Long Time Behavior of 2d Water Waves

by

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ABSTRACT

In this thesis, we study two problems concerning the long time behavior of the two dimensional water waves.

In the first project, we study the motion of the two dimensional inviscid incompressible, infinite depth water waves with point vortices in the fluid. We show that the Taylor sign condition $-\frac{\partial P}{\partial n} \geqslant 0$ can fail if the point vortices are sufficiently close to the free boundary, so the water waves can be subject to the Taylor instability. Assuming the Taylor sign condition, we prove that the water wave system with point vortices is locally wellposed in Sobolev spaces. Moreover, we show that if the water waves is initially symmetric with a certain symmetric vortex pair traveling downwards, then the free interface remains smooth for a long time, and for initial data of size $\epsilon \ll 1$, the lifespan of the solution is at least $O(\epsilon^{-2})$.

In the second project, we rigorously justify the Peregrine soliton from the full water waves. The Peregrine soliton $Q(x,t) = e^{it}(1 - \frac{4(1+2it)}{1+4x^2+4t^2})$ is an exact solution of the 1d focusing nonlinear schrödinger equation (NLS) $iB_t + B_{xx} = -2|B|^2B$, having the feature that it decays to e^{it} at the spatial and time infinities, and with a peak and troughs in a local region. It is considered as a prototype of the rogue waves by the ocean waves community. The 1D NLS is related to the full water wave system in the sense that asymptotically it is the envelope equation for the full water waves. In this project, working in the framework of water waves which decay non-tangentially, we give a rigorous justification of the NLS from the full water waves equation in a regime that allows for the Peregrine soliton. As a byproduct, we prove long time existence of solutions for the full water waves equation with small initial data in space of the form $H^s(\mathbb{R}) + H^{s'}(\mathbb{T})$, where $s \geqslant 4, s' > s + \frac{3}{2}$.

CHAPTER I

Introduction

1.1 Background

The two dimensional inviscid incompressible infinite depth water waves without surface tension is described by the free boundary Euler equations (It's called the water wave equations)

$$\begin{cases} v_t + v \cdot \nabla v = -\nabla P - (0,1) & on\ \Omega(t), \quad t \geqslant 0 \\ div\ v = 0, \quad curl\ v = \omega, & on\ \Omega(t), \quad t \geqslant 0 \\ P\Big|_{\Sigma(t)} \equiv 0, & t \geqslant 0 \\ (1,v) \ \text{ is tangent to the free surface } (t, \Sigma(t)). \end{cases} \tag{1.1}$$

Here $\Omega(t)$ is the fluid region, with a free interface $\Sigma(t)$, such that $\Sigma(t)$ separates the fluid region with density one from the air with density zero. v is the fluid velocity, and P is the pressure. In the irrotational case, $\omega \equiv 0$.

The Euler equations and its variants (notably, the Navier and Stokes equations) are the basic models used by scientists to study the motion of fluids. To understand the evolution of the fluid, it's natural to study the fundamental questions of the existence, uniqueness, regularity, stability, formation of singularity, and asymptotic behavior of solutions of the water wave equations (1.1). Since the boundary is unknown and the momentum equation and the boundary conditions are strongly nonlinear, the water wave equations (1.1) pose severe challenges for both rigorous mathematical analysis and numerical simulation. Even without the free boundaries, the closely related Millennium Prize Problem–the global regularity of the Navier-Stokes equations, is still open.

Because of its physical importance and the great difficulty, the mathematical theory of water waves has been a fascinating subject that has attracted the attention of scientists for centuries. For early works, see Newton [48], Stokes[56], Levi-Civita[43], and G.I. Taylor [57]. Nalimov [47], Yosihara[74] and Craig [20] proved local well-posedness for 2d water waves equation (1.79) for small initial data. In S. Wu's breakthrough works [69][70], she proved that for $n \geqslant 2$ the important strong Taylor sign condition

$$-\frac{\partial P}{\partial n}\Big|_{\Sigma(t)} \geqslant c_0 > 0 \tag{1.2}$$

always holds for the infinite depth water wave system (1.79), as long as the interface is non-self-intersecting and smooth, and she proved that the initial value problem for (1.79) is locally well-posed in $H^s(\mathbb{R}), s \geqslant 4$ without smallness assumption. Since then, a lot of interesting local well-posedness results were obtaind, see for example [2], [5], [15], [17], [36], [42], [44], [49], [54], [76]. Recently, almost global and global well-posedness for water waves (1.79) under irrotational assumption have also been proved, see [71], [72], [29], [39], [1], and see also [32] and [33]. More recently, there are strong interests in understanding the singularities of water waves, see for example [41], [66], [67], [68]. For the formation of splash singularities, see for example [11][12][18] [19] .

Note that most of the aforementioned works are done in irrotational setting. Also, the aforementioned works assume that the fluid is at rest at spatial infinity. However, rotational fluids are commonplace in nature, and there are many important water waves that are not at rest at ∞, so it's necessary to relax the aforementioned assumptions, which is the goal of the current thesis. My thesis consists of two projects. The first project concerns the long time behavior of water waves with non-constant vorticity, more specifically, the long time behavior of water waves with point vortices. The second project concerns water waves with nonvanishing boundary behavior, in particular, the justification of the Peregrine soliton from the full water waves.

1.1.1 Background of *Project 1*: Rotational water waves

Despite immense progress on Cauchy problem of irrotatoinal water waves, much less rigorous mathematical analysis have been done for rotational water waves. For vorticity ω that is a smooth function, Iguchi, Tanaka, and Tani [37] proved the local wellposedness of the free boundary problem for an incompressible ideal fluid in two space dimensions without surface tension. Ogawa and Tani [49] generalized Iguchi, Tanaka, and Tani's work to the case with surface tension. In [50], Ogawa and Tani generalized the wellposedness result to the finite depth case. In [15], Chritodoulou and Lindblad obtained a priori energy estimates of n dimensional incompressible fluid, without assuming irrotationality condition in a bounded domain without gravity. In particular, the authors introduce a geometrical point of view, estimating quantities such as the second fundamental form and the velocity of the free surface. For the same problem as in [15], H. Lindblad proved local wellposedness in [44]. The local wellposedness of rotational 3d infinite depth, inviscid incompressible water waves is proved by P. Zhang and Z. Zhang [76]. All the aforementioned existence results for rotational water waves are locally in time, and under the assumption that the strong Taylor sign condition holds.

Regarding the long time behavior, the only existing work is [34], in which Ifrim and Tataru proved cubic lifespan for 2d inviscid incompressible infinite depth water waves when the vorticity is a constant in the entire fluid domain. Assume the volocity field is (u, v). Assume the vorticity is $\omega = v_x - u_y \equiv c$, where c is a constant, the main idea of [34] is to replace the velocity field (u, v) by $(u + cy, v)$, then the problem is reduced to irrotational incompressible water waves, and from which the long time existence is proved. In [8], Bieri, Miao, Shahshahani, and Wu prove cubic lifespan for the motion of a self-gravitating incompressible fluid in a bounded domain with free boundary for small initial data for the irrotational case and for the case of constant vorticity. The case of constant vorticity is reduced to the irrotational one by working in a rotating framework with constant angular velocity.

Our goal in the first project is to understand the long time behavior of rotational water waves with non-constant vorticity. For water waves with non-constant vorticity, there is no obvious transformation to reduce the problem to the irrotational one, and the study of the long time behavior of the full water waves with non-constant vorticity requires new ideas.

One way to handle the vorticity is to discretize and assume it takes the form $\omega \approx \sum_{j=1}^{N} \psi_j$, where each ψ_j is a bump centered at some point $z_j \in \Omega(t)$. A useful choice of functions ψ_j is

$$\psi_j(z) = \lambda_j \delta^{-2} 1_{B_\delta(z_j)}(z), \qquad B_\delta(z_j) := \{z \in \mathbb{R}^2 : \quad |z - z_j| \leqslant \delta\}, \qquad \delta \textit{ small}.$$

Each $B_\delta(z_j)$ is called a vortex patch, and λ_j is the strength of the vorticity near z_j. This approximation is particular relevant when the fluid is highly rotational in a few small localized regions. A typical example is the water waves with submerged bodies. In this case, the vorticity is generated by the submerged obstacles. If the obstacles are sufficiently small, then we can assume the vorticity is supported at N distinct points, i.e.,

$$\omega(\cdot, t = 0) = \sum_{j=1}^{N} \lambda_j \delta_{z_j}(\cdot). \tag{1.3}$$

Each z_j is called a point vortex, with strength λ_j. The point vortex z_j generates a rotational velocity field $\frac{\lambda_j i}{2\pi} \frac{1}{\overline{z - z_j(t)}}$. Here, $z = x + iy$, $z_j(t) = x_j(t) + iy_j(t)$, and $\mathbb{R}^2$ is identified with $\mathbb{C}$, i.e., $(x, y) \mapsto x + iy$, and $\overline{z - z_j(t)}$ represents the complex conjugate of $z - z_j(t)$.

The water waves with point vortices describe the motion of submerged bodies, and it's believed to give insight to the study of turbulence. We are interested in the following question:

Question I.1. *If the interface of the water wave with point vortices is a small perturbation of the horizontal line initially, will it remain a small perturbation for a very long time? And how long will it remain small?*

In the first project of this dissertation, we'll give an affirmative answer to Question I.1. See Theorem I.10 for more details.

1.1.2 Background of *Project 2*: the Peregrine soliton

Another important research direction concerns the behavior of the water waves in various asymptotic regimes, see for example [21][52][4]. The 1d cubic NLS

$$iu_t + u_{xx} = -2|u|^2 u \tag{1.4}$$

is relevant in the deep water regime. It is completely integrable, and has many exact solutions. The 1d NLS is related to the full water wave system, in the sense that asymptotically it is the envelope equation for the free interface of the water waves. If one performs multiscale analysis to determine the modulation approximations to the solution of the finite or infinite depth 2d water waves equations, i.e., a solution $z(\alpha, t)$ of the parametrized free interface which is to the leading order a wave packet of the form

$$W(\alpha,t) := \alpha + \epsilon B(X,T)e^{i(k\alpha+\gamma t)} \quad (\epsilon \textit{ small}, \quad k, \gamma : \textit{constant}), \tag{1.5}$$

then B solves the 1d focusing cubic NLS. In infinite depth case, $X = \epsilon(\alpha + \frac{1}{2\gamma}t), T = \epsilon^2 t$, and $\gamma = \sqrt{k}$. So the envelope B is a profile that travels at the group velocity $\frac{1}{2\gamma} = \frac{d\gamma}{dk}$ determined by the dispersion relation of the water wave equations on time scale $O(\epsilon^{-1})$, and evolves according to the NLS on time scale $O(\epsilon^{-2})$.

This discovery was derived formally by Zakharov [75] for the infinite depth case, and by Hasimoto and Ono [30] for the finite depth case. In [22], Craig, Sulem, and Sulem applied modulation analysis to the finite depth 2D water wave equation, derived an approximate solution of the form of a wave packet and showed that the modulation approximation satisfies the 2D finite depth water wave equation to the leading order. In [53], Schneider and Wayne justified the NLS as the modulation approximation for a quasilinear model that captures some of the main features of the water wave equations.

The rigorous justification of the NLS for the full water waves was given by Totz and Wu [63] in infinite depth case, and the justification in a canal of finite depth was proved by Düll, Schneider and Wayne [25] . See also [35]. All of these works assume the data vanish at spatial infinity. However, there are many important solitons of NLS that are neither periodic nor vanishing at ∞. One such important example is the Peregrine soliton discovered by Peregrine in 1983 [51], which is defined by

$$Q(x,t) = e^{it}\left(1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right). \tag{1.6}$$

Plug Q in (1.5), one observes that W a weakly oscillatory periodic wave at the time and spatial infinity, but W has peaks and troughs at a local region. The Peregrine soliton is important to the ocean waves because the feature of the corresponding wave packet W is consistent with the qualitative description of a rogue wave in the ocean. We call the wave packet corresponding to the Peregrine soliton just by the Peregrine soliton. Indeed, the Peregrine soliton is conjectured to be one of the mechanisms for the formation of rogue waves by the ocean waves community, see [55] for more details. In 2010, the Peregrine soliton was observed in fibre optics [40], which shows that the Peregrine soliton is a nature phenomenon rather than just a mathematical prediction! Stimulated by this discovery, there have been a lot of efforts to produce the Peregrine solitons in other backgrounds, for example, in [13], the authors carried out the first experiment to observe Peregrine-type breather solutions in a water tank. These experiments suggest the Peregrine soliton is a plausible description of the formation of rogue waves. So it's desirable to have a mathematical theory to justify that the Peregrine soliton can be developed in water waves. Since the motion of the water waves is governed by the water wave equations, while the Peregrine soliton is an exact solution of the NLS, we ask the following question:

Question I.2. *Is there any solution $z(\alpha, t)$ to the system (1.81) with its envelope looks like the Peregrine soliton?*

Since the leading order of the envelope of the wave packet $W(\alpha,t) = \alpha + \epsilon B(X,T)e^{i(k\alpha+\gamma t)}$ evolves according to the NLS on time scale $O(\epsilon^{-2})$, in order to observe the evolution of the wave packet, the observer must focus on the water waves on time scale $O(\epsilon^{-2})$. So a more precise formulation of Question I.2 is as follows:

Question I.3. *Is there any solution $z(\alpha, t)$ to (1.81) such that*

$$\sup_{t\in[0,O(\epsilon^{-2})]} \|(z - W, \quad z_t - W_t, \quad z_{tt} - W_{tt})\| = o(\epsilon)? \tag{1.7}$$

Here $\|\cdot\|$ denotes some norm.

Since B is neither periodic nor vanishing at ∞, the framework in [63] or [25] cannot be applied to justify the Peregrine soliton from the full water wave equations. In the second project of this dissertation, we give an affirmative answer to Question I.3. See Theorem I.16 for more details.

1.2 *Project 1*: Settings, main results, and strategy

In this project, we investigate the two dimensional inviscid incompressible infinite depth water wave system with point vortices in the fluid. This system arises in the study of submerged bodies in a fluid (see for example [14],[24] and the references therein). It's also believed to give some insight into the problem of turbulence ([45], chap 4, §4.6). In idealized situation, such water waves are described by assuming the vorticity is a Dirac delta measure, i.e., the vorticity is given by $\omega(\cdot,t) = \sum_{j=1}^{N} \lambda_j \delta_{z_j(t)}(\cdot)$, where each $z_j \in \Omega(t)$, $\lambda_j \in \mathbb{R}$ represents the position and the strength of the j-th point vortex, respectively. It's well-known that each point vortex z_j generates a velocity field $\frac{\lambda_j}{2\pi}\frac{i}{\overline{z-z_j(t)}}$, which is purely rotational. We assume $z_j(0) \neq z_k(0)$, for $j \neq k$. Then the motion of the fluid is described by

$$\begin{cases} \left.\begin{array}{l} v_t + v\cdot\nabla v = -\nabla P - (0,1) \\ div\ v = 0 \\ curl\ v = \omega = \sum_{j=1}^{N}\lambda_j\delta_{z_j(t)} \end{array}\right\} \quad \Omega(t) \\ \frac{d}{dt}z_j(t) = (v - \frac{\lambda_j i}{2\pi}\frac{1}{\overline{z-z_j(t)}})\Big|_{z=z_j} \\ z_j(t)\in\Omega(t), \quad j=1,...,N \\ P\Big|_{\Sigma(t)} \equiv 0 \\ (1,v) \text{ is tangent to the free surface } (t,\Sigma(t)). \end{cases} \tag{1.8}$$

Here, $z = x+iy, z_j(t) = x_j(t)+iy_j(t)$.

Formally, this system is obtained by neglecting the self-interaction of the point vortices: intuitively, if we pretend that the velocity field v is well-defined at $z = z_j(t)$, then the motion of the fluid particle $z_j(t)$ is given by

$$\frac{d}{dt}z_j(t) = v(z_j(t),t) = (v - \frac{\lambda_j i}{2\pi}\frac{1}{\overline{z-z_j(t)}})\Big|_{z=z_j} + \frac{\lambda_j i}{2\pi}\frac{1}{\overline{z-z_j(t)}})\Big|_{z=z_j}.$$

We assume that the only singularities of v are at the point vortices, so $v - \frac{\lambda_j i}{2\pi}\frac{1}{\overline{z-z_j(t)}}$ is smooth, while $\frac{\lambda_j i}{2\pi}\frac{1}{\overline{z-z_j(t)}}$ is not defined at $z = z_j$. However, since the velocity field $\frac{\lambda_j i}{2\pi}\frac{1}{\overline{z-z_j(t)}}\Big|_{z\neq z_j}$ is purely rotational around $z_j(t)$, it won't move the center $z_j(t)$ at all, which means

$$\frac{d}{dt}z_j(t) = (v - \frac{\lambda_j i}{2\pi}\frac{1}{\overline{z-z_j(t)}})\Big|_{z=z_j}.$$

For a rigorous justification of this derivation, see [45] (Theorem 4.1, 4.2, chapter 4) for the fixed boundary case.

The system (1.8) has attracted a lot of attention from both mathematics and physics communities, and there have been a lot of numerical study of the system (1.8). See for example [23], [31], [27], [46], [61], [65] and the references therein for some numerical investigations. However, the rigorous mathematical analysis for water waves with point vortices is still missing.

1.2.1 Governing equation for the free boundary

It's easy to see that the system (1.8) is completely determined by the free surface $\Sigma(t)$, the velocity and the acceleration along the free surface, and the position of the point vortices.

