Representation Stability, Configurations Spaces, and Deligne–Mumford Compactifications

by

Philip Tosteson

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Doctoral Committee:
- Professor Andrew Snowden, Chair
- Professor Karen Smith
- Professor David Speyer
- Assistant Professor Jennifer Wilson
- Associate Professor Venky Nagar

Philip Tosteson

ptoste@umich.edu

orcid.org/0000-0002-8213-7857

Dedication

To Pete Angelos.

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Table of Contents

Abstract

We study the homology of configuration spaces and Deligne–Mumford compactifications using tools from the area representation stability.

In the case of configuration spaces, we show that for topological spaces X that are ≥ 2 dimensional in some sense, the cohomology of $\mathrm{Conf}_n(X)$ is a finitely generated module over the category of finite sets and injections, **FI**. From this, we deduce homological stability results for unordered configuration space.

In the case of Deligne–Mumford compactifications, we show that the homology is a finitely generated module over the opposite of the category of finite sets and surjections, $\mathbf{FS}^{\mathrm{op}}$. As a consequence, we show that for every i, g the sequence $\{b_i(\overline{M_{g,n}})\}_{n\in\mathbb{N}}$ eventually agrees with a sum of the form $\sum_{r=1}^{j} p_r(n)r^n$, where p_r is a polynomial.

Chapter 1

Introduction

1.1 Representation Stability

Many important spaces in topology and algebraic geometry come in sequences that depend on a parameter $n \in \mathbb{N}$. Simple examples are projective spaces $\mathbb{P}^n$ and Grassmannians $\mathbf{Gr}(k, n)$. More complicated examples include the classifying spaces of the linear and symmetric groups $\mathrm{B}\mathbf{GL}(n, \mathbb{Z})$, $\mathrm{B}\mathbf{S}_n$, as well as M_n, the moduli space of curves of genus n.

Homology is one of the most fundamental invariants of a topological space, so when we have a sequence of spaces $Y_n, n \in \mathbb{N}$, a natural question is:

Question 1.1.1. *How do the homology groups $H_i(Y_n)$ behave asymptotically for $n \gg 0$?*

Classical results in *homological stability* show that, in many cases, the groups $H_i(Y_n)$ stabilize to a group $H_i(Y_\infty)$ that is independent of n. Remarkably, these stable homology groups tend to have a simpler structure than the original groups. For instance the cohomology ring of the Grassmannian stabilizes to a polynomial ring $\mathbb{Z}[h_1, \ldots, h_k]$. For $\mathbf{GL}_n(\mathbb{Z})$, the stable dimensions of the groups are related to spaces of automorphic forms [2]. And for M_n, the stable homology was computed in Madsen-Weiss's proof of the Mumford conjecture [28].

However, when the spaces Y_n have a symmetry group G_n that grows in n, this symmetry can prevent the homology from stabilizing. For example, if X is a topological space, the configuration space $\mathrm{Conf}_n X$ parameterizes labelled points moving around in X. The symmetric group $\mathbf{S}_n$ acts on $\mathrm{Conf}_n X$ by relabelling points, and the dimensions of the homology groups of $\mathrm{Conf}_n X$ grow with n.

To get around this problem, Church and Farb [7] introduced a notion of *representation stability*, which describes what it means for representations V_n of different groups G_n to stabilize in n. And Church [3] proved that the $\mathbf{S}_n$ representations $H_i(\mathrm{Conf}_n(M), \mathbb{Q})$ stabilize, for any connected manifold of dimension ≥ 2.

Still, Church and Farb's definition of representation stability was partly ad hoc, and depended on consistent labelling schemes for irreducible representations of $\mathbf{S}_n$. Why would this particular phenomenon occur?

Church, Ellenberg and Farb [5] gave a conceptual explanation of representation stability—in terms of algebraic structures called **FI** *modules*. They showed that homology groups $H_i(\mathrm{Conf}_n(M))$ carry the structure of an **FI** module. The fact that this **FI** module is *finitely generated*, combined with the general theory of **FI** modules, gives a new proof of Church's theorem.

Simultaneously, in commutative algebra, structures related to **FI** modules arose in the work of Snowden [41], and were studied by Sam and Snowden in [38]. Since then, there has been an explosion of interest in the representation theory of so-called *combinatorial categories*, and their applications to topology and commutative algebra. Broadly, this area of study is referred to as *representation stability*.

1.2 Main Results

The main results of this thesis are in representation stability. We focus on applications to sequences of topological spaces Y_n, which carry an action of $\mathbf{S}_n$. Our main results concern the case where $Y_n = \mathrm{Conf}_n X$, the configuration space in a topological space X, and the case where $Y_n = \overline{\mathcal{M}}_{g,n}$, the Deligne-Mumford compactification of the moduli space of curves. In both of these situations we ask:

1. Is there a combinatorial category C that acts on the homology $H_i(Y_n)$?
2. If so, is $H_i(Y_n)$ finitely generated by the action of C?
3. What is the representation theory of C? How does this representation theory govern the behavior of $H_i(Y_n)$ for $n \gg 0$?

The following subsections give the flavor of our results in both of these cases. In the first case we use the category **FI**, and in the second case we use the category $\mathbf{FS}^{\mathrm{op}}$.

1.2.1 Configuration spaces of non-Manifolds

Given a topological space X, its *configuration space*

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \forall i \neq j\},$$

is the space of n ordered points in X, with the restriction that pairs of points are not allowed to collide.

The homotopy type of the configuration space $\mathrm{Conf}_n(X)$ is sensitive to the local structure of X, and not just its homotopy type. Because of this, when X is not a manifold, less can be said about the cohomology of its configuration space. Our first main result concerns an extension of Church's theorem to a wide class of topological spaces.

We say that $p \in X$ is a *roadblock* if there is a contractible neighborhood $U \ni p$ such that $U - p$ is disconnected.

In Chapter 5, we will use sheaf theory to construct a spectral sequence that computes the cohomology $\mathrm{Conf}_n(X)$ from the relative cohomology of $(X^n, X^n - X)$. With this tool, we show:

Theorem 1.2.1. *Let X be a finite CW complex or an algebraic variety that is connected and has no roadblocks. Then, for every i, $H^i(\mathrm{Conf}_n(X))$ has the structure of a finitely generated* **FI** *module. In particular, the cohomology of the unordered configuration space $H^i(\mathrm{Conf}_n(X)/\mathbf{S}_n, \mathbb{Q})$, stabilizes.*

In addition to proving representation stability for $H^i(\mathrm{Conf}_n(X))$, this proves rational homological stability for these unordered configuration spaces. A result that builds on our work is that that the betti numbers of $\mathrm{Conf}_n(X)$ agree with a polynomial for $n \geq 2i$ [9].

1.2.2 Deligne-Mumford compactifications

The algebraic variety $M_{g,n}$ parameterizes the set of genus g curves with n distinct marked points. The variety $\overline{M}_{g,n}$ is an important compactification introduced by Deligne and

Mumford, which adds new points corresponding to degenerate marked curves. The group $\mathbf{S}_n$ acts on these spaces by relabelling points.

Although the homology of $H_i(M_{g,n})$ and $H_i(\overline{M}_{g,n})$ has been studied intensively from many perspectives, its structure remains mysterious for g small relative to i. Paradoxically, the Euler characteristic of M_g grows *exponentially* in g and is often negative—implying an exponential quantity of odd degree homology—but there is no stable homology in odd degrees. This phenonomon is sometimes referred to as the *problem of dark matter*.

In Chapter 6 we fix i and g and study the asymptotics of $H_i(\overline{M}_{g,n})$ for $n \gg 0$ as an $\mathbf{S}_n$ representation. Unlike the situation for configuration spaces, the dimensions of these homology groups grow exponentially in n and the $\mathbf{S}_n$ representations do not stabilize. Thus techniques using **FI** modules are bound to fail.

Accordingly, we construct an action of the *opposite of the category of finite sets and surjections*, $\mathbf{FS}^{\mathrm{op}}$, on the homology of $\overline{M}_{g,n}$ and prove that it is finitely generated. This result places constraints on the homology of $\overline{M}_{g,n}$ for $n \gg 0$. In particular, it has the following consequence:

Theorem 1.2.2. *The generating function of the homology of $\overline{M}_{g,n}$ is rational and takes the following form*

$$\sum_n \dim H_i(\overline{M}_{g,n}) t^n = \frac{q(t)}{(1-t)^{i_1}(1-2t)^{i_2}\ldots(1-jt)}.$$

Thus for $n \gg 0$ we have $\dim H_i(\overline{M}_{g,n}) = p_1(n) + p_2(n)2^n + \cdots + j^n$, where the p_i are polynomials.

1.3 Structure of the Thesis

1.3.1 Species, Operads, Wiring Categories, and Koszul Duality

In Chapter 2, we give an introduction to the category of *linear species*, $\mathcal{S}$. A linear species is simply another name for a sequence of symmetric group representations. One can think of the category $\mathcal{S}$ as an enhancement of the ring of symmetric functions. The category of linear species, together with its product and composition structure, is a fundamental setting in which to study the representation theory of the symmetric and general linear group.

We introduce the notion of an *operad*, which a sort of algebraic structure built out out of composition of species. We use operads to construct categories, by associating to every operad P its *wiring category* $\mathbf{W}_P$, also known as the **PROP** of P.

Many combinatorial categories C which have been studied in representation stability take the form $C^{\mathrm{op}} = \mathbf{W}_P$. In particular $C = \mathbf{FS}^{\mathrm{op}}$ is of this form, for $P = \mathrm{Com}$ the commutative operad. We show that when this occurs, modules over C^{op} are equivalent to right modules over the operad.

One benefit of constructing categories in this way is that there is a well-developed theory of free resolutions of modules over operads, known as *Koszul duality*. If a category C^{op} is of the form $\mathbf{W}_P$, where P is a Koszul operad, then we can completely describe minimal free resolutions of simple C modules. In particular, we use this technique to construct resolutions of modules over $\mathbf{FS}^{\mathrm{op}}$.

Another benefit is that since the early 90's, algebraic structures on moduli spaces of curves have been studied using operads [29]. From this point of view, our results on the structure of the $\mathbf{FS}^{\mathrm{op}}$ action on the homology of $\overline{M}_{g,n}$ are a continuation of this study.

1.3.2 Poset Topology and Sheaf Theory

In Chapter 4, we give background on the homology of posets. Every poset Q is in particular a category, and so the notion of a representation of Q makes sense. Homology groups associated to Q modules can be defined as Ext groups in the category $\mathrm{Rep}\, Q$.

The standard approach to poset homology uses a simplicial complex $|Q|$. From the perspective of $\mathrm{Rep}\, Q$, the simplicial complex $|Q|$ corresponds to computing Ext groups using *bar resolution* of modules over Q. The bar resolution is the best available construction that works for arbitrary posets, and it is useful for making general arguments. However, for a specific posets Q, there are smaller complexes available and these can be useful for making computations.

The most important poset for us is the lattice of set partitions $\mathrm{P}(n)$. Its homology is closely connected to the species of Lie representations, which we define in Chapter 2. As a category, the partition lattice $\mathrm{P}(n)$ is equivalent to the over-category $[n]/\mathbf{FS}$ of surjections $[n] \twoheadrightarrow x$. Because of this, the structure of free resolutions over $\mathrm{P}(n)$ is closely related to that free resolutions over $\mathbf{FS}^{\mathrm{op}}$. They are both governed by the Lie operad via Koszul duality, and so can apply techniques from Chapter 3 to construct resolutions of $\mathrm{P}(n)$.

The representation-theoretic approach on poset homology is well–adapted for applications to sheaf cohomology. Here is the general idea of these applications.

When a topological space Y is stratified by a poset Q, there are associated representations $Q \to \mathrm{Sh}(Y)$. Given a sheaf $\mathcal{F}$ on Y, we can use our knowledge of free resolutions over Q to construct a resolution $\mathcal{G}_\bullet \xrightarrow{\sim} \mathcal{F}$, and then use this resolution to compute the cohomology of $\mathcal{F}$.

A filtration F_i of the resolution $\mathcal{G}_\bullet$ produces a spectral sequence that converges to the homology of $\mathcal{F}$. Often, this spectral sequence is independent of $\mathcal{G}_\bullet$, in the sense that if $\mathcal{G}'_\bullet$ is a different resolution (of the same type as $\mathcal{G}_\bullet$, in some sense), then there exists a filtration F'_i on $\mathcal{G}'_\bullet$ that gives the same spectral sequence. Because of this, for the purpose of constructing spectral sequences, it is possible to choose an inefficient resolution $\mathcal{G}'_\bullet$, such as the one associated to the Bar construction of Q. But to construct small chain complexes whose cohomology computes $H^\bullet(\mathcal{F})$, it is helpful to be able to choose the smallest possible resolution.

Along these lines, our main result in Chapter 5 is the construction of resolutions of $j_*\mathbb{Z}$, where $j : U \to Y$ is the top stratum of a space stratified by Q. These resolutions yield a spectral sequence that computes $H^\bullet(U)$ in terms of $H^\bullet(Q)$ and $H^\bullet(Y, Y - Z_q)$, where $Z_q \subset Y$ is the closed stratum corresponding to $q \in Q$.

1.3.3 Configuration Spaces

In Chapter 5 we study the cohomology of configuration spaces, making use of the techniques from Chapter 4.

For X a Hausdorff topological space, the product space X^n is stratified by $\mathrm{P}(n)$. The closed stratum corresponding to a partition p is

$$Z_p = \{(x_i)_{i\in[n]} \in X^n | x_i = x_j \text{ if } i \sim_p j\} = X^{\#p}.$$

The top open stratum is precisely the configuration space $\mathrm{Conf}_n X$. Thus from Chapter 4, we obtain a spectral sequence converging to $H^\bullet(\mathrm{Conf}_n(X))$ that is built out of terms involving $H^\bullet(X^n, X^n - X^{\#p})$ and $H^\bullet(\mathrm{P}(n))$.

This spectral sequence is compatible with the action of $\mathbf{FI}^{\mathrm{op}}$ on the stratified space $X^n, \mathrm{P}(n)$, and so it becomes a spectral sequence of **FI** modules. By bounding the terms of the spectral sequence, establish our main results on configuration spaces.

1.3.4 Moduli Spaces

In Chapter 6, we study the homology $H_i(\overline{M}_{g,n}, \mathbb{Q})$. The operadic structure on $\overline{M}_{g,n}$ shows that $H_i(\overline{M}_{g,n})$ is a right module over the commutative operad, and thus gives it the structure of an $\mathbf{FS}^{\mathrm{op}}$ module by the Morita equivalence proved in Chapter 2.

The space $\overline{M}_{g,n}$ is stratified by a poset of stable graphs, but this stratification is not compatible with the action of $\mathbf{FS}^{\mathrm{op}}$. We construct a coarsening of this stratification that is compatible with the $\mathbf{FS}^{\mathrm{op}}$ action, in some sense. Together with combinatorial arguments that bound the number of graphs with prescribed properties, we use this stratification to prove that the homology of $\overline{M}_{g,n}$ is a finitely generated $\mathbf{FS}^{\mathrm{op}}$ module.

Chapter 2

Species, Operads, and Combinatorial Categories

2.1 Linear Species and Symmetric Functions

In this section, we describe the category of linear species, which plays a central role in the representation theory of S_n, the representation theory of the general linear group and the theory of symmetric functions. Let k be a commutative ring.

Definition 2.1.1 (Category of linear species)**.** Let $\mathcal{S}_k$, be the category of k-species. Objects of $\mathcal{S}_k$ are functors from the category of finite sets and bijections to the category of k modules. We use $\mathcal{S}$ to refer to $\mathcal{S}_{\mathbf{Z}}$. □

Remark 2.1.2. The category $\mathcal{S}_k$ has many names, our terminology follows Joyal [22], who used the term species to denote the category of functors Bij $\to$ Set. □

More generally, the basic structures we consider will make sense for functors Bij $\to \mathcal{V}$ where $(\mathcal{V}, \otimes)$ is a symmetric monoidal category. We denote this category of functors by $\mathcal{S}_{\mathcal{V}}$. Relevant examples are $\mathcal{V}$ = Ch the category of chain complexes, $\mathcal{V}$ = Top the category of topological spaces, and various categories of algebro-geometric spaces (varieties, schemes, algebraic stacks, etc).

The category $\mathcal{S}$ is equivalent to the category of sequences of symmetric group representations ($\prod_n \operatorname{Rep} \mathbf{S}_n$) by evaluating on $[n]$ the set $\{1, \ldots, n\}$. An inverse functor is given by taking a sequence of $\mathbf{S}_n$ representations $M(n)$ to the functor

$$X \mapsto \bigoplus_n k \operatorname{Bij}(X, [n]) \otimes_{\mathbf{S}_n} M(n).$$

We will pass back and forth across this equivalence freely.

Many examples of linear species come from linearizing a combinatorial structure.

Example 2.1.3. The functor

$$X \mapsto \mathbb{Z}\{\text{partitions of } X \text{ into 3 subsets}\},$$

is a $\mathbb{Z}$-linear species, which is obtained by linearizing the Set species

$$X \mapsto \{\text{partitions of } X \text{ into 3 subsets}\}.$$

□

We can also construct linear species by "linearizing" topological species. The following example is especially relevant for us:

Example 2.1.4. For any topological space M, we have the topological species

$$X \mapsto \operatorname{Conf}_X(M) := \{\text{injections } X \hookrightarrow M\},$$

where the set of injections is topologized as a subspace of the product space M^X. Then, for any i, $X \mapsto H_i(\operatorname{Conf}_X(M), \mathbb{Z})$ is a $\mathbb{Z}$ linear species. □

Another basic example is given by the indicator species on representations of $\mathbf{S}_n$.

Definition 2.1.5. Let $n \in \mathbb{N}$. We define $\mathbb{Z}\mathbf{S}_n$ to be the species $X \mapsto \mathbb{Z} \operatorname{Bij}(X, [n])$. Similarly, for any $\mathbf{S}_n$ representation M, we define $M(X) = M \otimes_{\mathbf{S}_n} \mathbb{Z} \operatorname{Bij}(X, [n])$. □

A final important example is the species of Lie representations, which are closely related to the homology of configuration spaces and partition lattices.

Definition 2.1.6. For any set S, we define $\mathrm{Lie}(S)$ to be the free abelian group on all bracketings of the elements of S modulo the anticommutativity and jacobi relations. The functor $S \mapsto \mathrm{Lie}(S)$ is the species of Lie representations. □

Example 2.1.7 (Lie representations)**.** To clarify the definition, the first 3 nonzero Lie representations are as follows.

As a vector space, we have a presentation of $\mathrm{Lie}(\{1,2\})$ as

$$\mathrm{Lie}(\{1,2\}) = \mathbb{Z}\{[12],[21]\}/[12] = -[21]$$

As an $\mathbf{S}_3$ representation we have the presentation, $\mathrm{Lie}(\{1,2,3\})$ as

$$\mathrm{Lie}(\{1,2,3\}) = \frac{\mathbb{Z}\mathbf{S}_3\{[[12]3],[1[23]]\}}{[[12]3] = -[[21]3],\ [[12]3] = -[3[12]],\ [1[23]] + [3[12]] + [2[31]] = 0}.$$

As a $\mathbf{S}_4$ representation, we ahve the presentation

$$\mathrm{Lie}(\{1,2,3,4\}) = \frac{\mathbb{Z}\mathbf{S}_4\{[[12]3]4],[[1[23]]4],[[12][34]],[1[[23]4]],[1[2[34]]]\}}{[[12]3]4]] = -[[21]3]4]],\dots,[[1[23]]4] + [[3[12]]4] + [[2[31]]4] = 0}.$$

These presentations are not very efficient for working with the $\mathrm{Aut}(S)$ representation $\mathrm{Lie}(S)$. It turns out that $\mathrm{Lie}(S)$ is an $(|S|-1)!$ dimensional torsion free abelian group, but any given basis of it will not behave as nicely under the action of $\mathbf{S}_n$ or under composition as the presentation of $\mathrm{Lie}(S)$ as brackets modulo Lie algebra relations. □

2.1.1 Schur–Weyl duality

Let M be a k-species. We can use M to define a functor $F_M : \mathrm{Mod}\,k \to \mathrm{Mod}\,k$ by $F_M(V) = \bigoplus_n M(n) \otimes_{\mathrm{S}_n} V^{\otimes n}$.

Definition 2.1.8. A *polynomial functor* is any functor $G : \mathrm{Mod}\,k \to \mathrm{Mod}\,k$ that is isomorphic to F_M for some $M \in \mathcal{S}_k$. □

Remark 2.1.9. This class of functors is called polynomial because they are subquotients of sums of functors of the form $V \mapsto V^{\otimes n}$, which is analogous to the function $x \mapsto x^n$. Note that our terminology allows infinite sums as polynomial functors. □

We are most interested in polynomial functors in the case $k = \mathbb{Q}$, due to the following theorem.

Theorem 2.1.10 (Schur-Weyl Duality)**.** *The category $\mathcal{S}_{\mathbb{Q}}$ of $\mathbb{Q}$-linear species is equivalent to the category of polynomial functors* $\mathrm{Vec}_{\mathbb{Q}} \to \mathrm{Vec}_{\mathbb{Q}}$*, under the map $M \mapsto F_M$.*

The adjoint functor to $M \mapsto F_M$ is given by $F(V) \mapsto \mathrm{Nat}_{\mathrm{Vec}}(V^{\otimes n}, F(V))$.

2.1.2 Generating functions and symmetric functions

To every species $M \in \mathcal{S}$ that is sufficiently finite, we can assign a symmetric function that records the S_n representation $M_n \otimes \mathbb{Q}$. This symmetric function specializes to a generating function that just records the ranks of M_n. Natural algebraic operations on species correspond to algebraic operations on the corresponding generating function.

Definition 2.1.11 (Symmetric function of a species)**.** Let M be a species. We define the symmetric function s_M by

$$s_M(x_1, \dots, x_d) = \mathrm{tr}(\mathrm{diag}(x_1, \dots x_d), F_M(\mathbb{Q}\{1, \dots, d\}) \in \Lambda_d.$$

These symmetric functions are compatible as d grows and so define an element $s_M \in \Lambda$ $\square$

By Schur-Weyl duality and the character theory of $\mathbf{GL}_d$, this symmetric function is a complete invariant of $M \otimes \mathbb{Q} \in \mathcal{S}_{\mathbb{Q}}$.

Definition 2.1.12. Let M be a species. The *exponential generating function of M* is $H_M(t) := \sum_n \dim M_n \frac{t^n}{n!}$. $\square$

The exponential generating function H_M is the specialization of s_M under the map $\Lambda \to \mathbb{Z}\{\frac{t^n}{n!}\}$ that takes the homogeneous symmetric function

$$h_n(x_1, x_2, \dots) := \sum_{i_1 \leq i_2 \cdots \leq i_n} x_{i_1} x_{i_2} \dots x_{i_n}$$

to $\frac{t^n}{n!}$.

Example 2.1.13. The species $\mathbb{Z}\mathbf{S}_n$ has associated symmetric function

$$s_{\mathbb{Z}\mathbf{S}_n} = (x_1 + x_2 \dots)^n = p_1^n,$$

The species $\operatorname{triv}(n)$, corresponding to the trivial representation of $\mathbf{S}_n$ has associated symmetric function

$$s_{\operatorname{triv}(n)} = h_n(x_1, x_2, \dots) = \sum_{i_1 \leq i_2 \cdots \leq i_n} x_{i_1} x_{i_2} \dots x_{i_n}$$

The species $\operatorname{sgn}(n)$, corresponding to the alternating representation of $\mathbf{S}_n$ has associated symmetric function

$$s_{\operatorname{sgn}(n)} = e_n(x_1, x_2, \dots) = \sum_{i_1 < i_2 \cdots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}.$$

The exponential generating functions corresponding to these three examples are $\frac{1}{1-x}, e^x, e^x$ respectively. □

2.1.3 Tensor product

Let M, N be species. We define

$$(M \otimes N)(X) := \bigoplus_{X = A \sqcup B} M(A) \otimes N(B).$$

Under Schur-Weyl duality, we have $F_{M \otimes N}(V) = F_M(V) \otimes F_N(V)$. Thus we have $s_{M \otimes N} = s_M \cdot s_N \in \Lambda$. So the tensor product of species corresponds both to multiplication of symmetric functions and to multiplication of generating functions.

This tensor product gives the category of linear species a monoidal structure. The unit is the species $\mathbb{Z}\mathbf{S}_0$, defined by

$$\mathbb{Z}\mathbf{S}_0(X) = \begin{cases} \mathbb{Z} & \text{if } X = \emptyset \\ 0 & \text{otherwise} \end{cases},$$

and the associativity isomorphism is simply the identity.

Example 2.1.14. Let $\mathbb{Z}*$ be the constant species $X \mapsto \mathbb{Z}, \forall X$. Then we have a natural isomorphism

$$(\mathbb{Z} * \otimes \mathbb{Z}*)(X) \cong \mathbb{Z}\{A \subset X\}.$$

Thus the symmetric function of the species of subsets ($X \mapsto \mathbb{Z}\{A \subset X\}$) is $(\sum_{m \geq 0} h_m)^2$. Therefore the generating function of the species of subsets is $\exp(x)^2 = \exp(2x) = \sum_n 2^n x^n/n!$. □

The monoidal category $\mathcal{S}, \otimes$ admits a very special involution.

Definition 2.1.15 (Transpose)**.** We let $\mathrm{sgn} : \mathbf{Fin}^{\sim} \to \mathrm{Ab}$ be the species defined by $\mathrm{sgn}(x) = \wedge^{|x|}\mathbb{Z}x$. For any species M, we let $\mathrm{sgn}\, M$ denote the species $\mathrm{sgn}\, M(x) := \mathrm{sgn}(x) \otimes M(X)$. □

Proposition 2.1.16. *The functor* $\mathrm{sgn} : \mathcal{S} \to \mathcal{S}$ *defined by* $M \mapsto \mathrm{sgn}\, M$ *extends to an equivalence of monoidal categories.*

Proof. We define an isomorphism $(\mathrm{sgn}\, M) \otimes (\mathrm{sgn}\, N) \xrightarrow{\sim} \mathrm{sgn}(M \otimes N)$:

$$\bigoplus_{x=a_1 \sqcup a_2} \mathrm{sgn}(a_1)M(a_1) \otimes \mathrm{sgn}(a_2)N(a_2) \xrightarrow{\sim} \mathrm{sgn}(x) \bigoplus_{x=a_1 \sqcup a_2} M(a_1) \otimes N(a_2).$$

It is given by the wedge map $\wedge : \mathrm{sgn}(a_1) \otimes \mathrm{sgn}(a_2) \to \mathrm{sgn}(x)$ that takes $i_1 \wedge \ldots i_r \otimes j_1 \wedge \cdots \wedge j_s \to i_1 \wedge \ldots i_r \otimes j_1 \wedge \cdots \wedge j_s$. This map commutes with the associator. □

We call this equivalence of categories the *transpose functor*. The transpose functor takes the trivial representation of $\mathbf{S}_n$ to the sign representation. At the level of symmetric functions, the transpose functor induces the unique ring homomorphism $\Lambda \to \Lambda$ that takes homogeneous symmetric functions to elementary symmetric functions, $h_n \to e_n$.

Although the transpose functor respects the monoidal structure of $\mathcal{S}$, it does not respect the "obvious" symmetric monoidal structure on $\mathcal{S}$. Thus there are *two different ways* of enhancing the monoidal structure $\mathcal{S}$ to a symmetric monoidal structure, corresponding to two different isomorphisms

$$\bigoplus_{x=a_1 \sqcup b_2} M(a_1) \otimes N(a_2) \xrightarrow{\sim} \bigoplus_{x=b_1 \sqcup b_2} N(b_1) \otimes M(b_2).$$

Both isomorphisms permute the direct sum decomposition and take the summand $M(a_1) \otimes N(a_2)$ to the summand $N(b_1) \otimes M(b_2)$, where $b_1 = a_2$ and $b_2 = a_1$.

1. The *ordinary symmetric monoidal structure* is given by the usual switching map: $m \otimes n \mapsto n \otimes m$.
2. The *transposed symmetric monoidal structure* is given by the signed switching map $m \otimes n \mapsto (-1)^{|a_1||a_2|} n \otimes m$.

The transpose functor induces an equivalence of symmetric monoidal categories between these two different symmetric monoidal structures.

2.1.4 Composition product

For M, N objects of $\mathcal{S}$, the composite product $M \circ N$ is defined by

$$M \circ N(x) = \bigoplus_{r \geq 0} M(r) \otimes_{\mathbf{S}_r} \left(\bigoplus_{x = a_1 \sqcup \cdots \sqcup a_r \text{ ordered set partition}} N(a_1) \otimes \cdots \otimes N(a_r) \right),$$

where the sum is over all ordered set partitions of n.

Under Schur-Weyl duality, we have $F_{M \circ N}(V) = F_M(F_N(V))$. For symmetric functions, we have $s_{M \circ N} = s_M \circ s_N$, where $\circ$ is the plethysm operation of symmetric functions. At the level of the dimension generating functions, this corresponds to composition.

We can write $M \circ N$ more simply using the tensor product of species, as $\oplus_n M(n) \otimes_{\mathbf{S}_n} N^{\otimes n}$, where we use the symmetric monoidal structure defined above to give an action of $\mathbf{S}_n$ on $N^{\otimes n}$.

Example 2.1.17. The species of perfect matchings is defined by

$$X \mapsto \oplus_r \mathbb{Z}\{X = i_1 \sqcup \cdots \sqcup i_r | \, |i_j| = 2\}/\mathbf{S}_r.$$

This species is isomorphic to $(\oplus_r \operatorname{triv}(r)) \circ \operatorname{triv}(2)$. Thus we have that the symmetric function of this species is $(\sum_r h_r)[h_2]$, and the exponential generating function is $\exp(x^2/2)$.

Similarly, the species of set partitions

$$X \mapsto \oplus_r \mathbb{Z}\{X = b_1 \sqcup \cdots \sqcup b_r | |b_i| \geq 1\}/\mathbf{S}_r,$$

is isomorphic to $(\oplus_{r\geq 0}\operatorname{triv}(r)) \circ (\oplus_{b\geq 1}\operatorname{triv}(b))$, and has associated generating function $\exp(e^x - 1)$. □

2.1.5 Degree-wise dual

Given a species $M \in \mathcal{S}$, we define its *degree-wise dual* to be the species $M(x) = \operatorname{Hom}(M(x), \mathbb{Z})$. If M is degree-wise finitely generated, then M and M^* have the same symmetric function. In the next section, will make use of a compatibility between the degree-wise dual and composite product, which we describe here.

Suppose that $N(0) = 0$, and $N(r), M(r)$ are finitely generated free abelian groups. Then

$$(M \circ N)^*(X) := \operatorname{Hom}\left(\bigoplus_{r\geq 0} M(r) \otimes_{\mathbf{S}_r} \bigoplus_{x=a_1\sqcup\cdots\sqcup a_r} N(a_1)\otimes\cdots\otimes N(a_r), \mathbb{Z}\right)$$

$$= \bigoplus_{r\geq 0}\left(M^*(r) \otimes \bigoplus_{x=a_1\sqcup\cdots\sqcup a_r} N^*(a_1)\otimes\cdots\otimes N^*(a_r)\right)^{\mathbf{S}_r}.$$

There is a canoncial norm map $(N\circ M)^* \longrightarrow N^*\circ M^*$, taking coinvariants to invariants. By our torsion free hypothesis on N, the $\mathbf{S}_r$ module $\oplus_{x=a_1\sqcup\cdots\sqcup a_r} N^*(a_1)\otimes\cdots\otimes N^*(a_r)$ is free. Therefore so is its tensor product with m. Thus the map from coinvariants to invariants is an isomorphism $(N\circ M)^* \cong N^*\circ M^*$.

2.1.6 Species in graded abelian groups

As indicated in the introduction, we may consider species in any symmetric monoidal category $\mathcal{V}, \otimes$. Linear species that arise as the homology groups of a topological space are most naturally considered in the category $\mathcal{V} = \mathrm{Ab}^{\mathbb{Z}}$ of graded abelian groups, considered with its *graded* symmetric moniodal structure.

Definition 2.1.18. An object of $\mathrm{Ab}^{\mathbb{Z}}$ is a sequence of abelian groups $\{H_i\}_{i\in\mathbb{Z}}$,

$$H_2 \;\; H_1 \;\; H_0 \;\; H_{-1} \;\; H_{-2}.$$

We also write H^i to denote the group H_{-i}. The tensor product $- \otimes - : \mathrm{Ab}^{\mathbb{Z}} \times \mathrm{Ab}^{\mathbb{Z}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$ is defined by $(A \otimes B)_k = \oplus_{i+j=k} A_i \otimes B_j$.

The *graded symmetric monoidal structure* is induced by the isomorphism $(A \otimes B)_k \xrightarrow{\sim} (B \otimes A)_k$ defined by

$$a \otimes b \in B_i \otimes A_j \mapsto (-1)^{ij} b \otimes a \in A_j \otimes B_i.$$

We write $\mathrm{Ab}_{\geq k} = \mathrm{Ab}^{\leq k}$ for the subcategory of sequences satisfying $H_i = 0$ for all $i < 0$. □

The signs in the definition of the symmetric monoidal structure change the definition of composite product.

Given a species in $M \in \mathcal{S}_{\mathrm{Ab}^{\mathbb{Z}}}$, such that each $M_i(x)$ is finitely generated we may form the *Poincaré* symmetric function

$$s_M(\underline{x}, t) := \sum_{i \in \mathbb{Z}} s_{M_i}(\underline{x}) t^i \in \Lambda[[t, t^{-1}]].$$

If M_i vanishes for all $i < 0$ ($i > 0$), then we can interpret s_M as an element of the ring $\Lambda[[t^{-1}]]$ (resp. $\Lambda[[t]]$). We can also specialize to a generating function.

Definition 2.1.19. Given any species M in graded abelian groups and $k \in \mathbb{Z}$, we write $M[k]$ for the *leftward shift of M by k*. By definition, we have $(M[k])_n = M_{n-k}$. □

We finish this chapter with a description of the homology of configurations in $\mathbb{R}^d$ as a species of graded abelian groups.

