Electrical Networks, Hyperplane Arrangements and Matroids

by

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Dedication

This dissertation is dedicated to my parents.

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Abstract

This thesis introduces a class of hyperplane arrangements, called *Dirichlet arrangements*, arising from electrical networks with Dirichlet boundary conditions. Dirichlet arrangements encode harmonic functions on electrical networks and generalize *graphic arrangements*, a fundamental class of hyperplane arrangements arising from finite graphs.

The first part of the thesis studies the main combinatorial properties of Dirichlet arrangements in detail. We characterize these in ways that directly generalize well-known results on graphic arrangements. Particular attention is paid to the matroids underlying Dirichlet arrangements, called *Dirichlet matroids*. We prove a number of results concerning Dirichlet matroids, including some on the half-plane property, Bergman fans, and duals of circular electrical networks. These results are applied to related objects and problems, including response matrices of electrical networks, order polytopes of finite posets, and graph coloring problems.

The latter part of the thesis studies two specific problems. First, we show that a given Dirichlet arrangement is supersolvable if and only if its Orlik-Solomon algebra is Koszul. This answers an open question in the special case of Dirichlet arrangements. Second, we establish a relationship between structural rigidity of graphs and topological complexity of complements of hyperplane arrangements. The notion of topological complexity originates from the motion planning problem in topological robotics.

Chapter 1

Introduction

The relationship between graphs, matroids and hyperplane arrangements is a driving force in combinatorics. Historically, many theorems on arrangements and their matroids have grown out of results from graph theory. Today, graphs serve as an important test case for difficult questions about arrangements.

The present thesis expands this relationship to graphs with Dirichlet-type boundary conditions, called *electrical networks*. Our aim is twofold. On the one hand, we seek to create the same kind of explicit analogies between electrical networks, matroids and hyperplane arrangements that graphs so productively enjoy. On the other, we wish to bring to light any new behavior that comes with the introduction of boundary conditions.

The thesis is separated into six chapters. In Chapter 2, we provide necessary background material on electrical networks, hyperplane arrangements and matroids.

In Chapter 3, we associate a real hyperplane arrangement, called a *Dirichlet arrangement*, with a given electrical network. Every Dirichlet arrangement is a restriction of a graphic arrangement to an affine subspace defined by the boundary conditions. The main result is that the harmonic functions on the underlying network are critical points of multivalued functions from mathematical physics, defined on the complements of complex Dirichlet arrangements. The critical sets of these functions play an important role in the *Bethe ansatz* of certain quantum integrable systems. In our setting, each critical set consists of the harmonic functions that dissipate a prescribed amount of energy on each edge. In this chapter we also characterize the basic combinatorial invariants of Dirichlet arrangements, including supersolvability. We apply these results to visibility polytopes of order posets, to ψ-*graphical arrangements*, and to the *Precoloring Extension Problem*, which generalizes Latin squares and Sudoku puzzles.

In Chapter 4, we focus on the matroids attached to Dirichlet arrangements, called *Dirichlet matroids*. A Dirichlet matroid is a complete principal truncation of a graphic matroid along a clique. We feature several main results. First, we prove that the real roots of the numerator and denominator of the response matrix of an electrical network interlace along any line with positive direction vector. This follows from the *half-plane property* for Dirichlet matroids. Second, we characterize the Bergman fan of a Dirichlet matroid as a subfan of a graphic Bergman fan. For "complete" networks, this fan can be identified with a space of phylogenetic trees. Third, we prove a Dirichlet analog of the duality theorem for planar graphic matroids. This theorem uses the notion of the dual of a *circular* network. We also prove results on the reduced characteristic polynomials and 3-connectedness of Dirichlet matroids. The latter gives a finite upper bound on the number of distinct networks having the same Dirichlet matroid under certain assumptions.

In Chapter 5, we prove that a Dirichlet arrangement is supersolvable if and only if its Orlik-Solomon algebra is Koszul. It is an open question whether this holds for all central hyperplane arrangements. Previously, the answer was known to be affirmative for four classes of arrangements, including graphic arrangements. We construct an infinite family of electrical networks whose associated Dirichlet arrangements are combinatorially distinct from these previous classes.

In Chapter 6, we connect the notion of *topological complexity* from topological robotics with the notion of structural rigidity for graphs. We show that a graph is rigid in the plane if and only if the associated complex graphic arrangement is *large*, a combinatorial sufficient condition for the arrangement complement to achieve maximum topological complexity. We extend this result to Dirichlet arrangements, using combinatorial moves on graphs from structural rigidity to provide sufficient conditions for a Dirichlet arrangement to be large. We also provide a partial converse to these conditions.

Each of chapters 3–6 can be read independently, although previous chapters might help to establish context. Chapters 3–5, with the exception of Section 4.4, have appeared previously as independent papers [54, 55, 56]. The material of Section 4.4 and Chapter 6 has not appeared previously.

Chapter 2

Background

We review the basic concepts needed to read subsequent chapters. Further details can be found in [23, 69, 87].

2.1 Electrical networks

An *electrical network* is a configuration of wires (or *resistors*), some of which meet at their ends. To each wire we associate a positive real number, called the *conductance*, that measures the ease with which electrical current may pass through the wire. The conductance of a wire depends on its material and dimensions.

A *node* is a site where wires meet, or where a single wire ends and does not meet any others. In the classical setting, two nodes are chosen as *boundary nodes*, and one of them is grounded. A one-volt battery is put across the boundary nodes, and electric current propagates through the network. Relative to the grounded node, we can speak of the *voltage* of any other node in the network, i.e. the difference in electric potential between that node and the grounded node. Thus the grounded node has voltage 0 and the other boundary node has voltage 1. Current flows from nodes of higher voltage to nodes of lower voltage. A fundamental question is: *What are the voltages of the remaining nodes?*

This question can be answered using *Kirchhoff's laws* and *Ohm's law*. Taken together, these laws state that for any interior (i.e., non-boundary) node i we have

$$\sum_{j\sim i} \gamma_{ij}(v_i - v_j) = 0, \tag{2.1}$$

where v_i is the voltage at node i, γ_{ij} is the conductance of the wire between nodes i and j, and $j \sim i$ indicates that j and i share a wire. In words, (2.1) says that the current across any interior node is 0.

More generally, one can consider a network with an arbitrary number of boundary nodes. A voltage is then imparted to each of these nodes, e.g. by multiple batteries. One can also remove the requirement that one of the nodes be grounded. The result is that each boundary node j receives a voltage v_j, and the equation (2.1) still holds for every interior node i. The problem of computing the interior voltages is called the *discrete Dirichlet problem*, in analogy with the classical Dirichlet problem on a continuous domain. The boundary voltages are sometimes called *Dirichlet boundary conditions*. The discrete Dirichlet problem is also called the *forward problem* on an electrical network [23, p. 1].

Problem 2.1.1 (Discrete Dirichlet problem)**.** *To each node i, associate a real number h_i in a way that satisfies the following conditions:*

$$\begin{cases} h_i = u_i & \text{for all boundary nodes } i \\ \sum_{j\sim i} \gamma_{ij}(h_i - h_j) = 0 & \text{for all interior nodes } i. \end{cases} \tag{2.2}$$

The discrete Dirichlet problem can be formulated and solved in terms of graph theory. Let $\Gamma = (V, E)$ be the undirected graph with a vertex for every node of the electrical network, and an edge for every wire. Let $\partial V \subseteq V$ correspond to the set of boundary nodes. We assume that Γ is a connected graph with no loops or multiple edges, and that there are no edges between elements of ∂V. The conductances can be viewed as a function $\gamma : E \to \mathbb{R}$, and the voltages as a function $u : \partial V \to \mathbb{R}$. We write $\gamma \in \mathbb{R}^E$ and $u \in \mathbb{R}^{\partial V}$, thinking of γ and u as vectors indexed by E and ∂V, respectively.

Let L be the matrix with rows and columns indexed by V and entries given by

$$L_{ij} = \begin{cases} \sum_{k \sim i} \gamma_{ik} & \text{if } i = j \\ -\gamma_{ij} & \text{if } i \sim j \\ 0 & \text{else.} \end{cases} \tag{2.3}$$

The matrix L is called the *weighted Laplacian matrix*. We will write it in block form as

$$L = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}, \tag{2.4}$$

where the rows and columns of A are indexed by ∂V.

For every function $f \in \mathbb{R}^V$ we obtain a function $Lf \in \mathbb{R}^V$. Solving the discrete Dirichlet problem is equivalent to finding a function $v \in \mathbb{R}^{V \setminus \partial V}$ such that

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \varphi \\ 0 \end{bmatrix} \tag{2.5}$$

for some $\varphi \in \mathbb{R}^{\partial V}$. The Matrix-Tree Theorem gives

$$\det D = \sum_{T \in \mathcal{T}} \prod_{e \in T} \gamma_e, \tag{2.6}$$

where $\mathcal{T}$ is the set of all spanning trees $T \subseteq E$ of the multigraph obtained from Γ by identifying all boundary nodes as a single vertex. Since the conductances are positive, it follows that D is invertible. Thus there is a unique such $v \in \mathbb{R}^{V \setminus \partial V}$, given by $v = -D^{-1}B^T u$. The function $h \in \mathbb{R}^V$ extending both u and v is called *harmonic*.

While the current across an interior node is necessarily 0 by (2.1), this is not the case for boundary nodes. The function φ in (2.5) gives the currents across the boundary nodes; for all $j \in \partial V$ we have

$$\varphi_j = \sum_{k \sim j} \gamma_{jk}(h_j - h_k), \tag{2.7}$$

where h is the harmonic function defined above. The map $u \mapsto \varphi$ is linear. Namely, we have $\varphi = \Lambda u$, where $\Lambda = A - BD^{-1}B^T$.

Definition 2.1.2. The matrix $\Lambda = A - BD^{-1}B^T$ is called the *response matrix*.

The *inverse problem* on an electrical network is to recover the conductances γ from the response matrix Λ [23, p. 1]. This is an interesting and subtle problem, but we will not address it further.

Once the harmonic function h is established, one can discuss the energy dissipated by each edge. Given an edge $ij \in E$, the *energy* of ij is given by

$$\varepsilon_{ij} = \gamma_{ij}(h_i - h_j)^2. \tag{2.8}$$

More compactly, we can write $\varepsilon = h^T L h$. The total energy dissipated by the network, also called the *Dirichlet norm*, is thus the sum of ε_{ij} over all $ij \in E$. Notice that all edge energies are positive, since all conductances are positive.

The discrete Dirichlet problem can be posed more generally. For example, instead of positive real conductances γ, one can associate a complex number to each edge, called the *admittance*. With this generalization we can describe networks consisting of coils and capacitors, in addition to resistors. The voltages on the interior nodes are then solved for exactly as above. The only difference is that the matrix D in (2.5) is not necessarily invertible. However, it is invertible as long as the admittances are sufficiently generic.

Over time, the term *electrical network* has come to refer to a number of related constructions. By an *electrical network* (or simply a *network*) we will mean one of the following objects, depending on the context: the pair $(\Gamma, \partial V)$, consisting of a graph with boundary; the pair (Γ, u), consisting of a graph with Dirichlet boundary conditions, leaving the boundary implicit; and the triple (Γ, u, γ), consisting of a graph with Dirichlet boundary conditions and edge weights.

Example 2.1.3. Consider the network in Figure 2.1. On the left side of the figure, we represent the network as a graph, where the edges are labeled with their conductances and the boundary nodes are marked in white. On the right, we represent the network as a *circuit diagram*, where the jagged edges are resistors and the symbol at the top denotes a battery across the boundary nodes. This network is called a *Wheatstone bridge*, after the work of C. Wheatstone [104].

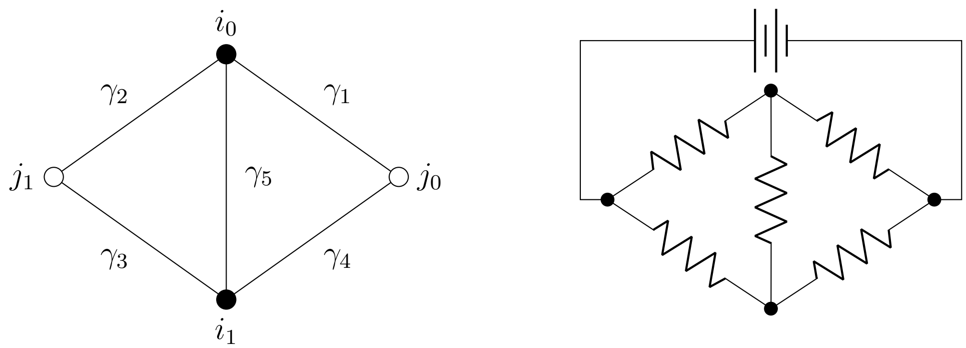


Figure 2.1: A Wheatstone bridge represented two ways.

Set boundary voltages $u(j_0) = 0$ and $u(j_1) = 1$. With respect to the vertex ordering j_0, j_1, i_0, i_1, the weighted Laplacian matrix is

$$L = \begin{bmatrix} \gamma_1+\gamma_4 & 0 & -\gamma_1 & -\gamma_4 \\ 0 & \gamma_2+\gamma_3 & -\gamma_2 & -\gamma_3 \\ -\gamma_1 & -\gamma_2 & \gamma_1+\gamma_2+\gamma_5 & -\gamma_5 \\ -\gamma_4 & -\gamma_3 & -\gamma_5 & \gamma_3+\gamma_4+\gamma_5 \end{bmatrix}.$$

Thus in the notation of (2.5) we have

$$A = \begin{bmatrix} \gamma_1+\gamma_4 & 0 \\ 0 & \gamma_2+\gamma_3 \end{bmatrix}, \quad B = \begin{bmatrix} -\gamma_1 & -\gamma_4 \\ -\gamma_2 & -\gamma_3 \end{bmatrix}, \quad D = \begin{bmatrix} \gamma_1+\gamma_2+\gamma_5 & -\gamma_5 \\ -\gamma_5 & \gamma_3+\gamma_4+\gamma_5 \end{bmatrix}.$$

Note that D is invertible if and only if

$$\det D = (\gamma_1+\gamma_2+\gamma_5)(\gamma_3+\gamma_4+\gamma_5) - \gamma_5^2$$

is nonzero. If D is invertible, then there is a unique harmonic function h given by

$h(j_0) = 0$, $h(j_1) = 1$,

$$h(i_0) = \frac{\gamma_1(\gamma_3 + \gamma_4 + \gamma_5) + \gamma_4\gamma_5}{\det D}, \quad h(i_1) = \frac{\gamma_4(\gamma_1 + \gamma_2 + \gamma_5) + \gamma_1\gamma_5}{\det D}.$$

For instance, if γ is a constant function, then $h(i_0) = h(i_1) = \frac{1}{2}$. The response matrix is given by

$$\Lambda = \frac{\alpha}{\det D} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where $\alpha = \gamma_1\gamma_2\gamma_3 + \gamma_1\gamma_2\gamma_4 + \gamma_1\gamma_2\gamma_5 + \gamma_1\gamma_3\gamma_4 + \gamma_1\gamma_3\gamma_5 + \gamma_2\gamma_3\gamma_4 + \gamma_3\gamma_4\gamma_5$. The energies can also be written as rational functions in the conductances. For example, writing ε_i for the energy of the edge with conductance γ_i, we have

$$\varepsilon_5 = \gamma_5 \left(\frac{\gamma_1\gamma_3 - \gamma_2\gamma_4}{\det D} \right)^2.$$

If γ is a constant function, then $\varepsilon_5 = 0$ and the energy of every other edge is $\frac{\gamma}{4}$. Thus the total energy dissipated is equal to γ in this case.

Example 2.1.4. Let Γ be a path graph. Let ∂V consist of both ends of the path. This graph is illustrated in Figure 2.2 with edges labeled by their conductances and boundary nodes marked by white circles.

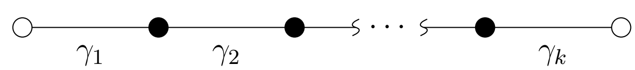


Figure 2.2: A path graph with edge weights labeled and boundary nodes marked in white.

Here the matrices L and D are symmetric and tridiagonal. For instance, when $d = 6$ we have

$$D = \begin{pmatrix} \gamma_1 + \gamma_2 & -\gamma_2 & & \\ -\gamma_2 & \gamma_2 + \gamma_3 & -\gamma_3 & \\ & -\gamma_3 & \gamma_3 + \gamma_4 & -\gamma_4 \\ & & -\gamma_4 & \gamma_4 + \gamma_5 \end{pmatrix}.$$

When all conductances are 1, we have

$$D = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

This matrix arises as the Cartan matrix of the root system A_n, and as the matrix of coupling coefficients of d harmonic oscillators in a linear chain (see, e.g., [44, §11.4] and [57, Exercise 4.2]). It also plays a role in other boundary value problems on path graphs [7, 15].

For general conductances, equation (2.6) gives

$$\det D = \sum_{i=1}^{d-1} \prod_{j \neq i} \gamma_j.$$

Set boundary voltages 0 and 1 for the left and right boundary nodes, respectively. Assuming that D is invertible, the values of the harmonic function can be computed using the recurrences in [95] for the inverse of a symmetric tridiagonal matrix. The formula for the energy of an edge is particularly simple, assuming that the conductances are nonzero. Writing ε_i for the energy of the edge with conductance γ_i, we have

$$\varepsilon_i = \left(\sum_{j=1}^{d-1} \frac{\gamma_i}{\gamma_j} \right)^{-2}$$

for all $i = 1, \dots, d-1$.

2.2 Hyperplane arrangements

Let $\mathbb{K}$ be a field and d a positive integer. A *hyperplane arrangement* (or simply an *arrangement*) in $\mathbb{K}^d$ is a finite set of affine hyperplanes of $\mathbb{K}^d$. We consider each arrangement $\mathcal{A}$ in $\mathbb{K}^d$ to be equipped with a set $\{f_H : H \in \mathcal{A}\}$ of affine functionals $f_H : \mathbb{K}^d \to \mathbb{K}$ such that $H = f_H^{-1}(0)$ for all $H \in \mathcal{A}$. The f_H are called *defining functions* of $\mathcal{A}$. Let $Q(\mathcal{A})$ denote the product of the functions f_H:

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H(x). \tag{2.9}$$

Note that the polynomial $Q(\mathcal{A})$ determines $\mathcal{A}$.

We write

$$T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H \tag{2.10}$$

if $\mathcal{A}$ is nonempty, and $T(\emptyset) = \mathbb{K}^d$. The arrangement $\mathcal{A}$ is *central* if $T(\mathcal{A})$ is nonempty and *essential* if the normal vectors of the hyperplanes in $\mathcal{A}$ span $\mathbb{K}^d$. We also write

$$U(\mathcal{A}) = \mathbb{K}^d \setminus \bigcup_{H \in \mathcal{A}} H. \tag{2.11}$$

The set $U(\mathcal{A})$ is called the *complement* of $\mathcal{A}$. When $\mathbb{K} = \mathbb{R}$, the connected components of $U(\mathcal{A})$ are called the *chambers* of $\mathcal{A}$.

The *intersection poset* is one of the main combinatorial objects associated to an arrangement. Namely, it is the set $L(\mathcal{A})$ of nonempty intersections of elements of $\mathcal{A}$, ordered by reverse inclusion. Thus $X \leq Y$ in $L(\mathcal{A})$ means $X \supseteq Y$. One can also assign a rank function to $L(\mathcal{A})$ by taking the rank of any element to be its codimension in $\mathbb{K}^d$. In general, $L(\mathcal{A})$ is a *meet-semilattice*; that is, every pair of elements of $L(\mathcal{A})$ has an infimum (called the *meet*). If $\mathcal{A}$ is central, then $L(\mathcal{A})$ is a *lattice*; every pair of elements has both an infimum and a supremum (called the *join*).

Given $X, Y \in L(\mathcal{A})$, the *closed interval* $[X, Y]$ is the set $[X, Y] = \{Z \in L(\mathcal{A}) : X \leq Z \leq Y\}$. Define an integer-valued function μ on the set of closed intervals

$[X, Y]$ of $L(\mathcal{A})$ by

$$\begin{aligned} &\mu(X, X) = 1 \text{ for all } X \in L(\mathcal{A}) \\ &\mu(X, Y) = -\sum_{X \leq Z < Y} \mu(X, Z) \text{ for all } X < Y \text{ in } L(\mathcal{A}). \end{aligned} \tag{2.12}$$

This is the *Möbius function* of the poset $L(\mathcal{A})$. The *characteristic polynomial* of $\mathcal{A}$ is the polynomial $\chi_{\mathcal{A}}$ with integer coefficients given by

$$\chi_{\mathcal{A}}(t) = \sum_{X \in L(\mathcal{A})} \mu(\hat{0}, X) t^{\dim(X)}, \tag{2.13}$$

where $\hat{0}$ is the minimal element of $L(\mathcal{A})$.

In case an arrangement is not central or essential, there is a "centralized" and "essentialized" version with very similar combinatorics. The *cone* $c\mathcal{A}$ over $\mathcal{A}$ is the central arrangement in $\mathbb{K}^{d+1}$ defined by

$$Q(c\mathcal{A}) = x_0 \prod_{H \in \mathcal{A}} f_H^h(x_0, \ldots, x_d), \tag{2.14}$$

where f_H^h is obtained from f_H by multiplying the constant term (possibly 0) in the formula for f_H by the new variable x_0. For example, if $f_H(x_1, x_2) = 5 - x_1 + 3x_2$, then $f_H^h(x_0, x_1, x_2) = 5x_0 - x_1 + 3x_2$.

The *essentialization* $\text{ess}(\mathcal{A})$ of $\mathcal{A}$ is the restriction of $\mathcal{A}$ to the subspace Y spanned by the normal vectors of all hyperplanes in $\mathcal{A}$, or equivalently the restriction of $\mathcal{A}$ to any translate of Y. Clearly $\text{ess}(\mathcal{A})$ is essential, and $L(\text{ess}(\mathcal{A})) \cong L(\mathcal{A})$.

Example 2.2.1. Consider the arrangement $\mathcal{A}$ consisting of all coordinate hyperplanes in $\mathbb{K}^d$. We have $Q(\mathcal{A}) = x_1 x_2 \cdots x_d$. Note that $T(\mathcal{A})$ consists of only the origin. Hence $\mathcal{A}$ is central and essential. The lattice $L(\mathcal{A})$ is isomorphic to the lattice of subsets of $\{1, 2, \ldots, n\}$, ordered by inclusion. It is not too hard to show that $\mu(\hat{0}, X) = (-1)^k$, where k is the rank of X in $L(\mathcal{A})$. From here it follows that $\chi_{\mathcal{A}}(t) = (t-1)^d$. If $\mathbb{K} = \mathbb{R}$, then the chambers of $\mathcal{A}$ are the (open) orthants of $\mathbb{R}^d$, of which there are 2^d.

Example 2.2.2. The *braid arrangement* $\mathcal{B}_d$ in $\mathbb{K}^d$ is defined by

$$Q(\mathcal{B}_d) = \prod_{1 \le i < j \le d} (x_i - x_j).$$

Thus $\mathcal{B}_n$ consists of $\binom{d}{2}$ hyperplanes. Note that $T(\mathcal{B}_d)$ is the set of points in $\mathbb{K}^d$ whose coordinates are all equal. Thus $\mathcal{B}_d$ is central, but not essential, since $\dim T(\mathcal{B}_d) = 1$. If $\mathbb{K} = \mathbb{R}$, then $U(\mathcal{B}_d)$ consists of $d!$ chambers. Indeed, every chamber corresponds to a linear ordering of the coordinates x_i for $1 \le i \le d$. It can be shown that $\chi_{\mathcal{B}_d}(t) = t(t-1)(t-2)\cdots(t-d+1)$.

2.2.1 Supersolvable arrangements

The characteristic polynomials in Examples 2.2.1 and 2.2.2 have all integer roots. This is not usually the case for a given arrangement. However, the roots of a *supersolvable* arrangement are always nonnegative integers. The supersolvable arrangements form a particularly nice class of arrangements with a number of desirable combinatorial, topological and geometric properties.

Suppose that the arrangement $\mathcal{A}$ is central, so that $L(\mathcal{A})$ is a lattice. Given $X, Y \in L(\mathcal{A})$, let $X \vee Y$ denote the join of X and Y in $L(\mathcal{A})$, and let $X \wedge Y$ denote their meet. Let $\operatorname{rk}(X)$ denote the rank of X. An element $X \in L(\mathcal{A})$ is *modular* in $L(\mathcal{A})$ if

$$\operatorname{rk}(X) + \operatorname{rk}(Y) = \operatorname{rk}(X \vee Y) + \operatorname{rk}(X \wedge Y) \tag{2.15}$$

for all $Y \in L(\mathcal{A})$. Equivalently, X is modular in $L(\mathcal{A})$ if for all $Y \in L(\mathcal{A})$ the Minkowski sum

$$X + Y = \{x + y \in \mathbb{K}^d : x \in X \text{ and } y \in Y\} \tag{2.16}$$

belongs to $L(\mathcal{A})$.

Definition 2.2.3. A central arrangement $\mathcal{A}$ is *supersolvable* if the intersection lattice $L(\mathcal{A})$ admits a maximal chain consisting of modular elements of $L(\mathcal{A})$. A non-central arrangement $\mathcal{A}$ is *supersolvable* if the cone $c\mathcal{A}$ over $\mathcal{A}$ is supersolvable.

Example 2.2.4. Consider the arrangement $\mathcal{A}$ from Example 2.2.1. Every $X \in L(\mathcal{A})$ corresponds to a set $S_X \subseteq \{1, \dots d\}$ such that X is the intersection of the hyperplanes $x_i = 0$ for all $i \in S_X$. We have $\mathrm{rk}(X) = n - |S_X|$, where $|S_X|$ is the cardinality of S_X. For any $X, Y \in L(\mathcal{A})$ we have $S_{X \vee Y} = S_X \cap S_Y$ and $S_{X \wedge Y} = S_X \cup S_Y$. Hence (2.15) follows from the inclusion-exclusion principle, proving that $\mathcal{A}$ is supersolvable. Alternatively, notice that $X + Y = X \wedge Y \in L(\mathcal{A})$.

2.2.2 Graphic arrangements

An important class of arrangements comes from graphs. Let $\Gamma = (V, E)$ be a finite connected simple graph on d vertices.

Definition 2.2.5. The *graphic arrangement* $\mathcal{A}(\Gamma)$ is the arrangement in $\mathbb{K}^d$ defined by

$$Q(\mathcal{A}(\Gamma)) = \prod_{ij \in E} (x_i - x_j), \tag{2.17}$$

where the coordinates of $\mathbb{K}^d$ are indexed by V.

Graphic arrangements are central, but not essential, since $\dim T(\mathcal{A}(\Gamma)) = 1$. It is possible to translate combinatorial and topological data from $\mathcal{A}(\Gamma)$, often intractable for general arrangements, into elementary graph-theoretic terms.

A *connected partition* of Γ is a partition π of V such that every set in π induces a connected subgraph of Γ. The connected partitions of Γ form a lattice Π_Γ in which $\pi \leq \rho$ if and only if π is a *refinement* of ρ, i.e. every set in π is a subset of a set in ρ. It is not immediately obvious that this defines a lattice. The join $\pi \vee \rho$ is the smallest partition of V of which π and ρ are both refinements. However, the meet $\pi \wedge \rho$ is not necessarily the common refinement of π and ρ, since the common refinement is not necessarily a connected partition. To obtain $\pi \wedge \rho$, we must further refine the common refinement of π and ρ into a connected partition. The intersection lattice $L(\mathcal{A}(\Gamma))$ is isomorphic to Π_Γ.

Let k be a positive integer. A *proper k-coloring* of Γ is a function $p : V \to \{1, \dots, k\}$ such that $p(i) \neq p(j)$ whenever $i \sim j$. The *chromatic polynomial* of Γ is a polynomial χ_Γ with integer coefficients such that for any k, $\chi_\Gamma(k)$ is the number of distinct proper k-colorings of Γ. It turns out that this is also the characteristic polynomial of $\mathcal{A}(\Gamma)$. That is, $\chi_{\mathcal{A}(\Gamma)} = \chi_\Gamma$.

An orientation of Γ is called *acyclic* if it contains no directed cycles. When $\mathbb{K} = \mathbb{R}$, the number of chambers of $\mathcal{A}(\Gamma)$ is the number of acyclic orientations of Γ.

Recall that the graph Γ is *chordal* (or *triangulated*) if no cycle of length greater than 3 is an induced subgraph of Γ.

Proposition 2.2.6 ([85, Proposition 2.8])**.** *The graphic arrangement $\mathcal{A}(\Gamma)$ is supersolvable if and only if Γ is chordal.*

Example 2.2.7. If $\Gamma = K_d$ is the complete graph on d vertices, then the graphic arrangement $\mathcal{A}(\Gamma)$ is the braid arrangement $\mathcal{B}_d$. Every partition of V is a connected partition of Γ, so $L(\mathcal{B}_d)$ is isomorphic to the lattice of partitions of $\{1, \ldots, d\}$, ordered by refinement.

Example 2.2.8. Let Γ be a d-cycle for $d \geq 3$. We have

$$Q(\mathcal{A}(\Gamma)) = (x_1 - x_d) \prod_{i=1}^{d-1} (x_i - x_{i+1}).$$

It can be shown that $\chi_\Gamma(t) = (-1)^d(t-1) + (t-1)^d$. All but 2 orientations of Γ are acyclic, so if $\mathbb{K} = \mathbb{R}$, then the number of chambers of $\mathcal{A}(\Gamma)$ is $2^d - 2$. If $d = 3$, then Γ contains no cycles of length greater than 3, so Γ is chordal. If $d \geq 4$, then Γ itself is a cycle of length greater than 3, so Γ is not chordal. Hence $\mathcal{A}(\Gamma)$ is supersolvable if and only if $d = 3$.

2.3 Matroids

One way to study the intersection lattice of a central arrangement is to study the associated *matroid*. Let E be a finite set. A *matroid* on E is a pair $M = (E, \mathcal{I})$, where $\mathcal{I}$ is a set of subsets of E satisfying

(i) $\emptyset \in \mathcal{I}$

(ii) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$

(iii) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there is $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

The set E is called the *ground set* of M. The elements of $\mathcal{I}$ are called *independent sets* of M (or simply *independent sets*). The subsets of E not belonging to $\mathcal{I}$ are called *dependent sets*. A matroid is *simple* if the cardinality of every dependent set is at least 3.

A *basis* of M is an independent set that is maximal (with respect to containment). A *circuit* of M is a maximal dependent set. Let $X \subseteq E$. The *rank* $\mathrm{rk}_M(X)$ of X is the maximal size of an independent set contained in X. The set X is a *flat* of M if $X = E$ or if $\mathrm{rk}_M(X \cup e) > \mathrm{rk}_M(X)$ for all $e \in E \setminus X$. The flats of M form a lattice $\mathcal{L}(M)$ ordered by inclusion. The lattice $\mathcal{L}(M)$ is called the *lattice of flats* of M.

Example 2.3.1. For positive integers $m \leq n$, one can define a matroid $U_{m,n}$ on an n-element set by taking the bases to be the m-element subsets. The matroids $U_{m,n}$ are called *uniform*. The circuits of $U_{m,n}$ are the $(m+1)$-element subsets. The rank of a subset S is given by $\min(|S|, m)$. Thus the proper flats of $U_{m,n}$ are the subsets with fewer than m elements. The matroid $U_{m,n}$ is simple if and only if $m \geq 2$.

A matroid is equivalently determined by its set of bases, by its set of circuits, by its rank function rk_M, and by its set of flats. The set $\{E \setminus B : B \text{ is a basis of } M\}$ is the set of bases of a matroid M^* called the *dual* of M. The circuits of M^* are called *cocircuits* of M. A matroid is also determined by its set of cocircuits.

If $X \subseteq E$, then there is a matroid $M|X$ whose independent sets are the elements of $\mathcal{I}$ contained in X. This matroid is called the *restriction* of M to X. For any $e \in E$ there is a matroid M/e on $E \setminus e$ whose independent sets are the sets X such that $X \cup e \in \mathcal{I}$. This matroid is the *contraction* of M by e. For any $Y \subseteq E$ we define a matroid M/Y on $E \setminus Y$ by contracting M successively by each element of Y in any order. This matroid is the *contraction* of M by Y. A *minor* of M is any matroid obtainable from M by a series of restrictions and contractions.

Any central hyperplane arrangement $\mathcal{A}$ defines a matroid $M(\mathcal{A})$ on $\mathcal{A}$, where a subset of $\mathcal{A}$ is independent if and only if the set of corresponding set of normal vectors is linearly independent. A matroid M is *representable* over $\mathbb{K}$ if $M = M(\mathcal{A})$

for some arrangement $\mathcal{A}$ in $\mathbb{K}^d$. A matroid is *binary* if it is representable over $\mathbb{F}_2$, and *regular* if it is representable over every field. The *characteristic polynomial* of $M(\mathcal{A})$ is the polynomial

$$\chi_{M(\mathcal{A})}(t) = t^{\mathrm{rk}(\mathcal{A})-d}\chi_{\mathcal{A}}(t), \tag{2.18}$$

where rk is the rank function for $\mathcal{A}$. Moreover we have $\mathcal{L}(M) \cong L(\mathcal{A})$; that is, the lattice of flats of $M(\mathcal{A})$ is isomorphic to the intersection poset of $\mathcal{A}$.

2.3.1 Graphic matroids

A graph $\Gamma = (V, E)$ defines a matroid $M(\Gamma)$ on E whose bases (resp., circuits) are the spanning trees (resp., the cycles) of Γ. A matroid M is *graphic* if $M = M(\Gamma)$ for some Γ. Graphic matroids are regular. The lattice of flats $\mathcal{L}(M(\Gamma))$ is isomorphic to the lattice Π_Γ of connected partitions of Γ. The characteristic polynomial of $M(\Gamma)$ is $t^{-1}\chi_\Gamma(t)$, where χ_Γ is the chromatic polynomial of Γ.

Example 2.3.2. Let $\Gamma = K_d$. There are d^{d-2} bases of $M(\Gamma)$, by Cayley's formula for the number of spanning trees of a complete graph. The lattice of flats is isomorphic to the lattice of partitions of $\{1, \ldots, d\}$ by Example 2.2.7. The characteristic polynomial is $\chi_{M(\Gamma)} = (t-1)(t-2)\cdots(t-d+1)$.