1.2.1.1 Lagrangian formulation

We parametrize the free surface by Lagrangian coordinates, i.e., let α be such that

$$z_t(\alpha,t) = v(z(\alpha,t),t). \tag{1.9}$$

We identify $\mathbb{R}^2$ with the complex plane. With this identification, a point (x,y) is the same as $x+iy$. Since $P\equiv 0$ along $\Sigma(t)$, we can write ∇P as $-iaz_\alpha$, where $a=-\frac{\partial P}{\partial n}\frac{1}{|z_\alpha|}$ is real valued. So the momentum equation along the free boundary becomes

$$z_{tt} - iaz_\alpha = -i. \tag{1.10}$$

Note that the second and the third equation in (1.8) imply that $\overline{v-\sum_{j=1}^N \frac{\lambda_j i}{2\pi(\overline{z-z_j(t)}})} = \bar{v}+\sum_{j=1}^N \frac{\lambda_j i}{2\pi(z-z_j(t))}$ is holomorphic in $\Omega(t)$ with the value at the boundary $\Sigma(t)$ given by $\bar{z}_t+\sum_{j=1}^N \frac{\lambda_j i}{2\pi(z(\alpha,t)-z_j(t))}$. Assume that $\bar{z}_t+\sum_{j=1}^N \frac{\lambda_j i}{2\pi(z(\alpha,t)-z_j(t))}\in L^2(\mathbb{R})$. We know that $\bar{z}_t+\sum_{j=1}^N \frac{\lambda_j i}{2\pi(z(\alpha,t)-z_j(t))}$ is the boundary value of a holomorphic function in $\Omega(t)$ if and only if

$$(I-\mathfrak{H})\Big(\bar{z}_t+\sum_{j=1}^N \frac{\lambda_j i}{2\pi(z(\alpha,t)-z_j(t))}\Big) = 0, \tag{1.11}$$

where $\mathfrak{H}$ is the Hilbert transform associated with the curve $z(\alpha,t)$, i.e.,

$$\mathfrak{H}f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{z_\beta}{z(\alpha,t) - z(\beta,t)} f(\beta) d\beta. \tag{1.12}$$

So the system (1.8) is reduced to a system of equations for the free boundary coupled with the dynamic equation for the motion of the point vortices:

$$\begin{cases} z_{tt} - iaz_\alpha = -i \\ \frac{d}{dt} z_j(t) = (v - \frac{1}{2\pi} \overline{\frac{\lambda_j i}{z - z_j}})\Big|_{z=z_j} \\ (I - \mathfrak{H})\Big(\bar{z}_t + \sum_{j=1}^N \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))}\Big) = 0. \end{cases} \tag{1.13}$$

Note that v can be recovered from (1.13). Indeed, we have

$$\bar{v}(z) + \sum_{j=1}^N \frac{\lambda_j i}{2\pi(z - z_j(t))} = \frac{1}{2\pi i} \int \frac{z_\beta}{z - z(\beta)} \Big(\bar{z}_t(\beta) + \sum_{j=1}^N \frac{\lambda_j i}{2\pi(z(\beta) - z_j(t))}\Big) d\beta. \tag{1.14}$$

So the system (1.8) and the system (1.13) are equivalent.

1.2.1.2 The Riemann mapping formulation and the Taylor sign condition.

Let $\mathbf{n}$ be the outward unit normal to the fluid-air interface $\Sigma(t)$. The quantity $-\frac{\partial P}{\partial \mathbf{n}}\Big|_{\Sigma(t)}$ plays an important role in the study of water waves.

Definition I.4. (The Taylor sign condition and the strong Taylor sign condition)

(1) If $-\frac{\partial P}{\partial \mathbf{n}}\Big|_{\Sigma(t)} \geqslant 0$ pointwisely, then we say the Taylor sign condition holds.

(2) If there is some positive constant c_0 such that $-\frac{\partial P}{\partial \mathbf{n}}\Big|_{\Sigma(t)} \geqslant c_0 > 0$ pointwisely, then we say the strong Taylor sign condition holds.

It is well known that when surface tension is neglected and the Taylor sign condition fails, the motion of the water waves can be subject to the Taylor instability [6],[10],[58],[26],[73].

For irrotational incompressible infinite depth water waves without surface tension, S. Wu [69][70] shows that the strong Taylor sign condition always holds provided that the interface is non self-intersecting and smooth.

For rotational water waves, by constructing explicit examples, we'll show that the Taylor sign condition can fail if the point vortices are sufficient close to the interface. We'll also give a criterion for the Taylor sign condition to hold. To calculate the important quantity $-\frac{\partial P}{\partial \mathbf{n}}$, we use the Riemann mapping formulation of the system (1.13), which we are to describe.

Let $\Phi(\cdot,t):\Omega(t)\to\mathbb{P}_-$ be the Riemann mapping such that $\Phi_z\to 1$ as $z\to\infty$. Let $h(\alpha,t):=\Phi(z(\alpha,t),t)$. Denote

$$Z(\alpha,t):=z\circ h^{-1}(\alpha,t),\qquad b=h_t\circ h^{-1},\qquad D_t:=\partial_t+b\partial_\alpha, \tag{1.15}$$

$$A:=(ah_\alpha)\circ h^{-1}. \tag{1.16}$$

In Riemann mapping variables, the system (1.13) becomes

$$\begin{cases}(D_t^2-iA\partial_\alpha)Z=-i\\ \frac{d}{dt}z_j(t)=(v-\dfrac{\lambda_j i}{2\pi(\overline{z-z_j})})\Big|_{z=z_j}\\ (I-\mathbb{H})(D_t\bar{Z}+\sum_{j=1}^N\dfrac{\lambda_j i}{2\pi(Z(\alpha,t)-z_j(t))})=0.\end{cases} \tag{1.17}$$

Here, $\mathbb{H}$ is the standard Hilbert transform which is defined by

$$\mathbb{H}f(\alpha):=\frac{1}{\pi i}p.v.\int_{-\infty}^{\infty}\frac{1}{\alpha-\beta}f(\beta)d\beta. \tag{1.18}$$

Denote

$$A_1:=A|Z_\alpha|^2. \tag{1.19}$$

Since $-\frac{\partial P}{\partial \mathbf{n}}\Big|_{\Sigma(t)} = \frac{A_1}{|Z_\alpha|}$, it's clear that the Taylor sign condition holds if and only if

$$\inf_{\alpha\in\mathbb{R}} \frac{A_1}{|Z_\alpha|} \geqslant 0, \tag{1.20}$$

and the strong Taylor sign condition holds if and only if

$$\inf_{\alpha\in\mathbb{R}} \frac{A_1}{|Z_\alpha|} > 0, \tag{1.21}$$

1.2.2 The main results of ***Project 1***

Our first result is a formula for the quantity A_1, from which we show that Taylor sign condition could fail if the point vortices are sufficient close to the interface. We also use this formula to find a criterion for strong Taylor sign condition to hold.

Let F be the holomorphic extension of $\bar{z}_t + \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \dfrac{1}{z(\alpha,t) - z_j(t)}$ in the domain $\Omega(t)$.

Theorem I.5. *Denote*

$$c_0^j := (\Phi^{-1})_z(\omega_0^j, t), \qquad \omega_0^j := \Phi(z_j(t), t). \tag{1.22}$$

$$\beta_0(t) := \inf_{\alpha\in\mathbb{R}} |Z_\alpha(\alpha,t)|, \qquad M_0(t) := \|F(\cdot,t)\|_\infty \tag{1.23}$$

$$\tilde{\lambda} =: \frac{\sum_{j=1}^N |\lambda_j|}{2\pi}, \qquad \tilde{d}_I(t) := \min_{1\leqslant j\leqslant N} \inf_{\alpha\in\mathbb{R}} |\alpha - \Phi(z_j(t),t)|, \qquad \tilde{d}_P(t) = \min_{j\neq k} |z_j(t) - z_k(t)|. \tag{1.24}$$

(1) (Formula for the Taylor sign condition) We have

$$A_1 = 1 + \frac{1}{2\pi}\int \frac{|D_tZ(\alpha,t) - D_tZ(\beta,t)|^2}{(\alpha-\beta)^2} d\beta - \sum_{j=1}^N \frac{\lambda_j}{\pi} Re\Big\{\frac{D_tZ - \dot{z}_j}{c_0^j(\alpha - w_0^j)^2}\Big\}, \tag{1.25}$$

(2) (Failure of the Taylor sign condition) Taylor sign condition could Fail if $\tilde{d}_I(t)$ is sufficiently small.

(3) (A criterion for strong Taylor sign condition) If

$$\frac{\tilde{\lambda}^2}{\tilde{d}_I(t)^3\beta_0} + \frac{\tilde{\lambda}^2}{\tilde{d}_I(t)^2\tilde{d}_P(t)} + \frac{2M_0\tilde{\lambda}}{\tilde{d}_I(t)^2} < \beta_0, \tag{1.26}$$

then the strong Taylor sign condition holds.

Part (1) is proved in Corollary II.20, part (2) is proved by examples from §2.3.3, §2.3.4, and part (3) is proved in Proposition II.25.

We obtain in our second result the locally wellposed in Sobolev spaces of the equation (1.13), provided that the strong Taylor sign condition holds initially. Let H^s represents the Sobolev space $H^s(\mathbb{R})$, which is define as

$$H^s(\mathbb{R}) := \Big\{ f \in L^2(\mathbb{R}) : \quad \|f\|_{H^s}^2 := \int_{-\infty}^{\infty} (1 + |2\pi\xi|^{2s})|\mathcal{F}f(\xi)|^2 d\xi < \infty \Big\}.$$

Here, $\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$ is the Fourier transform of f. If s is a nonnegative integer, then

$$\|f\|_{H^s}^2 \leqslant \sum_{k=0}^{s} \left\| \partial_\alpha^k f \right\|_{L^2}^2. \tag{1.27}$$

Decompose z_t as

$$\bar{z}_t(\alpha, t) = f + p, \qquad \text{where} \quad p = -\sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{z(\alpha, t) - z_j(t)}. \tag{1.28}$$

Note that p and p_t are determined by $z(\alpha, t)$ and f.

• <u>A discussion on the initial data</u> Assume that the initial value for (1.13) is given by

$$\xi_0(\alpha) := z(\alpha, t = 0) - \alpha, \qquad v_0 := z_t(t = 0), \qquad w_0 := z_{tt}(t = 0), \tag{1.29}$$

and denote

$$a_0 := a(t = 0). \tag{1.30}$$

ξ_0, v_0, w_0, a_0 must satisfy

$$w_0 - ia_0(\partial_\alpha \xi_0 + 1) = -i, \tag{1.31}$$

where a_0 is determined by

$$a_0|\partial_\alpha \xi_0 + 1| = |w_0 + i|. \tag{1.32}$$

v_0 satisfies

$$(I - \mathfrak{H}_0)\bar{v}_0 = -\sum_{j=1}^{N} \frac{\lambda_j i}{\pi} \frac{1}{\xi_0(\alpha) + \alpha - z_j(0)}, \tag{1.33}$$

where $\mathfrak{H}_0$ is the Hilbert transform associated with the curve $z(\alpha, 0) = \xi_0 + \alpha$.

Denote

$$d_I(t) := \min_{1 \leqslant j \leqslant N} \{d(z_j(t), \Sigma(t))\}, \qquad d_P(t) := \min_{\substack{1 \leqslant i,j \leqslant N \\ i \neq j}} \{d(z_i(t), z_j(t))\}. \tag{1.34}$$

Remark I.6. $d_I(t)$ represents the distance of the point vortices to the free boundary, the $'I'$ means interface. $d_P(t)$ represents the distance among the point vortices, $'P'$ means point vortices.

Let $\delta > 0$. Let $|D|^\delta$ be defined by

$$\mathcal{F}|D|^\delta f(\xi) = (2\pi|\xi|)^\delta \mathcal{F}f(\xi). \tag{1.35}$$

Theorem I.7. *(The local wellposedness) Assume* $s \geqslant 4$. *Assume* $(|D|^{1/2}\xi_0, v_0, w_0) \in H^s \times H^{s+1/2} \times H^s$, *satisfying (1.31), (1.32), (1.33), and*

(H1) Strong Taylor sign assumption. There is some $\alpha_0 > 0$ *such that*

$$\inf_{\alpha \in \mathbb{R}} a(\alpha, 0)|z_\alpha(\alpha, t = 0)| \geqslant \alpha_0 > 0. \tag{1.36}$$

(H2) Chord-arc assumption. There are constants $C_1, C_2 > 0$ *such that*

$$C_1|\alpha - \beta| \leqslant |z(\alpha, 0) - z(\beta, 0)| \leqslant C_2|\alpha - \beta|. \tag{1.37}$$

Then exists $T_0 > 0$ such that (1.13) admits a unique solution

$$(|D|^{1/2}(z-\alpha), z_t, z_{tt}) \in C([0,T_0]; H^s \times H^{s+1/2} \times H^s),$$

with T_0 depends on $\|(\partial_\alpha \xi_0, v_0, w_0)\|_{H^{s-1}\times H^s \times H^s}$, $d_I(0)^{-1}, d_P(0)^{-1}, C_1, C_2, \alpha_0$, s, and

$$\inf_{t\in[0,T_0]} \inf_{\alpha\in\mathbb{R}} a(\alpha,t)|z_\alpha(\alpha,t)| \geqslant \alpha_0/2. \tag{1.38}$$

Moreover, if T_0^* is the maximal lifespan, then either $T_0^* = \infty$, or $T_0^* < \infty$, but

$$\lim_{T\to T_0^*-} \|(z_t, z_{tt})\|_{C([0,T];H^s\times H^s)} + \sup_{t\to T_0^*} (d_I(t)^{-1} + d_P(t)^{-1}) = \infty. \tag{1.39}$$

or

$$\lim_{t\to T_0^*-} \inf_{\alpha\in\mathbb{R}} a(\alpha,t)|z_\alpha(\alpha,t)| \leqslant 0, \tag{1.40}$$

or

$$\sup_{\substack{\alpha\neq\beta \\ 0\leqslant t<T_0^*}} \left|\frac{\alpha-\beta}{z(\alpha,t)-z(\beta,t)}\right| + \sup_{\substack{\alpha\neq\beta \\ 0\leqslant t<T_0^*}} \left|\frac{z(\alpha,t)-z(\beta,t)}{\alpha-\beta}\right| = \infty. \tag{1.41}$$

Remark I.8. (H1) is the strong Taylor sign condition. By Theorem I.5, the Taylor sign condition (1.36) does not always hold. As was explained before, if surface tension is neglected and Taylor sign condition fails, the motion of the water waves could be subject to the Taylor instability. In order for the system (1.13) to be wellposed in Sobolev spaces, we need to assume that the Taylor sign condition (1.36) to hold.

Our third result is concerned with the long time behavior of the water waves with point vortices. We show that if the water wave is symmetric with a symmetric vortex pair traveling downward initially, then the free interface remains smooth for a long time, and for initial data satisfying

$$\|(|D|^{1/2}(z(\alpha,0)-\alpha), f, f_t)\|_{H^s\times H^{s+1/2}\times H^s} \leqslant \epsilon \ll 1,$$

the lifespan is at least $\delta_0\epsilon^{-2}$, for some $\delta_0 > 0$. Define[1]

$$\widehat{d_I}(t) := \min_{j=1,2} \inf_{\alpha\in\mathbb{R}} Im\{z(\alpha,t) - z_j(t)\}. \tag{1.42}$$

We make the following assumptions:

(H3) Vortex pair assumption. Assume $N = 2$, i.e., there are two point vortices, with positions $z_1(t) = x_1(t) + iy_1(t))$, $z_2(t) = x_2(t) + iy_2(t)$, strength λ_1, λ_2, respectively. Assume further that $z_1(t)$ and $z_2(t)$ are symmetric about the y-axis, i.e.,

$$x_1(t) = -x_2(t) = -x(t) < 0, \quad y_1(t) = y_2(t) := y(t) < 0,$$

and assume $\lambda_1 = -\lambda_2 := \lambda < 0$.

(H4) Symmetry assumption. Assume that velocity field $v = v_1 + iv_2$ satisfies: v_1 odd in x, v_2 even in x, and the free boundary $\Sigma(t)$ is symmetric about the y-axis.

(H5) Smallness assumption. Assume that at $t = 0$,

$$\|(|D|^{1/2}\xi_0, f(t=0), f_t(t=0))\|_{H^s\times H^{s+1/2}\times H^s} \leqslant \epsilon, \qquad \lambda^2 + |\lambda x(0)| \leqslant c_0\epsilon,$$

for some constant $c_0 = c_0(s)$. We can take $c_0 = \frac{1}{((s+12)!)^2}$.

(H6) Vortex-vortex interaction. Assume $\frac{|\lambda|}{x(0)} \geqslant M\epsilon$ for some constant $M \gg 1$ (say, $M = 200\pi$).

(H7) Vortex-interface interaction. Assume $\widehat{d_I}(0) \geqslant 1$. Assume $|\lambda| + x(0) \leqslant 1$.

Remark I.9. Assume (H3)-(H4) holds at $t = 0$, then by the uniqueness of the solutions to the system (1.8), (H3)-(H4) holds for all $t \in [0, T]$ when the solution exists.

[1] We use the notation $\widehat{d_I}$ to distinguish it from d_I. Please keep in mind that $\widehat{d_I}$ is not the Fourier transform of d_I.

Theorem I.10 (Long time behavior). *Let $s \geqslant 4$. Assume (H3)-(H7). There exists $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for all $0 < \epsilon \leqslant \epsilon_0$, the lifespan T_0^* of the solution to (1.8) satisfies $T_0^* \geqslant \delta_0 \epsilon^{-2}$. Moreover, there exists a constant C_s only depends on s such that*

$$\sup_{t \in [0, \delta_0 \epsilon^{-2}]} \left\| (|D|^{1/2}(z(\alpha,t)-\alpha), f(\cdot,t), f_t(\cdot,t)) \right\|_{H^s \times H^{s+1/2} \times H^s} \leqslant C_s \epsilon. \tag{1.43}$$

Here, δ_0 is an absolute constant independent of ϵ and s.

