Weyl's Law for Singular Algebraic Varieties

by

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ABSTRACT

It is a classical result that the spectrum of the Laplacian on a compact Riemannian manifold forms a sequence going to positive infinity and satisfies an asymptotic growth rate known as Weyl's law determined by the volume and dimension of the manifold. Weyl's law motivated Kac's famous question, "Can one hear the shape of a drum?" which asks what geometric properties of a space can be determined by the spectrum of its Laplacian? I will show Weyl's law also holds for the non-singular locus of embedded, irreducible, singular projective algebraic varieties with the metric inherited from the Fubini-Study metric of complex projective space. This non-singular locus is a non-complete manifold with finite volume that comes from a very natural class of spaces which are extensively studied and used in many different disciplines of mathematics. Since the volume of a projective variety in the Fubini-Study metric is equal to its degree times the volume of the complex projective space of the same dimension, the result of this thesis shows the algebraic degree of a projective variety can be "heard" from its spectrum. The proof follows the heat kernel method of Minakshisundaram and Pleijel using heat kernel estimates of Li and Tian. Additionally, the eigenfunctions of the Laplacian on a singular variety will also be shown to satisfy a bound analogous to the known bound for the eigenfunctions of the Laplacian on a compact manifold.

CHAPTER I

Introduction

Weyl's law dates back to Weyl's 1911 paper "Ueber die asymptotische Verteilung der Eigenwerte" (About the Asymptotic Distribution of Eigenvalues), in which Weyl proved in [Wey] Theorem X

$$\lim_{i\to\infty} \frac{i}{\lambda_i} = \frac{\mathrm{Area}(\Omega)}{4\pi}$$

for any bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where λ_i is the i-th eigenvalue of the Laplacian with Dirichlet boundary condition. Weyl also stated that his method can be generalized to higher dimensions. Minakshisundaram and Pleijel proved Weyl's law for any compact Riemannian manifold M^n (see pages 244 and 255-256 of [Min].) If M^n has a boundary, then Weyl's law holds for both the Dirichlet and Neumann boundary conditions. In these cases,

$$\lim_{i\to\infty} \frac{i}{\lambda_i^{n/2}} = \frac{\boldsymbol{\omega}_n V(M)}{(2\pi)^n}$$

where $\boldsymbol{\omega}_n$ is the volume of the unit ball in $\mathbb{R}^n$. This can be reformulated as

$$\sum_{\lambda_i\le\lambda} 1 \sim c_M \cdot \lambda^{n/2} \text{ as } \lambda \to \infty, \quad c_M = \frac{\boldsymbol{\omega}_n V(M)}{(2\pi)^n}.$$

Here, c_M is a constant which depends only on n and $V(M)$, and $f(\lambda) \sim g(\lambda)$ as $\lambda \to a$ is the equivalence relation meaning $\lim_{\lambda\to a} \frac{f(\lambda)}{g(\lambda)} = 1$. Observe that if $\mathrm{Spec}(\Delta) := \{\lambda_i\}$

is known, then Weyl's law determines first the dimension of M from the exponenent, and then $V(M)$ from c_M. This suggests one can 'hear' the volume of M. This is one of the first answers to the following question: What properties of a domain are determined by the spectrum of its Laplacian? Mark Kac [Kac] famously phrased this as "Can One Hear the Shape of a Drum?" The field of inverse problems in spectral geometry attempts to answer this question. In general, it is known that the spectrum cannot completely determine the Riemannian manifold. A counterexample was given by Milnor [Mil] in 1964 (this paper is one page.)

Singular, projective algebraic varieties give examples of non-compact manifolds by considering the non-singular locus. In fact, with the Fubini-Study metric inheirited from an embedding in $\mathbb{CP}^d$, the non-singular locus is not even a complete Riemannian manifold. Still, this metric and the corresponding Laplacian share many properties with those on compact manifolds. The volume is finite, and the spectrum of the unique self-adjoint extension of Laplacian in $L^2(M)$ is discrete (see Theorems 4.1 and 5.3 of [LT].) The main result of this thesis is that Weyl's law holds for this non-singular locus with the Fubini-Study metric. Weyl's law is already known in the special case when the singular variety is a curve (complex dimension 1) due to Brüning and Lesch's paper [BL2].

Theorem I.1. ***(Weyl's Law)***

Let $V \subset \mathbb{CP}^d$ be an irreducible, projective variety of complex dimension $\frac{n}{2}$, and define $N := V \setminus Sing(V)$ with the Fubini-Study metric from $\mathbb{CP}^d$. Define the counting function $N(\lambda) = \sum_{\lambda_i \leq \lambda} 1$. Then

$$N(\lambda) \sim \frac{\boldsymbol{\omega}_n V(N) \lambda^{\frac{n}{2}}}{(2\pi)^n} \text{ as } \lambda \to \infty,$$

hence

$$(\lambda_i)^{\frac{n}{2}} \sim \frac{(2\pi)^n i}{\boldsymbol{\omega}_n V(N)} \text{ as } i \to \infty.$$

As will be shown, the irreducible condition can be relaxed, so long as all of the irreducible components are the same dimension. The proof comes from comparing local asymptotic properties of the heat kernel to a global description.

Additionally, the eigenfunctions of the Laplacian on a singular, projective algebraic variety are bounded and satisfy the same bound in terms of the eigenvalues which the eigenfunctions of the Laplacian of a compact Riemannian manifold are known to satisfy (see equation (10.7) of [Li]).

Theorem I.2. *Let $\{\phi_i\}_{i=0}^{\infty}$ be an orthonormal basis of $L^2(N)$ consisiting of eigenfuctions of Δ with $\Delta\phi_i = \lambda_i\phi_i$ and $\lambda_{i+1} \geq \lambda_i$. There exists a constant C depending only on N such that for all $i > 0$,*

$$||\phi_i||_\infty \leq C \cdot \lambda_i^{\frac{n}{4}}.$$

CHAPTER II

The Laplacian on singular varieties

Let $V \subset \mathbb{CP}^d$ be an $\frac{n}{2}-$(complex) dimensional, irreducible, singular subvariety. Let N denote the non-singular locus of V, so N is an $n-$(real) dimensional, smooth, oriented, connected submanifold of $\mathbb{CP}^d$. The Fubini-Study metric on $\mathbb{CP}^d$ restricts to a Riemannian metric on N, making it a Kähler manifold. This restricted metric is also referred to as the Bergmann metric. With this metric, it is easy to see the volume of N is finite, as will be shown using a key volume estimate in Appendix C. Let

$$\Delta = (d + d^t)^2 = dd^t + d^t d = d^t d = -\mathrm{div} \circ \nabla$$

be the Laplacian on functions. Here, $d^t := - * d *$ is the formal adjoint of d, following the notation of Grieser and Lesch [GL]. In local coordinates (see page 5 of [Ch]), Δ is given by

$$\Delta u = -\frac{1}{\sqrt{g}} \sum_{j,k} \partial_j (g^{jk} \sqrt{g}) \partial_k u$$

where g_{jk} is the metric in local coordinates, g^{jk} is its inverse, and $\sqrt{g}$ is the square root of the determinant of this matrix.

This definition makes Δ a positive operator since

$$\langle \Delta u, u \rangle_2 = \langle (d + d^t)u, (d + d^t)u \rangle_2 = ||(d + d^t)u||_2 \geq 0, \qquad u \in C_0^\infty(N).$$

Further, Δ is an elliptic operator with principal symbol

$$\sigma_\Delta \in C^\infty(N, S^2(TN) \otimes \mathrm{Hom}(\mathbb{R}, \mathbb{R})), \quad \sigma_\Delta(\xi_x)(a) := -\frac{1}{2}\Delta(\phi^2 \cdot u)(x) = -||\xi_x||^2 a.$$

where $\xi_x \in T_x^* N, a \in \mathbb{R}$, and $u \in C^\infty(N)$ such that $u(x) = a$. Here, the norm on the cotangent bundle is the one induced from the Fubini-Study metric on the tangent bundle (see page 126 of [Voi] and page 183 of [BB].)

Because the manifold N is not compact, a priori there are multiple closed extensions of $\Delta : C_0^\infty(N) \to C_0^\infty(N)$ in $L^2(N)$. This is discussed extensively in [GL]. Following [LT], we define the domain of Δ to be the following:

$$\mathcal{D}(\Delta) := \{u \in C^2(N) \cap L^2(N) : d(u) \in L^2(N, T^*N) \text{ and } \Delta(u) \in L^2(N)\} \subset L^2(N; \mathbb{R}).$$

This can be stated in terms of domains of d and d^t by defining the domains

$$\mathcal{D}(d) = \{u \in C^1(N) \cap L^2(N) : d(u) \in L^2(N, T^*N)\},$$
$$\mathcal{D}(d^t) = \{u \in C^1(N, T^*N) \cap L^2(N, T^*N) : d^t(u) \in L^2(N)\},$$

so

$$\mathcal{D}(\Delta) = \{u \in C^2(N) \cap L^2(N) : u \in \mathcal{D}(d) \text{ and } d(u) \in \mathcal{D}(d^t)\}.$$

This definition makes Δ an essentially self-adjoint operator as proven by Li and Tian in Theorem 4.1 of [LT]. In the process of Li and Tian's proof, they prove an important integration by parts result known as the L^2 Stokes Theorem (L^2ST) which is a condition which is the topic of investigation of the paper [GL]. This theorem states the following.

Theorem II.1. *(L^2ST) (Li-Tian)*

$$W_0^{1,2}(N) = W^{1,2}(N)$$

and

$$\langle du, w\rangle_2 = \langle u, d^t w\rangle_2$$

for all $u \in W^{1,2}(N), w \in W^{1,2}(N, T^*N)$.

Here $W^{1,2}(M)$ denotes the first Sobolev space. As a vector space,

$$W^{1,2}(M) := \{u \in L^2(M) \text{ such that } d_{\text{weak}}u \in L^2(M, T^*M)\}$$
$$W^{1,2}(M, T^*M) := \{u \in L^2(M, T^*M) \text{ such that } d^t_{\text{weak}}u \in L^2(M)\}.$$

In particular, note that $\mathcal{D}(d) \subset W^{1,2}(N)$ and $\mathcal{D}(d^t) \subset W^{1,2}(N, T^*N)$. Therefore,

$$\langle \Delta u, u\rangle_2 \geq 0, \qquad u \in \mathcal{D}(\Delta),$$

and if $\Delta\phi = \lambda\phi$, then $\lambda||\phi||_2^2 \geq 0$, so $\lambda \geq 0$.

This choice of domain is natural in the following sense. By Lemma 3.1 of Brüning and Lesch [BL1], the closure of Δ is the Friedrichs extension of Δ with domain $C_0^\infty(N)$. In fact, the closure of Δ is the unique closed extension of Δ in $L^2(N)$. In order to show this, we will need some facts about Δ proven by Li and Tian in [LT].

In addition to being essentially self adjoint, this choice of Δ behaves in many ways like the Laplacian on a compact manifold. This is explicitly stated in the following theorem due to Li and Tian [LT] Thereoms 4.1 and 5.3 and Lemma 5.2 in the case $\frac{n}{2} > 1$ and Theorems 1.1 and 1.2 of Brüning and Lesch [BL2] together with Theorem 4.1 of [LT] when $\frac{n}{2} = 1$. Li and Tian's arguments use a Sobolev inequality that requires $\frac{n}{2} > 1$, but there is a similar Sobolev inequality for the case $\frac{n}{2} = 1$ (see Appendix B.)

Theorem II.2. *(Li-Tian, Brüning-Lesch) Let N be as above. Then*

1. $\Delta : \mathcal{D}(\Delta) \to L^2(N)$ *has a unique self-adjoint extension in* $L^2(N)$.

2. $(-\Delta - 1)^{-1} : L^2(N) \to W^{1,2}(N)$ *is a bounded operator.*

3. *The inclusion* $W^{1,2}(N) \hookrightarrow L^2(N)$ *is compact.*

4. *Spec*(Δ) = *Spec*$_p(\Delta) := \{L^2$*-eigenvalues* $\}$ *is discrete:*

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty$$

and the eigenfunctions form an orthonormal basis of $L^2(N)$.

Note that items 2 and 3 imply item 4 by the usual theory of symmetric compact operators, and the existence of an orthonormal basis of eigenfunctions implies item 1. By elliptic regularity, the eigenfunctions are smooth.

This theorem shows the statement of Weyl's law makes sense. Note that Brüning and Lesch prove Weyl's law in the case $\frac{n}{2} = 1$ in [BL2], so this case will not be needed for the result of this paper.

The heat equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) u(x,t) = 0, \quad \lim_{t \to 0} u(x,t) = u_0(x)$$

is closely related to the Laplacian. Note the signs because Δ has been defined to be a positive operator. Formally, the heat operator

$$e^{-\Delta t} : u_0(x) \mapsto u(x,t)$$

solves the heat equation with initial data $u_0(x) \in L^2(N)$. By constructing a kernel function $H(x,y,t)$ called the *heat kernel* such that

$$u(x,t) := \int_N H(x,y,t) u_0(y) dy,$$

solves the heat equation with initial data $u_0(x)$, Li and Tian show such a heat operator exists for N in Lemma 3.1 of [LT]. This heat kernel has many special properties described in Theorem II.3 below.

The heat kernel on N is constructed in the following way from the Dirichlet heat kernels of compact, smooth domains contained in N. Let T_j be the tubular neighborhood of radius 2^{-j} of $\mathrm{Sing}(V)$ in N. Then $N \setminus T_j$ is a compact, smooth manifold with boundary. Note that it is compact since it is a closed subset of V which is closed in the compact space $\mathbb{CP}^d$. Let $H_j(x, y, t)$ denote the Dirichlet heat kernel on $N \setminus T_j$. Then by the maximum principle,

$$0 < H_j(x, y, t) \leq H_{j+1}(x, y, t)$$

on their common domain. The main result (Theorem 2.1) of Li and Tian's paper [LT] states that the (Dirichlet) heat kernel of any algebraic submanifold of $\mathbb{CP}^d$ of real dimension n is bounded above by the transplanted heat kernel of $\mathbb{CP}^{n/2}$. This heat kernel of complex projective space will be denoted by $\tilde{H}(x, y, t)$ or $\tilde{H}(d(x, y), t))$ since it only depends on the distance between the points x and y. Li and Tian's theorem then states that for all j we have

$$H_j(x, y, t) \leq \tilde{H}(d(x, y), t))$$

So, we can define

$$H(x, y, t) := \sup_j H_j(x, y, t).$$

This gives the heat kernel on N. The full details of the construction can be found in pages 865 to 866 of Li and Tian [LT]. The following is a summery of the properties of $H(x, y, t)$ found in Lemma 3.1, Lemma 5.2, and Theorem 5.3 of [LT].

Theorem II.3. *(Li-Tian) Let $\frac{n}{2} > 1$, then*

1. $H(x,y,t) \in C^\infty(N \times N \times (0,\infty))$
2. $H(x,y_0,t_0) \in W^{1,2}(N)$ *for all* $y_0 \in N$ *and* $t_0 > 0$
3. $H(x,y,t) = H(y,x,t)$
4. $H(x,y,t) > 0$
5. $\left(\frac{\partial}{\partial t} + \Delta_y\right) H(x,y,t) = 0$
6. $\lim_{t\to 0} \int_M H(x,y,t)u_0(y)dy = u_0(x)$ *pointwise for* $u_0 \in C^0(N) \cap L^2(N)$
7. $e^{-\Delta t}u_0 := \int_N H(x,y,t)u_0(y)dy$ *is smooth and solves the heat equation with initial value* $u_0(x) \in L^2(N)$
8. $\int_N H(x,y,t)dy = 1$ *for all* $x \in N$ *and* $t > 0$
9. $H(x,y,t)$ *is the unique function on* N *satisfying 1-6*

We can then define the heat operator

$$e^{-\Delta t}(u_0)(x) := \int_N H(x,y,t)u_0(y)dy.$$

As previously mentioned, the main result of [LT] is that the heat kernel of N is bounded by the uniform heat kernel of complex projective space given in [LT] Theorem 2.1 with the remark on page 866.

Theorem II.4. *(Li-Tian) Let $\tilde{H}(x,y,t) = \tilde{H}(d(x,y),t))$ be the heat kernel of $\mathbb{CP}^{\frac{n}{2}}$. Then*

$$H(x,y,t) \leq \tilde{H}(d(x,y),t).$$

It should be noted that the upper bound is given by the heat kernel of the complex projective space of the same dimension as N with the Fubini-Study metric. Since $\tilde{H}(d(x,y),t)$ is a smooth function on a compact manifold, it is uniformly bounded in x and y for any given t.

