 1.2])**.** *Let $(A, \mathfrak{m}, k)$ be a noetherian complete local ring. Then, every pure ring homomorphism $A \to B$ splits as an A-module homomorphism.*

Proof. Let $f\colon A \to B$ be a pure ring homomorphism. We claim we have the following commutative diagram with exact rows, where the vertical homomorphisms are isomorphisms:

$$
\begin{tikzcd}
\operatorname{Hom}_A\bigl(B \otimes_A E_A(k), E_A(k)\bigr) \arrow[r, "(f\otimes \mathrm{id}_{E_A(k)})^*"] \arrow[d, "\wr"] & \operatorname{Hom}_A\bigl(A \otimes_A E_A(k), E_A(k)\bigr) \arrow[r] \arrow[d, "\wr"] & 0\\
\operatorname{Hom}_A\bigl(B, \operatorname{Hom}_A\bigl(E_A(k), E_A(k)\bigr)\bigr) \arrow[r, "f^*"] & \operatorname{Hom}_A\bigl(A, \operatorname{Hom}_A\bigl(E_A(k), E_A(k)\bigr)\bigr) \arrow[r] & 0\\
\operatorname{Hom}_A(B, A) \arrow[r, "f^*"] \arrow[u, "\wr"'] & \operatorname{Hom}_A(A, A) \arrow[r] \arrow[u, "\wr"'] & 0
\end{tikzcd}
$$

The top row is the Matlis dual of the map $f \otimes \mathrm{id}_{E_A(k)}\colon A \otimes_A E_A(k) \to B \otimes_A E_A(k)$, and the second row is obtained from the first by tensor-hom adjunction. The last row is obtained from the isomorphism $\operatorname{Hom}_A(E_A(k), E_A(k)) \simeq A$, which holds by the completeness of A [Mat89, Thm. 18.6(iv)]. Since the last row is surjective, we can choose $g \in \operatorname{Hom}_A(B, A)$ such that $f^*(g) = g \circ f = \mathrm{id}_A$. $\square$

We can now show Theorem B.2.3.

Proof of Theorem B.2.3. By the gamma construction (Theorem B.1.6), there exists a faithfully flat ring extension $A \hookrightarrow A^\Gamma$ such that $R^\Gamma \coloneqq R \otimes_A A^\Gamma$ is strongly F-regular (resp. F-pure) and F-finite. By F-finiteness, the ring R^Γ is split F-regular by [Has10a, Lem. 3.9] (resp. F-split by [HR76, Cor. 5.3]). Now consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & R \arrow[r, "F^e_R"] \arrow[d, hook] & F^e_* R \arrow[r, "F^e_*(-\cdot c)"] \arrow[d, hook] & F^e_* R \arrow[d, hook] \\
A^\Gamma \arrow[r] & R^\Gamma \arrow[r, "F^e_{R^\Gamma}"] & F^e_* R^\Gamma \arrow[r, "F^e_*(-\cdot(c\otimes 1))"] & F^e_* R^\Gamma
\end{tikzcd}$$

for every $c \in R^\circ$ and every integer $e > 0$, where the left square is cocartesian. Note that if $c \in R^\circ$, then $c \otimes 1 \in (R^\Gamma)^\circ$, since $R \to R^\Gamma$ satisfies going-down [Mat89, Thm. 9.5].

Since the inclusion $A \hookrightarrow A^\Gamma$ is faithfully flat, it is pure, hence splits as an A-module homomorphism by Lemma B.2.4. By base change, this implies the inclusion $R \hookrightarrow R^\Gamma$ splits as an R-module homomorphism. For both split F-regularity and F-splitting, it then suffices to note that if $F^e_*(- \cdot (c \otimes 1)) \circ F^e_{R^\Gamma}$ splits for some $c \in R^\circ$ and for some $e > 0$, then composing this splitting with a splitting of $R \hookrightarrow R^\Gamma$ gives a splitting of $F^e_*(- \cdot c) \circ F^e_R$ by the commutativity of the diagram above. $\square$

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