Remark I.11. If the initial data is sufficiently localized, then we can prove global wellposedness and modified scattering. A brief discussion of the main idea will be given in §1.2.4. We will give the full details of the proof in a forthcoming paper.

Remark I.12. The assumption $\frac{|\lambda|}{x(0)} \geqslant M\epsilon$ ensures that the point vortices travel downward at $t = 0$. In the proof of Theorem I.10, we will show that when $\frac{|\lambda|}{x(0)} = M\epsilon$, the velocity of the point vortices is comparable to ϵ, which is slow in some sense. Theorem I.10 demonstrates that even if the point vortices moves at an initial velocity as slow as $M\epsilon$, the water waves still remain smooth and small for a long time.

Remark I.13. Assumption (H7) implies that the strong Taylor sign condition holds initially. The assumption $\widehat{d}_I(0) \geqslant 1$ can be relaxed. To avoid getting into too many technical issues, we simply assume $\widehat{d}_I(0) \geqslant 1$. The assumption $|\lambda| + x(0) \leqslant 1$ is not an essential assumption. We assume this merely for convenience.

Remark I.14. The assumptions (H5), (H6), (H7) do allow $x(0)$ to be as small as we want. So $\frac{|\lambda|}{x(0)}$ can be very large.

A discussion on initial data. We need to show that initial data for this system satisfy the assumptions of Theorem I.10 exist. As before, denote

$$\xi_0(\alpha) = z(\alpha,0) - \alpha, \quad z_0 = \alpha + \xi_0, \quad v_0 = z_t(\alpha,0), \quad w_0 = z_{tt}(\alpha,0).$$

We need ξ_0, v_0, w_0 satisfy (1.31), (1.32), and (1.33). We need also the symmetry condition

$$Re\{v_0\} \text{ is odd in } \alpha, \qquad Im\{v_0\} \text{ is even in } \alpha, \qquad Re\{\xi_0\} \textit{ is odd} \qquad Im\{\xi_0\} \textit{ is even}. \tag{1.44}$$

Denote the Hilbert transform associates to $z_0(\alpha)$ by $\mathfrak{H}_0$. We'll use the following lemma.

Lemma I.15. *Let $Im\{\xi_0\}$ be even, $Re\{\xi_0\}$ be odd. Let $f = f_1 + if_2$ be such that f_1 is odd and f_2 is even. Then $Re\{\mathfrak{H}_0 f\}$ is odd and $Im\{\mathfrak{H}_0 f\}$ is even.*

Proof. This is proved by direct calculation. □

Given ξ_0 be such that $Re\{\xi_0\}$ odd and $Im\{\xi_0\}$ even, and given any real valued odd function f, if we let v_0 be such that

$$\bar{v}_0 = \frac{1}{2}(I + \mathfrak{H}_0)f - \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{z_0(\alpha) - z_j(0)}, \tag{1.45}$$

then by lemma I.15, we have $Re\{v_0\}$ is odd and $Im\{v_0\}$ is even, and satisfying the compatiability condition

$$(I - \mathfrak{H}_0)(\bar{v}_0 + \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{z_0(\alpha) - z_j(0)}) = 0. \tag{1.46}$$

1.2.3 Strategy of proof

we illustrate the strategy of proving the main theorems in this subsection. The first two theorems are more or less routine, while Theorem I.10 requires some new idea of controlling the motion of the point vortices.

1.2.3.1 The Taylor sign condition: proof of Theorem I.5

We follow S. Wu's work [69] to calculate the Taylor sign condition. Using Riemann variables, the momentum equation is written as $(D_t^2 + iA\partial_\alpha)\bar{Z} = i$. Recall that $A_1 := A|Z_\alpha|^2$. Multiply $(D_t^2 + iA\partial_\alpha)\bar{Z} = i$ by Z_α, apply $I - \mathbb{H}$ on both sides of the resulting equation, then take imaginary part. By using the facts

$$(I-\mathbb{H})(Z_\alpha-1)=0, \qquad (I-\mathbb{H})(D_t\bar{Z}+\sum_{j=1}^N \frac{\lambda_j i}{2\pi}\frac{1}{Z(\alpha,t)-z_j(t)})=0, \tag{1.47}$$

we obtain

$$A_1=1+\frac{1}{2\pi}\int \frac{|D_tZ(\alpha,t)-D_tZ(\beta,t)|^2}{(\alpha-\beta)^2}d\beta - Im\Big\{\sum_{j=1}^N \frac{\lambda_j i}{2\pi}\Big((I-\mathbb{H})\frac{Z_\alpha}{(Z(\alpha,t)-z_j(t))^2}\Big)(D_tZ-\dot{z}_j(t))\Big\}.$$

Then use some tools from complex analysis, we obtain (1.25).

To construct examples for which the Taylor sign condition fails, we consider initially still water waves with its motion purely generated by the point vortices. We are able to derive a formula for A_1 in terms of the intensity and location of these point vortices, from which we can see that Taylor sign condition could fail if the point voritces are close to the interface.

1.2.3.2 Local wellposedness:proof of Theorem I.24

If there is no point vortices in the water waves, S. Wu observes that one can obtain quasilinearization of the system (1.13) by taking one time derivative to the momentum equation. It turns out that this is still true for water waves with point vortices: take ∂_t on both sides of $(\partial_t^2+ia\partial_\alpha)\bar{z}=i$, we obtain

$$(\partial_t^2+ia\partial_\alpha)\bar{z}_t=-ia_t\bar{z}_\alpha. \tag{1.48}$$

In (1.28), we decompose $\bar{z}_t$ as $\bar{z}_t=f+p$, where $p=-\sum_{j=1}^N \frac{\lambda_j i}{2\pi}\frac{1}{z(\alpha,t)-z_j(t)}$. A key observation is that $(\partial_t^2+ia\partial_\alpha)p$ consists of lower order terms. Apply $I-\mathfrak{H}$ on both sides of equation (1.48), we obtain

$$-i(I-\mathfrak{H})a_t\bar{z}_\alpha=g_1+g_2, \tag{1.49}$$

where

$$g_1 := 2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 \bar{z}_{t\beta} d\beta. \tag{1.50}$$

$$g_2 := \frac{i}{\pi}\sum_{j=1}^{N} \lambda_j\Big(\frac{2z_{tt} + i - \partial_t^2 z_j}{(z(\alpha,t) - z_j(t))^2} - 2\frac{(z_t - \dot{z}_j(t))^2}{(z(\alpha,t) - z_j(t))^3}\Big). \tag{1.51}$$

So $a_t\bar{z}_\alpha$ is of lower order. The quasilinear system

$$\begin{cases} (\partial_t^2 + ia\partial_\alpha)\bar{z}_t = -ia_t\bar{z}_\alpha \\ \dot{z}_j(t) = (v - \frac{\lambda_j i}{2\pi}\frac{1}{z(\alpha,t) - z_j(t)})\Big|_{z=z_j(t)} \\ (I - \mathfrak{H})(\bar{z}_t + \sum_{j=1}^N \frac{\lambda_j i}{2\pi}\frac{1}{z(\alpha,t) - z_j(t)}) = 0. \end{cases} \tag{1.52}$$

is of hyperbolic type as long as the Taylor sign condition $a|z_\alpha| \geqslant \alpha_0 > 0$ holds. The local wellposedness is obtained by energy method.

1.2.3.3 Long time behavior: proof of Theorem I.10

To illustrate the idea of studying long time behavior, we begin with the following toy model.

Toy model: Consider

$$u_{tt} + |D|u = u_t^p + \frac{C}{(\alpha + it)^m}, \qquad p \geqslant 2, m \geqslant 2. \tag{1.53}$$

for some constant C such that $|C| \lesssim \epsilon$. Define an energy

$$E_s(t) = \sum_{k\leqslant s}\int |\partial_\alpha^k u_t(\alpha,t)|^2 + ||D|^{1/2}\partial_\alpha^k u|^2 d\alpha. \tag{1.54}$$

Then we have

$$\frac{d}{dt}E_s(t) \lesssim E_s(t)^{(p+1)/2} + \epsilon(1+|t|)^{-(m-1/2)}E_s(t)^{1/2}. \tag{1.55}$$

Assume $E_s(0) \lesssim \epsilon^2$. By the bootstrap argument, we can prove

$$E_s(t) \lesssim \epsilon^2, \qquad \forall\ t \lesssim \epsilon^{1-p}. \tag{1.56}$$

If the nonlinearity is at least cubic, i.e., $p \geqslant 3$, then the lifespan is at least ϵ^{-2}.

The water waves: If we can find θ, $\theta \approx z_t$, such that $(\partial_t^2 + |D|)\theta = F(z_t, |D|z, z_{tt}) + O(\frac{1}{(\alpha+it)})^m)$, where F is at least cubic and $m \geqslant 2$, then use an argument similar to that for the toy model, we expect cubic lifespan. Note that the nonlinearity of the quasilinear system (1.52) is quadratic, so it does not directly lead to lifespan of order $O(\epsilon^{-2})$. In the irrotational cases, S. Wu[71] found that the fully nonlinear transform $\theta := (I - \mathfrak{H})(z - \bar{z})$ satisfies

$$(\partial_t^2 - ia\partial_\alpha)\theta = -2[z_t, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}]z_{t\alpha} + \frac{1}{\pi i}\int_{-\infty}^{\infty}\Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2(z - \bar{z})_\beta d\beta := g. \tag{1.57}$$

g is cubic, while $a-1$ contains first order terms, so $(\partial_t^2 + |D|)\theta$ contains quadratic terms, which does not imply cubic lifespan. To resolve the problem, S. Wu considered change of variables $\kappa : \mathbb{R} \to \mathbb{R}$. She let $\zeta = z \circ \kappa^{-1}$, $b = \kappa_t \circ \kappa^{-1}$, $A = (a\kappa_\alpha) \circ \kappa^{-1}$. In new variables, the system (1.13) is written as (with $\lambda_j = 0$ for irrotational case)

$$\begin{cases} (D_t^2 - iA\partial_\alpha)\zeta = -i \\ (I - \mathcal{H})D_t\bar{\zeta} = 0, \end{cases} \tag{1.58}$$

and (1.57) becomes

$$(D_t^2-iA\partial_\alpha)\theta\circ\kappa^{-1} = -2[D_t\zeta,\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta+\frac{1}{\pi i}\int_{-\infty}^{\infty}\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta, \tag{1.59}$$

where

$$\mathcal{H}f(\alpha):=\frac{1}{\pi i}p.v.\int_{-\infty}^{\infty}\frac{\zeta_\beta}{\zeta(\alpha,t)-\zeta(\beta,t)}f(\beta)d\beta. \tag{1.60}$$

She realized that there exists a change of variables κ such that

$$(I-\mathcal{H})b=-[D_t\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}, \tag{1.61}$$

$$(I-\mathcal{H})(A-1)=i[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}+i[D_t^2\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}. \tag{1.62}$$

So b, $A-1$ are quadratic. Using this, S. Wu was able to prove the almost global existence for the irrotational water waves with small localized initial data. The method implies lifespan of order $O(\epsilon^{-2})$ for nonlocalized data of size $O(\epsilon)$ for irrotational water waves.

Assume there are point vortices in the fluid. We use S. Wu's change of variables, by taking $\kappa:\mathbb{R}\to\mathbb{R}$, satisfying that for $\zeta=z\circ\kappa^{-1}$,

$$(I-\mathcal{H})(\bar{\zeta}-\alpha)=0. \tag{1.63}$$

In new variables, by direct calculation, we have

$$(D_t^2-iA\partial_\alpha)\tilde{\theta}=G_c+G_d, \tag{1.64}$$

where $\tilde{\theta}=(I-\mathcal{H})(\zeta-\bar{\zeta})$, $A=(a\kappa_\alpha)\circ\kappa^{-1}$, $b=\kappa_t\circ\kappa^{-1}$, $D_t=\partial_t+b\partial_\alpha$. Let $\mathfrak{F}=f\circ\kappa^{-1}$, $q=p\circ\kappa^{-1}$. We have

$$G_c:=-2[\bar{\mathfrak{F}},\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}]\bar{\mathfrak{F}}_\alpha+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta. \tag{1.65}$$

$$G_d := -2[\bar{q}, \mathcal{H}]\frac{\partial_\alpha \bar{\mathfrak{F}}}{\zeta_\alpha} - 2[\bar{\mathfrak{F}}, \mathcal{H}]\frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 2[\bar{q}, \mathcal{H}]\frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 4D_t q. \tag{1.66}$$

$$(I - \mathcal{H})b = -[D_t\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha,t) - z_j(t)}. \tag{1.67}$$

$$(I - \mathcal{H})A = 1 + i[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha \mathfrak{F}}{\zeta_\alpha} + i[D_t^2\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - (I - \mathcal{H})\frac{1}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta(\alpha,t) - \dot{z}_j(t))}{(\zeta(\alpha,t) - z_j(t))^2}. \tag{1.68}$$

To control the acceleration $D_t^2\zeta$, we consider $\tilde{\sigma} = (I - \mathcal{H})D_t\tilde{\theta}$. We have

$$(D_t^2 - iA\partial_\alpha)\tilde{\sigma} = \tilde{G}, \tag{1.69}$$

where

$$\begin{aligned}\tilde{G} =&(I - \mathcal{H})(D_t G + i\frac{a_t}{a}\circ\kappa^{-1}A((I - \mathcal{H})(\zeta - \bar{\zeta}))_\alpha) - 2[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha D_t^2(I - \mathcal{H})(\zeta - \bar{\zeta})}{\zeta_\alpha}\\ &+ \frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t) - D_t\zeta(\beta,t)}{\zeta(\alpha,t) - \zeta(\beta,t)}\Big)^2\partial_\beta D_t(I - \mathcal{H})(\zeta - \bar{\zeta})d\beta\end{aligned} \tag{1.70}$$

The difficulty:

(1) G_c is cubic, while G_d consists of quadratic and first order terms. G_d is the contribution from the point vortices. Similarly, b, $A-1$ contains first order terms due to the presence of point vortices.

(2) Each term of G_d contains factors of the form $\sum_{j=1}^{N}\frac{\lambda_j}{(\zeta(\alpha,t)-z_j(t))^k}$ for some $k \geqslant 1$. It's possible that the point vortices travel upward and get closer and closer to the free interface. In that case, G_d becomes very large.

(3) The strong interaction between the point vortices could excite the water waves and make it significantly larger in a short time. Assume two point vortices $z_1(t) = -x +$

$iy, z_2(t) = x + iy$, with strength λ_1 and λ_2, respectively. Then the velocity of z_1 is

$$\dot{z}_1 = -\frac{\lambda_2 i}{2\pi(z_2 - z_1)} + \bar{F}(z_1(t), t), \qquad \dot{z}_2 = \frac{\lambda_1 i}{2\pi(z_2 - z_1)} + \bar{F}(z_2(t), t).$$

Roughly speaking, if $\frac{|\lambda|}{2\pi|z_2 - z_1|}$ is large, then $|\dot{z}_j(t)|$ is large. Since

$$p_t = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{z_t - \dot{z}_j(t)}{(z(\alpha,t) - z_j(t))^2},$$

in general, $\|p_t\|_{H^s}$ could be large as well. Therefore, small data theory does not directly apply in such situation. Moreover, $\tilde{G}$ contains the term $\sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(\dot{z}_j)^2}{(\zeta(\alpha,t) - z_j(t))^3}$, which is even worse than p_t if $\dot{z}_j(t)$ is large. In theorem I.10, we do allow $|\frac{\lambda}{z_1 - z_2}|$ large. See Remark I.14.

(4) If the point vortices collide, after the collision, we cannot use the same system to describe the motion of the fluid anymore, for the reason that the vorticity after the collision is not the same as that before it, which violates the conservation of vorticity.

The idea: Intuitively, if each point vortices $z_j(t)$ moves away from the free boundary rapidly, with the factor $\frac{1}{z(\alpha,t) - z_j(t)}$ decaying in time at least at a linear rate, then we could overcome the difficulties (1) and (2). We will show that this indeed is true if (H3)-(H6) holds initially. To overcome the difficulty (3), we use $\lambda_1 = -\lambda_2$ from assumption (H3), and by direct calculation, $\|p_t\|_{H^s}$ does not depend on $\dot{z}_1, \dot{z}_2$, resolving difficulty (3). Also, although the term $\left\|\sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(\dot{z}_j)^2}{(z(\alpha,t) - z_j(t))^3}\right\|_{H^s}$ could be large at time $t = 0$, yet its long time effect remains small, i.e.,

$$\int_0^T \left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(\dot{z}_j)^2}{(z(\alpha,t) - z_j(t))^3} \right\|_{H^s} dt \leqslant C'\epsilon,$$

for some constant C' which is independent of $\frac{|\lambda|}{4\pi x(0)}$. So we are able to overcome the difficulty of large vortex-vortex interaction, provided that the point vortices keeps traveling away from the free interface.

However, the motion of point vortices interferes with the motion of the water waves, it's not obvious at all that why the point vortices should escape to the deep water (toward $y = -\infty$). Indeed, in some cases, they can travel upwards toward the interface. Recall that the velocity of z_j is given by

$$\dot{z}_j(t) = (v - \frac{\lambda_j i}{2\pi}\frac{1}{\overline{z - z_j(t)}})\Big|_{z=z_j(t)} \tag{1.71}$$

The point vortices could interact with each other and with the water waves. In general, it's not always true that $Im\{\dot{z}_j(t)\} < 0$ (i.e., travels downward).

In the following, we discuss how the number of point vortices, the sign of λ_j, and the strength of $|\lambda_j|$ affect the motion of point vortices.

(1) For $N = 1$, the motion of the point vortex is hard to predict except for some special cases.

(2) For $N = 2$, $\lambda_1 = \lambda_2$. The two point vortices is more likely to rotate about each other and excite the fluid.