The relationship of the eigenvalues of Δ and the eigenvalues of $e^{-\Delta t}$ with $t > 0$ fixed will play a crucial role in the proof of Weyl's law.

Theorem II.5. *Let $\frac{n}{2} > 1$, then*

1. $\int_N H(x,z,t)H(z,y,s)dz = H(x,y,t+s)$ *for all* $s,t > 0$ *(semigroup property.)*
2. $e^{-\Delta t} : L^2(N) \to L^2(N)$ *is symetric.*
3. $e^{-\Delta t} : L^2(N) \to L^2(N)$ *is compact for* $t > 0$.
4. $\Delta\phi = \lambda\phi$ *if and only if* $e^{-\Delta t}\phi = e^{-\lambda t}\phi$.
5. *The eigenfunctions form an orthonormal basis of* $L^2(N)$.
6. *The heat operator* $e^{-\Delta t}$ *is trace class for each* $t > 0$.

Further, if

$$\{\phi_i(x)\}_{i=0}^{\infty}, \quad \Delta(\phi_i(x)) = \lambda_i \phi_i(x)$$

is such an orthonormal basis of $L^2(N)$, *then*

$$\int_N H(x,x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} = Tr(e^{-\Delta t}).$$

Remark. It is a result of Mercer in [Mer] that

$$H(x,y,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

with convergence uniform on compact sets. This will not be needed here, but I will later prove this limit with convergence uniform on all of N.

Proof. The proof of the semigroup property follows from the semigroup property of the Dirichlet heat kernels and dominated convergence. Since

$0 \le H_j(x,z,t)H_j(z,y,s) \le \tilde{H}(x,z,t)\tilde{H}(z,y,s)$ which is bounded for fixed z,y,t,s, and hence integrable on the finite volume manifild N, by dominated convergence

(where H_j is extended to all of N by 0)

$$\begin{aligned}\int_N H(x,z,t)H(z,y,s)dz &= \lim_{j\to\infty}\int_N H_j(x,z,t)H_j(z,y,s)dz \\ &= \lim_{j\to\infty}\int_{N\setminus T_j} H_j(x,z,t)H_j(z,y,s)dz \\ &= \lim_{j\to\infty} H_j(x,y,t+s) \\ &= H(x,y,t+s).\end{aligned}$$

To see that the heat operator is symmetric, compute using Fubini's theorem noting that $V(N) < \infty$ and $H(x,y,t) \in L^\infty(N\times N)$ by Theorem II.4, so $H(x,y,t)u(x)v(y) \in L^1(N \times N)$ for all t, where $u(x), v(y) \in L^2(N) \subset L^1(N)$. We have

$$\begin{aligned}\langle e^{-\Delta t}u, v\rangle_2 &= \int_N\int_N H(x,y,t)u(y)dyv(x)dx \\ &= \int_N\int_N H(x,y,t)u(y)v(x)dydx \\ &= \int_N\int_N H(x,y,t)u(y)v(x)dxdy \\ &= \int_N u(y)\int_N H(x,y,t)v(x)dxdy \\ &= \langle u, e^{-\Delta t}v\rangle_2.\end{aligned}$$

The proof of compactness is similar to the proof of Lemma 5.2 of [LT]. By Lemma 5.1 of [LT] we have the following bound on the heat operator:

$$||e^{-\lambda t}u||_2 \leq ||u||_2, \qquad t > 0, u \in L^2(N).$$

Using the integration by parts equality given by (3.2) on page 867 of [LT], we have the following equality from page 872 of [LT]:

$$\int_N |\nabla e^{-\Delta t} u|^2 dy = \int_N \int_N \int_N H(x,y,t) \Delta H(z,y,t) dy f(x) f(z) dx dz$$
$$\leq ||e^{-\lambda t} u||_2 \cdot || \int \Delta H(z,y,t) u(z) dz ||_2$$
$$= ||e^{-\lambda t} u||_2 \cdot || \int -\frac{\partial}{\partial t} H(z,y,t) u(z) dz ||_2$$
$$\leq ||u||_2 \cdot \left|\left| ||\frac{\partial}{\partial t} H(z,y,t)||_2 ||u||_2 dz \right|\right|_2 .$$

By Lemma 3.3 of [LT],

$$\int_N |\frac{\partial}{\partial t} H(z,y,t)|^2 dy \leq C t^{-2} \int_N H(z,y,t/2) dy,$$

so

$$\int_N |\nabla e^{-\Delta t} u|^2 dy \leq ||u||_2 \cdot \left|\left| \left(C t^{-2} \int_N H(z,y,t/2) dy \right)^{1/2} ||u||_2 dz \right|\right|_2$$
$$\leq C t^{-2} V(N) ||\tilde{H}(z,y,t/2)||_\infty \left(||u||_2\right)^2,$$

therefore

$$||\nabla e^{-\Delta t} u||_2 dy \leq C(t) ||u||_2,$$

and $e^{-\Delta t} : L^2(N) \to W^{1,2}(N)$ is bounded for all $t > 0$. Since the inclusion $W^{1,2}(N) \hookrightarrow L^2(N)$ is compact, $e^{-\Delta t} : L^2(N) \to L^2(N)$ is compact for $t > 0$, the spectrum of $e^{-\Delta t}$ is discrete, and the eigenfunctions of $e^{-\Delta t}$ form an orthonormal basis of $L^2(N)$ for each $t > 0$.

Now, the semigroup property can be used to relate the spectrum of the heat operator to the spectrum of the Laplacian. First, I claim if $e^{-\Delta 1}\phi = \kappa\phi$, then

$$e^{-\Delta t}\phi = \kappa^t \phi, \qquad t > 0.$$

To prove this, first consider the case when $t > 0$ is fixed, $e^{-\Delta \cdot t}\phi = \kappa(t)\phi$ and $k \in \mathbb{N}$. Then by the semigroup property and Fubini's theorem,

$$\begin{aligned} e^{-\Delta kt}\phi \int_N H(x,y,kt)\phi(y)dy &= \int_N \int_N H(x,z,(k-1)t)H(z,y,t)dz\phi(y)dy \\ &= \int_N \kappa(t)\phi(z)H(x,z,(k-1)t)dz \\ &= \kappa(t)\int_N \phi(z)H(x,z,(k-1)tdz, \end{aligned}$$

so by induction on k, ϕ is an eignunfunction of $e^{-\Delta kt}$ with eigenvalue $\kappa(t)^k$. Since both $e^{-\Delta kt}$ and $e^{-\Delta t}$ have an orthonormal eigenbasis of $L^2(N)$, their eigenfunctions coincide. Since $t > 0$ was arbitrary, the eigenvalues of $e^{-\Delta t}$ are the squares of the eigenvalues of $e^{-\Delta \frac{t}{2}}$, so in particular all eigenvalues are non-negative.

Now, if $\frac{p}{q} \in \mathbb{Q}$ is a positive rational number with $p, q \in \mathbb{N}$, then the eigenfunctions of $e^{-\Delta \frac{p}{q}}$ are the same as the eigenfunctions of $e^{-\Delta q \cdot \frac{p}{q}}$, which are the same as those of $e^{-\Delta 1}$. Further, if κ is an eigenvalue of $e^{-\Delta 1}$ with $e^{-\Delta 1}\phi = \kappa\phi$, then

$$e^{-\Delta p}\phi = \kappa^p \phi,$$

and since there is some eigenvalue $\kappa(\frac{p}{q})$ such that

$$e^{-\Delta \frac{p}{q}}\phi = \kappa(\tfrac{p}{q})\phi,$$

and

$$\kappa(\tfrac{p}{q})^q = \kappa^p,$$

we have

$$\kappa(\tfrac{p}{q}) = \kappa^{\frac{p}{q}}.$$

To show continuity in t, dominated convergence can be used since for $t \in [t_0-\epsilon, t_0+\epsilon]$,

$$|H(x,y,t)\phi(y)| \leq ||\tilde{H}(x,y,s)||_L^\infty(N \times N \times [t_0 - \epsilon, t_0 + \epsilon])|\phi(y)| \in L^2(N) \subset L^1(N),$$

so if r_j is a sequence of rational numbers approaching $t_0 > 0$, then by the continuity of $H(x,y,t)$,

$$\begin{aligned} e^{-\Delta t_0}\phi &= \int_N \lim_{j\to\infty} H(x,y,r_j)\phi(y)dy \\ &= \lim_{j\to\infty} \int_N H(x,y,r_j)\phi(y)dy \\ &= \lim_{j\to\infty} \kappa^{r_j}\phi \\ &= \kappa^{t_0}\phi. \end{aligned}$$

Note that by the same argument, instead taking the limit in x instead of t, $\phi(x)$ is continuous since $H(x,y,t)$ is continuous in x.

Now that the claim has been proven, if $\kappa = 0$, then

$$\begin{aligned} \phi(x) &= \lim_{t\to 0} H(x,y,t)\phi(y)dy \\ &= \lim_{t\to 0} e^{-\Delta t}\phi \\ &= \lim_{t\to 0} 0^t\phi = 0, \end{aligned}$$

so 0 is not an eigenvalue of $e^{-\Delta 1}$ or $e^{-\Delta t}$ for any $t > 0$, and the spectrum is strictly positive. In particular (by compactness of the operator), each eigenspace is finite dimensional. Applying the heat equation shows

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t} + \Delta\right) e^{-\Delta t}\phi \\ &= \left(\frac{\partial}{\partial t} + \Delta\right) \kappa^t\phi \\ &= \ln(\kappa)\kappa^t\phi + \kappa^t\Delta\phi, \end{aligned}$$

therefore

$$\Delta\phi = -\ln(\kappa)\phi.$$

Again, since $e^{-\Delta 1}$ and Δ both have an orthonormal eigenbasis of $L^2(N)$, their eigenfunctions coincide and (4) is proven.

To summarize, $e^{-t\Delta}$ is a positive, symmetric operator for each $t > 0$, and $H(x, y, t) > 0$. For each integer $L \geq 0$, define a difference operator $R_{t,L} : L^2(N) \to L^2(N)$.

$$R_{t,L}(u)(x) := \int_N \left(H(x, y, t) - \sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x) \phi_i(y) \right) u(y) dy.$$

Note that because $\phi_i \in L^2(N)$, $R_{t,L}$ is well defined. $R_{t,L}$ is symmetric by Fubini's theorem since both $H(x, y, t)$ and $\sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$ are symmetric in x and y. Further, it is bounded since $H(x, y, t) \in L^\infty(N)$ and the integral opertor with kernel $\sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$ is the direct sum of a linear map over the finite dimensional vector space spanned by the first $L + 1$ eigenvectors and the zero map on the compliment and hence is bounded. Because ϕ_i are the eigenfuctions of $e^{-t\Delta}$ with eigenvalues $e^{-\lambda_i t}$,

$$R_{t,L}(\phi_i)(x) = \begin{cases} e^{-\lambda_i t} \phi(x) - e^{-\lambda_i t} \phi(x) = 0 & \text{if } i \leq L \\ e^{-\lambda_i t} \phi(x) - 0 = e^{-\lambda_i t} \phi(x) & \text{if } i > L \end{cases}$$

Hence, $\{\phi_i\}_{i=0}^{\infty}$ is an orthonormal basis of $L^2(N)$ consisting of eigenfunctions of $R_{t,L}$, and therefore $R_{t,L}$ is a positive operator. Since $H(x, y, t) - \sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$ is continuous, it must be non-negative on the diagonal, otherwise, by continuity it would negative on a small neighborhood in $N \times N$, and then $\langle R_{t,L}(u), u \rangle_{L^2(N)}$ would be negative for u supported on the intersection of the two projections of that small neighborhood in $N \times N$ onto N. Note that the intersection is a non-empty since it contains a point on the diagonal. Explicitly,

$$\begin{aligned}
\langle R_{t,L}(u), u\rangle_{L^2(N)} &= \int_N \int_N \left(H(x,y,t) - \sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x)\phi_i(y) \right) u(y)dyu(x)dx \\
&= \int_{\mathrm{supp}(u)} \int_{\mathrm{supp}(u)} \left(H(x,y,t) - \sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x)\phi_i(y) \right) u(y)u(x)dydx \\
&\leq c \int_{\mathrm{supp}(u)} \int_{\mathrm{supp}(u)} u(y)u(x)dydx, \qquad \text{for some } c < 0 \\
&= c \int_{\mathrm{supp}(u)} u(y)dy \int_{\mathrm{supp}(u)} u(x)dx < 0.
\end{aligned}$$

Hence,

$$0 \leq \sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x)^2 \leq H(x,x,t) \leq \tilde{H}(0,t), \qquad \text{for all } L.$$

Notice that this implies that $\phi_i \in L^\infty(N)$ with bound

$$||\phi_i||_\infty \leq \sqrt{\tilde{H}(0,t)} e^{\lambda_i t/2}, \quad t > 0.$$

Since $e^{-\lambda_i t}\phi_i(x)^2 \geq 0$, there exists a pointwise limit

$$\sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x)^2 \to \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x)^2 \leq H(x,x,t) \qquad \text{pointwise.}$$

To see the existence of a limit off of the diagonal:

$$\sum_{i=0}^{L} e^{-\lambda_i t} \phi_i(x)\phi_i(y) \to \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y) \qquad \text{pointwise,}$$

use Cauchy-Schwarz as follows to show the sequence is Cauchy:

$$\begin{aligned}
\left| \sum_{i=L+1}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y) \right| &\leq \sum_{i=L+1}^{\infty} \left| e^{-\lambda_i t} \phi_i(x)\phi_i(y) \right| \\
&\leq \left(\sum_{i=L+1}^{\infty} \left(e^{-\lambda_i t/2}\right)^2 \phi_i(x)^2 \right)^{\frac{1}{2}} \left(\sum_{i=L+1}^{\infty} \left(e^{-\lambda_i t/2}\right)^2 \phi_i(y)^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=L+1}^{\infty} e^{-\lambda_i t} \phi_i(x)^2 \right)^{\frac{1}{2}} \left(\sum_{i=L+1}^{\infty} e^{-\lambda_i t} \phi_i(y)^2 \right)^{\frac{1}{2}} \to 0.
\end{aligned}$$

On the other hand, since $H(x,y,t) \in L^2(N)$ as a function of y for each fixed x and t, and the coefficients of its expansion in the orthonormal basis $\{\phi_i\}_{i=0}^\infty$ are given by

$$\int_N H(x,y,t)\phi_i(y)dy = e^{-\lambda_i t}\phi_i(x),$$

$$\sum_{i=0}^{L} e^{-\lambda_i t}\phi_i(x)\phi_i(y) \to H(x,y,t) \text{ in } L^2(N) \text{ as } L \to \infty$$

as a function of y with x and t fixed. Combining this with the pointwise limit,

$$H(x,y,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y) \quad a.e.$$

as a function of y for fixed x and t.

Now, by the semigroup property, for each fixed x and t,

$$\begin{aligned}
H(x,x,t) &= \int_N H(x,y,t/2)^2 dy \\
&= \int_N \left(\sum_{i=0}^{\infty} e^{-\lambda_i t/2}\phi_i(x)\phi_i(y) \right)^2 dy \\
&= \int_N \lim_{L\to\infty} \left(\sum_{i=0}^{L} e^{-\lambda_i t/2}\phi_i(x)\phi_i(y) \right)^2 dy \\
&= \lim_{L\to\infty} \int_N \left(\sum_{i=0}^{L} e^{-\lambda_i t/2}\phi_i(x)\phi_i(y) \right)^2 dy \\
&= \lim_{L\to\infty} \sum_{i=0}^{L} e^{-\lambda_i t} \left(\phi_i(x)\right)^2 \\
&= \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)^2.
\end{aligned}$$

Note that I used dominated convergence since $\left(\sum_{i=0}^{L} e^{-\lambda_i t/2}\phi_i(x)\phi_i(y)\right)^2$ is dominated by $H(x,x,t/2)H(y,y,t/2) \in L^1(N)$ as a function of y using the Cauchy-Schwarz argument above. I also used that the $\phi_i(x)$ form an orthonormal basis of $L^2(N)$.