Example 2.3.3. Let Γ be a cycle on $d \geq 3$ vertices. The only circuit of $M(\Gamma)$ is E. Every other subset of E is independent. Thus the bases are the sets of cardinality $d-1$. It follows that $M(\Gamma)$ is the uniform matroid $U_{d-1,d}$.

Example 2.3.4. Let Γ be a tree. The only basis of $M(\Gamma)$ is E. There are no dependent sets. Thus $M(\Gamma)$ is the uniform matroid $U_{d,d}$, called a *free* matroid.

2.3.2 Complete principal truncations

Given a flat F of M, one can define a matroid $T_F(M)$ on E with rank function

$$\mathrm{rk}_{T_F(M)}(X) = \begin{cases} \mathrm{rk}_M(X) - 1 & \text{if } \mathrm{rk}_M(X) = \mathrm{rk}_M(X \cup F) \\ \mathrm{rk}_M(X) & \text{else.} \end{cases} \tag{2.19}$$

The set F is again a flat of $T_F(M)$, so this construction can be iterated i times to obtain $T_F^i(M)$. The *complete principal truncation* of M along F is the matroid $\overline{T}_F(M) = T_F^i(M)$, where $i = \mathrm{rk}_M(F) - 1$. Geometrically, $\overline{T}_F(M)$ is obtained by freely adding a $(\mathrm{rk}_M(F) - 1)$-element set S to F and contracting S.

Chapter 3

Electrical networks and hyperplane arrangements

This chapter introduces *Dirichlet arrangements*, a generalization of graphic arrangements arising from electrical networks. We describe the main combinatorial features of Dirichlet arrangements, including supersolvability, in ways that directly generalize theorems on graphic arrangements. Dirichlet arrangements have been studied previously as *ψ-graphical arrangements*, with an equivalent but materially different definition. The definition of Dirichlet arrangements leads to clearer parallels between these arrangements and graphic arrangements. We obtain applications to electric networks with prescribed edge energies, and to *order polytopes* of finite posets.

3.1 Main definition and examples

By a *graph* we will mean one that is finite, connected and undirected with no loops or multiple edges. Let $\Gamma = (V, E)$ denote a graph on d vertices and k edges. Recall that the *graphic arrangement* $\mathcal{A}(\Gamma)$ of Γ over a field $\mathbb{K}$ is the arrangement in $\mathbb{K}^d$ defined by

$$Q(\mathcal{A}(\Gamma)) = \prod_{ij \in E} (x_i - x_j).$$

Fix a set $\partial V \subsetneq V$ of ≥ 2 vertices, no two of which are adjacent, and an injective function $u : \partial V \to \mathbb{K}$. We call ∂V the *boundary* of Γ and $V^\circ = V \setminus \partial V$ the *interior* of Γ. We call u the *boundary data* and the scalars $u(j) \in \mathbb{K}$ the *boundary values*. The elements of ∂V are called *boundary nodes*. Write $m = |\partial V|$ and $n = |V^\circ|$, so $d = m + n$. Whenever the vector spaces $\mathbb{K}^d$ and $\mathbb{K}^n$ appear, we consider their coordinates to be indexed by V and V°, respectively.

Definition 3.1.1. Let Γ be a graph as above, with boundary ∂V and boundary data $u : \partial V \to \mathbb{K}$. Let $\mathcal{X}$ be the affine subspace of $\mathbb{K}^d$ given by

$$\mathcal{X} = \{x \in \mathbb{K}^d : x_j = u(j) \text{ for all } j \in \partial V\} \tag{3.1}$$

Let $\overline{\mathcal{A}}(\Gamma, u)$ denote the arrangement in the space $\mathcal{X} \cong \mathbb{K}^n$ of hyperplanes $H \cap \mathcal{X}$ for all $H \in \mathcal{A}(\Gamma)$. An arrangement $\mathcal{A}$ is *Dirichlet* if $\mathcal{A} = \overline{\mathcal{A}}(\Gamma, u)$ for some (Γ, u).

This definition can be modified to accommodate repeated boundary values and edges between boundary nodes. In case of repeated boundary values, one can identify all vertices on which u takes the same value, removing any duplicate edges. In case ∂V is not an independent set, one can simply remove all edges between boundary nodes, assuming that the resulting graph is connected.

Recall that $m = |\partial V|$. If $m = 0$, then $\overline{\mathcal{A}}(\Gamma, u) = \mathcal{A}(\Gamma)$ is a graphic arrangement. If $m = 1$, then $\overline{\mathcal{A}}(\Gamma, u)$ is the essentialization of $\mathcal{A}(\Gamma)$. We assume that $m \geq 2$ to distinguish from these cases. When $m \geq 2$, the arrangement $\overline{\mathcal{A}}(\Gamma, u)$ is not central. We will occasionally prefer to work with the centralized version, e.g. the cone over $\overline{\mathcal{A}}(\Gamma, u)$.

Definition 3.1.2. Let $\mathcal{A}(\Gamma, u)$ denote the cone over the Dirichlet arrangement $\overline{\mathcal{A}}(\Gamma, u)$.

For the remainder of the chapter we will assume that $\mathbb{K} = \mathbb{R}$. This is a natural base field for Dirichlet arrangements, since we will typically think of the boundary data $u : \partial V \to \mathbb{K}$ as voltages, which are real numbers. We will see in Corollary 3.3.3 that the intersection poset of $\overline{\mathcal{A}}(\Gamma, u)$ is independent of u, and hence independent of $\mathbb{K}$.

We think of $\overline{\mathcal{A}}(\Gamma, u)$ as an n-dimensional affine slice of $\mathcal{A}(\Gamma)$, where for all $i \in V^\circ$ the coordinates x_i of $\mathcal{X} \cong \mathbb{R}^n$ are inherited from $\mathbb{R}^d$, and for all $j \in \partial V$ the coordinate x_j is specialized to the boundary value $u(j)$.

Example 3.1.3 (Wheatstone bridge)**.** Consider the graph Γ on the left side of Figure 3.1, where the boundary nodes j_1 and j_2 are marked by white circles. This is the Wheatstone bridge from Example 2.1.3.

Fix boundary values $u(j_0) = 0$ and $u(j_1) = 1$. This corresponds to placing a 1-volt battery between the boundary nodes. Writing $V^\circ = \{i_0, i_1\}$, the Dirichlet arrangement $\overline{\mathcal{A}}(\Gamma, u)$ is defined by

$$Q(\overline{\mathcal{A}}(\Gamma, u)) = (x_{i_0}^2 - 1)(x_{i_1}^2 - 1)(x_{i_0} - x_{i_1}).$$

The bounded chambers of $\overline{\mathcal{A}}(\Gamma, u)$ are the open triangles shaded on the right-hand side of Figure 3.1.

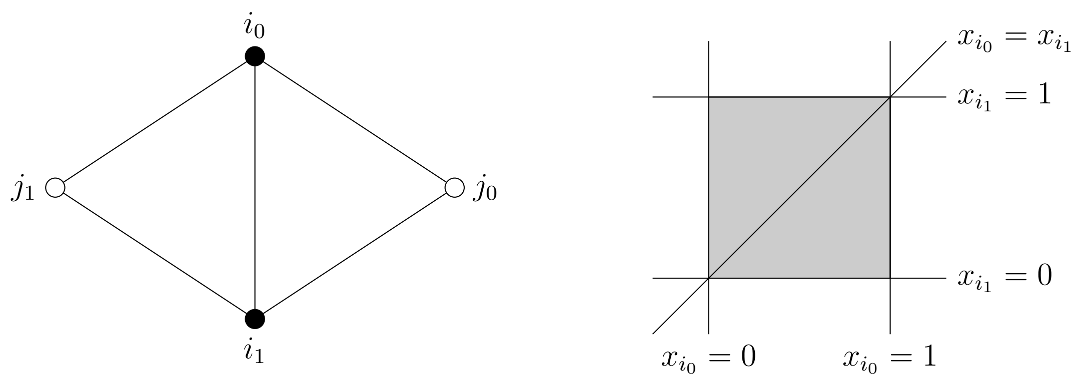


Figure 3.1: A Wheatstone bridge and a corresponding Dirichlet arrangement.

Example 3.1.4 (Discriminantal arrangements)**.** Suppose that V° is a clique, and that every vertex in ∂V is adjacent to every vertex in V°. We denote this graph with specified boundary by

$$\Gamma_{m,n} = (\Gamma, \partial V). \tag{3.2}$$

For instance, the Wheatstone bridge in Example 3.1.3 is $\Gamma_{2,2}$. The case $\Gamma_{5,4}$ is illustrated in Figure 3.2 with boundary nodes marked by white circles. If $(\Gamma, \partial V) =$

$\Gamma_{m,n}$ for some m and n, then $\overline{\mathcal{A}}(\Gamma, u)$ is called a *discriminantal arrangement* [77]. Discriminantal arrangements are studied for their connections to Lie algebras and mathematical physics. Every Dirichlet arrangement is a subset of a discriminantal arrangement.

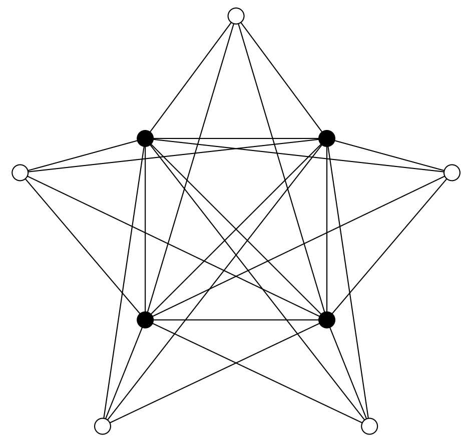


Figure 3.2: The network $\Gamma_{5,4}$ with boundary nodes marked in white.

Example 3.1.5 (Visibility arrangements of order polytopes)**.** Let $\mathcal{P}$ be a convex d-dimensional polytope in $\mathbb{R}^d$. The d-dimensional faces of P are called *facets*. The *visibility arrangement* vis($\mathcal{P}$) of $\mathcal{P}$, defined by Stanley [89], is the arrangement in $\mathbb{R}^d$ whose elements are the affine spans of all facets of $\mathcal{P}$. It is so named because the chambers of vis($\mathcal{P}$) correspond to the sets of facets of $\mathcal{P}$ visible from different points in $\mathbb{R}^P$. The unbounded chambers of vis($\mathcal{P}$) correspond to the sets of facets visible from arbitrarily far away. For example, if $\mathcal{P}$ is a regular hexagon in $\mathbb{R}^2$, then the visbility arrangement vis($\mathcal{P}$) is illustrated in Figure 3.3. There are 7 bounded chambers, including the interior of the hexagon, and 12 unbounded chambers.

We are interested in a certain polytope associated to a finite poset P. The *order polytope* $\mathcal{O}(P)$ of P is the set of all order-preserving functions $P \to [0, 1]$. Clearly $\mathcal{O}(P)$ is a convex polytope in $\mathbb{R}^P$. A *facet* of a polytope is a top-dimensional face. The *visibility arrangement* vis($\mathcal{O}(P)$) of $\mathcal{O}(P)$, defined by Stanley [89], is the arrangement in $\mathbb{R}^P$ whose elements are the affine spans of all facets of $\mathcal{O}(P)$. It is so named because the chambers of vis($\mathcal{O}(P)$) correspond to the sets of facets of

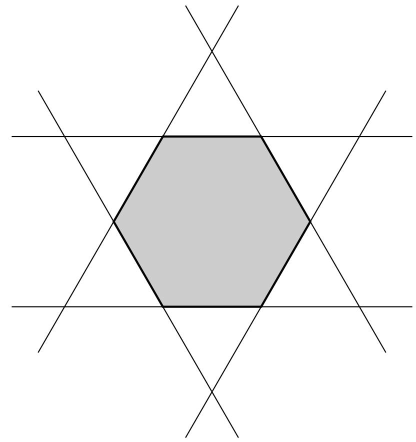


Figure 3.3: A regular hexagon in $\mathbb{R}^2$, shaded, and the associated visibility arrangement.

$\mathcal{O}(P)$ visible from different points in $\mathbb{R}^P$. The unbounded chambers of $\operatorname{vis}(\mathcal{O}(P))$ correspond to the sets of facets visible from arbitrarily far away.

Recall that the *Hasse diagram* of P is the graph with a vertex for every element of P and an edge ij whenever $i \leq j$ in P. Let H be the Hasse diagram of P. Let Γ be the connected graph obtained by adding 2 vertices j_0 and j_1 to H, with $j_0 \sim i$ if i is minimal in P and $j_1 \sim i$ if i is maximal in P. Let $\partial V = \{j_0, j_1\}$, and let $u : \partial V \to \mathbb{R}$ be given by $u(j_0) = 0$ and $u(j_1) = 1$. Then $\overline{\mathcal{A}}(\Gamma, u) = \operatorname{vis}(\mathcal{O}(P))$ (see [89, Theorem 4]).

Example 3.1.6 (Linear order polytope)**.** Let $P = \{1, \ldots, \ell\}$ with the usual linear ordering. The weakly increasing maps $P \to [0, 1]$ correspond to points $x \in \mathbb{R}^\ell$ with $0 \leq x_1 \leq \cdots \leq x_\ell \leq 1$. Thus the order polytope $\mathcal{O}(P)$ is an ℓ-simplex in $\mathbb{R}^\ell$. Every proper subset of the $\ell + 1$ facets of $\mathcal{O}(P)$ is a visibility set; by Corollary 3.2.3 we must have $\frac{1}{2}\alpha(\widehat{\Gamma}) = 2^{\ell+1} - 1$, where $\widehat{\Gamma}$ (as defined in Example 3.1.5) is a cycle graph on $\ell + 2$ vertices. The empty set is only visible from the interior of $\mathcal{O}(P)$, and is the only set not visible from far away.

3.2 Main results

The Dirichlet arrangement $\overline{\mathcal{A}}(\Gamma, u)$ is not a restriction of $\mathcal{A}(\Gamma)$ in the traditional sense, since (3.1) is not an intersection of elements of $\mathcal{A}(\Gamma)$. However, the following theorem shows that $\overline{\mathcal{A}}(\Gamma, u)$ preserves a good deal of graphic structure. An *order ideal* I of $L(\mathcal{A})$ is a subset such that if $X \in I$ and $Y \leq X$ in $L(\mathcal{A})$, then $Y \in I$.

Theorem 3.2.1. *Let $\widehat{\Gamma}$ be the graph obtained from Γ by adding an edge between each pair of boundary nodes. The following hold:*

(i) The intersection poset $L(\overline{\mathcal{A}}(\Gamma, u))$ of a Dirichlet arrangement is the order ideal of $L(\mathcal{A}(\Gamma))$ consisting of all connected partitions π of Γ such that no set in π contains more than one boundary node

(ii) The characteristic polynomial of the Dirichlet arrangement $\overline{\mathcal{A}}(\Gamma, u)$ is the quotient of the chromatic polynomial of $\widehat{\Gamma}$ by a falling factorial

(iii) If $\widehat{\Gamma}$ is 2-connected and $\mathbb{K} = \mathbb{R}$, then the bounded chambers of the Dirichlet arrangement $\overline{\mathcal{A}}(\Gamma, u)$ correspond bijectively to the possible orientations of current flow through Γ respecting the voltages u and in which the current flowing through each edge is nonzero.

Each part of Theorem 3.2.1 generalizes a key theorem on graphic arrangements. As corollaries, we obtain a formula for the number of orientations in part (iii), and we show that the coefficients of a chromatic polynomial remain log-concave after "modding out" by a clique of the graph.

We also characterize supersolvable Dirichlet arrangements. Stanley [85] showed that the graphic arrangement $\mathcal{A}(\Gamma)$ is supersolvable if and only if the graph Γ is chordal. We prove the following characterization of supersolvable arrangements $\overline{\mathcal{A}}(\Gamma, u)$, which is directly analogous to the graphic case. This answers a question of Stanley [88].

Theorem 3.2.2. *The Dirichlet arrangement $\overline{\mathcal{A}}(\Gamma, u)$ is supersolvable if and only if the graph $\widehat{\Gamma}$ from Theorem 3.2.1 is chordal.*

For an application of our results, let P be a finite poset and $\mathcal{O}(P)$ be the convex polytope in $\mathbb{R}^P$ of all order-preserving functions $P \to [0, 1]$. The polytope $\mathcal{O}(P)$ is called the *order polytope* of P [86]. Consider the sets of facets of $\mathcal{O}(P)$ visible from different points in $\mathbb{R}^P$, called *visibility sets* of $\mathcal{O}(P)$. In general not all visibility sets of $\mathcal{O}(P)$ are visible from far away, since certain obstructions are eliminated by viewing $\mathcal{O}(P)$ "from infinity."

The *beta invariant* of a graph is the nonnegative integer $\beta(\Gamma) = |\chi'_\Gamma(1)|$. One has $\beta(\Gamma) > 0$ if and only if Γ is 2-connected. Write $\alpha(\Gamma)$ for the number of acyclic orientations of Γ.

Corollary 3.2.3. *Let P be a finite poset. There is a graph G such that $\mathcal{O}(P)$ has exactly $\frac{1}{2}\alpha(G)$ visibility sets, of which exactly $\frac{1}{2}\alpha(G) - \beta(G)$ are visible from far away.*

Another application involves electrical networks with Dirichlet boundary conditions. For generic $\gamma \in \mathbb{C}^E$ the triple (Γ, u, γ) determines a unique harmonic function $h : V \to \mathbb{C}$ extending u (see section 2.1). The *energy* dissipated by a resistor $ij \in E$ is given by

$$\varepsilon_{ij} = \gamma_{ij}(h(i) - h(j))^2. \tag{3.3}$$

In the other direction, suppose that we are given a network (Γ, u) and fixed energies $\varepsilon \in \mathbb{C}^E$. It is natural to ask which conductances $\gamma \in \mathbb{C}^E$ produce the energies ε. Abrams and Kenyon [2] posed the equivalent problem of describing the set of harmonic functions associated to these γ, called *ε-harmonic functions* on (Γ, u).

We describe the ε-harmonic functions on (Γ, u) as critical points of *master functions* of $\overline{\mathcal{A}}(\Gamma, u)$ in the sense of Varchenko [98]. The *master function* of an arrangement $\mathcal{A}$ in $\mathbb{C}^d$ with weights $a \in \mathbb{C}^{\mathcal{A}}$ is the multivalued function $\Phi : U(\mathcal{A}) \to \mathbb{C}$ given by

$$\Phi(x) = \sum_{H \in \mathcal{A}} a_H \log f_H(x), \tag{3.4}$$

where the f_H are the defining functions of $\mathcal{A}$. Broadly speaking, master functions generalize logarithmic barrier functions, and their critical points generalize analytic centers of systems of linear inequalities [10, Sections 8.5.3 and 11.2.1].

Theorem 3.2.4. *The ε-harmonic functions on (Γ, u) are the critical points of the master function of $\overline{\mathcal{A}}(\Gamma, u)$ with weights ε.*

Theorem 3.2.4 connects electrical networks, a subject with a vast literature [13, 23, 72, 80, 83, 103], to critical points of master functions, an active area of research with applications to Lie algebras, physics, integrable systems, and algebraic geometry [20, 41, 61, 99, 100]. We obtain results of Abrams and Kenyon [2] as corollaries of Theorem 3.2.4. Combining these with Theorem 3.2.1 yields the following.

Corollary 3.2.5. *For generic ε, the number of ε-harmonic functions is*

$$\frac{\beta(\widehat{\Gamma})}{(|\partial V| - 2)!}. \tag{3.5}$$

The chapter is organized as follows. In Sections 3.3.1–3.3.3 we prove the three parts of Theorem 3.2.1. In Section 3.4 we prove Theorem 3.2.2 and relate Dirichlet arrangements to previous work [58, 89, 90]. In Section 3.5 we prove Theorem 3.2.4. In Section 3.6 we give alternate proofs of certain results using a different construction of $\overline{\mathcal{A}}(\Gamma, u)$. In Section 3.7 we exhibit an action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{tr}}/\mathbb{Q})$ on the critical points of any master function with positive rational weights, where $\mathbb{Q}^{\mathrm{tr}}$ is the field of totally real numbers.

3.3 Combinatorics of Dirichlet arrangements

Let $\Gamma = (V, E)$ be a graph on d vertices and k edges with boundary $\partial V \subseteq V$ and boundary data $u : \partial V \to \mathbb{R}$. Again we write $m = |\partial V| \geq 2$ and $n = |V^\circ|$ for the number of boundary nodes and interior vertices, respectively.

Graphic arrangements are well studied because of the ability to translate between properties of $\mathcal{A}(\Gamma)$ and corresponding properties of Γ [29, 30, 45, 59, 71, 78]. The following theorem is the graphic version of Theorem 3.2.1. For proofs, see [87].

Theorem 3.3.1. *For any graph Γ, the following hold:*

(i) The intersection poset $L(\mathcal{A}(\Gamma))$ is isomorphic to the lattice of connected partitions of Γ

(ii) The characteristic polynomial $\chi_{\mathcal{A}(\Gamma)}$ is the chromatic polynomial of Γ

(iii) The chambers of $\mathcal{A}(\Gamma)$ correspond bijectively to the acyclic orientations of Γ.

3.3.1 Intersection poset and connected partitions

Recall from section 2.2.2 that the connected partitions of Γ form a lattice ordered by refinement. That is, $\pi \leq \rho$ in Π_Γ if and only if π is a refinement of ρ. Recall that a set $I \subseteq \Pi_\Gamma$ is an *order ideal* if, whenever $\rho \in I$ and $\pi \leq \rho$ in Π_Γ, we have $X \in S$.

Definition 3.3.2. A connected partition of Γ is *boundary separating* if it belongs to

$$\Pi_{\Gamma,\partial V} = \{\pi \in \Pi_\Gamma : |P \cap \partial V| \leq 1 \text{ for all } P \in \pi\}. \tag{3.6}$$

Proof of Theorem 3.2.1(i). Let $X \in L(\overline{\mathcal{A}}(\Gamma, u))$ and $x \in X$. The coordinates of x are indexed by V; let x_i denote the coordinate indexed by $i \in V$. For each $i \in V$ let $S_i \subseteq V$ be the set of $j \in V$ for which there exists a path P from i to j such that x_v is the same for all $v \in P$. We obtain an element $\lambda_X = \{S_i : i \in V\}$ of Π_Γ. No distinct boundary nodes j and j' can belong to a single block S_i, as this would imply that $u(j) = u(j')$. Hence $\lambda_X \in \Pi_{\Gamma,\partial V}$.

Now suppose that $\pi \in \Pi_{\Gamma,\partial V}$. We reverse the above construction. For every block $B \in \pi$, let $E_B \subseteq E$ be the subset of edges with both ends in B. These define an element

$$Y_\pi = \bigcap_{B\in\pi} \bigcap_{e\in E_B} H_e$$

of $L(\overline{\mathcal{A}}(\Gamma, u))$, where each $H_e \in \overline{\mathcal{A}}(\Gamma, u)$ is the hyperplane corresponding to e. It is not hard to see that $Y_{\lambda_X} = X$ and $\lambda_{Y_\pi} = \pi$. Moreover, for $X, X' \in L(\overline{\mathcal{A}}(\Gamma, u))$ we have $X \subseteq X'$ if and only if $\pi_{X'} \leq \pi_X$. The result follows. □

Corollary 3.3.3. *The intersection poset $L(\overline{\mathcal{A}}(\Gamma, u))$ depends only on $(\Gamma, \partial V)$.*

Example 3.3.4. Let $(\Gamma, \partial V)$ be the Wheatstone bridge from Example 3.1.3 with any boundary data u. The Hasse diagram of $L(\mathcal{A}(\Gamma))$ is drawn in Figure 3.4, where

$L(\overline{\mathcal{A}}(\Gamma, u))$ is the order ideal consisting of the blue elements. Each element is labeled by the corresponding connected partition of Γ. For example, $a|bcd$ denotes the partition $\{\{a\}, \{b, c, d\}\}$.

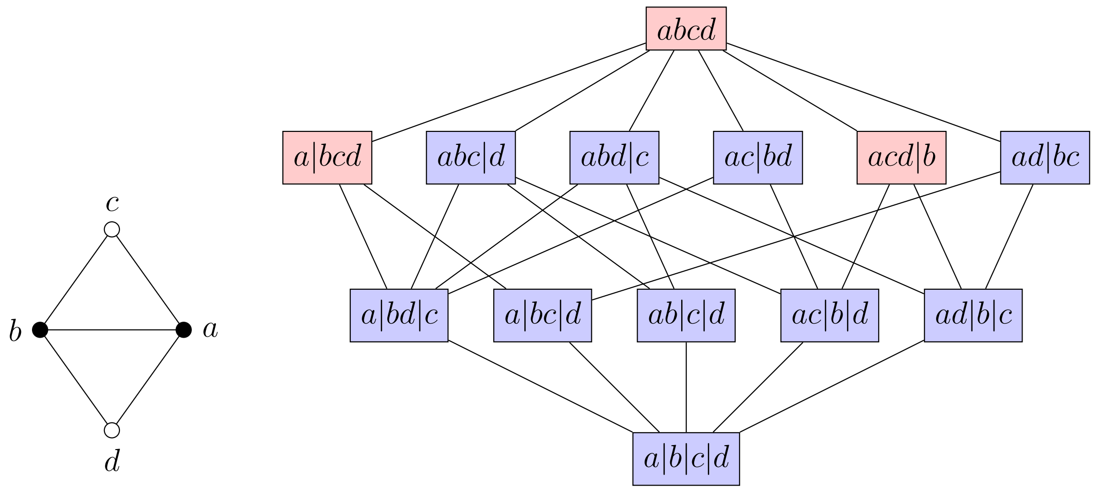


Figure 3.4: A Wheatstone bridge $(\Gamma, \partial V)$ with boundary nodes marked in white and the associad Hasse diagram of $L(\mathcal{A}(\Gamma))$ with the order ideal $L(\overline{\mathcal{A}}(\Gamma, u))$ marked in blue.

3.3.2 Characteristic polynomial and precolorings

For positive integers λ, write $[\lambda] = \{1, \ldots, \lambda\}$. Recall that a (proper) λ-*coloring* of Γ is a function $V \to [\lambda]$ taking distinct values on adjacent vertices. The *chromatic polynomial* χ_Γ of Γ is a polynomial with integer coefficients such that $\chi_\Gamma(\lambda)$ is the number of λ-colorings of Γ for all integers $\lambda \geq 1$.

Let $c : \partial V \to [m]$ be a bijection. Herzberg and Murty [39] exhibited a polynomial $\chi_{\Gamma,\partial V}$ with integer coefficients such that

$$\chi_{\Gamma,\partial V}(\lambda) = |\{\widehat{c} : V \to [p] \mid \widehat{c} \text{ is an } \lambda\text{-coloring of } \Gamma \text{ that extends } c\}| \qquad (3.7)$$

for all integers $\lambda \geq m$. The polynomial $\chi_{\Gamma,\partial V}$ is the basic object of the *Precoloring Extension Problem* [9, 17], which generalizes Latin squares and Sudoku puzzles [39].

Definition 3.3.5. We call $\chi_{\Gamma,\partial V}$ the *precoloring polynomial* of $(\Gamma, \partial V)$.

The following result is due implicitly to Crapo and Rota [22, Section 17] and was isolated later by Athanasiadis [4]. The resulting *Finite Field Method* is a powerful means of computing characteristic polynomials of arrangements. Note that if $\mathcal{A}$ is an arrangement over a finite field, then the complement $U(\mathcal{A})$ is a finite set. Thus the cardinality $|U(\mathcal{A})|$ is a nonnegative integer.

Proposition 3.3.6 ([4, Theorem 2.2])**.** *Suppose that $\mathcal{A}$ is an arrangement in $\mathbb{R}^d$ defined over $\mathbb{Z}$. Fix a prime $p \in \mathbb{Z}$, and let $\mathcal{A}^p$ be the arrangement in $\mathbb{F}_p^d$ obtained by reducing the defining equations of $\mathcal{A}$ mod p. If p is sufficiently large, then $\chi_{\mathcal{A}}(p) = |U(\mathcal{A}^p)|$.*

Proposition 3.3.7. *The characteristic polynomial of $\overline{\mathcal{A}}(\Gamma, u)$ is the precoloring polynomial $\chi_{\Gamma,\partial V}$.*

Proof. Fix a bijection $c : \partial V \to [m]$, and set boundary data $u = c$. Corollary 3.3.3 implies that $\chi_{\overline{\mathcal{A}}(\Gamma,u)}$ is unaffected by the choice of u. Consider $\mathbb{F}_p^n$ as the set $[p]^n$. We can assign to any point $x \in U(\overline{\mathcal{A}}(\Gamma, u)^p)$ an element of

$$\{\widehat{c} : V \to [p] \mid \widehat{c} \text{ is a } p\text{-coloring of } \Gamma \text{ that extends } c\}$$

by setting $\hat{c}(i) = x_i$ for all $i \in V^\circ$. This assignment is easily seen to be a bijection, whence $\chi_{\Gamma,\partial V}(p) = |U(\overline{\mathcal{A}}(\Gamma, u)^p)|$. The result now follows from Proposition 3.3.6 and the fact that $\chi_{\Gamma,\partial V}$ is a polynomial, since $\chi_{\Gamma,\partial V}(p) = |U(\overline{\mathcal{A}}(\Gamma, u)^p)|$ for infinitely many p. $\square$

Example 3.3.8. Let $(\Gamma, \partial V) = \Gamma_{m,n}$ as in Example 3.1.4. Fix a bijection $c : \partial V \to [m]$ and an integer $\lambda \geq d$. To extend c to a λ-coloring of Γ, we must choose for every interior vertex a color that has not yet been used. This accounts for $(\lambda - m)!/(\lambda - d)!$ possible extensions of c, and there are no others. Hence

$$\chi_{\Gamma,\partial V}(t) = (t - m)(t - m - 1) \cdots (t - d + 1).$$

In Example 3.3.8 the precoloring polynomial $\chi_{\Gamma,\partial V}(t)$ divides the chromatic polynomial $\chi_{K_d}(t)$, where K_d is the complete graph on d vertices. This is a consequence of the following proposition.

Proposition 3.3.9. *Let $\widehat{\Gamma}$ be the graph obtained from Γ by adding an edge between each pair of boundary nodes. The precoloring polynomial $\chi_{\Gamma,\partial V}$ satisfies*

$$\chi_{\widehat{\Gamma}}(t) = (t)_m \cdot \chi_{\Gamma,\partial V}(t), \tag{3.8}$$

where $\cdot$ denotes multiplication and $(t)_m = t(t-1)(t-2)\cdots(t-m+1)$ denotes a falling factorial.

Proof. Fix $\lambda \geq d$. We count the number of λ-colorings of $\widehat{G}$ vertex-by-vertex, starting with the boundary nodes. Since ∂V is a clique in $\widehat{G}$, there are λ ways to color the first boundary node, $\lambda - 1$ ways to color the second, and $\lambda - r + 1$ ways to color the rth. Once all the boundary nodes are colored, the number of ways to color the interior vertices is $\chi_{\Gamma,\partial V}(\lambda)$. Thus $\chi_{\widehat{\Gamma}}(t) = (t)_m \cdot \chi_{\Gamma,\partial V}(t)$ holds for infinitely many t, so it holds in general. □

Proof of Theorem 3.2.1(ii). The result follows from Propositions 3.3.7 and 3.3.9. □

Example 3.3.10. Let Γ be the path graph on $d \geq 3$ vertices, and let ∂V consist of both ends of the path. We have $\widehat{\Gamma} = C_d$, the d-cycle. Using the elementary formula $\chi_{C_d}(t) = (t-1)^d + (-1)^d(t-1)$, we obtain

$$\begin{aligned}\chi_{\Gamma,\partial V}(t) &= \frac{\chi_{C_d}(t)}{t(t-1)} \\ &= \prod_{r=1}^{n}(t + \zeta^r - 1),\end{aligned}$$

where $\zeta \in \mathbb{C}$ is any primitive kth root of unity.

Recall that a sequence $a_0, \ldots, a_n$ of nonnegative numbers is *log-concave* if

$$a_r^2 \geq a_{r-1}a_{r+1} \tag{3.9}$$

for all $r \in [n-1]$. A log-concave sequence is necessarily *unimodal*; i.e., there exists $s \in \{0, 1, \ldots, n\}$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{s-1} \leq a_s \geq a_{s+1} \geq \cdots \geq a_n. \tag{3.10}$$

Huh [40, Theorem 3] proved that the coefficients of a matroid representable over a field of characteristic 0 form a log-concave sequence. Phrased in terms of hyperplane arrangements, we have the following.

Proposition 3.3.11 ([40])**.** *Let $\mathcal{A}$ be a real or complex hyperplane arrangement, and write*

$$\chi_{\mathcal{A}}(t) = a_0t^n - a_1t^{n-1} + \cdots + (-1)^n a_n.$$

The sequence $a_0, \ldots, a_n$ is log-concave.

We can apply the previous result to Dirichlet arrangements:

Proposition 3.3.12. *Suppose that Γ contains a clique on ℓ vertices, and write*

$$\chi_\Gamma(t)/(t)_\ell = a_0t^n - a_1t^{n-1} + \cdots + (-1)^n a_n.$$

The sequence $a_0, \ldots, a_n$ is log-concave.

Proof. This follows from Propositions 3.3.9 and 3.3.11. □

Proposition 3.3.12 can help determine whether a given polynomial is chromatic. The following corollary gives a small example.

Corollary 3.3.13. *If $\chi_\Gamma(t)$ is divisible by $(t)_3$ but the coefficients of $\chi_\Gamma(t)/(t)_3$ do not form a log-concave sequence, then Γ contains a simple chordless cycle of length at least* 5.