Example 2.1.20 (Configurations in Euclidean Space)**.** Let $\mathrm{Lie}(n)$ be the representation of $\mathbf{S}_n$ on the abelian group of Lie brackets of n elements. Let Com be the species $\oplus_{n \geq 1} \mathrm{triv}(n)$. Consider the species in graded abelian groups $H_*(\mathrm{Conf}_\bullet(\mathbb{R}^d))$ defined by

$$X \mapsto \{H_i(\mathrm{Conf}_X(\mathbb{R}^d), \mathbb{Z})\}_{i \in \mathbb{Z}}.$$

Then we have

$$H_*(\mathrm{Conf}_\bullet(\mathbb{R}^d)) \cong \mathrm{Com} \circ \left(\mathrm{sgn}^{d-1} \bigoplus_n \mathrm{Lie}(n)[n(d-1)] \right).$$

In the case $d = 2$, the class in $H_1(\mathrm{Conf}_2(\mathbb{R}^2))$ corresponding to two points rotating about each other corresponds to the generator of $\mathrm{sgn}\,\mathrm{Lie}(2)$, denoted $1 \wedge 2[12]$. The group $\mathrm{sgn}\,\mathrm{Lie}(3)[3]$ is generated by the Lie brackets $s[[12]3], s[[13]2], s[[23]1]$, where $s = 1 \wedge 2 \wedge 3$. Up to signs, the class $s[[ij]k] \in H_2(\mathrm{Conf}_3\mathbb{R}^2)$ is the class of the 2 dimensional submanifold spanned by the points i, j rotating about each other as they both orbit k.[1]

This description gives a generating function identity

$$\sum_{i,n} \dim H_i(\mathrm{Conf}_n\mathbb{R}^d) t^i \frac{x^n}{n!} = \exp\left(\frac{-\log(1 - xt^d)}{t^d} \right),$$

where we have used the fact that $\dim \mathrm{Lie}(n) = (n-1)!$ to rewrite the generating function of $\oplus_n \mathrm{sgn}\,\mathrm{Lie}[n(d-1)]$ as $\sum_n \frac{(n-1)! x^n t^{d(n-1)}}{n!} = \frac{-\log(1-xt^d)}{t^d}$ □

This sort of description goes back at least to the work of Cohen, using May's theory of operads [30]. A good reference is [18][Section 1].

2.2 Operads and Combinatorial Categories

2.2.1 Operads

We will introduce operads and their associated wiring categories. The wiring category of an operad is often called the **PROP** associated to the operad, or the category of opeartors associated to the operad.

Definition 2.2.1 (Operad)**.** A *(linear) operad P* is a monoid in the monoidal category $(\mathcal{S}, \circ)$. Explicitly, an operad consists of $P \in \mathcal{S}$ together with a multiplication map $m : P \circ P \to P$

[1] Think of the earth, the moon, and the sun moving in the orbital plane.

and a unit $u : \mathbb{Z}\mathbf{S}_1 \to P$ which satisfy the associativity

$$\begin{tikzcd}
P \circ P \circ P \arrow[r, "P \circ m_P"] \arrow[d, "m_P \circ P"] & P \circ P \arrow[d, "m_P"] \\
P \circ P \arrow[r, "m_P"] & P
\end{tikzcd}$$

and unitality laws

$$\begin{tikzcd}
P \arrow[r, "u \circ \mathrm{id}_P"] \arrow[dr, "\mathrm{id}_P"'] & M \circ P \arrow[d, "m_P"] & P \arrow[l, "\mathrm{id}_P \circ u"'] \arrow[dl, "\mathrm{id}_P"] \\
& P &
\end{tikzcd},$$

expressed by the commutativity of the above diagrams. □

Remark 2.2.2. We can also consider operads in other categories of species. There are set operads, topological operads, chain complex operads (also known as *dg* operads), operads in algebraic varieties, etc. □

Definition 2.2.3. Let P be an operad. We say that the $\mathbf{S}_n$ representation $P(n)$ is the *collection of n-ary operations of P*. □

Remark 2.2.4. The data of the composition map $m : P \circ P \to P$ is encoded in a map

$$m_{r;n_1,\ldots,n_r} : P(r) \otimes P(n_1) \otimes \ldots P(n_r) \subset P(r) \otimes_{\mathbf{S}_r} \mathrm{Ind}_{S_{n_1} \times \cdots \times S_{n_r}}^{S_{\sum_i n_i}} P(n_1) \otimes \cdots \otimes P(n_r) \to^m P(\sum_{i=1}^{r} n_i).$$

The rest of the data of the composition map is determined by $\mathbf{S}_n$ equivariance. $\mathbf{S}_n$ equivariance also imposes conditions on the map $m_{r;n_1,\ldots,n_r}$. This condition, and the condition of associativity of $m : P \circ P \to P$ are encoded in identities satisfied by the maps $m_{r;n_1,\ldots,n_r}$. We will not make these identities explicit here, however May's original definition of operad is in terms of these [30]. □

We can graphically represent an operad by drawing its n-ary operations, and using insertion of one into the other to represent composition.

Let P be an operad. Although our main focus will be on the wiring categories of P, operads were originally introduced in order to define and study algebras over P.

Definition 2.2.5. Let P be a linear operad. An algebra over P is an abelian group A equipped with a map

$$m_A : P \circ A = \oplus_n P(n) \otimes_{\mathbf{S}_n} A^{\otimes n} \to A,$$

such that $(u_P \circ \mathrm{id}_A)m_A = \mathrm{id}_A$ and $(m_P \circ \mathrm{id}_A)m_A = (\mathrm{id}_P \circ m_A)m_A$. □

Informally, if A is an algebra over P, then every element $p \in P(n)$ defines an "n-ary operation" $m_A(p) : A^{\otimes n} \to A$ and the composition of n-ary operations reflects the composition law of P.

The collection of algebras over A forms a category in the natural way. Thus to every linear operad P corresponds to a type of algebraic structure on abelian groups. We often name an operad P by the algebraic structure it corresponds to: there is the associative operad As, the commutative operad Com, the lie algebra operad Lie, etc.

Example 2.2.6 (Associative operad $\mathrm{As}^{\mathrm{un}}$)**.** The unital associative operad $\mathrm{As}^{\mathrm{un}}$, in this example denoted by As, has underlying species $\mathrm{As}(n) = \mathbb{Z}\mathbf{S}_n$ or equivalently $\mathrm{As}(X) = \mathbb{Z}\{\text{total orderings of } X\}$. The unit of As is $\mathrm{id} : \mathbb{Z}\mathbf{S}_1 \to \mathrm{As}(1)$. The multiplication

$$m_{\mathrm{As}} : \mathrm{As} \circ \mathrm{As}(Y) = \bigoplus_{Y = X_1 \sqcup \cdots \sqcup X_r} \mathrm{As}(X_1) \otimes \cdots \otimes \mathrm{As}(X_r) \to \mathrm{As}(Y)$$

is given by concatenation of total orders.

Heuristically, we make this definition because given an associative algebra A, and a collection of elements of $\{a_x\}_{x \in X}$ the abelian group $\mathrm{As}(X)$ parameterizes the different ways we can multiply the variables a_x to form an element of of A. For instance the ordering $2 \le 3 \le 1 \in \mathrm{As}([3])$ corresponds to the multiplication $(a_1, a_2, a_3) \mapsto a_2 a_3 a_1$. Similarly, the concatenation of total orders corresponds to the substitution of these operations into one another.

The formalization of this heuristic is that there is a canonical equivalence between category of algebras over As and the category of associative algebras: If A is an associative algebra, we define $m_A : \mathrm{As}(A) = \bigoplus_{n \ge 0} A^{\otimes n} \to A$ by $a_1 \otimes \cdots \otimes a_n \mapsto a_1 \ldots a_n$. Conversely, if A is an algebra over As, then we define a multiplication and unit $A \otimes A \to A$ and $\mathbb{Z} \to A$ from the data of the map $\mathrm{As}(2) \circ A \to A$ and $\mathrm{As}(0) \circ A \to A$. □

After introducing the notion of a wiring category in the next section, we will give examples that correspond to other algebraic structures.

2.2.2 Wiring categories and examples

To every operad, we associate a wiring category. Informally, the wiring category of P is the category whose objects are finite sets and whose morphisms are finite set maps with fibers decorated by the operations of P.

Definition 2.2.7 (Wiring Categories)**.** The *wiring category* of an operad P, is denoted by $\mathbf{W}_P$. The objects of $\mathbf{W}_P$ are indexed by natural numbers.

If P is an operad in sets, then a map $m \to n \in \mathbf{W}_P(m,n)$ is given by a set map $m \to n$ and a $\#f^{-1}(i)$-ary operation of P for each $i \in \mathbb{N}$

$$\mathbf{W}_P(m,n) = \{(f, \{o_i\}_{i\in[m]} \mid f : [n] \to [m], o_i \in P(f^{-1}(i))\}.$$

Composition is defined by

$$(f, \{o_i\})(g, \{p_i\}) = (fg, \{m(o_i; \{p_j\}_{j\in f^{-1}(i)})\}.$$

If P is an operad in abelian groups, then $\mathbf{W}_P(m,n)$ is an abelian group defined by

$$\mathbf{W}_P(m,n) := \bigoplus_{f\in\mathbf{Fin}(m,n)} \bigotimes_{i\in[m]} P(f^{-1}(i)).$$

Composition is defined on simple tensors via the above formulas and then extended linearly. □

Remark 2.2.8. The wiring category of an operad P is often referred to as the **PROP** or the *category of operators* associated to P. The notion of a **PROP** was introduced by MacLane and Adams [27] as a categorical way of describing algebraic structures. □

We can represent morphisms in the wiring category graphically, using wiring diagrams.

Example 2.2.9 ($\mathrm{Com}^{\mathrm{un}}$ and **Fin**)**.** The unital commutative operad $\mathrm{Com}^{\mathrm{un}}$ is given by $\mathrm{Com}^{\mathrm{un}}(X) = \mathbb{Z}*, \forall X$. Thus $\mathrm{Com}^{\mathrm{un}}(n)$ is the trivial representation for all $n \geq 0$. The unit $u_{\mathrm{Com}^{\mathrm{un}}} : \mathbb{Z}\mathbf{S}_1 \to \mathrm{Com}^{\mathrm{un}}(1)$ is $\mathrm{id}_{\mathbb{Z}\mathbf{S}_1}$. The multiplication map

$$m(r, i_1, \ldots, i_r) : \mathbb{Z}\mathbf{S}_1 \otimes \bigotimes_{i=1}^{r} \mathbb{Z}\mathbf{S}_1 \to \mathbb{Z}\mathbf{S}_1$$

is the identity.

Algebras over $\mathrm{Com}^{\mathrm{un}}$ are unital commutative algebras. This corresponds to the fact that in a commutative algebra there is only one way (up to scaling) of multiplying n elements together.

The operad $\mathrm{Com}^{\mathrm{un}}$ is the linearization of a set operad with underlying species $X \mapsto *$. Accordingly $\mathbf{W}_{\mathrm{Com}^{\mathrm{un}}}$ is the linearization of an ordinary category. Since there is a unique i-ary operation for every i, we simply have $\mathbf{W}_{\mathrm{Com}^{\mathrm{un}}} = \mathbb{Z}\mathbf{Fin}$, where **Fin** is the category of all finite sets and maps between them. □

Example 2.2.10 ($\mathrm{As}^{\mathrm{un}}$ and **NCFin**)**.** The operad $\mathrm{As}^{\mathrm{un}}$ is also the linearization of a set operad, $X \mapsto \{$ total orderings on $X\}$. We let **NCFin**, the category of *non-commutative finite sets* be the category whose objects are finite sets, and whose morphisms are finite set maps with a total ordering on the fiber. Then $\mathbf{W}_{\mathrm{As}^{\mathrm{un}}} = \mathbb{Z}\mathbf{NCFin}$.

The category **NCFin** was used by Ellenberg and Wiltshire-Gordon [12] to study configurations on manifolds admitting a non-vanishing vector field. □

Example 2.2.11 (Com and **FS**)**.** We let Com denote the operad of non-unital commutative algebras. This operad defined to be the suboperad of $\mathrm{Com}^{\mathrm{un}}$ defined by $\mathrm{Com}(X) = \mathbb{Z}*$, if $|X| > 0$ and $\mathrm{Com}(\emptyset) = 0$.

The wiring category $\mathbf{W}_{\mathrm{Com}}$ is again the linearization of an ordinary category, **FS**, which is the wiring category of a set operad $\mathrm{Com}_{\mathrm{Set}}$. The objects of **FS** are finite sets. Since $\mathrm{Com}_{\mathrm{Set}}$ has no 0-ary operations, and there is a unique n-ary operation for all $n \geq 0$, the morphisms of $\mathbf{FS}([n],[m])$ are precisely the surjections $[n] \twoheadrightarrow [m]$. Thus **FS** is the category of finite sets and surjections. □

Remark 2.2.12. Similarly, there is a non-unital associative operad $\mathrm{As}^{\mathrm{nu}}$, and $\mathbf{W}_{\mathrm{As}^{\mathrm{nu}}}$ is the linearization of the category of non-commutative finite surjections **NCFS**. □

We can also define operads by generations and relations, and use these to define wiring categories.

Example 2.2.13 (Free(2) and the category of binary planar trees)**.** We define $F := \mathrm{Free}(2)$ to be the free operad generated by a binary relation in degree 2. This means that F satisfies the universal property in the category of operads that morphisms from $F \to P$ are in bijection with elements of $P(2)$. We can construct F as a species of binary planar trees. That is, we let

$$F(X) = \mathbb{Z}\{\text{ planar trees with leaves labelled by } X \text{ }\}.$$

To clarify our conventions: we have $F(3)$ is spanned by 12 different planar trees, corresponding to 2 distinct combinatorial types and 6 possible labellings of the leaves by $\{1, 2, 3\}$. The multiplication map $F \circ F \to F$ is given by grafting trees onto one another, and the unit is given by the inclusion of the tree with 1 leaf.

Algebras over Free(2) are just abelian groups A with a binary operation $A \otimes A \to A$. The wiring category of F is the linearization of a category of planar binary rooted forests, **PBT**. Objects of the category are finite sets, and $\mathbf{PBT}(X, Y)$ is the set of binary planar forests with leaves labelled by X and roots labelled by Y. Composition is given by grafting forests together, according to the labelling. □

Remark 2.2.14. Similarly, there is a free operad Free(n) for any $n \in \mathbb{N}$, which is a linearization of a set operad consisting of planar n-valent trees. More generally, for any species $M \in \mathcal{S}$, there is a free operad Free(M) consisting of trees with vertices decorated by elements of M. The case Free(n) corresponds to $M = \mathbb{Z}\mathbf{S}_n$: the total ordering on the vertices gives the planar structure of the tree. □

Example 2.2.15 (Lie and $\mathbf{W}_{\mathrm{Lie}}$)**.** The Lie operad is the quotient of Free(2) by relations corresponding to anti-symmetry and the Jacobi identity. Concretely, we may describe Lie(X) as a quotient of the Aut(X) representation of planar trees with leaves labelled by X, by an antisymmetry relation for each vertex of the tree, and a Jacobi relation for each pair of adjacent vertices.

The operad Lie is our main example of an operad which is *not* the linearization of a set operad, because the Jacobi relation is intrinsically linear. The wiring category $\mathbf{W}_{\mathrm{Lie}}$ consists of finite set maps with the fibers decorated by Lie brackets. $\square$

It is also possible to obtain **FI**-like categories that appear in the representation stability literature as wiring categories of operads.

Example 2.2.16 (The category **FI**). Let P be the linear species $\mathbb{Z}\mathbf{S}_0 \oplus \mathbb{Z}\mathbf{S}_1$. Then $P \circ P = \mathbb{Z}\mathbf{S}_0 \oplus (\mathbb{Z}\mathbf{S}_0 \oplus \mathbb{Z}\mathbf{S}_1)$, and we define a multiplication $P \circ P \to P$ by

$$\begin{pmatrix} \mathrm{id}_{\mathbf{S}_0} & \mathrm{id}_{\mathbf{S}_0} & 0 \\ 0 & 0 & \mathrm{id}_{\mathbf{S}_1} \end{pmatrix}.$$

Informally, P consists of a unary operation u and a 0-ary operation (otherwise known as a constant), 1. These operations satisfy $m_P(u \circ 1) = 1$ and $m_P(u \circ u) = u$. Since every operad contains a unit u, we see that P is the free operad on a single 0-ary operation. An algebra over P consists of an abelian group A and a distinguished element $a_1 \in A$.

Again, P is the linearization of an operad that is defined in the category $\mathcal{S}_{\mathrm{Set}}$. The wiring category of P is $\mathbb{Z}\mathbf{FI}$, where **FI** denotes the category of finite sets and injections. $\square$

Example 2.2.17 ($\mathbf{FI}_d$). Let P be the linear species $\mathbb{Z}\mathbf{S}_0^{\oplus d} \oplus \mathbb{Z}\mathbf{S}_1$. As in the previous example, we can define an operadic multiplication on P so that P becomes the free operad generated by d 0-ary operations. An algebra over P is an abelian group A with d distinguished elements $a_1, \ldots, a_d$.

The wiring category of P is $\mathbb{Z}\mathbf{FI}_d$, where $\mathbf{FI}_d$ is the category whose objects are finite sets, and whose morphisms $\mathbf{FI}_d([m], [n])$ consist of a finite injection together with a d-coloring on the complement of the image. $\square$

Remark 2.2.18. In general, we might ask which categories C can we exhibit as wiring categories $C = \mathbf{W}_P^{\mathrm{op}}$ for some Set operad P? Heuristically, the answer is categories C of the form "finite set maps with extra structure on the fiber." The extra structure may be data, such as a total order (in the case of **NCFin**) or it may be the property of being nonempty (in the case of **FS**).

One possible formalization of this heuristic is to note that there is a functor of categories $C \to \mathbf{Fin}$ whose fibers satisfy a monoidal property: the fiber $C/[n]$ is equivalent to $\prod_{i=1}^{n} C/[1]$. This approach is closely tied to the *Feynman categories* of Kaufmann and Ward [25], which generalize wiring categories of operads and other related algebraic structures that appear in mathematical physics. □

2.2.3 Morita Equivalence

There is a canonical equivalence between representations of $\mathbf{W}_P^{\mathrm{op}}$ and right modules over P.

Definition 2.2.19. Let P be an operad. A *right module over* P is a species $M \in \mathcal{S}$ together with a map $a_M : M \circ P \to M$ that satisfies associativity and unitality. In other words, the associativity diagram

$$\begin{tikzcd} M \circ P \circ P \arrow[r, "M \circ m_P"] \arrow[d, "a_M \circ P"] & M \circ P \arrow[d, "a_M"] \\ M \circ P \arrow[r, "a_M"] & M \end{tikzcd}$$

and the corresponding diagram for unitality commute.

The collection of all right modules over P forms a category, whose morphisms are maps of species $M \to N$ which commute with the structure maps. We denote the *category of right modules* by $\operatorname{Mod} P$. □

Definition 2.2.20. For any N in $\mathcal{S}$, we have that $N \circ P$ is a right P module, using the multiplication map $N \circ P \to P$. We call $N \circ P$, the free right module on N. □

We also recall the defintion of a representation of a linear category C.

Definition 2.2.21. Let C be a *linear category* over a ring k. By this we mean that C has a class of objects, also denoted by C, a group of morphisms $C(c, d) \in \operatorname{Mod} k$ for every $c, d \in C$, an identity element $k \to C(c, c)$ for all $c \in C$, and a composition law $C(c, d) \otimes C(d, e) \to C(c, e)$ for all $c, d, e \in C$. Together, this data is required to satisfy associativity and unitality.

Given two linear categories, C, D a *linear functor* $F : C \to D$ between them is a map of classes $F : C \to D$ and a map of k modules $C(c_1, c_2) \to D(Fc_1, Fc_2)$ for all c_1, c_2, which respects the composition law and the identity maps.

If C is a k linear category, a *representation of* C in $\operatorname{Mod} k$, or a C *module* is a linear functor $C \to \operatorname{Mod} k$. If C is an ordinary category, then a representation of C is simply an ordinary functor $C \to \operatorname{Mod} k$. □

Theorem 2.2.22. *The category of right modules over an operad P is canonically equivalent to the category of* $\mathbf{W}_P^{\mathrm{op}}$ *modules.*

Proof. A $\mathbf{W}_P^{\mathrm{op}}$ module M consists of a sequence of abelian groups M_n and maps $M_n \otimes \mathbf{W}_P^{\mathrm{op}}(n, m) \to M_m$ satisfying associativity and unitality. Equivalently, it consists of a sequence of $\mathbf{S}_n$ representations M_n, where the $\mathbf{S}_n$ structure is given by $\mathbb{Z}\mathbf{S}_n \subset \mathbf{W}_P(n, n)$ and maps of $\mathbf{S}_m$ representations $M_n \otimes_{\mathbf{S}_n} \mathbf{W}_P^{\mathrm{op}}(n, m) \to M_m$ satisfying associativity and unitality.

Now we have

$$M_n \otimes_{\mathbf{S}_n} \mathbf{W}_P(m, n) = M_n \otimes_{\mathbf{S}_n} \left(\bigoplus_{f : m \to n} \bigotimes_{i \in m} P(f^{-1}(i)) \right) = M_n \otimes_{\mathbf{S}_n} \left(\bigoplus_{m = a_1 \sqcup \cdots \sqcup a_n} P(a_i) \otimes \cdots \otimes P(a_m) \right),$$

which is $(M_n \circ P)(m)$. And by definition, a right P module M consists of a sequence of $\mathbf{S}_n$ representations $M(n)$ and maps $(M(n) \circ P)(m) \to M(m)$, which satisfy associativity and unitality. Thus a sequence of $\mathbf{S}_n$ representations is a right module over $\mathbf{W}_P^{\mathrm{op}}$ if and only if it defines a right module over P. □

In particular, we have that $\operatorname{Mod} \mathrm{Com}$ is equivalent to $\operatorname{Rep}(\mathbf{FS}^{\mathrm{op}})$, and $\operatorname{Mod} \mathrm{As}^{\mathrm{nu}}$ is equivalent to $\operatorname{Rep} \mathbf{NCFin}^{\mathrm{op}}$.

Chapter 3

Koszul Duality and Wiring Categories

In this chapter, we use the relationship between modules over operads and representations of wiring categories to describe the Koszul dual of a wiring category.

3.1 Koszul duality of Linear Categories

There are different points of view of Koszul duality. We will take the perspective that Koszul duality is a way of systematically describing the structure of free resolutions of modules over a graded ring R in terms of a Koszul dual graded ring $R^!$. Thus focus will be on describing this ring $R^!$, and on the explicit constructions of free resolutions over R via this description.

In contrast, one common definition is to say that R is *Koszul* if the free resolutions of simple representations are linear. The general theory of Koszul duality then implies the existence of an algebra $R^!$ which governs the structure of minimal free resolutions[1], and this theory can be used to construct a presentation of $R^!$ by generators and relations. However, in special situations it is possible to describe $R^!$ more directly.

Koszul duality is often studied in the context of graded modules over a non-negatively graded ring with semi-simple degree 0 part. We will work in the setting of modules over a non-negatively graded category, which is a mild generalization of the graded ring setting.

[1] $R^!$ is given as an Ext algebra between simple representations

Definition 3.1.1. An *non-negatively graded category* is a linear category C, with set of objects O, and a function $g : \{(x, y) \in \operatorname{Ob} C \mid C(x, y) \neq 0\} \to \mathbb{N}$ satisfying $g(x, y)+g(y, z) = g(x, z)$. $\square$

This setting (with additional finiteness conditions) was used in a paper of Gan and Li [16], which provided a criterion to prove Koszulness and proved that many of the categories appearing in the representation stability literature are Koszul, in the sense that their simple representations admit linear minimial free resolutions.

3.1.1 Weighted Monoidal Abelian Categories

Both ordinary Koszul duality (of graded algebras) and of graded categories can be subsumed under a general notion of Koszul duality of an algebra object A in a weighted monoidal abelian category. We avoid this extra layer of formalism in the succeeding sections, and indicate in remarks how the arguments generalize to this setting.

Definition 3.1.2. A *weighted monoidal abelian category* $\mathcal{A}$, is an abelian category with a monoidal structure $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and a decomposition $\mathcal{A} = \prod_{n\in\mathbb{N}} \mathcal{A}_n$, such that the tensor product factors into components $\otimes : \mathcal{A}_n \times \mathcal{A}_m \to \mathcal{A}_{n+m}$. We also assume that the tensor product preserves direct sums and cokernels in both factors. $\square$

Here is how this definition includes both settings.

- In the ordinary setting, $\mathcal{A} = \mathrm{Ab}^{\mathbb{N}}$ and the tensor product is induced by the ordinary tensor product $\mathrm{Ab}_m \otimes \mathrm{Ab}_n \to \mathrm{Ab}_{n+m}$. An associative algebra in $\mathcal{A}$ is simply a graded algebra.

- Let O be a set, and let $R \subset O \times O$ be a relation that defines a poset structure on O. Let $g : R \to \mathbb{N}$ be function that satisfies $g(x, y) + g(y, z) = g(x, z)$ for all $x \leq_R y \leq_R z \in O$. Then we grade the abelian category Ab^R by

$$\mathrm{Ab}^R = \prod_{n\in\mathbb{N}} \prod_{(x,y)\in R \ \mid \ g(x,y)=n} \mathrm{Ab}\,.$$

An object C of $\mathcal{A}$ is described by a set of abelian groups $C = \{C(x, y)\}_{(x,y)\in R}$. We define a tensor product $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by

$$C \otimes D = \{\oplus_{x\leq_R y\leq_R z} C(x, y) \otimes D(y, z)\}_{x,z}.$$

This tensor product is graded, by our assumption on the function g.

An algebra object in $\mathcal{A}, \otimes$ is the same as a non-negatively graded linear category C with object set O, satisfying $C(x, y) = 0$ unless $x \leq_R y$.

There is a further example which is relevant to us, because it is the natural setting to work with wiring categories.

Example 3.1.3. Let $\mathcal{A} = \prod_{l\leq m\in\mathbb{N}} \mathrm{Ab}^{\mathbf{S}_l\times\mathbf{S}_m}$. An object of $\mathcal{A}$ is a sequence $M = \{M(l, m)\}_{l\leq m}$ of $\mathbf{S}_l \times \mathbf{S}_m$ representations. We grade $\mathcal{A}$ by $\deg(l, m) = m - l$, i.e. by the decomposition $\mathcal{A} = \prod_{n\geq 0} \prod_{l\geq 0} \mathrm{Ab}^{\mathbf{S}_l\times\mathbf{S}_{l+n}}$. The tensor product is defined as

$$M \otimes N := \left\{\bigoplus_r M(l, r) \otimes_{\mathbf{S}_r} M(r, m)\right\}_{l\leq m}.$$

Algebra objects in $\mathcal{A}$ are linear categories C whose objects are indexed by natural numbers, together with functor $\sqcup_n \mathbb{Z}\mathbf{S}_n \to C$.

Let $P \in \mathcal{S}$ be an operad, such that $P(0) = 0$. Then the wiring category $\mathbf{W}_P^{\mathrm{op}}$ defines an algebra in $\mathcal{A}$. □

3.1.2 Dual Cocategory

Definition 3.1.4. A *linear cocategory* D, with object set O, is a coalgebra in the monoidal category $\mathrm{Ab}^{O\times O}, \otimes$, with tensor product defined as in the previous section by $\mathbb{Z}(x, y) \otimes \mathbb{Z}(y, z) = \mathbb{Z}(x, z)$.

More explictly, a cocategory D with object set O consists of a collection of abelian groups $\{D(x, y)\}_{x,y\in O}$, counits $\eta_x : D(x, x) \to \mathbb{Z}$, and comultiplication maps

$$\Delta_{D,x,z} : D(x, z) \to \bigoplus_{y\in O} D(x, y) \otimes D(y, z),$$

which satisfy associativity and unital identities.

We say the cocategory D is *graded* if we have the additional data of a grading function

$$g : \{(x,y) \in O \times O \mid D(x,y) \neq 0\} \to \mathbb{N},$$

such that $g(x,y) + g(y,z) = g(x,z)$. □

Thus a graded linear cocategory is the same as a coalgebra in the weighted monoidal abelian category defined in the previous section.

Definition 3.1.5 (Koszul Duality)**.** Let C, O be a graded linear category, such that $C(x,x) = \mathbb{Z}\mathrm{id}_C$. Assume that there exists a linear cocategory $C^{¡}, O$, and a natural isomorphism

$$t_{x,y} : C(x,y) \cong C^{¡}(x,y) \text{ for all } x, y \in O \text{ such that } g(x,y) = 1.$$

Define the map $d_r(t)$ by

$$d_r(t) : C \otimes C^{¡} \to^{\mathrm{id}_C \otimes \Delta_{C^{¡}}} C \otimes C^{¡} \otimes C^{¡} \to^{\mathrm{id}_C \otimes t \otimes \mathrm{id}_{C^{¡}}} C \otimes C \otimes C^{¡} \to^{m_C \otimes \mathrm{id}_{C^{¡}}} C \otimes C^{¡},$$

where t is the map $C^{¡}(x,y) \to C(x,y)$ defined by

$$\begin{cases} t = t_{x,y} & \text{if } g(x,y) = 1 \\ t = 0 & \text{otherwise.} \end{cases}.$$

If $d_r(t)^2 = 0$, then we say that t is a *twisting morphism*. In this case we have a chain complex $C \otimes C^{¡}$, defined using the grading that places $C^{¡}(x,y)$ in homological degree $g(x,y)$.

If this chain complex satisfies the exactness property:

$$H_i(C \otimes C^{¡})(x,y) = \begin{cases} \mathbb{Z}\mathrm{id}_C & \text{if } i = 0 \text{ and } x = y \\ 0 & \text{otherwise} \end{cases},$$

we say that t is a *Koszul twisting morphism*. In this case, we say that C is *Koszul*, and cocategory $C^{¡}$ is called the *Koszul dual* of C. □

Remark 3.1.6. In the above definition, we follow the terminology of Loday and Valette [26]. The presentation here is based on the formalism of their book.

To extend to the case of weighted monoidal category $1_{\mathcal{A}}$, we need to assume that $1_{\mathcal{A}} \to C$ is an isomorphism onto the degrree 0 compnent of C, and the exactness property needs to be modified to the hypothesis that $C \otimes C^{¡} \to C/C_{\geq 1} = 1_{\mathcal{A}}$ is a quasi-isomorphism. □

Remark 3.1.7. According to our definitions, it is not clear the Koszul dual of $C^{¡}$ and the twisting morphism $t : C^{¡} \to C$ are unique up to isomorphism. This is true, but we will not show it here, and instead simply abuse terminology. □

Example 3.1.8. Let O be the set $\{00, 01, 10, 11\}$, and let C be the linearization of the category corresponding to the commutative diagram.

$$\begin{tikzcd}
 & 11 & \\
10 \arrow[ur, "x"] & & 01 \arrow[ul, "y"'] \\
 & 00 \arrow[ul, "z"] \arrow[ur, "w"'] &
\end{tikzcd}$$

Thus C is generated by the morphisms x, y, z, w with relation $zx = wy$. This category admits a grading such that $\deg(x) = \deg(y) = \deg(z) = \deg(w) = 1$.

We define a linear category $C^!$, which is generated by x^*, y^*, z^*, w^*, subject to the anti-commutativity relation $x^*z^* = -w^*y^*$.

$$\begin{tikzcd}
 & 11 & \\
10 \arrow[ur, "x^*"] & - & 01 \arrow[ul, "y^*"'] \\
 & 00 \arrow[ul, "z^*"] \arrow[ur, "w^*"'] &
\end{tikzcd}$$

And we define $C^{¡}$ to be the dual cocategory $(C^!)^*$. For instance, we have $C^{¡}(00,01) = \mathbb{Z}x$, and

$$\Delta : C^{¡}(00,01) \to C^{¡}(00,01) \otimes C^{¡}(01,01) \oplus C^{¡}(00,00) \otimes C^{¡}(01,01),$$

is given by $\Delta(x) = x \otimes \mathrm{id}_{01} + \mathrm{id}_{01} \otimes x$. We can compute a more interesting coproduct by choosing a basis element r for $C(00,11)^!$, say $r = z^*x^* = -w^*y^*$. Then we have that

$$\Delta(r^*) = \mathrm{id}_{00} \otimes r^* + z \otimes x - w \otimes y + r^* \otimes \mathrm{id}_{11}.$$

There is a canonical twisting morphism $t : C^{¡}(ij,kl) \to C(ij,kl)$, given by taking $x \mapsto x, y \mapsto y, z \mapsto z, w \mapsto w$. The resulting candidate complex $(C \otimes C^{¡})$ has $(C \otimes C^{¡})(00,11)$ component given by

$$\mathbb{Z}\{\mathrm{id}_{00} \otimes r^*\} \to \mathbb{Z}\{z \otimes x, w \otimes y\} \to \mathbb{Z}\{zx \otimes \mathrm{id}_{11}\},$$

where the far left term is in homological degree 2. We have $d(\mathrm{id}_{00} \otimes r^*) = z \otimes x - w \otimes y$, because these are the only two terms in the coproduct of r^* which do not vanish after applying t. Similarly, we have

$$d(z \otimes x) = zx \otimes \mathrm{id}_{11} = wy \otimes \mathrm{id}_{11} = d(w \otimes y).$$

We see that the $(C \otimes C^{¡})(00,11)$ component is indeed a complex and is exact. The 4 components corresponding to length 1 intervals are also exact complexes. The only homology is in the components $(C \otimes C^{¡})(ij,ij) = \mathbb{Z}\mathrm{id}_{ij}$, and therefore $C^{¡}$ is the Koszul dual of C. □

Proposition 3.1.9. *In the setting of Definition 3.1.5, the following are equivalent.*

1. *The map $t : C \to C^!$ is a twisting morphism.*
2. *The map $t * t$, defined by*

$$C^! \to^{\Delta} C^! \otimes C^! \to^{t \otimes t} C \otimes C \to^{m} C$$

equals zero.