Finally,

$$\begin{aligned}\int_N H(x,x,t)dx &= \int_N \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)^2 \\ &= \lim_{L\to\infty} \int_N \sum_{i=0}^{L} e^{-\lambda_i t}\phi_i(x)^2 \\ &= \sum_{i=0}^{\infty} e^{-\lambda_i t}.\end{aligned}$$

□

Remark. The result of Mercer can be shown in this special case by using $H(x,x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)^2$ to show that the right side is in fact continuous. Dini's theorem can then be applied to show the convergence of this limit is uniform on compact sets. The Cauchy-Schwarz estimate can then be used to show uniform convergence of

$$H(x,y,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

on compact sets.

CHAPTER III

The heat kernel approach to Weyl's Law

There are a number of different methods of proving Weyl's law for compact manifolds. The proof I will provide for singular algebraic varieties follows the heat kernel method given on page 155 of [Ch] for compact manifolds. The idea of the heat kernel approach to Weyl's law is to relate local information about $H(x, y, t)$ to the sequence of eigenvalues by using the trace formula

$$\int_N H(x, x, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}$$

from Theorem II.5. In this way, local information about the heat kernel is equated to global information, ie. the spectrum. This is a commonly used idea which can be used to prove one of the most powerful local to global theorems, the Atiyah-Singer index theorem. The asymptotics in the variable t of the series $\sum_{i=0}^{\infty} e^{-\lambda_i t}$ can then be related to the asymptotics of the sequence of eigenvalues using the following Tauberian theorem.

Theorem III.1. ***(Tauberian theorem)*** *Let* $\{\lambda_i\}$ *be a sequence monotonically increasing to infinity and* $\rho > 0$. *Define* $N(\lambda) = \sum_{\lambda_i \leq \lambda} 1$. *If there exists a constant* C *such that*

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \sim Ct^{-\rho} \text{ as } t \to 0^+,$$

then

$$N(\lambda) \sim \frac{C\lambda^{\rho}}{\Gamma(\rho+1)} \text{ as } \lambda \to \infty.$$

This theorem is attributed to Hardy and Littlewood [HL] and also to Karamata [Kar] who provided a simpler proof of Hardy and Littewood's tauberian theorem. This Tauberian theorem and its generalizations are very deep and powerful theorems in analysis with a long history going back to Abel. This history will be discussed and a proof will be provided in Appendix A.

Now, Weyl's law is reduced to showing the following asymptotic property of the heat kernel which comes from its local description.

Theorem III.2. *The heat kernel of the nonsingular locus of a projective algebraic varity of complex dimension* $\frac{n}{2}$ *has the asymptotics*

$$\int_N H(x,x,t) \sim \frac{V(N)}{(4\pi t)^{\frac{n}{2}}} \text{ as } t \to 0.$$

The proof of this theorem will be the focus of Section V.

Applying Theorem III.1 to the result of Theorem III.2, one obtains Weyl's law for singular algebraic varieties with the Fubini-Study metric.

Theorem III.3. ***(Weyl's Law)***

Let $V \subset \mathbb{CP}^d$ *be an irreducible, projective variety of complex dimension* $\frac{n}{2}$, *and define* $N := V \setminus Sing(V)$ *with the Fubini-Study metric from* $\mathbb{CP}^d$. *Define the counting function* $N(\lambda) = \sum_{\lambda_i \leq \lambda} 1$. *Then*

$$N(\lambda) \sim \frac{\boldsymbol{\omega}_n V(N) \lambda^{\frac{n}{2}}}{(2\pi)^n} \text{ } as\ \lambda \to \infty,$$

hence

$$(\lambda_i)^{\frac{n}{2}} \sim \frac{(2\pi)^n i}{\boldsymbol{\omega}_n V(N)} \text{ } as\ i \to \infty.$$

Proof of Theorem III.3:

First, note that if V is non-singular, then $N = V$ is compact, and the theorem is classical. The case $\frac{n}{2}$ is due to Theorem 1.2 of Brüning and Lesch [BL2]. So, we can reduce to the case when V is singular, and $\frac{n}{2} > 1$.

The desired asymptotics of the heat trace is obtained by applying the local to global heat trace result of Theorem II.5 to the asymptotic results of Theorem III.2:

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} = \int_N H(x,x,t) \sim \frac{V(N)}{(4\pi)^{\frac{n}{2}}} \cdot t^{-\frac{n}{2}} \text{ as } t \to 0.$$

Now, an application of the Tauberian theorem (Theorem III.1) shows

$$N(\lambda) \sim \frac{V(N)}{(4\pi)^{\frac{n}{2}}} \cdot \frac{1}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{V(N)}{(4\pi)^{\frac{n}{2}}} \cdot \frac{\boldsymbol{\omega}_n \lambda^{\frac{n}{2}}}{\pi^{\frac{n}{2}}} = \frac{\boldsymbol{\omega}_n V(N) \lambda^{\frac{n}{2}}}{(2\pi)^n} \text{ as } \lambda \to \infty.$$

The expression for $\Gamma\left(\frac{n}{2}+1\right)$ when n is a positive integer can be computed using spherical coordinates and induction (for example, see [Ch].)

The second expression can be seen by noticing that $N(\lambda_i) = i+1$, and $\lambda_i \to \infty$ as $i \to \infty$, hence

$$(i+1) \sim \frac{\boldsymbol{\omega}_n V(N) \lambda_i^{\frac{n}{2}}}{(2\pi)^n} \text{ as } i \to \infty.$$

Since $\lim_{i\to\infty} \frac{i+1}{i} = 1$,

$$i \sim \frac{\boldsymbol{\omega}_n V(N) \lambda_i^{\frac{n}{2}}}{(2\pi)^n} \text{ as } i \to \infty.$$

In other words,

$$\lim_{i\to\infty} \frac{i(2\pi)^n}{\boldsymbol{\omega}_n V(N) \lambda_i^{\frac{n}{2}}} = 1,$$

or

$$(\lambda_i)^{\frac{n}{2}} \sim \frac{(2\pi)^n i}{\boldsymbol{\omega}_n V(N)} \text{ as } i \to \infty.$$

$\square$

Proposition III.4. *The condition that V be irreducible is not necessary. As long as all of the irreducible components of V are the same dimension, then Weyl's law still holds.*

Observe that if the irreducible components of V are not the same dimension, then Weyl's law doesn't make sense as written.

Proof. Let $V = \bigcup_j V_j$ be the decomposition of V into irreducible components, with $\dim V_j = \frac{n}{2}$ for every j. Note that there are only finitely many V_j. Let $N_j = V_j \setminus \mathrm{Sing}(V_j)$. If $x \in V_j \cap V_k$, then $x \in \mathrm{Sing}(V)$. On the other hand, if x is in a unique V_{j_0}, then so is a neighborhood (since V_j are closed), therefore $T_x V_{j_0} = T_x N$ and $x \in \mathrm{Sing}(V)$ if and only if $x \in \mathrm{Sing}(N)$. Ie. if

$$W := \{x \in V : x \in V_j \cap V_k \text{ for some } j \neq k\},$$

then

$$N = \coprod_j (N_j \setminus W),$$

where $V_j \setminus (N_j \cup W) \subset V_j$ is a proper, closed subvariety for every j.

I claim that each $(N_j \setminus W)$ is connected in the Euclidean topology. First, $(N_j \setminus W)$ is an irreducible quasiprojective variety since it is an open subspace of V_j in the Zariski topology. In particular, $(N_j \setminus W)$ is connected in the Zariski topology, and hence it is connected in the Euclidean topology (see Section 2 of Chapter 7 of [Sh].)

Therefore, the connected components of N in the Euclidean topology are $\{N_j \setminus W\}$. By Theorem III.3, Weyl's law holds on each N_j. $N_j \cap W \subset N_j$ has measure 0 because

$V_j \cap W$ is a proper subvariety of V_j, and thus has dimension strictly less than $\frac{n}{2}$. Therefore, the volume of $N_j \setminus W$ is equal to the volume of N_j, and the L^2-eigenfunctions of $N_j \setminus W$ are in a one-to-one correspondence with the L^2-eigenfunctions of N_j given by restriction of the functions. In particular, the eigenvalues of $N_j \setminus W$ and N_j are the same. Thus, Weyl's law holds on each $N_j \setminus W$, the connected components of N.

The volumes of the connected components sum to give the volume of N, and the the eigenspaces of Δ on the connected components direct sum to give the eigenspaces of Δ on N. To be precise, let

$$B_j(\lambda) := \{f_1^j, \dots, f_{l_j}^j\}$$

be an basis of the λ-eigenspace of Δ on $N_j \setminus W$. Define

$$f_{j,k}(x) := \begin{cases} f_k^j(x) & \text{if } x \in N_j \setminus W \\ 0 & \text{if } x \in N \setminus (N_j \setminus W) \end{cases}$$

the extension by 0 of f_k^j to N. Denote

$$B_j'(\lambda) := \{f_{j,1}, \dots, f_{j,l_j}\}.$$

Then,

$$\bigcup_j B_j'(\lambda)$$

forms an basis of the λ-eigenspace of Δ on N. To see this, note that each $\Delta f_{j,k} = \lambda f_{j,k}$, and $f_{j,k} \in L^2(N)$ since $||f_{j,k}|| = ||f_k^j||$, so each $f_{j,k}$ is in the eigenspace. They clearly form a linearly independent set. In fact, without loss of generality, taking f_k^j to be orthonormal, then $f_{j,k}$ are orthonormal. To see that they span, if $g \in L^2(N) \setminus \{0\}$ with $\Delta g = \lambda g$, then g restricts to an eigenfunction on each connected component, so $g = \sum_k a_k^j \cdot f_k^j$ on $N_j \setminus W$ where $a_k^j \in \mathbb{R}$, so $g = \sum_j \sum_k a_k^j \cdot f_k^j$ on N. Hence, the counting functions of the connected components sum to give the counting function of N. Since both the volume and counting function are additive, Weyl's law holds on N. $\square$

CHAPTER IV

Minakshisundaram-Pleijel asymptotic expansion

The local asymptotics of the heat kernel on a compact manifold were first investigated by Minakshisundaram and Pleijel in their 1949 paper [Min]. The idea is that since a Riemannian metric locally written in normal coordinates looks like a smooth perturbation of the Euclidean metric, the heat kernel will locally look like a smooth perturbation of the Euclidean heat kernel. The Euclidean heat kernel is given by

$$H_{\mathcal{E}}(x,y,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

and its properties are described in Appendix D.

Minakshisundaram and Pleijel used functions recursively defined in polar normal coordinates to construct a family of parametrix of the heat equation on a compact manifold. A parametrix can be thought of as a smooth perturbation of the heat kernel. A precise definition of a parametrix can be found on page 151 of [Ch], but will not be needed here. These parametrix can in fact be used to construct the heat kernel on a compact manifold without a priori knowledge of the spectrum of the Laplacian. This construction using the parametrix was origianlly done in [Min].

The heat kernel on a compact manifold can also be constructed in a completely different manner using the eigenfunctions and eigenvalues of the Laplacian obtained from elliptic theory. This construction is given in Theorem 10.1 on page 98 of [Li].

Now, let us look in detail at the ideas described above. The presentation given in this paper is drawn from the orinal presentation given in [Min] as well as those given Chapter IV of [Ch] and Chapter 11 of [Li].

Theorem IV.1. *Let M be a Riemannian manifold of dimension n with or without boundary. Let $h(x, y, t) : M \times M \times (0, \infty) \to \mathbb{R}$ be a smooth function satisfying*

- $h(x, y, t) = h(y, x, t)$
- $h(x, y, t) > 0$
- $\left(\frac{\partial}{\partial t} + \Delta_y\right) h(x, y, t) = 0$
- $\lim_{t \to 0} \int_M h(x, y, t) f_0(y) dy = f_0(x)$ *pointwise for* $f_0 \in C^0(M) \cap L^2(M)$
- $\int_M h(x, y, t) dy \leq 1$ *for all* x, t.

Let $A \subset M \setminus \partial M$ be compact. Then there exists $\rho > 0$ such that for each $x \in A$ the injectivity radius of x in M satsifies $Inj(x) > 3\rho$. For each $x \in A$ there exist smooth functions $u_j(x, y)$ defined for $y \in B_x(\rho)$ with $u_0(x, x) = 1$ such that the functions defined on $A \times M \times (0, \infty)$ by

$$H_k(x, y, t) := \eta(d(x, y)) H_{\mathcal{E}}(x, y, t) \sum_{j=0}^{k} t^j u_j(x, y)$$

for $k \geq 0$ satisfy

$$|h(x, y, t) - H_k(x, y, t)| \leq C t^{k+1-n/2}$$

for all $x, y \in A$ and $t \in (0, 1]$ for some constant $C > 0$, where $d(x, y)$ is the Riemannian distance function and $\eta(d)$ is a smooth bump function satisfying $\eta(d) = 1$ for $d < \rho$, $\eta(d) = 0$ for $d \geq 2\rho$, and $|\nabla \eta| \leq \frac{5}{\rho}$. There is a compact set $A' \subset M$ such that $H_k(x, y, t) = 0$ for all $y \in M \setminus A' \subset M, x \in A, t > 0$.

Proof. First, note the existence of such a $\rho > 0$ is justified by Proposition 2.1.10 on page 131 of [Kl]. The functions $u_j(x, y)$ will be constructed recursively. To construct $u_0(x, y)$, fix $x \in A$ and consider the polar normal coordinates centered at x. In particular, note that $B_{2\rho+\epsilon}(x)$ is contained in these coordinates for some $\epsilon > 0$. Define the function $\phi(x, y)$ to be the volume element at y in the (non-polar) Riemann normal coordinates centered at x. In particular, since the metric is given by the identity matrix at x in these coordinates,

$$\phi(x, x) = 1, \qquad \phi(x, y) > 0,$$

and $\phi(x, y)$ defines a smooth function in both variables. Note the smoothness in x follows from the smothness of the change of coordinates of the exponential map centered at one point to the exponential map centered at another point from the smooth dependence on parameters of the solutions of the geodesic flow.

Since the change of variables from normal coordinates to polar normal coordinates is induced from the composition with the change of coordinates from Euclidean to polar (spherical) coordinates on $\mathbb{R}^n$, the volume form changes by a factor of $r^{-(n-1)}$. Hence, in polar normal coordinates,

$$\phi(x, y) = \sqrt{g(y)} \cdot r(y)^{-(n-1)}$$

where $r(y) = d(x, y)$ is the polar radial coordinate, and

$$\sqrt{g(y)} = \sqrt{\det(g_{ij}(y))}$$

where g_{ij} is the coefficients of the Riemannian metric in the polar normal coordinates.

For convenience, define

$$u_{-1}(x, y) \equiv 0,$$

and define

$$
\begin{aligned}
u_0(x,y) &:= \phi^{-1/2}(x,y), \\
u_j(x,y) &:= -\phi^{-1/2}(x,y) r^{-j} \int_0^r s^{j-1}\phi^{1/2}(x, \exp_x(s\nu))\Delta_y u_{j-1}(x, \exp_x(s\nu)) ds, \\
&\quad = \phi^{-1/2}(x,y) \int_0^1 \tau^{j-1}\phi^{1/2}(x, \exp_x(r\tau\nu))\Delta_y u_{j-1}(x, \exp_x(r\tau\nu)) d\tau \\
&\quad \text{for } j > 0 \text{ where } \exp_x(r\nu) = y, \qquad s = r\tau.
\end{aligned}
$$

From the previous discussion, u_0 is a well defined, smooth, positive function with $u_0(x,x) = 1$. $u_j(x,y)$ can be shown to be smooth for all j by induction as follows: since the integrand above is smooth by the inductive hypothesis that u_{j-1} is smooth, the integral over the compact interval can be interchanged with derivatives, so by induction, $u_j(x,y)$ is also smooth.

The existence of the compact set A' follows since the points within distance 2ρ of A is compactly contained in M because A is covered by finitely many balls of radius 2ρ, and each such ball of radius 3ρ is homeomorphic to a Euclidean ball by the exponential map. These $u_j(x,y)$ satisfy some partial differential equations that are given in the following lemma.