Proof. Proposition 3.3.12 implies that Γ contains no cycle of length 3. Since $\chi_\Gamma(t)$ is divisible by $(t)_3$, we have $\chi_\Gamma(2) = 0$, so Γ is not 2-colorable. In other words, Γ is not bipartite, so it contains an odd cycle. This cycle must be of length at least 5, and must be simple and chordless, or else Γ would contain a cycle of length 3. $\square$

Example 3.3.14. Consider the polynomial $f(t) = t^5 - 4t^4 + 7t^3 - 8t^3 + 4t$. Note that $f(t)$ is a monic polynomial whose coefficients alternate in sign and form a log-concave sequence. However, the coefficients of $f(t)/(t)_3 = t^2 - t + 2$ do not form a log-concave sequence. Corollary 3.3.13 implies that if $f = \chi_\Gamma$, then Γ contains a simple chordless cycle of length at least 5. Since $\deg(f) = 5$, we must have $|V| = 5$. Hence Γ is necessarily the 5-cycle C_5. But $\chi_{C_5}(t) = (t)_3 \cdot (t^2 - 2t + 2) \neq f(t)$, so f is not chromatic.

3.3.3 Chambers and compatible orientations

Given a real arrangement $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ and $\overline{\mathcal{C}}(\mathcal{A})$ the sets of chambers and bounded chambers, resp., of $\mathcal{A}$. There is a bijection between the chambers of $\mathcal{C}(\mathcal{A}(\Gamma))$ and set of the acyclic orientations of Γ due to Greene [37]. Namely, to any $C \in \mathcal{C}(\mathcal{A}(\Gamma))$ we take $x \in C$ and assign the orientation $o(C)$ of Γ with $\vec{ij}$ if and only if $x_i > x_j$ for all $ij \in E$.

Definition 3.3.15. We say that an orientation σ of Γ *respects* u if for any path $i \to j$ in σ between boundary nodes i and j we have $u(i) > u(j)$.

Definition 3.3.16. Let $\mathcal{O}_{\Gamma,u}$ denote the set of acyclic orientations of Γ that respect u. Let $\overline{\mathcal{O}}_{\Gamma,u} \subseteq \mathcal{O}_{\Gamma,u}$ be the subset of those orientations with no sinks or sources in V°. The orientations in $\mathcal{O}_{\Gamma,u}$ and $\overline{\mathcal{O}}_{\Gamma,u}$ are called *semicompatible* and *compatible*, respectively.

Consider the edges of Γ as resistors with arbitrary conductances $\gamma \in (0, \infty)^k$, and suppose that every vertex in V° lies on a simple path in Γ between distinct boundary nodes. Current flows from vertices of higher voltage to vertices of lower voltage. As γ varies, the compatible orientations are the orientations of all current flows through Γ that respect the boundary voltages u and in which the current across every edge is nonzero. This point of view is reinforced by Proposition 3.3.20 below, which generalizes [53, Theorem a.1].

Proposition 3.3.17. *There is a bijection from the set of chambers (resp., bounded chambers) of $\overline{\mathcal{A}}(\Gamma, u)$ to the set of semicompatible (resp., compatible) orientations of (Γ, u).*

Proof. Recall that for any acyclic orientation $C \in \mathcal{C}(\mathcal{A}(\Gamma))$, the orientation $o(C)$ is obtained by taking $x \in C$ and directing $\vec{ij}$ if and only if $x_i > x_j$ for all $ij \in E$. We first show that the function o restricts to a bijection $\mathcal{C}(\overline{\mathcal{A}}(\Gamma, u)) \to \mathcal{O}_{\Gamma,u}$. Suppose that $C \in \mathcal{C}(\overline{\mathcal{A}}(\Gamma, u))$, and let $x \in C$. Since $x_i = u(i)$ for all $i \in \partial V$, $o(C)$ respects u. Since $U(\overline{\mathcal{A}}(\Gamma, u)) \subseteq U(\mathcal{A}(\Gamma))$, $o(C)$ is acyclic. Hence $o(C) \in \mathcal{O}_{\Gamma,u}$. Clearly o is injective.

Now suppose that $\sigma \in \mathcal{O}_{\Gamma,u}$, and note that u defines a total order on ∂V. Since σ is acyclic, we obtain a partial order on V by setting $j \leq i$ if and only if $\vec{ij} \in \sigma$. Extend this order to a total order on V; such an extension also extends the total order on ∂V. Thus we can take $y \in U(\overline{\mathcal{A}}(\Gamma, u))$ whose entries respect the total order on V. Write $o^{-1}(\sigma)$ for the chamber of $\overline{\mathcal{A}}(\Gamma, u)$ containing y. We have $o(o^{-1}(\sigma)) = \sigma$, so o is a bijection $\mathcal{C}(\overline{\mathcal{A}}(\Gamma, u)) \to \mathcal{O}_{\Gamma,u}$, as desired.

We must now show that $\sigma \in \overline{\mathcal{O}}_{\Gamma,u}$ if and only if $o^{-1}(\sigma) \in \overline{\mathcal{C}}(\overline{\mathcal{A}}(\Gamma, u))$. For the "if" direction, suppose that $\sigma \in \mathcal{O}_{\Gamma,u} \setminus \overline{\mathcal{O}}_{\Gamma,u}$, and suppose without loss of generality that $i \in V^\circ$ is a source of σ. Let $x \in o^{-1}(\sigma)$, and let $y \in \mathbb{R}^n$ be the standard basis element corresponding to i. Let $t > 0$ be large enough that $x + ty \in U(\overline{\mathcal{A}}(\Gamma, u))$, and let $C \in \mathcal{C}(\overline{\mathcal{A}}(\Gamma, u))$ be the chamber containing $x + ty$. Clearly i is a source of $o(C)$, and in fact $\sigma = o(C)$. Hence $o^{-1}(\sigma)$ is unbounded, proving the "if" direction.

For the "only if" direction, suppose that $\sigma \in \overline{\mathcal{O}}_{\Gamma,u}$. Let $f \in o^{-1}(\sigma)$, and let $\mathcal{X}$ be as in Definition 3.1.1. We show that any ray in $\mathcal{X}$ originating at f is not contained in the convex set $o^{-1}(\sigma)$. Let $g \in \mathbb{R}^n \setminus \{0\}$ with $g_i = 0$ for all $i \in \partial V$, and suppose without loss of generality that $g_v > 0$ for some $v \in V^\circ$. For large enough $t > 0$ we have $f + tg \in U(\overline{\mathcal{A}}(\Gamma, u))$ and $f_v + tg_v > u(w)$ for all $w \in \partial V$. If $C \in \mathcal{C}(\overline{\mathcal{A}}(\Gamma, u))$ is the chamber containing $f + tg$, then $o(C)$ has a source in V°. Hence $C \neq o^{-1}(\sigma)$. Since the direction of the ray in $\mathcal{X}$ was arbitrary, we conclude that $o^{-1}(\sigma)$ is bounded. □

Proof of Theorem 3.2.1(iii). The result follows from Propositions 3.3.20 and 3.3.17. □

Zaslavsky [109] expressed the numbers of chambers and bounded chambers of a real arrangement $\mathcal{A}$ in terms of the characteristic polynomial $\chi_{\mathcal{A}}$. We are particularly interested in counting the bounded chambers of $\overline{\mathcal{A}}(\Gamma, u)$ because of their role in Section 3.5.3.

Proposition 3.3.18 ([109, Theorems A and C])**.** *If $\mathcal{A}$ is a real arrangement, then the number of chambers of $\mathcal{A}$ is $|\chi_{\mathcal{A}}(-1)|$, and the number of bounded chambers is $|\chi_{\mathcal{A}}(1)|$.*

Proposition 3.3.18 gives $|\mathcal{C}(\overline{\mathcal{A}}(\Gamma, u))|$ and $|\overline{\mathcal{C}}(\overline{\mathcal{A}}(\Gamma, u))|$ in terms of the precoloring polynomial $\chi_{\Gamma,\partial V}$. The next theorem gives these counts in terms of a genuine chromatic polynomial. Recall that the *beta invariant* of Γ is the nonnegative integer $\beta(\Gamma)$ given by

$$\beta(\Gamma) = |\chi'_{\Gamma}(1)|, \tag{3.11}$$

where χ'_{Γ} is the derivative of χ_{Γ}. We have $\beta(\Gamma) > 0$ if and only if Γ is 2-connected [6, 67].

Theorem 3.3.19. *Let $\widehat{\Gamma}$ be the graph obtained from Γ by adding an edge between each pair of boundary nodes. The number of semicompatible orientations of (Γ, u) is*

$$|\mathcal{O}_{\Gamma,u}| = \frac{\alpha(\widehat{\Gamma})}{m!}, \tag{3.12}$$

where $\alpha(\widehat{\Gamma})$ is the number of acyclic orientations of $\widehat{\Gamma}$. The number of compatible orientations is

$$|\overline{\mathcal{O}}_{\Gamma,u}| = \frac{\beta(\widehat{\Gamma})}{(m-2)!}, \tag{3.13}$$

where $\beta(\widehat{\Gamma})$ is the beta invariant of $\widehat{\Gamma}$.

Proof. Proposition 3.3.9 says that

$$\chi_{\widehat{\Gamma}}(t) = (t)_m \cdot \chi_{\Gamma,\partial V}(t). \tag{3.14}$$

Evaluating both sides of (3.14) at $t = -1$ and rearranging gives $|\chi_{\Gamma,\partial V}(-1)| = |\chi_{\widehat{\Gamma}}(-1)|/m!$. Proposition 3.3.18 implies that $|\chi_{\widehat{\Gamma}}(-1)| = \alpha(\widehat{\Gamma})$. Now (3.12) follows from Proposition 3.3.17.

Taking derivatives of both sides of (3.14), evaluating at $t = 1$ and rearranging, we have $|\chi_{\Gamma,\partial V}(1)| = |\chi'_{\widehat{\Gamma}}(1)|/(m-2)!$. Thus (3.13) follows from Propositions 3.3.17 and 3.3.18. □

The identity $|\mathcal{O}_{\Gamma,u}| = |\chi_{\Gamma,\partial V}(-1)|$ was obtained by Jochemko and Sanyal [48, Corollary 4.5], who used a combinatorial reciprocity for $\chi_{\Gamma,\partial V}$. Equation (3.13) seems to be the first analogous treatment of $\chi_{\Gamma,\partial V}(1)$. From Theorem 3.3.19 we obtain the following result, which reinforces the point of view of compatible orientations as orientations of current flow.

Proposition 3.3.20. *Let $\widehat{\Gamma}$ be the graph obtained from Γ by adding an edge between each pair of boundary nodes. The following are equivalent:*

(i) *$\widehat{\Gamma}$ is 2-connected*

(ii) *(Γ, u) admits a compatible orientation for any boundary data u*

(iii) *Every interior vertex of Γ lies on a simple path in Γ between distinct boundary nodes.*

Proof. The equivalence of (i) and (ii) follows from Theorem 3.3.19. We prove the equivalence of (i) and (iii).

Suppose that (i) holds. Let $i \in V^\circ$. If there is no simple path in Γ connecting i to ∂V, then $\widehat{\Gamma}$ is disconnected, a contradiction. Suppose instead that there is a simple path in Γ connecting i to a boundary node j, but that there is no simple path containing i and two distinct boundary nodes. Notice that $\widehat{\Gamma} \setminus j$ is disconnected, so $\widehat{\Gamma}$ is not 2-connected, a contradiction. Hence (iii) holds.

Now suppose that (i) does not hold. Let $i \in V$ be such that $\widehat{\Gamma} \setminus i$ is disconnected. Since ∂V forms a clique in $\widehat{\Gamma}$, all boundary nodes remaining in $\widehat{\Gamma} \setminus i$ belong to the same component X of $\widehat{\Gamma} \setminus i$. Let j be a vertex of $\widehat{\Gamma} \setminus i$ not in X. Any path in Γ that contains j and begins and ends at distinct boundary nodes j must contain at least 2 edges (with multiplicity) incident to i. Such a path is not simple, so (iii) does not hold. □

Example 3.3.21. Suppose that $\partial V = \{i, j\}$ with any boundary data u. Here the orientations in $\overline{\mathcal{O}}_{\Gamma,u}$ are called *ij-bipolar* and have applications to graph drawing [24]. If one considers the edges of Γ as resistors with arbitrary positive conductances, then the ij-bipolar orientations of Γ are the possible orientations of current flow through Γ in which the current flowing through each resistor is nonzero after a battery is put across i and j. In this case, the formula (3.13) was observed by Abrams and Kenyon [2].

Proof of Corollary 3.2.3. This follows from Example 3.1.5 and Theorem 3.3.19. □

Question 3.3.22. *Let Δ be the graph with a vertex for every bounded chamber of $\overline{\mathcal{A}}(\Gamma, u)$ and an edge whenever the associated chambers share a facet. O. de Mendez [25] showed that in the case of 2 boundary nodes, if $\widehat{\Gamma}$ is 3-connected, then Δ is connected (see [24, Theorem 7.1]). Does this result hold in the case of 3 or more boundary nodes?*

3.4 Supersolvability and psi-graphical arrangements

Stanley [89] introduced the following class of arrangements to study visibility arrangements of order polytopes (see Example 3.1.5).

Definition 3.4.1. Denote the power set of $\mathbb{R}$ by $\mathcal{P}(\mathbb{R})$, and let $\psi : V \to \mathcal{P}(\mathbb{R})$ be such that $|\psi(i)| < \infty$ for all $i \in V$. Let $\mathcal{A}(\Gamma, \psi)$ be the arrangement in $\mathbb{R}^n$ of hyperplanes $\{x_i = x_j\}$ for all $ij \in E$ and $\{x_i = \alpha\}$ for all $i \in V$ and $\alpha \in \psi(i)$. An arrangement is called *ψ-graphical* if it is of the form $\mathcal{A}(\Gamma, \psi)$ for some pair (Γ, ψ).

It turns out that every Dirichlet arrangement can be realized as a ψ-graphical arrangement, and vice versa. We prove this equivalence. The main benefit of Defintion 3.1.1 over Definition 3.4.1 is that it leads to clearer parallels between Dirichlet arrangements and graphic arrangements. Theorem 3.2.2 and the results of Section 3.3 are just a few examples of this.

Proposition 3.4.2. *The classes of Dirichlet arrangements and ψ-graphical arrangements coincide.*

Proof. Let $\Gamma = (V, E)$ be a graph with boundary ∂V and boundary data u. Let Γ° be the subgraph of Γ induced by V°, and let $\psi^\circ : V^\circ \to \mathcal{P}(\mathbb{R})$ be given by $\psi^\circ(i) = \{u(j) : j \sim i \text{ and } j \in \partial V\}$. We have $\overline{\mathcal{A}}(\Gamma, u) = \mathcal{A}(\Gamma^\circ, \psi^\circ)$. Hence every Dirichlet arrangement is ψ-graphical.

Now consider a ψ-graphical arrangement $\mathcal{A}(\Gamma, \psi)$, and let $S = \bigcup_{i \in V} \psi(i)$. Let $V' = V \cup \{j_s : s \in S\}$ and $E' = \{ij_s : i \in V \text{ and } s \in \psi(i)\}$. Also let $\Gamma' = (V', E')$, and let $u' : \{j_s : s \in S\} \to \mathbb{R}$ be given by $u(j_s) = s$ for all $s \in S$. It is not hard to see that $\mathcal{A}(\Gamma, \psi) = \overline{\mathcal{A}}(\Gamma', u')$. Hence every ψ-graphical arrangement is Dirichlet. $\square$

Example 3.4.3. Consider the path graph Γ on the left side of Figure 3.5. Let $\psi : V \to \mathcal{P}(\mathbb{R})$ be given by the vertex labels, so if i is the top vertex for example, then $\psi(i) = \{0, 2, 3\}$. The pair (Γ, ψ) corresponds to the graph on the right side of Figure 3.5 with boundary vertices, marked in white, corresponding to the elements of $\bigcup_{i \in V} \psi(i) = \{0, 1, 2, 3\}$. We draw an edge between a black vertex i and a white vertex j whenever the number associated to j belongs to $\psi(i)$. For example, the top black vertex is incident to the boundary vertices corresponding to 0, 2 and 3.

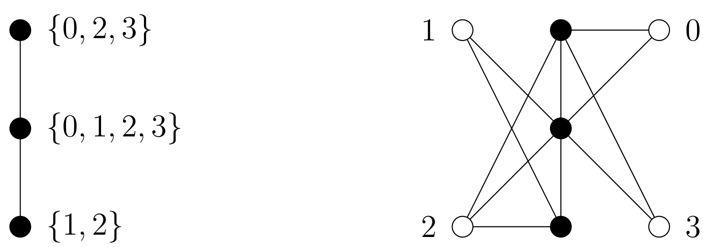


Figure 3.5: An illustration of the proof of Proposition 3.4.2.

The literature on ψ-graphical arrangements has focused on questions of supersolvability. Recall that $\overline{\mathcal{A}}(\Gamma, u)$ is not central, i.e. the intersection of its elements is empty. Definition 2.2.3 states that a non-central arrangement $\mathcal{A}$ is *supersolvable* if $L(c\mathcal{A})$ admits a maximal chain consisting of modular elements, where $c\mathcal{A}$ is the cone over $\mathcal{A}$.

Recall that Γ is *chordal* if it contains no induced cycles of length greater than 3. Alternatively, Γ is chordal if and only if it admits a *perfect elimination ordering.* A *perfect elimination ordering* of Γ is an ordering $i_1, \ldots, i_n$ of V such that for every $s = 1, \ldots, n$ the neighbors of i_s in $\{i_s, \ldots, i_n\}$ form a clique.

Proposition 2.2.6 states that a graphic arrangement $\mathcal{A}(\Gamma)$ is supersolvable if and only if Γ is chordal. Our result, Theorem 3.2.2, is a direct generalization for Dirichlet arrangements. This generalization builds on work of Mu–Stanley and Suyama–Tsujie [58, 90].

Lemma 3.4.4 ([75, p. 603])**.** *If Γ is chordal and $C \subseteq V$ is a clique of Γ, then there is a perfect elimination ordering of Γ ending with the elements of C.*

Proof of Theorem 3.2.2. Let Γ° denote the subgraph of Γ induced by V°, and let $\psi^\circ : V^\circ \to 2^{\mathbb{R}}$ be given by

$$\psi^\circ(i) = \{u(j) : j \sim i \text{ and } j \in \partial V\}. \tag{3.15}$$

Notice that $\overline{\mathcal{A}}(\Gamma, u) = \mathcal{A}(\Gamma^\circ, \psi^\circ)$. A *weighted elimination ordering* of $(\Gamma^\circ, \psi^\circ)$ is a perfect elimination ordering $i_1, \ldots, i_{n-m}$ of Γ° such that if $i_r \sim i_s$ with $r < s$, then $\psi^\circ(i_r) \subseteq \psi^\circ(i_s)$. We show that $(\Gamma^\circ, \psi^\circ)$ admits a weighted elimination ordering if and only if $\widehat{\Gamma}$ is chordal. Theorem 3.2.2 will then follow from [90, Theorem 2.2].

Suppose that $i_1, \ldots, i_n$ is a weighted elimination ordering of $(\Gamma^\circ, \psi^\circ)$. We claim that $i_1, \ldots, i_n, j_1, \ldots, j_m$ is a perfect elimination ordering of $\widehat{\Gamma}$ for any ordering $j_1, \ldots, j_m$ of ∂V. Suppose that $i_r \sim i_s$ and $i_r \sim i_t$ for $r < s, t$. Clearly the same adjacencies hold in $\widehat{\Gamma}$. Now suppose that $i_r \sim i_s$ and $i_r \sim j$ for some $j \in \partial V$. Since $r < s$ we have $\psi_u(i_r) \subseteq \psi_u(i_s)$, so $u(j) \in \psi_u(i_s)$. Hence $i_s \sim j$ in $\widehat{\Gamma}$. Now suppose without loss of generality that $i_r \sim j_1$ and $i_r \sim j_2$. Since ∂V is a clique in $\widehat{\Gamma}$, we have $j_1 \sim j_2$. The claim follows, proving that $\widehat{\Gamma}$ is chordal.

Conversely, suppose that $\widehat{\Gamma}$ is chordal. Since ∂V is a clique of $\widehat{\Gamma}$, Lemma 3.4.4 gives a perfect elimination ordering of $\widehat{\Gamma}$ whose last m vertices are the elements of ∂V.

The first n vertices of this perfect elimination ordering form a weighted elimination ordering of $(\Gamma^\circ, \psi^\circ)$. $\square$

Example 3.4.5. Let $(\Gamma, \partial V) = \Gamma_{m,n}$ with any boundary data u. Since $\widehat{\Gamma}$ is complete, any ordering of V is a perfect elimination ordering of $\widehat{\Gamma}$. Hence $\overline{\mathcal{A}}(\Gamma, u)$ is supersolvable.

More generally, suppose that V° is a clique in Γ, but make no assumptions about which boundary nodes and which interior vertices are adjacent. Write $\partial V = \{j_1, \ldots, j_m\}$ and $V^\circ = \{i_1, \ldots, i_{n-m}\}$ so that if i_r is adjacent to a boundary node and $r < s$, then i_s is adjacent to a boundary node. Notice that $i_1, \ldots, i_{n-m}, j_1, \ldots, j_m$ is a perfect elimination ordering of $\widehat{\Gamma}$. Hence $\overline{\mathcal{A}}(\Gamma, u)$ is supersolvable for any boundary data u.

Example 3.4.6. In this example $\mathcal{A}(\Gamma)$ is supersolvable but $\overline{\mathcal{A}}(\Gamma, u)$ is not. Let Γ be a path graph on $n \geq 3$ vertices with ∂V consisting of both ends of the path. In Example 3.3.10 we computed

$$\chi_{\Gamma,\partial V}(t) = \prod_{r=1}^{k-1}(t + \zeta^r - 1),$$

where $\zeta \in \mathbb{C}$ is a primitive kth root of unity. At most one root of $\chi_{\Gamma,\partial V}$ is a positive integer. Hence when $n \geq 4$, $\overline{\mathcal{A}}(\Gamma, u)$ is not supersolvable for any boundary data u.

3.5 Master functions and electrical networks

Given an arrangement $\mathcal{A}$ in $\mathbb{R}^d$, let $\mathcal{A}_{\mathbb{C}}$ be the arrangement in $\mathbb{C}^d$ defined by the polynomial

$$Q(\mathcal{A}_{\mathbb{C}}) = Q(\mathcal{A}),$$

where $Q(\mathcal{A}_{\mathbb{C}})$ is considered as a polynomial over $\mathbb{C}$. In other words, $\mathcal{A}_{\mathbb{C}} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$ is the *complexification* of $\mathcal{A}$. We think of $U(\mathcal{A}) = U(\mathcal{A}_{\mathbb{C}}) \cap \mathbb{R}^d$ as the set of real points of $U(\mathcal{A}_{\mathbb{C}})$.

Definition 3.5.1. Let $\mathcal{A}$ be an arrangement of k hyperplanes in $\mathbb{R}^d$. The *master function* of $\mathcal{A}$ with weights $\varepsilon \in \mathbb{C}^k$ is the multivalued function $\Phi^{\varepsilon}_{\mathcal{A}} : U(\mathcal{A}_{\mathbb{C}}) \to \mathbb{C}$ given by

$$\Phi^{\varepsilon}_{\mathcal{A}}(x) = \sum_{r=1}^{k} \varepsilon_r \log f_r(x), \tag{3.16}$$

where the f_r are the defining functions of $\mathcal{A}$.

Definition 3.5.2. A point $x \in U(\mathcal{A}_{\mathbb{C}})$ is a *critical point* of the master function $\Phi^{\varepsilon}_{\mathcal{A}}$ if $\nabla\Phi^{\varepsilon}_{\mathcal{A}}(x) = 0$. That is,

$$\sum_{r=1}^{k} \frac{\partial f_r}{\partial x_i} \frac{\varepsilon_r}{f_r(x)} = 0 \tag{3.17}$$

for all $i = 1, \ldots, d$.

The critical points of $\Phi^{\varepsilon}_{\mathcal{A}}$ are independent of the choices of f_r. As i ranges over $1, \ldots, d$, the equations (3.17) are sometimes called the *Bethe Ansatz equations* for $\Phi^{\varepsilon}_{\mathcal{A}}$ (see [84, Section 12.1]). We denote the set of critical points of $\Phi^{\varepsilon}_{\mathcal{A}}$ by $\mathcal{V}(\mathcal{A}, \varepsilon)$.

The term *master function* sometimes refers to the product $x \mapsto \exp(\Phi^{\varepsilon}_{\mathcal{A}}(x))$ of powers of affine functionals. Proposition 3.5.3 below is due to Varchenko [97] and is foundational in the study of master functions. Given $S \subseteq \mathbb{C}^d$ and a list of mutually disjoint sets $A_1, \ldots, A_\ell \subset \mathbb{C}^d$, we say that the elements of S form a *system of distinct representatives* for the sets $A_1, \ldots, A_\ell$ if $|S| = \ell$ and $S \cap A_r$ is nonempty for all $r = 1, \ldots, \ell$.

Proposition 3.5.3 ([97, Theorem 1.2.1])**.** *Let $\mathcal{A}$ be a real essential arrangement and $\varepsilon \in (0, \infty)^{\mathcal{A}}$. The critical points of the master function $\Phi^{\varepsilon}_{\mathcal{A}}$ form a system of distinct representatives for the bounded chambers of $\mathcal{A}$.*

For a short, elementary proof of Proposition 3.5.3, see [84, §9.2].

3.5.1 Laplacians and master functions

If every hyperplane in $\mathcal{A}$ contains the origin, then the defining functions f_r of $\mathcal{A}$ are homogeneous. In this case we let $L = L(\gamma)$ denote the $d \times d$ matrix in the usual basis of $\mathbb{R}^d$ with

$$x^T Lx = \sum_{r=1}^{k} \gamma_r f_r(x)^2 \tag{3.18}$$

for all $x \in \mathbb{R}^d$, where x^T is the transpose of x. We call L the *Laplacian matrix* of $\mathcal{A}$ with weights γ. Our terminology is explained by Example 3.5.4 below, which features in the remainder of this section.

Example 3.5.4. We let $L_\Gamma = L_\Gamma(\gamma)$ denote the Laplacian matrix of the graphic arrangement $\mathcal{A}(\Gamma)$ with weights γ. Here L_Γ is just the weighted Laplacian matrix of Section 2.1, where each edge e is weighted by γ_e. The quadratic form associated with L_Γ is the Dirichlet norm:

$$x^T L_\Gamma x = \sum_{ij \in E} \gamma_{ij}(x_i - x_j)^2 \tag{3.19}$$

for all $x \in \mathbb{R}^d$. If Γ as an electrical network with conductances $\gamma \in (0, \infty)^k$, and voltages x, then $x^T L_\Gamma x$ is the total energy dissipated by the network.

One can think of $\Phi^\varepsilon_\mathcal{A}$ as a (weighted) *logarithmic barrier function.* Hessian matrices of logarithmic barrier functions play an important role in interior point methods (see, e.g., [63]). The next proposition connects Laplacian matrices of an arrangement $\mathcal{A}$ to gradients and Hessian matrices of master functions of $\mathcal{A}$. Let $\Psi_\mathcal{A} : \mathbb{C}^k \times \mathbb{C}^d \to \mathbb{C}^k$ be given by

$$\Psi_\mathcal{A}(\gamma, x) = (\gamma_1 f_1(x)^2, \ldots, \gamma_k f_k(x)^2). \tag{3.20}$$

We write $\Psi = \Psi_\mathcal{A}$. For suitable functions g, we let $\mathrm{H}_g(x)$ denote the Hessian matrix of g, evaluated at x.

Proposition 3.5.5. *If every hyperplane in $\mathcal{A}$ contains the origin and $\varepsilon = \Psi(\gamma, x)$ for some $x \in U(\mathcal{A}_\mathbb{C})$, then $\nabla\Phi^\varepsilon_\mathcal{A}(x) = Lx$ and $\mathrm{H}_{\Phi^\varepsilon_\mathcal{A}}(x) = -L$.*

Proof. First, notice that

$$L_{ij} = \sum_{r=1}^{k} \frac{\partial f_r}{\partial x_i} \frac{\partial f_r}{\partial x_j} \gamma_r. \tag{3.21}$$

If $\varepsilon = \Psi(\gamma, x)$ for some $x \in U(\mathcal{A}_\mathbb{C})$, then

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Phi^\varepsilon_\mathcal{A}(x) &= \sum_{r=1}^{k} \frac{\partial f_r}{\partial x_i}\frac{\varepsilon_r}{f_r(x)} \\
&= \sum_{r=1}^{k} \frac{\partial f_r}{\partial x_i}\gamma_r f_r(x) \\
&= \sum_{r=1}^{k} \frac{\partial f_\ell}{\partial x_i}\gamma_r \left(\sum_{j=1}^{d} \frac{\partial f_r}{\partial x_j} x_j\right) \\
&= \sum_{j=1}^{d} \left(\sum_{r=1}^{k} \frac{\partial f_r}{\partial x_i}\frac{\partial f_r}{\partial x_j}\gamma_r\right) x_j,
\end{aligned}$$

so $\nabla\Phi^\varepsilon_\mathcal{A}(x) = Lx$ by (3.21). By a similar argument we also have

$$\frac{\partial^2}{\partial x_i \partial x_j}\Phi^\varepsilon_\mathcal{A}(x) = -L_{ij},$$

as desired. □

3.5.2 Discrete harmonic functions

Let $\gamma, \varepsilon \in \mathbb{C}^k$ be indexed by E. We adopt the language of Section 2.1, calling γ the *conductances* and ε the *energies*. We also refer to the entries γ_{ij} (resp., ε_{ij}) collectively as the *conductances* (resp., *energies*).

Recall the weighted Laplacian matrix of Γ from Section 2.1 and Example 3.5.4. Let $L_{\Gamma,\partial V} = L_{\Gamma,\partial V}(\gamma)$ denote the submatrix of L_Γ obtained by deleting all rows and columns indexed by ∂V. Define $x \in \mathbb{C}^V$ to be a *harmonic function* on (Γ, u, γ) if x extends $u \in \mathbb{R}^{\partial V}$ and

$$\sum_{j \sim i} \gamma_{ij}(x_i - x_j) = 0 \tag{3.22}$$

When there is no ambiguity, we say simply that x is *harmonic*.

If a harmonic function exists, then it is unique; we denote it by $h(\gamma)$. A harmonic function exists unless $L_{\Gamma,\partial V}$ is singular, which occurs only for γ in a proper algebraic subset of $\mathbb{C}^k$ by (2.6). We say that γ is *generic* if $L_{\Gamma,\partial V}$ is nonsingular. In particular, γ is generic whenever all γ_e have positive real parts.

3.5.3 Fixed-energy harmonic functions

We now prove Theorem 3.2.4 and obtain results of Abrams and Kenyon [2] as corollaries. Recall that $\mathcal{V}(\mathcal{A},\varepsilon)(\mathcal{A},\varepsilon)$ is the critical set of the master function $\Phi^{\varepsilon}_{\mathcal{A}}$ of $\mathcal{A}$. Also recall the function $\Psi_{\mathcal{A}}$ defined in (3.20). Given $\varepsilon \in \mathbb{C}^k$, we write

$$\begin{aligned} \Psi_{\Gamma,u} &= \Psi_{\overline{\mathcal{A}}(\Gamma,u)} \\ \Phi_{\Gamma,u} &= \Phi^{\varepsilon}_{\overline{\mathcal{A}}(\Gamma,u)} \\ \mathcal{V}_{\Gamma,u} &= \mathcal{V}(\overline{\mathcal{A}}(\Gamma,u),\varepsilon). \end{aligned} \tag{3.23}$$

We continue to assume that Γ is connected and ∂V is nonempty. If γ is generic in the sense of Section 3.5.2, then we write

$$\Psi_{\Gamma,u}(\gamma) = \Psi_{\Gamma,u}(\gamma, h(\gamma)). \tag{3.24}$$

Example 3.5.6. Let $(\Gamma,\partial V) = \Gamma_{m,n}$ as in Example 3.1.4. Fix positive integers ℓ_j for all $j \in \partial V$. Let $\varepsilon \in \mathbb{C}^k$ be given for all $ij \in E$ by

$$\varepsilon_{ij} = \begin{cases} 2 & \text{if } i,j \in V^{\circ} \\ -\ell_j & \text{if } j \in \partial V. \end{cases}$$

Here we have

$$\Phi_{\Gamma,u}(x) = \sum_{\{i,j\}\subset V^{\circ}} 2\log(x_i - x_j) - \sum_{\substack{i\in V^{\circ}\\ j\in\partial V}} \ell_j \log(x_i - u(j)).$$

This master function plays a crucial role in the construction of hypergeometric solutions of the $\mathfrak{sl}_2$ Knizhnik–Zamolodchikov equations [60, 77, 79]. Since the components of ε are not all positive, the structure of $\mathcal{V}_{\Gamma,u}$ is not settled by Proposition 3.5.3. In fact, the qualitative behavior of the critical points of $\Phi_{\Gamma,u}$ changes as n, m and ε are allowed to vary. This is shown in [79] by characterizing the critical points of $\Phi_{\Gamma,u}$ in terms of polynomial solutions of Fuchsian differential equations.

Proof of Theorem 3.2.4. Let $z \in \mathbb{C}^n$ extend u, and fix $\varepsilon \in (0,\infty)^k$. We must show that z is ε-harmonic on (Γ, u) if and only if $z \in \mathcal{V}_{\Gamma,u}$. Suppose first that z is ε-harmonic on (Γ, u). Then there is $\gamma \in \mathbb{C}^k$ such that $h(\gamma) = z$ and $\Psi_{\Gamma,u}(\gamma) = \varepsilon$. Thus for all $i \in V^\circ$ we have

$$0 = \sum_{j\sim i} \gamma_{ij}(z_i - z_j) = \sum_{j\sim i} \frac{\varepsilon_{ij}}{z_i - z_j} = \frac{\partial}{\partial x_i}\Phi_{\Gamma,u}(z), \tag{3.25}$$

where we have used the definition of $\Psi_{\Gamma,u}$. Hence $z \in \mathcal{V}_{\Gamma,u}$.

Conversely, suppose that $z \in \mathcal{V}_{\Gamma,u}$, and let $\gamma \in \mathbb{C}^k$ be given by $\gamma_{ij} = \varepsilon_{ij}/(z_i - z_j)^2$ for all $ij \in E$. It is not hard to see that (3.25) holds again for all $i \in V^\circ$, so $h(\gamma) = z$ and moreover $\Psi_{\Gamma,u}(\gamma) = \varepsilon$. Hence z is ε-harmonic on (Γ, u). □

It seems likely that the following corollary is known in some form, given the extensive literature on electrical networks. However, we have not seen it stated as such.