(3) $N = 2$, $\lambda_1 = -\lambda_2 = \lambda > 0$. In this case, the point vortices are more likely to move upward and getting closer and closer to the interface and hence cause the Taylor sign condition to fail.

(4) $N = 2$, $\lambda_1 = -\lambda_2 = \lambda < 0$ and $\frac{|\lambda|}{|z_1 - z_2|}$ is relatively large (say, $\frac{|\lambda|}{|z_1 - z_2|} \gg \epsilon$, where ϵ is the size of the initial data), then the point vortices moving straight downward rapidly and hence the term $\frac{1}{z(\alpha,t) - z_j(t)}$ decays like t^{-1}.

(5) $N \geqslant 3$. This is far beyond understood. Indeed, even for 2d Euler equations (fixed boundary) with point vortices, the problem is still not fully understood. When $N = 3$, the problem resembles the three-body problem .

(6) Vortex pairs. With $2N$ point vortices, denoted by $\{(z_{i1}, z_{i2}\}_{i=1}^N$. Let the corresponding strength be $\lambda_{i1}, \lambda_{i2}$, respectively. Assume $\lambda_{i1} = -\lambda_{i2} < 0$. Assume $|z_{i1} - z_{i2}|$ is

sufficiently small, while different pairs are sufficiently far away from each other. Then the point vortices travel downward, at least for a short time. It's likely that the factor $\frac{1}{z(\alpha,t)-z_{ij}}$ decays linearly in time. So long time existence will not be a surprise. For brevity, we consider only one vortex pair.

Therefore, from the above discussion, if we assume $N=2$ and $\lambda_1=-\lambda_2<0$, we expect that the point vortices keep traveling downward at a speed comparable to its initial speed, hence

$$|\zeta(\alpha,t)-z_j(t)|^{-1}=O(\frac{1}{\alpha+i\frac{|\lambda|}{x(0)}t}), \tag{1.72}$$

and we can expect to have

$$(D_t^2-iA\partial_\alpha)\tilde{\theta}=G_c+O(\frac{1}{(\alpha+i\frac{|\lambda|}{x(0)}t)^2}),$$

and

$$(D_t^2-iA\partial_\alpha)\tilde{\theta}=(\partial_t^2+|D|)\tilde{\theta}+cubic+O(\frac{1}{(\alpha+i\frac{|\lambda|}{x(0)}t)^2},$$

hence

$$(\partial_t^2+|D|)\tilde{\theta}=cubic+O(\frac{1}{(\alpha+i\frac{|\lambda|}{x(0)}t)^2}).$$

Similarly, at least formally, we have

$$(\partial_t^2+|D|)\tilde{\sigma}=cubic+O(\frac{1}{(\alpha+i\frac{|\lambda|}{x(0)}t)^2}).$$

From the discussion on the Toy model, we expect to prove lifespan of order $O(\epsilon^{-2})$ for small nonlocalized data of size $O(\epsilon)$.

Let's summarize our previous discussion in a more precisely way as the following.

Step 1. Change of variables. Let κ be the change of variables such that $(I-\mathcal{H})(\bar{\zeta}-\alpha)=0$, where $\zeta=z\circ\kappa^{-1}$. Then we derive the formula (1.97) for the quantity b and the formula

(1.98) for the quantity $A-1$.

Step 2. Nonlinear transform. Let $\tilde{\theta}=\theta\circ\kappa^{-1}, \tilde{\sigma}=D_t\tilde{\theta}$, where $\tilde{\theta}=(I-\mathcal{H})(\zeta-\bar{\zeta})$. Then we derive water wave equations (1.64)-(1.66) for $\tilde{\theta}$ and water wave equations (1.69)-(1.70) for $\tilde{\sigma}$.

Step 3. Bootstrap assumption. Assume that on $[0,T]$, we have

$$\|\zeta_\alpha-1\|_{H^s}\leqslant 5\epsilon,\qquad \|\mathfrak{F}\|_{H^{s+1/2}}\leqslant 5\epsilon,\qquad \|D_t\mathfrak{F}\|_{H^s}\leqslant 5\epsilon\qquad ,\forall\ t\in[0,T], \tag{1.73}$$

where $\mathfrak{F}=f\circ\kappa^{-1}$.

Step 4. Control the motion of $z_j(t)$. Under the bootstrap assumption (1.73), we show that for any $t\in[0,T]$,

$$\frac{1}{2}\leqslant\frac{x(t)}{x(0)}\leqslant 2. \tag{1.74}$$

In another words, the trajectory of the point vortices are almost parallel to each other. Therefore, we obtain decay estimate

$$d_I(t)^{-1}=\Big(\min_{j=1,2}\inf_{\alpha\in\mathbb{R}}|\zeta(\alpha,t)-z_j(t)|\Big)^{-1}\leqslant(1+\frac{|\lambda|}{20\pi x(0)}t)^{-1}, \tag{1.75}$$

Step 5. Energy estimates. Denote $\theta_k=(I-\mathcal{H})\partial_\alpha^k\tilde{\theta}$, $\sigma_k=(I-\mathcal{H})\partial_\alpha^k\tilde{\sigma}$. Define energy

$$E(t)=\sum_{0\leqslant k\leqslant s}\Big\{\int\frac{1}{A}|D_t\theta_k|^2+\int\frac{1}{A}|D_t\sigma_k|^2+i\int\theta_k\overline{\partial_\alpha\theta_k}+i\int\sigma_k\overline{\partial_\alpha\sigma_k}\Big\}. \tag{1.76}$$

By energy estimates, use the time decay of $\frac{1}{\zeta(\alpha,t)-z_j(t)}$, we obtain control of $D_t\tilde{\theta}$ and $D_t\tilde{\sigma}$. As a consequence, by bootstrap argument, we show that $T^*\geqslant\delta\epsilon^{-2}$.

Step 6. Change of variables back to lagrangian coordinates and completes the proof.

1.2.4 A remark on global existence and modified scattering.

If the initial data is suffciently localized similar to those considered by Ionescu & Pusateri in [39] and Alazard & Delort in [1], then we expect to prove that the system (1.8) (or equivalently, the system (1.13)) is globally wellposed and there is modified scattering. In this subsection, we explain why this should be true.

Let's consider localized initial data of size ϵ (in weighted Sobolev spaces), with ϵ small. Let's assume the assumptions (H3)-(H7) (The smallness assumption in Sobolev spaces is replaced by smallness assumption in weighted Sobolev spaces). By Theorem I.10, the system (1.13) admits a unique solution on $[0, \delta\epsilon^{-2}]$. For $t \in [0, \delta\epsilon^{-2}]$, the position of the point vortices $z_j = x_j + iy_j$ satisfies

$$\frac{1}{2} \leqslant \frac{|x_j(t)|}{|x_j(0)|} \leqslant 2, \qquad y_j(t) \leqslant -\frac{|\lambda|}{20\pi x(0)} t. \tag{1.77}$$

Therefore, as long as the growth of the water waves (measured by $\zeta_\alpha - 1, f, f_t$) is slow, the vortex pair will keep travelling downward at a speed comparable to its initial speed, and therefore the effect of the point vortices will keep small for all time. Indeed, we have

$$\int_0^T \|G_d(\cdot, t)\|_{H^s} \, dt \leqslant C'\epsilon, \tag{1.78}$$

for some absolute constant $C' > 0$ on any time $[0, T]$ on which the solution exists. From this point of view, the vortex pair is a globally small perturbation of the irrotational flow. If the initial data is sufficiently localized, then the effect from the vortex pair is also localized, and a similar argument as in [39] or [1] will give global existence and modified scattering. We'll prove this in a forthcoming paper.

1.3 *Project 2*: Settings, main results, and strategy

In *Project 2*, we aim to rigorously justify the Peregrine soliton from the full water waves. We consider the two dimensional inviscid incompressible irrotational infinite depth water waves without surface tension, which is described by the system (1.1) with $\omega \equiv 0$, i.e.,

$$\begin{cases} v_t + v \cdot \nabla v = -\nabla P - (0,1) & on\ \Omega(t), \quad t \geqslant 0 \\ div\ v = 0, \quad curl\ v = 0, & on\ \Omega(t), \quad t \geqslant 0 \\ P\Big|_{\Sigma(t)} \equiv 0, & t \geqslant 0 \\ (1,v) \text{ is tangent to the free surface } (t, \Sigma(t)). \end{cases} \tag{1.79}$$

It implies from $div\ v = 0$ and $curl\ v = 0$ that $\bar{v}$ is holomoprhic in $\Omega(t)$, so v is completely determined by its boundary value on $\Sigma(t)$. Let the interface $\Sigma(t)$ be given by $z = z(\alpha, t)$, with $\alpha \in \mathbb{R}$ the Lagrangian coordinate, so that $z_t(\alpha,t) = v(z(\alpha,t),t)$, and $v_t + v \cdot \nabla v\Big|_{\Sigma(t)} = z_{tt}$. Because $P(z(\alpha,t),t) \equiv 0$, we can write $\nabla P\Big|_{\Sigma(t)} = -iaz_\alpha$, where $a := -\frac{\partial P}{\partial n}\frac{1}{|z_\alpha|}$ is a real valued function. So the momentum equation $v_t + v \cdot \nabla v = -(0,1) - \nabla p$ along $\Sigma(t)$ can be written as

$$z_{tt} - iaz_\alpha = -i. \tag{1.80}$$

Since $\bar{z}_t$ is the boundary value of $\bar{v}$, the water wave equations (1.79) is equivalent to

$$\begin{cases} z_{tt} - iaz_\alpha = -i \\ \bar{z}_t \ is\ holomorphic. \end{cases} \tag{1.81}$$

Here, by $\bar{z}_t$ holomorphic, we mean that there is a holomorphic function $\Phi(\cdot,t)$ on $\Omega(t)$ such that $\bar{z}_t(\alpha,t) = \Phi(z(\alpha,t),t)$.

In this project, we give an affirmative answer to *Question 2'*. Denote

$$1 + H^s := \{f = 1 + g : g \in H^s\}. \tag{1.82}$$

Notation. Denote $\mathbb{T} := [-\pi, \pi]$.

Let $f = f_0 + f_1$, where $f_0 \in H^{s+s_0}(\mathbb{T})$, $f_1 \in H^s(\mathbb{R})$, where $s_0 > 3/2$. Define

$$\|f\|_{X^s} := \|f_0\|_{H^{s+s_0}(\mathbb{T})} + \|f_1\|_{H^s(\mathbb{R})}. \tag{1.83}$$

The main result of *Project 2* is the following theorem, which gives a rigorous justification of the NLS with nonzero boundary values at spatial infinity from the full water waves.

Theorem I.16. *Let $M_0 > 0$, $s \geqslant 4$ and $\mathcal{T} > 0$ be given. And let $k > 0$ be a given integer* [2]*, $B^0 \in 1 + H^{s+7}(\mathbb{R})$. Denote by $B(X,T)$ the solution of the NLS: $2iB_T + \frac{1}{4k^{3/2}}B_{XX} + k^{5/2}|B|^2B = 0$ with initial data $B(X, T = 0) = B^0$. Assume that $B \in C([0, \mathcal{T}]; X^{s+7})$. And assume $\zeta^{(1)}(\alpha, t) = B(X,T)e^{i\phi}$, where $X = \epsilon(\alpha + \frac{1}{2\sqrt{k}}t)$, $T = \epsilon^2 t$, and $\phi = k\alpha + \sqrt{k}t$. Then there exists a constant $\epsilon_0 = \epsilon_0(k, s, M_0, \|B^0 - 1\|_{H^{s+7}}, \mathcal{T}) > 0$ such that for all $\epsilon < \epsilon_0$, there exists initial data $(z(\cdot, 0), z_t(\cdot, 0), z_{tt}(\cdot, 0))$ to the water wave system (1.81) such that*

$$\begin{aligned}&\|(z_\alpha(\cdot, 0) - 1, z_t(\cdot, 0), z_{tt}(\cdot, 0)) - \epsilon(\partial_\alpha \zeta^{(1)}(0), \partial_t \zeta^{(1)}(0), \partial_t^2 \zeta^{(1)}(0))\|_{X^{s-1/2} \times X^{s+1/2} \times X^s}\\ &\leqslant M_0 \epsilon^{3/2}.\end{aligned} \tag{1.84}$$

Moreover there exists a constant $C = C(k, s, M_0, \mathcal{T}, \|B(0) - 1\|_{H^{s+7}}) > 0$ such that for all initial data satisfying (1.84), the water waves system has a unique solution with

$$(z_\alpha(\cdot, t) - 1, z_t, z_{tt}) \in C([0, \mathcal{T}\epsilon^{-2}]; X^{s-1/2} \times X^{s+1/2} \times X^s)$$

[2]Indeed, we need only $k > 0$. However, if $k \neq \mathbb{N}$, then $e^{ik\alpha} \notin H^s([-\pi, \pi])$, instead, it is in $H^s([0, \frac{2\pi}{k}])$. For simplicity, we take $k \in \mathbb{N}$. The same argument applies directly to the cases that $k \neq \mathbb{N}$

satisfying

$$\sup_{0\leqslant t\leqslant \mathcal{T}\epsilon^{-2}} \|(Im\{z_\alpha - 1\}, z_t, z_{tt}) - \epsilon(Im\{\partial_\alpha\zeta^{(1)}\}, \zeta_t^{(1)}, \zeta_{tt}^{(1)})\|_{X^{s-1/2}\times X^{s+1/2}\times X^s} \leqslant C\epsilon^{3/2}, \quad (1.85)$$

Remark I.17. Theorem I.16 gives rigorous justification of the NLS with nonzero boundary values at ∞ in Lagrangian coordinates. In (1.85), please note that we only justify the modulation approximation for the imaginary part of $z_\alpha - 1$,, i.e.,

$$\sup_{t\in[0,\mathcal{T}\epsilon^{-2}]} \Big\| Im\{z_\alpha - 1\} - \epsilon\,\mathrm{Im}\{\partial_\alpha\zeta^{(1)}\}\Big\|_{X^{s-1/2}} \lesssim \epsilon^{3/2}. \quad (1.86)$$

In Theorem III.5 (See §3.10.2), we give a full justification of the Peregrine soliton from full water waves in a different coordinates. We do not prove

$$\sup_{t\in[0,\mathcal{T}\epsilon^{-2}]} \Big\| Re\{z_\alpha - 1\} - \epsilon Re\{\partial_\alpha\zeta^{(1)}\}\Big\|_{X^{s-1/2}} \lesssim \epsilon^{3/2}$$

in the Lagrangian coordinates because there is no good control of the change of variables on time scale $O(\epsilon^{-2})$, please see Theorem III.5 and Remark III.52 for the details.

Remark I.18. Let B solves $iB_T + B_{XX} = -2|B|^2B$. Then

$$U(X,T) := \frac{2}{\sqrt{k^{5/2}}}B(\sqrt{8k^{3/2}}X, T)$$

solves $2iU_T + \frac{1}{4k^{3/2}}U_{XX} + k^{5/2}U|U|^2 = 0$.

Applying Theorem I.16 to $U(X,0) = \frac{2}{\sqrt{k^{5/2}}}Q(\sqrt{8k^{3/2}}X, 0)$ gives an affirmative answer to Question I.3.

Remark I.19. $s_0 > 3/2$ is of course not optimal. We take $s_0 > 3/2$ to avoid getting into too many technical issues.

1.3.1 Challenges of the problem and the strategy.

1.3.1.1 First difficulty: find a right class of water waves to work with.

Suppose B is the Peregrine soliton, then the wave packet W is nonvanishing. As a consequence, in order to justify the Peregrine soliton from the full water waves, we need to show that water waves with nonvanishing data of size $O(\epsilon)$ exist on time scale $O(\epsilon^{-2})$. In [3], Alazard, Burq, and Zuily proved local wellposedness of (1.79) with nonvanishing data in Kato's uniform local space $H^s_{ul}(\mathbb{R})$. Their result implies that for initial data of size $O(\epsilon)$, the lifespan of the solution is at least of order $O(\epsilon^{-1})$, which is not enough for justifying the Peregrine soliton. Even though the long time existence has been well-known for periodic waves and localized waves, to the author's best of knowledge, for nonvanishing water waves, no long time existence results with lifespan of the solution longer than the order of $O(\epsilon^{-1})$ exist, and the analytical tools developed for the vanishing or periodic data cannot be directly used in this setting.

In order to prove long time existence of the water wave system, one needs to find a cubic structure for the water wave equations. More precisely, we need to find some quantity θ such that $\partial_t\theta \approx z_t$ and

$$(\partial_t^2 - ia\partial_\alpha)\theta = F, \tag{1.87}$$

with F consists of cubic and higher order nonlinearities. For water waves with data in Sobolev spaces, there are two ways of doing this. The first one is the fully nonlinear transform constructed by S. Wu. In [71], S. Wu considered $\theta := (I - \mathfrak{H})(z - \bar{z})$ and showed that $(\partial_t^2 - ia\partial_\alpha)\theta = cubic$. Here,

$$\mathfrak{H}f(\alpha) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{z_\beta}{z(\alpha,t) - z(\beta,t)} f(\beta)d\beta \tag{1.88}$$

is the Hilbert transform associated with the free interface labeled by $z(\alpha, t)$. Using this fully nonlinear transform, S. Wu was able to prove the almost global existence for the irrotational water waves with small localized initial data. The method implies lifespan of order $O(\epsilon^{-2})$ for nonlocalized data of size $O(\epsilon)$ in Sobolev spaces. This nonlinear transform is also used in [72][63][62][39]. See also [32][33] for similar ideas. The second way is to use the normal form transformation to construct a cubic structure, see for example [1][39][64][38][7]. These two methods work well for water waves with periodic data or with data in Sobolev spaces.

However, for nonvanishing water waves, such a cubic structure was unclear for both methods. The first difficulty we confront is to find a right class of water waves that we can work with. This class of water waves must be non-vanishing at spatial infinity along the free interface. However, if the water waves have too many activities at infinity, then it's not obvious at all that why the water waves should exist for a long time.