Lemma IV.2. *The $u_j(x,y)$ defined above satisfy*

$$
\left(\frac{\partial}{\partial t} + \Delta_y\right) H_{\mathcal{E}}(x,y,t) \sum_{j=0}^{k} t^j u_j(x,y) = H_{\mathcal{E}}(x,y,t) t^k \Delta_y u_k(x,y). \tag{4.3}
$$

In polar normal coordinates,

$$
-\frac{1}{2}\left(r\Delta r - n + 1\right) u_j + r\left\langle \nabla r, \nabla u_j \right\rangle + j u_j = -\Delta_y u_{j-1}, \tag{4.4}
$$

or equivalently

$$
\frac{r}{2\phi}\frac{\partial \phi}{\partial r} + r\frac{\partial u_j}{\partial r} + j u_j = -\Delta_y u_{j-1}. \tag{4.5}
$$

Proof. The product rule is used to compute

(4.6)

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\Delta_y\right)H_{\mathcal{E}}(x,y,t)\sum_{j=0}^{k}t^j u_j(x,y) &= H_{\mathcal{E}}(x,y,t)\left(\frac{\partial}{\partial t}+\Delta_y\right)\sum_{j=0}^{k}t^j u_j(x,y)\\ &\quad+\sum_{j=0}^{k}t^j u_j(x,y)\left(\frac{\partial}{\partial t}+\Delta_y\right)H_{\mathcal{E}}(x,y,t)\\ &\quad-2\langle\nabla_y H_{\mathcal{E}}(x,y,t),\nabla_y\sum_{j=0}^{k}t^j u_j(x,y).\end{aligned}$$

Note that $H_{\mathcal{E}}$ satisfies the Euclidean heat equation, not the Riemannian one. The chain rule gives

$$\begin{aligned}\Delta_y H_{\mathcal{E}}(x,y,t) &= (4\pi t)^{-n/2}\Delta_y\exp\left(-\frac{r^2}{4t}\right)\\ &= (4\pi t)^{-n/2}\left(-\exp\left(-\frac{-r^2}{4t}\right)\frac{|-\nabla r^2|^2}{4t}+\exp\left(-\frac{-r^2}{4t}\right)\Delta\left(\frac{-r^2}{4t}\right)\right)\\ &= H_{\mathcal{E}}(x,y,t)\left(\frac{-2r\Delta r+2|\nabla r|^2}{4t}-\frac{|2r\nabla r|}{4t}\right)\\ &= H_{\mathcal{E}}(x,y,t)\left(\frac{-r\Delta r+1}{2t}-\frac{|2r\nabla r|}{4t}\right),\end{aligned}$$

where in the last step $|\nabla r|^2 = dr/dr = 1$ was used, and

$$\begin{aligned}\nabla_y H_{\mathcal{E}}(x,y,t) &= 4\pi t)^{-n/2}\exp\left(-\frac{r^2}{4t}\right)\cdot\left(\frac{-2r\nabla r}{4t}\right)\\ &= H_{\mathcal{E}}(x,y,t)\cdot\left(\frac{-r\nabla r}{2t}\right).\end{aligned}$$

The time derivative is

$$\begin{aligned}\frac{\partial}{\partial t}H_{\mathcal{E}}(x,y,t) &= -2\pi\cdot(4\pi t)^{-n/2-1}\exp\left(-\frac{r^2}{4t}\right)+(4\pi t)^{-n/2}\exp\left(-\frac{r^2}{4t}\right)\frac{r^2}{4t^2}\\ &= H_{\mathcal{E}}(x,y,t)\left(-\frac{n}{2t}+\frac{r^2}{4t^2}\right).\end{aligned}$$

Combining these computations with (4.6) we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\Delta_y\right) H_{\mathcal{E}}(x,y,t)\sum_{j=0}^{k} t^j u_j(x,y)\\
&= H_{\mathcal{E}}\Big[\left(\frac{-r\nabla r+1}{2t}-\frac{r^2}{4t^2}\right)\sum_{j=0}^{k} t^j u_j\\
&\quad+\sum_{j=0}^{k} t^j\Delta_y u_j+\left(-\frac{n}{2t}+\frac{r^2}{4t^2}\right)\sum_{j=0}^{k} t^j u_j+\sum_{j=0}^{k} j t^{j-1}u_j\\
&\quad+\frac{r}{t}\sum_{j=0}^{k} t^j\langle\nabla r,\nabla_y u_j\rangle\Big].
\end{aligned}$$

Now, if (4.4) is shown, then (4.3) follows as

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\Delta_y\right) H_{\mathcal{E}}(x,y,t)\sum_{j=0}^{k} t^j u_j(x,y)\\
&= H_{\mathcal{E}}\Big[\sum_{j=0}^{k} t^{j-1}\left(-\frac{1}{2}\left(r\Delta r-n+1\right)u_j+r\left\langle\nabla r,\nabla u_j\right\rangle+ju_j\right)+\sum_{j=0}^{k} t^j\Delta_y u_j\Big]\\
&= H_{\mathcal{E}}\Big[\sum_{j=0}^{k} -t^{j-1}\Delta_y u_{j-1}+\sum_{j=0}^{k} t^j\Delta_y u_j\Big]\\
&= H_{\mathcal{E}}(x,y,t)t^k\Delta_y u_k(x,y).
\end{aligned}$$

Equation (4.4) can be rewritten to (4.5) by using

$$\langle\nabla r,\nabla_y u_j\rangle=\frac{\partial u_j}{\partial r}$$

and

$$\frac{-r\Delta r-n+1}{2}=\frac{r}{2\phi}\frac{\partial\phi}{\partial r}. \tag{4.7}$$

This last equation can be computed in polar normal coordinates using the local coordinate description of Δ where the first coordinate index is the radial coordinate as follows.

$$\begin{aligned}
\Delta r &= -\frac{1}{\sqrt{g}} \sum_{j,k} \partial_j (g^{jk} \sqrt{g}) \partial_k r \\
&= -\frac{1}{\sqrt{g}} \sum_{j} \partial_j (g^{j1} \sqrt{g}) \partial_1 r \\
&= -\frac{1}{\sqrt{g}} \partial_1 (g^{11} \sqrt{g}) \partial_1 r \\
&= -\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r}.
\end{aligned}$$

Here I used the fact that the inverse of a block diagonal matrix is the block diagonal matrix of the inverses of the blocks. (g_{jk} is block diagonal with a block of (1) correponding to the radial direction.) This can then be used to compute (4.7):

$$\begin{aligned}
\frac{r}{2\phi}\frac{\partial \phi}{\partial r} &= \frac{r^n}{2\sqrt{g}} \frac{\partial(\sqrt{g} r^{-n+1})}{\partial r} \\
&= \frac{r^n}{2\sqrt{g}} \cdot \frac{\partial_r(\sqrt{g}) \cdot r^{n-1} - (n-1) r^{n-2} \sqrt{g}}{r^{2n-2}} \\
&= \frac{r \partial_r(\sqrt{g})}{2\sqrt{g}} - \frac{n-1}{2} \\
&= \frac{-r\Delta r - n + 1}{2}.
\end{aligned}$$

Finally, (4.5) can be directly computed from the definitions of the u_js. For $j > 0$,

$$\begin{aligned}
r\frac{\partial u_j}{\partial r} &= r\frac{\partial}{\partial r}\left(-\phi^{-1/2} r^{-j} \int_0^r s^{j-1} \phi^{1/2}(x, \exp_x(s\nu)) \Delta_y u_{j-1}(x, \exp_x(s\nu)) ds \right) \\
&= r\Big[\frac{1}{2}\phi^{-3/2} \frac{\partial \phi}{\partial r} r^{-j} \int_0^r s^{j-1} \phi^{1/2}(x, \exp_x(s\nu)) \Delta_y u_{j-1}(x, \exp_x(s\nu)) ds \\
&\quad + j r^{-j-1} \phi^{-1/2} \int_0^r s^{j-1} \phi^{1/2}(x, \exp_x(s\nu)) \Delta_y u_{j-1}(x, \exp_x(s\nu)) ds \\
&\quad - \phi^{-1/2} r^{-j} \left(r^{j-1} \phi^{1/2}(x,y) \Delta_y u_{j-1}(x,y) \right) \Big] \\
&= -\frac{r}{2\phi}\frac{\partial \phi}{\partial r} u_j - j u_j - \Delta_y u_{j-1}
\end{aligned}$$

For $j = 0$,

$$\frac{r}{2\phi}\frac{\partial\phi}{\partial r} + r\frac{\partial u_j}{\partial r} + ju_j = \frac{r}{2\phi}\frac{\partial\phi}{\partial r} + r\frac{\partial\phi^{-1/2}}{\partial r} + 0\cdot\phi^{-1/2}$$
$$= 0 = -\Delta_y u_{-1}.$$

$\square$

Now, we return to the proof of the theorem. The fundamental theorem of calculus is utilized to compute $h(x,y,t) - H_k(x,y,t)$ as follows: for $t > 0$, $x \in A$,

$$\begin{aligned}
(4.8)\qquad & \int_0^t \frac{\partial}{\partial s}\int_M h(z,y,t-s)H_k(x,z,s)dzds \\
&= \lim_{s\to t^-}\int_M h(z,y,t-s)H_k(x,z,s) - \lim_{s\to 0^+}\int_M h(z,y,t-s)H_k(x,z,s) \\
&= \lim_{s\to t^-}\int_M h(z,y,t-s)H_k(x,z,t) - \lim_{s\to 0^+}\int_M h(z,y,t)H_k(x,z,s) \\
&= h(x,y,t) - H_k(x,y,t).
\end{aligned}$$

The second and third lines above need justification. First let us investigate the first term of the second line.

$$\begin{aligned}
&\left|\lim_{s\to t^-}\int_M h(z,y,t-s)H_k(x,z,s) - \lim_{s\to t^-}\int_M h(z,y,t-s)H_k(x,z,t)\right| \\
&\quad\le \left|\int_M h(z,y,t-s)H_k(x,z,s) - \int_M h(z,y,t-s)H_k(x,z,t)\right| \\
&\quad + \left|\lim_{s\to t^-}\int_M h(z,y,t-s)H_k(x,z,s) - \int_M h(z,y,t-s)H_k(x,z,s)\right| \\
&\quad + \left|\lim_{s\to t^-}\int_M h(z,y,t-s)H_k(x,z,t) - \int_M h(z,y,t-s)H_k(x,z,t)\right|,
\end{aligned}$$

for all $s < t$, where the last two terms go to 0 as $s \to t^-$. The remaining term satisfies the inequality

$$\begin{aligned}
&\left|\int_M h(z,y,t-s)H_k(x,z,s) - \int_M h(z,y,t-s)H_k(x,z,t)\right| \\
&\quad\le \sup_{z\in M} |H_k(x,z,t)-H_k(x,z,s)| \int_M |h(z,y,t-s)|\,dz \\
&\quad\le \sup_{z\in A'} |H_k(x,z,t)-H_k(x,z,s)| \int_M h(z,y,t-s)dz \\
&\quad\le \sup_{z\in A'} |H_k(x,z,t)-H_k(x,z,s)| \text{ for all } s<t
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Since $H_k(x,z,s)$ is continuous on the compact set $(z,s) \in A' \times [t/2,t]$, for each $z \in A'$, there is a neighborhood of the form $U_z \times (a_z, t]$ where $U_z \subset A'$ such that

$$|H_k(x,z,t) - H_k(x,z_0,s_0)| \le \frac{\epsilon}{2} \text{ for all } z_0 \in U_z, s_0 \in (a_z,t].$$

By the triangle inequality,

$$|H_k(x,z_0,t) - H_k(x,z_0,s_0)| \le \epsilon \text{ for all } z_0 \in U_z, s_0 \in (a_z,t].$$

Since the compact set A' is covered by finitely many such U_z, if $a_{\max}$ the maximum of the corresponding a_z, $a_{\max} < t$, and

$$|H_k(x,z,t) - H_k(x,z,s_0)| \le \epsilon \text{ for all } z \in A', s_0 \in (a_{\max},t];$$

in other words

$$\lim_{s\to t^-} \sup_{z\in A} |H_k(x,z,t) - H_k(x,z,s)| = 0,$$

therefore

$$\lim_{s\to t^-} \int_M h(z,y,t-s)H_k(x,z,s) = \lim_{s\to t^-} \int_M h(z,y,t-s)H_k(x,z,t).$$

The same argument can be used to justify the second term of the second line of (4.8) using the following estimate for $s \leq 1$

$$\begin{aligned}
\int_M |H_k(x,z,s)|\, dz &\leq \sum_{j=0}^{k} \int_M \eta(d(x,z)) H_{\mathcal{E}}(x,z,s) s^j \,|u_j(x,z)|\, dz \\
&\leq \sum_{j=0}^{k} \int_M \eta(d(x,z)) H_{\mathcal{E}}(x,z,s) \,|u_j(x,z)|\, dz \\
&\leq C_1 \int_{\mathbb{R}^n} H_{\mathcal{E}}(x,z,s) dz \\
&= C_1
\end{aligned}$$

where C_1 is the max of the continuous function $\eta(d(x,z)) \sum_{j=0}^{k} |u_j(x,z)|$ on the compact set A'. Note that $\int_{\mathbb{R}^n} H_{\mathcal{E}}(x,z,s)dz = 1$ for all $s > 0$ from Appendix D.

To justify the first term of the third line of (4.8), notice $H_k(x,z,t)$ is in $L^2(M)$ as a function of z because it is continuous with compact support, so the initial condition assumption of $h(x,y,t)$ immediately gives the conclusion.

The second term of the third line is justified using the initial condition of the Euclidean heat kernel. From Appendix D,

$$\begin{aligned}
&\lim_{s\to 0^+} \int_M h(z,y,t)\eta(d(x,z)) H_{\mathcal{E}}(x,z,s) u_j(x,z) dz \\
&\quad = \lim_{s\to 0^+} \int_{\mathbb{R}^n} H_{\mathcal{E}}(x,z,s) h(z,y,t)\eta(d(x,z)) u_j(x,z) dz \\
&\quad = \eta(d(x,x)) h(x,y,t) u_j(x,x) \\
&\quad = h(x,y,t) u_j(x,x),
\end{aligned}$$

since $h(z,y,t)\eta(d(x,z))u_j(x,z)$ is compactly supported in z in the polar normal neighborhood of x. Therefore,

$$\begin{aligned}\lim_{s\to 0^+}\int_M h(z,y,t)H_k(x,z,s) &= \sum_{j=0}^{k}\lim_{s\to 0^+} s^j\int_M h(z,y,t)\eta(d(x,z))H_{\mathcal{E}}(x,z,s)u_j(x,z)dz\\ &= \sum_{j=0}^{k} h(x,y,t)u_j(x,x)\lim_{s\to 0^+} s^j\\ &= h(x,y,t)u_0(x,x)\\ &= h(x,y,t).\end{aligned}$$

The time derivative in (4.8) can be brought inside the integral because the integrand is smooth with compact support.

$$\begin{aligned}&\frac{\partial}{\partial s}\int_M h(z,y,t-s)H_k(x,z,s)dz\\ &\quad= \int_M \frac{\partial}{\partial s}\left[h(z,y,t-s)H_k(x,z,s)\right]dz\\ &\quad= \int_M -\frac{\partial h}{\partial s}(z,y,t-s)H_k(x,z,s)dz + \int_M h(z,y,t-s)\frac{\partial H_k}{\partial s}(x,z,s)dz\\ &\quad= \int_M \Delta_z h(z,y,t-s)H_k(x,z,s)dz + \int_M h(z,y,t-s)\frac{\partial H_k}{\partial s}(x,z,s)dz\\ &\quad= \int_M h(z,y,t-s)\left(\Delta_z + \frac{\partial}{\partial s}\right)H_k(x,z,s)dz,\end{aligned}$$

where the last line is from the integration by parts given by Stokes' theorem. Now, we can finially estimate the difference

$$
\begin{aligned}
&|h(x,y,t) - H_k(x,y,t)| \\
&\quad = \left| \int_0^t \int_M h(z,y,t-s) \left(\Delta_z + \frac{\partial}{\partial s} \right) H_k(x,z,s) dz ds \right| \\
&\quad \leq \left| \int_0^t \int_{B_x(\rho)} h(z,y,t-s) \left(\Delta_z + \frac{\partial}{\partial s} \right) H_k(x,z,s) dz ds \right| \\
&\qquad + \left| \int_0^t \int_{B_x(2\rho) \setminus B_x(\rho)} h(z,y,t-s) \left(\Delta_z + \frac{\partial}{\partial s} \right) H_k(x,z,s) dz ds \right| \\
&\qquad + \left| \int_0^t \int_{M \setminus B_x(2\rho)} h(z,y,t-s) \left(\Delta_z + \frac{\partial}{\partial s} \right) H_k(x,z,s) dz ds \right| \\
&\quad \leq \int_0^t \int_{B_x(\rho)} h(z,y,t-s) H_{\mathcal{E}}(x,z,s) s^k \left| \Delta_z u_k(x,z) \right| dz ds \\
&\qquad + \int_0^t \int_{B_x(2\rho) \setminus B_x(\rho)} h(z,y,t-s) \left| \left(\Delta_z + \frac{\partial}{\partial s} \right) H_k(x,z,s) \right| dz ds.
\end{aligned}
$$