Corollary 3.5.7. *Every point in every bounded chamber of* $\overline{\mathcal{A}}(\Gamma, u)$ *is a harmonic function on* (Γ, u, γ) *for some choice of conductances* $\gamma \in (0,\infty)^E$.

Proof. This is an application of [73, Theorem 3.3] to Theorem 3.2.4. □

For a fixed $\varepsilon \in \mathbb{C}^E$, as γ ranges over the generic conductances with $\Psi_{\Gamma,u}(\gamma) = \varepsilon$, we call the functions $h(\gamma)$ the *ε-harmonic functions* on (Γ, u). The results of Abrams and Kenyon [2] now follow:

Corollary 3.5.8 ([2, Theorems 1–3])**.** *Fix energies* $\varepsilon \in (0,\infty)^k$*, and let* $C(\varepsilon)$ *be the set of all generic conductances* $\gamma \in \mathbb{C}^k$ *for which* $\Psi_{\Gamma,u}(\gamma) = \varepsilon$*. The following hold:*

(i) $C(\varepsilon) \subset (0,\infty)^k$

(ii) The ε-harmonic functions on (Γ, u) form a system of distinct representatives for the bounded chambers of $\overline{\mathcal{A}}(\Gamma, u)$.

Proof. First we prove (i). Suppose that $\gamma \in C(\varepsilon)$. For all $ij \in E$ we have $\varepsilon_{ij} = \gamma_{ij}(h_i(\gamma) - h_j(\gamma))^2 > 0$, where we write $h_i(\gamma)$ for the ith component of $h(\gamma)$. Since $h(\gamma)$ is a real point, it follows that $\gamma_{ij} > 0$, proving (i).

To prove (ii), we show that $\overline{\mathcal{A}}(\Gamma, u)$ is essential. The result then follows from Theorem 3.2.4 and Proposition 3.5.3. For each $e \in E$ let H_e be the corresponding element of $\overline{\mathcal{A}}(\Gamma, u)$ with normal vector v_e of the form $x_i - x_j$ or $x_i - u(j)x_j$. Since Γ is connected, for any $i \in V^\circ$ there is a path $P \subseteq E$ with one endpoint i and the other endpoint in ∂V. We have $\sum_{e \in P} v_e = x_i$, replacing some v_e with $-v_e$ if necessary. It follows that the normal vectors span $\mathbb{K}^n$, so $\overline{\mathcal{A}}(\Gamma, u)$ is essential. □

In fact, there is a bijection from $C(\varepsilon)$ to the set of bounded chambers of $\overline{\mathcal{A}}(\Gamma, u)$ for all ε outside a proper algebraic subset of $\mathbb{C}^k$. This follows, for instance, from a generalization of Proposition 3.5.3 due to Orlik and Terao [66]. Corollary 3.5.8 has applications in rectangular tilings, as the ε-harmonic functions correspond to the possible tilings of a square by k rectangles with areas given by $\varepsilon \in (0, \infty)^k$ (see [2]).

Proof of Corollary 3.2.5. This follows from Theorem 3.3.19 and Corollary 3.5.8(ii). □

3.6 Dirichlet arrangements as modular fibers

A different realization of Dirichlet arrangements yields alternate proofs of some of our results, including Theorem 3.2.2. Given a central arrangement $\mathcal{A}$ in $\mathbb{R}^d$ and $X \in L(\mathcal{A})$, write $\mathcal{A}_X = \{H \in \mathcal{A} : H \supseteq X\}$. Let $\pi : \mathbb{R}^d \to \mathbb{R}^d/X$ be the natural projection. For every $H \in \mathcal{A}_X$ the image $\pi(H)$ is a hyperplane of $\mathbb{R}^d/X$. Thus we can consider the arrangement $\pi(\mathcal{A}_X)$ and its complement $U(\pi(\mathcal{A}_X))$. For every $v \in U(\pi(\mathcal{A}_X))$ let $\mathcal{A}_v$ be the restriction of $\mathcal{A}$ to the fiber $\pi^{-1}(v)$. In general $\mathcal{A}_v$ is not central, so we will consider the cone $c\mathcal{A}_v$ over $\mathcal{A}_v$ instead.

Recall from Section 2.3 that a central arrangement $\mathcal{A}$ defines a matroid $M(\mathcal{A})$ on $\mathcal{A}$. Also recall the definition of a *complete principal truncation* from Section 2.3.2. The set $X \in L(\mathcal{A})$ corresponds to aflat of the matroid $M(\mathcal{A})$. Thus we can speak of the complete principal truncation of $M(\mathcal{A})$ along X. We say that $X \in L(\mathcal{A})$ is *modular* if $X + Y \in L(\mathcal{A})$ for all $Y \in L(\mathcal{A})$, where $X + Y$ denotes the Minkowski sum. Equivalently, X is modular if $\mathrm{rk}(X \vee Y) + \mathrm{rk}(X \wedge Y) = \mathrm{rk}(X) + \mathrm{rk}(Y)$ for all $Y \in L(\mathcal{A})$, where rk is the rank function of $L(\mathcal{A})$.

Proposition 3.6.1 ([32, Theorem 2.4]). *If $\mathcal{A}$ is central and $X \in L(\mathcal{A})$ is modular, then the matroid $M(c\mathcal{A}_v)$ is isomorphic to $\overline{T}_X(M(\mathcal{A}))$, the complete principal truncation of $M(\mathcal{A})$ along X.*

Arrangements of the form $\mathcal{A}_v$ enjoy a number of desirable properties when X is modular in $L(\mathcal{A})$ [26, 32, 92]. In particular we have the following.

Proposition 3.6.2. *Suppose that X is a modular element of $L(\mathcal{A})$.*

(i) The intersection lattice $L(c\mathcal{A}_v)$ is independent of $v \in M(\pi(\mathcal{A}_X))$

(ii) We have $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}_X}(t) \cdot \chi_{\mathcal{A}_v}(t)$

(iii) If $L(\mathcal{A})$ admits a maximal chain of modular elements including X, then $\mathcal{A}_v$ is supersolvable.

(iv) If $\mathcal{A}_X$ and $\mathcal{A}_v$ are supersolvable, then $\mathcal{A}$ is supersolvable.

Proof. Item (i) follows from [32, Theorem 2.4]. Item (ii) then follows from the main result in [14].

To prove (iii) and (iv) we think of the elements of $L(\mathcal{A})$, $L(\mathcal{A}_X)$ and $L(c\mathcal{A}_v)$ as subsets of E by identifying them with the flats of the underlying matroids (see Section 2.3). From this point of view, the elements of $L(c\mathcal{A}_v)$ are precisely the elements of $L(\mathcal{A})$ that contain X or are disjoint from X (see [74, pp. 365–366]). Denote the rank functions of $L(\mathcal{A})$ and $L(c\mathcal{A}_v)$ by rk and $\overline{\mathrm{rk}}$, respectively. By Proposition 3.6.1 and induction on (2.19) we have

$$\overline{\mathrm{rk}}(F) = \begin{cases} \mathrm{rk}(F) - \mathrm{rk}(X) + 1 & \text{if } F \supseteq X \\ \mathrm{rk}(F) & \text{if } X \cap F = \emptyset. \end{cases} \tag{3.26}$$

To prove (iii), suppose that $\emptyset \subseteq Y_1 \subseteq \cdots \subseteq Y_{d-1}$ is a maximal chain of modular elements of $L(\mathcal{A})$ with $Y_{m-1} = X$. We claim that $\emptyset \subseteq Y_{m-1} \subseteq Y_{d-1}$ is a maximal chain of modular elements of $L(c\mathcal{A}_v)$. Let $F \in L(c\mathcal{A}_v)$. Since $X_i \vee F \supseteq X$ for all i, we have $\overline{\mathrm{rk}}(X_i \vee F) = \mathrm{rk}(X_i \vee F) - \mathrm{rk}(X) + 1$. If $F \supseteq X$, then $X \wedge F \supseteq X$, so $\overline{\mathrm{rk}}(X \wedge F) = \mathrm{rk}(X \wedge F) - \mathrm{rk}(X) + 1$. If $X \cap F = \emptyset$, then $(X_i \wedge F) \cap X = \emptyset$, so $\overline{\mathrm{rk}}(X_i \wedge F) = \mathrm{rk}(X_i \wedge F)$. In either case we have

$$\overline{\mathrm{rk}}(X_i \vee F) + \overline{\mathrm{rk}}(X_i \wedge F) = \mathrm{rk}(X_i \vee F) + \mathrm{rk}(X_i \wedge F) = \mathrm{rk}(X_i) + \mathrm{rk}(F),$$

proving the claim.

To prove (iv), suppose that $\emptyset \subseteq X_1 \subseteq \cdots \subseteq X_{m-1}$ and $\emptyset \subseteq Z_1 \subseteq \cdots \subseteq Z_{n+1}$ are maximal chains of modular elements of $L(\mathcal{A}_X)$ and $L(c\mathcal{A}_v)$, respectively. In this case $\emptyset \subseteq X_1 \subseteq \cdots \subseteq X_m \subseteq X \cup Z_1 \subseteq \cdots \subseteq X \cup Z_n$ is a maximal chain of modular elements of $L(\mathcal{A})$, where all but one of these containments are proper. The argument is similar to the the one in the previous paragraph. □

Dirichlet arrangements can be realized as arrangements of the form $\mathcal{A}_v$. For the remainder of the section, let $\mathcal{A} = \mathcal{A}_{\widehat{\Gamma}}$ and

$$X = \{x \in \mathbb{R}^V : x_i = x_j \text{ for all } i, j \in \partial V\}. \tag{3.27}$$

Here we have $\pi(\mathcal{A}_X) = \mathrm{ess}(\mathcal{A}(K_m))$, so $L(\mathcal{A}_X) \cong L(\mathcal{A}(K_m))$. Let $\widetilde{u} \in \mathbb{R}^V$ be any point extending u, and let $v = \widetilde{u} + X \in \mathbb{R}^V/X$. Then $\overline{\mathcal{A}}(\Gamma, u) = \mathrm{ess}(A_v)$, so $L(\overline{\mathcal{A}}(\Gamma, u)) \cong L(\mathcal{A}_v)$.

Lemma 3.6.3. *The set X defined in* (3.27) *is a modular element of $L(\mathcal{A})$.*

Proof. The subspace X corresponds to the connected partition σ of $\widehat{\Gamma}$ whose only non-singleton part is ∂V. If $Y \in L(\mathcal{A})$ has corresponding connected partition τ, then $X+Y$ is the subspace corresponding to the common refinement of τ and σ. Since ∂V is a clique, the common refinement is a connected partition. Hence $X+Y \in L(\mathcal{A})$. □

Corollary 3.3.3 now follows from Proposition 3.6.2(i). Theorem 3.2.1(ii) follows from Proposition 3.6.2(ii), since $\chi_{\mathcal{A}} = \chi_{\widehat{\Gamma}}$, $\chi_{A_v}(t) = t \cdot \chi_{\overline{\mathcal{A}}(\Gamma,u)}(t)$ and $t \cdot \chi_{\mathcal{A}_X}(t) = (t)_m$ by (2.13). Theorem 3.2.2 can be proven using Proposition 3.6.2 as follows.

Alternate proof of Theorem 3.2.2. Suppose that $\widehat{\Gamma}$ is chordal. By Proposition 2.2.6, $\mathcal{A} = \mathcal{A}(\widehat{\Gamma})$ is chordal. Since ∂V is a clique of $\widehat{\Gamma}$, Lemma 3.4.4 implies that $L(\mathcal{A})$ admits a maximal chain of modular elements including the set X defined in (3.27). Proposition 3.6.2(iii) says that $\mathcal{A}_v$ is supersolvable, so $\overline{\mathcal{A}}(\Gamma, u)$ is supersolvable.

Conversely, suppose that $\overline{\mathcal{A}}(\Gamma, u)$ is supersolvable. Since K_m is chordal, $\mathcal{A}_X$ is supersolvable. Proposition 3.6.2(iv) then implies that $\mathcal{A}$ is supersolvable, as desired. □

In Chapter 4 we define a *Dirichlet arrangement* to be a matroid of the form $M(\mathcal{A}(\Gamma, u))$, where $\mathcal{A}(\Gamma, u) = c\overline{\mathcal{A}}(\Gamma, u)$ is the cone over $\overline{\mathcal{A}}(\Gamma, u)$. The following theorem identifies these matroids as complete principal truncations.

Theorem 3.6.4. *The Dirichlet matroid $M(\mathcal{A}(\Gamma, u))$ is the complete principal truncation of the graphic matroid $M(\widehat{\Gamma})$ along the flat X consisting of all edges of $\widehat{\Gamma}$ between boundary nodes.*

Proof. This follows from Proposition 3.6.1 and the discussion preceding Lemma 3.6.3. □

3.7 Galois actions on critical sets

We prove Theorem 3.7.1 below, which generalizes an observation of Abrams and Kenyon [2, Corollary 5]. An algebraic number in $\mathbb{R}$ is called *totally real* if all of its Galois conjugates over $\mathbb{Q}$ are real. The set $\mathbb{Q}^{\mathrm{tr}}$ of all totally real numbers is a subfield of $\mathbb{R}$, and the (infinite) extension $\mathbb{Q}^{\mathrm{tr}}/\mathbb{Q}$ is Galois.

Theorem 3.7.1. *If $\mathcal{A}$ is an essential real arrangement defined over $\mathbb{Q}$ and $\varepsilon \in (0, \infty)^{\mathcal{A}}$ is a rational point, then* $\mathrm{Gal}(\mathbb{Q}^{\mathrm{tr}}/\mathbb{Q})$ *acts on the set of critical points of the* *master function* $\Phi_{\mathcal{A}}^{\varepsilon}$ *coordinatewise:*

$$\sigma \cdot (x_1, \ldots, x_n) = (\sigma(x_1), \ldots, \sigma(x_n)). \tag{3.28}$$

Hence $\mathrm{Gal}(\mathbb{Q}^{\mathrm{tr}}/\mathbb{Q})$ *acts on the set of bounded chambers of* $\mathcal{A}$*.*

Proof. Let $k = |\mathcal{A}|$. We have $x \in \mathcal{V}(\mathcal{A}, \varepsilon)$ if and only if x satisfies (3.17) for all $i = 1, \ldots, d$. Clearing denominators in (3.17) gives a system of polynomial equations over $\mathbb{Q}$:

$$\sum_{r=1}^{k} \frac{\partial f_r}{\partial x_i} \varepsilon_r \prod_{s \neq r} f_s(x) = 0. \tag{3.29}$$

By Proposition 3.5.3, the system has only finitely many solutions $x \in M(\mathcal{A}_{\mathbb{C}})$, so each solution is an algebraic point.

Let $\mathbb{K}$ be the field generated over $\mathbb{Q}$ by x_i, as x ranges over $\mathcal{V}(\mathcal{A}, \varepsilon)$ and i ranges over $1, \ldots, d$. Replace $\mathbb{K}$ by a Galois closure if necessary, and let $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. Clearly if x is a solution of the system (3.29), then $\sigma(x)$ is also a solution. Hence $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ acts on $\mathcal{V}(\mathcal{A}, \varepsilon)$. Moreover, Proposition 3.5.3 says that all solutions of (3.29) are real, so $\mathbb{K} \subset \mathbb{Q}^{\mathrm{tr}}$. The result follows. □

When $\mathcal{A} = \overline{\mathcal{A}}(\Gamma, u)$, Theorem 3.7.1 gives an action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{tr}}/\mathbb{Q})$ for each rational point $\varepsilon \in (0, \infty)^k$ on the set of ε-harmonic functions of (Γ, u) (or equivalently on the set of compatible orientations). Abrams and Kenyon conjectured in this case that if Γ is 3-connected, then the action is transitive given sufficiently general choices of u and ε [2, Conjecture 1]. Theorem 3.7.1 suggests that a similar statement might hold for any sufficiently "robust" arrangement $\mathcal{A}$.

Proposition 3.7.2 below describes an example in which Γ is 3-connected but the corresponding action is *not* transitive. A *wheel graph* on d vertices consists of a $(d-1)$-cycle and an additional vertex that is adjacent to every vertex in the $(d-1)$-cycle. The wheel graph on 15 vertices appears in Figure with vertex labels and directed edges.

Proposition 3.7.2. *Let* Γ *be a wheel graph on* $d \equiv 3 \pmod 4$ *vertices, and let* ∂V *consist of 2 opposite vertices on the outer cycle of the wheel. Fix rational boundary* *data u, and let $\varepsilon \in \mathbb{C}^k$ be identically 1. If $d > 3$, then the action of* $\mathrm{Gal}(\mathbb{Q}^{\mathrm{tr}}/\mathbb{Q})$ *on the set of ε-harmonic functions on* (Γ, u) *is not transitive.*

Proof. Label the outer vertices of Γ in a cycle by $i_0, \dots, i_{d-2}$, and write $d = 4\ell - 1$. Without loss of generality, suppose that $\partial V = \{i_0, i_{2\ell-1}\}$ with boundary values $u(i_0) = 1$ and $u(i_{2\ell-1}) = -1$. We exhibit an ε-harmonic function $f \in \mathbb{Q}^{n-2}$ on (Γ, u). Such a function is necessarily fixed by the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{tr}}/\mathbb{Q})$. Theorem 3.3.19 gives $|\overline{\mathcal{O}}_{\Gamma,u}| = (2\ell - 1)^2$, so the result will follow.

For $r = 1, \dots, \ell - 1$ let

$$f(i_r) = \prod_{s=0}^{r-1} \frac{2(\ell - 2s) - 1}{2(\ell - 2s) + 1}. \tag{3.30}$$

For $r = \ell, \dots, 2\ell - 2$ let $f(i_r) = -f(i_{2\ell-r-1})$, and for $r = 2\ell, \dots, 4\ell - 3$ let $f(i_r) = f(i_{4\ell-r-2})$. Finally, let f be 0 at the center of the wheel. This defines a function $f \in \mathbb{Q}^n$. It is routine to verify that f is ε-harmonic on (Γ, u). The case $d = 15$ is illustrated in Figure 3.6. □

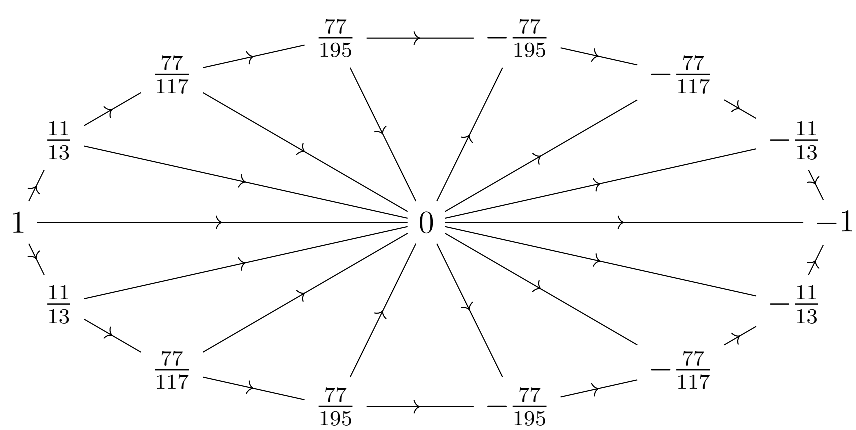


Figure 3.6: A harmonic function on a network and the associated compatible orientation.

Chapter 4

Matroids of Dirichlet arrangements

In this chapter we study the matroids underlying Dirichlet arrangements, called *Dirichlet matroids*. Theorem 3.6.4 characterizes Dirichlet matroids as complete principal truncations of graphic matroids along cliques. While graphic matroids are well studied, relatively little is known about the operation of complete principal truncation. We study the effect of this operation on various key properties of graphic matroids, including the half-plane property, planar duality and Bergman fans. In doing so, we relate Dirichlet matroids to other objects of interest, including response matrices of electrical networks, biased graphs and phylogenetic trees. Electrical networks are important objects in matroid theory, giving rise to positroids, Rayleigh matroids, log-concavity results, and related objects, such as electroids [18, 50, 72, 101]. Dirichlet matroids have not been previously connected to electrical networks.

4.1 Main results

Let Γ be a graph with an independent set $\partial V \subseteq V$ called the *boundary* and an injective function $u : \partial V \to \mathbb{K}$. Recall that the *Dirichlet arrangement* $\overline{\mathcal{A}}(\Gamma, u)$ is the restriction of the graphic hyperplane arrangement $\mathcal{A}(\Gamma)$ over $\mathbb{K}$ to the affine subspace

$$\{x \in \mathbb{K}^V : x_j = u(j) \text{ for all } j \in \partial V\}.$$

The cone $\mathcal{A}(\Gamma, u)$ over $\overline{\mathcal{A}}(\Gamma, u)$ defines a matroid. It follows from Corollary 3.3.3 that this matroid depends only on $N = (\Gamma, \partial V)$. This is the *Dirichlet matroid* associated to N, denoted by $M(N)$. Thus $M(N) = M(\mathcal{A}(\Gamma, u))$.

Our first result concerns the real roots of polynomials associated with the response Λ, which captures the electrical properties of the network N on the boundary (see Section 2.1). The entries of Λ are rational functions in the variables x_e for all $e \in E$ that encode the pairwise relationships between the boundary nodes.

Theorem 4.1.1. *Write* $\operatorname{tr}\Lambda = f/g$ *in lowest terms. The roots of* f *and* g *interlace along any line* $x + ty$ *with* $(x, y) \in \mathbb{R}^E \times (0, \infty)^E$.

We show that Theorem 4.1.1 follows from the half-plane property for Dirichlet matroids. A matroid M with set $\mathcal{B}$ of bases has the *half-plane property* if the polynomial $\sum_{B\in\mathcal{B}} \prod_{e\in B} x_e$ has no root with every x_e in the upper half-plane of $\mathbb{C}$. It is a folklore result from electrical engineering that every graphic matroid has the half-plane property [101, p. 4].

Our second result characterizes the Bergman fan $\widetilde{\mathcal{B}}(N)$ of $M(N)$, i.e., the tropical variety defined by the circuits of $M(N)$. The Bergman fan of a representable matroid determines the tropical compactification of the complement of the associated projective arrangement. We identify $\widetilde{\mathcal{B}}(N)$ as a subfan of $\widetilde{\mathcal{B}}(\widehat{\Gamma})$, the Bergman fan of the graphic matroid $M(\widehat{\Gamma})$, where $\widehat{\Gamma}$ is the graph obtained from Γ by adding an edge between each pair of boundary nodes.

Theorem 4.1.2. *There is a linear homeomorphism*

$$\widetilde{\mathcal{B}}(N) \to \widetilde{\mathcal{B}}(\widehat{\Gamma})_{\partial V}, \tag{4.1}$$

where $\widetilde{\mathcal{B}}(\widehat{\Gamma})_{\partial V}$ *is the subfan of* $\widetilde{\mathcal{B}}(\widehat{\Gamma})$ *consisting of all points constant on* $E(\widehat{\Gamma}) \setminus E(\Gamma)$.

Ardila and Klivans [3] showed that the Bergman fan $\widetilde{\mathcal{B}}(K_n)$ is homeomorphic to the space of phylogenetic n-trees. As a corollary of Theorem 4.1.2, we show that if $N = \Gamma_{m,n}$ as in Example 3.1.4, then $\widetilde{\mathcal{B}}(N)$ is homeomorphic to the space of phylogenetic trees with leaf set V in which every pair of leaves in ∂V has the same most recent common ancestor.

Our third result is a Dirichlet generalization of the result that $M^*(\Gamma) \cong M(\Gamma^*)$ when Γ is planar and Γ^* is any dual. When N is circular (i.e., when Γ can be embedded into a closed disk with ∂V lying on the boundary circle), there is a corresponding notion of a dual circular network N^*. We prove the following duality theorem for Dirichlet matroids.

Theorem 4.1.3. *Suppose that N is circular, that no vertex has degree 1, and that no vertex in $V \setminus \partial V$ has degree 2. Let $\overline{\Gamma}$ be the planar graph obtained from Γ by identifying all boundary nodes as a single vertex. If C is a cocircuit of $M(N)$, then one of the following holds:*

(i) C is a circuit of $M(\overline{\Gamma}^*)$

(ii) C can be written as a union of k distinct circuits of $M(N^*)$, where the minimum such k is less than $\frac{1}{2}|\partial V| + 1$ but not less than $\frac{1}{4}|\partial V| + \frac{1}{2}$.

In particular, if $|\partial V| = 2$, then $M^*(N) \cong M(N^*)$.

Thus the cocircuits of $M(N)$ are either cocycles of a related planar graph, or they are the union of a small number of circuits of $M(N^*)$. We give explicit families of examples to show that the bounds on k in part (ii) are tight.

Our next result concerns the reduced characteristic polynomial $\overline{\chi}_{M(N)}$, i.e., the *precoloring polynomial* of N defined in Section 3.3.2. We apply the broken circuit theorem from matroid theory to bound the coefficients of this polynomial.

Theorem 4.1.4. *Let $d = |V|$ and $n = |V \setminus \partial V|$. Write the chromatic polynomial of Γ and the precoloring polynomial of N as*

$$\begin{aligned} \chi_\Gamma(\lambda) &= a_0\lambda^d - a_1\lambda^{d-1} + \cdots + (-1)^d a_d \\ \overline{\chi}_{M(N)}(\lambda) &= b_0\lambda^n - b_1\lambda^{n-1} + \cdots + (-1)^n b_n. \end{aligned} \tag{4.2}$$

We have $a_i \geq b_i$ for all $i = 0, \dots, n$, with $a_i = b_i$ if i is less than the minimum number of edges in a path in Γ between distinct boundary nodes.

Our final result in this chapter is a simple characterization of 3-connected Dirichlet matroids.

Theorem 4.1.5. *The Dirichlet matroid $M(N)$ is 3-connected if and only if $\Gamma \setminus \partial V$ is connected and $\widehat{\Gamma}$, the graph obtained from Γ by adding a clique on ∂V, is 3-connected.*

Whitney [105] showed that 3-connected graphs are isomorphic if and only if the associated graphic matroids are isomorphic. It follows from Theorem 4.1.5 and a result of [27] that if $\Gamma \setminus \partial V$ is connected and $\widehat{\Gamma}$ is 3-connected, then there are at most 27 other networks with Dirichlet matroids isomorphic to $M(N)$.

4.2 Hyperplane, bias and matrix representations

We show how to represent Dirichlet matroids by Dirichlet arrangements and biased graphs. We also characterize the fields over which a given Dirichlet matroid is representable. We assume familiarity with basic matroid theory; our terminology follows [69].

4.2.1 Dirichlet arrangements and matroids

Let $\Gamma = (V, E)$ be a finite connected undirected graph with no loops or multiple edges. Let $\partial V \subseteq V$ be a set called the *boundary* that consists of ≥ 2 vertices, called *boundary nodes*. We call the pair $N = (\Gamma, \partial V)$ a *network*. Let $\partial E \subseteq E$ be the set of edges meeting ∂V. Recall that $\mathcal{A}(\Gamma, u)$ denotes the cone over the Dirichlet arrangement $\overline{\mathcal{A}}(\Gamma, u)$.

Definition 4.2.1. A matroid M is *Dirichlet* if $M \cong M(\mathcal{A}(\Gamma, u))$ for some pair (Γ, u).

The resulting matroid does not depend on u. We write a Dirichlet matroid as $M(N)$, where $N = (\Gamma, \partial V)$ is the associated network. Definition 4.2.1 can be extended to include $|\partial V| \leq 1$, in which case the associated Dirichlet matroid $\mathcal{A}$ is isomorphic to $M(\Gamma)$. We assume that $|\partial V| \geq 2$.

Example 4.2.2. If $|\partial V| = 2$, then $M(N) \cong M(\widehat{\Gamma})$ is graphic, where $\widehat{\Gamma}$ is the graph obtained from Γ by adding an edge between the two boundary nodes (see Proposition 4.2.22).

Example 4.2.3. Let Γ be a star graph on 4 vertices, with ∂V consisting of the 3 leaves. Let $u : \partial V \to \mathbb{K}$ be injective. The Dirichlet arrrangement $\mathcal{A}(\Gamma, u)$ consists of 3 points in $\mathbb{K}$. The Dirichlet matroid $M(N)$ is the uniform matroid $U_{2,4}$, i.e., the 4-pointed line.

Example 4.2.4. Let P be a finite poset. Recall that the *order polytope* $\mathcal{O}(P)$ of P is the set of all order-preserving functions $P \to [0, 1]$. Also recall that the *visibility arrangement* $\mathrm{vis}(\mathcal{O}(P))$ of $\mathcal{O}(P)$ is the arrangement in $\mathbb{R}^P$ whose elements are the affine spans of all facets of $\mathcal{O}(P)$.

Let Γ, ∂V and u be as in Example 3.1.5, so that $\overline{\mathcal{A}}(\Gamma, u) = \mathrm{vis}(\mathcal{O}(P))$. Example 4.2.2 implies that $M(N) \cong M(\widehat{\Gamma})$ is graphic.

4.2.2 Background on biased graphs

A *theta graph* is a graph consisting of 2 "terminal" vertices and 3 internally vertex-disjoint paths between the terminals. In other words, a theta graph resembles the symbol θ (see Figure 4.1). A *circle* of Γ is the edge set of a simple cycle of Γ. We say that a set $\mathcal{B}$ of circles of Γ is a *linear subclass* of Γ if, for any 2 distinct circles in $\mathcal{B}$ belonging to a theta subgraph H of Γ, the third circle of H also belongs to $\mathcal{B}$.

A *biased graph* is a pair $\Omega = (\Gamma, \mathcal{B})$ where $\mathcal{B}$ is a linear subclass of Γ. If a circle of Γ belongs to $\mathcal{B}$, then it is *balanced*; otherwise it is *unbalanced*. An edge set or subgraph X is *balanced* if every circle of Γ contained in X is balanced; otherwise X is *unbalanced*.

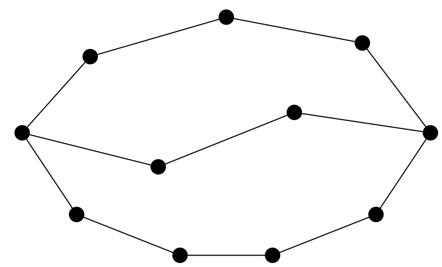


Figure 4.1: A theta graph.

There are three matroids associated to a biased graph $\Omega = (\Gamma, \mathcal{B})$, introduced by Zaslavsky [112]. The *bias matroid* $G(\Omega)$ is a matroid on E in which a set is independent if and only if each component contains either no circles, or exactly one circle, which must be unbalanced.

Let e_0 be an element not in E, and write

$$E_0 = E \cup e_0. \tag{4.3}$$

For $X \subseteq E$, let $c(X)$ denote the number of connected components of the graph (V, X). For $X \subseteq E_0$ with $e_0 \in X$, write $c(X) = c(X \setminus e_0)$. The *complete lift matroid* $L_0(\Omega)$ is the matroid on E_0 with rank function given by

$$\mathrm{rk}_{L_0(\Omega)}(X) = \begin{cases} |V| - c(X) & \text{if } X \subseteq E \text{ is balanced} \\ |V| - c(X) + 1 & \text{if } X \subseteq E \text{ is unbalanced or } e_0 \in X, \end{cases} \tag{4.4}$$

The *lift matroid* $L(\Omega)$ is the restriction of $L_0(\Omega)$ to E. Thus the circuits of $L(\Omega)$ are all the circuits of $L_0(\Omega)$ contained in E. A subset of E is independent in the lift matroid $L(\Omega)$ if and only if it contains no circles, or at most one circle, which must be unbalanced. This differs from the definition of independent sets in $G(\Omega)$ because there must be at most one circle overall, instead of one circle per connected component. The complete lift matroid is obtained by adding the unbalanced loop e_0 to $L(\Omega)$.

4.2.3 Biased graphs and networks

Let $\Omega(\Gamma)$ be the biased graph with underlying graph Γ whose linear subclass consists of all circles of Γ. Given $\Delta \subseteq E$, let Γ/Δ be the graph obtained by contracting all edges in Δ. Thus the edge set of Γ/Δ is $E \setminus \Delta$. Let Ω/Δ be the biased graph with underlying graph $\overline{\Gamma} := \Gamma/\Delta$ and linear subclass

$$\{C \in \mathcal{B} : C \subseteq E \setminus \Delta \text{ and } C \text{ is a circle of } \Gamma/\Delta\} \\ \cup \{C \setminus \Delta : C \in \mathcal{B} \text{ and } C \cap \Delta \text{ is a simple path}\}. \quad (4.5)$$

Let $\widehat{\Gamma}$ be the graph obtained from Γ by adding edges between every pair of boundary nodes, and let $\widehat{E}$ be the set of added edges. The pair $N = (\Gamma, \partial V)$ is called a *network*. We associate to N the biased graph $\Omega(N) = \Omega(\widehat{\Gamma})/\widehat{E}$. We write $G(N) = G(\Omega(N))$, and similarly for $L_0(N)$ and $L(N)$.

Definition 4.2.5. A *crossing* $C \subseteq E$ of N is a minimal path meeting 2 boundary nodes.

A circle C of $\Omega(N)$ is unbalanced if and only if C is a crossing of N.

Example 4.2.6. Consider the network N whose interior vertices form a cycle and whose boundary nodes are pendants, with each interior vertex adjacent to exactly 1 boundary node. The case $|\partial V| = 6$ is illustrated in Figure 4.2. In this example there is only one balanced circle of $\Omega(N)$, and it is the unique circle of Γ.

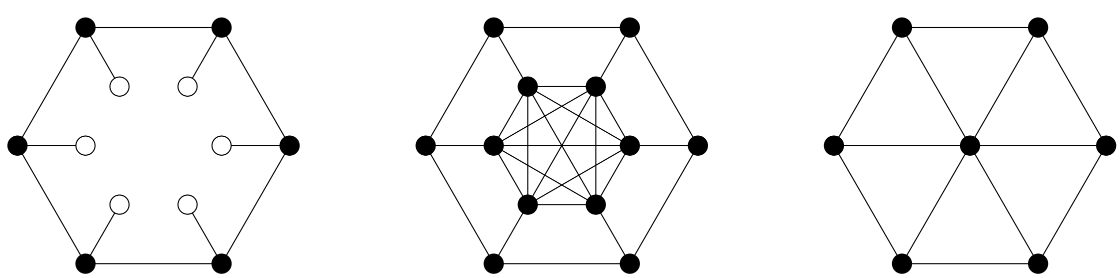


Figure 4.2: Left to right: a network N with boundary nodes marked in white, the graph $\widehat{\Gamma}$, and the graph $\overline{\Gamma}$.