1.3.1.2 The idea of resolving the first difficulty: water waves that decays non-tangentially

Let B be the Peregrine soliton, then the wave packet W can be decomposed as

$$W = W_0 + W_1, \qquad W_0 = \epsilon e^{i\epsilon^2 t} e^{i\phi}, \qquad W_1 = \alpha - \epsilon e^{i\epsilon^2 t} \frac{1 + 8i\epsilon^2 t}{1 + 4(\epsilon\alpha)^2 + 4(\epsilon^2 t)^2} e^{i\phi} \tag{1.89}$$

Note that W_0 is periodic, $W_1 - \alpha$ vanishes at infinity, therefore, we consider water waves which is a superposition of periodic waves and waves which vanish at infinity. Moreover, since $W_0 \in C^\infty(\mathbb{T})$, we can assume that the periodic waves has more regularity than the localized waves. This motivates us to work in the function space $X^s := H^s(\mathbb{R}) + H^{s+s_0}(\mathbb{T})$, where $s_0 > 3/2$.

Key observation: Although the velocity v is nonvanishing along the free interface, however, away from the free interface, v can vanish at spatial infinity. In other words, although the water waves have a lot of activity at spatial infinity along the free interface, however, away from the interface, the water waves can be at rest at infinity. This observation suggests that, away from the free interface, the interaction between the periodic waves and the localized waves is weak.

To make the above discussion precise, we use the notion of *decay nontangentially.*

Definition I.20 (Cone)**.** Let $z_0 \in \mathbb{C}$. Let $\theta_0 \in (0, \pi/2)$. Denote

$$C_{\theta_0}(z_0) := \Big\{ z \in \mathbb{C} : \Big| \frac{\operatorname{Re}(z - z_0)}{\operatorname{Im}(z - z_0)} \Big| \leqslant \tan\theta_0 \quad \& \quad \operatorname{Im} z \leqslant \operatorname{Im} z_0 \Big\}.$$

That is, $C_{\theta_0}(z_0)$ is the cone with vertex z_0 and angle θ_0.

Definition I.21 (Decay nontangentially)**.** Let $\phi(z)$ be a function in $\Omega(t)$. Let $z_0 \in \mathbb{C}$ be a fixed point. We say that $\phi(z) \to 0$ nontangentially as $z \to \infty$ if for any $0 < \theta < \pi/2$,

$$\lim_{\substack{z \in \Omega(t) \cap C_{\theta_0}(z_0) \\ |z| \to \infty}} \phi(z) = 0. \tag{1.90}$$

Remark I.22*.* Note that the definition above is invariant if we use different z_0. As a consequence, we choose $z_0 = 0$ and write $C_{\theta_0}(0)$ as C_{θ_0}.

Remark I.23*.* If ϕ is a periodic function in $\Omega(t)$, and

$$\lim_{\operatorname{Im} z \to -\infty} \phi(z) = 0,$$

then $\phi(z)$ decays nontangentially.

It turns out that the *decay nontangentially* is the right setting for nonvanishing water waves. If we assume the velocity field v decays nontangentially, follow S. Wu's method in [71], at least formally (in BMO sense, because the Hilbert transform $\mathfrak{H}$ maps L^∞ to BMO), we can show that the quantity $(I - \mathfrak{H})(z - \bar{z})$ satisfies

$$(\partial_t^2 - ia\partial_\alpha)(I - \mathfrak{H})(z - \bar{z}) \tag{1.91}$$

$$= -2[z_t, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}]z_{t\alpha} + \frac{1}{\pi i}\int_{-\infty}^{\infty} \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 (z - \bar{z})_\beta d\beta := g. \tag{1.92}$$

Formally, g is cubic, while $a-1$ contains first order terms, so $(\partial_t^2+|D|)\theta$ contains quadratic terms, which does not imply cubic lifespan. To resolve the problem, we follow S. Wu's idea and consider the change of variables $\kappa:\mathbb{R}\to\mathbb{R}$ such that $\bar{\zeta}-\alpha$ is boundary value of a holomorphic function which decays nontangentially, where $\zeta=z\circ\kappa^{-1}$. Denote $b=\kappa_t\circ\kappa^{-1}$, $A=(a\kappa_\alpha)\circ\kappa^{-1}$. In new variables, the system (1.81) is written as

$$\begin{cases}(D_t^2-iA\partial_\alpha)\zeta=-i\\ (I-\mathcal{H}_\zeta)D_t\bar{\zeta}=0,\end{cases}\tag{1.93}$$

and we have

$$(D_t^2-iA\partial_\alpha)(I-\mathcal{H}_\zeta)(\zeta-\bar{\zeta})\tag{1.94}$$

$$=-2[D_t\zeta,\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta+\frac{1}{\pi i}\int_{-\infty}^{\infty}\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta,\tag{1.95}$$

where

$$\mathcal{H}_\zeta f(\alpha):=\frac{1}{\pi i}p.v.\int_{-\infty}^{\infty}\frac{\zeta_\beta}{\zeta(\alpha,t)-\zeta(\beta,t)}f(\beta)d\beta.\tag{1.96}$$

Moreover, we have

$$(I-\mathcal{H})b=-[D_t\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha},\tag{1.97}$$

$$(I-\mathcal{H})(A-1)=i[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}+i[D_t^2\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}.\tag{1.98}$$

So b, $A-1$ are quadratic. Therefore, at least formally, we have

$$(\partial_t^2+|D|)(I-\mathcal{H}_\zeta)(\zeta-\bar{\zeta})=cubic.\tag{1.99}$$

If $\zeta-\alpha, D_t\zeta$ decay[3] at spatial infinity or periodic, then use S. Wu's method, we could prove that (1.79) is wellposed on time scale $O(\epsilon^{-2})$.

[3] In this paper, by a function f decays at ∞, we mean that $f\in H^s(\mathbb{R})$ for some $s\geqslant 0$, even though it could be possible that $\lim_{x\to\infty}f(x)$ does not exist.

1.3.1.3 The second difficulty

If the water wave is nonvanishing, then it's difficult to define an energy associated with (1.94) which still preserves the cubic structure. Indeed, because $\theta \notin L^q(\mathbb{R})$ for any $q \neq \infty$, we cannot estimate θ in $H^s(\mathbb{R})$. If we estimate θ in $W^{s,\infty}(\mathbb{R})$, as is explained by Alazard, Burq and Zuily in [3], there is loss of derivative in such spaces. One might try to estimate θ in Kato's uniform local Sobolev spaces as in [3]. Assume $\{\chi_n\}$ is a partition of unity of $\mathbb{R}$. One needs to consider the quantity $\chi_n\theta$. It turns out that $\mathcal{P}\chi_n\theta$ has first and quadratic nonlinearities, which are difficult to get rid of.

1.3.1.4 Idea of resolving the second difficulty

To resolve this problem, we note that if $\zeta_\alpha - 1 \in X^s$, then ζ can be decomposed uniquely as

$$\zeta = \omega + \xi_1, \tag{1.100}$$

where $\omega - \alpha$ is periodic, and ξ_1 decays at spatial infinity. Let b_0 and A_0 be determined by

$$(I - \mathcal{H}_p)b_0 = -[D_t^0\omega, \mathcal{H}_p]\frac{\bar{\omega}_\alpha - 1}{\omega_\alpha}, \tag{1.101}$$

$$(I - \mathcal{H}_p)(A_0 - 1) = i[(D_t^0)^2\omega, \mathcal{H}_p]\frac{\bar{\omega}_\alpha - 1}{\omega_\alpha} + i[(D_t^0)\omega, \mathcal{H}_p]\frac{\partial_\alpha(D_t^0)\bar{\omega}}{\omega_\alpha}. \tag{1.102}$$

where

$$\mathcal{H}_p f(\alpha) := \frac{1}{2\pi i}p.v.\int_{\mathbb{T}} \omega_\beta(\beta)\cot(\frac{1}{2}(\omega(\alpha) - \omega(\beta))f(\beta)d\beta. \tag{1.103}$$

Denote $D_t^0 := \partial_t + b_0\partial_\alpha$, then ω satisfies

$$(D_t^0)^2\omega - iA_0\omega_\alpha = -i. \tag{1.104}$$

It has been well known that the periodic water waves with initial data of size $O(\epsilon)$ exists on lifespan of order at least $O(\epsilon^{-2})$. So it suffices to control ξ_1 and $D_t\zeta - D_t^0\omega$ on time scale

$O(\epsilon^{-2})$. In BMO sense, we have

$$((D_t^0)^2 - iA_0\partial_\alpha)(I - \mathcal{H}_\omega)(\omega - \bar{\omega}) = cubic, \tag{1.105}$$

where $\mathcal{H}_\omega$ is the Hilbert transform associated with ω, i.e.,

$$\mathcal{H}_\omega f(\alpha, t) = p.v.\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\omega_\beta(\beta, t)}{\omega(\alpha, t) - \omega(\beta, t)} f(\beta, t) d\beta. \tag{1.106}$$

Now consider the quantity

$$\lambda := (I - \mathcal{H}_\zeta)\Big((I - \mathcal{H}_\zeta)(\zeta - \alpha) - (I - \mathcal{H}_\omega)(\omega - \alpha)\Big). \tag{1.107}$$

Then $\lambda \approx \xi_1$. Moreover, we can prove that

$$(D_t^2 - iA\partial_\alpha)\lambda = cubic. \tag{1.108}$$

Since λ is in Sobolev space, we can use energy method to prove the following result:

Theorem I.24. *Let $s \geqslant 4$. Let $\|(z_\alpha(\cdot, 0) - 1, z_t(t = 0), z_{tt}(t = 0))\|_{X^{s-1/2}\times X^{s+1/2}\times X^s} \leqslant \epsilon$. Assume $\bar{z}_t(t = 0) \in \mathcal{H}ol_{\mathcal{N}}(\Omega(0))$. Then there exists $\epsilon_0 = \epsilon_0(s) > 0$ sufficiently small and a constant $C_1 = C_1(s) > 0$ such that for all $0 \leqslant \epsilon \leqslant \epsilon_0$, the water wave equations (1.81) admit a unique solution $(z_\alpha(\cdot, t) - 1, z_t(\cdot, t), z_{tt}(\cdot, t)) \in C([0, C_1\epsilon^{-2}]; X^{s-1/2} \times X^{s+1/2} \times X^s)$. Moreover,*

$$\sup_{0\leqslant t\leqslant C_1\epsilon^{-2}} \|(z_\alpha - 1, z_t, z_{tt})\|_{X^{s-1/2}\times X^{s+1/2}\times X^s} \leqslant C\epsilon, \tag{1.109}$$

for some constant $C = C(s)$.

To our best knowledge, Theorem I.24 is the first long time existence for nonvanishing water waves. More importantly, using this long time existence result, we are able to justify the NLS from the full water waves in a regime that allows for Peregrine solitons, and prove Theorem

I.16.

1.3.1.5 Rigorous justification of the Peregrine soliton from water waves .

We prove Theorem I.16 through the following steps.

Step 1. Construction of approximate solution.

Let B be a solution to the NLS. Consider interface of the form

$$\zeta(\alpha,t)=\alpha+\sum_{n=1}^{\infty}\epsilon^n\zeta^{(n)}. \tag{1.110}$$

By multiscale analysis, we can choose $\zeta^{(n)}, n=1,2,3$ be such that $\zeta^{(1)}=B(X,T)e^{i\phi}$ and $\zeta^{(2)}$, $\zeta^{(3)}$ depend on B and ϕ only. We define an approximate solution $\tilde{\zeta}$ to ζ by

$$\tilde{\zeta}:=\alpha+\sum_{n=1}^{3}\epsilon^n\zeta^{(n)}, \tag{1.111}$$

then formally[4],

$$|\zeta-\tilde{\zeta}|=O(\epsilon^4). \tag{1.112}$$

Similarly, we approximate $D_t\zeta$, $D_t^2\zeta$ by some appropriate functions $\tilde{D}_t\tilde{\zeta}, \tilde{D}_t^2\tilde{\zeta}$ such that

$$|D_t\zeta-\tilde{D}_t\tilde{\zeta}|=O(\epsilon^4),\qquad |D_t^2\zeta-\tilde{D}_t\tilde{\zeta}|=O(\epsilon^4). \tag{1.113}$$

Denote

$$r:=\zeta-\tilde{\zeta}=r_0+r_1, \tag{1.114}$$

where r_0 is the periodic part of r, and r_1 decays at spatial infinity. Denote $\tilde{\xi}_0$ the periodic part of $\tilde{\zeta}-\alpha$. Denote

$$\tilde{\omega}=\alpha+\tilde{\xi}_0,\qquad \tilde{\xi}_1:=\tilde{\zeta}-\tilde{\omega}. \tag{1.115}$$

[4]This is in ∞-norm sense, i.e., $\|\zeta-\tilde{\zeta}\|_{W^{s,\infty}}=O(\epsilon^4)$. In X^s norm, $\|\zeta-\tilde{\zeta}\|_{X^s}=O(\epsilon^{7/2})$

To rigorous justify the NLS from the water waves, we need to control the error r on time scale $O(\epsilon^{-2})$.

Step 2. A priori error estimates for the periodic part

For data of the form (3.38), we show that

$$((D_t^0)^2 - iA_0\partial_\alpha)(I-\mathcal{H}_p)r_0 = \text{fourth order}. \tag{1.116}$$

So we can obtain

$$\sup_{t\in[0,O(\epsilon^{-2})]} \|(\partial_\alpha r_0, D_t^2 r_0, (D_t^0)^2 r_0)\|_{H^{s+s_0}(\mathbb{T})\times H^{s+s_0+1/2}(\mathbb{T})\times H^{s+s_0}(\mathbb{T})} \leqslant C\epsilon^{3/2}. \tag{1.117}$$

Step 3. A priori error estimates for the vanishing part

Consider the quantity

$$\rho_1 := (I-\mathcal{H}_\zeta)\Big\{\Big((I-\mathcal{H}_\zeta)\xi - (I-\mathcal{H}_\omega)\xi_0\Big) - \Big((I-\mathcal{H}_{\tilde{\zeta}})\tilde{\xi} - (I-\mathcal{H}_{\tilde{\omega}})\tilde{\xi}_0\Big)\Big\}. \tag{1.118}$$

We remind the readers that $\mathcal{H}_{\tilde{\zeta}}$ and $\mathcal{H}_{\tilde{\omega}}$ are the Hilbert transforms associated with $\tilde{\zeta}$ and $\tilde{\omega}$, respectively. We can show that $\rho_1 \approx r_1$.

By exploring the structure of $\tilde{\zeta}$, we show that

$$(D_t^2 - iA\partial_\alpha)\rho_1 = \text{fourth order}. \tag{1.119}$$

So we can obtain

$$\sup_{t\in[0,O(\epsilon^{-2})]} \|(\partial_\alpha r_1, D_t r_1, D_t^2 r_1)\|_{H^s(\mathbb{R})\times H^{s+1/2}(\mathbb{R})\times H^s(\mathbb{R})} \leqslant C\epsilon^{3/2}. \tag{1.120}$$

Step 4. In Step 1, we've constructed an approximate solution $(\tilde{\zeta}, \tilde{D}_t\tilde{\zeta}, \tilde{D}_t^2\tilde{\zeta})$ which exists on time scale $O(\epsilon^{-2})$. In Step 2 and Step 3, we obtain a priori bound on the energy for the remainder r on long time scale $O(\epsilon^{-2})$. However, since $\tilde{\zeta}$ does not in general satisfy the water wave equations, the wave packet like data $(\tilde{\zeta}(0), \tilde{D}_t\tilde{\zeta}(0), \tilde{D}_t^2\tilde{\zeta})(0)$ cannot be taken as the initial data of the water wave equations. Similar to that in [63], we show that there is initial data for the water wave system that is within $O(\epsilon^{3/2})$ to the wave packet $(\tilde{\zeta}(0), \tilde{D}_t\tilde{\zeta}(0), \tilde{D}_t^2\tilde{\zeta})(0)$. By long time existence of (1.93) with initial data $(\partial_\alpha(\zeta - \alpha), D_t\zeta, D_t^2\zeta)$ of size ϵ in $X^s \times X^{s+1/2} \times X^s$, the solution of the system (1.93) exists on time scale $O(\epsilon^{-2})$. The a priori bound on r gives the estimate of the error between ζ and the wave packet $\tilde{\zeta}$ on the order $O(\epsilon^{3/2})$ for time on the $O(\epsilon^{-2})$ scale. The appropriate wave packet approximation to z is then obtained upon changing coordinates back to the Lagrangian variable.

1.4 Outline of this thesis

1.4.1 Outline of Chapter 2

In §2.1, we introduce some basic notation and convention. Further notation and convention will be made throughout the paper if necessary. In §2.2 we will provide some analytical tools that will be used in later sections. In §2.3, we give a systematic investigation of the Taylor sign condition. We give examples that Taylor sign condition fails. We also give a sufficient condition which implies the strong Taylor sign condition . In §2.4, we prove Theorem I.24. In §2.5, we prove Theorem I.10.

1.4.2 Outline of Chapter 3

In §3.1, we introduce some basic notation and convention. Further notation and convention will be made throughout the paper if necessary. In §3.2 we will provide some analytical tools and the basic definitions that will be used in later sections. In section §3.3, we sketch a proof of long time existence of the periodic water waves system, which we will use in later sections. In Section §3.4, we set up the water waves system with data in X^s, derive formula for the corresponding quantities, and then prove long time existence of water waves in the function space X^s. In Section §3.5, we formally derive NLS with non-vanishing boundary value at ∞ from non-vanishing water waves system that we set up in Section §3.4, and obtain an approximation $\tilde{\zeta}$ to water waves system. In Section §3.6, we derive governing equations for r_0, then we show that r_0 remains small on time scale $O(\epsilon^{-2})$. In Section §3.7, we derive governing equations for r_1, and define corresponding energies that could be used to control norms of r_1. In Section 3.8, we obtain a priori bounds of a list of quantities that appear in the energy estimates, and in Section §3.9, we obtain energy estimates on time scale $O(\epsilon^{-2})$. As a consequence of the energy estimates, we prove our Main Theorem I.16 in Section §3.10. In the appendix, we show that $e^{-ik\alpha}$ cannot be the boundary value of a holomorphic function in the region below the curve $\{\omega(\alpha,t) = \alpha + c(t)e^{ik\alpha}\}$.