Set

$$
C_2 := \sup_{(x,z) \in A \times A'} \left| \Delta_z u_k(x,z) \right|.
$$

This is a non-negative finite number since it is the supremum of a continuous function on a compact set. Note that $B_x(\rho) \subset A'$ for $x \in A$. For $k > n/2$, $x, y \in A$ and $t > 0$,

$$
\begin{aligned}
&\int_0^t \int_{B_x(\rho)} h(z,y,t-s) H_{\mathcal{E}}(x,z,s) s^k \left| \Delta_z u_k(x,z) \right| dz ds \\
&\quad \leq C_2 \int_0^t s^k \frac{1}{(4\pi s)^{n/2}} \exp(0) \int_M h(z,y,t-s) dz ds \\
&\quad \leq \frac{C_2}{(4\pi)^{n/2}} \int_0^t s^{k-n/2} ds \\
&\quad = \frac{C_2}{(4\pi)^{n/2}} t^{k+1-n/2}
\end{aligned}
$$

Write

$$
\tilde{H}_k(x,y,t) := H_{\mathcal{E}}(x,y,t) \sum_{j=0}^{k} t^j u_j(x,y).
$$

For $x \in A$, $z \in B_x(2\rho) \setminus B_x(\rho)$, the product rule and the lemma show

$$\begin{aligned}
&\left|\left(\Delta_z + \frac{\partial}{\partial s}\right) H_k(x,z,s)\right| \\
&\quad = \Big|\eta(d(x,z))H_{\mathcal{E}}(x,z,s)s^k\Delta_z u_k(x,z) \\
&\qquad\quad + \tilde{H}_k(x,z,s)\Delta_z(\eta(d(x,z))) - 2\langle\nabla_z\eta(d(x,z)), \nabla_z\tilde{H}_k(x,z,s)\rangle\Big| \\
&\quad \le \left|\eta(d(x,z))H_{\mathcal{E}}(x,z,s)s^k\Delta_z u_k(x,z)\right| \\
&\qquad\quad + \left|\tilde{H}_k(x,z,s)\Delta_z(\eta(d(x,z)))\right| \\
&\qquad\quad + 2\left|\nabla_z\eta(d(x,z))\right| \left|\nabla_z\tilde{H}_k(x,z,s)\right| \\
&\quad \le H_{\mathcal{E}}\Bigg(|s^k\Delta u_k| + \left|\sum_{j=0}^k s^j u_j\right| |\Delta\eta| \\
&\qquad\quad + \frac{10}{\rho}\left|\nabla\left(\sum_{j=0}^k s^j u_j\right)\right|\Bigg) + \frac{5}{\rho}\left|\sum_{j=0}^k s^j u_j\right| |\nabla H_{\mathcal{E}}| \\
&\quad = H_{\mathcal{E}}\left(|s^k\Delta u_k| + \left|\sum_{j=0}^k s^j u_j\right| |\Delta\eta| + \frac{10}{\rho}\left|\left(\sum_{j=0}^k s^j \nabla u_j\right)\right| + \frac{5r}{2\rho s}\left|\sum_{j=0}^k s^j u_j\right|\right)
\end{aligned}$$

where the last two inequalities follow from the computations

$$\nabla\tilde{H}_k(x,z,s) = \left(\sum_{j=0}^k s^j u_j\right)\nabla H_{\mathcal{E}} + H_{\mathcal{E}}\nabla\left(\sum_{j=0}^k s^j u_j\right),$$

and

$$\begin{aligned}
\nabla H_{\mathcal{E}}(x,z,s) &= \nabla_z\left((4\pi s)^{-n/2}\exp\left(-\frac{d(x,z)^2}{4s}\right)\right) \\
&= -\frac{r}{2s}H_{\mathcal{E}}\nabla_z d(x,z) \\
&= -\frac{r}{2s}H_{\mathcal{E}}\nabla r,
\end{aligned}$$

where $|\nabla r| = 1$.

Now, we can get an estimate for s near 0. For $s \le 1$,

$$\begin{aligned}
&\left|\left(\Delta_z + \frac{\partial}{\partial s}\right) H_k(x,z,s)\right| \\
&\quad \le s^{-1}H_{\mathcal{E}}\cdot\left(|\Delta u_k| + \left|\sum_{j=0}^k u_j\right| |\Delta\eta| + \frac{10}{\rho}\left|\left(\sum_{j=0}^k \nabla u_j\right)\right| + \frac{5r}{2\rho}\left|\sum_{j=0}^k u_j\right|\right),
\end{aligned}$$

where the right hand quantity is uniformly bounded by $C_3 > 0$ for $(x, z) \in A \times A'$. Therefore, for $k > n/2$, $t \in (0, 1]$, and $x, y \in A$,

$$
\begin{aligned}
&|h(x,y,t) - H_k(x,y,t)| \\
&\quad \le \frac{C_2}{(4\pi)^{n/2}} t^{k+1-n/2} + \frac{C_3}{(4\pi)^{n/2}} \int_0^t s^{-n/2-1} \exp\left(-\rho^2 4s\right) \int_M h(z,y,t-s) dz ds \\
&\quad \le \frac{C_2}{(4\pi)^{n/2}} t^{k+1-n/2} + \frac{C_3}{(4\pi)^{n/2}} \int_0^t s^{-n/2-1} \exp\left(-\rho^2 4s\right) ds
\end{aligned}
$$

Substituting $w = s^{-1}$ and letting $w \to \infty$ and applying L'Hospital's rule shows $s^{-n/2-1} \exp(-\rho^2 4s) \to 0$ as $s \to 0^+$. Therefore, the integrand remains bounded on the finite volume domain of integration, and we conclude there exists $C_4 > 0$ such that

$$
\begin{aligned}
|h(x,y,t) - H_k(x,y,t)| &\le \frac{C_2}{(4\pi)^{n/2}} t^{k+1-n/2} + C_4 \\
&\le \left(\frac{C_2}{(4\pi)^{n/2}} + C_4 \right) t^{k+1-n/2}
\end{aligned}
$$

for all $k > n/2$, $t \in (0, 1]$, and $x, y \in A$.

Finally, if $k \le n/2$,

$$
\begin{aligned}
|h(x,y,t) - H_k(x,y,t)| &= \left| h(x,y,t) - \eta(d(x,y)) H_{\lceil n/2 \rceil + 1}(x,y,t) + H_{\mathcal{E}} \sum_{j=k+1}^{\lceil n/2 \rceil + 1} t^j u_j \right| \\
&\le \left(\frac{C_2}{(4\pi)^{n/2}} + C_4 \right) t^{\lceil n/2 \rceil + 2 - n/2} \\
&\qquad + (4\pi t)^{-n/2} \exp(0) \sum_{j=k+1}^{\lceil n/2 \rceil + 1} t^j |u_j| \\
&\le C t^{k+1-n/2}.
\end{aligned}
$$

□

CHAPTER V

The asymptotics of $\int_N H(x,x,t)dx$

The goal of this section is to prove Theorem III.2. By Theorems II.3 and IV.1, for $j > 0$,

$$|H(x,x,t) - (4\pi t)^{-n/2}| = |H(x,x,t) - H_{\mathcal{E}}(x,x,t)u_0(x,x)| \le C(j)t^{1-n/2},$$

$$x \in N \setminus T_j, t \in (0,1].$$

Taking the integral over $N \setminus T_j$,

$$\left| \int_{N\setminus T_j} H(x,x,t)dx - \int_{N\setminus T_j} (4\pi t)^{-n/2} dx \right| \le C(j)t^{1-n/2} V(N \setminus T_j), t \in (0,1],$$

so

$$\left| (4\pi t)^{n/2} \int_{N\setminus T_j} H(x,x,t)dx - V(N \setminus T_j) \right| \le C(j)t^V(N \setminus T_j), t \in (0,1].$$

Letting $t \to 0$,

$$\lim_{t\to 0}(4\pi t)^{n/2} \int_{N\setminus T_j} H(x,x,t)dx = V(N \setminus T_j).$$

On the other hand, using the volume estimate from Appendix C and the upper bound from Theorem II.4, there exists a constant $C_1 > 0$ independent of j such that

$$\begin{aligned}(4\pi t)^{n/2} \int_{T_j} H(x,x,t)dx &\le (4\pi t)^{n/2} \int_{T_j} \tilde{H}(0,t)dx \\ &= (4\pi t)^{n/2} V(T_j)\tilde{H}(0,t) \\ &\le (4\pi t)^{n/2} C_1 \cdot (2^{-j})^2 \tilde{H}(0,t)\end{aligned}$$

Now, apply Thereom IV.1 on the compact manifold $\mathbb{CP}^{\frac{n}{2}}$ and take the limit as $t \to 0$ to obtain

$$\lim_{t\to 0}(4\pi t)^{n/2}\tilde{H}(0,t) = 1,$$

so

$$\begin{aligned}
&\lim_{t\to 0}(4\pi t)^{n/2}\int_N H(x,x,t)dx \\
&\quad = \lim_{t\to 0}(4\pi t)^{n/2}\int_{N\setminus T_j} H(x,x,t)dx + \lim_{t\to 0}(4\pi t)^{n/2}\int_{T_j} H(x,x,t)dx \\
&\quad \le V(N\setminus T_j) + C_1\cdot(2^{-j})^2.
\end{aligned}$$

Finally, let $j \to \infty$ to obtain

$$\begin{aligned}
\lim_{t\to 0}(4\pi t)^{n/2}\int_N H(x,x,t)dx &\le \lim_{j\to\infty} V(N\setminus T_j) + C_1\cdot(2^{-j})^2 \\
&= V(N).
\end{aligned}$$

The lower bound is obtained using $H(x,x,t) > 0$, so

$$\begin{aligned}
&\lim_{t\to 0}(4\pi t)^{n/2}\int_N H(x,x,t)dx \\
&\quad = \lim_{t\to 0}(4\pi t)^{n/2}\int_{N\setminus T_j} H(x,x,t)dx + \lim_{t\to 0}(4\pi t)^{n/2}\int_{T_j} H(x,x,t)dx \\
&\quad \ge V(N\setminus T_j) + 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0}(4\pi t)^{n/2}\int_N H(x,x,t)dx &\ge \lim_{j\to\infty} V(N\setminus T_j) \\
&= V(N).
\end{aligned}$$

This proves the desired result:

$$\lim_{t\to 0}(4\pi t)^{n/2}\int_N H(x,x,t)dx = V(N).$$

□

CHAPTER VI

Eigenfunction bounds

Recall from the proof of Theorem II.5 that $\phi_i \in L^\infty(N)$ with bound

$$||\phi_i||_\infty \leq \sqrt{\tilde{H}(0,t)}e^{\lambda_i t/2}, \quad t > 0.$$

In this section, I will show that there is infact a polynomial bound similar to the bounds of the eigenfunctions of the Laplacian on a compact manifold.

Theorem VI.1. *Let $\{\phi_i\}_{i=0}^\infty$ be an orthonormal basis of $L^2(N)$ consisiting of eigenfuctions of Δ with $\Delta\phi_i = \lambda_i\phi_i$ and $\lambda_{i+1} \geq \lambda_i$. There exists a constant C depending only on N such that for all $i > 0$,*

$$||\phi_i||_\infty \leq C \cdot \lambda_i^{\frac{n}{4}}.$$

Proof. The proof is inspired by Peter Li's proof of a similar result for compact manifolds with boundary and the Dirichlet boundary condition on pages 99-100 of [Li]. Since the volume of N is finite, the constant functions are in $L^2(N)$, so $\phi_0 \equiv V(N)^{-\frac{1}{2}}$. On the other hand, if $\Delta(\phi) = 0$, then since $(\Delta + 1)^{-1}$ factors through $W^{1,2}(N)$ by Lemma 5.2 of [LT], $\phi \in W^{1,2}(N)$, so by the L^2ST condition proved in the proof of Theorem 4.1 of [LT] we can integrate by parts to obtain

$$0 = \int_N \phi\Delta\phi = \int_N |\nabla\phi|^2.$$

Since $\phi \in C^\infty(N)$ by elliptic regularity, $\nabla\phi$ is the 0-section, so ϕ must be a constant. Hence $\lambda_1 > 0$. For the case when $i > 0$ we will we will want to approximate $|\phi_i|$ by positive, smooth functions. Let

$$\phi_{i,\tau}(x) := \sqrt{\phi_i^2(x) + \tau^2}, \quad \tau > 0.$$

Note as $\tau \to 0$, $\sqrt{\phi_i^2(x) + \tau^2} \to |\phi_i|$ uniformly on N because

$$|\phi_i| \leq \sqrt{\phi_i^2(x) + \tau^2} \leq |\phi_i| + \tau.$$

This also shows $\phi_{i,\tau} \in L^\infty(N)$ because $\phi_i \in L^\infty(N)$. By elliptic regularity, $\phi_i \in C^\infty(N)$, so $\phi_{i,\tau} \in C^\infty(N)$. We have

$$\int_N \phi_{i,\tau}^2 = \int_N (\phi_i^2 + \tau^2) = 1 + \tau^2 V(N) < \infty,$$

hence $\phi_{i,\tau} \in L^2(N)$. By Lemma 5.2 of [LT], $\phi_i \in W^{1,2}(N)$. To see $\phi_{i,\tau} \in W^{1,2}(N)$, compute

$$\begin{aligned}\left\|\nabla\left(\sqrt{\phi_i^2+\tau^2}\right)\right\|_{L^2} &= \left\|\frac{\phi_i}{\sqrt{\phi_i^2+\tau^2}}\cdot\nabla\phi_i\right\|_{L^2} = \left(\int_N \frac{\phi_i^2}{\phi_i^2+\tau^2}\cdot|\nabla\phi_i|^2\right)^{\frac{1}{2}} \\ &\leq \left(\int_N |\nabla\phi_i|^2\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

The following lemma shows that $\Delta\phi_{i,\tau}$ has a bound similar to that of the eigenfunction.

Lemma VI.2. *For $\tau > 0$, $\Delta\phi_{i,\tau} \in L^1(N)$ and*

$$\Delta(\phi_{i,\tau}) \leq \lambda_i \phi_{i,\tau}$$

Remark. $\Delta\phi_{i,\tau}$ may not be bounded from below, and it may not even be in $L^2(N)$; hence, $\phi_{i,\tau}$ may not be in $\mathcal{D}(\Delta)$. If $|\nabla\phi_i|^2 \in L^2(N)$, then $\Delta\phi_{i,\tau} \in L^2(N)$, which together with $\phi_{i,\tau} \in W^{1,2}(N)$ would show $\phi_{i,\tau} \in \mathcal{D}(\Delta)$. Further, if $|\nabla\phi_i| \in L^\infty(N)$, so is $\Delta\phi_{i,\tau}$.