There is a characterization of the biased graphs $\Omega(N)$ by Zaslavsky [110]. For $i \in V$ let $\Omega \setminus i$ be the biased graph obtained by deleting i and all edges incident to i. If Ω is unbalanced but $\Omega \setminus i$ is balanced, then i is called a *balancing vertex* of Ω. A biased graph with a unique balancing vertex is called *almost balanced*.

Proposition 4.2.7 ([110, Proposition 1])**.** *A biased graph Ω is almost balanced if and only if $\Omega = \Omega(N)$ for some network N.*

4.2.4 Equivalence of hyperplane and bias representations

A *gain graph* is a triple $\Phi = (\Gamma, \varphi, \mathfrak{G})$ consisting of a graph Γ, a group $\mathfrak{G}$ called the *gain group* and a function $\varphi : V \times V \to \mathfrak{G}$ called the *gain function* such that $\varphi(i,j) = \varphi(j,i)^{-1}$ for all (i,j). If $ij \in E$, then we consider (i,j) to be the edge ij oriented from i to j.

For any circle C of Γ, order the vertices of C in a cycle as $i_1, \dots, i_\ell = i_1$, and write

$$\varphi(C) = \varphi(i_1, i_2)\varphi(i_2, i_3) \cdots \varphi(i_{\ell-1}, i_\ell). \tag{4.6}$$

In general the element $\varphi(C)$ depends on the choice of starting vertex and direction, unless $\varphi(C)$ is the identity. Let

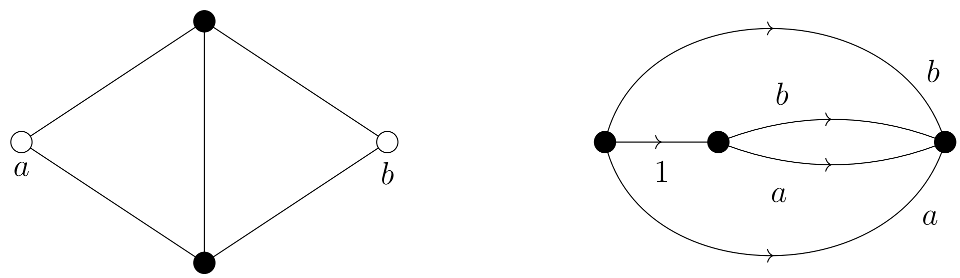
$$\mathcal{B} = \{C \subseteq E : C \text{ is a circle of } \Gamma \text{ with } \varphi(C) \text{ the identity of } \mathfrak{G}\}. \tag{4.7}$$

The set $\mathcal{B}$ is a linear subclass of Γ. Thus every gain graph defines a biased graph whose set of balanced circles is $\mathcal{B}$.

Let $u : \partial V \to \mathbb{K}$. Let $\Phi(\Gamma, u)$ be the gain graph with underlying graph $\overline{\Gamma}$; gain group $\mathbb{K}$, considered as an additive group; and gain function $\varphi : V \times V \to \mathbb{K}$ given by

$$\varphi(i,j) = \begin{cases} u(j) & \text{if } ij \in \partial E \text{ with } j \in \partial V \\ -u(i) & \text{if } ij \in \partial E \text{ with } i \in \partial V \\ 0 & \text{else.} \end{cases} \tag{4.8}$$

Example 4.2.8. Consider the graph Γ on the left side of Figure 4.2.8 with boundary nodes marked in white and values of u labeled. The associated gain graph $\Phi(\Gamma, u)$ is illustrated on the right side of Figure 4.2.8. An edge oriented from i to j with label k means that $\varphi(i,j) = k$.



Definition 4.2.9. Let $i \in V \setminus \partial V$. The *block* of N containing i is the set $U \subseteq V$ of all vertices j such that there exists a path $i_1 \cdots i_k$ in Γ with $i_1 = i$, $i_k = j$, and $i_1, \ldots, i_{k-1} \in V \setminus \partial V$.

Definition 4.2.10. The function $u : \partial V \to \mathbb{K}$ is *block injective* if the restriction of u to $U \cap \partial V$ is injective for every block U of N.

A circle C of $\Phi(\Gamma, u)$ is unbalanced if and only if C is a crossing of N between boundary nodes on which u takes distinct values, so $\Phi(\Gamma, u)$ is independent of u, as long as u is block injective. Write $\Phi(N) = \Phi(\Gamma, u)$, where u is any block-injective function.

Proposition 4.2.11. *If u is block injective, then $L_0(N) \cong M(c\mathcal{A}(\Gamma, u))$. In particular, $L_0(N) \cong M(N)$.*

Proof. Suppose that u is block injective. A circle C of $\Phi(\Gamma, u)$ is unbalanced if and only if C is a crossing of N. The discussion before Definition 4.2.5 implies that $\Phi(N) = \Omega(N)$ as biased graphs. We have $L_0(\Phi(N)) \cong M(N)$ by [113, Theorem 4.1(a)], so the result follows. $\square$

4.2.5 Independent sets, bases and circuits

A forest $F \subseteq E$ is a *grove* of N if F meets every vertex in $V \backslash \partial V$ and every component of F meets at least one boundary node.

Definition 4.2.12. Let Σ_1 be the set of all groves F of N that contain exactly 1 crossing of N. Let Σ_0 be the set of all groves F of N that contain no crossing of N.

Following Proposition 4.2.11, we take $E_0 = E \cup e_0$ to be the ground set of $M(N)$.

Proposition 4.2.13. *A set $X \subseteq E_0$ is independent in $M(N)$ if and only if one of the following holds:*
(A) $X \subseteq F$ for some $F \in \Sigma_1$
(B) $X \subseteq F \cup e_0$ for some $F \in \Sigma_0$.
Equivalently, X is dependent in $M(N)$ if and only if one of the following holds:
(C) X contains a cycle of Γ
(D) X contains 2 crossings
(E) X contains e_0 and a crossing.

Proof. This follows from [112, Theorem 3.1(c)]. □

Proposition 4.2.14. *A set $X \subseteq E_0$ is a basis of $M(N)$ if and only if one of the following holds:*
(A) $X \in \Sigma_1$
(B) $X = Y \cup e_0$ for some $Y \in \Sigma_0$.

Proof. This follows from [112, Theorem 3.1(g)]. □

Proposition 4.2.15. *A set $C \subseteq E_0$ is a circuit of $M(N)$ if and only if one of the following holds:*
(A) $C = X \cup e_0$ for some crossing X
(B) $C \subseteq E$ is a cycle of Γ meeting at most 1 boundary node
(C) $C \subseteq E$ is a minimal set containing 2 distinct crossings and no circuits of type (B).

Proof. This follows from [112, Theorem 3.1(e)]. □

Example 4.2.16. This example generalizes Example 4.2.3. Let N be a network with a single interior vertex. We call N the *star network* on $|V|$ vertices. Here $\overline{\Gamma}$ consists of 2 vertices connected by $|\partial V|$ edges, and every circle is unbalanced in $\Omega(N)$. A set $X \subseteq E_0$ is independent if and only if $|X| \leq 2$. Hence $M(N)$ is the $|V|$-pointed line $U_{2,|V|}$.

4.2.6 Matrix representations

The following theorem characterizes the fields over which a Dirichlet matroid $M(N)$ is representable. We deduce that most Dirichlet matroids are not graphic, since graphic matroids are regular.

Theorem 4.2.17. *The Dirichlet matroid $M(N)$ is representable over $\mathbb{K}$ if and only if*

$$|\mathbb{K}| \geq \max |U \cap \partial V|, \tag{4.9}$$

where the maximum runs over all blocks U of N, defined in Definition 4.2.9.

Lemma 4.2.18. *If $e \in E$, then $M(N)/e = L_0(\Omega(N)/e)$, where $\Omega(N)/e$ is the biased graph with underlying graph $\overline{\Gamma}/e$ and in which a circle $C \subseteq E \setminus e$ of $\overline{\Gamma}/e$ is balanced if and only if $C \cup e$ is a balanced circle of $\Omega(N)$.*

Proof. The result follows from the discussion in [111, p. 38]. □

Proof of Theorem 4.2.17. Let $s = \max |U \cap \partial V|$. If $|\mathbb{K}| \geq s$, then there exists a block-injective function $u : \partial V \to \mathbb{K}$. Thus $M(N)$ is representable over $\mathbb{K}$ by Proposition 4.2.11, since any hyperplane representation over $\mathbb{K}$ gives a matrix representation over $\mathbb{K}$.

Now suppose that $|\mathbb{K}| < s$, and let U be a block with $s = |U \cap \partial V|$. Let $F \subseteq E$ be the set of all edges with both endpoints in U. Let $\partial F \subseteq F$ be the set of all edges with one endpoint in $U \cap \partial V$. Deleting all edges in $E \setminus F$ and contracting all edges in $F \setminus \partial F$ yields the star network N' on $s + 1$ vertices (see Example 4.2.16). Since $M(N') \cong U_{2,s+1}$, we obtain $U_{2,s+1}$ as a minor of $M(N)$ by Lemma 4.2.18. But $U_{2,s+1}$ is not a minor of any matroid representable over $\mathbb{K}$ [69, Corollary 6.5.3]. □

Corollary 4.2.19. *The matroid $M(N)$ is representable over $\mathbb{K}$ if and only if $|\mathbb{K}|$ is at least the chromatic number of the graph with vertex set ∂V and edge set*

$$\{ij : P \cap \partial V = \{i, j\} \text{ for some path } P \subseteq V \text{ in } \Gamma\}. \tag{4.10}$$

Corollary 4.2.20. *The following are equivalent:*

(a) $M(N)$ *is binary*

(b) $M(N)$ *is regular*

(c) $|U \cap \partial V| \leq 2$ *for all blocks* U *of* N*.*

A matroid is called a $\sqrt[6]{1}$*-matroid* if it can be represented over $\mathbb{C}$ by a matrix A such that $\eta^6 = 1$ for every minor η of A.

Corollary 4.2.21. *The matroid $M(N)$ is a $\sqrt[6]{1}$-matroid if and only if $|U \cap \partial V| \leq 3$ for all blocks U of N.*

Proof. A matroid M is a $\sqrt[6]{1}$-matroid if and only if M is representable over $\mathbb{F}_3$ and $\mathbb{F}_4$ [106, Theorem 1.2]. Hence the result follows from Theorem 4.2.17. □

Proposition 4.2.22. *If $|\partial V| = 2$, then $M(N) \cong M(\widehat{\Gamma})$ is graphic.*

Proof. Suppose that $m = 2$, and let e be the edge of $\widehat{\Gamma}$ between the boundary nodes. Swapping e_0 and e gives an explicit isomorphism of matroids. □

Alternative proof. Suppose that $m = 2$. Assign an orientation of $\widehat{\Gamma}$, and let A be the associated vertex-edge incidence matrix, so that A represents $M(\widehat{\Gamma})$ over $\mathbb{K}$. Write $\partial V = \{i, j\}$, and suppose that $e = ij$ is oriented from i to j, so that $A_{j,e} = 1$. The sum of all rows of A is 0, so deleting the ith row of A does not affect the matroid represented by A. But the columns of the resulting matrix A' are normal vectors of the elements of $\mathcal{A}(N, u)$, where $u : \partial V \to \mathbb{K}$ is given by $u(i) = 0$ and $u(j) = 1$. Hence A' represents $M(\mathcal{A}(N, u))$ over $\mathbb{K}$, and the result follows from Theorem 4.2.17. □

4.3 Half-plane property and the response matrix

Let S be a finite set. For any set $\mathcal{T}$ of subsets of S, define a polynomial $w(\mathcal{T})$ over $\mathbb{C}$ by

$$w(\mathcal{T})(x) = \sum_{T \in \mathcal{T}} \prod_{s \in T} x_s, \tag{4.11}$$

where the variables x_s are indexed by S and x denotes the tuple of all x_s. The *basis generating polynomial* of a matroid M is $w(\mathcal{B})$, where $\mathcal{B}$ is the set of bases of M.

Given a complex number, vector, or matrix z, let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of z, resp. Let $\mathbb{R}^n_+$ denote the (strictly) positive orthant in $\mathbb{R}^n$. A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is *stable* if f has no roots x with $\mathrm{Im}(x) \in \mathbb{R}^n_+$.

Definition 4.3.1. A matroid M is *HPP* (short for *half-plane property*) if the basis generating polynomial of M is stable.

Stable polynomials and HPP matroids are well studied [11, 12, 19, 101, 102]. For a list of known HPP and non-HPP matroids, see [28]. The next proposition describes a fundamental family of examples (see, e.g., [19, Theorem 1.1]).

Proposition 4.3.2. *Every graphic arrangement is HPP.*

4.3.1 Laplacian and response matrices

Let $x \in \mathbb{C}^E$, and let $L = L(x)$ be the weighted Laplacian matrix with weights x, defined in Section 2.1. Write L in block form as

$$L = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}, \tag{4.12}$$

where A is the submatrix of L with rows and columns indexed by ∂V. If D is invertible, then recall that the *response matrix* of N is the $\partial V \times \partial V$ matrix $\Lambda = \Lambda(x)$ given by

$$\Lambda = A - BD^{-1}B^T. \tag{4.13}$$

If N is considered as an electrical network with edge conductances $x \in \mathbb{R}_+^E$ and voltages $u \in \mathbb{R}^{\partial V}$ applied to the boundary, then Λu is the vector of resulting currents across the boundary nodes.

Lemma 4.3.3. *Suppose that D is invertible. Let $u \in \mathbb{R}^{\partial V}$, $v = -D^{-1}B^T u$ and $\varphi = \Lambda u$. We have*

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \varphi \\ 0 \end{bmatrix}. \tag{4.14}$$

Proof. This is verified by direct computation. □

Lemma 4.3.4. *If* $\mathrm{Re}(x) \in \mathbb{R}_+^E$*, then* $\mathrm{Re}(\Lambda)$ *is positive semidefinite.*

Proof. Suppose that $\mathrm{Re}(x) \in \mathbb{R}_+^E$, and let $u \in \mathbb{R}^{\partial V}$. Let $f \in \mathbb{C}^V$ be the column vector on the left side of (4.14). Order the boundary nodes $1, \ldots, m$ and the interior vertices $m+1, \ldots, d$. We have

$$u^T \mathrm{Re}(\Lambda) u = \sum_{i,j=1}^{m} u_i \mathrm{Re}(\Lambda)_{ij} u_j = \sum_{i,j=1}^{m} \mathrm{Re}(u_i \overline{\Lambda_{ij} u_j}) = \sum_{i=1}^{m} \mathrm{Re}(u_i \overline{[\Lambda u]_i}).$$

Lemma 4.3.3 implies that $Lf|_{\partial V} = \Lambda u$ and $Lf|_{V \setminus \partial V} = 0$, so

$$\sum_{i=1}^{m} \mathrm{Re}(u_i \overline{[\Lambda u]_i}) = \sum_{i=1}^{d} \mathrm{Re}(f_i \overline{[Lf]_i}).$$

Write $x_{ij} = 0$ for all non-adjacent $i, j \in V$. Direct computation gives

$$[Lf]_i = \sum_{j=1}^{d} x_{ij}(f_i - f_j),$$

so

$$\begin{aligned}
\sum_{i=1}^{d} \mathrm{Re}(f_i\overline{[Lf]_i}) &= \sum_{i,j=1}^{d} \mathrm{Re}(f_i\overline{x_{ij}(f_i - f_j)}) \\
&= \sum_{1\le i<j\le d} \mathrm{Re}(f_i\overline{x_{ij}(f_i - f_j)}) + \sum_{1\le j<i\le d} \mathrm{Re}(f_i\overline{x_{ij}(f_i - f_j)}) \\
&= \sum_{1\le i<j\le d} \mathrm{Re}(f_i\overline{x_{ij}(f_i - f_j)}) - \sum_{1\le i<j\le d} \mathrm{Re}(f_j\overline{x_{ij}(f_i - f_j)}) \\
&= \sum_{1\le i<j\le d} \mathrm{Re}((f_i - f_j)\overline{x_{ij}(f_i - f_j)}) \\
&= \sum_{1\le i<j\le d} \mathrm{Re}(x_{ij})|f_i - f_j|^2
\end{aligned}$$

is positive. The result follows. □

4.3.2 Basis generating polynomial

We establish formulas for the basis generating polynomial of $M(N)$ and use them to prove that every Dirichlet arrangement has the half-plane property. This can also be proven using the results of Section 3.6 and [19, Proposition 4.11], but the connection to the response matrix is lost.

Let P denote the basis generating polynomial of $M(N)$, and for $i = 0, 1$ write

$$P_i = w(\Sigma_i), \tag{4.15}$$

where w is as defined in (4.11), and Σ_i are the sets of groves from Definition 4.2.12. Proposition 4.2.14 implies that

$$P(x, x_0) = P_1(x) + x_0 P_0(x) \tag{4.16}$$

for all $(x, x_0) \in \mathbb{C}^E \times \mathbb{C}$, where x_0 is the variable corresponding to e_0. Let $\operatorname{tr}\Lambda$ denote the trace of the response matrix Λ.

Lemma 4.3.5. *For all* $(x, x_0) \in \mathbb{C}^E \times \mathbb{C}$ *with* $\mathrm{Im}(x) \in \mathbb{R}_+^E$*, the basis generating polynomial of* $M(N)$ *is given by*

$$P(x, x_0) = P_0(x)\left(x_0 + \frac{1}{2}\operatorname{tr}\Lambda\right). \tag{4.17}$$

Proof. For all distinct boundary nodes i and j let

$$\Sigma_{ij} = \{F \in \Sigma_1 : F \text{ contains a path from } i \text{ to } j\}.$$

Let $P_{ij} = w(\Sigma_{ij})$, so that $P_1 = \frac{1}{2}\sum_{i \neq j} P_{ij}$. The Principal Minors Matrix-Tree Theorem implies that $\det D = P_0$, where D is the matrix defined in (4.12) (see [16]). Since P_0 is the basis generating polynomial of $M(\overline{\Gamma})$, Proposition 4.3.2 implies that Λ is well defined whenever $\mathrm{Im}(x) \in \mathbb{R}_+^E$. Thus if $\mathrm{Im}(x) \in \mathbb{R}_+^E$, then for all $i \neq j$ we have

$$-\Lambda_{ij} = \frac{P_{ij}}{P_0} \tag{4.18}$$

(see, e.g., [49, Proposition 2.8]). It is not hard to see that Λ is symmetric, and that every row sum of Λ is zero [23, p. 3]. Hence $\sum_{i \neq j} \Lambda_{ij} = -\operatorname{tr}\Lambda$. The result now follows from (4.16) and (4.18). □

Proposition 4.3.6. *Every Dirichlet matroid has the half-plane property.*

Proof. Let $(x, x_0) \in \mathbb{C}^E \times \mathbb{C}$ with $\mathrm{Re}(x) \in \mathbb{R}_+^E$ and $\mathrm{Re}(x_0) > 0$. Since P is homogeneous, it suffices to show that $P(x, x_0) \neq 0$. Since P_0 is the basis generating polynomial of $M(\overline{\Gamma})$, Proposition 4.3.2 implies that $P_0(x) \neq 0$. Thus by Lemma 4.3.5 it suffices to show that $\mathrm{Re}(\operatorname{tr}\Lambda(x)) \geq 0$ whenever $\mathrm{Re}(x) \in \mathbb{R}_+^n$. This is the content of Lemma 4.3.4. □

Corollary 4.3.7. *The bias matroid* $G(N)$ *is HPP.*

Proof. The set of bases of $G(N)$ is Σ_1 by [112, Theorem 2.1(g)]. Hence P_1 is the basis generating polynomial of $G(N)$. The result follows from [19, Proposition 2.1], since $P_1 = P(x, 0)$. □

4.3.3 Interlacing roots

We prove Theorem 4.1.1. For $1 \leq i \leq n$, the *Wronskian* with respect to x_i is the bilinear map W_{x_i} on $\mathbb{R}[x_1, \ldots, x_n]$ given by

$$W_{x_i}(f, g) = f \cdot \partial_i g - \partial_i f \cdot g. \tag{4.19}$$

If $x, y \in \mathbb{R}^n$, then $W_t(f(x + ty), g(x + ty))$ is a univariate polynomial in t. Two polynomials $f, g \in \mathbb{R}[x_1, \ldots, x_n]$ are in *proper position*, written $f \ll g$, if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n_+$ we have

$$W_t(f(x + ty), g(x + ty))(t) \geq 0 \tag{4.20}$$

for all $t \in \mathbb{R}$. For technical reasons we also declare that $0 \ll f$ and $f \ll 0$ for all f. If $f \ll g$, then for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n_+$ the real zeros of $f(x + ty)$ and $g(x + ty)$ interlace (see [11]).

Proposition 4.3.8 ([11, Corollary 5.5])**.** *Let $f, g \in \mathbb{R}[x_1, \ldots, x_n]$. We have $g \ll f$ if and only if $f + x_0 g \in \mathbb{R}[x_0, \ldots, x_n]$ is stable.*

Proof of Theorem 4.1.1. The result follows from (4.16), Proposition 4.3.6, and Proposition 4.3.8. □

4.3.4 Rayleigh monotonicity

For all $1 \leq i, j \leq n$ let $\mathcal{E}_{ij}$ be the bilinear map on $\mathbb{R}[x_1, \ldots, x_n]$ given by

$$\mathcal{E}_{ij}(f, g) = \partial_i f \cdot \partial_j g + \partial_j f \cdot \partial_i g - f \cdot \partial_i \partial_j g - \partial_i \partial_j f \cdot g, \tag{4.21}$$

where $\partial_i = \frac{\partial}{\partial x_i}$ etc. Also let

$$\Delta_{ij}(f) = \frac{1}{2}\mathcal{E}_{ij}(f, f) = \partial_i f \cdot \partial_j f - f \cdot \partial_i \partial_j f. \tag{4.22}$$

The polynomial f is *multiaffine* if the power of every x_i is at most 1 in every term of f.

Proposition 4.3.9 ([11, Theorem 5.10])**.** *A multiaffine polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is stable if and only if $\Delta_{ij}(f)(x) \geq 0$ for all $x \in \mathbb{R}^n$.*

Consider a matroid M on $E(M)$ with basis generating polynomial f. For any $i, j \in E(M)$, the *Rayleigh difference* of i and j in M is the polynomial

$$\Delta_{ij}(M) = \Delta_{ij}(f). \tag{4.23}$$

Proposition 4.3.9 implies that M is HPP if and only if $\Delta_{ij}(M)(x) \geq 0$ for all $i, j \in E(M)$ and all $x \in \mathbb{R}^{E(M)}$.

Consider N as a network of resistors with conductances $x \in \mathbb{R}_+^E$. If $|\partial V| = 2$, then $\operatorname{tr} \Lambda$ is the *effective conductance* between the two boundary nodes. *Rayleigh's Monotonicity Law* is the classical result that if a single conductance x_e increases while all other conductances remain constant, then the effective conductance between the two boundary nodes does not decrease. Thus the following proposition generalizes Rayleigh's Monotonicity Law.

Proposition 4.3.10. *If $x \in \mathbb{R}^E$ with $P_0(x) \neq 0$, then $\operatorname{tr} \Lambda(x)$ does not decrease when a single x_f increases and x_e remains constant for all $e \neq f$.*

Proof. We have

$$\Delta_{e_0 f}(P) = P_0 \cdot (\partial_f P_1 + x_0 \partial_f P_0) - (P_1 + x_0 P_0) \cdot \partial_f P_0 = P_0 \cdot \partial_f P_1 - P_1 \cdot \partial_f P_0.$$

Hence

$$\frac{1}{2} \partial_f \operatorname{tr} \Lambda = \frac{P_0 \cdot \partial_f P_1 - P_1 \cdot \partial_f P_0}{(P_0)^2} = \frac{\Delta_{e_0 f}(P)}{(P_0)^2} = \frac{\Delta_{e_0 f}(M(N))}{(P_0)^2},$$

since $\operatorname{tr} \Lambda = P_1 / P_0$. On the other hand, Theorem 4.3.6 and Proposition 4.3.9 imply that $\Delta_{e_0 f}(M(N))$ is nonnegative on $\mathbb{R}^E \times \mathbb{R}$. The result follows. □

The next proposition gives a Cauchy–Schwarz-type characterization of multivariate polynomials in proper position, and seems to be new.

Proposition 4.3.11. *We have $g \ll f$ for $f, g \in \mathbb{R}[x_1, \ldots, x_n]$ if and only if f or g is stable,*

$$W_{x_i}(f, g) \leq 0 \tag{4.24}$$

for all $1 \leq i \leq n$ and all $x \in \mathbb{R}^n$, and

$$\mathcal{E}_{ij}(f, g)^2 \leq \mathcal{E}_{ij}(f, f) \cdot \mathcal{E}_{ij}(g, g) \tag{4.25}$$

for all $1 \leq i, j \leq n$ and all $x \in \mathbb{R}^n$.

Proof. Let $h(x) = f(x) + x_0 g(x)$. Direct computation gives

$$\Delta_{ij}(h)(x, x_0) = x_0^2 \cdot \Delta_{ij}(g)(x) + x_0 \cdot \mathcal{E}_{ij}(f, g)(x) + \Delta_{ij}(f)(x),$$

which is nonnegative for all $(x, x_0) \in \mathbb{R}^n \times \mathbb{R}$ by Proposition 4.3.9 and, considered as a polynomial in x_0, has discriminant

$$\mathcal{E}_{ij}(f, g)^2 - 4\Delta_{ij}(f) \cdot \Delta_{ij}(g) = \mathcal{E}_{ij}(f, g)^2 - \mathcal{E}_{ij}(f, f) \cdot \mathcal{E}_{ij}(g, g).$$

The result follows. □

Proposition 4.3.2 is equivalent to the statement that $\Delta_{ij}(P_0)$ is nonnegative on $\mathbb{R}^E$. The next result strengthens Proposition 4.3.2 by giving a nontrivial lower bound for $\Delta_{ij}(P_0)$.

Corollary 4.3.12. *For all $e, f \in E$ and all $x \in \mathbb{R}^E$ that are not roots of $\Delta_{ef}(P_1)$ we have*

$$\Delta_{ef}(M(\overline{\Gamma}))(x) \geq \frac{\mathcal{E}_{ef}(P_0, P_1)(x)^2}{4\Delta_{ef}(P_1)(x)} \geq 0. \tag{4.26}$$

Proof. Let $e, f \in E$, and let $x \in \mathbb{R}^E$ be such that $\Delta_{ef}(P_1)(x) \neq 0$. Theorem 4.3.6 and Proposition 4.3.11 imply that

$$\mathcal{E}_{ef}(P_0, P_1)(x)^2 \leq 4\Delta_{ef}(P_0)(x) \cdot \Delta_{ef}(P_1)(x) = 4\Delta_{ef}(M(\overline{\Gamma}))(x) \cdot \Delta_{ef}(P_1)(x). \tag{4.27}$$

Proposition 4.3.2 implies that $\Delta_{ef}(M(\overline{\Gamma}))(x) \geq 0$. Since P_0 and P_1 are nonzero, (4.27) implies that $\Delta_{ef}(P_1)(x) > 0$. The result follows after dividing (4.27) through by $4\Delta_{ef}(P_1)(x)$. □

4.4 Bergman fans

We prove Theorem 4.1.2. Given a ground set E and a subset $X \subseteq E$, let $V(X)$ be the set of points $w \in \mathbb{R}^E$ such that the minimum of the set $\{w_e : e \in X\}$ is achieved at least twice. For a set $\mathcal{X}$ of subsets of E, let $V(\mathcal{X}) = \bigcap_{X \in \mathcal{X}} V(X)$. The *Bergman fan* of a matroid M on E is the set $\widetilde{\mathcal{B}}(M) = V(\mathcal{C})$, where $\mathcal{C}$ is the set of circuits of M. If M is representable, then $\widetilde{\mathcal{B}}(M)$ determines the tropical compactification of the complement of the associated projective arrangement [34].

Write $\widetilde{\mathcal{B}}(\Gamma) = \widetilde{\mathcal{B}}(M(\Gamma))$ and $\widetilde{\mathcal{B}}(N) = \widetilde{\mathcal{B}}(M(N))$. The Bergman fan of a matroid is the support of a polyhedral fan, i.e. a polyhedral complex in which every polyhedron is a cone. Theorem 4.1.2 identifies $\widetilde{\mathcal{B}}(N)$ as a subfan of $\widetilde{\mathcal{B}}(\widehat{\Gamma})$. Bergman fans of matroid truncations appear in the proof of Rota's log-concavity conjecture for representable matroids by Huh and Katz [42]. Bergman fans of graphic matroids have also been studied, and have been shown to realize the space of phylogenetic trees [3, 107].

A *tropical basis* of M is any set $\mathcal{X}$ of subsets of E such that $V(\mathcal{X}) = \widetilde{\mathcal{B}}(M)$. We prove Theorem 4.1.2 by computing a suitable tropical basis of the Dirichlet matroid $M(N)$. This tropical basis happens to be minimal (with respect to inclusion). First we need a basic lemma. *Pasting* two circuits means taking their symmetric difference. Since the symmetric difference operation is commutative and associative, we can paste any finite number of sets.

Lemma 4.4.1. *If Y is obtained by pasting elements of $\mathcal{X}$, then $V(\mathcal{X}) \subseteq V(Y)$.*

Proof. Suppose that Y is obtained by pasting elements X_1 and X_2 of $\mathcal{X}$. Let $w \in V(\mathcal{X})$. Suppose without loss of generality that $\min\{w_e : e \in X_1\} \leq \min\{w_e : e \in X_2\}$. If $\min\{w_e : e \in X_1\}$ is achieved on $X_1 \cap X_2$, then it equals $\min\{w_e : e \in X_2\}$, so

$\min\{w_e : e \in Y\}$ is achieved at least once on each of X_1 and X_2. If not, then $\min\{w_e : e \in Y\}$ is achieved at least twice on X_1. In either case we have $w \in V(Y)$. $\square$

A *chord* of a circuit C is any element i such that there exist circuits C_1 and C_2 with $C_1 \cap C_2 = i$ and $C_1 \triangle C_2 = C$. Recall the 3 types of circuits of $M(N)$ described in Proposition 4.2.15. A chord of a circuit $X \cup e_0$ of type (A) is any edge in $E \setminus X$ joining two vertices met by X. A chord of a circuit C of type (B) is any edge in $E \setminus C$ joining two vertices met by C.

Proposition 4.4.2. *There is a minimal tropical basis of $M(N)$ consisting of all chordless circuits of types (A) and (B) in Proposition 4.2.15.*

Proof. Suppose that a circuit $C \subseteq E$ of type (B) admits a chord j. Then there is a set $F \subseteq C$ such that $F \cup j$ and $(C \setminus F) \cup j$ are circuits of type (B). Pasting these two circuits yields C. Iterating this argument gives C as a pasting of chordless cycles of type (B).

Let $\overline{X} = X \cup e_0$ be a circuit of type (A) for some $X \subseteq E$. If i is a chord of $\overline{X}$, then $X \cup i$ contains a single circuit Z of type (B), and the set $(X \setminus Z) \cup i$ is a crossing. Pasting Z and $(X \setminus Z) \cup i$ yields $\overline{X}$. Iterating this argument and the argument from the first paragraph gives $\overline{X}$ as a pasting of chordless cycles of types (A) and (B).

Let $\mathcal{B}$ be the set of all chordless circuits of types (A) and (B). Let Y be a circuit of type (C). We claim that $V(\mathcal{B}) \subseteq V(Y)$. Let C_1 and C_2 be distinct crossings contained in Y, so that $Y = C_1 \cup C_2$. Let $w \in V(\mathcal{B})$. Suppose without loss of generality that $\min\{w_e : e \in Y \cup e_0\}$ occurs on $C_1 \cup e_0$. If C_1 and C_2 are disjoint and $w_{e_0} = \min\{w_e : e \in Y\}$, then this minimum is achieved at least once on each of C_1 and C_2. If C_1 and C_2 are disjoint and $w_{e_0} \neq \min\{w_e : e \in Y\}$, then this minimum is achieved at least twice on C_1. Therefore $w \in V(Y)$ in this case.

Suppose now that C_1 and C_2 are not disjoint, so that Y contains a third crossing C_3. If $\min\{w_e : e \in C_2\} > \min\{w_e : e \in C_1 \cup e_0\}$, then the latter must occur twice on C_1; otherwise, $\min\{w_e : C_2 \cup e_0\}$ occurs only once, on e_0, contradicting $w \in V(\mathcal{B})$. Hence $\min\{w_e : e \in Y\}$ occurs twice, on C_1. If $\min\{w_e : e \in C_2\} = \min\{w_e : e \in C_1 \cup e_0\}$ and these are both equal to w_{e_0}, then $\min\{w_e : e \in C_3 \cup e_0\}$ occurs only once, on e_0, a contradiction. Hence if $\min\{w_e : e \in C_2\} = \min\{w_e : e \in C_1 \cup e_0\}$, then these minima are less than w_{e_0}, and the minimum $\min\{w_e : e \in Y\}$ occurs at least 3 times. Therefore $w \in V(Y)$ again, proving the claim. It follows that $\mathcal{B}$ is a tropical basis of $M(N)$.

We show that $\mathcal{B}$ is inclusion minimal. Suppose that the circuit C from above is chordless. For some $e \in C$, let $w \in \mathbb{R}^{E_0}$ be 1 on $C \setminus e$ and 0 on the rest of E_0. Any circuit in $\mathcal{B} \setminus C$ must contain at least two elements of E_0 not in C. The point w achieves its minimum at least twice on such a circuit. Hence $w \in V(\mathcal{B} \setminus C) \setminus V(\mathcal{B})$, proving that $\mathcal{B} \setminus C$ is not a tropical basis.