3. *The map*

$$d_l(t) : C^{\text{i}} \otimes C \to^{\Delta\otimes \mathrm{id}_C} C^{\text{i}} \otimes C^{\text{i}} \otimes C \to^{\mathrm{id}_{C^{\text{i}}}\otimes t\otimes \mathrm{id}_C} C^{\text{i}} \otimes C \otimes C \to^{\mathrm{id}_{C^{\text{i}}}\otimes m} C^{\text{i}} \otimes C,$$

squares to zero.

Proof. Using the associativity of m_C and the coassociativity of $\Delta_{C^{\text{i}}}$, we have that $d_r(t)^2$ is equal to

$$C \otimes C^{\text{i}} \to^{\mathrm{id}_C\otimes\Delta} C \otimes C^{\text{i}} \otimes C^{\text{i}} \to^{\mathrm{id}_C\otimes(t*t)\otimes \mathrm{id}_{C^{\text{i}}}} C \otimes C \otimes C^{\text{i}} \to^{m\otimes \mathrm{id}_{C^{\text{i}}}} C \otimes C^{\text{i}}.$$

Thus if $t * t = 0$ then $d_r(t)^2 = 0$. Conversely, precomposing $d(t)^2$ by $u \otimes \mathrm{id}_{C^!}$ and postcomposing by $\mathrm{id}_C \otimes \eta$, we obtain $t * t$. Thus if $d(t)^2 = 0$, then $t * t = 0$. This shows $1 \iff 2$. The equivalence of 2 and 3 is identical. □

3.1.3 Koszul Resolution

Now if C is a Koszul category with dual C^{i}, then using C^{i} we may construct free resolutions of C modules.

Definition 3.1.10. A *left C module* is a linear functor $M : C \to \mathrm{Ab}$. A *right module* is a linear functor $M : C^{\mathrm{op}} \to \mathrm{Ab}$. □

Warning 3.1.11. This notion is not the same as a module over C in the monoidal category $\mathrm{Ab}^{O\times O}, \otimes$. In terms of monoidal structures, a module over C can be defined as follows. The category Ab^{O} is a module category over the monoidal category $\mathrm{Ab}^{O\times O}$, with module structure defined by

$$\{C(x, y)\}_{x,y\in O} \otimes \{M(x)\}_{x\in O} = \{\oplus_y M(y) \otimes C(y, z)\}_{z\in O}.$$

Then a C module M is defined with respect to this module structure as an object of Ab^{O} together with an action $C \otimes M \to M$ that satisfies unitality and associativity.

Thus, to axiomatize this situation, we need a weighted monoidal abelian category $\mathcal{A}, \otimes$, a weighted left module category $\mathcal{M}$, an algebra object $A \in \mathcal{A}$ and a module over it $M \in \mathcal{M}$. □

Definition 3.1.12. Let C be a graded linear category, with Koszul dual $C^{¡}$ and twisting morphism $t : C^{¡} \to C$. Then for any left C module M, and right C module N we define a chain complex $\mathbf{K}(C, M)$ of C modules, as follows.

The underlying C module of $\mathbf{K}(C, M)$ is

$$C \otimes C^{¡} \otimes M \;=\; c \mapsto \bigoplus_{d,e \in O} C(c,d) \otimes C^{¡}(d,e) \otimes M(e) \in \operatorname{Mod} C.$$

The homological grading is defined by $\deg(C(c,d) \otimes C^{¡}(d,e) \otimes M(e)) = g(d,e)$. The differential $d_{\mathbf{K}(C,M)}$, is defined in terms of the maps

$$\begin{aligned} d_r : C \otimes C^{¡} \otimes M &\to^{\mathrm{id}_C \otimes \Delta \otimes \mathrm{id}_M} C \otimes C^{¡} \otimes C^{¡} \otimes M \\ &\to^{\mathrm{id}_C \otimes \mathrm{id}_{C^{¡}} \otimes t \otimes \mathrm{id}_M} C \otimes C^{¡} \otimes C \otimes M \to^{\mathrm{id}_C \otimes \mathrm{id}_{C^{¡}} \otimes m_M} C \otimes C^{¡} \otimes M, \end{aligned}$$

and

$$\begin{aligned} d_l : C \otimes C^{¡} \otimes M &\to^{\mathrm{id}_C \otimes \Delta \otimes \mathrm{id}_M} C \otimes C^{¡} \otimes C^{¡} \otimes M \\ &\to^{\mathrm{id}_C \otimes t \otimes \mathrm{id}_{C^{¡}} \otimes \mathrm{id}_M} C \otimes C^{¡} \otimes C \otimes M \to^{m_C \otimes \mathrm{id}_{C^{¡}} \otimes \mathrm{id}_M} C \otimes C^{¡} \otimes M. \end{aligned}$$

The maps d_l, d_r commute, and by the second identity of Prop 3.1.9 they both square to zero.[2] Thus they define a bi-complex $d_{\mathbf{K}(C,M)} = d_l \pm d_r$. We refer to $\mathbf{K}(C, M)$ as the *Koszul complex of M*. □

Remark 3.1.13. In the above definition, we can replace C by any left C module N, and define a chain complex $\mathbf{K}(N, M) \in \mathrm{Ab}^{O \times O}$ with underlying abelian group

$$\mathbf{K}(N,M)(x,y) = \bigoplus_{x \le z_1 \le z_2 \le y} N(x) \otimes C^{!}(x,y) \otimes M(y).$$

To generalize this construction to the abstract setting of the previous section, we would need the notion of a "pairing" between left and right weighted module categories over $\mathcal{A}$.

[2]For d_r we also need to use associativity of $C \otimes M \to M$, to apply the identity.

Since we do not know the correct definition of a pairing between module categories, we restrict to the case $\mathbf{K}(C, M)$, which suffices to give us free resolutions. $\square$

First we show that the Koszul complex is usually a resolution.

Proposition 3.1.14. *Suppose that C is Koszul and either the underlying abelian group of M, or of $C \otimes C^{¡}$ is torsion free. Then $\mathbf{K}(C, M)$ is a resolution of M.*

Proof. We filter $\mathbf{K}(C, M)$ by defining a function ν on the summands $C(x, y) \otimes C^{!}(y, z) \otimes M(x)$ of $\mathbf{K}(C, M)$. We let $\nu(C(x, y) \otimes C^{!}(y, z) \otimes M(x)) = g(x, y) + g(y, z)$. The filtered piece $F_i\mathbf{K}(C, M)$ is defined to be the subgroup consisting of summands S with $\nu(S) \leq i$.

The map d_l decreases the value of ν by one and d_r preserves it. Thus $F_i\mathbf{K}(C, M)$ is indeed a subcomplex, and further the associated graded complex is simply $C \otimes C^{!} \otimes M$, with differential d_r. This complex factors as $(C \otimes C^{!}) \otimes M$, and by hypothesis $(C \otimes C^{!})$ is exact, except in degree 0 where it isomorphic to the unit of $\mathcal{A}, \otimes$. Since M is flat, $(C \otimes C^{!}) \otimes M$ has homology quasi-isomorphic to M. Thus the spectral sequence associated to the filtration shows that $\mathbf{K}(C, M)$ is exact, except in degree 0. $\square$

Remark 3.1.15. To extend Proposition 3.1.14 to the setting of weighted monoidal categories, we need the hypothesis that either $M \in \mathcal{M}$ or $C \otimes C^{¡} \in \mathcal{A}$ is *flat* in the sense that that they define exact functors $- \otimes M : \mathcal{A} \to \mathcal{M}$ and $C \otimes C^{¡} : \mathcal{M} \to \mathcal{M}$. $\square$

3.2 Koszul Operads and Wiring Categories

3.2.1 Dual Cooperad

The definition of Koszul operads parallels the definition of Koszul categories. First, we need to define cooperads, which are the obvious dualization of operads.

Definition 3.2.1. A *cooperad Q* is a linear species $Q \in \mathcal{S}$, together with a comultiplication $\Delta_Q : Q \to Q \circ Q$ and a counit $\eta_Q : Q \to \mathbb{Z}\mathbf{S}_1$, that satisfy associativity and unitality identities.

We say that Q is *graded* if it is a cooperad in the category $\mathbf{S}_{\mathrm{Ab}^{\mathbb{N}}}$ of species in *homologically graded abelian groups*. Here the category $\mathrm{Ab}^{\mathbb{N}}$ is equipped with the homologically graded symmetric monoidal structure, so that the isomorphism $V_n \otimes W_m \to W_m \otimes V_n$ is given by $v \otimes w \mapsto (-1)^{nm} w \otimes v$. □

Next, we define what it means for an operad to be Koszul.

Definition 3.2.2. Let $P \in \mathcal{S}$ be an operad such that $P(0) = 0$ and $P(1) = \mathbb{Z}\mathbf{S}_1$. We say that P is *binary Koszul* if there exists a graded cooperad $P^{¡}$, and a degree 1 *Koszul twisting morphism* $P^{¡}(2) \cong P(2)$ such that

$$d_t : P^{¡} \circ P \to^{\overline{\Delta} \circ \mathrm{id}_P} P^{¡} \circ P^{¡} \circ P \to^{\mathrm{id}_{P^{¡}} \circ t \circ \mathrm{id}_P} P^{¡} \circ P \circ P \to^{\mathrm{id}_{P^{¡}} \circ m} P^{¡} \circ P$$

satisfies $d_t^2 = 0$, and using the grading of $P^{¡}$ to define a chain complex, we have that the canonical map $P^{¡} \circ P \to P(1) = \mathbb{Z}\mathbf{S}_1$ is a quasi-isomorphism.

Here $\overline{\Delta} : P^{¡} \to P^{¡} \circ P^{¡}$ is Δ postcomposed with the projection map onto the sub-species of

$$(P^{¡} \circ P^{¡})(x) = \bigoplus_r P^{¡}(i_r) \otimes_{\mathbf{S}_r} \left(\bigoplus_{x = i_1 \sqcup \cdots \sqcup i_r} P^{¡}(i_1) \otimes \cdots \otimes P^{¡}(i_r) \right),$$

consisting of summands $x = i_1 \sqcup \cdots \sqcup i_r$ such that at most one block i_k has $|i_k| > 1$.

We say that $P^{¡}$ is the *Koszul dual cooperad* to P. □

Remark 3.2.3. We can *attempt* to give $\mathcal{S}_{\geq 1} = \mathrm{Rep} \sqcup_{n \geq 1} \mathbf{S}_n$ the structure of a weighted monoidal abelian category, as follows. The grading is the decomposition $\mathcal{S}_{\geq 1} \simeq \prod_{n \in \mathbb{N}} \mathrm{Rep}(\mathbf{S}_{n+1})$. (An $\mathbf{S}_n$ representation has degree $n - 1$). The monoidal structure is defined to be the composite product $\circ : \mathcal{S}_{\geq 1} \times \mathcal{S}_{\geq 1} \to \mathcal{S}_{\geq 1}$.

However, this attempt fails in an important way: the composition product $\circ$ is not bilinear, in other words $M \circ -$ does not preserve direct sums. This issue is related to the rather ad hoc definition of $\overline{\Delta}$. There is a variant of the composite product, denoted $- \circ_{(1)} -$ which is linear. See [26][Section 6.1] for the definition of this product, and a definition of the complex above that is closer to the definition in the previous section. The treatment in Loday–Vallette is more general, in that they do not treat only binary Koszul duality. □

We can construct cooperads by dualizing opeards.

Proposition 3.2.4. *Assume we have a cooperad $Q(n)$ such that $Q(n)$ is finitely generated and torsion free for all n. Then its dual Q^* is a operad.*

Similarly, if P is an operad such that $P(n)$ is finitely generated and torsion free, then P^* is a cooperad.

Proof. This follows from Section 2.1.5, since the comultiplication map dualizes to $Q^* \circ Q^* \to Q^*$, and similarly for the counit. □

The main result of this section is an extension of the equivalence between right modules over P and right modules over $\mathbf{W}_P^{\mathrm{op}}$ to include Koszul duality. We consider $\mathbf{W}_P^{\mathrm{op}}$ as an algebra in the weighted symmetric monoidal category $\mathcal{A} = \prod_{m \leq n} \operatorname{Rep} \mathbf{S}_m \times \mathbf{S}_n$, defined in Section 3.1.1. Our first step is to define a coalgebra in this category, associated to a co-operad.

Definition 3.2.5. Let Q be a graded cooperad. We define the *cowiring category* as a coalgebra object in $\mathcal{A} = \prod_{m \leq n} \operatorname{Rep} \mathbf{S}_m \times \mathbf{S}_n$. We let

$$\mathbf{C}_Q(m,n) := \bigoplus_{m = b_1 \sqcup \ldots b_n} Q(b_1) \otimes \cdots \otimes Q(b_n) = (Q)^{\otimes n}(m) = \mathbb{Z}\mathbf{S}_n \circ Q$$

as an $\mathbf{S}_m \times \mathbf{S}_n$ representation. The coproduct is defined by

$$\mathbf{C}_Q(m,n) \cong (\mathbb{Z}\mathbf{S}_n \circ Q)(m) \to^{\mathrm{id} \circ \Delta_Q} (\mathbb{Z}\mathbf{S}_n \circ Q \circ Q)(m) \cong \bigoplus_{m \leq l \leq n} \mathbf{C}_Q(m,l) \otimes_{\mathbf{S}_l} \mathbf{C}_Q(l,n),$$

and the counit is given by the counit of Q

$$\mathbf{C}_Q(n,n) = \bigoplus_{n = b_1 \sqcup \ldots b_n} Q(b_i) \to \bigoplus_{n = b_1 \sqcup \ldots b_n,\ |b_i| = 1} Q(1)^{\otimes n} \to \bigoplus_{n = b_1 \sqcup \cdots \sqcup b_n,\ |b_i| = 1} \mathbb{Z} = \mathbb{Z}\mathbf{S}_n.$$

□

Theorem 3.2.6. *Let P be a Koszul operad, with Koszul dual cooperad $P^{¡}$. Assume that both $P(n)$ and $P^{¡}(n)$ are torsion free and finitely generated for all n. Then the opposite wiring category of P is Koszul dual to the opposite cowiring category of $P^{¡}$.*

Proof. By definition, we have a twisting map $t : P^{¡} \to P$. The associated complex $P^{¡} \circ P$ resolves $\mathbb{Z}\mathbf{S}_1$. Since the induction product of species is exact, the complex $\bigotimes^n (P^{¡} \circ P) = \mathbb{Z}\mathbf{S}_n \circ P^{¡} \circ P$ resolves $(\mathbb{Z}\mathbf{S}_1)^{\otimes n} = \mathbb{Z}\mathbf{S}_n$.

Note that for any $M \in \mathcal{S}$, we have

$$(M \circ P^{¡})(x) = \bigoplus_{r \in \mathbb{N}} M(r) \otimes_{\mathbf{S}_r} \left(\bigoplus_{x = a_1 \sqcup \cdots \sqcup a_r} P^{¡}(a_1) \otimes \cdots \otimes P^{¡}(a_r) \right) = \bigoplus_{r \in \mathbb{N}} M(r) \otimes_{\mathbf{S}_r} \mathbf{C}_{P^{¡}}(m, r),$$

and there is a similar identity for $N \circ P$. Thus the $\mathbf{S}_m$ representation $(\mathbb{Z}\mathbf{S}_n \circ P^{¡} \circ P)(m)$, is canonically isomorphic to

$$\bigoplus_{l} \mathbf{C}_{P^{¡}}(l, n) \otimes_{\mathbf{S}_l} \mathbf{W}_P(m, l).$$

Considering $\mathbf{W}_P^{\mathrm{op}}$ and $\mathbf{C}_{P^{¡}}^{\mathrm{op}}$ as objects of $\mathcal{A}$, the above group is $(\mathbf{W}_P^{\mathrm{op}} \otimes_{\mathcal{A}} \mathbf{C}_{P^{¡}}^{\mathrm{op}})(n, m)$. Further, the twisting morphism $P^{¡}(2) \to P(2)$ induces a morphism

$$t_{\mathbf{W}} : \mathbf{C}_{P^{¡}}^{\mathrm{op}}(n, n+1) = \oplus_{i<j \in [n]} P^{¡}(\{i, j\}) \to^{t} \oplus_{i<j \in [n]} P(\{i, j\}) = \mathbf{W}_P^{\mathrm{op}}(n, n+1).$$

The corresponding endomorphism

$$d^r(t_{\mathbf{W}}) : (\mathbf{W}_P^{\mathrm{op}} \otimes_{\mathcal{A}} \mathbf{C}_{P^{¡}}^{\mathrm{op}})(n, m) \to (\mathbf{W}_P^{\mathrm{op}} \otimes_{\mathcal{A}} \mathbf{C}_{P^{¡}}^{\mathrm{op}})(n, m),$$

equals the differential on $(\mathbb{Z}\mathbf{S}_n \circ P^{¡} \circ P)(m)$. Thus $d^r(t_{\mathbf{W}})$ squares to zero and gives a resolution of $(1_{\mathcal{A}})(n, n) = \mathbb{Z}\mathbf{S}_n$. By definition, this means that $\mathbf{C}_{P^{¡}}^{\mathrm{op}}$ is Koszul dual to $\mathbf{W}_P^{\mathrm{op}}$. □

3.2.2 Free Resolutions of $\mathbf{FS}^{\mathrm{op}}$ modules

We can use the formalism of the preceding sections to describe the free resolutions of $\mathbf{FS}^{\mathrm{op}}$ modules. Recall that $\mathbf{FS}^{\mathrm{op}}$ is associated to the commutative operad. The key non-trivial input is that the commutative operad is Koszul, and that its dual cooperad is $\mathrm{Com}^{¡} = (\Sigma^{-1} \mathrm{Lie})^*$.

Definition 3.2.7. Let P be a graded operad. We define a new graded operad ΣP as follows. As a graded abelian group we let $\Sigma P(n) := \mathbb{Z}\{s\}^* \otimes \mathbb{Z}\{s\}^{\otimes n} \otimes P(n)$, where $\mathbb{Z}\{s\}$ is the homologically graded abelian group generated by s, with $\deg(s) = 1$.

The grading and the $\mathbf{S}_n$ action on $\Sigma P(n)$ is determined by the tensor product of graded vector spaces. Thus the underlying $\mathbf{S}_n$ representation is isomorphic to $\operatorname{sgn} P(n)$.[3] And we have $\deg(s^* s^{n-1} p) = n - 1 + \deg p$.

The operad structure is defined as follows. The unit element of $\Sigma P(1)$ is $s^* \otimes s \otimes u_P$. If we write an n-ary operation of ΣP in the form $s^* s^n p$, then

$$m_{\Sigma P}(s^* s^r p; s^* s^{i_1} q_1, \dots, s^{i_r} q_r) = s^* s^{\sum_j i_j} m_P(p; q_1, \dots, q_r).$$

This operation has an inverse Σ^{-1}, where we use the abelian group $\mathbb{Z}\{s^{-1}\}$ with $\deg(s^{-1}) = -1$. We can perform an similar operation for cooperads, and we have $(\Sigma^{-1} P)^* = \Sigma P^*$. □

Theorem 3.2.8. *[15][Theorem 6.8] The twisting morphism $t : (\Sigma^{-1} \mathrm{Lie})^*(2) \to \mathrm{Com}(2)$, that takes $s^* s^2 [12] \to (12)$ exhibits $(\Sigma^{-1} \mathrm{Lie})^*$ as the Koszul dual cooperad of* Com.

Remark 3.2.9. We will use Theorem 3.2.8 to resolve $\mathbf{FS}^{\mathrm{op}}$ modules. It can also be used to compute the homology of the partition lattice, but this argument has the potential to be circular, since in characteristic $\neq 0$, the easiest way to prove Theorem 3.2.8 it is via the homology of the partition lattice. □

Since $\mathbb{Z}\mathbf{FS}^{\mathrm{op}}$ is the wiring category of Com, Theorem 3.2.6 implies that $\mathbb{Z}\mathbf{FS}^{\mathrm{op}}$ is Koszul dual to $(\mathbf{C}^{\mathrm{op}}_{(\Sigma^{-1} \mathrm{Lie})^*})$.

Write $D := \mathbf{C}_{(\Sigma^{-1} \mathrm{Lie})}$. by Proposition 3.1.14, we can resolve $\mathbf{FS}^{\mathrm{op}}$ modules using D. [4]

Let $n \mapsto M(n)$ be an $\mathbf{FS}^{\mathrm{op}}$ module. Then the Koszul resolution takes the form

$$\mathbf{FS}^{\mathrm{op}} \otimes D \otimes M := x \mapsto \left(\oplus_{n,m} \mathbb{Z}\mathbf{FS}(x, n) \otimes D(n, m) \otimes M(m)\right).$$

[3] This is because we have an isomorphism of $\mathbf{S}_n$ representations $\mathbb{Z}\{s\}^{\otimes n} \cong \operatorname{sgn}(n)$.

[4] Notice that $D(n, m)$ is a free $\mathbf{S}_m$ module, so the flatness hypothesis of Proposition 3.1.14 is satisfied.

Using the grading to write this as a chain complex, we have

$$\bigoplus_n \mathbb{Z}\mathbf{FS}(-,n) \otimes_{\mathbf{S}_n} M(n) \leftarrow \bigoplus_n \mathbb{Z}\mathbf{FS}(-,n+1) \otimes_{\mathbf{S}_{n+1}} D(n+1,n) \otimes_{\mathbf{S}_n} M(n)$$
$$\leftarrow \bigoplus_n \mathbb{Z}\mathbf{FS}(-,n+2) \otimes_{\mathbf{S}_{n+2}} D(n+2,n) \otimes_{\mathbf{S}_n} M(n) \leftarrow \ldots,$$

which resolves the $\mathbf{FS}^{\mathrm{op}}$ module M.

In the case $M = \mathbb{Z}\mathbf{S}_d$, this simplifies to a minimal free resolution.

$$\mathbb{Z}\mathbf{FS}(-,d) \leftarrow \mathbb{Z}\mathbf{FS}(-,d+1) \otimes_{\mathbf{S}_{d+1}} D(d+1,d) \leftarrow \mathbb{Z}\mathbf{FS}(-,d+2) \otimes_{\mathbf{S}_{d+2}} D(d+2,d) \tag{3.2.10}$$
$$\leftarrow \mathbb{Z}\mathbf{FS}(-,d+3) \otimes_{\mathbf{S}_{d+3}} D(d+3,d) \ldots$$

Proposition 3.2.11. *There are canonical isomorphisms of graded* $\mathbf{S}_n \times \mathbf{S}_m$ *represenatiosn*

$$\mathrm{Ext}^{\bullet}_{\mathbf{FS}^{\mathrm{op}}}(\mathbb{Z}\mathbf{S}_n, \mathbb{Z}\mathbf{S}_m) \cong \mathbf{C}^{\mathrm{op}}_{(\Sigma^{-1}\,\mathrm{Lie})^*}(n,m)^*$$

and

$$\mathrm{Tor}^{\mathbf{FS}}_{\bullet}(\mathbb{Z}\mathbf{S}_n, \mathbb{Z}\mathbf{S}_m) \cong \mathbf{C}^{\mathrm{op}}_{\Sigma^{-1}\,\mathrm{Lie}}(n,m).$$

Proof. By construction, the resolution 3.2.10 is minimal in the sense that the differential of a generator $\mathrm{id}_n \otimes g \in \mathbb{Z}\mathbf{FS}(n,n) \otimes_{\mathbf{S}_n} D(n,d)$ has positive degree. Using the resolution to compute Tor and Ext, we get a complex with 0 differential, and homology given as in the statement of the proposition. □

We will consider the case $d = 2$. We write (12) for the element $t^*tt[12] \in \Sigma^{-1}\,\mathrm{Lie}(2)$, where $t = s^{-1}$. We can extend this notation using the composition of the operad. Thus ((12)(34)) denotes the element

$$m((12);(12),(34)) = t^*t^4[[12][34]] \in \Sigma^{-1}\,\mathrm{Lie}(4).$$

The group $D^*(3,2) = \mathbf{W}_{\Sigma\,\mathrm{Lie}}(3,2)$ is spanned by

$$(12) \otimes 3, (13) \otimes 2, (23) \otimes 1, 3 \otimes (12), 2 \otimes (13), 1 \otimes (23)$$

The group $D^*(4,2)$ is $8+8+6$ dimensional, and spanned by

$$((12)3) \otimes 4,\ ((13)2) \otimes 4,\ ((12)4) \otimes 3,\ ((14)2) \otimes 3, ((13)4) \otimes 2,\ ((14)3) \otimes 2,\ ((23)4) \otimes 1,\ ((24)3) \otimes 1$$
$$4 \otimes ((12)3),\ 4 \otimes ((13)2),\ 3 \otimes ((12)4),\ 3 \otimes ((14)2),\ 2 \otimes ((13)4),\ 2 \otimes ((14)3),\ 1 \otimes ((23)4),\ 1 \otimes ((24)3)$$
$$(12) \otimes (34),\ (13) \otimes (24),\ (14) \otimes (23),\ (34) \otimes (12),\ (24) \otimes (13),\ (23) \otimes (14)$$

Thus the resolution of $\mathbb{Z}\mathbf{S}_2$ begins

$$\mathbb{Z}\mathbf{S}_2 \leftarrow \mathbb{Z}\mathbf{FS}(-,2) \leftarrow \mathbb{Z}\mathbf{FS}(-,3) \otimes_{\mathbf{S}_3} \mathbb{Z}^{\oplus 6} \leftarrow \mathbb{Z}\mathbf{FS}(-,4) \otimes_{\mathbf{S}_4} \mathbb{Z}^{\oplus 22} \leftarrow \dots.$$

To compute the differential from the $\mathbb{Z}\mathbf{FS}(-,4)$ term to the $\mathbb{Z}\mathbf{FS}(-,3)$ term, we need to determine $d(\mathrm{id}_{\mathbf{S}_4} \otimes_{\mathbf{S}_3} f)$ for $f \in D(4,2)$. Recall that the differential in degree 4 is given by

$$\mathbf{FS}(4,4) \otimes_{\mathbf{S}_4} D(4,2) = D(4,2) \to^{\Delta} D(4,3) \otimes_{\mathbf{S}_3} D(3,2) \to^{t \otimes \mathrm{id}} \mathbf{FS}(4,3) \otimes_{\mathbf{S}_3} D(3,2).$$

Here Δ is the dual of $D(4,3) \otimes_{\mathbf{S}_3} D(3,2) \to D(4,2)$, post-composed with the norm isomorphism between invariants and coinvariants.

Chapter 4

Poset Homology and Stratifed Spaces

4.1 Poset Resolutions

In this section, we introduce the homological algebra of representations of posets. In the next sections, we will be considering representations P in the category of sheaves of abelian groups on a topological space X, as well as representations of P in abelian groups. Thus we fix a symmetric monoidal abelian category $\mathcal{A}, \otimes$, which admits arbitrary sums. We let $k \in \mathcal{A}$ denote the unit object.

Definition 4.1.1. Let $A \in \mathcal{A}$, and $X \in \text{Set}$. We denote by AX the object $\oplus_{x \in X} A \in \mathcal{A}$, which has basis X.

Let V be an abelian group, we denote by $V \otimes A$ the object of $\mathcal{A}$ satsifying the universal property $\mathcal{A}(V \otimes A, B) \cong \text{Ab}(V, \mathcal{A}(A, B))$. This object exists and can be constructed from a presentation $V = \text{coker}(\mathbb{Z}X \to \mathbb{Z}Y)$ as $V \otimes A = \text{coker}(AX \to AY)$. □

We also give our homological conventions.

Definition 4.1.2 (Homological Conventions)**.** We let $\text{Ch}(\mathcal{A})$ denote the category of chain complexes in $\mathcal{A}$. A chain complex $C \in \text{Ch}(\mathcal{A})$ consists of a sequence $\{C_i\}_{i \in \mathbb{Z}}$ of objects in

$\mathcal{A}$, together with differentials $d : C_i \to C_{i-1}$, which satisfy $d^2 = 0$.[1]

$$\cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to C_{-2} \to \ldots$$

Given a chain complex $C \in \mathrm{Ch}(\mathcal{A})$ and $i \in \mathbb{Z}$, we write C^i for the group C_{-i}. We let $\mathrm{Ch}(\mathcal{A})^{\leq k} = \mathrm{Ch}(\mathcal{A})_{\geq k}$ denote the full subcategory of complexes such that satisfying $C_i = 0$ for all $i < k$. The category $\mathrm{Ch}(\mathcal{A})^{\geq k} = \mathrm{Ch}(\mathcal{A})_{\leq k}$ is defined similarly.

The *kth shift* of a chain complex, denoted $C[k]$, is a chain complex with the same underling differential and all groups shifted k steps to the left. Thus $(C[k])_i := C_{i-k} = C^{i+k}$

A *bicomplex* is an object of $B \in \mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$. Thus B consists $\{B_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$, and commuting horizontal and vertical differentials $d_h : B_{i,j} \to B_{i-1,j}$ and $d_v : B_{i,j} \to B_{i,j-1}$ respectively. Given a bicomplex, its *totalization* $\mathrm{Tot}(B) \in \mathrm{Ch}(\mathcal{A})$ is the complex with $\mathrm{Tot}(B)_n = \bigoplus_{i+j=n} B_{i,j}$ and differential defined by $d_{\mathrm{Tot}}(b_{i,j}) = d_h(b_{i,j}) + (-1)^i d_v(b_{i,j})$

The *cone* of a map of chain complexes $f : C_i \to D_i$ is denoted $\mathrm{cone}(C \to^f D)$. It is defined to be $\mathrm{Tot}(B)$, where bi-complex B defined by $B_{0,i} = D_i$ and $B_{1,i} = C_i$, and all other groups zero. The horizontal differential of B is induced by f, and the vertical differential of B is induced by the differential of D and C. □

Definition 4.1.3. Let P be a finite poset. We think of P as a category, as follows. The objects of P are the elements of P, and there is a unique morphism from $p \in P$ to $q \in P$ whenever $p \leq q$. We denote this morphism simply by $p \leq q$. The identity morphism is $p \leq p$. For $p, q \in P$, we write $P(p, q)$ for the set of morphisms $p \to q$. Thus

$$\begin{cases} * & \text{if } p \leq q \\ \emptyset & \text{otherwise} \end{cases}$$

The assignment $q \mapsto P(p, q)$ defines a functor $P \to \mathrm{Set}$, which we denote by $P(p, -)$. □

Definition 4.1.4. A *representation of P in $\mathcal{A}$* is a functor from $M : \widehat{P} \to \mathcal{A}$. Thus M consists of

1. $M_p \in \mathcal{A}$ for each $p \in P$,

[1]Equivalently $\mathrm{Ch}(\mathcal{A})$ is the category of representations of a linear category $\mathbf{D}$ whose objects are $i \in \mathbb{Z}$, and whose only nonzero groups of morphisms are $\mathbf{D}(i, i-1) = \mathbb{Z}d$ and $\mathbf{D}(i, i) = \mathbb{Z}\mathrm{id}_{\mathbf{D}}$.

2. a map $M_{pq} : M_p \to M_q$, for each $p \leq q \in P$

and this data satisfies the conditions $M_{pq}M_{qr} = M_{pr}$, and $M_{pp} = \mathrm{id}$. We will synonymoulsy refer to M a P *module*, or an object of the functor category $\mathcal{A}^P$.

□

We can draw the category P and its representations using the Hasse Diagram of the poset.

Example 4.1.5. For the partition lattice on 3 elements, P(3), the category associated to P(3) is

$$\begin{tikzcd}
& \{1\}\{2\}\{3\} & \\
\{12\}\{3\} \arrow[ur] & \{13\}\{2\} \arrow[u] & \{23\}\{1\} \arrow[ul] \\
& \{123\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

And a representation of P(3) is a commutative diagram:

$$\begin{tikzcd}
& M_{\{1\}\{2\}\{3\}} & \\
M_{\{12\}\{3\}} \arrow[ur] & M_{\{13\}\{2\}} \arrow[u] & M_{\{23\}\{1\}} \arrow[ul] \\
& M_{\{123\}} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

□

Definition 4.1.6. The *free module* on an object p of P is denoted $kP(p, -)$. It is defined by $kP(p, -)_q := kP(p, q)$. Thus if $p \leq q$ we have $kP(p, q) = k$, and $kP(p, q) = 0$ otherwise.

We say that a module $M \in \mathcal{A}^P$ is *free* if it is isomorphic to a direct sum of modules of the form $kP(p, -)$. □

We will often be most concerned with free modules over the opposite poset P^{op}.

Example 4.1.7. For $\mathrm{P}(3)^{\mathrm{op}}$, we have the free module $k\mathrm{P}(3)^{\mathrm{op}}(\{1\}\{2\}\{3\}, -) = k\mathrm{P}(3)(-, \{1\}\{2\}\{3\})$

$$\begin{tikzcd}
& k \arrow[dl, "\mathrm{id}_k"'] \arrow[d, "\mathrm{id}_k"'] \arrow[dr, "\mathrm{id}_k"] & \\
k \arrow[dr, "\mathrm{id}_k"'] & k \arrow[d, "\mathrm{id}_k"'] & k \arrow[dl, "\mathrm{id}_k"] \\
& k &
\end{tikzcd}$$

And the free module $k\mathrm{P}(3)^{\mathrm{op}}(\{13\}\{2\}, -) = k\mathrm{P}(3)(-, \{13\}\{2\})$ is:

$$\begin{tikzcd}
& 0 \arrow[dl] \arrow[d] \arrow[dr] & \\
0 \arrow[dr] & k \arrow[d, "\mathrm{id}_k"'] & 0 \arrow[dl] \\
& k &
\end{tikzcd}$$

where of course, all of the maps between the zero object are zero. □

A map of P modules $f : kP(-, p) \to N$ is uniquely determined by where it sends the identity element $p \leq p \in kP(p, p)$. In other words, free modules satisfy the following universal property.

Proposition 4.1.8 (Yoneda Lemma)**.** *Let N be a P^{op} representation. Then* $\mathrm{Hom}_{P^{\mathrm{op}}}(kP(-, p), N) = \mathrm{Hom}_{\mathcal{A}}(k, N_p)$. .

When $p \leq q$ we refer to the map $kP(-, p) \to kP(-, q)$ corresponding to $p \leq q \in kP(p, q)$ as *multiplication by $p \leq q$*. If we apply $\mathrm{Hom}_{P^{\mathrm{op}}}(-, N)$ we get the map $N_{pq} : N_q \to N_p$.