Proof. If $i = 0$, the function is constant, and the equality is clear. Assume $i > 0$, so $\lambda_i > 0$. From the usual computation using $\Delta = -\mathrm{div} \circ \nabla$ in a local geodesic frame of the tangent bundle of N, for any $f \in C^\infty(\mathbb{R})$ and $g \in C^\infty(N)$ we have

$$\Delta(f(g(x))) = -f''(g(x))|\nabla g(x)|^2 + f'(g(x))\Delta g(x).$$

Applying this to $f(x) = \sqrt{g(x)^2 + \tau^2}$ and $g = \phi_i$, we have

$$f'(x) = \frac{x}{\sqrt{x^2 + \tau^2}} \text{ and } f''(x) = \frac{\tau^2}{(x^2 + \tau^2)^{3/2}},$$

so

$$\begin{aligned} \Delta\left(\sqrt{\phi_i^2 + \tau^2}\right) &= \frac{-\tau^2}{(\phi_i^2 + \tau^2)^{3/2}} \cdot |\nabla \phi_i|^2 + \frac{\phi_i}{\sqrt{\phi_i^2 + \tau^2}} \cdot \Delta\phi_i \\ &= \frac{-\tau^2}{(\phi_i^2 + \tau^2)^{3/2}} \cdot |\nabla \phi_i|^2 + \frac{\lambda_i \phi_i^2}{\sqrt{\phi_i^2 + \tau^2}}. \end{aligned}$$

The first term on the right is non-positive, and the second term on the right is non-negative; therefore,

$$\Delta\phi_{i,\tau} \geq \frac{-\tau^2}{(\phi_i^2 + \tau^2)^{3/2}} \cdot |\nabla \phi_i|^2 \geq -\tau^{-1}|\nabla \phi_i|^2 \in L^1(N),$$

and

$$\begin{aligned} \Delta\phi_{i,\tau} &\leq \frac{\lambda_i \phi_i^2}{\sqrt{\phi_i^2 + \tau^2}} = \frac{\phi_i^2}{\phi_i^2 + \tau^2} \cdot \lambda_i \sqrt{\phi_i^2 + \tau^2} \\ &\leq \lambda_i \sqrt{\phi_i^2 + \tau^2} = \lambda_i \phi_{i,\tau} \in L^2(N) \subset L^1(N); \end{aligned}$$

therefore,

$$\Delta\phi_{i,\tau} \in L^1(N).$$

□

Now, let $k \geq 2$. From the lemma,

$$\lambda_i \int_N \phi_{i,\tau}^k \geq \int_N \phi_{i,\tau}^{k-1} \Delta\phi_{i,\tau}.$$

To integrate by parts, for $j \in \mathbb{N}$, let η_j be a compactly supported smooth bump function with

$$\eta_j(x) = 1, \text{ for } x \in N \setminus T(2^{-(j-1)}), \quad \eta_j(x) = 0, \text{ for } x \in T(2^{-j}), \quad |\nabla \eta_j| \leq 5 \cdot 2^j.$$

Here, $T(r)$ denotes the tubular neighborhood of the singular locus of radius r. Since

$$|\eta_j \phi_{i,\tau}^{k-1} \Delta \phi_{i,\tau}| \leq \left| ||\phi_{i,\tau}||_\infty^{k-1} \Delta \phi_{i,\tau} \right| \in L^1(N), \quad j \in \mathbb{N}$$

by Lebesgue dominated convergence,

$$\begin{aligned}
\int_N \phi_{i,\tau}^{k-1} \Delta \phi_{i,\tau} &= \int_N \lim_{j \to \infty} \eta_j \phi_{i,\tau}^{k-1} \Delta \phi_{i,\tau} \\
&= \lim_{j \to \infty} \int_N \eta_j(x) \phi_{i,\tau}^{k-1} \Delta \phi_{i,\tau} \\
&= \lim_{j \to \infty} \int_N \langle \nabla(\eta_j \phi_{i,\tau}^{k-1}), \nabla \phi_{i,\tau} \rangle \\
&= \lim_{j \to \infty} \left(\int_N \langle \eta_j \nabla(\phi_{i,\tau}^{k-1}), \nabla \phi_{i,\tau} \rangle + \int_N \langle \phi_{i,\tau}^{k-1} \nabla \eta_j, \nabla \phi_{i,\tau} \rangle \right) \\
&:= \lim_{j \to \infty} (I) + (II).
\end{aligned}$$

It should be noted that the sign of integration by parts is opposite of the usual convention because of the choice of sign in Δ. Now, using the fact that $\nabla \eta_j$ is supported in $T_j := T(2^{-(j-1)}) \setminus T(2^{-j})$ and the estimate of $V(T_j) \leq C_1 \cdot (2^{-j})^2$ for some constant $C_1 > 0$ depending only on N from Appendix C, we can estimate (II)

with Hölder's inequality.

$$\begin{aligned}\left|\int_N \langle \phi_{i,\tau}^{k-1}\nabla\eta_j, \nabla\phi_{i,\tau}\rangle\right| &\leq \int_{T_j} \left|\langle \phi_{i,\tau}^{k-1}\nabla\eta_j, \nabla\phi_{i,\tau}\rangle\right| \\ &\leq \int_{T_j} \phi_{i,\tau}^{k-1}|\nabla\eta_j||\nabla\phi_{i,\tau}| \\ &\leq 5\cdot 2^j \int_{T_j} \phi_{i,\tau}^{k-1}|\nabla\phi_{i,\tau}| \\ &\leq 5\cdot 2^j ||\phi_{i,\tau}||_\infty^{k-1} V(T_j)^{\frac{1}{2}} ||\nabla\phi_{i,\tau}||_2^{\frac{1}{2}} \\ &\leq 5\cdot 2^j ||\phi_{i,\tau}||_\infty^{k-1} C_1^{\frac{1}{2}} \cdot 2^{-j} \left(\int_{T_j} |\nabla\phi_{i,\tau}|^2\right)^{\frac{1}{2}} \\ &= 5C_1^{\frac{1}{2}} ||\phi_{i,\tau}||_\infty^{k-1} \left(\int_{T_j} |\nabla\phi_{i,\tau}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\phi_{i,\tau} \in W^{1,2}(N)$,

$$\lim_{j\to\infty} \int_{T_j} |\nabla\phi_{i,\tau}|^2 = 0,$$

therefore,

$$\lim_{j\to\infty} (II) = \lim_{j\to\infty} \int_N \langle \phi_{i,\tau}^{k-1}\nabla\eta_j, \nabla\phi_{i,\tau}\rangle = 0.$$

The limit of (I) can be computed using dominated convergence because

$$\langle \eta_j \nabla(\phi_{i,\tau}^{k-1}), \nabla\phi_{i,\tau}\rangle = \eta_j\langle (k-1)\phi_{i,\tau}^{k-2}\nabla\phi_{i,\tau}, \nabla\phi_{i,\tau}\rangle = (k-1)\eta_j\phi_{i,\tau}^{k-2}|\nabla\phi_{i,\tau}|^2,$$

and

$$\left|(k-1)\eta_j\phi_{i,\tau}^{k-2}|\nabla\phi_{i,\tau}|^2\right| \leq (k-1)||\phi_{i,\tau}||_\infty^{k-2}|\nabla\phi_{i,\tau}|^2 \in L^1(N),$$

hence

$$\begin{aligned}\lim_{j\to\infty} (I) &= \lim_{j\to\infty} \int_N \langle \eta_j \nabla(\phi_{i,\tau}^{k-1}), \nabla\phi_{i,\tau}\rangle \\ &= \int_N \lim_{j\to\infty} \langle \eta_j \nabla(\phi_{i,\tau}^{k-1}), \nabla\phi_{i,\tau}\rangle \\ &= \int_N \lim_{j\to\infty} (k-1)\eta_j\phi_{i,\tau}^{k-2}|\nabla\phi_{i,\tau}|^2 \\ &= \int_N (k-1)\phi_{i,\tau}^{k-2}|\nabla\phi_{i,\tau}|^2.\end{aligned}$$

Combining (I) and (II), the 'integration by parts' equality

$$\int_N \phi_{i,\tau}^{k-1}\Delta\phi_{i,\tau} = \int_N (k-1)\phi_{i,\tau}^{k-2}|\nabla\phi_{i,\tau}|^2$$

is obtained. Using

$$\nabla(\phi_{i,\tau}^{\frac{k}{2}}) = \frac{k}{2}\phi_{i,\tau}^{\frac{k-2}{2}}\nabla\phi_{i,\tau},$$

gives the equality

$$\int_N (k-1)\phi_{i,\tau}^{k-2}|\nabla\phi_{i,\tau}|^2 = \int_N \frac{4(k-1)}{k^2}|\nabla(\phi_{i,\tau}^{\frac{k}{2}})|^2.$$

The inequalities of this section together show the $W^{1,2}(N)$ norm of $\phi_{i,\tau}^{\frac{k}{2}}$ is bounded by its $L^2(N)$ norm in the following way:

$$\int_N |\nabla(\phi_{i,\tau}^{\frac{k}{2}})|^2 \leq \frac{\lambda_i k^2}{4(k-1)}\int_N \phi_{i,\tau}^k = \frac{\lambda_i k^2}{4(k-1)}\int_N |\phi_{i,\tau}^{\frac{k}{2}}|^2 < \infty.$$

Here, the last integral is finite because $\phi_{i,\tau} \in L^\infty(N)$ and $V(N) < \infty$, hence $\phi_{i,\tau}^{\frac{k}{2}} \in W^{1,2}(N)$.

Applying the Sobolev ineqality in Theorem B.1, produces the inequality

$$\begin{aligned}\left(\int_N |\phi_{i,\tau}^{\frac{k}{2}}|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}} &\leq C_{Sob}\left(\int_N |\nabla(\phi_{i,\tau}^{\frac{k}{2}})|^2 + \int_N |\phi_{i,\tau}^{\frac{k}{2}}|^2\right)\\ &\leq C_{Sob}\left(\frac{\lambda_i k^2}{4(k-1)} + 1\right)\int_N |\phi_{i,\tau}^{\frac{k}{2}}|^2,\end{aligned}$$

where $m = n/2$. Utilizing $2(k-1) > k$ and $1 \leq \frac{k}{2}$ for $k \geq 2$, one obtains

$$\left(\int_N |\phi_{i,\tau}|^{k\frac{m}{m-1}}\right)^{\frac{m-1}{m}} \leq kC_{Sob}\lambda_i\int_N |\phi_{i,\tau}|^k.$$

Writing $\beta = \frac{m}{m-1} \in (1,2]$ and taking $k = 2\beta^j$ for $j \in \mathbb{N}$ produces the inequality

$$\left(||\phi_{i,\tau}||_{2\beta^{j+1}}\right)^{2\beta^j} \leq 2\beta^j C_{Sob}\lambda_i\left(||\phi_{i,\tau}||_{2\beta^j}\right)^{2\beta^j},$$

therefore

$$||\phi_{i,\tau}||_{2\beta^{j+1}} \leq \left(2\beta^j C_{Sob}\lambda_i\right)^{\frac{1}{2\beta^j}}||\phi_{i,\tau}||_{2\beta^j}.$$

Now, by induction on j,

$$
\begin{aligned}
||\phi_{i,\tau}||_{2\beta^{j+1}} &\leq \left(\prod_{l=0}^{j} \left(2\beta^l C_{Sob} \lambda_i \right)^{\frac{1}{2\beta^l}} \right) ||\phi_{i,\tau}||_2 \\
&= \left(\prod_{l=0}^{j} \left(2\beta^l C_{Sob} \lambda_i \right)^{\frac{1}{2\beta^l}} \right) \left(1 + \tau^2 V(N) \right)^{\frac{1}{2}}, \quad j \in \mathbb{N}.
\end{aligned}
$$

To compute the limit, the product can be converted to a sum with the logarithmic function.

$$
\begin{aligned}
\lim_{j\to\infty} \log \left(\prod_{l=0}^{j} \left(2\beta^l C_{Sob} \lambda_i \right)^{\frac{1}{2\beta^l}} \right) &= \lim_{j\to\infty} \sum_{l=0}^{j} \frac{1}{2\beta^l} \log \left(2\beta^l C_{Sob} \lambda_i \right) \\
&= \lim_{j\to\infty} \sum_{l=0}^{j} \frac{1}{2\beta^l} \left(l \log(\beta) + \log(2C_{Sob}\lambda_i) \right) \\
&= \frac{\log(\beta)}{2} \lim_{j\to\infty} \sum_{l=0}^{j} \frac{l}{\beta^l} + \frac{1}{2} \frac{\beta}{\beta - 1} \log(2C_{Sob}\lambda_i) \\
&= C_2(m) + \log((2C_{Sob}\lambda_i)^{\frac{m}{2}}),
\end{aligned}
$$

where

$$
C_2(m) = \frac{\log(\beta)}{2} \lim_{j\to\infty} \sum_{l=0}^{j} \frac{l}{\beta^l} < \infty
$$

by the ratio test. Using the continuity of the exponential function,

$$
\begin{aligned}
\exp \left(\lim_{j\to\infty} \log \left(\prod_{l=0}^{j} \left(2\beta^l \lambda_i C_{Sob} \right)^{\frac{1}{2\beta^l}} \right) \right) &= \lim_{j\to\infty} \exp \circ \log \left(\prod_{l=0}^{j} \left(2\beta^l \lambda_i C_{Sob} \right)^{\frac{1}{2\beta^l}} \right) \\
&= \lim_{j\to\infty} \prod_{l=0}^{j} \left(2\beta^l \lambda_i C_{Sob} \right)^{\frac{1}{2\beta^l}},
\end{aligned}
$$

so

$$
\lim_{j\to\infty} \prod_{l=0}^{j} \left(2\beta^l \lambda_i C_{Sob} \right)^{\frac{1}{2\beta^l}} = e^{C_2(m)} (2C_{Sob})^{\frac{m}{2}} \cdot \lambda_i^{\frac{m}{2}}.
$$

Using the monotone norms,

$$
\begin{aligned}
||\phi_{i,\tau}||_\infty &= \lim_{j\to\infty} V(N)^{\frac{-1}{2\beta^j}} ||\phi_{i,\tau}||_{2\beta^j} \\
&= \lim_{j\to\infty} ||\phi_{i,\tau}||_{2\beta^j} \\
&\leq C\lambda_i^{\frac{m}{2}} \left(1 + \tau^2 V(N) \right)^{\frac{1}{2}}, \quad \tau > 0.
\end{aligned}
$$

Therefore,

$$||\phi_i||_\infty \leq ||\phi_{i,\tau}||_\infty \leq C\lambda_i^{\frac{m}{2}} \left(1 + \tau^2 V(N)\right)^{\frac{1}{2}}, \quad \tau > 0.$$

Finally, letting $\tau \to 0$, shows

$$||\phi_i||_\infty \leq C\lambda_i^{\frac{m}{2}}.$$

CHAPTER VII

An application

Let N be the non-singular locus of an embedded, irreducible, complex projective varitety of complex dimension $m = \frac{n}{2} > 1$, and let $\{\phi_i\}_{i=0}^{\infty}$ be an orthonormal basis of $L^2(N)$ consisiting of smooth eigenfuctions of Δ with $\Delta\phi_i = \lambda_i\phi_i$ and $\lambda_{i+1} \geq \lambda_i$.

Theorem VII.1.

$$H(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

with convergence uniform on $N \times N \times [a, \infty)$ *for every* $a > 0$.

Proof. By the above eigenfunction bound and Weyl's law, there exists constants $C_1 > 0$, $C_2 > 0$, and $C_3 > 0$ such that for large enough L,

$$\begin{aligned}
\sum_{i=L}^{\infty} e^{-\lambda_i t}\phi_i(x)^2 &\leq \sum_{i=L}^{\infty} C_1 e^{-\lambda_i a}\lambda_i^{n/2} \\
&\leq \sum_{i=L}^{\infty} C_1 C_3 e^{-C_2 i^{2/n} a} i \\
&= \sum_{i=L}^{\infty} Ai \cdot B^{i^{2/n}}
\end{aligned}$$

for some constants $A > 0$, $B \in (0, 1)$. Since

$$\lim_{i\to\infty}\left(Ai \cdot B^{i^{2/n}}\right)^{1/i} = \lim_{i\to\infty} B^{i^{2/n}-1} = 0,$$

this series converges by the root test. Therefore, $\sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)^2$ is uniformly Cauchy, and hence uniformly convergent. Using the Cauchy-Schwarz estimate from the proof of Theorem II.5,

$$\sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

is also uniformly Cauchy and hence uniformly convergent. Therefore, since each of the finite sums is continuous, the limit is also continuous. Since the limit agrees with the continuous function $H(x, y, t)$ almost everywhere, the pointwise limit is $H(x, y, t)$. $\square$

APPENDICES

APPENDIX A

Tauberian theorem

The Tauberian theorem (Theorem A.2) is a deep and powerful theorem that took a lot of work by many mathemeticians to prove. It has a long history encompassing the theorems known as the Abelian and Tauberian theorems. Theorem III.1 which is used to prove Weyl's law falls under the cases called Tauberian theorems which provide partial converses to the various Abelian theorems named after Abel's theorem for power series which is about the continuity of a power series on the boundary of its disk of convergence. It is stated as the following:

Theorem A.1. ***(Abel's theorem)***

If $\sum_{k=0}^{\infty} a_k$ is convergent in $\mathbb{C}$, then

$$\lim_{x \to 1^-} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

Proofs of this theorem can be found in undergraduate analysis books such as on page 174 of [Ru].