CHAPTER II

Long time behavior of 2D water waves with point vortices

2.1 Notation and convention

Throughout chapter, we assume that the velocity field $|v(z,t)| \to 0$ as $|z| \to \infty$ and $z(\alpha,t) - \alpha \to 0$ as $|\alpha| \to \infty$. We use $C(X_1, X_2, ..., X_k)$ to denote a positive constant C depends continuous on the parameters $X_1, ..., X_k$. Such constant $C(X_1, ..., X_k)$ could be different even we use the same letter C. The commutator $[A, B] = AB - BA$. Given a function $g(\cdot, t) : \mathbb{R} \to \mathbb{R}$, the composition $f(\cdot, t) \circ g := f(g(\cdot, t), t)$. For a function $f(\alpha, t)$ along the free surface $\Sigma(t)$, we say f is holomorphic in $\Omega(t)$ if there is some holomorphic function $F : \Omega(t) \to \mathbb{C}$ such that $f = F|_{\Sigma(t)}$. We identify the $\mathbb{R}^2$ with the complex plane. A point (x, y) is identified as $x + iy$. For a point $z = x + iy$, $\bar{z}$ represents the complex conjugate of z.

2.2 Preliminaries and basic analysis

Lemma II.1 (Sobolev embedding)**.** *Let $s > 1/2$. Let $f \in H^s(\mathbb{R})$. Then $f \in L^\infty$, and*

$$\|f\|_\infty \leqslant C\|f\|_{H^s},$$

where $C = C(s)$. If $s = 1$, we can take $C = 1$.

2.2.1 Hilbert transform, layer potentials

Definition II.2 (Hilbert transform)**.** We define the Hilbert transform associates with a curve $z(\alpha, t)$ as

$$\mathfrak{H}f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{z_\beta(\beta, t)}{z(\alpha, t) - z(\beta, t)} f(\beta, t) d\beta. \tag{2.1}$$

The standard Hilbert transform is the one associated with $z(\alpha) = \alpha$, which is denoted by

$$\mathbb{H}f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{1}{\alpha - \beta} f(\beta) d\beta. \tag{2.2}$$

It is well-known that the following holds:

Lemma II.3. *Let $f \in L^2(\mathbb{R})$. Then f is the boundary value of a holomorphic function in $\Omega(t)$ if and only if $(I - \mathfrak{H})f = 0$. f is the boundary value of a holomorphic function in $\Omega(t)^c$ if and only if $(I + \mathfrak{H})f = 0$.*

Because of the singularity of the velocity at the point vortices, we don't have $(I - \mathfrak{H})\bar{z}_t = 0$. However, the following lemma asserts that $\bar{z}_t$ is almost holomorphic, in the sense that $(I - \mathfrak{H})\bar{z}_t$ consists of lower order terms.

Lemma II.4 (Almost holomorphicity)**.** *We have*

$$(I - \mathfrak{H})\bar{z}_t = -\frac{i}{\pi} \sum_{j=1}^{N} \frac{\lambda_j}{z(\alpha, t) - z_j(t)}. \tag{2.3}$$

Proof. Since $\bar{z}_t + \sum_{j=1}^N \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))}$ is the boundary value of a holomorphic function in $\Omega(t)$, by lemma II.3,

$$(I - \mathfrak{H})\Big(\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}\Big) = 0,$$

hence

$$(I - \mathfrak{H})\bar{z}_t = -\sum_{j=1}^{N} (I - \mathfrak{H}) \frac{\lambda_j i}{2\pi(z(\alpha, t) - z_j(t))}. \tag{2.4}$$

Since $\frac{1}{z(\alpha,t)-z_j(t)}$ is the boundary value of the holomorphic function $\frac{1}{z-z_j(t)}$ in $\Omega(t)^c$, by lemma II.3 again, we have

$$(I-\mathfrak{H})\frac{1}{z(\alpha,t)-z_j(t)}=\frac{2}{z(\alpha,t)-z_j(t)}. \tag{2.5}$$

(2.4) together with (2.5) complete the proof of the lemma. □

Definition II.5 (Double layer potential)**.**

$$\mathfrak{K}f(\alpha):=p.v.\int_{-\infty}^{\infty}Re\{\frac{1}{\pi i}\frac{z_\beta}{z(\alpha,t)-z(\beta,t)}\}f(\beta)d\beta. \tag{2.6}$$

Definition II.6 (Adjoint of double layer potential)**.**

$$\mathfrak{K}^*f(\alpha):=p.v.\int_{-\infty}^{\infty}Re\{-\frac{1}{\pi i}\frac{z_\alpha}{|z_\alpha|}\frac{|z_\beta|}{z(\alpha,t)-z(\beta,t)}\}f(\beta)d\beta. \tag{2.7}$$

Lemma II.7. *Let $z(\alpha)$ be a chord-arc curve such that*

$$\beta_0|\alpha-\beta|\leqslant|z(\alpha)-z(\beta)|\leqslant\beta_1|\alpha-\beta|,\qquad\forall\ \alpha,\beta\in\mathbb{R}. \tag{2.8}$$

Then we have

$$\|\mathfrak{H}f\|_{L^2}\leqslant C(\beta_0,\beta_1)\|f\|_{L^2}. \tag{2.9}$$

$$\|\mathfrak{K}f\|_{L^2}\leqslant C(\beta_0,\beta_1)\|f\|_{L^2}. \tag{2.10}$$

$$\|\mathfrak{K}^*f\|_{L^2}\leqslant C(\beta_0,\beta_1)\|f\|_{L^2}. \tag{2.11}$$

$$\left\|(I\pm\mathfrak{K})^{-1}f\right\|_{L^2}\leqslant C(\beta_0,\beta_1)\|f\|_{L^2}. \tag{2.12}$$

$$\|(I\pm\mathfrak{K}^*)^{-1}f\|_{L^2}\leqslant C(\beta_0,\beta_1)\|f\|_{L^2}. \tag{2.13}$$

For proof, see for example [16], [60].

2.2.2 Commutator estimates

Denote

$$S_1(A,f) = p.v. \int \prod_{j=1}^{m} \frac{A_j(\alpha) - A_j(\beta)}{\gamma_j(\alpha) - \gamma_j(\beta)} \frac{f(\beta)}{\gamma_0(\alpha) - \gamma_0(\beta)} d\beta. \tag{2.14}$$

$$S_2(A,f) = \int \prod_{j=1}^{m} \frac{A_j(\alpha) - A_j(\beta)}{\gamma_j(\alpha) - \gamma_j(\beta)} f_\beta(\beta) d\beta. \tag{2.15}$$

We have the following comutator estimates, which can be found in [63], [71].

Lemma II.8. *(1) Assume each γ_j satisfies the chord-arc condition*

$$C_{0,j}|\alpha - \beta| \leqslant |\gamma_j(\alpha) - \gamma_j(\beta)| \leqslant C_{1,j}|\alpha - \beta|. \tag{2.16}$$

Then both $\|S_1(A,f)\|_{L^2}$ and $\|S_2(A,f)\|_{L^2}$ are bounded by

$$C \prod_{j=1}^{m} \left\| A_j' \right\|_{X_j} \|f\|_{X_0},$$

where one of the $X_0, X_1, ...X_m$ is equal to L^2 and the rest are L^∞. The constant C depends on $\left\| \gamma_j' \right\|_{L^\infty}^{-1}, j = 1, .., m$.

(2) Let $s \geqslant 3$ be given, and suppose chord-arc condition (3.33) holds for each γ_j, then

$$\|S_2(A,f)\|_{H^s} \leqslant C \prod_{j=1}^{m} \left\| A_j' \right\|_{Y_j} \|f\|_Z,$$

where for all $j = 1, ..., m$, $Y_j = H^{s-1}$ or $W^{s-2,\infty}$ and $Z = H^s$ or $W^{s-1,\infty}$. The constant C depends on $\left\| \gamma_j' \right\|_{H^{s-1}}, \|\gamma_j\|_\infty^{-1}, j = 1, ..., m$.

As a consequence of lemma III.18, we have the following commutator estimates.

Lemma II.9. *Let $k \geqslant 1$. Assume $z(\alpha, t)$ satisfies chord-arc condition*

$$C_1|\alpha - \beta| \leqslant |z(\alpha, t) - z(\beta, t)| \leqslant C_2|\alpha - \beta|, \tag{2.17}$$

and $z_\alpha - 1 \in H^{k-1}$. *Then*

$$\left\| [\partial_\alpha^k, \mathfrak{H}] f \right\|_{L^2} \leqslant C \|\partial_\alpha f\|_{H^{k-1}}, \tag{2.18}$$

where the constant $C = C(\|z_\alpha - 1\|_{H^{k-1}}, C_1, C_2)$.

Proof. Use

$$[\partial_\alpha^k, \mathfrak{H}] = \sum_{l=0}^{k} \partial_\alpha^l [\partial_\alpha, \mathfrak{H}] \partial_\alpha^{k-l}$$

Then use induction to complete the proof. □

2.2.3 Some estimates involving point vortices

In this subsection we estimate some integrals involving the point vortices.

Lemma II.10. *Assume* $z(\alpha, t)$ *satisfies the same condition as in lemma II.9. Let* $k > 1$. *Then*

$$\int_{-\infty}^{\infty} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta \leqslant C d_I(t)^{-k+1}, \tag{2.19}$$

where $C = 4C_0^{-1} + \frac{4C_1^{k-1}}{(k-1)C_0^{k-1}}$.

Proof. We may assume that $d_I(t) = d(z_j(t), z(0, t))$.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta \\
=& \int_{|z(0,t) - z(\beta,t)| \leqslant 2d_I(t)} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta + \int_{|z(0,t) - z(\beta,t)| \geqslant 2d_I(t)} \frac{1}{|z_j(t) - z(\beta, t)|^k} d\beta \\
:=& I + II.
\end{aligned}$$

Denote

$$E := \{\beta : |z(0, t) - z(\beta, t)| \leqslant 2d_I(t)\}.$$

Since

$$C_0 |\beta - 0| \leqslant |z(\beta, t) - z(0, t)|,$$

we have for $\beta \in E$,

$$|\beta - 0| \leqslant \frac{2}{C_0} d_I(t).$$

Therefore

$$I \leqslant 4C_0^{-1} d_I(t)^{-k+1}.$$

For $\beta \in E^c$, use the chord-arc condition (2.17), we have

$$|z(\beta,t) - z(0,t) - d_I(t)| \geqslant |z(\beta,t) - z(0,t)| - d_I(t) \geqslant \frac{1}{2}|z(\beta,t) - z(0,t)| \geqslant \frac{C_0}{2}|\beta - 0|. \quad (2.20)$$

Also, we have

$$C_1|\beta - 0| \geqslant |z(\beta,t) - z(0,t)| \geqslant 2d_I(t). \quad (2.21)$$

So

$$|\beta| \geqslant \frac{2}{C_1} d_I(t) \quad (2.22)$$

Therefore, for II, we have

$$II \leqslant \frac{2^k}{C_0^k} \int_{|\beta| \geqslant \frac{2}{C_1} d_I(t)} |\beta|^{-k} d\beta = 2\frac{2^k}{(k-1)C_0^k} \frac{C_1^{k-1}}{2^{k-1}} d_I(t)^{-k+1} = \frac{4C_1^{k-1}}{(k-1)C_0^{k-1}} d_I(t)^{-k+1}.$$

□

Corollary II.11. *Assume $z(\alpha,t)$ satisfies the same condition as in lemma II.9. Given $m \geqslant 2$, there exist $C = (k+m)!C(C_0, C_1, \|z_\alpha - 1\|_{H^{m-1}})$ such that*

$$\left\| \frac{1}{(z(\alpha,t) - z_j(t))^k} \right\|_{H^m} \leqslant C(d_I(t)^{-k+1/2} + d_I(t)^{-k-m+1/2}) \quad (2.23)$$

In particular, if $d_I(t) \geqslant 1$, then we have

$$\left\| \frac{1}{(z(\alpha,t) - z_j(t))^k} \right\|_{H^m} \leqslant C d_I(t)^{-k+1/2}. \quad (2.24)$$

2.2.4 Basic identities

Lemma II.12. *Assume* $z_t, z_\alpha - 1 \in C^1([0,T]; H^1(\mathbb{R}))$*,* $f \in C(\mathbb{R}\times[0,T])$ *and* $f_\alpha(\alpha,t) \to 0$ *as* $|\alpha| \to \infty$*. We have*

$$[\partial_t, \mathfrak{H}]f = [z_t, \mathfrak{H}]\frac{f_{t\alpha}}{z_\alpha} \tag{2.25}$$

$$[\partial_t^2, \mathfrak{H}]f = [z_{tt}, \mathfrak{H}]\frac{f_\alpha}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i}\int (\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)})^2 f_\beta(\beta,t)d\beta \tag{2.26}$$

$$[a\partial_\alpha, \mathfrak{H}]f = [az_\alpha, \mathfrak{H}]\frac{f_\alpha}{z_\alpha}, \qquad \partial_\alpha \mathfrak{H} f = z_\alpha \mathfrak{H}\frac{f_\alpha}{z_\alpha} \tag{2.27}$$

$$[\partial_t^2 - ia\partial_\alpha, \mathfrak{H}]f = 2[z_t, \mathfrak{H}]\frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i}\int (\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)})^2 f_\beta(\beta,t)d\beta \tag{2.28}$$

For proof, see [71].

Lemma II.13. *Let* $D_t = \partial_t + b\partial_\alpha$*, then*

$$[D_t^2, \partial_\alpha] = -D_t(b_\alpha)\partial_\alpha - b_\alpha D_t\partial_\alpha - b_\alpha\partial_\alpha D_t \tag{2.29}$$

$$\begin{aligned}[D_t^2, \partial_\alpha^k] = \sum_{m=0}^{k-1} \Big[&\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m} + \partial_\alpha^m(b_\alpha \partial_\alpha^{k-m} D_t) + \partial_\alpha^m(b_\alpha[b\partial_\alpha, \partial_\alpha^{k-m}]) + \partial_\alpha^m b_\alpha \partial_\alpha^{k-m} D_t \\ &+ \partial_\alpha^m b_\alpha [b\partial_\alpha, b]\partial_\alpha^{k-m-1}\Big]\end{aligned} \tag{2.30}$$

Proof. It's easy to see that

$$\begin{aligned}[D_t^2, \partial_\alpha^k] &= -\sum_{m=0}^{k-1}\Big[\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m} + \partial_\alpha^m(b_\alpha D_t \partial_\alpha^{k-m}) + \partial_\alpha^m b_\alpha \partial_\alpha D_t \partial_\alpha^{k-m-1}\Big] \\ &= -\sum_{m=0}^{k-1}\Big[\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m} + \partial_\alpha^m(b_\alpha \partial_\alpha^{k-m} D_t) + \partial_\alpha^m(b_\alpha[b\partial_\alpha, \partial_\alpha^{k-m}]) + \partial_\alpha^m b_\alpha \partial_\alpha^{k-m} D_t \\ &\quad + \partial_\alpha^m b_\alpha[b\partial_\alpha, b]\partial_\alpha^{k-m-1}\Big]\end{aligned}$$

□

2.2.5 Preservation of symmetries.

The water waves with point vortices preserve the symmetry (H4). Such symmetry is well-known if there is no point vortex.

Lemma II.14 (Preservation of symmetries)**.** *Let* $\Omega(0) \in \mathbb{R}^2$ *be symmetric about* $x = 0$*. Let* $Re\{F\}$ *be odd in* x*, and* $Im\{F\}$ *be even in* x *at time* $t = 0$*. Suppose the solution to the system (1.8) exists on* $[0, T_0]$*. Then* $Re\{F\}$ *remains odd in* x*, and* $Im\{F\}$ *remains even in* x *for all* $t \in [0, T_0]$*.*

This is the consequence of the uniqueness of the solutions to equation (1.8).

2.3 Taylor sign condition

In this section we give a systematic study of the Taylor sign condition. We derive the formula (1.25), and then use this formula to show that the Taylor sign condition could fail if the point vortices are sufficiently close to the free interface. We also obtain a criterion for the Taylor sign condition to hold.