Definition 4.1.9. Let $S(q)$ denote the representation of P defined by

$$S(q)_p := \begin{cases} k & \text{for } p = q \\ S(q)_p = 0 & \text{otherwise.} \end{cases}$$

For all $p < q$ the map $S(q)_{pq}$ is 0.

The same definition gives a representation of P^{op}, and we will refer to both of these representations by the same name. □

When $\mathcal{A} = \operatorname{Mod} k$ for k a field, the $\{S(q)\}_{q \in P}$ are exactly the simple representations of P.

Next we define the tensor product operation, which takes a P^{op} representation in Ab and a P representation in $\mathcal{A}$ and produces an object of $\mathcal{A}$.

Definition 4.1.10 (Tensor products)**.** Let $M \in \mathrm{Ab}^{P^{\mathrm{op}}}$, and let $N \in \mathcal{A}^P$. The *tensor product of M and N over P* is the coequalizer

$$M \otimes_P N := \operatorname{coeq}\left(\bigoplus_{p_1 \le p_2 \in P} M_{p_2} \otimes N_{p_1} \underset{1 \otimes N_{p_1 p_2}}{\overset{M_{p_2 p_1} \otimes 1}{\rightrightarrows}} \bigoplus_{p \in P} M_p \otimes N_p \right)$$

where the first map takes the summand $M_{p_2} \otimes N_{p_1}$ to $M_{p_1} \otimes N_{p_1}$ by multiplication on the right, and the second takes $M_{p_1} \otimes N_{p_2}$ to $M_{p_2} \otimes N_{p_2}$ by multiplication on the left.

Suppose that M, N are chain complexes in $\mathrm{Ch(Ab)}^{P^{\mathrm{op}}}$ and $\mathrm{Ch}(\mathcal{A})^P$ respectively. Then their tensor product makes sense as a chain complex, where the symbol $\otimes$ in the above definition is interpreted as the tensor product of chain complexes $\mathrm{Ch(Ab)} \otimes \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$. □

Tensoring with a free module on an element q of $\widehat{P}^{\mathrm{op}}$ is the same as evaluating at that element:

Example 4.1.11. We have the co-Yoneda identity

$$kP(-, q) \otimes_P N = N_q.$$

If we apply $- \otimes_P N$ to the the multiplication by $p \leq q$ map $kP(-, p) \to kP(-, q)$ we get the map $N_{pq} : N_p \to N_q$, the action of $p \leq q$. □

The previous example and the right exactness of $- \otimes_{\widehat{P}} N$ gives a way of computing tensor products that is often easier than Definition 4.1.10.

Example 4.1.12 (Tensor products from presentations)**.** If we have a presentation of the $\widehat{P}^{\mathrm{op}}$ module M as the cokernel of a map of free modules

$$f : \bigoplus_p V_p \otimes kP(-, p) \to \bigoplus_q V_q \otimes kP(-, q),$$

which determined by maps $f_{pq} : V_p \to V_q$ for $p \leq q$, then

$$M \otimes_{\widehat{P}} N = \operatorname{coker}(\bigoplus_p V_p \otimes N_p \to \bigoplus_q V_q \otimes N_q),$$

with the map $a(p, q) : V_p \otimes N_p \to V_q \otimes N_q$ defined by

$$a(p, q) = \begin{cases} f_{pq} \otimes N_{pq} & \text{if } p \leq q \\ 0 & \text{otherwise.} \end{cases}$$

□

Many functors $\mathcal{A}^P \to \mathcal{A}$ are given by tensoring with a P^{op} representation. For instance when $\mathcal{A} = \mathrm{Ab}$, all left adjoint functors $\mathrm{Ab}^P \to \mathrm{Ab}$ arise uniquely in this way. Colimits are another example:

Example 4.1.13 (Colimits as tensor products)**.** Let $\mathbb{Z}P$ denote the representation of P which is the linearization of the terminal functor $P \to \mathrm{Set}$, $p \mapsto *$. Then $\mathbb{Z}P \otimes_P M = \operatorname{colim}_P M$. □

We use the derived functors of $A \otimes_P - : \mathcal{A}^P \to \mathcal{A}$ for A in $\mathrm{Ab}^{P^{\mathrm{op}}}$. In the next definition we give our conventions for these derived tensor products. In summary, although $\mathcal{A}^P$ does not have necessarily have enough projectives, $\mathrm{Ab}^{P^{\mathrm{op}}}$ does. Thus A admits a resolution by free modules in, which we use to define models of $A \otimes^{\mathrm{L}}_P M$.

Definition 4.1.14 (Derived functors of $A \otimes -$)**.** A resolution of $A \in \mathrm{Ab}^{P^{\mathrm{op}}}$ by a chain complex of free P^{op} representations, $F \xrightarrow{\sim} A \in \mathrm{Ch}(\mathrm{Ab})$, yields a resolution $F \otimes k \xrightarrow{\sim} A \otimes k \in \mathrm{Ch}(\mathcal{A}^{P^{\mathrm{op}}})$.

For $M \in \mathrm{Ch}(\mathcal{A}^P)$, we say that $F \otimes_P M \in \mathrm{Ch}(\mathcal{A})$ is a *model* for $A \otimes^{\mathrm{L}}_P M$. If $\mathcal{A} = \mathrm{Ab}$ then we write $\mathrm{Tor}^P_i(A, B)$ for the abelian group $H_i(F \otimes_P M)$.

Given any two free resolutions F, F' of $A \in \mathrm{Ab}^{P^{\mathrm{op}}}$, there exists a quasi-isomorphism $x : F \to F'$ that extends the identity map on A, and which is unique up to homotopy. Let x' be a homotopy inverse to x. Then $x \otimes 1 : F \otimes_P M \to F' \otimes_P M \in \mathrm{Ch}(\mathcal{A})$ has a homotopy inverse $x' \otimes 1$, and so in particular it is a quasi-isomorphism. Thus given any two models of $A \otimes^{\mathrm{L}}_P M$ in our sense, we can construct a quasi-isomorphism between them and each of these morphisms is homotopic.

Given a morphism of posets $g : Q \to P$, a free resolution $F \xrightarrow{\sim} A$ over P and a free resolution $G \xrightarrow{\sim} g^*A$ over Q, there is a quasi-isomorphism $g^*F \xrightarrow{\sim} g^*A$, and hence a map $j : G \to g^*F$ extending the identity on g^*A, which is unique up to homotopy.

Given $M \in \mathrm{Ch}(\mathcal{A})^P$, $N \in \mathrm{Ch}(\mathcal{A})^Q$, and a map $a : N \to g^*M$, we obtain a map $G \otimes_Q N \to F \otimes_P M$, which *models* $g^*A \otimes^{\mathrm{L}}_Q N \to A \otimes^{\mathrm{L}}_P M$. It given by the composition

$$G \otimes_Q N \to^{j \otimes a} g^*F \otimes_Q g^*M \to F \otimes_P N.$$

For any other two resolutions F', G', any map $j' : G' \to g^*F'$ lifting id_{g^*A}, there exist comparison maps, $x : F \to F'$ and $y : G \to G'$ as above, and we have that $j \circ y$ is homotopic to $g^*x \circ j$, by the uniqueness of j up to homotopy. □

Suppose that $\hat{P}$ is a poset with top element $\hat{1} \in \widehat{P}$. The next example suggests how the homology groups $\mathrm{Tor}^{\widehat{P}}_\bullet(S(\hat{1}), S(p))$ are related to configuration spaces in the case $\widehat{P} = \mathrm{P}(3)$.

Example 4.1.15 (Partition lattice)**.** Consider $Q = \mathrm{P}(3)$, the partition lattice on 3 elements. The top element $\hat{1}$ is the discrete partition $\{1\}\{2\}\{3\}$.

The following module is the $\mathrm{P}(3)^{\mathrm{op}}$ representation $S(\hat{1})$.

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & \\
0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[dl] \\
& 0 &
\end{tikzcd}
$$

We would like to resolve this module by free modules. The free module:

$$
\begin{tikzcd}
& \mathbb{Z} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{Z} \arrow[dr] & \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}
$$

surjects onto $S(\hat{1})$. The kernel

$$
\begin{tikzcd}
& 0 \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{Z} \arrow[dr] & \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}
$$

is generated by 3 elements, and so is surjected onto by the following sum of free modules.

$$
\begin{tikzcd}
& 0 \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{Z} \arrow[dr] & 0 \arrow[d] & 0 \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}
\quad \bigoplus \quad
\begin{tikzcd}
& 0 \arrow[dl] \arrow[d] \arrow[dr] & \\
0 \arrow[dr] & \mathbb{Z} \arrow[d] & 0 \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}
\quad \bigoplus \quad
\begin{tikzcd}
& 0 \arrow[dl] \arrow[d] \arrow[dr] & \\
0 \arrow[dr] & 0 \arrow[d] & \mathbb{Z} \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}
$$

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[d] \arrow[dr] & \\ 0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[dl] \\ & \mathbb{Z}^{\oplus 2} & \end{tikzcd}$$

And the kernel is the following free module.
Thus we have a resolution of $S(\{1\}\{2\}\{3\})$ given by

$$\mathbb{Z}Q(-,\{1\}\{2\}\{3\}) \xleftarrow{d_0} \mathbb{Z}Q(-,\{1\}\{23\}) \oplus \mathbb{Z}Q(-,\{2\}\{13\}) \mathbb{Z}Q(-,\{3\}\{12\}) \xleftarrow{d_1} \mathbb{Z}Q(-,\{123\}) \otimes \mathbb{Z}^2.$$

The differential d_0 takes the summand $\mathbb{Z}P(-,\{r\}\{st\})$ to $\mathbb{Z}P(-,\{123\})$ by multiplication by $\{r\}\{st\} \leq \hat{1}$. In other words, it is given by the row vector:

$$d_0 = \begin{pmatrix} \{1\}\{23\} \leq \hat{1} & \{2\}\{13\} \leq \hat{1} & \{3\}\{12\} \leq \hat{1} \end{pmatrix}.$$

The differential d_1 is given by the matrix:

$$\begin{pmatrix} \{123\} \leq \{1\}\{23\} & 0 \\ -\{123\} \leq \{2\}\{13\} & \{123\} \leq \{2\}\{13\} \\ 0 & -\{123\} \leq \{3\}\{12\} \end{pmatrix}.$$

We note that the 0th free module in this resolution has rank 1, the first has rank 3 and the second has rank 2. We may assemble this data into a Poincaré polynomial $1 + 3t + 2t^2$. This agrees with the Poincaré polynomial of $\mathrm{Conf}_3(\mathbb{R}^2)$!

The symmetric group $\mathbf{S}_3$ acts on P, so we may consider P modules with an $\mathbf{S}_3$ action that is compatible with the action on $\widehat{P}$. If we let $\mathbf{S}_3$ act trivially on $S(\{1\}\{2\}\{3\})$ we may upgrade the free resolution to one that is equivariant for the $\mathbf{S}_3$ action. Then $\mathbf{S}_3$ acts trivally on the generators of the 0th free module; the generators of the first free module are the permutation representation of $\mathbf{S}_3$; and the generators of the second free module are the standard representation of $\mathbf{S}_3$. This corresponds to the equivariant Poincaré polynomial $s_3 + (s_3 + s_{2,1})t + s_{2,1}t^2$. This is agrees with the equivariant Poincaré polynomial of

$\mathrm{Conf}_3(\mathbb{R}^2)$!

To obtain a model for $S(\{1\}\{2\}\{3\}) \otimes^{\mathrm{L}}_{\mathrm{P}(3)} S(\{123\})$ from this free resolution, we tensor it with $S(\{123\})$. All of the terms in the free resolution except for the last become zero, and we have that $\mathrm{Tor}_2(S(\{1\}\{2\}\{3\}), S(\{123\})) = \mathbb{Z}^2$. Considered $\mathbf{S}_3$ equivariantly, this group is the standard representation $(2, 1)$. $\square$

Let $\widehat{P}$ be a poset with top element $\hat{1}$. The next example shows that a uniform resolution of $S(\hat{1})$ exists.

Example 4.1.16 (Reduced bar resolution and strict functoriality)**.** Let C^s_p be an object of $\mathrm{Ch}(\mathcal{A}^{\widehat{P}})$. We have a uniform construction of $S_{\hat{1}} \otimes^{\mathrm{L}}_{\widehat{P}} C$ coming from the reduced bar resolution of $S(\hat{1})$:

$$\mathbb{Z}\widehat{P}(-, 1) \leftarrow \bigoplus_{p_1 < \hat{1}} \mathbb{Z}\widehat{P}(-, p_1) \leftarrow \bigoplus_{p_2 < p_1 < \hat{1}} \mathbb{Z}\widehat{P}(-, p_2) \leftarrow \dots$$

The ith term of this resolution, $\overline{B}_i$, is the sum over all strict length i chains:

$$\overline{B}_i = \bigoplus_{p_i < p_{i-1} < \dots < p_1 < \hat{1}} \mathbb{Z}\widehat{P}(-, p_i)$$

and the differential $d_i = \sum_{i=0}^{r} (-1)^r \delta^i_r$ comes from a semi-simplical $\mathbb{Z}$ module. When $r = 0$, define $\delta_r = 0$. When $0 < r < i$, the map $\delta^i_r : \overline{B}_i \to \overline{B}_{i-1}$ takes summand corresponding $p_i < \dots < p_r < \dots < \hat{1}$ to the summand $p_i < \dots \widehat{p_r} < \dots < \hat{1}$ by the identity map. When $r = i$, the map δ^i_r takes the summand $p_i < \dots < \hat{1}$, $\mathbb{Z}\widehat{P}(-, p_i)$, to the summand $p_{i-1} < \dots < \hat{1}, \mathbb{Z}\widehat{P}(-, p_{i-1})$, by the multiplication by $p_{i-1} \leq p_i$

This resolution is exact because it is the reduced chain complex of the simplical $\mathbb{Z}$ module B , with $B_i = \{p_1 \leq \dots \leq p_i \leq 1\}$. The bar construction B is contractible, as in [44] Chapter 8.

Tensoring with C we get a model for $S(\hat{1}) \otimes^{\mathrm{L}}_{\widehat{P}} C$, the totalization of the bicomplex:

$$(s, -t) \mapsto \bigoplus_{p_t < p_{t-1} < \dots < p_1 < \hat{1}} C^s_{p_t},$$

with differential in the s variable given by d_C and differential in the t variable $d_t = \sum_{i=0}^{r}(-1)^r\delta_r^t$. When $r = 0$, $\delta_0 = 0$; for $0 < r < t$, δ_r^t takes the summand corresponding $p_t < \cdots < p_r < \cdots < \hat{1}$ to the summand $p_t < \ldots \widehat{p_r} < \cdots < \hat{1}$ by the identity map; and when $r = t$, the map δ_t^t takes the summand $p_t < \cdots < \hat{1}$, M_{p_t} to the summand $p_{i-1} < \cdots < \hat{1}$, $M_{p_{i-1}}$ by the action of $p_{i-1} \leq p_i$

Take a $f : \widehat{Q} \to \widehat{P}$ a map between posets with top element such that $f^{-1}(\{\hat{1}_P\}) = \{\hat{1}_Q\}$, D a chain complex of Q representations, and $x : D \to f^*C$. We obtain a representative for $S(\hat{1}) \otimes_Q^{\mathrm{L}} D \to S(\hat{1}) \otimes_{\widehat{P}}^{\mathrm{L}} C$ from the map

$$\bigoplus_{q_r<\cdots<q_1<\hat{1}} D_{q_r}^i \to \bigoplus_{q_r<\cdots<q_1<\hat{1}} C_{fq_r}^i \to \bigoplus_{p_r<\cdots<p_1<\hat{1}} C_{p_r}^i,$$

where the last map takes the summand corresponding to a chain $q_r < \cdots < q_1 < \hat{1}$ to $fq_r < \cdots < fq_1 < \hat{1}$ if $fq_i \neq fq_j$ for any distinct $i, j \in \{1, \ldots, r\}$ and takes the summand to 0 otherwise. □

We may use the bar complex to compare the derived tensor product with poset homology.

Proposition 4.1.17 (Comparison with poset homology)**.** *When $q < \hat{1}$, the reduced bar complex for $S_{\hat{1}} \otimes_{\widehat{P}}^{\mathrm{L}} S_q[2]$ is isomorphic to the complex of chains $C_*((q, 1))$, see [43]. In particular, we have* $\mathrm{Tor}_i(S(\hat{1}), S_q) = \widetilde{H}_{i-2}((q, 1), k)$ *where $\widetilde{H}_*$ is the reduced poset homology.*

For specific posets, there often are smaller resolutions than the bar complexes that give models for $S_{\hat{1}} \otimes^{\mathrm{L}} -$.

Example 4.1.18 (Products)**.** Let $\widehat{P}, \widehat{Q}$ be posets with top elements $\hat{1}_P, \hat{1}_Q$. Given a representation of $\widehat{P}$ and a representation of $\widehat{Q}$ we can take the *external tensor product* and form a representation of $\widehat{P} \times \widehat{Q}$. We denote this tensor product by $\boxtimes$. Then $S(\hat{1}_{P\times Q}) = S(\hat{1}_P) \boxtimes_k S(\hat{1}_Q)$. For any two elements p, q we have $k(\widehat{P} \times \widehat{Q})(-, p \times q) = k\widehat{P}(-, p) \boxtimes k\widehat{Q}(-, q)$. So from free resolutions $F \xrightarrow{\sim} S(\hat{1}_P)$ and $G \xrightarrow{\sim} S(\hat{1}_Q)$ we obtain a map $F \boxtimes G \xrightarrow{\sim} S(\hat{1}_{P\times Q(}$. This map is a quasi-isomorphism because the underlying k modules are free. □

Example 4.1.19 (Boolean poset)**.** Let $\mathrm{B}(n)$ be the poset of subsets of $[n] = \{1, \ldots, n\}$ ordered by reverse inclusion. Giving a $\mathrm{B}(n)$ representation is the same as giving a $\mathbb{N}^n$ graded

$k[x_1, \ldots, x_n]$ module, concentrated in degrees $\{0,1\}^n$. We have that $\mathrm{B}(n)$ is isomorphic to the n fold product of the two element poset $0 \leq 1$. For $0 \leq 1$, the simple $S(\hat{1})$ is resolved by $k\widehat{P}(-,1) \leftarrow k\widehat{P}(-,0)$. From the above, we get a Koszul resolution of $S(\hat{1})$ for $\mathrm{B}(n)$.

If C_I^s is a chain complex of $\mathrm{B}(n)$ representations, then $S(\hat{1}) \otimes_{\mathrm{B}(n)} C$ is the totalization of the bicomplex:

$$(s, -t) \mapsto \bigoplus_{I \subset [n],\ |I| = t} j_{I*} C_I^s|_{U_I}$$

with $d_t = \sum_{i=1}^n (-1)^i x_i$ and $d^s = d_C^s$.

□

Definition 4.1.20. We call a free resolution *finite* when it has finitely many terms and each is finitely generated. □

The following proposition gives an algorithm that useful for computing small resolutions. For the boolean poset and the partition poset, this process gives free resolutions isomorphic to the ones described above.

Proposition 4.1.21 (Minimal Free Resolutions)**.** *Let P^{op} be a finite poset. When k is a field, every representation $A \in \mathrm{Mod}\, k^{P^{\mathrm{op}}}$ admits a finite free resolution $F \xrightarrow{\sim} A$ such the differential of the chain complex $F \otimes_P S(p)$ vanishes for any $p \in P$.*

Proof. Mimicing the situation of modules over a graded ring, we may construct a "minimal" free resolution. Let NA denote the vector space, graded by the elements of P such that in degree p we have

$$(NA)_p = \frac{A}{\sum_{pq} \operatorname{im} A_{pq}},$$

or in other words $NA = \bigoplus_{p \in \widehat{P}} A \otimes_{\widehat{P}} S(p)$, where $S(p)$ is the simple P rep which is one dimensional at p and zero elsewhere. Pick a basis for this vector space $\{e_i^p\}_{p \in P, i=1,\ldots,n_p}$, and then choose representatives $\{a_i^p\}$ for each basis vector in A. These generate A, and so we get a surjection $\oplus_{i,p} kP(-,p) \twoheadrightarrow A$. This is the first step in the free resolution. If $A \neq 0$ then the kernel of this surjection K_1 must be zero in at least one degree where A was nonzero (any degree q such that $A_{q'} = 0$ for all $q' \geq q$), and so the process terminates. □

There is also a dual theory for minimal injective resolutions of poset representations over a field.

Let $\widehat{P}$ be poset with top element $\hat{1}$. In addition to the derived tensor product with $S(\hat{1})$, we will use the derived functors of the colimit over $P = \widehat{P} - \hat{1}$.

Definition 4.1.22 (Homotopy colimit)**.** Let M be an object of $\mathrm{Ch}(\mathcal{A})^{\widehat{P}}$. Let $\mathbb{Z}P \in \mathrm{Ab}^{\widehat{P}}$ be the module defined by

$$(\mathbb{Z}P)_p \begin{cases} 0 & \text{if } p = \hat{1} \\ \mathbb{Z} & \text{if } p \in P = \widehat{P} - 1 \end{cases}.$$

For every $p \leq q \neq \hat{1}$, the map $\mathbb{Z}P_{pq}$ is the identity. We have that $\mathbb{Z}P \otimes_{\widehat{P}} M = \operatorname{colim}_P M|_P$.

Thus given a model for $\mathbb{Z}P \otimes^{\mathrm{L}}_{\widehat{P}} M$, we say that it is a *model for the homotopy colimit*, $\mathrm{Lcolim}_P M$.

For any such model, there is a map $\mathrm{Lcolim}_P M \to M_1$ given by $\mathrm{Lcolim}_{\widehat{P}-\hat{1}} M \to \operatorname{colim}_P M|_P \to M_{\hat{1}}$ and we say that the representation M is $\widehat{P}$*-exact*, if this map is a quasi-isomorphism. □

Proposition 4.1.23. *We have that* $S(\hat{1}) \otimes^{L}_{\widehat{P}} M = \operatorname{cone}(\mathrm{Lcolim}_P M \to M_1)$.

More precisely for any free resolution F of $\mathbb{Z}P$ with canonical map $F \otimes_{\widehat{P}} M \to$ $\operatorname{colim}_P M \to M_1$ *, we have that* $\operatorname{cone}(F \otimes_{\widehat{P}} M \to M_1)$ *is a model for* $S(\hat{1}) \otimes^{L}_{\widehat{P}} M$*. In particular, M is an exact representation if and only if* $S(\hat{1}) \otimes^{\mathrm{L}}_{\widehat{P}} M \simeq 0$.

Proof. We have a short exact sequence of $\widehat{P}^{\mathrm{op}}$ representations: $0 \to \mathbb{Z}P \to \mathbb{Z}\widehat{P}(-,1) \to S(\hat{1}) \to 0$. So for any resolution F of $\mathbb{Z}P$, the cone of $F \to \mathbb{Z}P \to \mathbb{Z}P(-,1)$ is a free resolution of $S(\hat{1})$. Tensoring with M gives $S(\hat{1}) \otimes^{\mathrm{L}} M \simeq \operatorname{cone}(F \to \mathbb{Z}P(-,1)) \otimes_{\widehat{P}} M = \operatorname{cone}(F \otimes_{\widehat{P}} M \to M_1)$ □

If $\widehat{Q}$ and $\widehat{P}$ are meet lattices, then a meet preserving homomorphism $gL\widehat{Q} \to \widehat{P}$ that takes $\hat{1}_Q$ to $\hat{1}_P$ has the property that models of $A \otimes^{\mathrm{L}}_{\widehat{P}} M$ pull back to models of $gA \otimes^{\mathrm{L}}_{\widehat{Q}} g^*C$.

Proposition 4.1.24 (Transfer)**.** *Let $g : \widehat{Q} \to \widehat{P}$ be a surjective meet-preserving morphism of meet-lattices . Let C be an object of* $\mathrm{Ch}(\mathcal{A})^{\widehat{P}}$ *and A of* $\mathrm{Ab}^{\widehat{P}^{op}}$*. Then $g^*A \otimes^{L}_{\widehat{Q}} g^*C \to A \otimes^{\mathrm{L}}_{\widehat{P}} C$ is a quasi-isomorphism for any choice of resolution of A and g^*A. In particular C is $\widehat{P}$-exact if and only if g^*C is $\widehat{Q}$-exact.*

Proof. It suffices to prove the statement for a particular choice of free resolutions for A and g^*A. First we note that the meet preserving map $g : \widehat{Q} \to \widehat{P}$ has an adjoint $f : \widehat{P} \to \widehat{Q}$ given by

$$fp = \wedge_{\{q',\ fq' \geq p\}}\ q'.$$

If $fp \leq q$ then applying g we have $p \leq (\wedge_{\{q',\ gq' \geq p\}} gq') \leq gq$. Conversely, if $p \leq gq$ then $fp = (\wedge_{\{q', gq' \geq p\}} q') \leq q$. Further because g is surjective, there is some q' with $gq' = p$, and hence $gfp = p$.

Thus $g^*\mathbb{Z}\widehat{P}(p, -) = \mathbb{Z}\widehat{P}(p, f-) = \mathbb{Z}\widehat{Q}(fp, -)$. Take a resolution of A by free modules: $K_i = \bigoplus_p V_i^p \otimes \mathbb{Z}\widehat{P}(-, p)$ where V_i^p is a finitely generated free $\mathbb{Z}$ module, and the differentials are given by maps $d_i^{p \leq q} : V_i^p \to V_i^q$. This pulls back to a resolution of g^*A by free modules $G^*K_i = \bigoplus_p V_i^p \otimes k\widehat{Q}(-, fp)$. Tensoring with C we get on the one hand a complex representing $A \otimes^L_{\widehat{Q}} C$, which is the totalization of the bicomplex:

$$(-i, j) \mapsto \bigoplus_p V_i^p \otimes C_p^j,$$

and differential in the i variable given by $d_i^{p \leq q} \otimes C_{pq}^j$, and the j variable given by d_C. On the other hand we get a complex representing $g^*A \otimes^{\mathrm{L}} g^*M$ which is the totalization of the bicomplex

$$(-i, j) \mapsto \bigoplus_p V_i^p \otimes C_{gfp}^j = \bigoplus_p V_i^p \otimes C_p^j,$$

with the same differentials, and the natural map between them is an isomorphism. The last statement of the theorem follows because $g^*S(\hat{1}) = S(\hat{1})$. □

We note that the above proposition can be also be seen as a simple corollary of the Quillen Fibre lemma.

4.1.1 Resolutions from subobject lattices

We have two propositions in this subsection: the first gives a criterion for the higher Tor's to vanish, and the second decribes how this gives you a free resolution based on the homology of the poset.

Proposition 4.1.25. *Let $\mathcal{A}$ be an abelian category, and let P be a finite meet lattice, with top element $\hat{1}$. Suppose that $M : P \to \mathcal{A}$ is a functor such that*

1. *Each $M_p \to M_{\hat{1}}$ is injective, and $M_p \cap M_q = M_{p\wedge q}$*
2. *For $p \in P$ and $S \subset P$ such that $p \notin S$, we have $M_p \cap \sum_{q\in S} M_q = \sum_{q\in S} M_p \cap M_q$*

Then the natural map

$$S(\hat{1}) \otimes^{\mathrm{L}}_P M \to S(\hat{1}) \otimes_P M = \operatorname{coker}(\bigoplus_{p<\hat{1}} M_p \to M_{\hat{1}})$$

is a quasi-isomorphism.

Remark 4.1.26. Condition 1 is equivalent to: For all p, q the map $M_{p\wedge q} \to M_p \times_{M_{\hat{1}}} M_q$ is an isomorphism.

Condition 2 can be weakened to: There is an ordering of the atoms of P: $a_1, \dots, a_n$ such that for all $i = 1 \dots, n$ we have $M_{a_i} \cap \sum_{j>i} M_{a_j} = \sum_{j>i} M_{a_i} \cap M_{a_j}$. □

Proof. By Proposition 4.1.24, it suffices to show this after pulling back to the boolean lattice on the atoms of P. We denote the atoms by $\{a_i\}_{i\in I}$. Then we have the Koszul complex

$$K = M_1 \leftarrow \bigoplus_i M_{a_i} \leftarrow \bigoplus_{i<j} M_{a_i\wedge a_j} \leftarrow \bigoplus_{i<j<k} M_{a_i\wedge a_j\wedge a_k} \leftarrow \dots$$

We argue that $H_i(K) = 0$ for $i > 0$ by induction on $|I|$. We note that K is the cone of the natural map from the complex

$$M_{a_1} \leftarrow \bigoplus_{1<i} M_{a_1\wedge a_i} \leftarrow i \bigoplus_{1<i<j} M_{a_1\wedge a_i\wedge a_j} \leftarrow \dots$$

to the complex

$$M_1 \leftarrow \bigoplus_{1<i} M_{a_i} \leftarrow \bigoplus_{1<i<j} M_{a_i\wedge a_j} \leftarrow \dots$$

Then by induction the homology of the second complex is concentrated in degree 0. And for $i > 0$ (since $M_{a_1} \to M_1$ is injective), the homology of the first complex agrees with the homology of

$$M_1 \leftarrow \bigoplus_{1<i} M_{a_1\wedge a_i} \leftarrow \bigoplus_{1<i<j} M_{a_1\wedge a_i\wedge a_j} \leftarrow \dots$$

vanishes in degrees > 0 again by induction. On H_0 the map from the first complex to the second is given by

$$\frac{M_{a_1}}{\sum_{i>1} M_{a_1} \cap M_{a_i}} \to \frac{M_1}{\sum_{i>1} M_{a_i}}$$

so the kernel is

$$\frac{M_{a_1} \cap \sum_i M_{a_i}}{\sum_{i>1} M_{a_1} \cap M_{a_i}} = 0,$$

by condition 2. Thus the homology of K is concentrated in degree 0 and $H_0(K) = M_1/\sum_i M_{a_i}$. ◻

Without condition 2, the proposition is false. Take $\mathcal{A} = \mathrm{Vec}_{\mathbb{R}}$ and the lattice of subobjects given by $\mathbb{R}e_1, \mathbb{R}e_2, \mathbb{R}e_1 + e_2$ inside of $\mathbb{R}e_1 \oplus \mathbb{R}e_2$.

Proposition 4.1.27. *Let k be a field and $\mathcal{A}$ a k-linear abelian category. Let P be a poset with top element $\hat{1}$, and $M : P \to \mathcal{A}$ a functor. If the natural map*

$$(S(\hat{1}) \otimes_{\mathbb{Z}} k) \otimes^{\mathrm{L}}_P M \to (S(\hat{1}) \otimes_{\mathbb{Z}} k) \otimes_P M = \operatorname{coker}(\bigoplus_{p<\hat{1}} M_p \to M_{\hat{1}})$$

is a quasi-isomorphism, then there is a complex

$$M_{\hat{1}} \leftarrow \bigoplus_p \widetilde{H}_{-1}((p,\widehat{1}), k) \otimes M_p \leftarrow \bigoplus_q \widetilde{H}_0((q,\hat{1}), k) \otimes M_q \leftarrow \bigoplus_r \widetilde{H}_1((r,\hat{1}), k) \otimes M_r \leftarrow \dots$$

that resolves $M_{\hat{1}}/(\sum_{p<\hat{1}} M_p)$. The differentials are linear combinations of the maps $M_{pq} : M_p \to M_q$.

Proof. Take the minimal free resolution of $k\hat{1}$ as a P^{op} representation and tensor it with M. The higher homology vanishes by hypothesis, and the terms are as stated because $\mathrm{Tor}_i^P(S(\hat{1}) \otimes k, S(p) \otimes k) = \widetilde{H}_{i-2}((p,\hat{1}), k)$. ◻

Further, if you know the minimal free resolution of $S(\hat{1})$, then you can describe the differentials in the complex explicitly.

Example 4.1.28 (Monomial Resolutions)**.** Let k be a field, and let $\mathcal{A}$ be the category of $R = k[x, y, z]$ modules. Consider the monomial ideal generated by (xy, yz, xz). Taking greatest common divisors, these monomials generate the following sub-lattice of monomials ordered by divisibility.

$$\begin{tikzcd}
 & 1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
xy \arrow[dr, dash] & xz \arrow[d, dash] & yz \arrow[dl, dash] \\
 & xyz &
\end{tikzcd},$$

where 1 is the empty GCD. To this lattice, we may associate a representation:

$$\begin{tikzcd}
 & R & \\
(xy)R \arrow[ur] & (xz)R \arrow[u] & (yz)R \arrow[ul] \\
 & (xyz)R \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd},$$

Because this representation is monomial, the intersections of these ideals are just intersections of sets of monomials. Thus this representation satisfies the properties of Proposition 4.1.25. Further, it is isomorphic to

$$\begin{tikzcd}
 & R & \\
R \arrow[ur, "xy"] & R \arrow[u, "xz"] & R \arrow[ul, "yz"'] \\
 & R \arrow[ul, "z"] \arrow[u, "y"] \arrow[ur, "x"'] &
\end{tikzcd}$$

and thus as in the proof of Proposition 4.1.27, we have a minimal free resolution

$$R/(xy, xz, yz) \leftarrow R \leftarrow R^{\oplus 3} \leftarrow R^{\oplus 2},$$

where the differentials are given by the matrices of the resolution of the partition lattice on 3 elements, Example 4.1.15, with the entries relabelled according to the diagram. For example, we replace $\{12\}\{3\} \geq \{123\}$ by z. □

This trick can be extended to reduce the computation of the minimal free resolution of an arbitrary monomial ideal $(m_1, m_2, \ldots, m_r)$ to the computation of a minimal resolution of $S(\hat{1})$ over the meet-lattice spanned by the GCD's of the monomials. In the case where this lattice L is the face poset of a simplicial complex or a CW complex, the minimal free resolution over L is multiplicity-free and can be read off directly from the faces.

4.1.2 Homology of the Partition Lattice

Definition 4.1.29. Let $\mathrm{P}(X)$ denote the *poset of partitions* of the set X ordered by refinement.