The Tauberian theorems are named after Alfred Tauber who proved a converse to Abel's theorem under certain conditions in his 1897 paper [Tau]. John Edensor Littlewood then strengthened Tauber's result in the 1911 paper [Lit] by weakening the condition required to prove the converse of Abel's theorem. Godfrey Harold

Hardy and Littlewood later significantly generalized Littlewoods result to the Hardy-Littlewood tauberian theorem in the 1914 paper [HL]. This theorem is Theorem A.13 in the case when $\rho = 1$ and $\lambda_k = k$. In the 1930 paper [Kar], Jovan Karamata gave a much simpler proof of the Hardy-Littlewood tauberian theorem which is less than a page of long. There are also other generalizations of Tauber's theorem such as Wiener's tauberian theorem. One generalization encompassing both the Abelian and Tauberian theorems is presented by Feller on page 443 of [Fe].

Theorem A.2. *(Theorem 1 of [Fe]) Let U be a measure with a Laplace transform $\Omega(s) := \int_0^\infty e^{-sx} dU(x)$ defined for $s > 0$. Let $\tau = t^{-1} \in (0, \infty)$. Then each of the relations*

$$\frac{\Omega(\tau s)}{\Omega(\tau)} \to s^{-\rho} \text{ as } \tau \to 0, \tag{A.3}$$

and

$$\frac{U(tx)}{U(t)} \to x^{\rho} \text{ as } t \to \infty \tag{A.4}$$

implies the other as well as

$$\Omega(\tau) \sim U(t)\Gamma(\rho + 1) \text{ as } t \to \infty.$$

Here, $U(t)$ means $U([0, t])$. Theorem III.1 can be seen as a direct consequence of this theorem by taking U to be the Lebesgue-Stieltjes measure associated with $N(\lambda) = \sum_{\lambda_i \leq \lambda} 1$, so $\Omega(t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}$. By assumption of Theorem III.1,

$$\lim_{\tau \to 0} \frac{C \cdot (\tau s)^{-\rho}}{\Omega(\tau s)} = 1, \quad s > 0.$$

Therefore,

$$\lim_{\tau \to 0} \frac{\Omega(\tau s)}{\Omega(\tau)} = \lim_{\tau \to 0} \frac{\Omega(\tau s)}{\Omega(\tau)} \cdot \frac{\Omega(\tau)}{C\tau^{-\rho}} \cdot \frac{C(\tau s)^{-\rho}}{\Omega(\tau s)} = \lim_{\tau \to 0} \frac{1}{s^{\rho}} = \frac{1}{s^{\rho}},$$

so, by Theorem A.2

$$N(\lambda) \sim \frac{\Omega(\lambda^{-1})}{\Gamma(\rho+1)} \sim \frac{C\lambda^{\rho}}{\Gamma(\rho+1)} \text{ as } \lambda \to \infty.$$

While Feller provides a stronger generalization of Theorem III.1, a proof can be found in [Fe], and I will not provide this proof here. Instead, I will provide Karamata's proof which I think is more insightful. Karamata's main result that he uses to prove the Hardy-Littlewood theorem is the following:

Theorem A.5. ***(Karamata's main result)*** *If*

$$\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty} a_k x^k = s \tag{A.6}$$

and

$$a_k \geq -M, \quad M \geq 0 \textit{ for all } k,$$

then

$$\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty} a_k g(x^k) x^k = s\cdot\int_0^1 g(t)dt \tag{A.7}$$

for all bounded Riemann integrable functions $g(t)$ on the interval $[0,1]$.

Proof. The proof is a translation from the original paper [Kar].

Without loss of generality take $a_k \geq 0$, otherwise the following can be applied to both of the sequences $a_k + M$ and M replacing a_k.

Replacing x by $x^{1+\alpha}$ where $\alpha \geq 0$ in (A.6) gives

$$\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty} a_k x^{k\alpha} x^k = \frac{s}{1+\alpha},$$

and therefore,

$$\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty} a_k P(x^k) x^k = s\cdot\int_0^1 P(t)dt$$

for every polynomial $P(t)$. Now, for every bounded Riemann integrable function $g(t)$ on the interval $[0, 1]$ and for every $\epsilon > 0$, one one can find two polynomials $p(t)$ and $P(t)$ such that

$$p(t) \leq g(t) \leq P(t), \qquad \text{for } 0 \leq t \leq 1$$

and

$$\int_0^1 (P(t) - p(t))dt \leq \epsilon.$$

Since the numbers $a_k \geq 0$, it follows that (A.7) is valid for every bounded and Riemann integrable function $g(t)$. □

I will now give a few comments to fill out the details of the proof a little bit more. First, I will explain the details of the without loss of generality statement. Under the assumptions of the theorem,

$$\lim_{x \to 1^-} (1 - x) \cdot \sum_{k=0}^{\infty} M x^k = \lim_{x \to 1^-} (1 - x) \frac{M}{1 - x} = M,$$

and

$$\begin{aligned} \lim_{x \to 1^-} (1 - x) \cdot \sum_{k=0}^{\infty} (a_k + M) x^k &= \lim_{x \to 1^-} (1 - x) \cdot \sum_{k=0}^{\infty} a_k x^k + \lim_{x \to 1^-} (1 - x) \cdot \sum_{k=0}^{\infty} M x^k \\ &= s + M. \end{aligned}$$

Applying the theorem to these two cases gives

$$\begin{aligned}
\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}a_n g(x^k)x^k &= \lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}(a_k+M-M)g(x^k)x^k \\
&= \lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}(a_k+M)g(x^k)x^k \\
&\quad - \lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}Mg(x^k)x^k \\
&= (s+M)\cdot\int_0^1 g(t)dt - M\cdot\int_0^1 g(t)dt \\
&= s\cdot\int_0^1 g(t)dt.
\end{aligned}$$

When x is replaced by $x^{1+\alpha}$ in (A.6), the new case is for $\alpha > 0$ where L'Hospital's rule can be applied as follows:

$$\begin{aligned}
s &= \lim_{x\to 1^-}(1-x^{1+\alpha})\cdot\sum_{k=0}^{\infty}a_k(x^{1+\alpha})^k \\
&= \lim_{x\to 1^-}\frac{1-x^{1+\alpha}}{1-x}\cdot(1-x)\cdot\sum_{k=0}^{\infty}a_k x^{k\alpha}x^k \\
&= \lim_{x\to 1^-}\frac{1-x^{1+\alpha}}{1-x}\cdot\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}a_k x^{k\alpha}x^k \\
&= \lim_{x\to 1^-}\frac{-(1+\alpha)\cdot x^{\alpha}}{-1}\cdot\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}a_k x^{k\alpha}x^k \\
&= (1+\alpha)\cdot\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}a_k x^{k\alpha}x^k,
\end{aligned}$$

so

$$\begin{aligned}
\lim_{x\to 1^-}(1-x)\cdot\sum_{k=0}^{\infty}a_k x^{k\alpha}x^k &= \frac{s}{1+\alpha} \\
&= s\cdot\left(\frac{1^{\alpha+1}}{1+\alpha}-\frac{0^{\alpha+1}}{1+\alpha}\right) \\
&= s\cdot\int_0^1 t^{\alpha}dt \qquad \text{for } \alpha \geq 0.
\end{aligned}$$

By the linearity of all terms involved, the equality holds for all polynomials.

As might be expected, the Weierstrass approximation theorem can be used to see the polynomial bounds. Let $g(t)$ and $\epsilon > 0$ be given as above. Since $g(t)$ is Riemann integrable, its set of discontinuities is of Lebesgue measure 0, so there is an open set E of Lebesgue measure less than $\frac{\epsilon}{8(||g||_\infty+1)}$ containing all of the discontinuities of $g(t)$. There is also an open set F of Lebesgue measure less than $\frac{\epsilon}{4(||g||_\infty+1)}$ containing $\overline{E}$. Choose a continuous partition of unity $\{\phi_1(t), \phi_2(t)\}$ subordinate to the open cover $\{F, \overline{E}^c\}$. Define

$$g_U(t) := \phi_2(t)g(t) + \phi_1(t)||g||_\infty,$$

$$g_L(t) := \phi_2(t)g(t) - \phi_1(t)||g||_\infty,$$

so $g_U(t), g_L(t)$ are continuous functions which agrees with $g(t)$ outside a set of measure $\frac{\epsilon}{4(||g||_\infty+1)}$ and satisfy

$$g_L(t) \leq g(t) \leq g_U(t).$$

Now, by Weierstrass approximation, there exist polynomials $p_U(t), p_L(t)$ such that

$$|p_U(t) - g_U(t)| < \frac{\epsilon'}{8} \text{ for all } t,$$

$$|p_L(t) - g_L(t)| < \frac{\epsilon'}{8} \text{ for all } t,$$

where

$$\epsilon' := \min(\epsilon, 4).$$

The bounding polynomials can be taken to be

$$P(t) := p_U(t) + \frac{\epsilon'}{8},$$

$$p(t) := p_L(t) - \frac{\epsilon'}{8},$$

so

$$p(t) \leq g(t) \leq P(t),$$

and

$$\begin{aligned}
\int_0^1 (P(t)-p(t))dt &= \int_F (P(t)-p(t))dt + \int_{F^c} (P(t)-p(t))dt \\
&\leq \int_F (||g||_\infty + 1) - (-||g||_\infty - 1)\, dt + \int_{F^c} \frac{\epsilon}{2} dt \\
&\leq 2(||g||_\infty + 1) \cdot \frac{\epsilon}{4(||g||_\infty + 1)} + \frac{\epsilon}{2} \cdot 1 \\
&\leq \epsilon.
\end{aligned} \tag{A.8}$$

Finally, these bounds are used to prove the result in the following way. Since the $a_k \geq 0$, for $x \in [0,1]$

$$(1-x) \cdot \sum_{k=0}^{\infty} a_k p(x^k) x^k \leq (1-x) \cdot \sum_{k=0}^{\infty} a_k g(x^k) x^k \leq (1-x) \cdot \sum_{k=0}^{\infty} a_k P(x^k) x^k,$$

and $s \geq 0$. This gives the inequalities

$$\begin{aligned}
s \cdot \int_0^1 p(t)dt &\leq \liminf_{x \to 1^-} (1-x) \cdot \sum_{k=0}^{\infty} a_k g(x^k) x^k \\
&\leq \limsup_{x \to 1^-} (1-x) \cdot \sum_{k=0}^{\infty} a_k g(x^k) x^k \leq s \cdot \int_0^1 P(t)dt,
\end{aligned}$$

and also

$$s \cdot \int_0^1 p(t)dt \leq s \cdot \int_0^1 g(t)dt \leq s \cdot \int_0^1 P(t)dt.$$

If A and B are two numbers satisfying

$$s \cdot \int_0^1 p(t)dt \leq A \leq B \leq s \cdot \int_0^1 P(t)dt$$

for all such $p(t)$ and $P(t)$, then

$$\begin{aligned}
|B - A| &\leq s \cdot \int_0^1 P(t)dt - s \cdot \int_0^1 p(t)dt \\
&= s \cdot \int_0^1 P(t) - p(t)dt \\
&\leq s \cdot \epsilon,
\end{aligned}$$

so $A = B$ since $\epsilon > 0$ was arbitrary.

Therefore, the lim inf, lim sup, and $\int_0^1 g(t)dt$ must all be equal so $\lim_{x \to 1^-}(1-x) \cdot \sum_{k=0}^{\infty} a_k g(x^k) x^k$ exists and is $\int_0^1 g(t)dt$.

Karamata's main result can be used to directly prove the Hardy-Littlewood theorem, but it needs to be slightly modified to prove the version of the Tauberian theorem that is used to prove Weyl's law. There are two modifications that must be made. The theorem must allow for sequences $\{\lambda_k\}$ monotonically increasing to infinity instead of only the sequence $\{k\}$ in the powers of x and also allow for an additional power ρ.

Theorem A.9. *Let $\{\lambda_k\}$ be a sequence monotonically increasing to infinity and $\rho > 0$. If*

$$\lim_{x \to 1^-} (1-x)^\rho \cdot \sum_{k=0}^{\infty} a_k x^{\lambda_k} = s$$

and

$$a_k \geq -M, \quad M \geq 0 \text{ for all } k,$$

then

$$\lim_{x \to 1^-} (1-x)^\rho \cdot \sum_{k=0}^{\infty} a_k g(x^{\lambda_k}) x^{\lambda_k} = \frac{s}{\Gamma(\rho)} \int_0^\infty g\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \tag{A.10}$$

for all bounded Riemann integrable functions $g(t)$ on the interval $[0, 1]$.

Proof. The proof is almost the same as Karamata's proof of Theorem A.5.

Again, without loss of generality take $a_k \geq 0$. There is an equality given by equation (1.3) on page 430 of [Fe] which will be used:

$$\frac{1}{(1+\alpha)^\rho} = \frac{1}{\Gamma(\rho)} \int_0^\infty e^{-(\alpha+1)t} t^{\rho-1} dt$$

Notice this is just a change of variables. Taking $u = (\alpha+1)t$, $du = (\alpha+1)dt$ gives

$$\begin{aligned}\int_0^\infty e^{-(\alpha+1)t} t^{\rho-1} dt &= \int_0^\infty e^{-u} \left(\frac{u}{\alpha+1}\right)^{\rho-1} \frac{1}{\alpha+1} du \\ &= \frac{1}{(\alpha+1)^\rho} \int_0^\infty e^{-u} u^{\rho-1} du \\ &= \frac{1}{(\alpha+1)^\rho} \Gamma(\rho).\end{aligned}$$

Replacing x by $x^{1+\alpha}$ where $\alpha \geq 0$ and applying L'Hosipital's rule as before and utilizing the above equality gives

$$\begin{aligned}\lim_{x\to 1^-} (1-x)^\rho \cdot \sum_{k=0}^\infty a_k x^{\lambda_k \alpha} x^{\lambda_k} &= \frac{s}{(1+\alpha)^\rho} \\ &= \frac{s}{\Gamma(\rho)} \int_0^\infty e^{-(\alpha+1)t} t^{\rho-1} dt \\ &= \frac{s}{\Gamma(\rho)} \int_0^\infty \left(e^{-t}\right)^\alpha e^{-t} t^{\rho-1} dt.\end{aligned}$$

By linearity

$$\lim_{x\to 1^-} (1-x)^\rho \cdot \sum_{k=0}^\infty a_k P(x^{\lambda_k}) x^{\lambda_k} = \frac{s}{\Gamma(\rho)} \int_0^\infty P\left(e^{-t}\right) e^{-t} t^{\rho-1} dt$$

for every polynomial $P(t)$. Now, a few modifications must be made to Karamata's polynomial bounding argument discussed above. If $g(t)$ is bounded and Riemann integrable on $[0,1]$ (in particular it is Lebesgue integrable), then by Hölder's inequality

$$\int_0^1 g(t)^2 \leq ||g||_\infty \cdot \int_0^1 |g(t)| dt < \infty$$

so $g(t)$ is square integrable on $[0,1]$. The change of variables $u = e^{-t}$ gives the following equality of improper integrals:

$$\int_0^\infty g\left(e^{-t}\right) e^{-t} t^{\rho-1} dt = \int_0^1 g(u)(-\log(u))^{\rho-1} du. \tag{A.11}$$

Note the improper integrals are the same as Lebesgue integrals since the integrands are positive (see for example page 84 of [WZ]). Take the polynomials $p(t)$ and $P(t)$

as before. There are now 3 cases: $\rho = 1$, $\rho > 1$, and $\rho \in (0,1)$. For the case $\rho = 1$, from (A.11)

$$\int_0^\infty g\left(e^{-t}\right) e^{-t} t^{\rho-1} dt = \int_0^1 g(u) du < \infty,$$

so this improper integral converges, and

$$\int_0^\infty P\left(e^{-t}\right) e^{-t} t^{\rho-1} dt - \int_0^\infty p\left(e^{-t}\right) e^{-t} t^{\rho-1} dt = \int_0^1 P(u) - p(u) du \leq \epsilon,$$

so the result follows as before.