2.3.1 The Taylor sign condition in Riemann variables

Recall that $\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha,t)-z_j(t))}$ is holomorphic, i.e., there is a holomorphic function $F(z,t)$ in $\Omega(t)$ such that

$$F(z(\alpha,t),t) = \bar{z}_t(\alpha,t) + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))}.$$

So we have

$$\begin{aligned} \bar{z}_{tt} =& \partial_t \bar{z}_t = \partial_t\Big(F(z(\alpha,t),t) - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))}\Big) \\ =& F_z(z(\alpha,t),t) z_t + F_t + \sum_{j=1}^{N} \frac{\lambda_j i(z_t(\alpha,t) - \dot{z}_j(t))}{2\pi(z(\alpha,t) - z_j(t))^2} \end{aligned} \tag{2.31}$$

Note that

$$\dot{z}_j(t) = (v - \frac{\lambda_j i}{2\pi(z - z_j(t))})\Big|_{z=z_j(t)} = \bar{F}(z_j(t), t) - \sum_{k:k\neq j} \frac{\lambda_k i}{2\pi \overline{z_k(t) - z_j(t)}} \tag{2.32}$$

Let $\Phi(\cdot, t) : \Omega(t) \to \mathbb{P}_-$ be the Riemann mapping such that $\Phi_z \to 1$ as $z \to \infty$. Let $h(\alpha, t) := \Phi(z(\alpha, t), t)$. Denote [1]

$$Z(\alpha, t) := z \circ h^{-1}(\alpha, t), \qquad b = h_t \circ h^{-1}, \qquad D_t := \partial_t + b\partial_\alpha, \tag{2.33}$$

$$A := (a h_\alpha) \circ h^{-1}. \tag{2.34}$$

Use (2.31), apply h^{-1} on both sides of $\bar{z}_{tt} + ia\bar{z}_\alpha = i$, we obtain

$$F_z \circ Z(\alpha, t) D_t Z + F_t \circ Z(\alpha, t) + \sum_{j=1}^{N} \frac{\lambda_j i (D_t Z(\alpha, t) - \dot{z}_j(t))}{2\pi (Z(\alpha, t) - z_j(t))^2} + iA\bar{Z}_\alpha = i. \tag{2.35}$$

Multiply by Z_α on both sides of (2.35), and denote

$$A_1 := A|Z_\alpha|^2,$$

we obtain

$$F_z \circ Z Z_\alpha D_t Z + F_t \circ Z Z_\alpha + \sum_{j=1}^{N} \frac{\lambda_j i D_t Z Z_\alpha - \lambda_j i \dot{z}_j(t) Z_\alpha}{2\pi (Z(\alpha, t) - z_j(t))^2} + iA_1 = iZ_\alpha. \tag{2.36}$$

Apply $I - \mathbb{H}$ on both sides of the above equation, then take imaginary parts, we obtain

$$A_1 = 1 - Im\Big\{(I - \mathbb{H})(F_z \circ Z Z_\alpha D_t Z) + (I - \mathbb{H}) \sum_{j=1}^{N} \frac{\lambda_j i (D_t Z Z_\alpha - \dot{z}_j(t) Z_\alpha)}{2\pi (Z(\alpha, t) - z_j(t))^2}\Big\} \tag{2.37}$$

[1] In §2.5, we also use the notation A, b, D_t. We'd like the readers to keep in mind that they are not the same. In §2.5, $A = (a\kappa_\alpha) \circ \kappa^{-1}, b = \kappa_t \circ \kappa^{-1}, D_t = \partial_t + \kappa_t \circ \kappa^{-1}\partial_\alpha$.

Note that

$$F_z \circ Z Z_\alpha = \partial_\alpha (D_t \bar{Z} + \sum_{j=1}^N \frac{\lambda_j i}{2\pi(Z(\alpha,t) - z_j(t))}) = \partial_\alpha D_t \bar{Z} - \sum_{j=1}^N \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha,t) - z_j(t))^2}$$

is holomorphic. So we have

$$\begin{aligned}(I-\mathbb{H})F_z \circ Z Z_\alpha D_t Z =& [D_t Z, \mathbb{H}](\partial_\alpha D_t \bar{Z} - \sum_{j=1}^N \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha,t) - z_j(t))^2}) \\ =& [D_t Z, \mathbb{H}]\partial_\alpha D_t \bar{Z} - \sum_{j=1}^N [D_t Z, \mathbb{H}] \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha,t) - z_j(t))^2})\end{aligned} \tag{2.38}$$

We know that

$$-Im[D_t Z, \mathbb{H}]\partial_\alpha D_t \bar{Z} = \frac{1}{2\pi}\int \frac{|D_t Z(\alpha,t) - D_t Z(\beta,t)|^2}{(\alpha-\beta)^2} d\beta \geqslant 0. \tag{2.39}$$

Also,

$$\begin{aligned}& - \sum_{j=1}^N [D_t Z, \mathbb{H}] \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha,t) - z_j(t))^2}) + (I-\mathbb{H}) \sum_{j=1}^N \frac{\lambda_j i (D_t Z Z_\alpha - \dot{z}_j(t) Z_\alpha)}{2\pi(Z(\alpha,t) - z_j(t))^2} \\ =& \sum_{j=1}^N [D_t Z, I-\mathbb{H}] \frac{\lambda_j i Z_\alpha}{2\pi(Z(\alpha,t) - z_j(t))^2}) + (I-\mathbb{H}) \sum_{j=1}^N \frac{\lambda_j i (D_t Z Z_\alpha - \dot{z}_j(t) Z_\alpha)}{2\pi(Z(\alpha,t) - z_j(t))^2} \\ =& \sum_{j=1}^N \frac{\lambda_j i}{2\pi} D_t Z (I-\mathbb{H}) \frac{Z_\alpha}{(Z(\alpha,t) - z_j(t))^2} - \sum_{j=1}^N \frac{\lambda_j i}{2\pi}(I-\mathbb{H}) \frac{\dot{z}_j(t) Z_\alpha}{(Z(\alpha,t) - z_j(t))^2} \\ =& \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \Big((I-\mathbb{H}) \frac{Z_\alpha}{(Z(\alpha,t) - z_j(t))^2} \Big) (D_t Z - \dot{z}_j(t))\end{aligned} \tag{2.40}$$

So we have

$$
\begin{aligned}
A_1 =& 1 + \frac{1}{2\pi}\int \frac{|D_tZ(\alpha,t) - D_tZ(\beta,t)|^2}{(\alpha-\beta)^2}d\beta \\
&- Im\Big\{\sum_{j=1}^N \frac{\lambda_j i}{2\pi}\Big((I-\mathbb{H})\frac{Z_\alpha}{(Z(\alpha,t)-z_j(t))^2}\Big)(D_tZ - \dot{z}_j(t))\Big\} \\
=& 1 + \frac{1}{2\pi}\int \frac{|D_tZ(\alpha,t) - D_tZ(\beta,t)|^2}{(\alpha-\beta)^2}d\beta \\
&- \sum_{j=1}^N \frac{\lambda_j}{2\pi} Re\Big\{\Big((I-\mathbb{H})\frac{Z_\alpha}{(Z(\alpha,t)-z_j(t))^2}\Big)(D_tZ - \dot{z}_j(t))\Big\}
\end{aligned}
\tag{2.41}
$$

To get an estimate as sharp as possible for the Taylor sign condition, we'd like to get rid of the Hilbert transform $\mathbb{H}$ in the formula above.

It's easy to see that, if $Z = \alpha$, then $\frac{Z_\beta}{(Z(\beta,t)-z_j(t))^2}(D_tZ - \dot{z}_j(t))$ is boundary value of a holomorphic function in $\mathbb{P}_+$, so

$$
\Big((I-\mathbb{H})\frac{Z_\alpha}{(Z(\alpha,t)-z_j(t))^2}\Big)(D_tZ - \dot{z}_j(t)) = 2\frac{Z_\alpha}{(Z(\alpha,t)-z_j(t))^2}(D_tZ - \dot{z}_j(t))
$$

For the general case, we use the following lemma.

Lemma II.15. *Let $z_0 \in \Omega(t)$. Then*

$$
(I-\mathbb{H})\frac{1}{Z(\alpha,t)-z_0} = \frac{2}{c_1(\alpha-w_0)}, \qquad c_1 = (\Phi^{-1})_z(w_0), \qquad w_0 = \Phi(z_0,t). \tag{2.42}
$$

Proof. Note that $Z(\alpha,t) = \Phi^{-1}(\alpha,t)$. So $Z(\alpha,t) - z_0$ is the boundary value of $\Phi^{-1}(z,t) - z_0$ in the lower half plane. Since Φ^{-1} is 1-1 and onto, $\Phi^{-1}(z,t) - z_0$ has a unique zero $w_0 := \Phi(z_0)$, so $\frac{1}{Z(\alpha,t)-z_0}$ has a exactly one pole of multiplicity one. For z near w_0, we have

$$
\Phi^{-1}(z,t) - z_0 = c_1(z-w_0) + \sum_{n=2}^\infty c_n(z-w_0)^n, \qquad where \quad c_1 = (\Phi^{-1})_z(w_0) \neq 0. \tag{2.43}
$$

Therefore, we have $\frac{1}{Z(\alpha,t)-z_0} - \frac{1}{c_1(\alpha-w_0)}$ is holomorphic in $\mathbb{P}_-$, and hence

$$(I-\mathbb{H})\Big(\frac{1}{Z(\alpha,t)-z_0} - \frac{1}{c_1(\alpha-w_0)}\Big) = 0. \tag{2.44}$$

Since $\frac{1}{c_1(\alpha-w_0)}$ is holomorphic in $\mathbb{P}_+$, we obtain

$$(I-\mathbb{H})\frac{1}{Z(\alpha,t)-z_0} = (I-\mathbb{H})\frac{1}{c_1(\alpha-w_0)} = \frac{2}{c_1(\alpha-w_0)}. \tag{2.45}$$

□

Corollary II.16. *We have*

$$(I-\mathbb{H})\frac{Z_\alpha}{(Z(\alpha,t)-z_j(t))^2} = \frac{2}{(\Phi^{-1})_z(\Phi(z_j(t)))(\alpha-\Phi(z_j(t)))^2} \tag{2.46}$$

Proof. We have

$$\begin{aligned}(I-\mathbb{H})\frac{Z_\alpha}{(Z(\alpha,t)-z_j(t))^2} &= -\partial_\alpha(I-\mathbb{H})\frac{1}{Z(\alpha,t)-z_j(t)}\\ &= -\partial_\alpha\frac{2}{(\Phi^{-1})_z(\Phi(z_j(t)))(\alpha-\Phi(z_j(t)))}\\ &= \frac{2}{(\Phi^{-1})_z(\Phi(z_j(t)))(\alpha-\Phi(z_j(t)))^2}.\end{aligned}$$

□

Corollary II.17. *We have*

$$A_1 = 1 + \frac{1}{2\pi}\int\frac{|D_tZ(\alpha,t)-D_tZ(\beta,t)|^2}{(\alpha-\beta)^2}d\beta - \sum_{j=1}^{N}\frac{\lambda_j}{\pi}Re\Big\{\frac{D_tZ-\dot{z}_j}{c_0^j(\alpha-w_0^j)^2}\Big\}, \tag{2.47}$$

where

$$c_0^j = (\Phi^{-1})_z(\omega_0^j), \qquad \omega_0^j = \Phi(z_j). \tag{2.48}$$

2.3.2 A formula for A_1 when $Z(\alpha,t)=\alpha$, $D_tZ=\sum_{j=1}^N \frac{\lambda_j i}{2\pi}\overline{\frac{1}{\alpha-z_j(t)}}$

If the point vortices are very close to the interface, then Taylor sign condition can fail. To see this, we study the special case when $Z(\alpha,t)=\alpha$ and $D_tZ(\alpha,t)=\sum_{j=1}^N \frac{\lambda_j i}{2\pi}\overline{\frac{1}{\alpha-z_j(t)}}$. Since the integral term of the formula (2.47) is nonlocal, in order to obtain a more convenient form of (2.47), we use residue theorem to calculate this integral. We'll use the following formula.

Lemma II.18. *Let $w_1, w_2 \in \mathbb{P}_-$. Then*

$$\int_{-\infty}^{\infty} \frac{1}{(\beta-w_1)(\beta-\overline{w_2})}d\beta = \frac{2\pi i}{\overline{w_2}-w_1} \tag{2.49}$$

Proof. $\overline{w_2}$ is the only residue of $\frac{1}{(\beta-w_1)(\beta-\overline{w_2})}$ in $\mathbb{P}_+$. By residue Theorem,

$$\int_{-\infty}^{\infty} \frac{1}{(\beta-w_1)(\beta-\overline{w_2})}d\beta = \frac{2\pi i}{\overline{w_2}-w_1}.$$

□

As a consequence, we have

Corollary II.19. *Assume $Z(\alpha,t)=\alpha$, $D_tZ=\sum_{j=1}^N \frac{\lambda_j i}{2\pi}\overline{\frac{1}{\alpha-z_j(t)}}$. We have*

$$\frac{1}{2\pi}\int \frac{|D_tZ(\alpha,t)-D_tZ(\beta,t)|^2}{(\alpha-\beta)^2}d\beta = \sum_{1\leqslant j,k\leqslant N} \frac{\lambda_j\lambda_k}{(2\pi)^2}\frac{1}{(\alpha-z_j)\overline{(\alpha-z_k)}}\frac{i}{\overline{z_k}-z_j}. \tag{2.50}$$

Proof. We have

$$D_tZ(\alpha,t)-D_t(\beta,t) = \sum_{j=1}^N \frac{\lambda_j i}{2\pi}\overline{\frac{\beta-\alpha}{(\alpha-z_j)(\beta-z_j)}}. \tag{2.51}$$

So we have

$$
\begin{aligned}
&\Big|\frac{D_tZ(\alpha,t)-D_t(\beta,t)}{\alpha-\beta}\Big|^2=\Big|\sum_{j=1}^N\frac{\lambda_j}{2\pi}\frac{1}{(\alpha-z_j)(\beta-z_j)}\Big|^2\\
=&\sum_{j=1}^N\sum_{k=1}^N\frac{\lambda_j\lambda_k}{(2\pi)^2}\frac{1}{(\alpha-z_j)(\beta-z_j)\overline{(\alpha-z_k)(\beta-z_k)}}
\end{aligned}
\tag{2.52}
$$

Apply lemma II.18, we have

$$
\int_{-\infty}^{\infty}\frac{1}{(\alpha-z_j)(\beta-z_j)\overline{(\alpha-z_k)(\beta-z_k)}}d\beta=\frac{1}{(\alpha-z_j)\overline{(\alpha-z_k)}}\frac{2\pi i}{\overline{z_k}-z_j}.
$$

So we have

$$
\begin{aligned}
\frac{1}{2\pi}\int\frac{|D_tZ(\alpha,t)-D_tZ(\beta,t)|^2}{(\alpha-\beta)^2}d\beta=&\sum_{1\leqslant j,k\leqslant N}\frac{\lambda_j\lambda_k}{(2\pi)^3}\frac{1}{(\alpha-z_j)\overline{(\alpha-z_k)}}\frac{2\pi i}{\overline{z_k}-z_j}\\
=&\sum_{1\leqslant j,k\leqslant N}\frac{\lambda_j\lambda_k}{(2\pi)^2}\frac{1}{(\alpha-z_j)\overline{(\alpha-z_k)}}\frac{i}{\overline{z_k}-z_j}.
\end{aligned}
$$

□

Corollary II.20. *Assume* $Z(\alpha,t)=\alpha$*,* $D_tZ=\sum_{j=1}^N\frac{\lambda_j i}{2\pi}\overline{\frac{1}{\alpha-z_j(t)}}$*. Then*

$$
A_1=1+\sum_{1\leqslant j,k\leqslant N}\frac{\lambda_j\lambda_k}{(2\pi)^2}\frac{1}{(\alpha-z_j)\overline{(\alpha-z_k)}}\frac{i}{\overline{z_k}-z_j}-\sum_{j=1}^N\frac{\lambda_j}{\pi}Re\Big\{\frac{D_tZ-\dot{z}_j}{(\alpha-z_j)^2}\Big\}.
\tag{2.53}
$$

In the following two subsections, we use Corollary II.20 to construct examples for which the Taylor sign condition fails.

2.3.3 One point vortex: An example that the Taylor condition fails

We have the following characterization. Recall that $\frac{A_1}{|Z_\alpha|}=-\frac{\partial P}{\partial \mathbf{n}}$.

Proposition II.21. *Assume that at time t, the interface is* $\Sigma(t)=\mathbb{R}$*, the fluid velocity is* $v(z,t)=\frac{\lambda i}{2\pi}\overline{\frac{1}{z-z_1(t)}}$*, i.e., it is generated by a single point vortex* $z_1(t):=x(t)+iy(t)$*. Then*

(1) If $\frac{\lambda^2}{|y|^3} < \frac{8\pi^2}{3}$, strong Taylor sign condition holds. We have

$$\inf_{\alpha\in\mathbb{R}} A_1(\alpha,t) \geqslant 1 - \frac{3}{8\pi^2}\frac{\lambda^2}{|y|^3} > 0. \tag{2.54}$$

(2) If $\frac{\lambda^2}{|y|^3} > \frac{8\pi^2}{3}$, Taylor sign condition fails, i.e., there exists $\alpha \in \mathbb{R}$ such that $A_1(\alpha,t) < 0$.

(3) If $\frac{\lambda^2}{|y|^3} = \frac{8\pi^2}{3}$, 'Degenerate Taylor sign condition' holds, i.e.,

$$A_1(\alpha,t) > 0, \quad \forall\ \alpha \neq x(t); \qquad \textit{and } A_1(x(t),t) = 0. \tag{2.55}$$

Remark II.22. The quantity $\frac{\lambda^2}{|y|^3}$ is a measurement of the interface-vortex interaction. λ is the intensity of the point vortex, and $|y|$ is the distance from the point vortex to the interface.

Proof. Note that

$$\dot{z}_1(t) = 0.$$

So we have

$$\begin{aligned} -\sum_{j=1}^{N}\frac{\lambda_j}{\pi}Re\Big\{\frac{D_tZ-\dot{z}_j}{(\alpha-z_j)^2}\Big\} =& -\frac{\lambda}{\pi}Re\Big\{\frac{D_tZ(\alpha,t)}{(\alpha-z_1(t))^2}\Big\} \\ =& -\frac{\lambda}{\pi}Re\Big\{\frac{1}{(\alpha-z_1(t))^2}\frac{\lambda i}{2\pi\overline{\alpha-z_1(t)}}\Big\} \\ =&\frac{\lambda^2}{2\pi^2}\frac{y}{|\alpha-z_1(t)|^4}. \end{aligned} \tag{2.56}$$

Therefore, by Corollary II.20, we have

$$A_1(\alpha,t) = 1 + \frac{\lambda^2}{4\pi^2}\frac{1}{|\alpha-z_1|^2}\frac{i}{-2iy} + \frac{\lambda^2}{2\pi^2}\frac{y}{|\alpha-z_1(t)|^4}.$$

Without loss of generality, we can assume $x = 0$. Setting $\partial_\alpha A_1(\alpha,t) = 0$, it's easy to see that $A_1(\alpha,t)$ admits a unique local minimum at $\alpha = x = 0$. Moreover, it's easy to see that

$$\lim_{\alpha\to\pm\infty} A_1(\alpha,t) = 1. \tag{2.57}$$

Therefore, $\inf_{\alpha\in\mathbb{R}} A(\alpha,t) \leqslant 0$ if and only if $A_1(0,t) \leqslant 0$.