Formally, the set of set partitions of X into r blocks is

$$\mathrm{P}(X, r) = \{ \text{ surjections } e : X \twoheadrightarrow [r]\}/\mathbf{S}_r,$$

and $\mathrm{P}(X) = \sqcup_{r \in \mathbb{N}} \mathrm{P}(X, r)$. Let $p, q \in \mathrm{P}(X)$ be represented by $f : X \twoheadrightarrow [r]$ and $g : X \twoheadrightarrow [s]$. We say that $p \geq q$ if there exists a surjection $h : [r] \twoheadrightarrow [s]$ such that $h \circ f = g$. This relation is well defined, and gives $\mathrm{P}(X)$ the structure of a poset.

We can represent a partition p by a collection of blocks $b_1 \sqcup \cdots \sqcup b_r$, where $b_i \subset X, b_i \neq \emptyset$ is a collection of mutually disjoint subsets. This data gives a unique map $f : X \twoheadrightarrow [r]$ such that $f^{-1}(i) = b_i$. The associated partition p only depends on the choice of subsets b_i, and not on their labelling. □

Partitions of X are closely related to surjections $X \twoheadrightarrow [n]$.

Definition 4.1.30. Let X be a finite set. The *undercategory* $X/\mathbf{FS}$ is the category whose objects are surjections $f : X \twoheadrightarrow Y$, and whose morphisms are $g : Y \to Y'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & Y \arrow[d, "g"] \\ & Y' \end{tikzcd}$$

commutes. □

Proposition 4.1.31. *The category* $X/\mathbf{FS}$ *is equivalent to* $\mathrm{P}(X)^{\mathrm{op}}$.

Proof. First we show that the isomorphism classes of $X/\mathbf{FS}$ are given by partitions. Since every finite set Y is isomorphic to $[n]$ for some n, every object of $X/\mathbf{FS}$ is isomorphic to one of the form $f : X \twoheadrightarrow [n]$. Two such objects f, f' are isomorphic if and only if there is a bijection $\sigma \in \mathbf{S}_n$ such that $\sigma \circ f = f'$. Thus the isomorphism classes are set partitions.

Next let $f : X \twoheadrightarrow [n], g : X \twoheadrightarrow [m]$ be two objects of $X/\mathbf{FS}$. There is at most one map $h : [n] \twoheadrightarrow [m]$ which makes the diagram commute, since the image of each element of X is determined by g and f is a surjection. Thus $\mathrm{Hom}(f, g) = *$ if there is a surjection $h : [n] \twoheadrightarrow [m]$ such that $h \circ f = g$, and is empty otherwise. This completes the proof. □

In fact, we can use this relationship to compare the homology of $\mathrm{P}(n)$ with the homology of $\mathbf{FS}^{\mathrm{op}}$.

Definition 4.1.32 (Ranking of Partition Lattice)**.** Given $p \leq q \in \mathrm{P}(n)$, we define

$$g(p, q) = \#\{ \text{ blocks of } q \} - \#\{ \text{ blocks of } p \}.$$

We let $r(p) = g(\hat{1}, p)$. □

We recall the definition of the tensor product of modules over a category.

Definition 4.1.33. For any small linear category C, we can define the tensor product of a C^{op} module M and a C module N, exactly as in the definition of the tensor product over a poset.

$$M \otimes_C N := \mathrm{coeq}\left(\bigoplus_{c_1, c_2 \in C} M_{c_2} \otimes C(c_1, c_2) N_{c_1} \underset{\mathrm{id}_M \otimes m_N}{\overset{m_M \otimes \mathrm{id}_N}{\rightrightarrows}} \bigoplus_{c \in C} M_c \otimes N_c \right).$$

The Tor groups $\mathrm{Tor}_i^C(M, N)$ can be computed by resolving either M or N by projective modules before tensoring. □

Theorem 4.1.34. *Let* $\mathrm{P}(n)$ *be the partition lattice on n elements. There is a canonical isomorphism of* $\mathbf{S}_m$ *representations*

$$\mathrm{Tor}^{\mathbf{FS}}_{\bullet}(\mathbb{Z}\mathbf{S}_n, \mathbb{Z}\mathbf{S}_m) \cong \bigoplus_{p,\ r(p)=n-m} \mathrm{Tor}^{\mathrm{P}(m)}_{\bullet}(S(\hat{1}), \mathbb{Z}\mathbf{S}_n \otimes S(p)).$$

Proof. Consider the representation $([m]/\mathbf{FS})^{\mathrm{op}} \to \operatorname{Mod}\mathbf{FS}$ given by

$$([m] \twoheadrightarrow [n]) \mapsto \mathbb{Z}\mathbf{FS}(n, -).$$

By Proposition 4.1.31, this gives a $\mathrm{P}(m)$ representation. This representation is isomorphic to the one which takes $p \in \mathrm{P}(m)$ to the submodule $M_p \subset \mathbb{Z}\mathbf{FS}(m, -)$ consisting of surjections that factor through p.

Because these submodules are the linearization of subsets, the functor M satisfies the distributivity hypothesis of Proposition 4.1.25. Further, the quotient of $\mathbb{Z}\mathbf{FS}(m, -)$ by the sum of the M_p, $p < \hat{1}$ is clearly $\mathbb{Z}\mathbf{S}_n$. Thus we have that any model for $S(\hat{1}) \otimes^{\mathrm{L}}_{\mathrm{P}(m)} M$ is quasi-isomorphic to $\mathbb{Z}\mathbf{S}_n$ as a chain complex of $\mathbf{FS}$ modules.

Thus given a free resolution $F \xrightarrow{\sim} S(\hat{1}) \in \mathrm{Ch}(\mathrm{Ab}^{\mathrm{P}(n)^{\mathrm{op}}})$, we have $F \otimes_{\mathrm{P}(m)} M$ is a chain complex of free $\mathbf{FS}$ modules, resolving $\mathbb{Z}\mathbf{S}_m$. Thus the homology of the complex

$$\mathbb{Z}\mathbf{S}_n \otimes_{\mathbf{FS}} (F \otimes_{\mathrm{P}(m)} M) \tag{4.1.35}$$

is $\mathrm{Tor}^{\mathbf{FS}}_{\bullet}(\mathbb{Z}\mathbf{S}_n, \mathbb{Z}\mathbf{S}_m)$.

On the other hand,

$$\mathbb{Z}\mathbf{S}_n \otimes_{\mathbf{FS}} (F \otimes_{\mathrm{P}(m)} M) \cong F \otimes_{\mathrm{P}(m)} \mathbb{Z}\mathbf{S}_n \otimes_{\mathbf{FS}} M.$$

And for every p, we have $M_p = \mathbb{Z}\mathbf{FS}([m]/\!\sim p, -)$ thus

$$\mathbb{Z}\mathbf{S}_n \otimes_{\mathbf{FS}} M_p = \begin{cases} \mathbb{Z}\mathbf{S}_n & \text{if } r(p) = m - n \\ 0 & \text{otherwise} \end{cases}.$$

Since only one row of the represenation $p \mapsto \mathbb{Z}\mathbf{S}_n \otimes_{\mathbf{FS}} M_p$, this representation takes the form $\bigoplus_{n-m=r(p)} \mathbb{Z}\mathbf{S}_n \otimes S(p)$. Thus the complex 4.1.35 also computes

$$\mathrm{Tor}_\bullet^{\mathrm{P}(m)}\left(S(\hat{1}), \bigoplus_{p\ r(p)=n-m} \mathbb{Z}\mathbf{S}_n \otimes S(p)\right),$$

these two groups are isomorphic. □

Thus in the special case $n = 1$, we have an isomorphism of graded $\mathbf{S}_m$ representations

$$\mathrm{Tor}_\bullet^{\mathrm{P}(m)}(S(\hat{1}), S(\hat{0})) \cong \mathrm{Tor}_\bullet^{\mathbf{FS}}(\mathbb{Z}\mathbf{S}_1, \mathbb{Z}\mathbf{S}_m) \cong \mathbf{C}^{\mathrm{op}}_{(\Sigma^{-1}\,\mathrm{Lie})^*}(1, m) \cong (\Sigma^{-1}\,\mathrm{Lie})^*(m)$$

by Proposition 3.2.11. Since all of the intervals $[p, q]$ in the partition lattice are isomorphic to products of $\mathrm{P}(n_i)$, this special case in fact determines all of the other Tor groups.

Remark 4.1.36. This is not the most efficient or geometrically enlightening way to compute the homology of the partition lattice. The computation relies on Proposition 3.2.11, which in turn relies on Fresse [15][Theorem 6.8], which is most easily proved using the homology of $\mathrm{P}(n)$. However, we hope it demonstrates the close interrelationship between the homology of the partition lattice, the commutative operad, and $\mathbf{FS}^{\mathrm{op}}$. □

4.1.3 Spectral sequences

We show how to obtain a spectral sequence from a model for $A \otimes^{\mathrm{L}}_P M$.

Suppose we have C in $\mathrm{Ch}(\mathcal{A})^{\widehat{P}}$ and we want to compute $H^i(\mathrm{Lcolim}_P C)$. There are two relevant spectral sequences: we could take the hypercohomology spectral sequence relating $\mathrm{Lcolim}_P H^i(C)$ to $H^i(\mathrm{Lcolim} C)$, or we could use the canonical filtration on $\widehat{P}$ representations to get spectral sequence relating $H^i(\mathrm{Lcolim}_P C_p)$ to $H^i(C)$. In fact, we obtain a spectral sequence that interpolates between these two, works also for $A \otimes^{L}_{\widehat{P}} -$ and is natural both in $\widehat{P}$ and in free resolutions $F \xrightarrow{\sim} A$. We give the filtration explicitly in the statement of the theorem:

Definition 4.1.37. Let P be finite poset. We say that a *non-degenerate ranking* of P is a function $r : P \to \mathbb{N}$ such that $p > p' \implies r(p) < r(p')$. We say that P is a *ranked poset* and say that r is the *rank function* of P.

A *morphism of ranked posets* $P \to Q$ is a map $f : P \to Q$ such that $r(f(p)) = r(q)$. $\square$

Proposition 4.1.38 (Spectral sequence for $A \otimes^{\mathrm{L}}_P -$)**.** *Let C be a left bounded chain complex of P representations. Let A be a P^{op} representation, with a finite free resolution $G_j = \bigoplus_p V^p_j \otimes kP(-, p)$, with differentials given by $d^{p\leq q}_i : V^p_i \to V^q_{i-1}$. Then we have a model for $kA \otimes_P C$ given by the totalization of the bicomplex*

$$(s, -t) \mapsto \bigoplus_{p \in P} C^s_p \otimes V^p_t,$$

with d_t given by $d^{p\leq q}_t \otimes C^s_{pq} : V^p_t \otimes C^s_p \to V^q_t \otimes C^s_q$, and $d_s = d_C$. And the filtration $F_i(G \otimes_P C) :=$

$$\bigoplus_{t\in\mathbb{N}, p\in P,\ s\in\mathbb{Z},\ 2r(p)+s<i} V^p_t \otimes C^s_p \oplus \bigoplus_{t\in\mathbb{N}, p\in P,\ s\in\mathbb{Z},\ 2r(p)+s=i} \ker(1 \otimes d^s_{p,C} : V^p_t \otimes C^s_p \to V^p_t \otimes C^{s+1}_p)$$

gives a convergent spectral sequence

$$E^{i,-j}_1 = \bigoplus_{s,t,\ s-t=i-j} \ \bigoplus_{p\in P,\ 2r(p)=i-s} k\,\mathrm{Tor}_t(A, H^s(C_p) \otimes S_p) \implies H^{i-j}(A \otimes^{\mathrm{L}}_P C).$$

Alternately, for $p \in P, s, t \in \mathbb{Z}$, the term $k\,\mathrm{Tor}_t(A, H^s(C_p) \otimes S_p)$ appears exactly once on the E_1 page, in the place $E^{i,-j}_1 = E^{2r(p)+s,-t-2r(p)}_1$.

Proof. This asserted filtration is increasing. The filtration is preserved by both differentials since the t differential only decreases $2r(p) + s$, d_C increases it by at most one, and if $2r(p) + s = i$ then d_C acts by 0 .

The associated graded is $F_iG/F_{i-1}G =$

$$\bigoplus_{t\in\mathbb{N}, p\in P,\ s\in\mathbb{Z},\ 2r(p)+s=i-1} V^p_t \otimes \frac{C^s_p}{\ker(d^s_{p,C})} \oplus \bigoplus_{t\in\mathbb{N}, p\in P,\ s\in\mathbb{Z},\ 2r(p)+s=i} V^p_t \otimes \ker(d^s_{p,C}).$$

On this page, if $p < q$, the map $d^{p\leq q}_t \otimes C^s_{pq}$ must act by zero, because it decreases $2r(p) + s$ by 2. So the d_t differential preserves the grading by $p \in P$, as does $d_s = d_C$ and so we get a direct sum decomposition of F_i/F_{i-1} over $p \in P$ as:

$$\bigoplus_{p\in P} \bigoplus_{t\in\mathbb{N}, p\in P,\ s\in\mathbb{Z},\ 2r(p)+s=i-1} V_t^p \otimes \operatorname{cone}\left(d_{C,p}^{s-1} : \frac{C_p^{s-1}}{\ker(d_{C,p}^{s-1})} \to \ker(d_{p,C}^{s})\right)$$

$$= \bigoplus_{p\in P} \operatorname{cone}\left(\frac{C_p^{i-2r(p)-1}}{\ker(d_{C,p}^{i-2r(p)-1})} \to \ker(d_{p,C}^{i-2r(p)})\right) \otimes (G \otimes_P S_p) \to^{\simeq} \bigoplus_{p\in P} H^{i-2r(p)}(C_p) \otimes (G \otimes_P S_p)$$

where the last line is the natural map to the cokernel, which is a quasi-isomorphism by the spectral sequence for a bicomplex. Taking cohomology yields the desired E_1 spectral sequence. ◻

Remark 4.1.39. We have the following functoriality for the spectral sequence of Proposition 4.1.38: let $f : Q \to P$ a map between ranked posets, D a chain complex of Q representations, B a Q^{op} module with free resolution K, and $u : D \to f^*C$, and $v : B \to f^*A$, and $\widetilde{v} : K \to f^*A$. Then the map $K \otimes_Q D \to F \otimes_P C$ preserves the filtration, and gives morphism of spectral sequences which on cohomology is $H^{i-j}(B \otimes_Q^{\mathrm{L}} D) \to H^{i-j}(A \otimes_P^{\mathrm{L}} C)$.

Because the rank function of Q is non-degenerate we have $f^*S(p) = \bigoplus_{q\in Q, fq=p} S(q)$, and we get a map $\bigoplus_{q\in Q, fq=p} \operatorname{Tor}_*^Q(B, S(q)) \to \operatorname{Tor}_*^P(A, S(p))$ induced by $\widetilde{v}$. Also, the map $D \to f^*C$ gives a map $\bigoplus_{q\in Q,\ fq=p} H^s(D_q) \to H^s(D_p)$. The action of $K \otimes_Q D \to F \otimes_P C$ on the E_1 page is given by $\bigoplus_{q\in Q, fq=p} H^s(C_q) \otimes \operatorname{Tor}_t^Q(A, S(q)) \to H^s(C_p) \otimes k \operatorname{Tor}_t^P(A, S(p))$ the diagonal of the tensor product of these two maps. ◻

4.2 Sheaves and Stratified Spaces

Let X be a topological space. We will use $\mathrm{Sh}(X)$ to denote the category of sheaves of abelian groups on X. The most important sheaf on X is the constant sheaf $\mathbb{Z}_X \in \mathrm{Sh}(X)$, defined to be the sheafification of the assignment $U \mapsto \mathbb{Z}(U)$. When the context is clear, simply write $\mathbb{Z}$.

4.2.1 Category of Stratified Spaces

Definition 4.2.1. Let P be a poset. The *order topology* on P is defined by

$$I \subset P \text{ closed } \iff i \in I, x \leq i \implies x \in I$$

□

Definition 4.2.2. Let X be a topological space. A stratification of X by a poset P is a continuous map $f : X \to P$.

Given a stratification, define the closed subset $Z_p \subset X$ by $Z_p := f^{-1}(P_{\leq p})$, the open subset $U_p \subset X$ by $f^{-1}(P_{\geq p})$, and the locally closed stratum $S_p \subset X$ by $S_p := f^{-1}(\{p\})$.

Equivalently, a stratification of X by a finite poset P can be defined by the data of a collection of closed sets $Z_p \subset X$, satisfying $Z_p \subset Z_q$ whenever $p \leq q$. The associated map $f : X \to P$ takes $Z_p - \cup_{q<p} Z_q$ to p. □

We define a category $\mathcal{X}$, whose objects consist of spaces stratified by ranked lattices, and sheaves on them.

Definition 4.2.3. Let $\mathcal{X}$ denote the category whose objects consist of the data $(X, \mathcal{F}, \widehat{P}, \{Z_p\}_{p\in\widehat{P}})$ where:

- X is a topological space,
- $\mathcal{F}$ is a sheaf on X,
- $\widehat{P}$ is a finite meet lattice, with non-degenerated rank function $r_{\widehat{P}} : \widehat{P} \to \mathbb{N}$
- $\{Z_p\}_{p\in\widehat{P}}$ is a collection of closed subsets indexed by $\widehat{P}$ such that $Z_p \cap Z_q = Z_{p\wedge q}$ and $Z_{\hat{1}} = X$.

A morphism from $(X, \mathcal{F}, \widehat{P}, \{Z_p\}_{p\in\widehat{P}})$ to $(Y, \mathcal{G}, \widehat{Q}, \{W_q\}_{q\in Q})$ is given by:

- A map $f : Y \to X$
- A ranked map of lattices $f : Q \leftarrow \widehat{P}$ such that $f^{-1}(Z_p) = W_{fq}$,

- A map of sheaves $y : f^*\mathcal{F} \to \mathcal{G}$.

If we have two maps $f : X' \to X$, $f : \widehat{P} \to \widehat{P'}$, $y : f^*\mathcal{F} \to \mathcal{F}'$ and $g : X'' \to X'$, $g : \widehat{P'} \to \widehat{P''}$, $y' : g^*\mathcal{F}' \to \mathcal{F}''$ then their composite is given by $f \circ g : X'' \to X$ and the map $y' \circ g^*y : (f^* \circ g^*)\mathcal{F} \simeq g^*f^*\mathcal{F} \to^{g^*y} g^*\mathcal{F}' \to^{y'} \mathcal{F}''$, where we use the natural isomorphism $f^*g^* \simeq (g \circ f)^*$. □

We also fix some notation in this context.

Definition 4.2.4. For an object $(X, \mathcal{F}, \widehat{P}, \{Z_p\}_{p\in\widehat{P}})$ of X, we write $i_p : Z_p \to X$ for inclusion, and $j_p : U_p \to X$ for the inclusion of $U_p := X - Z_p$. We define $U = X - \bigcup_{p\in P} Z_p$ and write $j : U \to X$ for the inclusion of U into X. □

4.2.2 Resolution of Sheaves and Spectral Sequences

The next property says that $\mathrm{R}\pi_*$ takes models of $A \otimes^{\mathrm{L}}_{\widehat{P}} M$ to models of $A \otimes^{\mathrm{L}}_{\widehat{P}} \mathrm{R}\pi_* M$.

Proposition 4.2.5. *Let $\mathcal{A}$ be an abelian category that has enough injectives, and let $\pi_* : \mathcal{A} \to \mathcal{B}$ be right adjoint functor between abelian categories. Let A be an object of $\mathrm{Ab}^{\widehat{P}^{\mathrm{op}}}$, and let M be an object of $\mathrm{Ch}(\mathcal{A})^{\widehat{P}}$. Then for a finite free resolution $F \xrightarrow{\sim} A$, and $M \xrightarrow{\sim} \mathcal{J}$ a resolution of M by a π_* acyclic complex, we have that $F \otimes_P \pi_* \mathcal{J} = \pi_* kF \otimes_P \mathcal{J}$, where the right hand side is a model for $\mathrm{R}\pi_*(kA \otimes^{\mathrm{L}}_{\widehat{P}} M)$ and the left hand side is a model for $A \otimes^{\mathrm{L}}_{\widehat{P}} \mathrm{R}\pi_* M$.*

Proof. Write $F_i = \bigoplus_{p\in\widehat{P}} V_i^p \otimes k\widehat{P}(-, p)$ where the multiplicity space V_i^p is a finitely generated free k module, and the differential $F_i \to F_{i-1}$ is given by maps $d_i^{p\leq q} : V_i^p \to V_{i-1}^q$. Then since π_* commutes with finite direct sums, we have that $\pi_* kF \otimes \mathcal{J}$ is the totalization of the bicomplex

$$(-i, j) \mapsto \bigoplus_{p\in\widehat{P}} V_i^p \otimes \pi_* \mathcal{J}_p^j$$

with differential in the i variable given by $\sum_{p\leq q} d^i_{p\leq q} \otimes \pi_* \mathcal{J}^j_{pq}$ and differential in the j variable given by $d_{\mathcal{J}}$. Since finite sums of π_* acyclics are acyclic, and $kF \otimes_{\widehat{P}} M \to kF \otimes_{\widehat{P}} \mathcal{J}$ is a quaisisomorphism, this complex is a model for $\mathrm{R}\pi_*(kA \otimes^{\mathrm{L}}_{\widehat{P}} M)$. And it is equal to $F \otimes \pi_* \mathcal{J}$, which is a model for $A \otimes^{\mathrm{L}}_{\widehat{P}} \mathrm{R}\pi_*(M)$. □

Let $B([n])$ be the boolean poset of all subsets $I \subset \{1, \ldots, n\}$, ordered by $I \leq J \iff I \supset J$.

Recall that restrictions and pushforwards of flabby sheaves along open inclusions are flabby.

Proposition 4.2.6 (Boolean case)**.** *Let $(X, \mathcal{F}, \widehat{P}, \{Z_p\}_{p \in \widehat{P}})$ be an object of $\mathcal{X}$ with $\widehat{P} = \mathrm{B}(n)$. Let $\mathcal{F} \to \mathcal{J}^\bullet$ be a flabby resolution. Let C be the chain complex in $\mathrm{Sh}(X)^{\mathrm{B}(n)}$ given by $C_I = j_* \mathcal{J}^\bullet|_{U_I}$ for $I \neq \emptyset$ and $C_\emptyset = j_* \mathcal{J}|_U$, with maps given by restriction. Then C is $\mathrm{B}(n)$-exact.*

Proof. To show that C is $\mathrm{B}(n)$-exact, we must show that the totalization of the bicomplex

$$(s, -t) \mapsto \bigoplus_{I \subset [n],\ |I| = t} j_{I*} \mathcal{J}^s|_{U_I},$$

with differential in the s variable given by $d_{\mathcal{J}}$ and differential in the $-t$ variable given by the alternating sum of the restriction maps, is exact. By the spectral sequence for a bicomplex, it suffices to show that for every flabby sheaf $\mathcal{I}$ on X, the cochain complex

$$C^{-t} = \bigoplus_{I \subset [n],\ |I| = t} j_{I*} \mathcal{I}|_{U_I},$$

with differential the alternating sum of the restriction maps, is exact. We do this by induction on n. In the case $n = 1$, we have $U_\emptyset = U_{\{1\}}$, and so the complex has two terms with its only differential the identity, and hence is exact. When $n > 1$ we have that $C^\bullet$ is the pushforward of a complex of acyclics from $U_{[n]} = \cup_{i \in [n]} U_i$, so we are reduced to showing the complex is exact there, and hence to showing that $C^\bullet|_{U_i}$ is exact for all i. Using that $j^* j_* \simeq id$, and that $U_I \cap U_i = U_i$ if $i \in I$ we have that

$$C^{-t}|_{U_i} = \bigoplus_{I \subset [n],\ |I| = t} j_{I*} \mathcal{I}|_{U_I \cap U_i} = \bigoplus_{J \subset [n],\ i \in J,\ |J| = t} \mathcal{I}|_{U_i} \oplus \bigoplus_{K \subset [n] - i, |K| = t-1} j_{i*} \mathcal{I}|_{\bigcup_{k \in K} (U_k \cap U_i)}.$$

The differential preserves the summand corresponding to subsets K not containing i, so a we get a two step filtration of $C^\bullet|_{U_i}$, with associated graded pieces corresponding to the two summands above. The first complex is an iterated cone of the the identity map of $\mathcal{I}|_{U_i}$, and so is exact.[2] And by induction applied to $[n] - \{i\}$ on $X = U_i$, the second summand is the pushforward of an exact complex of acyclics, and hence is exact. $\square$

The following theorem is the source of our spectral sequences. It gives a more formal refinement of Theorem

Theorem 4.2.7 (Resolution)**.** *Let $(X, \mathcal{F}, \widehat{P}, \{Z_p\}_{p \in \widehat{P}})$ be an object of $\mathcal{X}$. Let $\mathcal{F} \xrightarrow{\sim} \mathcal{J}$ be a flabby resolution. Define $j_{*\bullet}\mathcal{J}|_{U_\bullet}$ to be object of $\mathrm{Ch}(\mathrm{Sh}(X))^{\widehat{P}}$ given by $p \mapsto j_{*p}\mathcal{J}|_p$, and $\mathcal{J}_\bullet$ to be the constant representation $p \mapsto \mathcal{J}$. Define $i_{*\bullet}i^!_\bullet\mathcal{J}$ to be $\mathrm{cone}(\mathcal{J}_\bullet \to j_{*\bullet}\mathcal{J}|_{U_\bullet})$. Then for any free resolution $G \xrightarrow{\sim} k\{*\}|_{\widehat{P}-\hat{1}}$ we construct a quasi-isomorphism*

$$j_*\mathcal{J}|_{X - \bigcup_{p \in P} Z_p} \xrightarrow{\sim} \mathrm{Lcolim}_P\, j_{*\bullet}\mathcal{J}|_{U_\bullet}$$

and a zig-zag of quasi-isomorphisms

$$j_*\mathcal{J}|_{X - \bigcup_{p \in P} Z_p} \xrightarrow{\sim}\xleftarrow{\sim} S(\hat{1}) \otimes^{\mathrm{L}}_{\widehat{P}} i_{*\bullet}i^!_\bullet\mathcal{J}.$$

Thus for any object $C_\bullet$ of $\mathrm{Ch}(\mathrm{Mod}\, k^{\widehat{P}})$ that is isomorphic to $\pi_* i^!_\bullet \mathcal{J}$ in $\mathbf{D}(\mathrm{Mod}\, k^{\widehat{P}})$ we have $H^i(S(\hat{1}) \otimes^{\mathrm{L}}_{\widehat{P}} C_\bullet) \cong H^i(X - \bigcup_{p \in P} Z_p, \mathcal{F})$.

Proof. For the first quasi-isomorphism, we show that the representation defined by $p \mapsto j_{*p}\mathcal{J}|_{U_p}$, $p \in P$ and $1 \mapsto \mathcal{J}$ is exact. Since $\mathrm{B}(n)$ is the free meet lattice, have a surjection from $\mathrm{B}(n) \to \widehat{P}$ where n is the number of atoms of $\widehat{P}$. Proposition 4.1.24 reduces its exactness to that of its pullback to $\mathrm{B}(n)$, and this is the case we showed in Proposition 4.2.6. Fix a free resolution $G \xrightarrow{\sim} k\{*\}_P$. Exactness implies that the map $\varphi : G \otimes_{\widehat{P}} j_{*\bullet}\mathcal{J}|_{U_\bullet} \to \mathrm{colim}_P\, j_{*\bullet}\mathcal{J}|_{U_\bullet} \to j_*\mathcal{J}|_U$ is a quasi-isomorphism.

Let $F = \mathrm{cone}(G \to kP(-,1))$. The map $F \xrightarrow{\sim} S(\hat{1})$ gives a free resolution of $S(\hat{1})$. We have that

$$F \otimes_{\widehat{P}} j_{*\bullet}\mathcal{J}|_{U_\bullet} = \mathrm{cone}(G \otimes_{\widehat{P}} j_{*\bullet}\mathcal{J}|_{U_\bullet} \to kP(-,1) \otimes_{\widehat{P}} j_{*\bullet}\mathcal{J}|_{U_\bullet}) \xrightarrow{\sim} \mathrm{cone}(j_*\mathcal{J}|_U \to 0) = j_*\mathcal{J}|_U[1].$$

[2] Alternately it comes from a constant representation of the boolean poset of subsets containing i, and so is exact.

Therefore

$$F \otimes_{\widehat{P}} i_{*\bullet} i^!_{\bullet} \mathcal{J} = \mathrm{cone}(F \otimes_{\widehat{P}} \mathcal{J}_\bullet \to F \otimes_{\widehat{P}} j_{*\bullet} \mathcal{J}|_{U_\bullet})[-1] \xleftarrow{\sim} \mathrm{cone}(0 \to j_* j^* \mathcal{J}[1])[-1] = j_* \mathcal{J}|_U.$$

The quasi-isomorphism $\xleftarrow{\sim}$ follows because the derived tensor product of $S(\hat{1})$ with any constant $\widehat{P}$ representation is zero. □

Example 4.2.8 (Goresky–MacPherson formula)**.** Suppose $Y = \mathbb{R}^n$, the closed subsets Z_p are linear subspaces, of codimension r_p, and $\mathcal{F}$ is the constant sheaf $\mathbb{Z}$. Then by Theorem 4.2.7, $\mathrm{R}\pi_* \mathbb{Z}|_U \simeq S(\hat{1}) \otimes^{\mathrm{L}}_{\widehat{P}} C_\bullet$, where the chain complex of $\widehat{P}$ representations $C_\bullet$, has cohomology $H^\bullet(C_p) \simeq H^\bullet(\mathbb{R}^n, \mathbb{R}^n - Z_p, \mathbb{Z}) = \mathbb{Z}[-r_p]$.

We claim that $C_\bullet$ is isomorphic to a direct sum of its homology groups in the derived category. Filter C by rank, so that $(F_i C)_p = C_p$ if $r_p \le i$ and $(F_i C)_p = 0$ otherwise. Then $F_i C / F_{i-1} C \cong \bigoplus_{p,\ r_p = i} \mathbb{Z}[-r_p]$. For $i > j$ we have that $\mathrm{Ext}^1_{\widehat{P}}(F_i C / F_{i-1} C, F_i C / F_{i-1} C) = 0$. Therefore the filtration splits and the claim follows. Thus for $i > 0$ we have

$$H^i(U, \mathbb{Z}) \;=\; H^i(S(\hat{1}) \otimes^{\mathrm{L}}_{\widehat{P}} \bigoplus_{p \in \widehat{P}} \mathbb{Z}[-r_p]) \;=\; \bigoplus_{p \in \widehat{P}} \mathrm{Tor}^{\widehat{P}}_{-i+r_p}(S(\hat{1}), S(p))$$

Here $\mathrm{Tor}^{\widehat{P}}_d(S(\hat{1}), S(p)) = H_d(S(\hat{1}) \otimes^{\mathrm{L}} S(p))$. By Proposition 4.1.17, $\mathrm{Tor}^{\widehat{P}}_d(S(\hat{1}), S(p)) = \widetilde{H}_{d-2}((p, 1), \mathbb{Z})$, and we obtain the classical statement [43]. □

Now we construct the spectral sequences that we will use to study configuration space. First, we recall the definition of relative cohomology.

Definition 4.2.9 (Relative cohomology)**.** Let X be a topological space, U an open subset and $\mathcal{F}$ a sheaf on X. For any flabby resolution $\mathcal{F} \xrightarrow{\sim} \mathcal{J}$ we define the *relative cohomology* $H^i(X, U, \mathcal{F})$ to be $H^i(\pi_* \,\mathrm{cone}(\mathcal{J} \to j_* \mathcal{J}|_U)[-1])$. □

Theorem 4.2.10. *Let $(X, \mathcal{F}, \widehat{P}, \{Z_p\}_{p \in \widehat{P}})$ be an object of $\mathcal{X}$. Then we construct spectral sequences*

$$E_1^{i,-j} = \bigoplus_{s,t,\ s-t=i-j} \; \bigoplus_{p \in \widehat{P},\ 2r(p)=i-s} H_t(\mathrm{Lcolim}_P H^s(U_p, \mathcal{F})) \implies H^{i-j}(X - \cup_{p \in P} Z_p, \mathcal{F})$$

and

$$E_1^{i,-j} = \bigoplus_{s,t,\ s-t=i-j} \ \bigoplus_{p\in P,\ 2r(p)=i-s} \mathrm{Tor}_t(S(\hat{1}), H^s(X, U_p, \mathcal{F})) \implies H^{i-j}(X - \cup_{p\in P} Z_p, \mathcal{F}),$$

where by convention, in the second spectral sequence we have $H^s(X, U_1, \mathcal{F}) := H^s(X, \emptyset, \mathcal{F}) = H^s(X, \mathcal{F})$. *These spectral sequences are natural in the sense that they define a functor from* $\mathcal{X}$ *to spectral sequences of* k *modules.*

Proof. To obtain strict functoriality we use Godement's canonical flabby resolution of $\mathcal{F}$, denoted $C(\mathcal{F}, X)$ [20]. To compute derived tensor products we use the reduced bar resolution $\overline{B}(S(\hat{1}))$ and the shift of its truncation that resolves $\mathbb{Z}P$, which we denote by $\overline{B}(\mathbb{Z}P)$. We form the chain complex of $\widehat{P}$ representations $p \mapsto \pi_* C(\mathcal{F}, X)|_{U_p}$, which models $\mathrm{R}\pi_*(\mathcal{F}|_{U_p})$ because restrictions of flabby sheaves are flabby. Then applying the construction of Proposition 4.2.5 to the constructions of Theorem 4.2.7, we obtain functorial quasi-isomorphisms

$$\overline{B}(\mathbb{Z}P) \otimes_{\widehat{P}} \pi_* C(\mathcal{F}, X)|_{U_p} \xrightarrow{\sim} \pi_*(C(\mathcal{F}, X)|_U)$$

and

$$\overline{B}(S(\hat{1})) \otimes_{\widehat{P}} \mathrm{cone}(C(\mathcal{F}, X) \to \pi_*(C(\mathcal{F}, X)|_{U_\bullet}) \xrightarrow{\sim} \pi_*(C(\mathcal{F}, X)|_U).$$

These quasi-isomorphisms give functorial spectral sequences by Proposition 4.1.38. □

4.2.3 Local Cohomology

Let X be a topological space, $i : Z \to X$ be a closed subset and $j = \bar{i} : U \to Z$ its open complement. In this subsection, we give the background we need on $H^\bullet(X, U, \mathcal{F})$, called the local cohomology of Z in X.