Suppose now that the circuit $\overline{X}$ from above is chordless. Let $x \in \mathbb{R}^{E_0}$ be 1 on X and 0 on $E_0 \setminus X$. Any circuit of type (A) in $\mathcal{B} \setminus \overline{X}$ must contain e_0 and at least one edge in $E \setminus X$. Any circuit of type (B) in $\mathcal{B} \setminus \overline{X}$ must contain at least two elements of $E \setminus X$. The point x achieves its minimum at least twice on any such circuit. Hence $\mathcal{B} \setminus \overline{X}$ is not a tropical basis, proving that $\mathcal{B}$ is inclusion minimal. □

We remarked in the introduction that $\widetilde{\mathcal{B}}(M)$ has the structure of a polyhedral fan. Two points w and z belong to the same cone of $\widetilde{\mathcal{B}}(M)$ if and only if the set of w-maximal bases is the set of z-maximal bases [34, Proposition 2.5]. By a *w-maximal* basis B we mean that $\sum_{i \in B} w_i$ is maximal among all bases B of M. This decomposition of $\widetilde{\mathcal{B}}(M)$ is sometimes called its *coarse subdivision*. A *subfan* of $\widetilde{\mathcal{B}}(M)$ is a polyhedral fan, each of whose cones is a cone of $\widetilde{\mathcal{B}}(M)$.

Lemma 4.4.3. *For any $w \in \widetilde{\mathcal{B}}(\widehat{\Gamma})$ and any w-maximal spanning tree T of $K_{\partial V}$, there is a w-maximal spanning tree of $\widehat{\Gamma}$ containing T.*

Proof. Let $w \in \widetilde{\mathcal{B}}(\widehat{\Gamma})$. We construct the desired tree with a greedy algorithm. Suppose that some number, possibly zero, of the edges of $\widehat{\Gamma}$ are colored red. Let C be a cycle of $\widehat{\Gamma}$. If C contains a red edge, then do nothing. If C contains no red edges, then color a w-minimal edge of C red. This procedure is called the *red rule*. Starting with all edges of $\widehat{\Gamma}$ uncolored and applying the red rule to all cycles of $\widehat{\Gamma}$ in any order yields a w-maximal spanning tree of $\widehat{\Gamma}$ consisting of the uncolored edges [91, Theorem 6.1].

Suppose that no edge in C is red, and that C contains exactly one edge not in $E(\Gamma)$. Since $w \in \widetilde{\mathcal{B}}(\widehat{\Gamma})$, there must be a w-minimal edge of C in $E(\Gamma)$. When applying the red rule to such a cycle, we stipulate that only edges in $E(\Gamma)$ may be colored red. We will call this the *modified red rule.*

Start with $\widehat{\Gamma}$ uncolored. First apply the red rule to all cycles of $\widehat{\Gamma}$ contained in $E(\Gamma)$. Next, apply the modified red rule to all cycles of $\widehat{\Gamma}$ containing exactly one edge not in $E(\Gamma)$. At this stage, if a cycle contains edges in $E(\Gamma)$, then it contains a red edge. Moreover all red edges are in $E(\Gamma)$. Let S be the set of uncolored edges in $E(\Gamma)$. It follows that if T is any w-maximal spanning tree of $K_{\partial V}$, then $S \cup T$ is a w-maximal spanning tree of $\widehat{\Gamma}$. □

Proof of Theorem 4.1.2. Let $f : \mathbb{R}^{E_0} \to \mathbb{R}^{E(\widehat{\Gamma})}$ be such that the eth coordinate of $f(x)$ is x_e if $e \in E(\Gamma)$ and x_{e_0} otherwise. Clearly f is linear and injective. We claim that $f(\widetilde{\mathcal{B}}(N)) = \widetilde{\mathcal{B}}(\widehat{\Gamma})_{\partial V}$.

Let $U \subseteq \mathbb{R}^{E(\widehat{\Gamma})}$ be the subspace of points constant on $E(K_{\partial V})$. Let $x \in \widetilde{\mathcal{B}}(\widehat{\Gamma}) \cap U$, and let $X \subseteq E_0$ be a circuit of $M(N)$. If $X \subseteq E(\Gamma)$, then X is also a circuit of $M(\widehat{\Gamma})$, so the minimum of $\{f(x)_e : e \in X\}$ is obtained at least twice. If instead $e_0 \in X$, then $X \setminus e_0$ is a crossing between two boundary nodes i and j. In this case $(X \setminus e_0) \cup ij$ is a circuit of $M(\widehat{\Gamma})$. Since $ij \in E(K_{\partial V})$ we have $x_{ij} = f(x)_{e_0}$, so again the minimum of $\{f(x)_e : e \in X\}$ is achieved at least twice. Hence $f(x) \in \widetilde{\mathcal{B}}(N)$, proving the claim.

It remains to be shown that $\widetilde{\mathcal{B}}(\widehat{\Gamma})_{\partial V}$ is a subfan of $\widetilde{\mathcal{B}}(\widehat{\Gamma})$. Let $w \in \widetilde{\mathcal{B}}(\widehat{\Gamma})$. The restriction $w|_{E(K_{\partial V})}$ of w to $E(K_{\partial V})$ belongs to $\widetilde{\mathcal{B}}(K_{\partial V})$. Since $K_{\partial V}$ is complete, this restriction defines a combinatorial type of phylogenetic m-tree (see Section 4.4.1). By [3, Proposition 3], the combinatorial type of tree determines and is determined by the set of $w|_{E(K_{\partial V})}$-maximal spanning trees of $K_{\partial V}$. In particular, the following are equivalent:

(i) This tree has no internal edges

(ii) Every spanning tree of $K_{\partial V}$ is $w|_{E(K_{\partial V})}$-maximal

(iii) $w \in U$.

Suppose that $w \in U$ and $z \in \widetilde{\mathcal{B}}(\widehat{\Gamma}) \setminus U$. Let T be a spanning tree of $K_{\partial V}$ that is not $z|_{E(K_{\partial V})}$-maximal. There is no z-maximal spanning tree of $\widehat{\Gamma}$ containing T. However, Lemma 4.4.3 implies that w-maximal spanning tree of $\widehat{\Gamma}$ containing T. Hence w and z belong to different cones of $\widetilde{\mathcal{B}}(\widehat{\Gamma})$, proving that $\widetilde{\mathcal{B}}(\widehat{\Gamma}) \cap U = \widetilde{\mathcal{B}}(\widehat{\Gamma})_{\partial V}$ is a subfan of $\widetilde{\mathcal{B}}(\widehat{\Gamma})$. □

4.4.1 Phylogenetic trees and discriminantal arrangements

We show that when $\widehat{\Gamma}$ is complete, $\widetilde{\mathcal{B}}(N)$ realizes a certain subclass of phylogenetic trees. Let T be a rooted tree with labeled leaves and a real-valued function ω on its edges. Suppose that the root is not a leaf, and that no non-root vertex has degree 2. The *distance* between any distinct vertices i, j of T is the (possibly negative) sum of $\omega(e)$ over the edges e in the unique path between i and j. The pair (T, ω) is called a *phylogenetic tree* if

(i) The distance between the root and any leaf is the same, and

(ii) $\omega(e) > 0$ for any edge e not incident to a leaf.

A phylogenetic tree with ℓ leaves is called a *phylogenetic ℓ-tree*.

The vertices of a phylogenetic tree form a poset in which the root is the unique minimal element and the leaves are the maximal elements. If two vertices i and j are adjacent with $i \leq j$, then j is the *child* of i. A phylogenetic tree is *binary* if every non-leaf vertex has exactly two children. The *most recent common ancestor* of two vertices i and j is their infimum. The *combinatorial type* of a phylogenetic tree (T, ω) is simply the tree T (along with its root and leaf labeling).

Billera, Holmes and Vogtmann [8] gave a geometric realization of the space of phylogenetic n-trees. To construct the space $\mathcal{T}_n$, one first takes $(n-2)$-dimensional orthants corresponding to the combinatorial types of binary phylogenetic n-trees. The $n-2$ facets of such an orthant correspond to the internal edges of the binary tree; a point in the facet represents a tree in which the corresponding edge has been contracted. Every contracted tree arises from exactly 2 binary trees. One then glues the facets of 2 orthants together when they represent the same combinatorial type of tree. The lower-dimensional faces represent further contractions and are glued together according to the same rule. Thus the common vertex of the orthants represents the tree with no internal edges. The space $\mathcal{T}_3$ is illustrated in Figure 4.3.

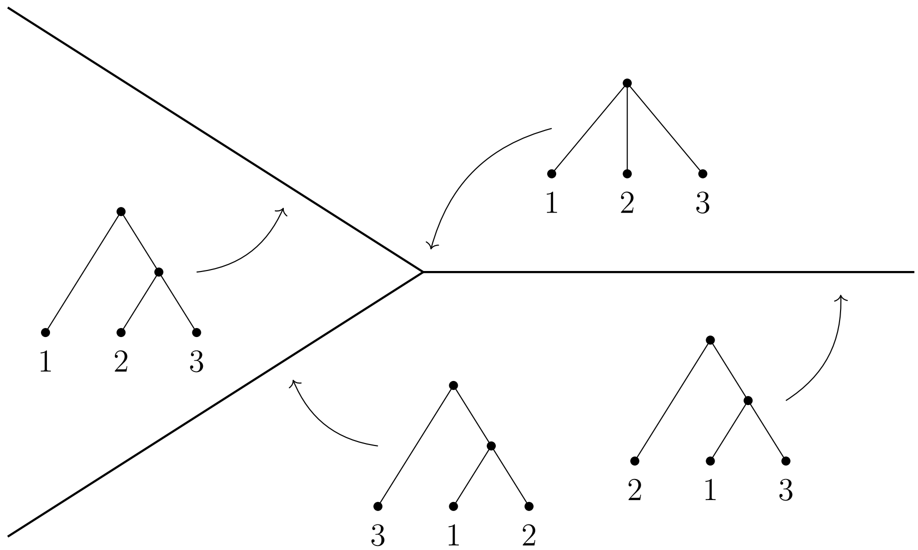


Figure 4.3: The space $\mathcal{T}_3$.

The space $\mathcal{T}_n \times \mathbb{R}$ parametrizing the phylogenetic n-trees. The factor of $\mathbb{R}$ keeps track of the distance between the root and any leaf, which is not fixed in the construction of [8]. Moreover $\mathcal{T}_n \times \mathbb{R}$ is a polyhedral fan whose cones correspond to the combinatorial types of the trees. Ardila and Klivans [3] showed that $\widetilde{\mathcal{B}}(K_n)$ and $\mathcal{T}_n \times \mathbb{R}$ realize the same polyhedral fan.

Proposition 4.4.4 ([3, Proposition 3])**.** *There is a piecewise-linear homeomorphism*

$$\mathcal{T}_n \times \mathbb{R} \to \widetilde{\mathcal{B}}(K_n) \tag{4.28}$$

that identifies the decomposition of $\mathcal{T}_n \times \mathbb{R}$ into combinatorial tree types with the coarse decomposition of $\widetilde{\mathcal{B}}(K_n)$.

If we identify the leaves of a phylogenetic tree $(T, \omega) \in \mathcal{T}_n \times \mathbb{R}$ with K_n, then the function f is easy to describe: $f(T)_{ij} = \omega(i,j)$ for all leaves i and j. We say that a set S of leaves of T is *equidistant* if every pair of leaves in S has the same most recent common ancestor. It is not hard to check that S is equidistant if and only if $\omega(i,j) = \omega(i,k)$ for all $i,j,k \in S$. Let $\mathcal{T}_{m,n} \times \mathbb{R}$ denote the space of phylogenetic

$(m+n)$-trees with a prescribed equidistant m-set. Up to permutation of coordinates, this space depends only on the size of the equidistant set.

Let $N = \Gamma_{m,n}$, so that $\widehat{\Gamma} = K_d$. Restricting $f : \mathcal{T}_d \times \mathbb{R} \to \widetilde{\mathcal{B}}(\widehat{\Gamma})$ to the set of trees with ∂V equidistant, we obtain the following.

Proposition 4.4.5. *There is a piecewise-linear homeomorphism*

$$\mathcal{T}_{m,n} \times \mathbb{R} \to \widetilde{\mathcal{B}}(\Gamma_{m,n}) \tag{4.29}$$

that identifies the decomposition of $\mathcal{T}_{m,n} \times \mathbb{R}$ into combinatorial tree types with the coarse decomposition of $\widetilde{\mathcal{B}}(\Gamma_{m,n})$.

4.5 Dual networks

We now prove Theorem 4.1.3. A network N is *circular* if there is an embedding of Γ into a closed disk D in the plane such that ∂V belongs to the boundary ∂D and V° belongs to the interior. In this section we assume that N is circular and equipped with such an embedding. We also assume that no vertex in V is of degree 1, and that no vertex in $V \setminus \partial V$ is of degree 2.

Let R be the set of components of $D \setminus \Gamma$, and let $\partial R \subset R$ be the set of components meeting ∂D. There is a circular network N^* whose vertices (resp., boundary nodes) correspond to the elements of R (resp., of ∂R) and in which two vertices are adjacent if and only if the corresponding elements of R are adjacent. Thus the edges of N^* correspond to the edges of N. The network N^* is called the *dual* of N. An example is illustrated in Figure 4.4.

The requirement that N have no vertices of degree 1 and no interior vertices of degree 2 ensures that N^* has no multiple edges or edges between boundary nodes. Moreover N^* has no vertices of degree 1 and no interior vertices of degree 2.

Definition 4.5.1. An *insulator* of N is a minimal set $Y \subset E$ containing paths between every pair of boundary nodes of N^*.

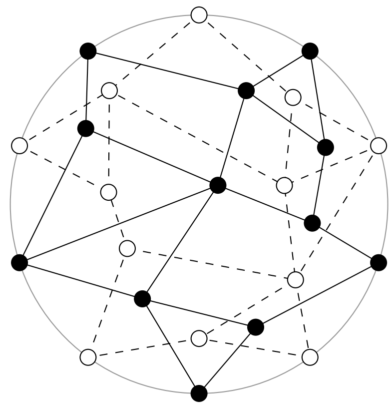


Figure 4.4: A circular network and its dual network, with one vertex set marked in white and the other in black.

Proposition 4.5.2. *A set $X \subset E_0$ is a cocircuit of $M(N)$ if and only if one of the following holds:*

(i) X is a cocircuit of $M(\overline{\Gamma})$

(ii) $X = Y \cup e_0$ for some insulator Y of N.

Proof. This follows from Proposition 4.2.14. □

Lemma 4.5.3. *If $|\partial V| = 2$, then $M^*(N) \cong M(N^*)$.*

Proof. Suppose that $|\partial V| = 2$. Proposition 4.2.22 says that $M(N) \cong M(\widehat{\Gamma})$. Since N is circular, $\widehat{\Gamma}$ is planar; let $\widehat{\Gamma}^*$ be its dual graph. Notice that $\widehat{\Gamma}^*$ is the graph obtained by adding an edge between the 2 boundary nodes of N^*. Hence $M(N^*) \cong M(\widehat{\Gamma}^*)$ again by Proposition 4.2.22. We have $M^*(\widehat{\Gamma}) \cong M(\widehat{\Gamma}^*)$ by [69, Lemma 2.3.7], so (i) follows. □

Lemma 4.5.4. *Any cocircuit C of $M(N)$ of type (ii) in Proposition 4.5.2 can be written as a union of distinct circuits of $M(N^*)$. The minimal number of circuits required is at least $\frac{1}{4}|\partial V| + \frac{1}{2}$ and at most $\frac{1}{2}|\partial V| + \frac{1}{2}$, with both extremes occurring for infinitely many values of $|\partial V|$.*

Proof. The case $|\partial V| = 2$ follows from Lemma 4.5.3. Suppose that $|\partial V| \geq 3$. Let $C_1, \ldots, C_k$ be distinct circuits of $M(N^*)$ such that $C = C_1 \cup \cdots \cup C_k$. Recall the 3 types of circuits of $M(N^*)$ from Proposition 4.2.15. If k is minimal, then one C_i is a circuit of type (iii), and the others are circuits of type (ii). A circuit of type (ii) meets exactly 3 nodes if it is connected and exactly 4 nodes if it is disconnected. A circuit of type (iii) meets exactly 2 nodes. Hence k is not less than $\frac{1}{4}|\partial V| + \frac{1}{2}$.

We now show by induction that k can be taken to be less than $\frac{1}{2}|\partial V| + \frac{1}{2}$. For the base case, let C_1' be a circuit of $M(N^*)$ of type (ii), so C_1' meets at least 3 boundary nodes of N^*. For the inductive step, suppose that $C_1', \ldots, C_{j-1}'$ are distinct circuits of $M(N^*)$ of type (ii). Let n_j be the number of boundary nodes of N^* not met by $U_j := C_1' \cup \cdots \cup C_{j-1}'$. If $n_j \leq 1$, then set $\ell = j$. If $n_j = 0$, then let C_j' be any circuit of $M(N^*)$ of type (iii). If $n_j = 1$, then let C_j' be a circuit of $M(N^*)$ of type (iii) such that $C_j \cap E$ is not contained in U_j. If $n_j \geq 2$, then let C_j' be a circuit of $M(N^*)$ of type (ii) meeting exactly 2 boundary nodes of N^* not met by U_j. Continue this procedure until a value of ℓ is reached. By construction we have $\ell = \lceil \frac{1}{2}|\partial V| \rceil \leq \frac{1}{2}|\partial V| + \frac{1}{2}$, as desired.

For proofs that both extreme values of k occur infinitely often, see Examples 4.5.5 and 4.5.6 below. □

Example 4.5.5. Consider the networks in Figure 4.5. From left to right, these are the *sunflower networks* on 4, 5 and 6 boundary nodes. We obtain a similar network on any number of boundary nodes.

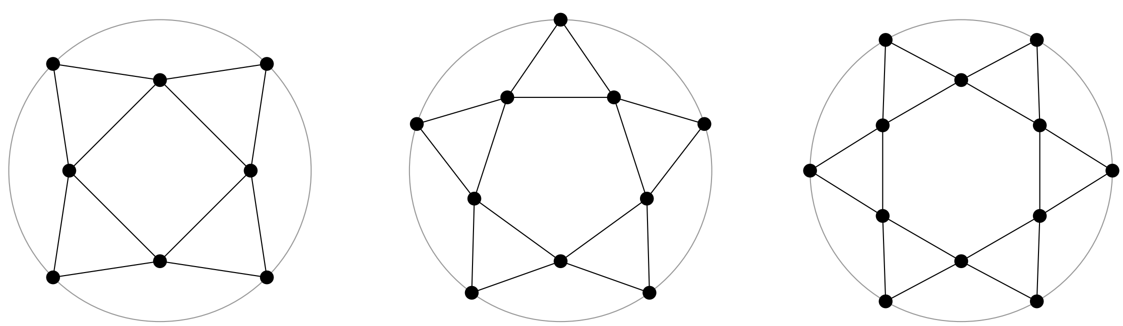


Figure 4.5: Three sunflower networks.

The left side of Figure 4.8 illustrates the sunflower network N with $|\partial V| = 5$ and its dual N^*. On the right side, an insulator Y of N is highlighted in blue. The minimum number of circuits of $M(N^*)$ whose union is $Y \cup e_0$ is 3. In a similar fashion we can construct an insulator of the sunflower network on any odd number m of boundary nodes. The corresponding minimum number of circuits is $\frac{1}{2}m + \frac{1}{2}$, achieving the upper bound for k in Lemma 4.5.4.

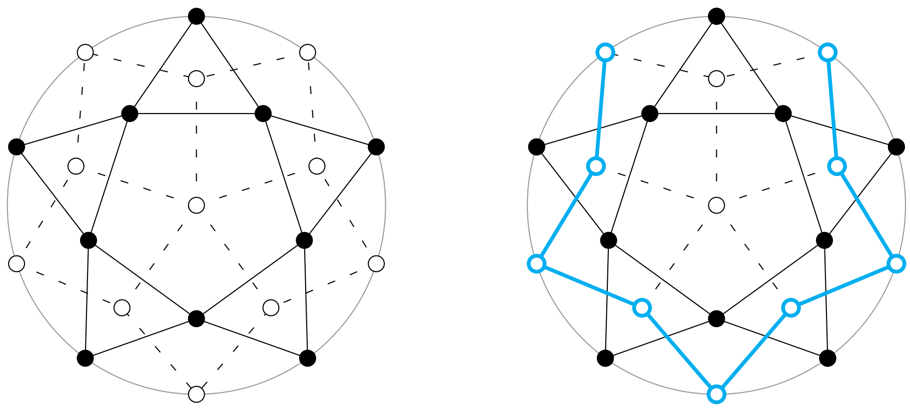


Figure 4.6: A sunflower network and its dual, left; an insulator in blue, right.

Example 4.5.6. Consider the networks in Figure 4.7. From left to right, these are the *double sunflower networks* on 4, 6 and 10 boundary nodes. We obtain a similar network on any even number of boundary nodes.

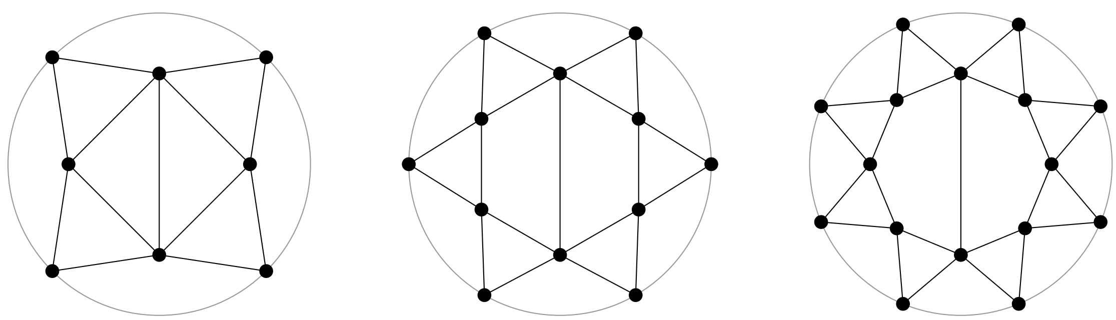


Figure 4.7: Three double sunflower networks.

The left side of Figure 4.8 illustrates the double sunflower network N with $|\partial V| = 6$ and its dual N^*. On the right side, an insulator Y of N is highlighted in blue.

The minimum number of circuits of $M(N^*)$ whose union is $Y \cup e_0$ is 2. In a similar fashion we can construct an insulator of the double sunflower network on any number $m \equiv 2 \pmod 4$ of boundary nodes. The corresponding minimum number of circuits is $\frac{1}{4}m + \frac{1}{2}$, achieving the lower bound for k in Lemma 4.5.4.

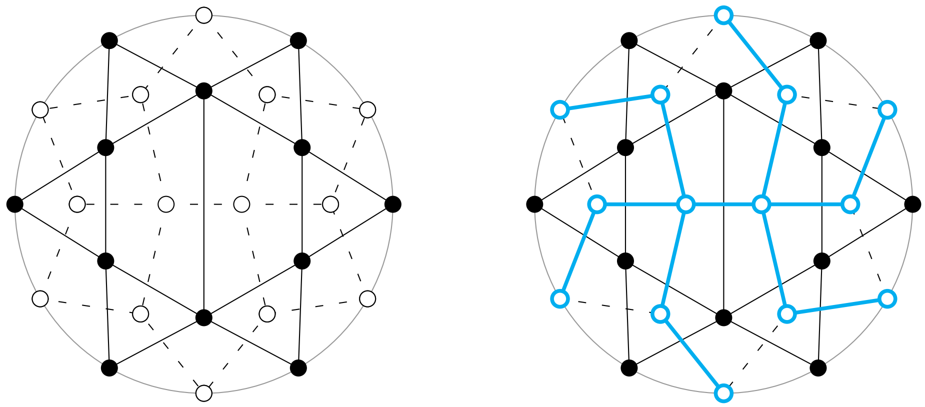


Figure 4.8: A double sunflower network and its dual, left; an insulator in blue, right.

Proof of Theorem 4.1.3. The result follows from Lemmas 4.5.3 and 4.5.4 and Proposition 4.5.2. □

4.6 Characteristic polynomials and graph colorings

4.6.1 Results from hyperplane arrangements

Given a matroid or hyperplane arrangement M, write χ_M for the characteristic polynomial of M. If M is a matroid, then write $\overline{\chi}_M$ for the *reduced characteristic polynomial* of M, given by

$$\overline{\chi}_M(\lambda) = (\lambda - 1)^{-1}\chi_M(\lambda).$$

Write χ_Γ for the chromatic polynomial of Γ, given by

$$\chi_\Gamma(\lambda) = \lambda \chi_{M(\Gamma)}(\lambda). \tag{4.30}$$

The *beta invariant* of a matroid M is

$$\beta(M) = (-1)^{\mathrm{rk}(M)+1} \chi'_M(1),$$

where χ'_M is the derivative of χ_M. Crapo showed that $\beta(M) \geq 0$, and that $\beta(M) > 0$ if and only if M is 2-connected [21, Theorem II].

The graph Γ is *2-connected* if Γ is connected and $\Gamma \setminus i$ is connected for all $i \in V$. The graph Γ is *chordal* if for any cycle Z of length ≥ 4 there is an edge not in Z meeting two vertices of Z. The next theorem summarizes results of Chapter 3 as they apply to $M(N)$.

Proposition 4.6.1. *Write $m = |\partial V|$, and let $\widehat{\Gamma}$ denote the graph obtained from Γ by adding an edge between each pair of boundary nodes. Then*

(i) The polynomial $\chi_{M(N)}$ can be written in terms of a chromatic polynomial:

$$\chi_{M(N)}(\lambda) = (\lambda)_m^{-1}(\lambda - 1)\chi_{\widehat{\Gamma}}(\lambda), \tag{4.31}$$

where $(\lambda)_m = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-m+1)$ is a falling factorial

(ii) The beta invariant of $M(N)$ divides the beta invariant of $M(\widehat{\Gamma})$:

$$\beta(M(N)) = \frac{\beta(M(\widehat{\Gamma}))}{(m-2)!} \tag{4.32}$$

(iii) The matroid $M(N)$ is 2-connected if and only if $\widehat{\Gamma}$ is 2-connected

(iv) The lattice of flats $\mathcal{L}(M(N))$ is supersolvable if and only if $\widehat{\Gamma}$ is chordal.

Example 4.6.2. Suppose that $N = \Gamma_{m,n}$ as in Example 3.1.4. Here $\widehat{\Gamma} = K_d$ is complete, so that $\chi_{\widehat{\Gamma}}(\lambda) = (\lambda)_d$. Theorem 4.6.1(i) gives

$$\overline{\chi}_{M(N)}(\lambda) = (\lambda)_m^{-1}(\lambda)_d = (\lambda - m)_{d-m}.$$

Example 4.6.3. Consider the network N from Example 4.2.6 and Figure 4.2. For all $m \geq 3$ let $\chi_m = \overline{\chi}_{M(N)}$. Let C be the unique circle of Γ. A deletion-contraction argument gives

$$\chi_m(\lambda+1) = \sum_{S \subsetneq C} (-1)^{|S|} \lambda^{m-|S|-|K(S)|} \prod_{K \in K(S)} (\lambda - |K|),$$

where $K(S)$ is the set of components of S. We propose the following closed form and recurrence relation for χ_m, which we have verified for $m \leq 11$ using `SageMath` [93].

Conjecture 4.6.4. *For $m \geq 3$ we have*

$$\chi_m(\lambda+1) = \frac{\omega_+^m + \omega_-^m}{2^m} + (-1)^m(\lambda - m - 1). \tag{4.33}$$

where $\omega_\pm = \lambda - 2 \pm \sqrt{\lambda^2+4}$. In particular, χ_m satisfies the recurrence

$$\begin{aligned} \chi_3(\lambda) &= \lambda^3 - 6\lambda^2 + 14\lambda - 13 \\ \chi_4(\lambda) &= \lambda^4 - 8\lambda^3 + 28\lambda^2 - 51\lambda + 41 \\ \chi_{m+2}(\lambda) &= (\lambda-1)\chi_{m+1}(\lambda) + (\lambda+1)\chi_m(\lambda) + (-1)^m(-2\lambda + m - 1). \end{aligned} \tag{4.34}$$

4.6.2 Broken circuits and the precoloring polynomial

We now prove Theorem 4.1.4. The proof uses the Broken Circuit Theorem of matroid theory, which we state below as Proposition 4.6.5. Fix an ordering of the ground set $E(M)$ of a matroid M. With respect to this ordering, a *broken circuit* of M is a set $C \setminus \min(C)$, where C is a circuit of M. The *broken circuit complex* of M is the set

$$\mathrm{BC}(M) = \{X \subset E(M) : X \text{ contains no broken circuit of } M\}. \tag{4.35}$$

We view $\mathrm{BC}(M)$ as a (simplicial) complex whose $(i-1)$-dimensional faces are the i-elements sets in $\mathrm{BC}(M)$. The complex $\mathrm{BC}(M)$ depends on the ordering of $E(M)$, but the number of faces of a given dimension does not:

Proposition 4.6.5 ([87, Theorem 4.12])**.** *Let M be a matroid on $E(M)$. With* *respect to any ordering of $E(M)$, the number of i-element sets in* $\mathrm{BC}(M)$ *is* $(-1)^i$ *times the coefficient of* $\lambda^{\mathrm{rk}(M)-i}$ *in* $\chi_M(\lambda)$.

We also consider the *reduced broken circuit complex* $\overline{\mathrm{BC}}(M)$, obtained from $\mathrm{BC}(M)$ by deleting the minimal element of $E(M)$ and all faces containing it. Every facet of $\mathrm{BC}(M)$ contains the minimal element of $E(M)$, so $\mathrm{BC}(M)$ is easily recovered from $\overline{\mathrm{BC}}(M)$.

Corollary 4.6.6. *With respect to any ordering of $E(M)$, the number of i-element sets in* $\overline{\mathrm{BC}}(M)$ *is* $(-1)^i$ *times the coefficient of* $\lambda^{\mathrm{rk}(M)-i-1}$ *in* $\overline{\chi}_M(\lambda)$.

Recall Proposition 4.2.15, which classifies the circuits of $M(N)$. Note that the circuits of type (C) come in two flavors: one contains 3 distinct crossings, while the other contains only 2. These are illustrated in Figure 4.9. Circuits of type (C) containing only 2 distinct crossings are either disconnected, as pictured, or connected with both crossings meeting at a single boundary node.

Figure 4.9: Two circuits of type (C) in Proposition 4.2.15.

Lemma 4.6.7. *Fix an ordering of E, and extend this ordering to E_0 by taking e_0 to be minimal. The reduced broken circuit complex* $\overline{\mathrm{BC}}(M(N))$ *is a subcomplex of* $\mathrm{BC}(M(\Gamma))$*:*

$$\overline{\mathrm{BC}}(M(N)) = \{X \in \mathrm{BC}(M(\Gamma)) : X \textit{ contains no crossing}\} \subset \mathrm{BC}(M(\Gamma)). \quad (4.36)$$

Proof. Let $X \in \overline{\mathrm{BC}}(M(N))$, so that $X \subset E$ contains no broken circuit of $M(N)$. Recall the 3 types of circuits of $M(N)$ from Proposition 4.2.15. A circle of Γ is a circuit of type (B) if it meets at most 1 boundary node, or of type (C) if it meets exactly 2 boundary nodes. If a circle C of Γ meets 3 or more boundary nodes, then every element of C is contained in a circuit $Y \subset C$ of type (C). Any broken circuit of $M(\Gamma)$ is a circle of Γ minus its minimal element. Thus X contains no broken circuit of $M(\Gamma)$. A broken circuit of $M(N)$ arising from a type (B) circuit is a crossing. Hence X contains no crossing.

Now suppose instead that $X \in \mathrm{BC}(M(\Gamma))$ contains no crossing. Since X contains no broken circuit of $M(\Gamma)$, it contains no broken circuit of $M(N)$ arising from a type (B) circuit. Since X contains no crossing, it contains no broken circuit of $M(N)$ arising from a circuit of type (A) or (C). Hence X contains no broken circuit of $M(N)$. □

Proof of Theorem 4.1.4. The result follows from Proposition 4.6.5, Corollary 4.6.6 and Lemma 4.6.7. □

4.7 3-Connectedness

We prove Theorem 4.1.5. Our main tools are characterizations by Slilaty and Qin [82] of 2- and 3-connected bias matroids $G(\Omega)$ in terms of the biased graph Ω. Given a biased graph Ω we write $E(\Omega)$ for the edge set of Ω.

Definition 4.7.1. For any integer $k \geq 1$, a *vertical k-biseparation* of a biased graph $\Omega = (\Gamma, \mathcal{B})$ is a partition (X, Y) of $E(\Omega)$ satisfying the following two conditions:

(i) $|X|, |Y| \geq k$

(ii) Each of X and Y meets a vertex not met by the other,

and any one of the following three conditions:

(iii) $\ell = k + 1$ with both X and Y balanced

(iv) $\ell = k$ with only one of X and Y balanced

(v) $\ell = k - 1$ with neither X nor Y balanced,

where ℓ is the number of vertices met by both X and Y.

We say that Ω is *vertically k-biconnected* if it admits no vertical r-biseparations for any $r < k$. We say that Ω is *simple* if it has no balanced circles of length 1 or 2 and no vertices incident to 2 or more unbalanced loops. A *balancing set* of Ω is an edge set S such that $\Omega \setminus S$ is balanced.

Proposition 4.7.2 ([82, Corollary 1.2])**.** *If Ω is a connected and unbalanced biased graph on at least 3 vertices, then $G(\Omega)$ is 2-connected if and only if Ω is vertically 2-biconnected and admits no balanced loops or balancing sets of rank 1.*

Proposition 4.7.3 ([82, Theorem 1.4])**.** *If Ω is a connected and unbalanced biased graph on at least 3 vertices, then $G(\Omega)$ is 3-connected if and only if Ω is simple and vertically 3-biconnected and admits no balancing sets of rank 1 or 2.*

Let $\Omega_0(N)$ be the biased graph obtained from $\Omega(N)$ by adding an unbalanced loop to v. Call this loop e_0, so that $E(\Omega_0(N)) = E_0$. Notice that $G(\Omega_0(N)) = M(N)$. Thus to prove Theorem 4.1.5 it suffices to show that $\Omega_0(N)$ satisfies the hypothesis of Proposition 4.7.3 if and only if $\Gamma \setminus \partial V$ is connected and $\widehat{\Gamma}$ is 3-connected.