For the case $\rho > 1$, apply the bound $t^{\rho-1} \leq B e^{\frac{t}{4(\rho-1)}}$ for $t \geq 0$ for some $B > 0$ and the change of variables of (A.11) to obtain

$$\int_0^\infty g\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \leq B \int_0^1 g(u) u^{\frac{-1}{4}} du. \tag{A.12}$$

Since

$$\int_0^1 u^{\frac{-1}{2}} du = \lim_{a \to 0} \left[2u^{\frac{1}{2}}\right]_a^1 = 2$$

and $g(t)$ is bounded, the improper integral in (A.10) converges. Apply (A.12) to $P(t) - p(t)$ and break up the integral as before where F is the open set of small Lebesgue measure used in (A.8) and $F^c = [0,1] \setminus F$. Then, use Cauchy-Schwarz to obtain

$$\begin{aligned}
&\int_0^\infty P\left(e^{-t}\right) e^{-t} t^{\rho-1} dt - \int_0^\infty p\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \\
&\quad \leq B \int_F (P(u) - p(u)) u^{\frac{-1}{4}} du + B \int_{F^c} (P(u) - p(u)) u^{\frac{-1}{4}} du \\
&\quad \leq B \left(\int_F (P(u) - p(u))^2 du \right)^{\frac{1}{2}} \left(\int_F u^{\frac{-1}{2}} du \right)^{\frac{1}{2}} \\
&\qquad + B \left(\int_{F^c} (P(u) - p(u))^2 du \right)^{\frac{1}{2}} \left(\int_{F^c} u^{\frac{-1}{2}} du \right)^{\frac{1}{2}} \\
&\quad \leq \sqrt{2} B \left(\left(\int_F (P(u) - p(u))^2 du \right)^{\frac{1}{2}} + \left(\int_{F^c} (P(u) - p(u))^2 du \right)^{\frac{1}{2}} \right) \\
&\quad \leq \sqrt{2} B \left(\left((2(||g||_\infty + 1))^2 \cdot \frac{\epsilon}{4(||g||_\infty + 1)} \right)^{\frac{1}{2}} + \left(\left(\frac{\epsilon}{2} \right)^2 \cdot 1 \right)^{\frac{1}{2}} \right) \\
&\quad \leq \sqrt{2} B \left(((||g||_\infty + 1)\epsilon)^{\frac{1}{2}} + \frac{\epsilon}{2} \right)
\end{aligned}$$

Since this goes to 0 as $\epsilon \to 0$, the result follows as before.

The final case is $\rho \in (0, 1)$. In this case, from the change of variables (A.11)

$$\int_1^\infty g\left(e^{-t}\right) e^{-t} t^{\rho-1} dt$$

converges, and

$$\int_0^1 g\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \leq ||g||_\infty \int_0^1 t^{\rho-1} dt = \frac{||g||_\infty}{\rho},$$

so the improper integral in (A.10) converges. This time, I will apply the Hölder inequality before changing variables. Notice $\frac{\rho-2}{2(\rho-1)} \in (1, \infty)$.

$$\begin{aligned}
&\int_0^\infty P\left(e^{-t}\right) e^{-t} t^{\rho-1} dt - \int_0^\infty p\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \\
&\quad = \int_0^1 (P-p)\left(e^{-t}\right) e^{-t} t^{\rho-1} dt + \int_1^\infty (P-p)\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \\
&\quad \leq \left(\int_0^1 \left((P-p)\left(e^{-t}\right) e^{-t}\right)^{\frac{\rho-2}{-\rho}} dt\right)^{\frac{-\rho}{\rho-2}} \left(\int_0^1 \left(t^{\rho-1}\right)^{\frac{\rho-2}{2(\rho-1)}} dt\right)^{\frac{2(\rho-1)}{\rho-2}} \\
&\qquad + \int_1^\infty (P-p)\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \\
&\quad = \left(\int_0^1 \left((P-p)\left(e^{-t}\right) e^{-t}\right)^{\frac{\rho-2}{-\rho}} dt\right)^{\frac{-\rho}{\rho-2}} \left(\frac{2}{\rho}\right)^{\frac{2(\rho-1)}{\rho-2}} + \int_1^\infty (P-p)\left(e^{-t}\right) e^{-t} t^{\rho-1} dt \\
&\quad = \left(\frac{2}{\rho}\right)^{\frac{2(\rho-1)}{\rho-2}} \int_{\frac{1}{e}}^1 q(u) du + \int_0^{\frac{1}{e}} (P(u) - p(u)(-\log(u))^{\rho-1} du
\end{aligned}$$

where $q(u) = (P(u) - p(u))^{\frac{\rho-2}{-\rho}} (u)^{\frac{-2\rho-2}{\rho}}$. Breaking up the above integral and bounding the various miscellaneous functions gives

$$= C_1(\rho) \cdot \left(\int_{F \cap \left[\frac{1}{e}, 1\right]} q_2(u) du + \int_{F^c \cap \left[\frac{1}{e}, 1\right]} q_2(u) du \right)^{\frac{-\rho}{\rho-2}} + e^{\rho-1} \cdot \int_0^{\frac{1}{e}} P(u) - p(u) du$$

where

$$C_1(\rho) = \left(\frac{2}{\rho}\right)^{\frac{2(\rho-1)}{\rho-2}} \cdot e^{\frac{-2\rho-2}{\rho-2}}, \qquad q_2(u) = (P(u) - p(u))^{\frac{\rho-2}{-\rho}}.$$

This is

$$\leq C_1(\rho) \left((2(||g||_\infty + 1))^{\frac{\rho-2}{-\rho}} \cdot \frac{\epsilon}{4(||g||_\infty + 1)} + \left(\frac{\epsilon}{2}\right)^{\frac{\rho-2}{-\rho}} \cdot \left(1 - \tfrac{1}{e}\right) \right)^{\frac{-\rho}{\rho-2}} + e^{\rho-1} \cdot \epsilon$$

which goes to 0 as $\epsilon \to 0$, and the result follows for this final case. $\square$

Theorem A.9 can be immediately applied to prove the Tauberian theorem used in the proof of Weyl's law.

Theorem A.13. *Let $\{\lambda_k\}$ be a sequence monotonically increasing to infinity, $a_k \geq -M$, for some $M \geq 0$, and $\rho > 0$. Define $\tilde{N}(\lambda) = \sum_{\lambda_k \leq \lambda} a_k$. If there exists a constant C such that*

$$\sum_{k=0}^{\infty} a_k e^{-\lambda_k t} \sim C t^{-\rho} \text{ as } t \to 0^+,$$

then

$$\tilde{N}(\lambda) \sim \frac{C\lambda^{\rho}}{\Gamma(\rho+1)} \text{ as } \lambda \to \infty.$$

Notice that Theorem III.1 is the special case when $a_k = 1$ for all k.

Proof. Setting $x = e^{-t}$,

$$\begin{aligned}
\lim_{x\to 1^-} (1-x)^{\rho} \cdot \sum_{k=0}^{\infty} a_k x^{\lambda_k} &= \lim_{t\to 0^+} (1-e^{-t})^{\rho} \cdot \sum_{k=0}^{\infty} a_k e^{-\lambda_k t} \\
&= \lim_{t\to 0^+} (t + O(t^2))^{\rho} \cdot \sum_{k=0}^{\infty} a_k e^{-\lambda_k t} \\
&= C.
\end{aligned}$$

Here, I used the fact that $\sim$ is an equivalence relation and $(t+O(t^2))^{-\rho} \sim t^{-\rho}$. The reflexive and symmetric properties of the equivalence relation are obvious, and the transitive property follows from the multiplication of limits. If $f(t) \sim g(t)$ and $g(t) \sim h(t)$ as $t \to 0^+$, then

$$\lim_{t\to 0^+} \frac{f(t)}{h(t)} = \lim_{t\to 0^+} \frac{f(t)}{g(t)} \cdot \frac{g(t)}{h(t)} = \lim_{t\to 0^+} \frac{f(t)}{g(t)} \cdot \lim_{t\to 0^+} \frac{g(t)}{h(t)} = 1 \cdot 1 = 1,$$

so $f(t) \sim h(t)$ as $t \to 0^+$. To see $(t+O(t^2))^{-\rho} \sim t^{-\rho}$, compute

$$\lim_{t\to 0^+} \frac{(t+O(t^2))^{-\rho}}{t^{-\rho}} = \lim_{t\to 0^+} \left(\frac{t+O(t^2)}{t}\right)^{-\rho} = \lim_{t\to 0^+} (1+O(t))^{-\rho} = 1.$$

Additionally, $(1-e^{-t}-t) \in O(t^2)$ as $t \to 0^+$ by L'Hospital's rule:

$$\lim_{t\to 0^+} \frac{1-e^{-t}-t}{t^2} = \lim_{t\to 0^+} \frac{-e^{-t}-1}{2t} = \lim_{t\to 0^+} \frac{e^{-t}}{2} = \frac{1}{2}.$$

Theorem A.9 can now be applied with the bounded integrable function

$$g(T) = \begin{cases} 0 & \text{if } 0 \leq T < e^{-1} \\ T^{-1} & \text{if } e^{-1} \leq T \leq 1 \end{cases}$$

to show

$$\lim_{x \to 1^-} (1-x)^\rho \cdot \sum_{k=0}^{\infty} a_k g(x^{\lambda_k}) x^{\lambda_k} = \frac{C}{\Gamma(\rho)} \int_0^\infty g\left(e^{-T}\right) e^{-T} T^{\rho-1} dT.$$

The right hand side is just

$$\frac{C}{\Gamma(\rho)} \int_0^1 e^T e^{-T} T^{\rho-1} dT = \frac{C}{\Gamma(\rho)} \frac{1}{\rho} = \frac{C}{\Gamma(\rho+1)},$$

and substituting $x = e^{-\frac{1}{\lambda}}$ into the left hand side gives

$$\begin{aligned} \lim_{\lambda \to \infty} \left(\tfrac{1}{\lambda} + O(\tfrac{1}{\lambda^2})\right)^\rho \cdot \sum_{k=0}^{\infty} a_k g\left(e^{-\frac{\lambda_k}{\lambda}}\right) e^{-\frac{\lambda_k}{\lambda}} &= \lim_{\lambda \to \infty} \left(\tfrac{1}{\lambda} + O(\tfrac{1}{\lambda^2})\right)^\rho \cdot \sum_{\lambda_k \leq \lambda} a_k e^{\frac{\lambda_k}{\lambda}} e^{-\frac{\lambda_k}{\lambda}} \\ &= \lim_{\lambda \to \infty} \left(\tfrac{1}{\lambda} + O(\tfrac{1}{\lambda^2})\right)^\rho \cdot \tilde{N}(\lambda) \\ &= \lim_{\lambda \to \infty} \left(\tfrac{1}{\lambda}\right)^\rho \cdot \tilde{N}(\lambda) \end{aligned}$$

so

$$\lim_{\lambda \to \infty} \frac{\tilde{N}(\lambda)}{\lambda^\rho} = \frac{C}{\Gamma(\rho+1)}$$

$\square$

APPENDIX B

Sobolev spaces and inequalities

The following Sobolev inequality of Michael-Simon found on page 874 of [LT] will be the key tool used in proving an upper bound on the eigenfunctions of the Laplacian.

Theorem B.1. *(Michael-Simon) For each N the non-singular locus of an irreducible projective variety of complex dimension $m = \frac{n}{2} > 1$, there exists a constant $C_{Sob} > 0$ such that for all $u \in W^{1,2}(N)$, we have*

$$\left(\int_N |u|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}} \leq C_{Sob}\left(\int_N |\nabla u|^2 + \int_N |u|^2\right).$$

This version for algebraic varieties is derived from the Sobolev inequality given for generalized submanifolds of $\mathbb{R}^d$ given in Theorem 2.1 of [MS] by the remarks on page 874 of [LT]. Additionally, note that since $W_0^{1,2}(N) = W^{1,2}(N)$, the inequality for all of $W^{1,2}(N)$ follows from the inequality for smooth, compactly supported functions.

This inquality should be compared with the Sobolev inequality for compact manifolds of real dimension $n > 2$ found in equation (10.5) on page 98 of [Li] which is given by

$$\left(\int_M |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_{Sob}\left(\int_N |\nabla u|^2\right).$$

In fact, Corollary 9.8 of [Li] states if M is compact of any dimension, for all $\alpha > 0$

(the above case is $\alpha = \frac{n}{n-1}$), there is $C_\alpha > 0$ such that

$$\left(\int_M |u|^{\frac{2\alpha}{2-\alpha}} \right)^{\frac{2-\alpha}{\alpha}} \leq C_\alpha \left(\int_N |\nabla u|^2 \right).$$

Therefore, in the compact case, for $n = 2$, the Sobolev inquality can be taken to be

$$\int_M |u|^2 \leq C_{Sob} \left(\int_N |\nabla u|^2 \right).$$

APPENDIX C

Volume estimate

The following volume estimate is used in [LT] to show various integration by parts formulas such as on pages 867 and 871 of [LT], and it is used throughout this thesis.

Theorem C.1. *Let $V \subset \mathbb{CP}^d$ be an irreducible, projective variety of complex dimension $\frac{n}{2}$, and define $N := V \setminus Sing(V)$ with the Fubini-Study metric from $\mathbb{CP}^d$. Let T_ϵ be the tubular neighborhood of radius ϵ of Sing(V) in N. Then there exists a constant $C_V > 0$ such that*

$$V(T_\epsilon) \leq C_V \epsilon^2.$$

A consequence of this is that N has finite volume (since the compliment of such a tubular neighborhood is compact.)

I am unable to track the origin of the esitimate, but a proof can be constructed from the ideas of the proof of Proposition 4.2 of [Bei]. Following [Bei], there exists a resolution of singularities

$$\pi : \hat{V} \to V$$

such that $\hat{V}$ is smooth and compact, π is holomorphic,surjective, and restricts to a biholmorphism on the compliment of the exceptional divisor $E = \pi^{-1}(\text{Sing}(V))$:

$$\pi|_{\hat{V} \setminus E} : \hat{V} \setminus E \xrightarrow{\sim} N.$$

Further, the exceptional divisor can be taken to be a normal-crossings divisor of complex codimension 1. Therefore, $E = \cup_{j=1}^{k} D_j$ where D_j are smooth, compact, and of codimension ≥ 1 in $\hat{V}$. Taking the composition $\hat{V} \to V \subset \mathbb{CP}^d$, the Fubini-Study metric can be pulled back to a symmetric tensor on $\hat{V}$ which agrees with the Fubini-Study metric of N on the compliment of the excepetional divisor. On the exceptional divisor, this tensor is not positive definite. It is degenerate in the directions along the exceptional divisor. Still, if h is any Riemannian metric on $\hat{V}$, and g is this pull back of the Fubini-Study metric to $\hat{V}$, since the unit tangent bundle on $\hat{V}$ is compact, there is a constant $C > 0$ such that $h \geq C \cdot g$. In particular, $V_h(T_\epsilon) \geq C^{n/2} V_g(T_\epsilon)$. ($T_\epsilon$ is defined in the Fubini-Study metric.) Since $\pi^{-1}(T_\epsilon)$ is contained in the tubular neighborhood of E in $\hat{V}$ of radius $C\epsilon$ in the metric h and this tubular neighborhood in $\hat{V}$ satisfies the desired volume estimate since the volume is bounded by a finite sum of volumes of tubular neighborhoods of compact submanifolds of real codimension ≥ 2, $V(T_\epsilon)$ must also satisfy the desired estimate with an appropriate constant dependent on C and k.

APPENDIX D

The Euclidean Heat Kernel

This appendix is a compilation of facts about the Euclidean heat kernel, which can be found in Section 2.3 of [Ev]. The heat kernel of R^n is given by

$$H_{\mathcal{E}}(x,y,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

This is clearly a smooth, positive and symmetric function, which satisfies the Euclidean heat equation. This kernel function allows the construction of solutions of the heat equation with any continuous and bounded initial value on $\mathbb{R}^n$. Precisely, if $u_0(x) \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the function defined by

$$u(x,t) := \int_{\mathbb{R}^n} H_{\mathcal{E}}(x,y,t) u_0(y) dy$$

is smooth on $R^n \times (0,\infty)$,

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) u(x,t) = 0,$$

and

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = u_0(x_0) \text{ for all } x_0 \in \mathbb{R}^n.$$

The integral of this heat kernel can be explicitely computed using a change of variables, Fubini's theorem, and the famous integral $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$. For each fixed

x and t,

$$\begin{aligned}\int_{\mathbb{R}^n} H_{\mathcal{E}}(x,y,t)dy &= (\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-|z|^2}dz \\ &= (\pi)^{-n/2}\prod_{j=1}^{n}\int_{\mathbb{R}} e^{-|z|^2}dz \\ &= 1.\end{aligned}$$

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