The functor $i^! : \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$ is often defined by

$$i^!(\mathcal{F})(V \cap Z) = \{s \in \mathcal{F}(V) \mid s \text{ is supported on } Z\}.$$

We will use an alternate definition of $i^!$ for chain complexes, from which the properties that we need follow easily. Recall that for any $\mathcal{F} \in \mathrm{Ch}(\mathrm{Sh}(X))$ there is a natural map $\mathcal{F} \to j_* j^* \mathcal{F}$.

Definition 4.2.11. Let $s : S \hookrightarrow X$ be the inclusion of a topological subspace of X. We define $s^! : \mathrm{Ch}(\mathrm{Sh}(X)) \to \mathrm{Ch}(\mathrm{Sh}(S))$ by

$$s^! \mathcal{F} = \mathrm{cone}[-1](s^* \mathcal{F} \to s^* \overline{s}_* \overline{s}_* \mathcal{F}),$$

where $\overline{s} : X - S \to X$ denotes the inclusion of the complement of s.

We say that $s^! \mathcal{F}$ is a *model for* $\mathrm{R}s^!$ if the complex of sheaves $\mathcal{F}$ is injective or flabby. If we take the cohomology of the global sections of a model of $\mathrm{R}s^! \mathcal{F}$ we get the relative sheaf cohomology of $X - S, X$:

$$H^i(\mathrm{R}\pi_* \mathrm{R}i^! \mathcal{F}) = H^i(X, X - S, \mathcal{F}).$$

□

We use the following version of the Kunneth formula:

Proposition 4.2.12 (Kunneth for local cohomology)**.** *Let X, X' be locally contractible topological spaces, let $\mathcal{F}, \mathcal{F}'$ sheaves of k modules , and let Z, Z' be closed subsets with inclusions i, i' and complements U, U'. On $T \times T'$ we have the closed subset $Z \times Z'$. Then* $\mathrm{R}(i \times i')^!(\mathcal{F} \boxtimes \mathcal{F}') \simeq (\mathrm{R}i^! \mathcal{F}) \boxtimes (\mathrm{R}i'^! \mathcal{F}'$. *In particular for k a field, then we have that* $H^\bullet(X \times X', X \times X' - Z \times Z', \mathcal{F} \boxtimes \mathcal{F}') = H^\bullet(X, U, \mathcal{F}) \otimes H^\bullet(X', U', \mathcal{F}')$

Proof. We have that $X \times X' - Z \times Z' = (U \times X') \cup (X \times U')$. So by Mayer-Vietoris and the Kunneth formula[3], we have an exact sequence:

$$\mathrm{R}j_*(\mathcal{F} \boxtimes \mathcal{F}')|_{X \times X' - Z \times Z'} \to (\mathrm{R}j_* \mathcal{F}|_U) \boxtimes \mathcal{F}' \oplus \mathcal{F} \boxtimes (\mathrm{R}j_* \mathcal{F}'|_{U'}) \to (\mathrm{R}j_* \mathcal{F}|_U) \boxtimes (\mathrm{R}j_* \mathcal{F}'|_{U'}),$$

so that

$$R(i \times i')^!(\mathcal{F} \boxtimes \mathcal{F})'[1] \simeq \mathrm{Tot}\left(\mathcal{F} \boxtimes \mathcal{F}' \to (\mathrm{R}j_* \mathcal{F}|_U) \boxtimes \mathcal{F}' \oplus \mathcal{F} \boxtimes (\mathrm{R}j_* \mathcal{F}'|_{U'}) \to (\mathrm{R}j'_* \mathcal{F}|_U) \boxtimes (\mathrm{R}j_* \mathcal{F}'|_{U'})\right)$$

[3]Here we use that X, X' are locally contractible to apply Kunneth.

$$= \mathrm{cone}(\mathcal{F} \to \mathrm{R}j_*\mathcal{F}|_U) \boxtimes \mathrm{cone}(\mathcal{F} \to \mathrm{R}j'_*\mathcal{F}'|_{U'}) = (\mathrm{R}i^!\mathcal{F}[1]) \boxtimes (\mathrm{R}i'^!\mathcal{F}'[1])$$

$\square$

It is also important, that $i^!$ is determined locally.

Proposition 4.2.13 (Localization)**.** *Let $V \subset X$ be an open subset and write $i|_V : Z \cap V \to V$ for the inclusion. Then $(\mathrm{R}i^!\mathcal{F})|_V = \mathrm{R}i^!|_V(\mathcal{F}|_V)$.*

Lastly, to restate the vanishing criterion, we use a Kunneth formula for the dualizing complex of a space:

Proposition 4.2.14 (Kunneth for Dualizing Complexes)**.** *Let $i_X : X \to \mathbb{R}^n$ and $i_Y : Y \to \mathbb{R}^m$ be two locally contractible topological spaces together with closed embeddings into euclidean space, j_X, j_Y the embeddings of their open complements. Let V, W be free k modules. We write $\pi_X : X \to *$ and $\pi_Y : Y \to *$ for the projections to a point. Then $\pi_X^! V \simeq Ri_X^! \underline{V}[n]$ and $\pi_Y^! W \simeq Ri_Y^! W[m]$ are isomorphic complexes in the derived category. And we have that $(\pi_X \times \pi_Y)^!(V \otimes W) \simeq \pi_X^! V \boxtimes \pi_Y^! W$*

Proof. By [24], Chapter 3, for any closed embedding $Z \hookrightarrow \mathbb{R}^l$, and vector space U, we have $\mathrm{R}\pi_X^! U \simeq \mathrm{R}i_Z^! \pi_{\mathbb{R}^m}^! U \simeq \mathrm{R}i_Z^! U[l]$. So for the closed embedding $Z = X \times Y \hookrightarrow \mathbb{R}^n \times \mathbb{R}^m$ we have that $\pi_{X\times Y}^! V \otimes W = \mathrm{R}(i_X^\times i_Y)^! V[n] \otimes W[m] = \mathrm{R}i_X^! V[n] \boxtimes \mathrm{R}i_Y^! W[m]$ by 4.2.12. $\square$

Proposition 4.2.15. *Let X be a locally contractible topological space, embeddable in $\mathbb{R}^l$, with diagonal embedding $\Delta_n : X \to X^n$. Then $\mathrm{R}\Delta_n^! \underline{k} = {\omega_X^*}^{\otimes n-1}$.*

Proof. Proposition 4.2.14 states that $\omega_{X^n} = \omega_X^{\boxtimes n}$. Verdier duality intertwines $f^!$ and f^* [24], so we have $\Delta^! \underline{k}^{\boxtimes n} = \mathrm{R}\underline{Hom}(\ \mathrm{R}\Delta^*\ \mathrm{R}\underline{Hom}(\underline{k}^{\boxtimes n}, \omega_{X^n}), \omega_X)$. Therefore we have $\mathrm{R}\Delta_n^! \underline{k} = \underline{Hom}(\omega_X^{\otimes n}, \omega_X) = (\omega_X^*)^{\otimes n-1}$. $\square$

Chapter 5

Configuration spaces

5.1 Results

Let X be a topological space and k be a field. In this chapter, we study the behavior of $H^\bullet(\mathrm{Conf}_n(X), k)$, the cohomology of configurations of n ordered points in X, for $n \gg 0$. We use the theory of **FI** modules, following the approach developed by Church, Ellenberg, and Farb [5]. **FI** is the category of finite sets and injections, and an **FI** module is a functor from **FI** to the category of k modules. An injection $[m] \hookrightarrow [n]$ yields a map $\mathrm{Conf}_m(X) \leftarrow \mathrm{Conf}_n(X)$ by forgetting and relabeling the points. This gives the cohomology of configuration space the structure of an **FI** module, $n \mapsto H^i(\mathrm{Conf}_n(X), k)$.

When M is an orientable manifold of dimension ≥ 2, Church, Ellenberg, and Farb proved that $H^i(\mathrm{Conf}_\bullet(M), \mathbb{Q})$ is a finitely generated **FI** module. The main purpose of this paper is to extend the finite generation of $H^i(\mathrm{Conf}_\bullet(X), k)$ to topological spaces X that are in some sense ≥ 2 dimensional.

It is important to show that $H^i(\mathrm{Conf}_\bullet(X), k)$ is finitely generated because finitely generated **FI** modules exhibit uniform behavior for large n. As shown in [5], finite generation of $H^i(\mathrm{Conf}_\bullet(X), \mathbb{Q})$ implies that the character of $H^i(\mathrm{Conf}_n(X), \mathbb{Q})$ as an $\mathbf{S}_n$ representation agrees with a character polynomial for $n \gg 0$. Thus finite generation is closely related to the representation stability results of [4, 8]. Taking the multiplicity of the trivial representation recovers the classical homological stability of unordered configuration space

[31, 39]. In positive characteristic, the work of Nagpal shows that finite generation of $H^i(\mathrm{Conf}_\bullet(X), \mathbb{F}_p)$ as an **FI** module implies periodicity in the cohomology of unordered configuration space [32].

First, we give an explicit criterion for the sheaf cohomology $H^i(\mathrm{Conf}_\bullet(X), k)$ to be a finitely generated **FI** module in terms of the vanishing of the relative cohomology of $(X^n, X^n - \Delta X)$.

Theorem 5.1.1 (Criterion for finite generation)**.** *Let k be a field and let X be a Hausdorff, connected, locally contractible topological space with $H^i(X, k)$ finite dimensional for $i \geq 0$. Let $c \in \mathbb{N}$. Write $\Delta : X \to X^n$ for the diagonal embedding. Suppose the following conditions hold:*

1. *For $n \geq 2$ and $i < n$, we have $H^i(X^n, X^n - \Delta X, k) = 0$,*
2. *For $n \gg 0$, we have $H^i(X^n, X^n - \Delta X, k) = 0$ for $i \leq n + c$,*

Then $H^i(\mathrm{Conf}_n(X), k)$ is a finitely generated **FI** *module for all $i < c$.*

Notice for $X = \mathbb{R}^d$ we have that $H^\bullet(\mathbb{R}^{dn}, \mathbb{R}^{dn} - \Delta\mathbb{R}^d, k)$ equals k in degree $(d-1)n$, thus $\mathbb{R}^d$ satisfies conditions (1) and (2) exactly when $d \geq 2$. When $d = 1$, we have $H^0(\mathrm{Conf}_n\mathbb{R}^1, k) = k[\mathbf{S}_n]$, which grows too quickly to be a finitely generated **FI** module.

Throughout this paper, we take $H^i(Y, k)$ to mean the sheaf cohomology of Y, which agrees with singular cohomology whenever Y is locally contractible [40]. When X is Hausdorff and locally contractible, $Y = \mathrm{Conf}_n(X)$ is also locally contractible.

Interpreting $H^i(X^n, X^n - \Delta X, k)$ as the hypercohomology of a complex of sheaves gives a corollary of Theorem 5.1.1 that is local on X. If X has an open cover by spaces of the form $V \times \mathbb{R}^2$ then the Kunneth formula and the vanishing of $H^i(\mathbb{R}^{2n}, \mathbb{R}^{2n} - \mathbb{R}^2, k)$ together imply that X satisfies this local criterion:

Corollary 5.1.2. *Let X be a Hausdorff, connected, locally contractible topological space with $H^i(X, k)$ finite dimensional. If every point p has a neighborhood $U_p \cong V_p \times \mathbb{R}^2$ for some space V_p, then $H^i(\mathrm{Conf}_\bullet(X), k)$ is a finitely generated* **FI** *module for all i.*

The following examples satisfy the hypotheses of Corollary 5.1.2. We believe they are new.

Example 5.1.3. Let W be connected topological space with finite dimensional cohomology. Suppose that W is either a topologically stratified space[42], or a Whitney stratified spacce [21]. If all the strata of W are ≥ 2 dimensional, then $H^i(\mathrm{Conf}_\bullet(W), k)$ is a finitely generated **FI** module. $\square$

Example 5.1.4. Let Y be a connected CW complex with finitely many cells. Then $H^i(\mathrm{Conf}_n(Y \times \mathbb{R}^2), k)$ is a finitely generated **FI** module. $\square$

The relative cohomology of $(X^n, X^n - \Delta X)$ can be expressed in terms of the Verdier dual of the tensor powers of ω_X, where ω_X is the dualizing complex of X, whenever the hypotheses of Verdier duality are satisfied [24]. Under this reformulation, if ω_X is concentrated in homological degree ≥ 2, then the two vanishing conditions of Theorem 5.1.1 follow. Further, the homology of the stalk of ω_X at p is $\widetilde{H}_{i-1}(U - p, k)$, where U is a contractible neighborhood of p. From this we obtain a criterion for finite generation that depends on whether, locally, removing a point disconnects X.

Corollary 5.1.5. *Let k be a field, and let X be a connected locally contractible topological space with $H^i(X, k)$ finite dimensional, and $H^r_c(X, k) = 0$ for $r \gg 0$. Assume that X admits a closed embedding into $\mathbb{R}^n$. If ω_X is supported in homological degree ≥ 2, then $H^i(\mathrm{Conf}_\bullet(X), k)$ is a finitely generated* **FI** *module for all i.*

Recall that the dualizing complex of a manifold M is its orientation sheaf, concentrated in homological degree $\dim M$. So the hypothesis that ω_X is supported in homological degree ≥ 2 extends the hypothesis that M has dimension ≥ 2.

Corollary 5.1.6. *Let k be a field and let X be a connected, locally contractible, closed subset of $\mathbb{R}^n$ such that $H^i(X, k)$ is finite dimensional and $H^r_c(X, k)$ vanishes for $r \gg 0$. If for all $p \in X$ there is a contractible neighborhood U of p such that $U - p$ is connected, then $H^i(\mathrm{Conf}_\bullet(X))$ is a finitely generated* **FI** *module for all i.*

Notice that ≥ 2 dimensional manifolds satisfy these hypotheses, while 1 dimensional manifolds do not. The following examples are applications of Corollary 5.1.6,

Example 5.1.7. Let G be a connected finite graph, considered as a CW complex. Then $H^i(\mathrm{Conf}_\bullet(G \times \mathbb{R}^1), k)$ is a finitely generated **FI** module for all i. $\square$

Example 5.1.8. Let X consist of two solid balls glued together along a closed interval on their boundary running from the north to south pole, $X = D^3 \cup_{[0,1]} D^3$. Then $H^i(\mathrm{Conf}_\bullet(X), k)$ is a finitely generated **FI** module for all i. $\square$

5.2 Criteria for representation stability

Let k be a field, and let X be a Hausdorff, locally contractible toplogical space. In this section, using the functoriality of the spectral sequence of Theorem 4.2.10, we obtain a spectral sequence of **FI** modules that converges to the cohomology of configuration space with coefficients in k. Then we give a necessary and sufficient criterion for the E_1 page to be finitely generated in cohomological degrees $\leq k$. This yields the criterion for the finite generation of cohomology, Theorem 5.1.1, since [6] shows that finite generation passes along spectral sequences. Along the way, we show that the E_1 page is a free **FI** module and give generators for it.

We define how **FI** acts on the spectral sequences contstructed in Chapter 4, describe the E_1 page as an **FI** module, and prove the criterion for finite generation as Theorem 5.2.13.

First we recall the definition of the category **FI**.

Definition 5.2.1. The category **FI** has objects finite sets, and maps $\mathbf{FI}(a, b) = \{$ injections $a \hookrightarrow b\}$. For k a commutative ring, an **FI** module is a functor from **FI** to the category of k modules. $\square$

The next definition explains how **FI** acts on the partition lattice $\mathrm{P}(n)$.

Definition 5.2.2. Given an injection $f : a \hookrightarrow b$ and a partition $p \in \mathrm{P}(a)$ we define $f_* p$ to be the partition of b such that $\mathrm{im}\, f$ is partitioned by p under its identification with a and such that no $i, j \in b - \mathrm{im}\, f$ are in the same block (i.e. the complement of $\mathrm{im}\, f$ is given the structure of a the discrete partition).

If $p \leq p'$ then clearly $f_* p \leq f_* p'$. Thus this construction defines a functor $\mathrm{P} : \mathbf{FI} \to \mathbf{Poset}$, given by $a \mapsto \mathrm{P}(a)$.

We say that a partition q of $[n]$ is *irreducible* if it is not equal to $f_* p$ for any inclusion $f : [n] \to [m]$ for $n < m$. Equivalently, q is irreducible if and only if every block of q has size ≥ 2.

The lattice $\mathrm{P}(n)$ is graded: we define the *rank of p* be $r(p) = n - \#\text{blocks}$. We have that $r(p) = r(f_* p)$. In other words, the action of **FI** preserves the rank.

For p of $[m]$ and q of $[n]$ there is a partition $p \sqcup q$ of $[m] \sqcup [n]$. □

Next we describe how **FI** acts on powers of X and hence on the spectral sequences from Chapter 6.

Definition 5.2.3 (Action of **FI**)**.** Let X be a topological space, and let $\mathcal{F}$ be a sheaf of k modules on X with a distinguished global section $s : k \to \mathcal{F}$. From this data, we describe a functor $\mathbf{FI} \to \mathcal{X}$.

First note that the n-fold product of X, is the same as the topological space of maps $[n] \to X$. For a partition p of n (given by $n \to m$), write Z_p for the collection of all maps that $n \to X$ that factor through p and U_p for $X^n - Z_p$. Thus our convention is that $Z_{[n]} = X^{[n]}$, and $U_{[n]} = \emptyset$. We also write $\mathcal{F}^{\boxtimes[n]}$ for the nth external tensor power of $\mathcal{F}$, a sheaf on $X^{[n]}$.

The functor $\mathbf{FI} \to \mathcal{X}$ takes $[n]$ to $(X^{[n]}, \{Z_p\}_{p \in \mathrm{P}(n)-1}, \mathrm{P}(n), \mathcal{F}^{\boxtimes n})$. An injection, $g : [n] \hookrightarrow [m]$ gives a projection map $g : X^{[m]} \to X^{[n]}$, and the preimage of Z_p under this map is Z_{gp}, where $g : \mathrm{P}(n) \to \mathrm{P}(m)$ is as described above. Using the natural isomorphism $g^* \mathcal{F}^{\boxtimes[n]} \simeq \mathcal{F}^{\boxtimes \operatorname{im} g} \boxtimes k^{\boxtimes[m]-\operatorname{im} g}$, we have g act by

$$1^{\boxtimes \operatorname{im} g} \boxtimes s^{\boxtimes[m]-\operatorname{im} g} : \quad \mathcal{F}^{\operatorname{im} g} \boxtimes k^{\boxtimes[m]-\operatorname{im} g} \to \mathcal{F}^{\boxtimes[m]}.$$

□

Let $\mathrm{Conf}_n(X)$ be the *ordered configuration space* of n points in X. Then $X^{[n]} - \bigcup_{p \in \mathrm{P}(n)} Z_p = \mathrm{Conf}_n(X)$

Example 5.2.4. The partitions of 3 are

$$\{1\}\{2\}\{3\},\ \{1\}\{23\},\ \{2\}\{13\},\ \{3\}\{12\},\ \text{and}\ \{123\}$$

with $\{1\}\{2\}\{3\} \geq \{12\}\{3\} \geq \{123\}$. If we take the inclusion $f : [3] \to [5]$ with $1 \mapsto 1, 2 \mapsto 5, 3 \mapsto 3$, then $f\{12\}\{3\} = \{15\}\{3\}\{2\}\{4\}$ The irreducible partitions of [4] are $\{12\}\{34\}$ and $\{1234\}$

For X a topological space, we have $Z_{\{12\}\{34\}} = \{(x_1, x_2, x_3, x_4) \in X^4 \mid x_1 = x_2 \text{ and } x_3 = x_4\}$ $\square$

From the functor $\mathbf{FI} \to \mathcal{X}$, and the fact that the partition poset is graded Cohen Macaulay [43] ($\mathrm{Tor}_i(S(\hat{1}), S(p))$ is only nonzero in degree $i = \operatorname{depth} p$), we obtain the following spectral sequence:

Proposition 5.2.5 (Spectral sequence of **FI** modules)**.** *Let k be a field. Then there is a spectral sequence of* **FI** *modules converging to the cohomology of configuration space as an* **FI** *module* $E_1^{i,-j} =$

$$\bigoplus_{s\in\mathbb{Z},\ p\in \mathrm{P}(\bullet),\ s-r(p)=i-j,\ 2r(p)=i-s} H^s(X^\bullet, U_p, \mathcal{F}) \otimes \mathrm{Tor}_t^{\mathrm{P}(\bullet)}(S(\hat{1}), S(p)) \implies H^{i-j}(X^\bullet - \cup_{p\in \mathrm{P}(\bullet)-1} Z_p, \mathcal{F}).$$

In other words, for each $p \in \mathrm{P}(n)$, $s \in \mathbb{Z}$, the term $H^s(X^n, U_p, \mathcal{F}) \otimes \mathrm{Tor}_{r(p)}(S(\hat{1}), S(p))$ appears exactly once on the E_1 page, in the place $E_1^{-i,j} = E_1^{2r(p)+s,-3r(p)}$.

Proof. This spectral sequence is the second spectral sequence of Theorem 4.2.10. Because k is a field,

$$\mathrm{Tor}_t^{\mathrm{P}(n)}(S(\hat{1}), H^s(X^\bullet, U_p, \mathcal{F}) \otimes S(p)) \cong H^s(X^\bullet, U_p, \mathcal{F}) \otimes \mathrm{Tor}_t^{\mathrm{P}(n)}(S(\hat{1}), S(p)).$$

And because $\mathrm{P}(n)$ is graded Cohen–Macaulay, $\mathrm{Tor}_t^{\mathrm{P}(n)}(S(\hat{1}), S(p))$ vanishes except for $t = r(p)$. With these simplifications, we obtain the desired spectral sequence. $\square$

In order to identify the action on the E_1 page, we use the Kunneth formula to describe the following **FI** module:

Proposition 5.2.6. *Let k be a field, and X be Hausdorff and locally contractible. Let $\mathcal{F}$ be a sheaf of k vector spaces on X such that $H^0(X, \mathcal{F}) = k$ with distinguished section x. For any p_0 a partition of n_0 and $d \in \mathbb{N}$, the* **FI** *module $H^d(X, U_{p_0}, \mathcal{F}^{\boxtimes [n]})$ defined by:*

$$n \mapsto \bigoplus_{\{g:[n_0]\hookrightarrow[n]\}/\mathbf{S}_{n_0}} H^d\left(X^{[n]}, U_{gp_0},\ \mathcal{F}^{\boxtimes n}\right),$$

is freely generated as follows: for each $r \leq d$ and integer partition of $d - r = l_1 + \cdots + l_m$, with $l_1 \geq \cdots \geq l_m \geq 1$, the $\mathbf{S}_{m+n_0}$ module

$$\bigoplus_{\{g:[m]\hookrightarrow[m]\sqcup[n_0]\}} H^r(X^{[m]\sqcup[n_0]-\operatorname{im} g}, U_{gp_0}, \mathcal{F}^{\boxtimes[n_0]}) \otimes \bigotimes_{y\in\operatorname{im} g} H^{l_{(g^{-1})y}}(X, \mathcal{F})$$

$$= \operatorname{Ind}_{\mathbf{S}_m\times\mathbf{S}_n}^{\mathbf{S}_{n+m}} \left(H^r(X^{n_0}, X^{n_0} - Z_\lambda, \mathcal{F}^{\boxtimes n_0}) \otimes \bigoplus_{\sigma\in\mathbf{S}_m} H^{l_{\sigma 1}}(X, \mathcal{F}) \otimes \cdots \otimes H^{l_{\sigma m}}(X, \mathcal{F}) \right)$$

generates an **FI** *submodule that is free (i.e. isomorphic to its induction to* **FI***), and these submodules give a direct sum decomposition of $H^d(\mathrm{R}i^!_{p_0}\mathcal{F}^{\boxtimes[n]})$.*

Proof. Note that $X^{[n]} - Z_{g_*p_0} = X^{[n]-\operatorname{im} g} \times (X^{[n_0]} - Z_{p_0})$, so the pair $(X^{[n]}, X^{[n]} - Z_{gp_0}) = (X, X)^{[n]-\operatorname{im} g} \times (X^{[n_0]}, Z_{p_0})$. Thus by the Kunneth formula, Proposition 4.2.12, and since k is a field, we have that

$$\bigoplus_{\{g:\hookrightarrow[n]\}/\mathbf{S}_{n_0}} H^\bullet\left(X^{[n]}, U_{g_*p_0},\ \mathcal{F}^{\boxtimes n}\right) = \bigoplus_{\{g:[n_0]\hookrightarrow[n]\}/\mathbf{S}_{n_0}} H^\bullet(X, \mathcal{F})^{\otimes[n]-\operatorname{im} g} \otimes H^\bullet(X^{n_0}, U_p, \mathcal{F}^{[n_0]})$$

The piece in cohomological degree d is

$$\bigoplus_{\{g:[n_0]\hookrightarrow[n]\}/\mathbf{S}_{n_0}} \quad \bigoplus_{c_t\in\mathbb{N},\ t\in[n]-\operatorname{im} g,\ r\in\mathbb{N},\ r+\sum_t c_t=d} H^r(X^{[n_0]}, U_{p_0}, \mathcal{F}^{\boxtimes n_0}) \otimes \bigotimes_{t\in n-\operatorname{im} g} H^{c_t}(X, \mathcal{F}).$$

An injection $h : n \hookrightarrow m$ acts by mapping the summand corresponding to g, r and $c_t, t \in n - \operatorname{im}(g)$ to the summand corresponding to $h \circ g, r$ and the sequence c_s defined by: $c_s = c_t$ if $s \in \operatorname{im} h - \operatorname{im}(g \circ h)$ and $c_s = 0$ if $s \in [m] - \operatorname{im} h$. Up to a sign, the maps uses the isomorphism $x : k \to H^0(X, \mathcal{F})$ tensored with the action of $\mathbf{S}_{n_0}$ on $H^r(X^{[n_0]}, U_{p_0}, \mathcal{F}^{\boxtimes[n_0]})$. The $\mathbf{S}_m$ module corresponding to $d - r = l_1 + \cdots + l_m$, is the $\mathbf{S}_m$ submodule of the degree $[m] \sqcup [n_0]$ piece which is spanned by the summand corresponding to the canonical $g = [n_0] \hookrightarrow [n_0] \sqcup [m]$ and $c_i = l_i$ for $i \in [n_0] \sqcup [m] - [n_0] = [m]$. We can generate every summand corresponding to the same integer partition padded by zeros. And since the action of **FI** preserves the nonzero part of the partition, the submodules corresponding to these summands intersect trivially. Finally, recall that the free **FI** module on an $\mathbf{S}_m$

module M is $t \mapsto M \otimes_{\mathbf{S}_m} k\{m \hookrightarrow t\} = \bigoplus_{\{m \hookrightarrow t\}/\mathbf{S}_m} M$, so that each submodule spanned is free. □

Remark 5.2.7. In particular, Proposition 5.2.6 describes the $\mathbf{S}_n$ representations appearing on the E_1 page of the spectral sequence. In fact, if we sum over all degrees n and we use the identification of the homology of $\mathrm{P}(n)$ with $\Sigma\,\mathrm{Lie}^*(m)$ given by Theorem 4.1.34 we have

$$\bigoplus_n \oplus_{i,j}(E_1^{i,j})_n \;=\; \bigoplus_n \bigoplus_{p=b_1\sqcup\cdots\sqcup b_r \in \mathrm{P}(n)} \bigotimes_{i=1}^{r} H^\bullet(X^{b_1}, X^{b_1} - X, \mathcal{F}^{\boxtimes b_1}) \otimes (\Sigma\,\mathrm{Lie}^*)(b_r).$$

In the terminology of Chapter 2, this is the k-linear species

$$\mathrm{Com} \circ \left(\bigoplus_{r \geq 0} H^\bullet(X^r, X^r - X, \mathcal{F}^{\boxtimes r}) \otimes (\Sigma\,\mathrm{Lie}^*)(r) \right),$$

since for any linear species M, we have that $(\mathrm{Com} \circ M)(n)$ is given

$$\bigoplus_{n=b_1\sqcup\cdots\sqcup b_r} M(b_1) \otimes \cdots \otimes M(b_r).$$

One can compare this computation to the case $X = \mathbb{R}^n$, from Example 2.1.20. □

It is essential for us that **FI** modules are Noetherian, which implies finite generation passes along spectral sequences. The following theorem appears in [6].

Theorem 5.2.8 (CEFN)**.** *Let k be a Noetherian ring. Then a submodule of a finitely generated* **FI** *module is finitely generated.*

Corollary 5.2.9 (CEFN)**.** *Suppose a spectral sequence of* **FI** *modules $E_1^{p,-q}$, converges to a graded* **FI** *module H^{p-q}. If each $\oplus_{p-q=i} E_1^{p,-q}$ is a finitely generated* **FI** *module for all i, then H^i is finitely generated for all i.*

The following is not a direct consequence of the above theorem but is easier to prove:

Proposition 5.2.10 (CEF)**.** *If M, N are finitely generated* **FI** *modules, then $M \otimes_k N$ is a finitely generated* **FI** *module.*

Proof. By right exactness, it suffices to show that the tensor product $k\mathbf{FI}(n,-) \otimes k\mathbf{FI}(m,-)$ is finitely generated. This is the **FI** module:

$$t \mapsto k\{\text{pairs } (f : m \hookrightarrow t, g : n \hookrightarrow t)\} = \bigoplus_{a \le \min(m,n)} \bigoplus_{\{\text{spans } m \hookleftarrow a \hookrightarrow n\}/\mathbf{S}_a} k\mathbf{FI}(m \sqcup_a n, t),$$

which is finitely generated □

Next we characterize when the degree $\le k$ piece of the spectral sequence is a finitely generated **FI** module, by way of an auxiliary function.

Definition 5.2.11. For any partition p of n, let $\mathrm{van}(n) = \max\{t \mid H^t(X^n, U_p, \mathcal{F}^{\boxtimes n}) = 0\}$. In particular, for the indiscrete partition $[n]$ corresponding to the diagonal embedding we have $\mathrm{van}([n]) = \max\{t \mid H^t(X^n, X^n - X, \mathcal{F}^{\boxtimes n}) = 0\}$. □

Proposition 5.2.12. *Let k be a field and let X be locally contractible and Hausdorff. Then the degree $\le k$ piece of the E_1 page of the* **FI** *module local cohomology spectral sequence in Proposition 5.2.5, $\bigoplus_{i-j\le k} E_1^{i,-j}$ is finitely generated if and only if* $\mathrm{van}([n]) - r([n]) \ge 0$ *and for every n and* $\mathrm{van}([n]) - r([n]) \ge k$ *for $n \gg 0$.*

Proof. As a graded **FI** module, the degree d piece of the E_1 page admits a direct sum decomposition:

$$\bigoplus_{n_0 \in \mathbb{N},\ \{p \in \widehat{P}(n_0) \text{ irreducible}\}/\mathbf{S}_{n_0},\ s \in \mathbb{N},\ s - r(p_0) = d} \left(\bigoplus_{\{f : n_0 \hookrightarrow n\}/\mathbf{S}_{n_0}} H^s(X^{[n]}, U_{f\lambda}, \mathcal{F}^{\boxtimes [n]}) \otimes_k \mathrm{Tor}_{r(\lambda)}(S_{\hat{1}}, S_{f\lambda}) \right).$$

We write $H^s(X^{n_0}, U_{p_0}) \otimes H_\lambda^{\mathrm{P}(n)}$ for the summand corresponding to s, p_0. We claim that the degree d piece is finitely generated if and only if there are only finitely many n_0, p_0, s with p_0 irreducible such that $s - r(p_0) = d$ and $H^s(X^{n_0}, U_{p_0}) \neq 0$. First, if infinitely many are nonzero then we get an infinite direct sum of free modules, and tensoring each with $\mathrm{Tor}_{r(p_0)}(S(\hat{1}), S(f_* p_0))$ gives another infinite sum of free modules by Proposition 5.2.6, and so we get an infinite sum on the E_1 page in degree d, which is not finitely generated. Conversely, we have that $\bigoplus_{\{f : n_0 \hookrightarrow \bullet\}/\mathbf{S}_n} \mathrm{Tor}_{r(\lambda)}(S(\hat{1}), S(f_*\lambda)) \subset H^{r(\lambda)}(\mathrm{Conf}_\bullet \mathbb{R}^2, k)$ is finitely generated. So if only finitely many n_0, p_0, s have $H^s(X^{n_0}, U_{p_0})$ nonzero, then tensoring each with $\bigoplus_{\{f:n_0 \hookrightarrow m\}/S_n} \mathrm{Tor}_{r(\lambda)}(S(\hat{1}), S(f_*\lambda))$ preserves finite generation, by Proposition 5.2.10.

Next, there are only finitely many n_0, p_0, s with p_0 irreducible and such that $s - r(p_0) = d$ and $H^s(X^{n_0}, U_{p_0}) \neq 0$ iff there are only finitely many finite natural number sequences $c_1 \geq \cdots \geq c_m \geq 1$ and $b \in \mathbb{N}$, p_0 a partition of n_0 such that $b - r(p_0) + \sum c_i = k$ and $H^{c_1}(X, \mathcal{F}) \otimes \cdots \otimes H^{c_m}(X, \mathcal{F}) \otimes H^b(X^{n_0}, U_{p_0}, \mathcal{F}) \neq 0$. This follows from the fact that $H^s(X^{n_0}, U_{p_0}) \neq 0$ iff there is some choice of $c_1 \geq \cdots \geq c_m \geq 1$ and b as before, and for each s, p_0, n_0 there are at most finitely many choices of c_i, b such that $b + \sum c_i - r(p_0) = k$.