Lemma 4.7.4. *The biased graph $\Omega_0(N)$ is connected, unbalanced and simple.*

Proof. Clearly $\Omega_0(N)$ is connected because Γ is connected; and $\Omega_0(N)$ is unbalanced because e_0 is unbalanced. It remains to show that $\Omega_0(N)$ is simple. The only circle of $\Omega_0(N)$ of length 1 is e_0, which is unbalanced. A balanced circle of length 2 would be a double edge in Γ, which we have excluded by assumption. Hence $\Omega_0(N)$ is simple. □

Lemma 4.7.5. *If $\Omega_0(N)$ admits a balancing set S of rank 2, then $m = 2$ and $S = \{e_0, e\}$ for some $e \in E$ such that $\Gamma \setminus e$ contains no crossing.*

Proof. A balancing set must contain e_0. Thus a balancing set of rank 2 is of the form $\{e, e_0\}$ for some $e \in E$ such that $\Gamma \setminus e$ contains no crossing. This is only possible if $m = 2$. □

Lemma 4.7.6. *No vertical k-biseparation of $\Omega_0(N)$ satisfies condition (iii) of Definition 4.7.1.*

Proof. Given any partition of E_0, the part containing e_0 is unbalanced. $\square$

Let S_i be the vertex sets of the components of $\Gamma \setminus \partial V$. For each i let T_i be the set of all edges meeting S_i. We call the T_i the *tracts* of N. Note that $\Gamma \setminus \partial V$ is connected if and only if E is the only tract of N. Also note that every vertex met by distinct tracts of N is a boundary node. Since Γ is connected and ∂V is an independent set, every tract meets at least 2 boundary nodes.

A *vertical k-separation* of a graph Γ is a partition (X, Y) of E such that $|X|, |Y| \geq k$ and exactly k vertices are met by both X and Y. Removing these k vertices disconnects Γ.

Lemma 4.7.7. *Let (X, Y) be a partition of E_0 satisfying conditions (i) and (ii) of Definition 4.7.1 with $k = 2$, and assume that $e_0 \in X$. Then (X, Y) satisfies (iv) if and only if there is a unique tract T of N containing Y and satisfying one of*

(iv†) $(X \cap T, Y)$ is a vertical 1-separation of $\Gamma(T)$ and Y meets exactly one boundary node

(iv‡) $(X \cap T, Y)$ is a vertical 2-separation of $\Gamma(T)$ and Y meets no boundary nodes,

where $\Gamma(T)$ is the subgraph of Γ induced by T.

Proof. It is easy to verify that either of (iv†) or (iv‡) implies (iv). We prove the opposite direction. Suppose that Y meets more than one boundary node. If Y contains a tract of N, then Y is balanced. If Y does not contain a tract of N, then $|S| > 2$, where $S \subset (V \setminus \partial V) \cup v$ is the set of all vertices met by both X and Y. In either case, (X, Y) does not satisfy (iv).

Now suppose that (X, Y) is a vertical 2-biseparation satisfying (iv), so that Y meets at most one boundary node. Note that Y must be contained in a tract T of N, since otherwise $|S| > 2$, a contradiction. Thus $S \cap V$ is the set of vertices of $\Gamma(T)$ met by both $X \cap T$ and Y. Since $|S \cap V|$ is 2 minus the number of boundary nodes met by Y, $(X \cap T, Y)$ is a vertical $|S \cap V|$-separation of $\Gamma(T)$. $\square$

Lemma 4.7.8. *A partition (X, Y) of E_0 satisfying conditions (i) and (ii) of Definition 4.7.1 with $k = 2$ satisfies (v) if and only if it satisfies* *(v†) $X \cap E$ and $Y \cap E$ are nonempty unions of tracts of N*

Proof. It is easy to check that (v†) implies (v). We prove the opposite direction. If Y is unbalanced, then it contains a path between distinct boundary boundary nodes. Thus if $\ell = 1$, then v is the only vertex of $\overline{\Gamma}$ met by both X and Y. It follows that Y is a nonempty union of tracts of N. The same follows for $X \setminus e_0 = E \setminus Y$. $\square$

Proof of Theorem 4.1.5. The networks N with $|V \setminus \partial V| = 1$ are the star networks of Example 4.2.16. In this case the uniform matroid $M(N) = U_{2,|V|}$ is 3-connected by [46, p. 312]. Suppose that $|V \setminus \partial V| \geq 2$ for the remainder of the proof.

We prove the "only if" direction first. Suppose that $M(N)$ is 3-connected. *A fortiori* $M(N)$ is 2-connected, so $\widehat{\Gamma}$ is 2-connected by Theorem 4.6.1(iii). Suppose that $\widehat{\Gamma} \setminus \{i, j\}$ is disconnected for some $i, j \in V$. Condition (v†) of Lemma 4.7.8 implies that there is only one tract of N, which must be E, so $\Gamma \setminus \partial V$ is connected. If $i, j \in \partial V$, then $\Gamma \setminus \partial V$ is disconnected, a contradiction. If $i, j \in V^\circ$ (resp., if $i \in V^\circ$ and $j \in \partial V$), then $\Omega_0(N)$ admits a vertical 2-biseparation satisfying (iv‡) (resp., (iv†)) of Lemma 4.7.7 with $T = E$, a contradiction. Hence $\widehat{\Gamma}$ is 3-connected.

Now we prove the "if" direction. Suppose that $\Gamma \setminus \partial V$ is connected and $\widehat{\Gamma}$ is 3-connected. *A fortiori* $\widehat{\Gamma}$ is 2-connected, so $\Omega_0(N)$ is vertically 2-biconnected and admits no balancing sets of rank 1 by Theorem 4.6.1(iii) and Proposition 4.7.2. If a balancing set of rank 2 existed, then Lemma 4.7.5 would imply that Γ is not 2-connected and $m = 2$. But then $\widehat{\Gamma}$ would not be 3-connected, a contradiction. Following Proposition 4.7.3, it remains to show that $\Omega_0(N)$ is vertically 3-biconnected. Suppose that $\Omega_0(N)$ admits a vertical 2-biseparation (X, Y). Lemma 4.7.6 then says that (X, Y) does not satisfy (iii). If (X, Y) satisfies either (iv†) or (iv‡), then $\widehat{\Gamma}$ is not 3-connected, a contradiction. Specifically, in case (iv†), we can disconnect $\widehat{\Gamma}$ by removing the boundary node met by Y and the vertex met by $X \setminus e_0$ and Y. In case (iv‡) we remove the two vertices met by both $X \setminus e_0$ and Y. Finally (X, Y) cannot satisfy (v†) since E is the only tract of N by assumption. $\square$

Matroids that are 3-connected enjoy nice structural properties. For example, if the graphic matroid $M(\Gamma)$ is 3-connected, then it is uniquely determined by Γ up to isomorphism. We have the following analog for Dirichlet matroids. Two networks $N = (\Gamma, \partial V)$ and $N' = (\Gamma', \partial V')$ are *isomorphic* if there is an isomorphism of Γ and Γ' that maps ∂V to $\partial V'$.

Corollary 4.7.9. *If* $\Gamma \backslash \partial V$ *is connected and* $\widehat{\Gamma}$ *is 3-connected, then up to isomorphism there are at most 27 networks* N' *such that* $M(N) \cong M(N')$.

Proof. This follows from Theorem 4.1.5, Proposition 4.2.7 and [27, Corollary 2]. □

We say that the network N is *Hamiltonian* if there is a circuit of $M(N)$ meeting every vertex of $\overline{\Gamma}$. Such a circuit is called *Hamiltonian*. For example, the star network N from Example 4.2.16 is Hamiltonian, and every circuit of $M(N)$ is Hamiltonian.

Corollary 4.7.10. *If* N *is Hamiltonian and*

$$\frac{1}{3}|E| \geq |V| - |\partial V| + 2, \tag{4.37}$$

then for any Hamiltonian circuit C *of* $M(N)$ *there is a circle* $C' \subset E$ *disjoint from* C *such that* $(\Gamma \setminus \partial V) \setminus C'$ *is connected and* $\widehat{\Gamma} \setminus C'$ *is 3-connected.*

Proof. This is an application of [68, Theorem 6.1] to Theorem 4.1.5. □

Chapter 5

Koszulness and supersolvability

In this chapter we characterize supersolvable Dirichlet arrangements in terms of their *Orlik-Solomon algebras*. The Orlik-Solomon algebra of a complex arrangement is a combinatorially defined graded algebra that is isomorphic to the cohomology ring of the complement of the arrangement. We show that a Dirichlet arrangement is supersolvable if and only if its Orlik-Solomon algebra is *Koszul*, an algebraic property that is slightly weaker than being semisimple. It is an open question whether a given arrangement is supersolvable if and only if its Orlik-Solomon algebra is Koszul. This question has been previously been answered for four other classes of arrangements. We exhibit an infinite family of Dirichlet arrangements that do not belong to any of these other four classes, showing that our results properly extend previous work.

5.1 Main results

A *Koszul algebra* is a graded algebra that is "as close to semisimple as it can possibly be" [5, p. 480]. Koszul algebras play an important role in the topology of complex hyperplane arrangements. For example, if $\mathcal{A}$ is such an arrangement and U its complement, then the Orlik-Solomon algebra $\mathrm{OS}(\mathcal{A})$ is Koszul if and only if U is a rational $K(\pi, 1)$-space. Also if $\mathrm{OS}(\mathcal{A})$ is Koszul and $G_1 \triangleright G_2 \triangleright \cdots$ denotes the lower central series of the fundamental group $\pi_1(U)$, defined by $G_1 = \pi_1(U)$ and

$G_{n+1} = [G_n, G_1]$, then the celebrated *Lower Central Series Formula* holds:

$$\prod_{k=1}^{\infty}(1 - t^k)^{\varphi_k} = P(U, -t), \tag{5.1}$$

where $P(U, t)$ is the Poincaré polynomial of U and $\varphi_k = \mathrm{rk}(G_k/G_{k+1})$.

It is natural to seek a combinatorial characterization of the arrangements $\mathcal{A}$ for which $\mathrm{OS}(\mathcal{A})$ is Koszul. Shelton and Yuzvinsky [81, Theorem 4.6] showed that if $\mathcal{A}$ is supersolvable, then $\mathrm{OS}(\mathcal{A})$ is Koszul. Whether the converse holds is unknown.

Question 5.1.1. *If the Orlik-Solomon algebra of a central hyperplane arrangement $\mathcal{A}$ is Koszul, then is $\mathcal{A}$ supersolvable?*

We answer this question affirmatively for cones (or centralizations) over *Dirichlet arrangements*, a generalization of graphic arrangements arising from electrical networks and order polytopes of finite posets (see Chapter 3).

Theorem 5.1.2. *The cone over a Dirichlet arrangement is supersolvable if and only if its Orlik-Solomon algebra is Koszul.*

Question 5.1.1 has been answered affirmatively for other classes of arrangements, including graphic arrangements [43, 47, 78, 96]. Our next theorem shows that Theorem 5.1.2 properly extends all previous results. We say that two central arrangements are *combinatorially equivalent* if the underlying matroids are isomorphic.

Theorem 5.1.3. *There are infinitely many cones over Dirichlet arrangements that are not combinatorially equivalent to any arrangement for which Question 5.1.1 has been previously answered.*

Dirichlet arrangements have also been called *ψ-graphical arrangements* [58, 89, 90]. It was conjectured in [58] and proven in [90] that the cone over a Dirichlet arrangement is supersolvable if and only if it is free.

5.2 Background

5.2.1 Orlik-Solomon algebras

Given an ordered central arrangement $\mathcal{A}$ over $\mathbb{K}$, let V be the $\mathbb{K}$-vector space with basis $\{e_a : a \in \mathcal{A}\}$. Let $\Lambda = \Lambda(V)$ be the exterior algebra of V. Write $xy = x \wedge y$ in Λ. The algebra Λ is graded by taking $\Lambda^0 = \mathbb{K}$ and Λ^p to be spanned by all elements of the form $e_{a_1} \cdots e_{a_p}$.

Let $\partial : \Lambda \to \Lambda$ be the linear map defined by $\partial 1 = 0$, $\partial e_a = 1$ for all $a \in \mathcal{A}$, and

$$\partial(xy) = \partial(x)y + (-1)^p x\partial(y) \tag{5.2}$$

for all $x \in \Lambda^p$ and $y \in \Lambda$.

The set X is *dependent* if the normal vectors of the hyperplanes in X are linearly dependent. A *circuit* is a minimal dependent set. If $X = \{a_1, \ldots, a_p\} \subseteq \mathcal{A}$, assuming the a_i are in increasing order, write $e_X = e_{a_1} \cdots e_{a_p}$ in Λ.

Definition 5.2.1. The *Orlik-Solomon algebra* $\mathrm{OS}(\mathcal{A})$ of a central arrangement $\mathcal{A}$ is the quotient of Λ by the *Orlik-Solomon ideal*

$$I = \langle \partial(e_C) : C \subseteq \mathcal{A} \text{ is a circuit}\rangle. \tag{5.3}$$

That is, $\mathrm{OS}(\mathcal{A}) = \Lambda/I$.

5.2.2 Koszul algebras

We briefly recall the definition of a Koszul algebra. For more detail, see [31, 36]. Let A be a Noetherian graded algebra over a field $\mathbb{K}$, with grading $A = \bigoplus_{i \geq 0} A_i$. Let M be an A-module. A *resolution* (or *projective resolution*) of M is an exact sequence of A-modules

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to M \to 0 \tag{5.4}$$

such that each P_i is a projective module.

Suppose that M is graded and finitely generated. In this situation M is projective if and only if it is free. Hence in any resolution of M, there are integers $b_i \geq 0$ such that $P_i = A^{b_i}$ for all i, where A^{b_i} is the b_i-fold direct sum of A. We obtain an exact sequence

$$\cdots \to A^{b_n} \to \cdots \to A^{b_1} \to A^{b_0} \to M \to 0 \tag{5.5}$$

of finitely generated graded A-modules.

Each map $A^{b_i} \to A^{b_{i-1}}$ is given by a $b_{i-1} \times b_i$ matrix T_i whose entries are homogeneous elements of A. Write $A_+ = \bigoplus_{i>0} A_i$. The resolution is *minimal* if the entries of all T_i belong to A_+. Up to isomorphism of chain complexes there is a unique minimal resolution of M. If the resolution (5.5) is minimal, then b_0 is the minimal number of generators of M.

Let $\varepsilon : A \to A/A_+ \cong \mathbb{K}$ be the quotient map, called the *augmentation map*, and consider $\mathbb{K}$ as an A-module by setting $a \cdot x = \varepsilon(a)x$ for all $a \in A$ and $x \in \mathbb{K}$. Consider the minimal resolution of $\mathbb{K}$:

$$\cdots \xrightarrow{T_{n+1}} A^{b_n} \xrightarrow{T_n} \cdots \xrightarrow{T_2} A^{b_1} \xrightarrow{T_1} A \to \mathbb{K} \to 0. \tag{5.6}$$

If all entries of the matrices T_i belong to A_1, then A is *Koszul*.

Example 5.2.2. Let V be the vector space over $\mathbb{K}$ with basis $\{x_1, \ldots, x_n\}$. Let $A = T(V)$ be the tensor algebra of V. The minimal resolution of $\mathbb{K}$ is

$$0 \to A^n \to A \to \mathbb{K} \to 0,$$

where the map $A^n \to A$ is given by the row vector $(x_1\ x_2\ \cdots\ x_n)$.

Quadraticity is a key property of Koszul algebras. A *minimal generator* of the Orlik-Solomon algebra I is an element of the form $\partial(e_C)$, where C is a circuit and

$$\partial(e_C) \notin \langle \partial(e_X) : X \subseteq \mathcal{A} \text{ is a circuit with } |X| < |C| \rangle. \tag{5.7}$$

If the minimal generators of I are of degree 2, then $\mathrm{OS}(\mathcal{A})$ is called *quadratic*.

Proposition 5.2.3 ([36, Definition-Theorem 1])**.** *If* $\mathrm{OS}(\mathcal{A})$ *is Koszul, then* $\mathrm{OS}(\mathcal{A})$ *is quadratic.*

5.3 Proof of Theorem 5.1.2

We prove the following theorem, which implies Theorem 5.1.2.

Theorem 5.3.1. *Let* $\widehat{\Gamma}$ *be the graph obtained from* Γ *by adding an edge between each pair of boundary nodes. The following are equivalent:*

(i) $\widehat{\Gamma}$ *is chordal*
(ii) $\mathcal{A}(\Gamma, u)$ *is supersolvable*
(iii) $\mathrm{OS}(\mathcal{A}(\Gamma, u))$ *is Koszul*
(iv) $\mathrm{OS}(\mathcal{A}(\Gamma, u))$ *is quadratic.*

We write x instead of $\{x\}$ for all single-element sets. Let e_0 be an element not in E, and let $E_0 = E \cup e_0$, so that $\mathcal{A}(\Gamma, u)$ is indexed by E_0. Fix an ordering of E_0 with e_0 minimal. We say that $C \subseteq E_0$ is a *circuit* if the corresponding subset of $\mathcal{A}$ is a circuit.

Definition 5.3.2. A set $X \subseteq E$ is a *crossing* if it is a minimal path between 2 distinct boundary nodes.

The following is a restatement of Proposition 4.2.15 for convenience.

Proposition 5.3.3. *A set* $C \subseteq E_0$ *is a circuit if and only if one of the following holds:*

(A) $C = X \cup e_0$ *for some crossing* X
(B) $C \subseteq E$ *is a cycle of* Γ *meeting at most 1 boundary node*
(C) $C \subseteq E$ *is a minimal set containing 2 distinct crossings and no circuit of type (B).*

The circuits of type (C) in Proposition 5.3.3 come in two flavors: one contains 3 distinct crossings, while the other contains only 2. These are illustrated in Figure

4.9. Circuits of type (C) containing only 2 distinct crossings are either disconnected, as pictured, or connected with both crossings meeting only on the boundary.

Taken together, the following 2 lemmas imply that circuits of type (C) do not contribute minimal generators to the Orlik-Solomon ideal I. When the usage is clear we will write $S = e_S$, so that S is considered as an element of Λ and a subset of E_0.

Lemma 5.3.4. *Let $C \subseteq E$ be a circuit containing distinct crossings X_1, X_2 and X_3. In Λ we have*

$$\partial(C) \in \langle \partial(e_0X_1), \partial(e_0X_2), \partial(e_0X_3)\rangle. \tag{5.8}$$

Proof. There are mutually disjoint paths $P_1, P_2, P_3 \subseteq E$ in Γ such that $C = P_1 \cup P_2 \cup P_3$ and $X_i = P_j \cup P_k$ for distinct i, j, k. Write $a_i = |P_i|$, and suppose without loss of generality that $X_1 = P_2P_3$, $X_2 = P_1P_3$ and $X_3 = P_1P_2$ in Λ. We have

$$\begin{aligned}
\partial(e_0X_3) &= P_1P_2 - e_0\partial(P_1)P_2 - (-1)^{a_1}e_0P_1\partial(P_2)\\
\partial(e_0X_2) &= P_1P_3 - e_0\partial(P_1)P_3 - (-1)^{a_1}e_0P_1\partial(P_3)\\
\partial(e_0X_1) &= P_2P_3 - e_0\partial(P_2)P_3 - (-1)^{a_2}e_0P_2\partial(P_3)
\end{aligned}$$

Thus

$$\begin{aligned}
\partial(P_3)\partial(e_0X_3) &= (-1)^{(a_1+a_2)(a_3-1)}(P_1P_2\partial(P_3) - e_0\partial(P_1)P_2\partial(P_3)\\
&\qquad - (-1)^{a_1}e_0P_1\partial(P_2)\partial(P_3))\\
\partial(P_2)\partial(e_0X_2) &= (-1)^{a_1(a_2-1)}(P_1\partial(P_2)P_3 - e_0\partial(P_1)\partial(P_2)P_3\\
&\qquad + (-1)^{a_1+a_2}e_0P_1\partial(P_2)\partial(P_3))\\
\partial(P_1)\partial(e_0X_1) &= \partial(P_1)P_2P_3 + (-1)^{a_1}e_0\partial(P_1)\partial(P_2)P_3\\
&\qquad + (-1)^{a_1+a_2}e_0\partial(P_1)P_2\partial(P_3)
\end{aligned}$$

Since $C = P_1P_2P_3$, we have

$$\partial(C) = \partial(P_1)P_2P_3 + (-1)^{a_1}P_1\partial(P_2)P_3 + (-1)^{a_1+a_2}P_1P_2\partial(P_3),$$

A computation now gives

$$\partial(C) = \partial(P_1)\partial(e_0X_1) + (-1)^{a_1a_2}\partial(P_2)\partial(e_0X_2) + (-1)^{(a_1+a_2)a_3}\partial(P_3)\partial(e_0X_3),$$

proving the result. □

Lemma 5.3.5. *Suppose that X_1 and X_2 are crossings such that no vertex in $V \setminus \partial V$ is met by both X_1 and X_2. In Λ we have*

$$\partial(C) \in \langle \partial(e_0X_1), \partial(e_0X_2) \rangle. \tag{5.9}$$

Proof. The proof is similar to that of Lemma 5.3.4. In particular, we have

$$\partial(X_1X_2) = \partial(X_1e_0)\partial(X_2) + \partial(X_1)\partial(X_2e_0),$$

proving the result. □

Let $C \subseteq E_0$ be a circuit. An element $i \in E_0$ is a *chord* of C if there exist circuits C_1 and C_2 such that $i = C_1 \cap C_2$ and $C = (C_1 \setminus C_2) \cup (C_2 \setminus C_1)$. If C admits no chord, then C is *chordless*.

Proposition 5.3.6. *The minimal generators of I are the elements of the form $\partial(C)$, where $C \subseteq E_0$ is a chordless circuit of type (A) or (B) in Proposition 5.3.3.*

Proof. Let J be the ideal of Λ generated by the elements of the form $\partial(C)$ for all circuits C of types (A) and (B) in Proposition 5.3.3. Note that any circuit of type (C) is described by either Lemma 5.3.4 or 5.3.5. It follows that $J = I$ is the Orlik-Solomon ideal.

Let $C \subseteq E_0$ be a circuit of type (A) or (B). It remains to show that $\partial(C)$ is a minimal generator of I if and only if C is chordless. Notice that a chord of C is any edge $i \in E$ connecting two vertices met by $E \cap C$.

Suppose first that C is of type (B), and write $C = \{e_1, \ldots, e_r\}$. We have

$$\partial(C) = \sum_{j=1}^{r} (-1)^{j-1} e_1 \cdots \widehat{e_j} \cdots e_r.$$

There is a chord i of C if and only if there is a circuit C' of with a term of $\partial(C')$ dividing $e_2 \cdots e_r$. Suppose that such a chord i exists, and partition C into two paths P_1 and P_2 such that $P_1 \cup i$ and $P_2 \cup i$ are cycles of Γ. Write $a_j = |P_j|$, and suppose without loss of generality that $C = P_1 P_2$ in Λ. We have

$$\partial(C) = \partial(P_1)\partial(iP_2) + (-1)^{a_1 a_2} \partial(P_2)\partial(iP_1),$$

so $\partial(C)$ is not a minimal generator. Thus if C is a cycle of Γ, then $\partial(C)$ is a minimal generator of I if and only if C is chordless.

Now suppose that $C = X \cup e_0$ for some crossing X. We have $\partial(C) = X - e_0\partial(X)$. There is a circuit C' with a term of $\partial(C')$ dividing X if and only if there is a chord i of X. Suppose that such a chord i exists. Partition X into two sets X_1 and X_2 such that $X_1 \cup i$ is a cycle of Γ and $X_2 \cup i$ is a crossing. Write $b_j = |X_j|$, and suppose without loss of generality that $X = X_1 X_2$ in Λ. We have

$$(-1)^{b_1}\partial(C) = \partial(X_1)\partial(e_0 i X_2) + (e_0\partial(X_2) + (-1)^{b_2} X_2)\partial(iX_1),$$

where $X_1 \cup i$ and $X_2 \cup \{e_0, i\}$ are circuits of smaller size than C. Hence $\partial(C)$ is not a minimal generator. Thus if $C = X \cup e_0$ for some crossing X, then $\partial(C)$ is a minimal generator of I if and only if C is chordless. The result follows. □

Proposition 5.3.7. *The graph $\widehat{\Gamma}$ is chordal if and only if there are no chordless circuits of type (A) or (B) in Proposition 5.3.3 having size ≥ 4.*

Proof. Let $\widehat{E}$ be the set of edges of $\widehat{\Gamma}$ not in E. Suppose that C is a chordless circuit of size $k \geq 4$. If $C = X \cup e_0$ is of type (A) for some crossing X, then there is $e \in \widehat{E}$ such that $X \cup e$ is a cycle of $\widehat{\Gamma}$ admitting no chord. If C is of type (B), then C is a cycle of Γ (and hence $\widehat{\Gamma}$) admitting no chord. The "only if" direction follows. Now suppose that $\widehat{\Gamma}$ has a cycle Z of size ≥ 4 admitting no chord. Then either $Z \subseteq E$, in which case Z is a circuit of type (B); or $Z \cap \widehat{E}$ consists of a single edge e, in which case $(Z \setminus e) \cup e_0$ is a circuit of type (A). □

Proof of Theorem 5.3.1. (i) ⇒ (ii): This follows from Theorem 3.2.2. (ii) ⇒ (iii): This follows from [81, Theorem 4.6]. (iii) ⇒ (iv): This is the content of Proposition 5.2.3. (iv) ⇒ (i): This follows from Propositions 5.3.6 and 5.3.7. □

5.4 An infinite family

We prove Theorem 5.4.3 below, which implies Theorem 5.1.3. There are four classes of arrangements for which Question 5.1.1 was previously answered:

(i) Graphic arrangements
(ii) Ideal arrangements
(iii) Hypersolvable arrangements
(iv) Ordered arrangements with disjoint minimal broken circuits.

See [43, 47, 78, 96] for individual treatments. A priori it is unclear how these classes overlap with cones over Dirichlet arrangements.

Given a central arrangement $\mathcal{A}$, let $M(\mathcal{A})$ be the usual matroid on $\mathcal{A}$, so X is independent in $M(\mathcal{A})$ if and only if the set of normal vectors of X is linearly independent. For more on matroids and central arrangements, see [87]. Recall that two central arrangements are called *combinatorially equivalent* if their underlying matroids are isomorphic.

Definition 5.4.1. Let $\chi(\Gamma, \partial V)$ denote the chromatic number of the graph with vertex set ∂V and an edge between i and j if and only if there is a crossing in Γ connecting i and j.

Example 5.4.2. Consider the graph Γ on the left side of Figure 5.1 with ∂V marked in white. On the right side is the graph with vertex set ∂V and an edge between i and j if and only if there is a crossing in Γ connecting i and j. This graph can

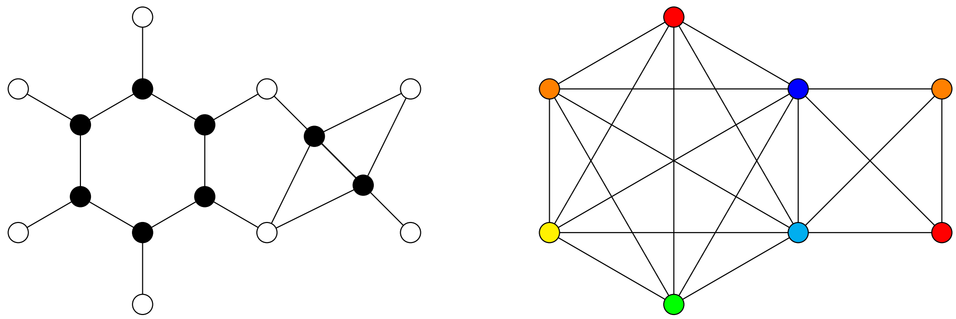


Figure 5.1: A graph with boundary nodes marked in white and an illustration of the associated number $\chi(\Gamma, \partial V)$.

be colored using 6 colors, as pictured, and no fewer, since it contains a clique on 6 vertices. Hence $\chi(\Gamma, \partial V) = 6$.

Theorem 5.4.3. *Suppose that $|E| \geq 240$ and $\chi(\Gamma, \partial V) \geq 4$, and that some vertex of Γ is adjacent to at least 3 boundary nodes. If $\Gamma \setminus \partial V$ contains the wheel graph on 5 vertices as an induced subgraph, then $\mathcal{A}(\Gamma, u)$ is not combinatorially equivalent to any graphic arrangement, ideal arrangement, hypersolvable arrangement, or ordered arrangement with disjoint minimal broken circuits.*

Example 5.4.4. Recall that the *join* $G + H$ of 2 graphs G and H is the disjoint union of G and H with edges added between every vertex of G and every vertex of H. The join of any finite number of graphs is defined by induction. Let $\overline{K}_n$ and K_n be the edgeless and complete graphs, resp., on n vertices. Let W_5 be the wheel graph on 5 vertices. The graph $\Gamma = \overline{K}_4 + K_{14} + W_5$ with boundary $\partial V = \overline{K}_4$ satisfies the hypothesis of Theorem 5.4.3 and does so with the minimum possible number of vertices. In particular we have $|E| = 245$, $\chi(\Gamma, \partial V) = 4$, and $|V| = 23$.

The proof of Theorem 5.4.3 can be found at the end of the section. First we need some preliminary results on the classes of arrangements (ii)–(iv).

5.4.1 Ideal arrangements

Let $\Phi \subseteq \mathbb{K}^n$ be a finite root system with set of positive roots Φ^+. A standard reference for root systems is [44]. The *Coxeter arrangement* associated to Φ is the set of normal hyperplanes of Φ^+. Every Coxeter arrangement associated to a classical root system A_n, B_n, C_n or D_n is a subset of an arrangement of the following type.

Definition 5.4.5. For all $n \geq 2$ let $\mathcal{B}_n$ be the arrangement in $\mathbb{K}^n$ of hyperplanes

$$\{x_i = x_j : 1 \leq i < j \leq n\} \cup \{x_i + x_j = 0 : 1 \leq i < j \leq n\} \cup \{x_i = 0 : 1 \leq i \leq n\}.$$

Proposition 5.4.6. *If $\chi(\Gamma, \partial V) \geq 4$ and $|E| \geq 240$, then $\mathcal{A}(\Gamma, u)$ is not combinatorially equivalent to any subarrangement of any Coxeter arrangement.*

Proof. The matroids $M(\mathcal{B}_n)$ are representable over any field $|\mathbb{K}|$ with $|\mathbb{K}| \geq 3$. However $M(\mathcal{A}(\Gamma, u))$ is not representable over $\mathbb{K}$ if $|\mathbb{K}| < \chi(\Gamma, \partial V)$ by Theorem 4.2.17. Hence if $\chi(\Gamma, \partial V) \geq 4$, then $\mathcal{A}(\Gamma, u)$ is not combinatorially equivalent to any subarrangement of $\mathcal{B}_n$.

The exceptional root systems E_6, E_7, E_8, F_4 and G_2 all have 240 or fewer elements. Hence no subarrangement of the associated Coxeter arrangements can have more than 240 elements. The result now follows from the classification of finite root systems. □

An *ideal arrangement* (or a *root ideal arrangement*) is a certain subarrangement of a Coxeter arrangement (see [1, 43]). Graphic arrangements are subarrangements of $\mathcal{B}_n$. Thus we have the following.

Corollary 5.4.7. *If $\chi(\Gamma, \partial V) \geq 4$ and $|E| \geq 240$, then $\mathcal{A}(\Gamma, u)$ is not combinatorially equivalent to any ideal arrangement or graphic arrangement.*

5.4.2 Hypersolvable arrangements

Let $\mathcal{A}$ be a central arrangement, and let $X \subseteq Y \subseteq \mathcal{A}$. The containment $X \subseteq Y$ is *closed* if $X \neq Y$ and $\{a, b, c\}$ is independent for all distinct $a, b \in X$ and $c \in Y \setminus X$. The containment $X \subseteq Y$ is *complete* if $X \neq Y$ and for any distinct $a, b \in Y \setminus X$ there is $\gamma \in X$ such that $\{a, b, \gamma\}$ is dependent.

If $X \subseteq Y$ is closed and complete, then the element γ is uniquely determined by a and b. Write $\gamma = f(a,b)$. The containment $X \subseteq Y$ is *solvable* if it is closed and complete, and if for any distinct $a,b,c \in Y \setminus X$ with $f(a,b)$, $f(a,c)$ and $f(b,c)$ distinct, the set $\{f(a,b), f(a,c), f(b,c)\}$ is dependent.

An increasing sequence $X_1 \subseteq \cdots \subseteq X_k = \mathcal{A}$ is called a *hypersolvable composition series* for $\mathcal{A}$ if $|X_1| = 1$ and each $X_i \subseteq X_{i+1}$ is solvable.

Definition 5.4.8 ([47, Definition 1.8])**.** The central arrangement $\mathcal{A}$ is *hypersolvable* if it admits a hypersolvable composition series.

There is an analog for graphs. Let $S \subseteq T \subseteq E$. We say that $S \subseteq T$ is *solvable* if it satisfies the following conditions:

(a) There is no 3-cycle in Γ with two edges from S and one edge from $T \setminus S$

(b) Either $T \setminus S = e$ with neither endpoint of e met by S, or there exist distinct vertices $v_1, \ldots, v_k, v$ met by T with $v_1, \ldots, v_k$ met by S such that

 (i) S contains a clique on $\{v_1, \ldots, v_k\}$, and

 (ii) $T \setminus S = \{vv_s \in E : s = 1, \ldots, k\}$.

An increasing sequence $S_1 \subseteq \cdots \subseteq S_k = E$ is called a *hypersolvable composition series* for Γ if $|S_1| = 1$ and each $S_i \subseteq S_{i+1}$ is solvable

Definition 5.4.9 ([70, Definition 6.6])**.** The graph Γ is *hypersolvable* if it admits a hypersolvable composition series.

Proposition 5.4.10. *If the graph Γ is hypersolvable, then so is any induced subgraph of Γ.*

Proof. Suppose that $S_1 \subseteq \cdots \subseteq S_k$ is a hypersolvable composition series for Γ, and let $\overline{\Gamma}$ be an induced subgraph of Γ with edge set $\overline{E} \subseteq E$. By eliminating empty sets and trivial containments in the sequence $S_1 \cap \overline{E} \subseteq \cdots \subseteq S_k \cap \overline{E}$ one obtains a hypersolvable composition series for $\overline{\Gamma}$. □

The following proposition generalizes half a result of Papadima and Suciu [70, Proposition 6.7], who showed that Γ is hypersolvable if and only if the associated graphic arrangement is hypersolvable.