Thus we have that the E_2 page is finitely generated in cohomological degree d for all $d \leq k$ iff there are only finitely many $c_1 \geq \cdots \geq c_m \geq 1, b, p_0$ such that $b - r(p_0) + \sum_i c_i \leq k$ and $H^{c_1}(X, \mathcal{F}) \otimes \cdots \otimes H^{c_m}(X, \mathcal{F}) \otimes H^b(X^{[n_0]}, U_{p_0}\mathcal{F}^{\boxtimes[n_0]} \neq 0$. This is true iff there are only finitely many b, p_0 such that $b - r(p_0) \leq k$ and $H^b(X^{[n_0]}, U_{p_0}\mathcal{F}^{\boxtimes[n_0]} \neq 0$, because each sequence c_i, b, p_0 gives such a b, p_0, each b, p_0 arises in this way, and for each b, p_0 there are only fintitely many positive sequences c_i that satisfy $b - r(p_0) + \sum_i c_i \leq k$.

For each p_0, there is a b such that $b - r(p_0) \leq k$ and $H^b(X^{[n_0]}, U_{p_0}, \mathcal{F}^{\boxtimes[n_0]}) \neq 0$ if and only if $\mathrm{van}(p_0) \leq k$, and for a fixed p_0 there at most finitely many $b \geq 0$ with $b \leq k + r(p_0)$. So the E_1 page is finitely generated in degree $\leq k$ iff there are finitely many irreducible partitions p_0 with $\mathrm{van}(p_0) - r(p_0) \leq k$.

Now for partitions p, q of a, b we have that $U_{p \sqcup q} = X^a \times X^b - Z_p \times Z_q$ so the Kunneth formula for local cohomology shows that $\mathrm{van}(p \sqcup q) = \mathrm{van}(p) + \mathrm{van}(q)$. Every irreducible parition p_0 is the disjoint union of discrete partitions $[n]$ for $n \geq 2$. So if we have $\mathrm{van}([n]) - r([n]) \leq k$ for only finitely many n, and $\mathrm{van}([n]) - r([n]) \geq 1$ for $n \geq 2$, then there are only finitely many ways of combining the blocks $[n]$, $n \geq 2$ to get an irreducible partition p_0 with $\mathrm{van}(p_0) - r(p_0) \leq k$. Conversely if there is an $[n]$ with $\mathrm{van}([n]) - r([n]) \leq 0$, then $\sqcup_{i=1}^M [n]$ gives an infinite sequence of irreducible partitions with $\mathrm{van}(p_0) - r(p_0) \leq k$, and if there are infinitely many n with $\mathrm{van}([n]) - r([n]) \leq k$ then there are trivially infinitely many irreducible partitions p_0 with $\mathrm{van}(p_0) - r(p_0) \leq k$. $\square$

Proposition 5.2.12 and 5.2.8 together imply our main theorem about configuration spaces, which gives a criterion for $H^{\leq c}$ to be a finitely generated **FI** module. Here we write it in terms of the auxiliary function $\mathrm{van}(n)$.

Theorem 5.2.13 (Criterion for representation stability)**.** *Let X be a Hausdorff, locally contractible topological space and $\mathcal{F}$ a sheaf of k vector spaces on X such that $H^0(X,\mathcal{F}) = k$ and $H^i(X,\mathcal{F})$ is finite dimensional for $i > 0$. If*

1. $\operatorname{van}(n) \geq n-1$ *for* $n \geq 2$ *and*
2. $\operatorname{van}(n) \geq n+c$ *for* $n \gg 0$*,*

then $H^i(\mathrm{Conf}_\bullet(X),\mathcal{F})$ is a finitely generated **FI** *module for all $i < c$.*

We give the proofs of corollaries from the introduction:

Proof of Corollary 5.1.2. Let Δ_n^X be the nth diagonal embedding of X. Then $H^i(X^n, X^n - \Delta_n^X X, k) = H^i(\mathrm{R}\pi_* \Delta_n^{X!} k)$. If a complex of sheaves C has homology (as a complex of sheaves) supported in cohomological degree $\geq i$, then so does $\mathrm{R}^i\pi_* C$. Thus it suffices for $\mathrm{R}\Delta_n^{X!}k$ to be supported in cohomological degree $\geq n$ when $n \geq 2$, and to have the range where it vanishes grow faster than $n+c$ for any constant c. Since $\mathrm{R}\Delta_n^! k$ is a complex of sheaves, this can be checked locally, and we have by the Kunneth formula, Proposition 4.2.12 and locality of $\mathrm{R}i^!$, Proposition 4.2.13, that $\mathrm{R}\Delta_n^{X!}k|_{U_p} = \mathrm{R}\Delta_n^{V_p!}k \boxtimes \mathrm{R}\Delta_n^{\mathbb{R}^2!}k$. But for $\mathbb{R}^2$ we have the diagonal embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^{2n}$, so taking the cone from the pushforward of the complement $\mathbb{R}^{2n-2}$ we see that $\mathrm{R}\Delta_n^{\mathbb{R}^2!}k = k[-(2n-2)]$ is concentrated in cohomological degree $\geq 2n-2$. So by the external tensor product, so is $\mathrm{R}\Delta_n^{X!}k|_{U_p}$. □

Proof of Corollary 5.1.6. Proposition 4.2.15 shows that $H^i(X^n, X, k) = H^i(\mathrm{R}\pi_* \mathrm{R}\operatorname{Hom}(\omega_X^{\otimes n}, \omega_X)$, for X locally contractible and embeddable into euclidean space. As in the proof of Corollary 5.1.2, to satisify the hypotheses of Theorem 5.1.1, it suffices for $\mathrm{R}\operatorname{Hom}(\omega_X^{\otimes n}, \omega_X)$ to be concentrated in cohomological degree $\geq 2n-2$, and thus for ω_X to be concentrated in homological degree ≥ 2.

The complex ω_X is concentrated in homological degree ≥ 2 if and only if every stalk of ω_X has homological degree ≥ 2. For a contractible neighborhood $j : U \to X$ of p we have that $i_p^* \pi_X^! k = i_p^* \pi_U^! k$, and for the inclusion $l : U - p \to U$ we have the triangle

$$l_! l^! \pi_U^! k \to \pi_U^! k \to i_! i^* \pi_U^! k,$$

and applying $\pi_{U!}$ we get a triangle involving the homology of $U - p$, the homology of U and $H_\bullet(i_p^* \omega_X)$. So we see that $H_i(i_p^*\omega_X) = \widetilde{H}_{i-1}(U-p, k)$. Thus if $U - p$ is connected, the homology of the stalk vanishes in homological degrees < 2. $\square$

Chapter 6

Moduli Spaces

6.1 Introduction to Moduli Spaces of Curves

The space $M_{g,n}$ is the moduli space of genus g complex curves, with n marked points. Informally this means that as a set

$$M_{g,n} = \frac{\{C \text{ complex genus } g \text{ curve}, p_1, \ldots, p_n \in C\}}{\{\text{ isomorphisms } C \xrightarrow{\sim} C' \text{ preserving } p_1, \ldots, p_n\}},$$

and $M_{g,n}$ is equipped with a topology so that as we vary the complex structure on C or move the marked points $p_1, \ldots, p_n$ we trace out a continuous path in $M_{g,n}$.

For many of our arguments, this heuristic picture, together with the belief that natural geometric operations on curves define continuous maps, is sufficient. To help establish this belief, we give a short introduction to the functor of points approach to $M_{g,n}$.

6.1.1 Functor of Points and Homology

One way to construct the topology on $M_{g,n}$ is to give it the structure of an algebraic variety. In this chapter, all algebraic varieties will be considered over $\mathbb{C}$, and homology will be the homology of the underlying topological space with $\mathbb{Q}$ coefficients. To characterize this structure, we can attempt to define what a map from an algebraic variety (or scheme) $X \to M_{g,n}$ is.

Definition 6.1.1. Given an scheme X, we define

$$M_{g,n}(X) = \frac{\{C \to X \text{ flat map with fibers } C_x \text{ smooth genus } g \text{ curves; } p_1, \ldots, p_n : *_X \to C \text{ markings}\}}{\{\text{ isomorphisms } C \xrightarrow{\sim} C' \text{ preserving marked points}\}},$$

where $*_X$ is the terminal scheme over X, i.e. $X \to X$, and $p_1, \ldots, p_n$ are maps of schemes over X. This construction defines a functor from schemes to sets, which we refer to as the *set-valued moduli problem of genus g curves and n marked points*. □

An element of $M_{g,n}(X)$ in particular gives a map of sets, $X \to M_{g,n}$. The map is defined by associating to each $x \in X$ the fiber C_x lying above it and the markings $p_1(x), \ldots, p_n(x)$. We can think of the sets map $f : X \to M_{g,n}$ that arise from $f \in M_{g,n}(X)$ as those maps which are algebraic.

The Yoneda lemma implies that if there were a scheme $Y_{g,n}$ together with a natural isomorphism $M_{g,n}(X) \simeq \mathrm{Sch}(X, Y_{g,n})$, then $Y_{g,n}$ would be uniquely characterized by this property. Unfortunately there is not such a scheme, but there is an algebraic variety $M_{g,n}$ which is the best approximation to the set-valued moduli problem, and this property uniquely characterizes $M_{g,n}$.

The variety $M_{g,n}$ is not smooth, essentially because we have modded out by isomorphisms. There is an enhancement of $M_{g,n}$ to a groupoid-valued moduli problem $\mathcal{M}_{g,n}$ such that $\mathcal{M}_{g,n}$ is smooth in some sense.

Definition 6.1.2. We define $\mathcal{M}_{g,n}(X)$ to be the groupoid whose objects are

$$\{C \to X \text{ flat map with fibers smooth genus } g \text{ curves } p_1, \ldots, p_n : *_X \to C\},$$

and whose morphisms are isomorphisms $C \to C'$ that preserve the marked points. This construction defines a functor from the category of schemes to the category of groupoids.[1]

□

[1] More precisely, this is a 2-functor to the 2-category of groupoids, which just means that if f_1, f_2 are maps of schemes, then there are natural isomorphisms $\alpha(f_1, f_2) : \mathcal{M}_{g,n}(f_1 f_2) \simeq \mathcal{M}_{g,n}(f_1)\mathcal{M}_{g,n}(f_1)$, which satisfy a compatibility condition (the pentagon identity). Any such 2-functor replaced by an equivalent one satisfying $\alpha(f_1, f_2) = \mathrm{id}$. So for our purposes we will ignore this distinction. This abuse is analogous to ignoring the associator in a monoidal category $\mathcal{A}, \otimes$, which we did implicitly in Chapter2.

There is an embedding of the category of schemes Sch into a larger category DM of Deligne–Mumford stacks, such that a map between two Deligne–Mumford stacks is a groupoid.[2] And there is an object $\mathcal{M}_{g,n} \in \mathrm{DM}$ and a natural equivalence $\mathrm{DM}(X, \mathcal{M}_{g,n}) \simeq \mathcal{M}_{g,n}(X)$, for all schemes X. This property uniquely characterizes the Deligne–Mumford stack $\mathcal{M}_{g,n}$. We say that $\mathcal{M}_{g,n}$ *represents* the functor.

Although $\mathcal{M}_{g,n}$ may seem esoteric, it is actually easier to make naive arguments about it than about $M_{g,n}$, because we know the universal property it satisfies. Even better, our focus is on the rational cohomology of $\mathcal{M}_{g,n}$ and $M_{g,n}$, and there is no difference between them.

Proposition 6.1.3. *[1][Proposition 36] For any Deligne–Mumford stack $\mathcal{X}$ with coarse moduli space X there is a natural map $\mathcal{X} \to X$, which induces an isomorphism on $H_i(\mathcal{X}, \mathbb{Q}) \xrightarrow{\sim} H_i(X, \mathbb{Q})$.*

Notice, we have not defined $H_i(\mathcal{X}, \mathbb{Q})$ here, so this theorem can be taken as a definition.[3] We have also not formally defined the notion of a coarse moduli space. It suffices to know that the coarse moduli space of $\mathcal{M}_{g,n}$ is $M_{g,n}$. In the next section, we will define the stack $\overline{\mathcal{M}}_{g,n}$; its coarse moduli space is $\overline{M}_{g,n}$.

The idea of the proof of the Proposition is that $\mathcal{X}$ and X differ only in that points of $x \in \mathcal{X}$ can have a finite automorphism group. Rationally, the homology of a finite group is the same as the homology of a point $H_i(\mathrm{Aut}(x), \mathbb{Q}) = H_i(*, \mathbb{Q})$.

6.1.2 Deligne–Mumford Compactifications

The space $\overline{\mathcal{M}}_{g,n}$ is a natural compactification of the moduli space $\mathcal{M}_{g,n}$ of smooth curves with n marked points, obtained by allowing families of smooth curves to degenerate to singular curves with double points.

Definition 6.1.4. Let X be algebraic variety. We say that $x \in X$ is a *double point* if a formal neighborhood of x is isomorphic to the formal neighborhood of two lines intersecting transversely, i.e. if there is an isomorphism of local rings $\widehat{\mathcal{O}}_x \cong \mathbb{C}[[u, v]]/uv$. □

[2]Thus, DM is really a 2-category.

[3]The cohomology of an algebraic stack $\mathcal{X}$ can be defined by covering $\mathcal{X}$ by a variety $Z \to \mathcal{X}$, and building a bicomplex out of the simplicial chain complex $n \mapsto C^*(Z^{\times_{\mathcal{X}} n})$

Definition 6.1.5. A *stable marked curve* is a proper, irreducible, one dimensional algebraic variety C, together with a collection of marked points $p_i \in C$ that satisfy

1. Every singular point $c \in C$ is a double point, and every marked point p_i is non-singular.
2. Each genus 0 irreducible component contains ≥ 3 marked or double points.
3. Each genus 1 irreducible component contains ≥ 1 marked or double points.

□

The second two conditions guarantee that C has finitely many automorphisms that preserve the markings.

We will consider the variety $\overline{M}_{g,n}$. Its set of points is

$$\overline{M}_{g,n} = \frac{\{C, p_1, \ldots, p_n \; C \text{ is a stable marked curve of genus } g\}}{C, \underline{p} \sim D, \underline{q} \text{ if } C, D \text{ are isomorphic by map preserving the markings}}.$$

There is a Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n} \in \mathrm{DM}$, which is related to $\overline{M}_{g,n}$ in the same way as $\mathcal{M}_{g,n}$ relates to $M_{g,n}$. Thus, there is a functor to groupoids $X \mapsto \overline{\mathcal{M}}_{g,n}(X)$. The groupoid $\overline{\mathcal{M}}_{g,n}(X)$ has objects

$$\{C \to X \text{ flat map with fibers stable genus } g \text{ curves } p_1, \ldots, p_n : *_X \to C\}.$$

The isomorphisms of $\overline{\mathcal{M}}_{g,n}(X)$ are isomorphisms of curves which preserve the marked points.

Remark 6.1.6. There are simple variants of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ where the points are labelled by any finite set X. We refer to these variants as $\mathcal{M}_{g,X}$ and $\overline{\mathcal{M}}_{g,X}$ respectively. □

6.1.3 Stable graphs

Unlike $\mathcal{M}_{g,n}$, different points in $C \in \overline{\mathcal{M}}_{g,n}$ correspond to curves with different underlying topological spaces. To each curve C, we can associate a graph that records its topological type and its marking. These graphs form a poset which stratifies $\overline{\mathcal{M}}_{g,n}$. We will use a modification of this stratification to bound $H_i(\overline{\mathcal{M}}_{g,n})$.

A *stable graph G of genus g, with n external edges* is a connected graph G: a subset of the univalent vertices of G are marked as external and are put in bijection with $\{1, \ldots, n\}$, and the other vertices v are assigned a genus $g(v) \in \mathbf{N}$. We call the edges adjacent to external vertices *external*: every external vertex corresponds to a unique external edge. For this reason we do not refer to external vertices directly—when we say that $v \in G$ is a vertex of G, we will always mean an internal vertex. This data is subject to the condition that $\dim H^1(G) + \sum_{v \in G} g(v) = g$, each genus 0 vertex is at least trivalent, and each genus 1 vertex is at least 1-valent. We write $g = g(G)$ and $n = n(G)$, and $n(v)$ for the valence of an vertex.

It is also convenient record the data of G in a different way.

Definition 6.1.7. We define the *set of half edges of G* to be

$$\mathrm{Half}(G) = \{(v, e) \mid v \in G \text{ is a vertex } e \text{ is an edge adjacent to } v\}.$$

The vertices of G define a set partition $V_G \in \mathrm{P}(\mathrm{Half}(G))$, such that each block of V_G is the set of half edges adjacent to a given vertex. [4]

We define an involution $\sigma_G : \mathrm{Half}(G) \to \mathrm{Half}(G)$ as follows. If $(e, v) \in \mathrm{Half}(G)$ and e is an internal edge connecting v to $\overline{v}$, then $\sigma(e, v) := (e, \overline{v})$. Otherwise, e is external and $\sigma(e, v) := (e, v)$.

Finally, there is a bijection $r_G : \mathrm{Fix}(\sigma_G) \to \{1, \ldots, n\}$ that labels the external half edges. □

The data of the set $\mathrm{Half}(G)$, the partition V_G, the involution σ_G, and the marking r_G, are sufficient to reconstruct G. We can also use this data to define isomorphisms of stable graphs.

Definition 6.1.8. An isomorphism of stable graphs G_1, G_2 is a bijection between $\mathrm{Half}(G_1)$ and $\mathrm{Half}(G_2)$ that preserves V_G, σ_G, and r_G. □

If G is a stable graph and G' is a quotient graph, such that none of the external edges of G are identified, then G' inherits a stable structure: the external vertices are the images

[4] Recall our convention is such that the half edges that are adjacent to external vertices do not contribute to $\mathrm{Half}(G)$.

of the external vertices under $\pi : G \to G'$ with the induced labelling, and the genus of a vertex w of G' is $\dim H^1(\pi^{-1}(G)) + \sum_{v \in \pi^{-1}(w)} g(v)$. We say that the stable graph G' is a quotient of G.

We write $\mathrm{Stab}(g,n)$ for the set of isomorphism classes of stable graphs of genus g with n external edges. Then $\mathrm{Stab}(g,n)$ is naturally a poset: we define $G \geq G'$ if G is a quotient of G'. And we say that $G \prec G'$ if G' can be obtained from G by quotienting a single internal edge.

The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ admits a stratification by the poset of stable graphs $\mathrm{Stab}(g,n)$: the stratum $\mathcal{M}_G$ for $G \in \mathrm{Stab}(g,n)$ is the locus of stable curves C such that the dual graph of C is isomorphic to G.

Here the *dual graph* of C is the stable graph that has an vertex v for each irreducible component of C, an edge between v and w for every a double point joining their irreducible components, and an external edge for each marked point of C adjacent to the vertex of the irreducible component that the point lies on. The genus $g(v)$ of a vertex is the genus of its irreducible component.

6.1.4 Operad Structure

The Deligne–Mumford compactifications of the moduli spaces of marked curves $\overline{\mathcal{M}}_{g,n}$ carry the structure of a modular operad [19]. In particular, the assignment

$$X \mapsto \mathcal{M}(X) := \begin{cases} \bigsqcup_{g \geq 0} \overline{\mathcal{M}}_{g,* \sqcup X} & \text{if } |X| \geq 2 \\ * \bigsqcup_{g \geq 1} \overline{\mathcal{M}}_{g,* \sqcup X} & \text{if } |X| = 1 \\ \emptyset & \text{if } X = \emptyset \end{cases}$$

is an operad in the category of Deligne–Mumford stacks, as follows.

The operad structure is defined using gluing maps. First we define a gluing map that glues two curves together.

Definition 6.1.9. Given stable marked curves $C \in \overline{\mathcal{M}}_{g,X}$ and $D \in \overline{\mathcal{M}}_{h,Y}$ and $i \in X$, $y \in Y$, then $C \sqcup_{i,j} D \in \overline{\mathcal{M}}_{g+h, X-j \sqcup Y-i}$ is the stable curve obtained by gluing C to D along the point $i \in C$ and $j \in D$.

The operation $- \sqcup_{i,j} -$ extends to a map

$$g(i,j) : \overline{\mathcal{M}}_{g,X} \times \overline{\mathcal{M}}_{h,Y} \to \overline{\mathcal{M}}_{g,X-j\sqcup Y-i}.$$

More formally, this map can be defined as a natural transformation between functors of points. □

Now we define the gluing maps that give the operad structure.

Definition 6.1.10 (Operad Structure)**.** Let $m(r; i_1, \ldots, i_r)\overline{\mathcal{M}}([r]) \times \overline{\mathcal{M}}([i_1]) \times \ldots \overline{\mathcal{M}}([i_r]) \to \overline{\mathcal{M}}([i_1] \sqcup \cdots \sqcup [i_r])$ be the map corresponding to the operation

$$C \times D_1 \times \cdots \times D_r \mapsto C \sqcup (D_1 \sqcup \cdots \sqcup D_r)/p_j \sim *_j,$$

where $p_j \in C$ is the jth marked point, and $*_j \in D_j$ is the point marked by $*$.

Let $u : * \to \overline{\mathcal{M}}$ be the inclusion of the basepoint in degree 1. Then m and u give $\mathcal{M}$ the structure of an operad in the category of Deligne-Mumford stacks. Since homology is a symmetric monoidal functor, $H_*(\overline{\mathcal{M}})$ is an operad in the symmetric monoidal category of graded vector spaces.[5] □

The sub-species $\overline{\mathcal{M}}_0 \subset \overline{\mathcal{M}}$ given by the genus zero components $* \sqcup \sqcup_{n\geq 2} \overline{\mathcal{M}}_{0,n}$, forms a suboperad. Thus in particular $H_*(\overline{\mathcal{M}})$ is a right module over $H_*(\overline{\mathcal{M}}_0)$.

Since $\overline{\mathcal{M}}_{0,n}$ is connected, we have that $H_0(\overline{\mathcal{M}}_{0,n}) = \mathbb{Q}$, and so $H_0(\overline{\mathcal{M}})$ is isomorphic to the commutative operad. In this way $H_*(\overline{\mathcal{M}})$ becomes a right module over the commutative operad. By Theorem 2.2.22, this implies that the functor $[n] \mapsto H_*(\overline{\mathcal{M}}([n]))$ is an $\mathbf{FS}^{\mathrm{op}}$ module. This $\mathbf{FS}^{\mathrm{op}}$ module structure on the homology of Deligne compactifications is closely related to the one that we define and study in the next section.

[5]Because DM is a 2-category, one has to be careful to make m and u satisfy the operadic identities coherently. This coherence issue does not affect the operad structure on homology, because equivalent natural transformations induce the same map on homology.

6.2 Statement of Results

We will study $H_i(\overline{\mathcal{M}}_{g,n})$, the homology of the Deligne–Mumford moduli space of stable marked curves, from the point of view of representation stability.

The symmetric group $\mathbf{S}_n$ acts on $\overline{\mathcal{M}}_{g,n}$ by relabeling the marked points, so that if we fix i and g we obtain a sequence of symmetric group representations $n \mapsto H_i(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$.

Question 6.2.1. *What is the asymptotic behavior of these $\mathbf{S}_n$ representations for $n \gg 0$?*

The following theorem is an application of our main result in this Chapter.

Theorem 6.2.2. *Let $i, g \in \mathbf{N}$. There is a constant $C = p(g,i)$, where p is a polynomial in g and i of order $O(g^2 i^2)$, such that the following hold.*

1. *The generating function for the dimension of $H_i(\overline{\mathcal{M}}_{g,n})$ is rational and takes the form*

$$\sum_n \dim H_i(\overline{\mathcal{M}}_{g,n}) t^n = \frac{f(t)}{\prod_{j=1}^{C} (1 - jt)^{d_j}}$$

   for some polynomial $f(t)$ and $d_j \in \mathbf{N}$.
   In particular, there exist polynomials $f_1(n), \ldots, f_C(n)$ such that for $n \gg 0$ we have

$$\dim H_i(\overline{\mathcal{M}}_{g,n}) = \sum_{j=1}^{C} f_j(n) j^n$$

2. *Let λ be an integer partition of n. If the irreducible $\mathbf{S}_n$ representation M_λ occurs in the decomposition of $H_i(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$, then λ has length $\leq C$. (The Young diagram of λ has $\leq C$ rows).*
3. *Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_C$ be an integer partition of k, and $\lambda + n$ be the partition $\lambda_1 + n \geq \lambda_2 \geq \cdots \geq \lambda_C$. The multiplicity of $\lambda + n$ in $H_i(\overline{\mathcal{M}}_{g,n+k})$,*

$$n \mapsto \dim \operatorname{Hom}_{\mathbf{S}_{n+k}}(M_{\lambda+n}, H_i(\overline{\mathcal{M}}_{g,n+k})),$$

   is bounded by a polynomial of degree $C - 1$.

To establish Theorem 6.2.2 we extend the action of the symmetric groups to the action of a category, and we prove that the homology groups are finitely generated under this action. Finite generation then constrains the behavior of $H_i(\overline{\mathcal{M}}_{g,n})$ for $n \gg 0$.

6.2.1 Main Result

Let $\mathbf{FS}^{\mathrm{op}}$ be the opposite of the category of finite sets and surjections. We give $n \mapsto H_\bullet(\overline{\mathcal{M}}_{g,n})$ the structure of an $\mathbf{FS}^{\mathrm{op}}$ module.

More concretely, let $[n]$ be the set $\{1, \ldots, n\}$. For every surjection $f : [n] \to [m]$, we define a map

$$f^* : H_\bullet(\overline{\mathcal{M}}_{g,n}) \leftarrow H_\bullet(\overline{\mathcal{M}}_{g,m}),$$

such that $(f \circ g)^* = g^* f^*$. We describe f^* in two special cases, which suffice to determine it in general. In these cases, f^* is the map on homology induced by a map of spaces, $F^* : \overline{\mathcal{M}}_{g,n} \leftarrow \overline{\mathcal{M}}_{g,m}$.

When f is a bijection, F^* is the map that takes a stable marked curve C and permutes its marked points by precomposing with f.

When $f : [n+1] \to [n]$ is the surjection defined by $f(n+1) = n$ and $f(i) = i$ otherwise, F^* is the map that glues a copy of $\mathbf{P}^1$ to the point $n \in C$ and marks $\mathbf{P}^1 \sqcup_{\{n\}} C$ by keeping the marked points $\{1, \ldots, n-1\} \subset C$ and marking two new points $\{n, n+1\} \subset (\mathbf{P}^1 \sqcup_{\{n\}} C) - C$.

Our main theorem states that this $\mathbf{FS}^{\mathrm{op}}$ module is finitely generated.

Theorem 6.2.3. *Let $g, i \in \mathbf{N}$. Then the $\mathbf{FS}^{\mathrm{op}}$ module*

$$n \mapsto H_i(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$$

is a subquotient of an $\mathbf{FS}^{\mathrm{op}}$ module that is finitely generated in degree $\leq p(g, i)$ where $p(g, i)$ is a polynomial in g and i of order $O(g^2 i^2)$.

Theorem 6.2.2 is implied by Theorem 6.2.3 and results on finitely generated $\mathbf{FS}^{\mathrm{op}}$ modules due to Sam and Snowden [37]. The polynomial of Theorem 6.2.3 is the same as the polynomial of Theorem 6.2.2.

Remark 6.2.4. The category $\mathbf{FS}^{\mathrm{op}}$ acts on the homology of $\overline{\mathcal{M}}_{g,n}$ through maps that glue on copies of $\mathbf{P}^1$ with two marked points. These maps are a very small part of the full operadic structure on $H_\bullet(\overline{\mathcal{M}}_{g,n})$ generated by all gluing maps. The *tautological ring* is the subring of $H^\bullet(\overline{\mathcal{M}}_{g,n})$ generated by the image of all of the fundamental classes $[\overline{\mathcal{M}}_{g,n}]$ under gluing maps and cup products. In some sense, Theorem 6.2.3 says that for i, g fixed, all of the classes in $H_i(\overline{\mathcal{M}}_{g,n})$ are tautological "relative" to a finite list of classes, using only maps that glue on copies of $\mathbf{P}^1$ with 2 marked points. □

6.2.2 Stability

Although the dimensions $\dim H_i(\overline{\mathcal{M}}_{g,n})$ grow exponentially in n, and therefore do not stabilize in the naive sense, Theorem 6.2.3 implies that there exists a constant N such that the $\mathbf{S}_n$ representations $H_i(\overline{\mathcal{M}}_{g,n})$ are completely determined by the vector spaces $\{H_i(\mathcal{M}_{g,m})\}_{m \leq N}$ and the algebraic structure they inherit from surjections $[m] \twoheadrightarrow [m']$.

For $r \in \mathbf{N}$, let $\mathbf{FS}_r^{\mathrm{op}}$ be the full subcategory of $\mathbf{FS}^{\mathrm{op}}$ spanned by sets of size $\leq r$. We may restrict an $\mathbf{FS}^{\mathrm{op}}$ module M to an $\mathbf{FS}_r^{\mathrm{op}}$ module, denoted $\mathrm{Res}_r\, M$. The functor Res_r has a left adjoint Ind_r, which takes an $\mathbf{FS}_r^{\mathrm{op}}$ module to the $\mathbf{FS}^{\mathrm{op}}$ module freely generated by it modulo relations from $\mathbf{FS}_r^{\mathrm{op}}$.

Theorem 6.2.5. *Let $i, g \in \mathbf{N}$. There exists $N \in \mathbf{N}$ such that the natural map of $\mathbf{FS}^{\mathrm{op}}$ modules*

$$\mathrm{Ind}_N \mathrm{Res}_N H_i(\overline{\mathcal{M}}_{g,-}) \to H_i(\overline{\mathcal{M}}_{g,-})$$

is an isomorphism. In particular, any presentation of the $\mathbf{FS}_N^{\mathrm{op}}$ module $\mathrm{Res}_N H_i(\overline{\mathcal{M}}_{g,-})$ gives a presentation of the $\mathbf{FS}^{\mathrm{op}}$ module $H_i(\overline{\mathcal{M}}_{g,-})$.

Since $\mathbf{FS}_N$ is a finite category and Ind_N can be described explicitly as a colimit, Theorem 6.2.5 gives a procedure to compute $H_i(\overline{\mathcal{M}}_{g,n})$ from a finite amount of algebraic data.

Theorem 6.2.5 follows from Theorem 6.2.3 and a Noetherianity result due to Sam and Snowden [37].

6.2.3 Relation to other work

Our work is motivated by the approach to representation stability introduced by Church, Ellenberg and Farb, which uses modules over **FI**, the category of finite sets and injections [5].

The theory of **FI** modules has been used by Jiménez Rolland to study the homology of $\mathcal{M}_{g,n}$ [35], and by Maya Duque and Jiménez Rolland to study the real locus of $\overline{\mathcal{M}}_{0,n}$ [36]. Because the homology of $\overline{\mathcal{M}}_{g,n}$ grows at an exponential rate, it cannot admit the structure of a finitely generated **FI** module, and so a larger category is needed to control the homology of the full compactification.

Using an explicit presentation of the cohomology ring $H^\bullet(\overline{\mathcal{M}}_{0,n})$ given in [13], Sam defined an action of $\mathbf{FS}^{\mathrm{op}}$ on the cohomology of $\overline{\mathcal{M}}_{0,n}$, and proved that it was finitely generated. Our work was motivated by his suggestion that there could exist a finitely generated $\mathbf{FS}^{\mathrm{op}}$ action on the cohomology of $\overline{\mathcal{M}}_{g,n}$ for general g.

Sam and Snowden showed that $\mathbf{FS}^{\mathrm{op}}$ is noetherian (submodules of finitely generated modules are finitely generated), and described the Hilbert series of finitely generated $\mathbf{FS}^{\mathrm{op}}$ modules [37]. We use their results to deduce concrete implications from Theorem 6.2.3.

Proudfoot and Young have used $\mathbf{FS}^{\mathrm{op}}$ modules to study the intersection cohomology of a space closely related to $\overline{\mathcal{M}}_{0,n}$ [34]. The $\mathbf{FS}^{\mathrm{op}}$ module they construct is closely related to our construction in the case $g = 0$. The statement of our Theorem 6.2.2 parallels their Theorem 4.3.

In order to produce non-tautological classes, Faber and Pandharipande [14] established restrictions on the $\mathbf{S}_n$ representations that appear in the tautological ring, which resemble the restrictions on $\mathbf{S}_n$ representations we obtain in Theorem 6.2.2. Our restrictions on representations are weaker, but they hold for all cohomology classes.

Kapranov–Manin [23] observed that $\bigoplus_{i,n} H_i(\overline{\mathcal{M}}_{g,n})$ is a right module over the hypercommutative opeard. This algebraic structure extends the action of $\mathbf{FS}^{\mathrm{op}}$ on $H_i(\overline{\mathcal{M}}_{g,n})$ for fixed i.

6.2.4 Action of binary tree category

Before passing to homology, the category $\mathbf{FS}^{\mathrm{op}}$ does not naturally act on $\overline{\mathcal{M}}_{g,n}$. Instead, we construct an action of a category of binary trees.

We define **BT** to be the category whose objects are natural numbers, and whose morphisms $m \to n$ are rooted binary forests with the leaves labelled by $[m]$ and the roots labelled by $[n]$. (Morphisms $n \to n$ are binary forests such that each tree is a single root).

Composition is defined by gluing the roots to the leaves. For a forest F, the function that takes each leaf to its root is a surjection $h_F : [n] \to [m]$. The assignment $F \mapsto h_F$ defines a functor $\mathbf{BT} \to \mathbf{FS}$ that realizes $\mathbf{FS}$ as a quotient of $\mathbf{BT}$.

The category $\mathbf{BT}^{\mathrm{op}}$ acts on $\overline{\mathcal{M}}_{g,n}$ by gluing on trees of marked projective lines. For each labelled binary forest $F \in \mathbf{BT}(m, n)$, there is a corresponding variety L_F, which is a disjoint union of trees and projective lines. The variety L_F has a component for each binary tree T of F: if T consists only of a root, then the component is a point; otherwise the component is a genus zero stable curve with dual graph isomorphic to T.[6] The variety L_F is naturally marked by $\{1, \ldots, m\}$ and $\{1, \ldots n\}$ in a way that is unique up to unique isomorphism.

From a stable curve $C \in \overline{\mathcal{M}}_{g,n}$ and a labelled rooted forest $F \in \mathbf{BT}(m, n)$, we obtain a new stable curve $F^*C \in \overline{\mathcal{M}}_{g,m}$ by gluing L_F to C along the marked points $\{1, \ldots, m\}$, and using the marking of L_F by $\{1, \ldots, n\}$ to mark F^*C. This defines a gluing map $F^* : \overline{\mathcal{M}}_{g,m} \to \overline{\mathcal{M}}_{g,n}$ for each $F \in \mathbf{BT}(n, m)$, and these maps define an action of $\mathbf{BT}^{\mathrm{op}}$ on $\overline{\mathcal{M}}_{g,n}$.

To see that the representation of $\mathbf{BT}^{\mathrm{op}}$, $n \mapsto H_i(\overline{\mathcal{M}}_{g,n})$, factors through $\mathbf{FS}^{\mathrm{op}}$, let F_1 and F_2 be two forests inducing the same surjection $h : [n] \to [m]$. There is a family of gluing maps from $\overline{\mathcal{M}}_{g,m}$ to $\overline{\mathcal{M}}_{g,n}$,

$$\overline{\mathcal{M}}_{g,m} \times \left(\prod_{i \in [m],\ h^{-1}(i) > 1} \overline{\mathcal{M}}_{0,\#h^{-1}(i)+1} \right) \to \overline{\mathcal{M}}_{g,n}.$$

The maps F_1^* and F_2^* correspond to evaluating at the points in the second factor defined by the components of L_{F_1} and L_{F_2}, respectively. Since the second factor is connected, F_1^* and F_2^* induce the same map on homology.

6.3 Proof of Theorem 6.2.3

First, we sketch the proof of Theorem 6.2.3.

[6]See Section 6.3.1 for definition of dual graph.

Informal Argument. We stratify $\overline{\mathcal{M}}_{g,n}$ by dual graphs G. To prove finite generation of $H_i(\overline{\mathcal{M}}_{g,n})$, it suffices to show that only finitely many strata $\mathcal{M}_G$ contribute $\mathbf{FS}^{\mathrm{op}}$ generators. The stratum $\mathcal{M}_G$ is a quotient of a product of moduli spaces $\prod_{v\in G}\mathcal{M}_{g(v),n(v)}$.

Since the homology of $\overline{\mathcal{M}}_{g,n}$ is pure, it suffices to consider the pure Borel–Moore homology of the strata: $\mathrm{W}_{-i}H_i^{\mathrm{BM}}(\mathcal{M}_G)$. By fibering $\mathcal{M}_{g,n}$ over $\mathcal{M}_{g,1}$ we show that the Borel–Moore homology of $\mathcal{M}_{g,n}$ vanishes for $n > i+3$, so only strata $\mathcal{M}_G$ for which $\sum_v \mathrm{val}(v) - 3 \le i$ contribute to $H_i(\overline{\mathcal{M}}_{g,n})$.

Thus for G ranging over all graphs that contribute to $H_i(\overline{\mathcal{M}}_{g,n})$, the number of vertices of G that have valence > 3 and genus > 0 is bounded by a function of g and i. So as $n \to \infty$, the number of trivalent genus 0 vertices of G must increase.

Say that a stable graph H *has an external* Y if it has a genus 0 trivalent vertex v which is adjacent to two external edges. Since $\mathbf{FS}^{\mathrm{op}}$ acts on graphs by gluing on trivalent vertices[7], if H has an external Y then the classes from $\mathcal{M}_H$ are pushed forward from lower degree. If G has two adjacent trivalent genus 0 vertices v_1, v_2 such that each v_i has an external edge, the cross ratio relation shows that the classes from $\mathcal{M}_G$ are homologous to classes from $\mathcal{M}_H$, where H has an external Y, and thus are also pushed forward from lower degree.

Therefore, to prove finite generation, it is enough to show that when the number of trivalent genus 0 vertices of G is large then either (1) G has an external Y, or (2) G has two adjacent trivalent genus 0 vertices v, v', each with an external edge. Each trivalent vertex with *no* external edges contributes $1/2$ to $-\chi(G)$. The bound $-\chi(G) \le g-1$ implies that, as the number of trivalent vertices increases, one of the two possibilities must occur. □

Unfortunately, in formalizing the above argument, we encounter the problem that $\mathbf{FS}^{\mathrm{op}}$ does not act on the Borel–Moore homology spectral sequence for the stable graph stratification, and the category of binary trees, $\mathbf{BT}^{\mathrm{op}}$, which does act, is not known to be noetherian.

Accordingly, in Section 6.3.1, we define a coarsening of the stable graph stratification for which $\mathbf{FS}^{\mathrm{op}}$ does act on the associated Borel–Moore homology spectral sequence. In Section 6.3.2, we prove two combinatorial lemmas about stable graphs. In Section 6.3.3, we combine the results from previous sections to prove Theorem 6.2.3.

[7]More accurately, $\mathbf{BT}^{\mathrm{op}}$ acts

6.3.1 Coarsening of the stable graph stratification

We construct a coarsening of the stratification by $\mathrm{Stab}(g, n)$, which has the property that $\mathbf{FS}^{\mathrm{op}}$ still acts on the corresponding Borel–Moore homology spectral sequence, to prove Theorem 6.2.3. This stratification is defined in terms of the operation in the next proposition:

Proposition 6.3.1. *Let G be a stable graph. Then there is a unique stable quotient $\overline{G} \geq G$ such that no two distinct genus 0 vertices of $\overline{G}$ are connected by an edge, and $\overline{G} \to G$ only identifies edges between genus 0 vertices.*

Proof. Consider the subgraph spanned by the genus 0 vertices of G and the edges between them. Choose a minimal spanning tree T_i for each connected component, and define H to be the quotient of G such that each T_i is identified to a point.

Any other quotient of G satisfying the hypotheses of the proposition must collapse a minimal spanning tree, and all quotients of minimal spanning trees are isomorphic. □

Notice that $\overline{\overline{H}} = \overline{H}$.

Warning 6.3.2. It is not true that if $G \leq H$, then $\overline{G} \leq \overline{H}$. □

Let $\mathrm{Q}(g, n)$ be the set of stable graphs $G \in \mathrm{Stab}(g, n)$ such that that no distinct genus 0 vertices are connected by an edge. The next proposition gives the stratification of $\overline{\mathcal{M}}_{g,n}$ by $\mathrm{Q}(g, n)$ that we use:

Proposition 6.3.3. *There is a poset structure on $\mathrm{Q}(g, n)$ such that the map $H \mapsto \overline{H}$ is a surjection of posets $\mathrm{Stab}(g, n) \to \mathrm{Q}(g, n)$. This surjection induces a stratification of $\overline{\mathcal{M}}_{g,n}$, where the stratum corresponding $G \in \mathrm{Q}(g, n)$ is $S(G) = \bigcup_{H,\, \overline{H}=G} \mathcal{M}_H$.*

Proof. Let $G, H \in \mathrm{Q}(g, n)$. We say that $G \leq_{\mathrm{Q}} H \iff$ there is sequence of stable graphs $G = G_0, G_1, \ldots, G_n = H$ such that either $G_i \prec G_{i+1}$, or $G_i \succ G_{i+1}$ and G_i is obtained from G_{i+1} by collapsing an edge between two genus 0 vertices.

By definition, $\leq_{\mathrm{Q}}$ is reflexive and transitive. To prove antisymmetry, suppose $G \leq_{\mathrm{Q}} H$ and $H \leq_{\mathrm{Q}} G$, and let $G = G_0, G_1, \ldots, G_n = H$ be as sequence exhibiting that $G \leq H$.

For a stable graph J, we define $s(J) \in \mathbf{N}^{\oplus\infty}$ to be the vector whose 0th entry is the number of edges between genus 0 vertices and genus ≥ 1 vertices, and whose ith entry for $i \geq 1$ is the number of vertices of genus i. Totally order these vectors reverse lexicographically. If J and J' are related by collapsing an edge between genus distinct genus 0 vertices, then $s(J) = s(J')$. If J and J' are related by collapsing a self edge of a genus 0 vertex, an edge between two genus ≥ 1 vertices, or an edge from a genus ≥ 1 vertex to a genus 0 vertex, then $s(J) < s(J')$. Thus $s(G_i) \leq s(G_{i+1})$ for all i, and we have $s(G) \leq s(H)$ and $s(H) \leq s(G)$.

For every i, the graphs G_i and G_{i+1} are related by collapsing edges between distinct genus zero vertices. Otherwise, we would have $s(G_i) < s(G_{i+1})$, contradicting $s(G) = s(H)$. Therefore $\overline{G_i} = \overline{G}_{i+1}$ for all i, and so $G = \overline{G} = \overline{H} = H$, establishing antisymmetry.

The map $H \mapsto \overline{H}$ is a map of posets since $H \leq G$ if and only if H and G are related by a sequence of edge collapses, and $H, \overline{H}$ and $G, \overline{G}$ are related by a sequence of collapses between distinct genus zero vertices. Composing these three sequences proves that $\overline{H} \leq_Q \overline{G}$.

Finally, the surjection $\mathrm{Stab}(g, n) \to \mathrm{Q}(g, n)$ shows that

$$Z(G) := \bigcup_{H \in \mathrm{Q}(g,n),\ J \in \mathrm{Stab}(g,n)\ \mid\ H \leq_{\mathrm{Q}} G,\ \overline{J} = H} \mathcal{M}_J$$

is a closed subset of $\overline{\mathcal{M}}_{g,n}$ and

$$S(G) := Z(G) - \left(\bigcup_{H \in \mathrm{Q}(g,n),\ H < G} Z(H) \right) = \bigcup_{\overline{J} = G} \mathcal{M}_G.$$

□

Definition 6.3.4. For $G \in \mathrm{Q}(g, n)$ define $\widetilde{S}(G)$ to be the space

$$\widetilde{S}(G) := \prod_{v \text{ vertex},\ g(v)=0} \overline{\mathcal{M}}_{e(v),n(v),0} \times \prod_{v \text{ vertex},\ g(v) \geq 1} \mathcal{M}_{g(v),n(v)},$$

where $e(v)$ is the number of self edges of v and $\overline{\mathcal{M}}_{e,n,0}$ is the moduli space of stable curves C of genus e and n marked points, such that all of the irreducible components of C have genus 0. □

There is a canonical map $\widetilde{S}(G) \to S(G)$ given by gluing a product of curves along the points joined by the graph.

To state the next proposition, we fix a graph $G \in \mathrm{Q}(g, n)$. Recall that the data of G can be encoded as a set of half edges, a partition that records which half edges are adjacent to the same vertex, and an involution that records which half edges are glued together.

We let A be the group of permutations of

$$\{h \in \mathrm{Half}(G) \mid h \text{ is not a part of self edge of a genus 0 vertex }\}.$$

Let $A_G \subset A$ be the subgroup which preserves the partition and involution (i.e. respects the graph structure). The group A_G acts on $\widetilde{S}(G)$ by relabeling.

Proposition 6.3.5. *If $G \in \mathrm{Q}(g, n)$, the map $\widetilde{S}(G) \to S(G)$ induces an isomorphism $\widetilde{S}(G)/A_G \cong S(G)$.*

Proof. We directly check that the map is an isomorphism on $\mathbb{C}$ points, then in formal neighborhoods by deformation theory, and finally show this suffices by covering by the Hilbert scheme.

On $\mathbb{C}$ points $\widetilde{S}(G)(\mathbb{C})$ is the groupoid of collections of marked curves $\{C_v\}_{v \in \mathrm{Vert}(G)}$, such that

- if $g(v) \geq 1$ then C_v is smooth of genus $g(v)$
- if $g(v) = 0$, then C_v is genus $e(v)$
- and each irreducible component of C_v has genus 0.

Similarly, $S(G)(\operatorname{Spec} \mathbb{C})$ is the groupoid of stable marked curves C whose dual graph H has $\overline{H} = G$.

The group A_G acts strictly on the groupoid $\widetilde{S}(G)(\operatorname{Spec} \mathbb{C})$, by taking $\sigma(\{C_v\}) = \{C_{\sigma v}\}$ and relabeling marked points according to the action of σ on half edges. There are canonical isomorphisms $\sigma : \mathrm{glue}(\{C_v\}) \xrightarrow{\sim} \mathrm{glue}(\{C_{\sigma v}\})$; any curve in $S(G)(\mathbb{C})$ is glued from a curve in $\widetilde{S}(G)(\mathbb{C})$; and any isomorphism $\mathrm{glue}(\{C_v\}) \to \mathrm{glue}(\{C'_v\})$ factors uniquely as a relabeling of the components of C_v by $\sigma \in A_g$, and isomorphisms between the components $f_v : C_{\sigma v} \to C'_{\sigma v}$. This shows that $S(G)(\mathbb{C})$ is the groupoid quotient of $\widetilde{S}(G)(\mathbb{C})$ by G.

Next we check that the map is an isomorphism in formal neighborhoods. More precisely, let $\{C_v\}_{v \in G} \in \widetilde{S}(G)(\mathbb{C})$ be a point, let $C := \mathrm{glue}(\{C_v\})$ have dual graph H, and let B be a local artinan $\mathbb{C}$ algebra. We claim that

$$\widetilde{S}(G)(B)/A_G \times_{\widetilde{S}(G)(\mathbb{C})} \{C_v\} \to S(G)(B) \times_{S(G)(\mathbb{C})} C$$

is an equivalence.

From the deformation theory of marked stable curves, we have that the groupoid $S(G)(B) \times_{S(G)(\mathbb{C})} C$ is naturally equivalent to $\mathrm{Hom}(\mathrm{Spec}\, B, X)$ where

$$X \subset \mathrm{Ext}^1(\Omega_C(\sum_i p_i), \mathcal{O}_C)$$

is a union of linear subspaces (here $\{p_i\}_{i=1}^n$ are the marked points of C). Let $H_0 \subset H$ be the subgraph spanned by genus 0 vertices. There is one linear subspace for each spanning forest of H_0. The subspace corresponding to a spanning forest F is

$$\ker\left(\mathrm{Ext}^1(\Omega_C(\sum_i p_i), \mathcal{O}_C) \to \prod_{e \in \mathrm{edge(H-F)}} \mathrm{Ext}^1(\Omega_{\hat{C}_e}, \mathcal{O}_{\hat{C}_e})\right)$$

where $\hat{C}_e$ is the formal neighborhood of the double point $e \in C$ corresponding to e.

Similarly we have $\widetilde{S}(G)(B)/A_G \times_{\widetilde{S}(G)(\mathbb{C})} \{C_v\} = \widetilde{S}(G)(B) \times_{\widetilde{S}(G)(\mathbb{C})} \{C_v\}$ is equivalent to $\mathrm{Hom}(\mathrm{Spec}\, B, \widetilde{X})$, where

$$\widetilde{X} \subset \prod_{v \in \mathrm{vert}(G)} \mathrm{Ext}^1(\Omega_{C_v}(\sum_{u_v} q_{u_v}), \mathcal{O}_{C_v})$$

is a union of linear subspaces (the sum is over half edges u_v adjacent to v). Again, $\widetilde{X}$ is a union of subspaces, one for each spanning forest of H_0. The subspace corresponding to F

is

$$\ker\left(\prod_{v\in \mathrm{vert}(G)} \mathrm{Ext}^1(\Omega_{C_v}(\sum_{u_v} q_{u_v}), O_{C_v}) \to \prod_{e\in \mathrm{edge}(H-H_0)} \mathrm{Ext}^1(\Omega_{\hat{C}_e}, O_{\hat{C}_e})\right).$$

From these identifications, and the fact that

$$\prod_{v\in \mathrm{vert}(G)} \mathrm{Ext}^1(\Omega_{C_v}(\sum_{u_v} q_{u_v}), O_{C_v}) = \ker\left(\mathrm{Ext}^1(\Omega_C(\sum_i p_i), O_C) \to \prod_{e\in \mathrm{edge}(G-H_0)} \mathrm{Ext}^1(\Omega_{\hat{C}_e}, O_{\hat{C}_e})\right)$$

it follows that the map is an isomorphism on formal neighborhoods.

Now, let $\mathcal{H}$ be the Hilbert scheme of tricanonically embedded stable curves of genus $g(G)$ with $n(G)$ marked points. Let $\mathcal{H}(G)$ be the stratum consisting of curves with dual graph J such that $\overline{J} = G$. Then $\mathcal{H}(G)$ is a finite type scheme which surjects onto $S(G)$, and the fibre product $\widetilde{\mathcal{H}}(G) := \mathcal{H}(G) \times_{S(G)} \widetilde{S}(G)/A_G$ is a locally closed subscheme of a product of hilbert schemes, hence also of finite type. Base changing, we see that $f : \widetilde{\mathcal{H}}(G)(\mathbb{C}) \to \mathcal{H}(G)(\mathbb{C})$ is an equivalence and

$$\widetilde{\mathcal{H}}(G)(B) \times_{\widetilde{\mathcal{H}}(G)(\mathbb{C})} x \to \mathcal{H}(G)(B) \times_{\mathcal{H}(G)(\mathbb{C})} f(x)$$

is an equivalence for all $x \in \widetilde{\mathcal{H}}(G)(\mathbb{C})$ and B as above. A map between finite type schemes that is an isomorphism on $\mathbb{C}$ points and formal neighborhoods is an isomorphism, thus $\widetilde{\mathcal{H}}(G) \to \mathcal{H}(G)$ is an isomorphism, and so $\widetilde{S}(G)/A_G \to S(G)$ is an isomorphism. $\square$

We extend the action of $\mathbf{BT}^{\mathrm{op}}$ $\overline{\mathcal{M}}_{g,n}$ to an action on $\sqcup_{G\in \mathrm{Q}(g,n)} S(G)$ and $\sqcup_{G\in \mathrm{Q}(g,n)} \widetilde{S}(G)$, as follows. The category $\mathbf{BT}^{\mathrm{op}}$ acts on the poset $\mathrm{Q}(g,n)$ by gluing on trees of stable graphs, and the gluing map $f : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ corresponding to $f \in \mathbf{BT}([m],[n])$ induces maps $S(G) \to S(f^*G)$ and $\widetilde{S}(G) \to \widetilde{S}(f^*G)$, which give maps on the disjoint union.

From the stratification of $\overline{\mathcal{M}}_{g,n}$ by $\mathrm{Q}(g,n)$, we obtain a Borel–Moore homology spectral sequence. The next proposition records how the isomorphism of Proposition 6.3.5 is compatible with the action of $\mathbf{FS}^{\mathrm{op}}$, and the spectral sequence of this stratification.

Proposition 6.3.6. *The actions of* $\mathbf{BT}^{\mathrm{op}}$ *on* $\sqcup_{G\in \mathrm{Q}(g,n)} S(G)$ *and* $\sqcup_{G\in \mathrm{Q}(g,n)} \widetilde{S}(G)$ *induce an*

$\mathbf{FS}^{\mathrm{op}}$ *module structure on Borel–Moore homology such that*

$$\bigoplus_{G \in \mathrm{Q}(g,n)} H_\bullet^{\mathrm{BM}}(S(G)) \to \bigoplus_{G \in \mathrm{Q}(g,n)} H_\bullet^{\mathrm{BM}}(\widetilde{S}(G))$$

is a map of $\mathbf{FS}^{\mathrm{op}}$ *modules, and the Borel–Moore homology spectral sequence*

$$\bigoplus_{G \in \mathrm{Q}(g,n)} H_\bullet^{\mathrm{BM}}(S(G)) \implies H_\bullet(\overline{\mathcal{M}}_{g,n})$$

is a spectral sequence of $\mathbf{FS}^{\mathrm{op}}$ *modules.*

Proof. By definition, the action of $\mathbf{BT}^{\mathrm{op}}$ preserves the stratification and acts through closed embeddings, and thus induces maps of spectral sequences. On Borel–Moore homology, the action factors through $\mathbf{FS}^{\mathrm{op}}$ because the gluing map for a tree T with n leaves $T : \widetilde{S}(G) \to \widetilde{S}(T^*G)$ extends to a map $S(G) \times \overline{\mathcal{M}}_{0,n} \to S(T^*G)$, the maps corresponding to different trees correspond to evaluating on different boundary strata of $\overline{\mathcal{M}}_{0,n}$. Since $\overline{\mathcal{M}}_{0,n}$ is smooth and proper, $H_0^{\mathrm{BM}}(\overline{\mathcal{M}}_{0,n}) = \mathbb{Q}$, and so every tree induces the same map on Borel-Moore homology. □

6.3.2 Bounding Graphs

The two lemmas in this subsection are combinatorial: the first says that there are only finitely many graphs in $\mathrm{Q}(g, n)$ with $\leq i + 1$ valent genus one vertices, and genus zero vertices obeying certain bounds.

Lemma 6.3.7. *Let $f(i, e, a)$ be a linear function $f(i, e, a) = ri + se + ta + u$ with $r, s, t \geq 0$. If G is a stable genus g graph such that:*

1. *There are no edges between distinct genus* 0 *vertices*
2. *Every genus* ≥ 1 *vertex has valence* $\leq i + 1$
3. *Every genus* 0 *vertex with e self edges and a edges to other vertices, has* $\leq f(i, e, a)$ *external edges.*

Then the total number of external edges of G is bounded by:

$$n(G) \leq (i + 1)g + (i + 1)g f(i, g, (i + 1)g).$$

Proof. For a vertex v, let $e(v)$ be the number of self edges and $a(v)$ be the number of edges to a distinct vertex.

By 2) and 3) we have

$$n(G) \leq \sum_{v, g(v) \geq 1} (i+1) + \sum_{v,\ g(v)=0} f(i, e(v), a(v)).$$

For every genus 0 vertex v, we have $e(v) \leq g$, because the genus is $\leq g$, and $a(v) \leq (i+1)g$ because any genus 0 vertex is adjacent to a vertex of genus ≥ 1. Thus $f(i, e(v), a(v)) \leq f(i, g, (i+1)g)$.

Now either G has only one genus 0 vertex, or every genus 0 vertex must be adjacent to a genus ≥ 1 vertex. In the first case $n(G) \leq f(i, g, 0)$, and in the second we have

$$\#\{\text{ genus 0 vertices }\} \leq (i+1)g$$

So either $n(G) \leq f(i, e, 0)$ or

$$n(G) \leq (i+1)g + (i+1)g f(i, g, (i+1)g)$$

which proves the claim. □

The next lemma says that stable graphs of genus g that correspond to strata of dimension $\leq i$ and only have genus 0 vertices must contain certain subgraphs when $n(H) \gg 0$.

Lemma 6.3.8. *Let J be the stable graph consisting of a single genus 0 vertex, e self edges, and $n-e$ labelled external edges, with a partition of the $[n-e]$ sets $[a]$ and $[b]$ of size a and b. If $b > 13a + 16i + 8e - 7$ then every stable graph H with $\overline{H} = J$ satisfies at least one of the following:*

1. *The sum $\sum_{v \in H}(n(v) - 3)$ is $> i$*
2. *There is a trivalent vertex v adjacent to 2 external edges in $[b]$*
3. *There are two adjacent trivalent vertices v, v' such that both v and v' are adjacent to external edges in $[b]$*

Proof. We prove the contrapositive. Let $a(v)$, $b(v)$, and $e(v)$ denote the number of edges adjacent to v that are respectively external in $[a]$, external in $[b]$, and self edges. Let H be a stable graph with $\overline{H} = G$ and which satisfies:

1. $\sum_{v \text{ vertex}}(n(v) - 3) \leq i$
2. Every trivalent vertex has $b(v) \leq 1$
3. There are no two adjacent trivalent vertices v, v' with $b(v), b(v') \geq 1$.

First, by (1), H has $\leq i$ vertices of valence > 3.

Next by retracting external edges and computing the (negative) Euler characteristic of H, we have

$$e - 1 = \sum_{v \in H} (1/2\ \#\{\text{non external edges of } v\} - 1).$$

We have that $\#\{\text{non external edges of } v\} = n(v) - a(v) - b(v)$. Then breaking up the sum by $n(v)$ and $a(v)$, we have:

$$\begin{aligned} e - 1 \ &= \sum_{v,\ n(v)>3} (1/2(n(v) - a(v) - b(v)) - 1) \ + \sum_{v,\ n(v)=3,\ a(v)\geq 1} (1/2(n(v) - a(v) - b(v)) - 1) \\ &\quad + \sum_{v,\ n(v)=3,\ a(v)=0,\ b(v)=1} 0 \ + \sum_{v,\ n(v)=3,\ a(v)=0,\ b(v)=0} 1/2 \\ &\geq \ -i - a + 0 + \sum_{v,\ n(v)=3,\ a(v)=0,\ b(v)=0} 1/2. \end{aligned}$$

Here we used 2) and the bound on vertices of valence > 3. Thus rearranging this inequality

$$\#\{\text{trivalent vertices } v, \text{ s.t. } a(v) = b(v) = 0\} \leq 2(i + a + e - 1)$$

Since there are at most a trivalent vertices with $a(v) > 0$, there are at most $\leq 2(i+a+e-1)+a$ trivalent vertices with $b(v) = 0$.

Next by (3) every trivalent vertex with $b(v) = 1$ satisfies at least one of the following:

A v has an edge in $[a]$

B v is adjacent to a trivalent vertex with $b(v) = 0$

C v is adjacent to a vertex of valence > 3

D v has a self edge and is the only vertex.

We have the following bounds on the number of trivalent vertices satsifying each of these cases

A $\leq a$

B $\leq 3\#\{$ genus 0 trivalent vertices with $b(v) = 0\} \leq 3(2(i + a + e - 1) + a)$

C

$$\leq \sum_{v,\ n(v)>3} n(v) \leq i + 3\ \#\{ > 3\text{-valent vertices}\} \leq i + 3i$$

D ≤ 1,

where in **C** we have rearranged (1) and used the bound on vertices of valence > 3. Combining these bounds, we have

$$\#\{\text{trivalent vertices with } b(v) = 1\ \} \leq a + 3(2(i + a + e - 1) + a) + 4i + 1 = 10a + 10i + 6e - 5$$

Thus in total, the valence 3 vertices contribute at most $13a + 12i + 8e - 7$ to b.

Lastly we have

$$\sum_{v,\ n(v)>3} b(v) \leq \sum_{v,\ n(v)>3} b(v) \leq i + 3i,$$

as above. So the vertices of valence > 3 contribute at most $4i$ to b. In total we see that $b \leq 13a + 16i + 8e - 7$, completing the proof. □

6.3.3 Proof of Finite Generation

Before proving finite generation, we record the following vanishing statement.

Proposition 6.3.9. *We have that* $H_i^{\mathrm{BM}}(\mathcal{M}_{g,n}) = 0$ *for* $i < n - 1$.

Proof. We will show $H_c^i(\mathcal{M}_{g,n}) = 0$ for $i < n + 1$. Consider the map $f : \mathcal{M}_{g,n} \to \mathcal{M}_{g,1}$ which forgets the last $n - 1$ points. By the Leray sequence for compact support and base change, it suffices to show that $H_c^i(f^{-1}(x))$ vanishes for $i < n - 1$. The fibers are topologically isomorphic to $\mathrm{Conf}_{n-1}(\Sigma_g - p)$ where $\Sigma_g - p$ is the genus g surface with one point removed. This vanishing follows from Getzler's [17] spectral sequence converging to the compactly supported cohomology of configuration space, or its generalization by Petersen [33]. As a graded vector space, the E_2 page is

$$\bigoplus_{0\le k\le n-2}(\mathbb{Q}[-1]\oplus\mathbb{Q}^{\oplus 2g}[-2])^{\otimes n-1-k}\otimes\mathbb{Q}^{\oplus c_{n,k}}[-k],$$

where $c_{n,k}$ is a certain unsigned Stirling number. Since the lowest degree term is in degree $n-1$, the result follows. □

Throughout this section, we let $f(i,e,a)=13a+16i+8e-7$ and $p(i,g)=(i+1)g+(i+1)gf(i,g,(i+1)g)$. Expanding, we have $p(i,g)=8g^2i^2+29g^2i+16gi^2+21g^2+10gi-6g$.

We now show that the $\mathbf{FS}^{\mathrm{op}}$ module $n\mapsto H_i(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ is a subquotient of one that is finitely generated in degrees $\le p(i,g)$. In particular, since Sam and Snowden[37] proved that submodules of finitely generated $\mathbf{FS}^{\mathrm{op}}$ modules are finitely generated, we see that the homology is finitely generated.

Proof of Theorem 6.2.3. We use the Borel–Moore homology spectral sequence for the stratification of $\overline{\mathcal{M}}_{g,n}$ by $\mathrm{Q}(g,n)$ defined in Section 6.3.1. Write $S(G)$ for the stratum corresponding to G and $\widetilde{S}(G)=\prod_{v\in G,\ g(v)=0}\overline{\mathcal{M}}_{0,n(v)}\times\prod_{v\in G,\ g(v)\ge 1}\mathcal{M}_{g(v),n(v)}$ for the cover of $S(G)$. Then by Proposition 6.3.6, this is a spectral sequence of $\mathbf{FS}^{\mathrm{op}}$ modules. And there is a homomorphism of $\mathbf{FS}^{\mathrm{op}}$ modules

$$\bigoplus_{G\in\mathrm{Q}(g,n)}H_\bullet^{\mathrm{BM}}(S(G))\to\bigoplus_{G\in\mathrm{Q}(g,n)}H_\bullet^{\mathrm{BM}}\left(\widetilde{S}(G)\right).$$

By Proposition 6.3.5, the left hand side is identified with the A_G invariants of the right hand side.

We use Deligne's theory of weights [11, 10]. Since the homology of $\overline{\mathcal{M}}_{g,n}$ is pure, to show that $H_i(\overline{\mathcal{M}}_{g,n})$ is finitely generated, it suffices to show that $\bigoplus_{G\in\mathrm{Q}(g,n)}W_{-i}H_i^{\mathrm{BM}}\left(\widetilde{S}(G)\right)$ is a finitely generated $\mathbf{FS}^{\mathrm{op}}$ module.

By Lemma 6.3.7, all graphs $G\in\mathrm{Q}(g,n)$ with

1. no genus 1 vertices of valence $> i+1$

2. no genus 0 vertices with e self edges, a edges to adjacent vertices and $> f(i,e,a)$ external edges,

have $n \leq p(i,g)$. Thus for the rest of the proof, we show that the $\mathbf{FS}^{\mathrm{op}}$ module defined by

$$\bigoplus_{G \in \mathrm{Q}(g,n)} W_{-i} H_i^{\mathrm{BM}}(\widetilde{S}(G))$$

is generated by the groups $W_{-i}H_i^{\mathrm{BM}}(\widetilde{S}(L))$ for graphs L satisfying 1 and 2.

Accordingly, let c be a pure class in $W_{-i}H_i^{\mathrm{BM}}(\widetilde{S}(G))$, where G is a graph that does not satisfy 1 and 2. We want to show that c is a linear combination of classes pushed forward from lower degrees. (In other words c is a linear combination of classes of the form $f^*(d)$ for some surjection $f : [n] \to [n-1]$).

We have that

$$W_{-i}H_i^{\mathrm{BM}}(\widetilde{S}(G)) = \bigoplus_{j:\mathrm{Vert}(G) \to \mathbf{N},\ \sum_v j(v) = i} \left(\bigotimes_{v,\ g(v)=0} W_{-j(v)} H_{j(v)}^{\mathrm{BM}}(\overline{\mathcal{M}}_0(v)) \otimes \bigotimes_{v,\ g(v)\geq 1} W_{-j(v)} H_{j(v)}^{\mathrm{BM}}(\mathcal{M}_{n(v),g(v)}) \right).$$

If G has a genus ≥ 1 vertex with valence $> i+1$, then by Proposition 6.3.9 we have that $W_{-j}H_j(\mathcal{M}_{g(v),n(v)}) = 0$ for all $j \leq i$. Thus $W_{-i}H_i^{\mathrm{BM}}(\widetilde{S}(G)) = 0$ and the claim follows.

So assume that w is a genus 0 vertex of G with e self edges, a edges to distinct vertices, and $> f(i,e,a)$ external edges. Taking linear combinations it suffices to show $c = f_*(d)$ for classes c of the form $c = c_w \otimes c'$, where $c_w \in H_{j(w)}(\overline{\mathcal{M}}_0(w))$ is a pure class for some $j(w) \leq i$ and

$$c \in W_{j(w)-i} H_{i-j(w)} \left(\prod_{v \neq w,\ g(v)=0} \overline{\mathcal{M}}_{e(v),n(v),0} \times \prod_{v,\ g(v)\geq 1} \mathcal{M}_{g(v),n(v)} \right)$$

Let J be the graph spanned by edges adjacent to v_0 and $b > 11i + 7e + 9a - 5$ external edges. The stable graph stratification $\overline{\mathcal{M}}_{e(v),n(v),0}$ has strata given by finite quotients of copies of $\mathcal{M}_{0,n}$, so the Borel–Moore homology spectral sequence shows that the pure Borel–Moore homology of $\overline{\mathcal{M}}_{e(v),n(v),0}$ is spanned by fundamental classes of the strata: $[\overline{\mathcal{M}_H}]$ for stable graphs H with $\overline{H} = J$ and $\dim \mathcal{M}_H \leq i$. Thus it suffices to show that

$[\overline{\mathcal{M}_H}] \otimes c' = f^*(d)$ for some surjection $f : [n] \to [n-1]$. By Lemma 6.3.8 at least one of the following occurs:

1. $\sum_{v \text{ vertex}}(n(v)-3)$ is $> i$, this contradicts the dimension of $\mathcal{M}_H$ and so does not hold.
2. There a trivalent vertex v adjacent to 2 external edges, labelled by $s < t \in [n]$. Let f be the surjection $[n] \to [n]/s \sim t \to [n-1]$, where we have fixed a bijection b between $[n]/s \sim t$ and $[n-1]$. Let K be the graph obtained from H by merging the edges labelled by s, t, labelled by $[n-1]$ via the bijection b. Then $f^*([\overline{\mathcal{M}_K}] \otimes c') = [\overline{\mathcal{M}_H}] \otimes c'$.
3. There are two adjacent trivalent vertices v_1, v_2 in H, such that v_1, v_2 both have external edges. Then applying the cross ratio relation to H at v_1, v_2 we have that $[\overline{\mathcal{M}_H}] = [\overline{\mathcal{M}_{H'}}]$ where H' has a vertex which is adjacent to two external edges. Now the same argument as 2) applies.

Therefore we have $[\overline{\mathcal{M}_H}] \otimes c' = f^*(d)$ for some surjection $f : [n] \to [n-1]$, and we are done. $\square$

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