Proposition 5.4.11. *If $\mathcal{A}(\Gamma, u)$ is hypersolvable, then the graph $\widehat{\Gamma}$, obtained from Γ by adding edges between every pair of boundary nodes, is hypersolvable.*

Proof. Let $\widehat{E}$ be the set of added edges, so that the edge set of $\widehat{\Gamma}$ is the disjoint union $E \cup \widehat{E}$. Write $\partial V = \{v_1, \ldots, v_m\}$. For $i = 1, \ldots, m-1$

$$T_i = \{v_r v_s \in \widehat{E} : r < s \leq i+1\},$$

so for example $T_{m-1} = \widehat{E}$.

Suppose that $X_1 \subseteq \cdots \subseteq X_k$ is a hypersolvable composition series for $\mathcal{A}(\Gamma, u)$. For each i let $S_i \subseteq E_0$ be the set corresponding to X_i. Let j be the smallest index for which $e_0 \in S_j$. Consider the increasing sequence

$$S_1 \subseteq \cdots \subseteq S_{j-1} \subseteq S_{j-1} \cup T_1 \subseteq \cdots \subseteq S_{j-1} \cup T_{m-1} \subseteq S_{j+1} \cup \widehat{E} \subseteq \cdots \subseteq S_k \cup \widehat{E},$$

omitting the initial portion $S_1 \subseteq \cdots \subseteq S_{j-1}$ if $j = 1$. It is routine to show that this sequence is a hypersolvable composition series for $\widehat{\Gamma}$. □

Example 5.4.12. Consider the network N on the left side of Figure 5.2. Here $\widehat{\Gamma} = W_5$ is the wheel graph on 5 vertices. An exhaustive argument shows that W_5 is not hypersolvable. Hence Proposition 5.4.11 implies that $\mathcal{A}(\Gamma, u)$ is not hypersolvable.

Figure 5.2: Left to right: a network N with boundary nodes marked in white and the associated graph $\widehat{\Gamma} = W_5$.

Question 5.4.13. *Does the converse of Proposition 5.4.11 hold? In other words, is $\mathcal{A}(\Gamma, u)$ hypersolvable whenever $\widehat{\Gamma}$ is hypersolvable?*

5.4.3 Disjoint minimal broken circuits

Fix an ordering of a central arrangement $\mathcal{A}$, and let $\min X$ denote the minimal element of any $X \subseteq \mathcal{A}$. The *broken circuits* of $\mathcal{A}$ are the sets $C \setminus \min C$ for all circuits C of $\mathcal{A}$. A broken circuit is *minimal* if it does not properly contain any broken circuits. Van Le and Römer [96, Theorem 4.9] answered Question 5.1.1 affirmatively for all ordered arrangements with disjoint minimal broken circuits. No matter the ordering, many Dirichlet arrangements do not satisfy this requirement, as the following proposition implies.

Proposition 5.4.14. *If there is an element of $V \setminus \partial V$ adjacent to at least 3 boundary nodes, then the minimal broken circuits of $\mathcal{A}(\Gamma, u)$ are not disjoint with respect to any ordering.*

Proof. Suppose that $i \in V \setminus \partial V$ is adjacent to distinct boundary nodes j_1, j_2 and j_3. Let e_r be the edge ij_r for $r = 1, 2, 3$. Fix an ordering of $\mathcal{A}(\Gamma, u)$ and suppose without loss of generality that $e_1 < e_2 < e_3$. We obtain circuits $\{e_0, e_1, e_3\}$ and $\{e_0, e_2, e_3\}$. The associated broken circuits are minimal, since there are no circuits of size ≤ 2. Moreover both broken circuits contain e_3. □

Proof of Theorem 5.4.3. Since $\chi(\Gamma, \partial V) \geq 4$ and $|E| \geq 240$, Corollary 5.4.7 says that $\mathcal{A}(\Gamma, u)$ is not combinatorially equivalent to any ideal arrangement or graphic arrangement. Since $\Gamma \setminus \partial V$ contains W_5 as an induced subgraph, $\widehat{\Gamma}$ also contains W_5 as an induced subgraph. Example 5.4.12 and Propositions 5.4.10 and 5.4.11 imply that $\mathcal{A}(\Gamma, u)$ is not hypersolvable, a property depending only on $M(\mathcal{A}(\Gamma, u))$. Finally Proposition 5.4.14 says that the broken circuits of $\mathcal{A}(\Gamma, u)$ are not disjoint with respect to any ordering. This property only depends on $M(\mathcal{A}(\Gamma, u))$, so the result follows. □

Chapter 6

Topological complexity and structural rigidity

In this chapter we relate the notion of *topological complexity*, originating in topological robotics, with the notion of *rigidity* of graphs in surfaces. Topological complexity measures the difficulty of planning a continuous motion through a topological space. A rigid graph is one for which, when its vertices are placed generically on a surface in $\mathbb{R}^3$, the only continuous motions preserving all edge lengths are rigid motions. We study *largeness* of a central complex arrangement, a combinatorial property guaranteeing that the complement of the arrangement is maximally topologically complex. First, we recast a result of Fieldsteel [35] to show that a graphic arrangement is large if and only if the underlying graph is rigid in the plane. We use combinatorial moves from structural rigidity to give a new proof of the "if" direction of the theorem. We then use these and other moves to identify a class of electrical networks whose associated Dirichlet arrangements are large. We provide a partial converse to this theorem as well. The electrical networks in question relate to rigid graphs in an infinite cylinder.

6.1 Main results

Our starting point is the problem of planning a continuous motion through a given topological space X. Let X^I denote the space of continuous paths $\gamma : [0,1] \to X$ equipped with the compact-open topology. For every integer $s \geq 2$ let X^s denote the s-fold Cartesian product of X, and let $\pi_s : X^I \to X^s$ be given by

$$\pi_s(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{s-1}\right), \ldots, \gamma\left(\frac{s-2}{s-1}\right), \gamma(1)\right). \tag{6.1}$$

The *higher topological complexity* $\mathrm{TC}_s(X)$ of X is the minimum number $|C|-1$ where C is an open cover of X^s such that π_s admits a continuous section on every set in C. If no such cover exists, then we set $\mathrm{TC}_s(X) = \infty$. This definition is due to Rudyak [76], generalizing work of Farber [33] on the motion planning problem in robotics. One can think of $\mathrm{TC}_s(X)$ as measuring the difficulty of planning a continuous motion through a given sequence of s points in X.

It is a natural problem to compute $\mathrm{TC}_s(X)$ for familiar classes of spaces in combinatorics, such as complements of hyperplane arrangements. If M is the complement of a central complex arrangement $\mathcal{A}$ of rank r, then

$$\mathrm{TC}_s(M) \leq rs - 1. \tag{6.2}$$

Yuzvinsky [108] gave a combinatorial condition on arrangements guaranteeing equality in (6.2). Arrangements satisfying the condition of Yuzvinsky are called *large*.

Our first result takes a characterization of large graphic arrangements due to Fieldsteel [35] and recasts it in terms of *structural rigidity*. Let $S \subseteq \mathbb{R}^3$ be a surface and $\Gamma = (V, E)$ a graph. A *framework* of Γ on S is a pair (Γ, p), where $p \in S^V$ is a tuple of points in S indexed by V. The framework (Γ, p) is *rigid* if the only continuous motions preserving the distance between p_i and p_j for all $ij \in E$ are the rigid motions in $\mathbb{R}^3$. The graph Γ is *generically rigid* in S if (Γ, p) is rigid whenever all coordinates of all p_i are algebraically independent.

Theorem 6.1.1. *The graphic arrangement $\mathcal{A}(\Gamma)$ is large if and only if Γ is generi-*

cally rigid in the plane.

Certain classes of rigid graphs admit constructions by sequences of combinatorial moves. A graph Γ is *minimally generically rigid* in S if Γ is rigid in S and removing any edge from Γ yields a graph that is not generically rigid in S. The minimally generically rigid graphs in the plane are the graphs obtainable from K_2 by any sequence of the two moves on the top of Figure 6.1, called *Henneberg moves*. We use this fact to give a new proof of the "if" direction of Theorem 6.1.1.

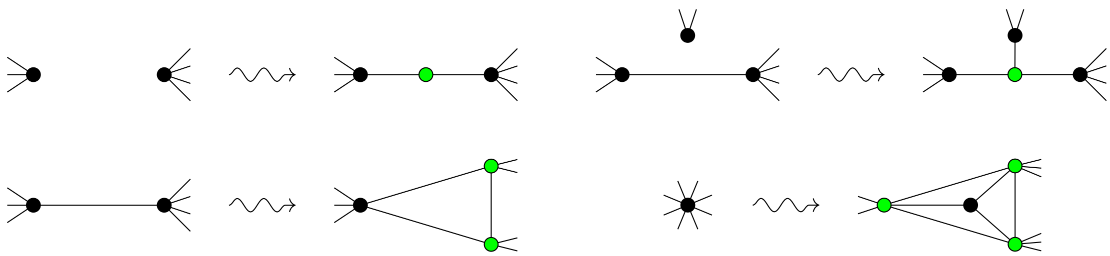


Figure 6.1: Four combinatorial moves on graphs.

There is a similar construction for a certain class of graphs that admit rigid frameworks in the infinite cylinder. These graphs are obtainable from K_4 by any sequence of Henneberg moves and the moves on the bottom of Figure 6.1, called *edge-to-*K_3 and *vertex-to-*K_4. Our main result, Theorem 6.1.2, uses these four moves to identify a class of large Dirichlet arrangements.

Given an independent set $\partial V \subseteq V$ and injective function $u : \partial V \to \mathbb{C}$, recall that the *Dirichlet arrangement* $\overline{\mathcal{A}}(\Gamma, u)$ in $\mathbb{C}^{V \setminus \partial V}$ is the set of hyperplanes given by

$$\begin{cases} x_i = x_j & \text{for all } ij \in E \text{ with } i, j \in V \setminus \partial V \\ x_i = u_j & \text{for all } ij \in E \text{ with } j \in \partial V. \end{cases}$$

We assume that $|\partial V| \geq 2$. Let $\mathcal{A}(\Gamma, u)$ denote the cone over $\overline{\mathcal{A}}(\Gamma, u)$. Let $\overline{\Gamma}$ denote the multigraph obtained from Γ by identifying all boundary nodes as a single vertex v_0. The multigraphs $\overline{\Gamma}$ in the following theorem admit rigid frameworks in the infinite cylinder, by results of Nixon and Owen [64, Theorem 1.5] and Nixon, Owen and Power [64, Theorem 5.4].

Theorem 6.1.2. *If there is a spanning subgraph of* $\overline{\Gamma}$ *obtainable from a* K_4 *containing* v_0 *by a sequence of edge-to-*K_3*, vertex-to-*K_4 *and Henneberg moves respecting* v_0*, then* $\mathcal{A}(\Gamma, u)$ *is large.*

The phrase *respecting* v_0 has a technical meaning explained in Section 6.5. Our final result is a partial converse to Theorem 6.1.2. A *large pair* of a central arrangement $\mathcal{A}$ is a pair of subsets of $\mathcal{A}$ satisfying certain properties with respect to a given ordering of $\mathcal{A}$. The arrangement $\mathcal{A}$ is *large* if there exists an ordering with respect to which $\mathcal{A}$ admits a large pair. We are interested in orderings of $\mathcal{A}(\Gamma, u)$ in which the cone hyperplane $x_0 = 0$ is minimal.

Theorem 6.1.3. *If* $\mathcal{A}(\Gamma, u)$ *admits a large pair with respect to an ordering in which the cone hyperplane is minimal, then there is a spanning subgraph* H *of* $\Gamma/\partial V$ *such that the following hold:*

(i) H *is rigid in the cylinder*

(ii) No subgraph of H *away from* v_0 *is rigid in the cylinder.*

6.2 Background

An *arrangement* $\mathcal{A}$ in $\mathbb{C}^s$ is a finite set of affine hyperplanes in $\mathbb{C}^s$. The *rank* of $\mathcal{A}$ is the dimension of the span of the normal vectors of elements of $\mathcal{A}$. The arrangement $\mathcal{A}$ is *central* if its elements all contain the origin. Any central arrangement $\mathcal{A}$ defines a matroid on $\mathcal{A}$ in which a set X is independent if and only if the normal vectors of the elements of X are linearly independent. A *basis* (resp., a *circuit*) of a matroid is a maximal independent (resp., minimal dependent) set.

6.2.1 Large arrangements

Let $\mathcal{A}$ be a central arrangement. Recall that a subset of $\mathcal{A}$ is *NBC* with respect to an ordering of $\mathcal{A}$ if it contains no broken circuits of the matroid $M(\mathcal{A})$.

Definition 6.2.1. A *basic pair* of $M(\mathcal{A})$ with respect to an ordering of $\mathcal{A}$ is an ordered pair (B, C) of disjoint NBC subsets of $\mathcal{A}$ such that B is a basis.

Definition 6.2.2. A *large pair* of $M(\mathcal{A})$ with respect to an ordering of $\mathcal{A}$ is a basic pair (B, C) with $|C| = r - 1$.

Definition 6.2.3. A central arrangement $\mathcal{A}$ is *large* if there exists an ordering with respect to which $M(\mathcal{A})$ admits a basic pair.

Proposition 6.2.4 ([108, pp. 125–126])**.** *The following hold for any large arrangement $\mathcal{A}$ of rank r:*

(i) If U is the complement of $\mathcal{A}$, then $\mathrm{TC}_s(U) = rs - 1$.

(ii) If $\mathcal{A}'$ is a central arrangement of rank r with $\mathcal{A} \subseteq \mathcal{A}'$, then $\mathcal{A}'$ is large.

6.2.2 Dirichlet arrangements and matroids

Let Γ, ∂V and u be as in the introduction. Let $\partial E \subseteq E$ be the set of edges meeting ∂V. Write $d = |V|$, $m = |\partial V|$ and $n = d - m$. We take the coordinates of $\mathbb{C}^{n+1}$ to be z_i for all $i \in V \setminus \partial V$ and an additional coordinate z_0. Thus $\mathcal{A}(\Gamma, u)$ consists of all hyperplanes in $\mathbb{C}^{n+1}$ of the forms $z_i = u(j)z_0$ for all $ij \in \partial E$ with $j \in \partial V$; $z_i = z_j$ for all $ij \in E \setminus \partial E$; and $z_0 = 0$.

Let $E_0 = E \cup e_0$, where e_0 is a new element, so that $\mathcal{A}(\Gamma, u)$ is indexed by E_0. The matroid on E_0 defined by $\mathcal{A}(\Gamma, u)$ depends only on the network $N = (\Gamma, \partial V)$. Denote this matroid by $M(N)$.

The next two propositions are restatements of Propositions 4.2.14 and 4.2.15. They describe the bases and circuits of $M(N)$. A forest $X \subseteq E$ is a *grove* of N if X meets every element of $V \setminus \partial V$ and every component of X meets at least one boundary node. A *crossing* of N is the edge set of a minimal path in Γ between boundary nodes. Let Σ_0 (resp., Σ_1) be the set of groves of N containing no crossings (resp., exactly 1 crossing).

Proposition 6.2.5. *A set $X \subseteq E_0$ is a basis of $M(N)$ if and only if one of the following holds:*

(i) $X = Y \cup \{e_0\}$ for some $Y \in \Sigma_0$

(ii) $X \in \Sigma_1$.

Proposition 6.2.6. *A set $X \subseteq E_0$ is a circuit of $M(N)$ if and only if one of the following holds:*

(i) $X = Z \cup \{e_0\}$ *for a crossing* $Z \subseteq E$ *of* N

(ii) $X \subseteq E$ *is a cycle of* Γ *meeting at most 1 boundary node*

(iii) $X \subseteq E$ *is a minimal acyclic set containing more than 1 crossing.*

Example 6.2.7. Suppose that $\overline{\Gamma} = K_4$, where $\overline{\Gamma}$ is the multigraph obtained by identifying all vertices of ∂V as a single vertex. There are two possibilities for N, both illustrated in Figure 6.2 with boundary nodes marked in white. The corresponding $\overline{\Gamma}$ is illustrated on the left of Figure 6.2.7 with v_0 marked in white. In the middle and on the right of Figure 6.2.7 are two spanning trees of $\overline{\Gamma}$. Let $B = T \cup e_0$, where T is the tree in the middle, and let C be the tree on the right. Take the ordering $e_0 < e_1 < \cdots < e_6$ of E_0. We claim that (B, C) is a large pair of $M(N)$.

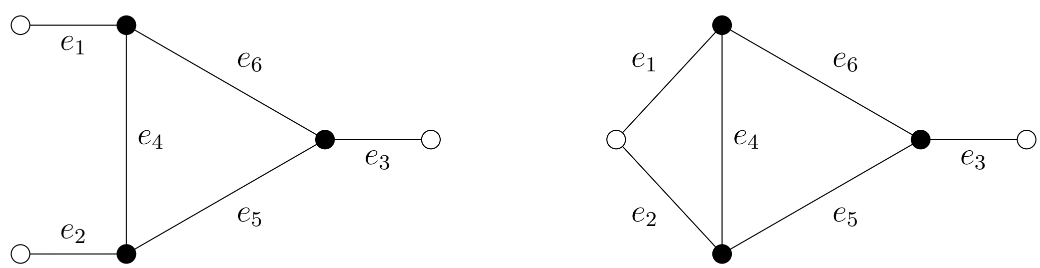


Figure 6.2: The two networks N with $\overline{\Gamma} = K_4$.

By Proposition 6.2.6 the subsets of B that are circuits of $M(N)$ minus one element are $\{e_0, e_1, e_4\}$, $\{e_0, e_1, e_4, e_5\}$ and $\{e_4, e_5\}$. The missing elements are e_2, e_3 and e_6, resp. None of these are minimal in the corresponding circuits, so B is NBC. If N is the network on the right of Figure 6.2, then $\{e_1, e_2\}$ is the only subset of C that is a circuit of $M(N)$ minus one element. The missing element is e_4, which is not minimal in the corresponding circuit. If N is the network on the left, then no subset of C is a circuit of $M(N)$ minus one element. In either case C is NBC. Hence (B, C) is a large pair of $M(N)$.

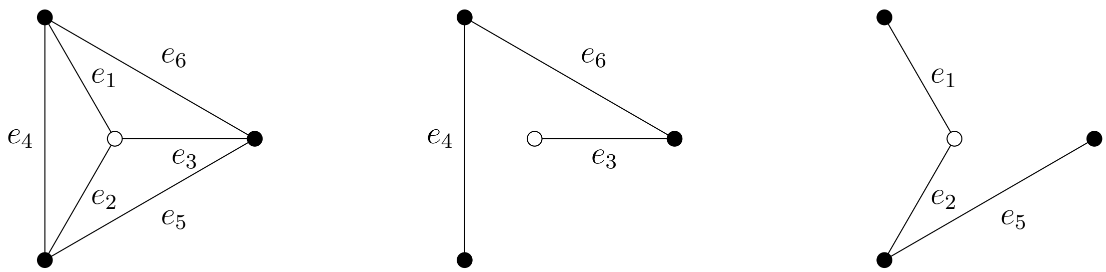


Figure 6.3: Disjoint spanning trees of K_4.

6.2.3 Tight graphs

The graph Γ is (k, ℓ)-*tight* if $|E| = k|V| - s$ and for every subgraph $H = (V_H, E_H)$ we have $|E_H| \leq k|V_H| - s$. This definition is due to Lee and Streinu [52]. A theorem of Nash-Williams and Tutte [62, 94] says that the (k, k)-tight graphs are precisely those that can be partitioned into k edge-disjoint spanning trees.

There are inductive constructions of the simple $(2, 3)$-tight and $(2, 2)$-tight graphs. We define four operations on graphs. A *Henneberg 1* move consists of adding a new vertex v and two edges vi and vj for distinct vertices i and j. A *Henneberg 2* move consists of deleting an edge ij, adding a new vertex v, and adding the edges vi, vj and vk for some other vertex k.

Proposition 6.2.8 ([38, 51])**.** *The following are equivalent for a simple graph* Γ*:*

(i) Γ *is* $(2, 3)$*-tight*

(ii) Γ *is minimally rigid in the plane*

(iii) Γ *can be obtained from* K_2 *by a sequence of Henneberg 1 and 2 moves.*

An *edge-to-*K_3 move consists of deleting an edge ij and splitting a vertex j into two vertices k and ℓ, reassigning its remaining incident edges between k and ℓ, and adding the edges ik, $i\ell$, and $k\ell$. A *vertex-to-*K_4 move consists of removing a vertex v and reassigning its incident edges among 3 new vertices i, j and k, adding a fourth new vertex, and drawing a K_4 on the new vertices. These operations are easier to understood visually; see Figure 6.1. We write $\Gamma \to \Delta$ if Δ is obtained from Γ by such a move.

Recall the definition of a rigid framework (Γ, p) in a surface $S \subseteq \mathbb{R}^3$ from Section 6.1. We say that (Γ, p) is *minimally rigid* in S if (Γ, p) is rigid in S and removing any edge of Γ yields a framework that is not rigid in S. One can also define a *completely regular* framework in S, but we do not do so here; we refer the reader instead to [65, Definition 3.3].

Proposition 6.2.9 ([64, Theorem 1.5] and [65, Theorem 5.4])**.** *The following are equivalent for a simple graph* Γ*:*

(i) Γ *is* $(2,2)$*-tight*

(ii) Γ *admits a minimally rigid completely regular framework on the infinite cylinder*

(iii) Γ *can be obtained from* K_4 *by a sequence of Henneberg, edge-to-*K_3 *and vertex-to-*K_4 *moves.*

6.3 Graphic arrangements

A result of Fieldsteel implies that the graphic arrangement $\mathcal{A}(\Gamma)$ is large if and only if Γ is $(2,3)$-tight. Proposition 6.2.8 says that this is equivalent to Theorem 6.1.1, and leads to the following proof of the "if" direction of the theorem. Given $X, Y \subseteq E$ and $e \in E$ we say that $X \subset E$ is an *e-broken circuit* of $M(N)$ (resp., an *e-broken cycle* of $\overline{\Gamma}$) in Y if $X \subseteq Y$, $e \notin Y$, and $X \cup e$ is a circuit of $M(N)$ (resp., a cycle of $\overline{\Gamma}$).

Lemma 6.3.1. *If* Γ *is obtained from* K_2 *by Henneberg 1 and 2 moves, then* $\mathcal{A}(\Gamma)$ *is large.*

Proof. Suppose that $\Gamma = K_2$. Take B to consist of the only edge and C to be empty. Then (B, C) is a large pair of Γ.

For the remainder of the proof suppose that Γ is any large graph, and let (B, C) be a large pair of Γ with respect to a fixed ordering of E. Suppose that $\Gamma \to \Gamma_0$ is a Henneberg 1 move. Let k be the added vertex, and ik and jk the added edges, as illustrated in Figure 6.3.1. Let $B_0 = B \cup ik$ and $C_0 = C \cup jk$. Any cycle of Γ_0

containing an element of $\{ik, jk\}$ must properly contain both. Thus by declaring ik and jk to be greater than any element of E we obtain an ordering of $E \cup \{ik, jk\}$ with respect to which (B_0, C_0) is a large pair of Γ_0.

Figure 6.4: A Henneberg 1 move.

Next suppose that $\Gamma \to \Gamma_1 = (V_1, E_1)$ is a Henneberg 2 move. Call the deleted edge ij and the added vertex v, so that the added edges are vi, vj, and vk for some other vertex k. This is illustrated in Figure 6.3.1.

Figure 6.5: A Henneberg 2 move.

Suppose without loss of generality that $ij \in B$. Let $B_1 = (B \setminus ij) \cup \{vi, vj\}$ and $C_1 = C \cup vk$. Consider the ordering of E as a string of strict inequalities. Replacing ij in this string with $vi < vk < vj$ gives an ordering of E_1. We claim that (B_1, C_1) is a large pair of Γ_1 with respect to this ordering.

Given a forest F and two vertices x and y, let P_F^{xy} denote the edge set of the unique path in F from x to y, assuming one exists. The only possible vi-broken cycle of G_1 in C_1 is $P_{C_1}^{iv} = P_C^{ik} \cup vk$. But $P_C^{ik} \cup vk$ is a vi-broken cycle of G_1 in C_1 only if

(i) $vi < e$ for all $e \in P_C^{ik} \cup vk$.

Similarly the only possible vj-broken cycle of G_1 in C_1 is $P_{C_1}^{jv} = P_C^{jk} \cup vk$, which is a vj-broken cycle of G_1 in C_1 only if

(ii) $vj < e$ for all $e \in P_C^{ik} \cup vk$.

Since B is a tree, one of P_B^{ik} and P_B^{jk} must be contained in the other. Suppose without loss of generality that $P_B^{ik} \subseteq P_B^{jk}$. The only possible vk-broken cycle of G_1 in B_1 is $P_{B_1}^{vk} = P_B^{ik} \cup vi$. But $P_B^{ik} \cup vi$ is a vk-broken cycle of G_1 in B_1 only if

(iii) $vk < vi$ and $vk < e$ for all $e \in P_B^{ik} \cup vi$.

There must be an edge $f \in P_C^{ik} \cup P_C^{jk}$ such that $ij > f$; otherwise $P_C^{ik} \cup P_C^{jk}$ is an ij-broken cycle of Γ in C, contradicting the assumption that C is NBC. Suppose without loss of generality that $f \in P^{ik}$. This violates condition (i), so there is no vi-broken cycle of Γ_1 in C_1. We have also declared $vk < vj$, violating (ii), so there is no vj-broken cycle in C_1. Since C is NBC, the only possible broken cycles of Γ_1 in C_1 are the vi- or vj-broken cycles. Hence C_1 is NBC.

We have declared $vi < vk$, violating (iii), so there are no vk-broken cycles of Γ_1 in B_1. By replacing ij in the ordering of E, we have ensured that $ij < e$ for $e \in E_0$ if and only if $vi < e$ and $vj < e$. Since B is NBC, it follows that there are no e-broken cycles of Γ_1 in B_1 for any $e \in C_1 \setminus v_k$. Hence B_1 is NBC and (B_1, C_1) is a large pair of Γ_1. □

6.4 Proof of Theorem 6.1.3

For this section, fix an ordering of E_0 with e_0 minimal. Theorem 6.1.3 follows from Lemmas 6.4.1 and 6.4.2 below. Recall that Σ_0 is the set of groves of N containing no crossing, defined in Definition 4.2.12.

Lemma 6.4.1. *If B and C are NBC subsets of E_0, then (B, C) is a large pair of $\mathcal{A}(\Gamma, u)$ if and only if $e_0 \in B$, $B \setminus e_0 \in \Sigma_0$ and $C \in \Sigma_0$.*

Proof. The "if" direction follows immediately from Proposition 6.2.5. We prove the "only if" direction. Suppose that e_0 is minimal in E_0 and that (B, C) is a large pair of E_0. Since B is a basis of $M(N)$, by Proposition 6.2.5 we must have $B \in \Sigma_1$ or $e_0 \in B$ and $B \setminus e_0 \in \Sigma_0$. If $B \in \Sigma_1$, then B contains a crossing. But since e_0 is minimal, any crossing is a broken circuit, a contradiction. Hence $e_0 \in B$ and $B \setminus e_0 \in \Sigma_0$. If a subset of $E_0 \setminus B$ of size n contains no circuits, then it must be a spanning tree of $\overline{\Gamma}$. Hence $C \in \Sigma_0$. □

Lemma 6.4.2. *If (B, C) is a large pair of $M(N)$, then $H = (B \setminus e_0) \cup C$ is a $(2,2)$-tight spanning subgraph of $\overline{\Gamma}$, and every $(2,2)$-tight subgraph of H contains v_0.*

Proof. Lemma 6.4.1 implies that H is a $(2,2)$-tight spanning subgraph. Let J be a $(2,2)$-tight subgraph of H. By restricting B and C to the edges of J, we obtain disjoint NBC spanning trees S and T of J. If J does not contain v_0, then these spanning trees must in fact contain no broken cycle. We claim that this is impossible. Let e be the minimal edge in J, and suppose that $e \in S$. There is a path P in T between the endpoints of e. Since e is minimal, P is an e-broken cycle of J, proving the claim. Hence J must contain v_0. □

6.5 Proof of Theorem 6.1.2

Continue to fix an ordering of E_0 with e_0 minimal. We call v_0 the *quotient vertex* of $\overline{\Gamma} = \Gamma/\partial V$. Let $N_i = (\Gamma_i, \partial V_i)$ be a network with $\overline{\Gamma}_i = \Gamma_i/\partial V_i$. We maintain this notation throughout. We say that $N \to N_i$ is a Henneberg 1, Henneberg 2, edge-to-K_3 or vertex-to-K_4 move if $\overline{\Gamma}_i$ is obtained from $\overline{\Gamma}$ by such a move that *respects* the quotient vertex. By this we mean that

(i) For a Henneberg 1 or 2 move, the quotient vertex in the original graph remains the quotient vertex in the new graph

(ii) For an edge-to-K_3 move, the quotient vertex is not the deleted vertex, and remains the quotient vertex in the new graph

(iii) For a vertex-to-K_4 move, the vertex in question is v_0, and the quotient vertex in the new graph is the "center" of the new K_4 (i.e., the only black vertex of the graph on the bottom-right of Figure 6.1).

Taken together, the following four lemmas imply Theorem 6.1.2.

Lemma 6.5.1. *Let $N \to N_0$ be a Henneberg 1 move. If $M(N)$ is large, then $M(N_0)$ is large.*

Proof. Suppose that (B, C) is a large pair of $M(N)$. Let k be the added vertex with ik and jk the added edges, as in Figure 6.3.1. Let $B_1 = B \cup ik$ and $C_1 = C \cup jk$. Declare ik and jk to be greater than every element in E_0. We claim that this gives an ordering with respect to which (B_1, C_1) is a large pair of $M(N_1)$. The only possible broken circuits introduced by the Henneberg 1 move are the ik- and jk-broken circuits. Any ik- or jk-broken circuit of $M(N_1)$ must contain an element of E. Such a broken circuit cannot be ik- or jk-broken, since these are greater than the element of E. The claim follows. □

Lemma 6.5.2. *Let $N \to N_2$ be a Henneberg 2 move. If $M(N)$ is large, then $M(N_2)$ is large.*

Proof. Suppose that (B, C) is a large pair of $M(N)$. Call the deleted edge ij and the added vertex v, so that the added edges are vi, vj, and vk for some other vertex k, as illustrated in Figure 6.3.1.

Suppose that $ij \in B$; the argument is similar if $ij \in C$ instead. Let $B_2 = (B \setminus ij) \cup \{vi, vj\}$ and $C_2 = C \cup vk$. Consider the ordering of E_0 as a string of inequalities $e_0 < \cdots < e_{|E|}$. Replacing ij in this string with $vi < vk < vj$ gives an ordering of $E_2 \cup e_0$. We claim that (B_2, C_2) is a large pair of $M(N_2)$ with respect to this ordering.

Given a tree T and two vertices x and y, let P_T^{xy} denote the edge set of the unique path in T from x to y. If $P_B^{ik} \cup \{vi, vk\}$, $P_C^{ik} \cup \{vi, vk\}$ and $P_C^{jk} \cup \{vj, ck\}$ are circuits of $M(N_2)$ (i.e. if they are are cycles of Γ_2), then the proof of Lemma 6.3.1 implies that B_2 and C_2 are NBC. If instead any of $P_B^{ik} \cup \{vi, vk\}$, $P_C^{ik} \cup \{vi, vk\}$ and $P_C^{jk} \cup \{vj, ck\}$ are not circuits of $M(N_2)$ (i.e., if any of these sets is a crossing of N_2), then since e_0 is minimal, Proposition 6.2.6 implies that we can ignore the corresponding condition (i), (ii) or (iii) in the proof of Lemma 6.3.1. Hence B_2 and C_2 are NBC, so (B_2, C_2) is a large pair of $M(N_2)$. □

Lemma 6.5.3. *Let $N \to N_3$ be an edge-to-K_3 move. If $M(N)$ is large, then $M(N_3)$ is large.*

Proof. Suppose that (B, C) is a large pair of $M(N)$. Call the deleted edge ij and the added vertices k and ℓ, as in Figure 6.5.3.

Suppose without loss of generality that $ij \in B$. Let $B_3 = (B \setminus ij) \cup \{ik, i\ell\}$ and $C_3 = C \cup k\ell$. Replace ij in the ordering of E_0 with $ik < i\ell < k\ell$. We claim that (B_3, C_3) is a large pair of $M(N_3)$ with respect to this ordering. First, note that

Figure 6.6: An edge-to-K_3 move.

$B_3 \setminus \{ik, i\ell\}$ cannot contain a $k\ell$-broken circuit of $M(N_3)$, since otherwise B contains a cycle of $\overline{\Gamma}$. Also, C_3 cannot contain an ik- or a $i\ell$-broken circuit, since otherwise C would contain an ij-broken circuit. The only possible remaining broken circuit introduced by the move is $\{ik, i\ell\}$, but the minimal element of the cycle $\{ik, i\ell, k\ell\}$ is ik. The claim follows. □

Lemma 6.5.4. *Let $N \to N_4$ be a vertex-to-K_4 move. If $M(N)$ is large, then $M(N_4)$ is large.*

Proof. Suppose that (B, C) is a large pair of $M(N)$. Label the edges of the new K_4 according to the left side of Figure 6.2. Let $B_4 = B \cup \{e_3, e_4, e_6\}$ and $C_4 = C \cup \{e_1, e_2, e_5\}$. Declare the new edges to be greater than e_0 but less than any element of E, and set $e_1 < e_2 < \cdots < e_6$. We claim that B_4 and C_4 contain no e-broken circuits of $M(N_4)$ for any new edge e. Such a broken circuit must consist of only new edges, since otherwise B or C contained a cycle of $\overline{\Gamma}$. Example 6.2.7 implies that there are no such broken circuits consisting of only new edges, proving the claim. Moreover B_4 and C_4 cannot contain any e-broken circuits of $M(N_4)$ for any $e \in E_0$, since otherwise B or C would have contained an e-broken circuit of $M(N)$. Hence B_4 and C_4 are NBC, so (B_4, C_4) is a large pair of $M(N_4)$. □

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