We have

$$A_1(0,t) = 1 - \frac{\lambda^2}{8\pi^2}\frac{1}{y^3} + \frac{\lambda^2}{2\pi^2}\frac{1}{y^3} = 1 - \frac{3\lambda^2}{8\pi^2}\frac{1}{|y|^3}.$$

If $\frac{\lambda^2}{|y|^3} < \frac{8\pi^2}{3}$, then $A(\alpha,t) \geqslant A(0,t) > 0$. If $\frac{\lambda^2}{|y|^3} > \frac{8\pi^2}{3}$, then $A_1(0,t) < 0$. If $\frac{\lambda^2}{|y|^3} = \frac{8\pi^2}{3}$, then $A_1(0,t) = 0$, and for $\alpha \neq 0$, $A_1(\alpha,t) > A_1(0,t) = 0$. □

2.3.4 Two point vortices: Another example that Taylor sign condition fails

We show that if the point vortices are too close to the interface, then the Taylor sign condition fails.

Assume that $Z(\alpha,t) = \alpha$. Assume $D_tZ = \sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\overline{\frac{1}{\alpha - z_j(t)}}$. Assume $z_1(t) = -x(t) + iy(t)$, $z_2(t) = x(t) + iy(t)$, with $x(t) > 0$, $y(t) < 0$, and $\lambda_1 = -\lambda_2 := \lambda$. Let's calculate $A_1(0,t)$.

We have

$$D_tZ(\alpha,t) = \sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\overline{\frac{1}{\alpha - z_j(t)}} = \frac{\lambda i}{2\pi}\overline{\frac{\overline{z_1(t) - z_2(t)}}{(\alpha - z_1(t))(\alpha - z_2(t))}}$$

Since $z_1 - z_2 = -2x$,

$$D_tZ(0,t) = \frac{\lambda i}{\pi}\frac{x}{x^2+y^2}.$$

At $\alpha = 0$, we have

$$\sum_{j=1}^2 \frac{\lambda_j}{\pi}\frac{1}{(0 - z_j(t))^2} = \frac{\lambda}{\pi}\frac{4xyi}{(x^2+y^2)^2}.$$

So

$$-\sum_{j=1}^{2}\frac{\lambda_j}{\pi}\frac{1}{(0-z_j(t))^2}D_tZ(0) = -\frac{\lambda i}{\pi}\frac{x}{x^2+y^2}\frac{\lambda}{\pi}\frac{4xyi}{(x^2+y^2)^2} = \frac{4\lambda^2x^2y}{\pi^2(x^2+y^2)^3} \tag{2.58}$$

We have

$$\dot{z}_1 = \frac{-\lambda i}{2\pi}\frac{1}{\overline{z_1-z_2}} = \frac{\lambda i}{4\pi}\frac{1}{x}.$$

$$\dot{z}_2 = \frac{\lambda i}{2\pi}\frac{1}{\overline{z_2-z_1}} = \frac{\lambda i}{4\pi}\frac{1}{x}.$$

So we have

$$\sum_{j=1}^{2}\frac{\lambda_j}{\pi}\frac{1}{(0-z_j)^2}\dot{z}_j(t) = \frac{\lambda i}{4\pi}\frac{1}{x}\sum_{j=1}^{2}\frac{\lambda_j}{\pi}\frac{1}{(0-z_j)^2} = \frac{\lambda i}{4\pi x}\frac{\lambda}{\pi}\frac{4xyi}{(x^2+y^2)^2} = -\frac{\lambda^2 y}{\pi^2(x^2+y^2)^2}.$$

So we have

$$-\sum_{j=1}^{N}\frac{\lambda_j}{\pi}Re\Big\{\frac{1}{(0-z_j(t))^2}(D_tZ(0,t)-\dot{z}_j(t))\Big\} = \frac{\lambda^2y(3x^2-y^2)}{\pi^2(x^2+y^2)^3}.$$

On the other hand, we have

$$\begin{aligned}
&\sum_{1\leqslant j,k\leqslant 2}\frac{\lambda_j\lambda_k}{(2\pi)^2}\frac{1}{(0-z_j)\overline{(0-z_k)}}\frac{i}{\overline{z_k}-z_j}\\
=&\frac{\lambda^2}{4\pi^2}\frac{1}{|z_1|^2}\frac{i}{\bar{z}_1-z_1}+\frac{\lambda^2}{4\pi^2}\frac{1}{|z_2|^2}\frac{i}{\bar{z}_2-z_2}-\frac{\lambda^2}{4\pi^2}\frac{1}{z_1\bar{z}_2}\frac{i}{\bar{z}_2-z_1}-\frac{\lambda^2}{4\pi^2}\frac{1}{z_2\bar{z}_1}\frac{i}{\bar{z}_1-z_2}\\
=&\frac{\lambda^2}{4\pi^2}\frac{1}{|z_1|^2}\frac{1}{|y|}+2Re\Big\{\frac{\lambda^2}{8\pi^2}\frac{i}{(x-iy)^3}\Big\}\\
=&\frac{\lambda^2}{4\pi^2}\frac{1}{x^2+y^2}\frac{1}{|y|}+\frac{\lambda^2}{4\pi^2}\frac{y^3-3x^2y}{(x^2+y^2)^3}\\
=&\frac{\lambda^2}{4\pi^2}\frac{x^4+5x^2y^2}{|y|(x^2+y^2)^3}.
\end{aligned}$$

Therefore, by Corollary II.20, we have

$$\begin{aligned} A_1(0,t) =& 1 + \frac{\lambda^2}{4\pi^2}\frac{x^4+5x^2y^2}{|y|(x^2+y^2)^3} + \frac{\lambda^2 y(3x^2-y^2)}{\pi^2(x^2+y^2)^3} \\ =& 1 + \frac{\lambda^2}{4\pi^2}\frac{x^4+5x^2y^2-12x^2y^2+4y^4}{|y|(x^2+y^2)^3} \\ =& 1 + \frac{\lambda^2}{4\pi^2}\frac{x^4+4y^4-7x^2y^2}{|y|(x^2+y^2)^3} \end{aligned} \tag{2.59}$$

So we have

Proposition II.23. *Assume that at time t, the interface is $\Sigma(t) = \mathbb{R}$, the fluid velocity is $v(z,t) = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \overline{\frac{1}{z - z_j(t)}}$, where $\lambda_1 = -\lambda_2 := \lambda$. Assume*

$$z_1(t) = -x + iy, \qquad z_2(t) = x + iy, \quad \textit{where} \quad x > 0, y < 0.$$

If

$$1 + \frac{\lambda^2}{4\pi^2}\frac{x^4+4y^4-7x^2y^2}{|y|(x^2+y^2)^3} < 0, \tag{2.60}$$

then Taylor sign condition fails.

Corollary II.24. *Under the assumption of Proposition II.23, if $|x| = |y|$ and $\frac{\lambda^2}{|y|^3} > 16\pi^2$, then the strong Taylor sign condition fails.*

2.3.5 A criterion that implies the strong Taylor sign condition

If the vortex-vortex, vortex-interface interaction is weak, then the Taylor sign condition holds. Let's recall that we denote F by

$$\bar{F}(z,t) := v(z,t) - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \overline{\frac{1}{z - z_j(t)}}. \tag{2.61}$$

We have the following.

Proposition II.25. *Assume* $\inf_{\alpha\in\mathbb{R}}|Z_\alpha| = \beta_0$, $\|F\|_\infty = M_0$. *Denote*

$$\tilde{\lambda} =: \frac{\sum_{j=1}^N |\lambda_j|}{\pi}, \qquad \tilde{d}_I(t) := \min_{1\leqslant j\leqslant N}\inf_{\alpha\in\mathbb{R}} |\alpha - \Phi(z_j(t))|, \qquad \tilde{d}_P(t) = \min_{j\neq k} |z_j(t) - z_k(t)|.$$

If

$$\frac{\tilde{\lambda}^2}{2\tilde{d}_I(t)^3\beta_0} + \frac{\tilde{\lambda}^2}{2\tilde{d}_I(t)^2\tilde{d}_P(t)} + \frac{2M_0\tilde{\lambda}}{\tilde{d}_I(t)^2} < \beta_0, \tag{2.62}$$

then the strong Taylor sign condition holds.

Proof. Use formula

$$A_1 = 1 + \frac{1}{2\pi}\int \frac{|D_tZ(\alpha,t) - D_tZ(\beta,t)|^2}{(\alpha-\beta)^2}d\beta - \sum_{j=1}^N \frac{\lambda_j}{\pi} Re\Big\{\frac{D_tZ - \dot{z}_j}{c_0^j(\alpha - \omega_0^j)^2}\Big\} \tag{2.63}$$

For $\sum_{j=1}^N \frac{\lambda_j}{\pi} Re\Big\{\frac{D_tZ - \dot{z}_j}{c_0^j(\alpha-\omega_0^j)^2}\Big\}$, we have

$$\Big|\sum_{j=1}^N \frac{\lambda_j}{\pi} Re\Big\{\frac{D_tZ - \dot{z}_j}{c_0^j(\alpha - \omega_0^j)^2}\Big\}\Big| \leqslant \frac{\sum_{j=1}^N |\lambda_j|}{\pi} \max_{1\leqslant j\leqslant N} |c_0^j|^{-1}(\|D_tZ\|_\infty + |\dot{z}_j|)(\inf_{\alpha\in\mathbb{R}} |\alpha - \Phi(z_j)|)^{-2}$$

Since $Z(\alpha) = \Phi^{-1}(\alpha)$, we have $\partial_\alpha\Phi^{-1}(\alpha) = Z_\alpha$ and $\partial_\alpha\Phi^{-1}(\alpha)$ is the boundary value of $(\Phi^{-1})_z$. Note that $(\Phi^{-1})_z$ never vanishes. By maximum modules principle of holomorphic functions (apply to $\frac{1}{(\Phi^{-1})_z}$),

$$|c_0^j| = |(\Phi^{-1})_z(\Phi(z_j))| \geqslant \inf_{\alpha\in\mathbb{R}} |Z_\alpha| \geqslant \beta_0. \tag{2.64}$$

Since

$$Z(\alpha,t) - z_j(t) = \Phi^{-1}(\alpha,t) - \Phi^{-1}(\omega_0^j) = \Phi_z^{-1}(z')(\alpha - \omega_0^j) \tag{2.65}$$

for some $z' \in \mathbb{P}_-$, so we have

$$|Z(\alpha,t) - z_j(t)| \geqslant \beta_0|\alpha - \omega_0^j|. \tag{2.66}$$

Therefore,

$$|D_tZ| \leqslant |F| + \sum_{j=1}^N \frac{|\lambda_j|}{2\pi\beta_0}\frac{1}{|\alpha-\omega_0^j|} = M_0 + \frac{\tilde{\lambda}}{2\tilde{d}_I(t)\beta_0}.$$

similarly,

$$|\dot{z}_j(t)| = |\bar{F} + \sum_{k\neq j}\frac{\lambda_k i}{2\pi}\overline{\frac{1}{z_j(t)-z_k(t)}}| \leqslant M_0 + \frac{\sum_{j=1}^N|\lambda_j|}{2\pi}\tilde{d}_P(t)^{-1} = M_0 + \frac{\tilde{\lambda}}{2\tilde{d}_P(t)}.$$

So we obtain

$$\begin{aligned}|\sum_{j=1}^N \frac{\lambda_j}{\pi}Re\Big\{\frac{D_tZ-\dot{z}_j}{c_0^j(\alpha-\omega_0^j)^2}\Big\}| \leqslant & \beta_0^{-1}\frac{\tilde{\lambda}}{\tilde{d}_I(t)^2}(M_0 + \frac{\tilde{\lambda}}{2\tilde{d}_I(t)\beta_0} + M_0 + \frac{\tilde{\lambda}}{2\tilde{d}_P(t)})\\ \leqslant & \beta_0^{-1}(\frac{\tilde{\lambda}^2}{2\tilde{d}_I(t)^3\beta_0} + \frac{\tilde{\lambda}^2}{2\tilde{d}_I(t)^2\tilde{d}_P(t)} + \frac{2M_0\tilde{\lambda}}{\tilde{d}_I(t)^2}).\end{aligned}$$

If (2.62) holds, then

$$|\sum_{j=1}^N \frac{\lambda_j}{\pi}Re\Big\{\frac{D_tZ-\dot{z}_j}{c_1^j(\alpha-\omega_0^j)^2}\Big\}| < 1.$$

Then $A_1 > 0$, so strong Taylor sign condition holds. □

In particular, if $Z_\alpha \sim 1$, $d_I(t) \gtrsim 1$, $M_0 \ll 1$, and $|\lambda| \ll 1$, then the strong Taylor sign condition holds.

2.4 Local wellposedness: proof of Theorem I.24

In this section we prove Theorem I.24, i.e., prove local wellposedness of water waves with general N point vortices. As was explained in the introduction, our strategy is to quasilinearize the system (1.13) by taking one time derivative of the momentum equation $(\partial_t^2 + ia\partial_\alpha)\bar{z} = i$, and then obtain a closed energy estimate.

Recall that we assume

$$\inf_{\alpha\in\mathbb{R}} a(\alpha,0)|z_\alpha(\alpha,0)| \geqslant \alpha_0 > 0. \tag{2.67}$$

$$C_1|\alpha-\beta| \leqslant |z(\alpha,0)-z(\beta,0)| \leqslant C_2|\alpha-\beta|. \tag{2.68}$$

Let $T_0 \geqslant 0$, we make the following a priori assumptions:

$$\inf_{t\in[0,T_0]}\inf_{\alpha\in\mathbb{R}} a(\alpha,t)|z_\alpha(\alpha,t)| \geqslant \frac{\alpha_0}{2}, \qquad \frac{1}{2}C_1|\alpha-\beta| \leqslant |z(\alpha,0)-z(\beta,0)| \leqslant 2C_2|\alpha-\beta|, \tag{2.69}$$

and

$$\sup_{t\in[0,T_0]} \|z_{tt}(\cdot,t)\|_{H^1} \leqslant 2\|w_0\|_{H^1}. \tag{2.70}$$

Without loss of generality, we assume $T_0 \leqslant 1$.

Remark II.26. The a priori assumptions (2.69) and (2.70) hold at $t=0$. For $0 < t \leqslant T_0$, (2.69) and (2.70) will be justified by a bootstrap argument.

2.4.1 Velocity and acceleration of the point vortices.

For water waves with point vortices, the motion of the point vortices affects the dynamics of the water waves in a fundamental way, so we need to have a good understanding of the velocity and acceleration of the point vortices. We decompose the velocity field $\bar{v}$ as

$$\bar{v}(z,t) = F(z,t) - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi}\frac{1}{z-z_j(t)}. \tag{2.71}$$

So F is holomorphic in $\Omega(t)$. We have the following estimate for the velocity and acceleration of the point vortices.

Lemma II.27. *Assume that the assumptions of Theorem I.24 hold, and assume the a priori assumptions (2.69) and (2.70). Then*

$$|\dot{z}_j(t)| + |\ddot{z}_j(t)| \leqslant C(N\lambda_{max}, \|z_t\|_{H^2}, \|z_{tt}\|_{H^1}, d_I(t)^{-1}, d_P(t)^{-1}, C_1). \tag{2.72}$$

where

$$\lambda_{max} = \max_{1\leqslant j\leqslant N} |\lambda_j|, \tag{2.73}$$

and $C : (\mathbb{R}_+ \cup \{0\})^6 \to \mathbb{R}_+ \cup \{0\}$ *is a polynomial with positive coefficients.*

Proof. The main tool is the maximum principle of holomorphic functions.

Estimate $\dot{z}_j$: By (II.27), we have

$$\dot{z}_j(t) = (v - \frac{\lambda_j i}{\overline{z - z_j(t)}})\Big|_{z=z_j(t)} = \sum_{1\leqslant k\leqslant N, k\neq j} \frac{\lambda_k i}{2\pi(\overline{z_j(t) - z_k(t)})} + \bar{F}(z_j(t)). \tag{2.74}$$

Note that $\bar{F}$ is an anti-holomorphic function with boundary value $z_t - \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{1}{\overline{z(\alpha,t) - z_j(t)}}$, by maximum principle, we have

$$|\bar{F}(z_j(t),t)| \leqslant \|F(\cdot,t)\|_{L^\infty(\Sigma(t))} = \left\| z_t - \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{1}{\overline{z(\alpha,t) - z_j(t)}} \right\|_\infty. \tag{2.75}$$

By Triangle inequality, we obtain

$$\begin{aligned} |\dot{z}_j(t)| \leqslant & \|z_t\|_\infty + \left\| \sum_{k\neq j} \frac{\lambda_k i}{2\pi(\overline{z_j(t) - z_k(t)})} \right\|_\infty + \left\| \sum_{k=1}^N \frac{\lambda_k i}{2\pi \overline{z(\alpha,t) - z_k(t)}} \right\|_\infty \\ \leqslant & \|z_t\|_{H^1} + N\lambda_{max}(d_P(t)^{-1} + d_I(t)^{-1}). \end{aligned} \tag{2.76}$$

Estimate $\ddot{z}_j(t)$: Take time derivative of both sides of $\dot{z}_j(t) = \sum_{k:k\neq j} \frac{\lambda_k i}{2\pi(\overline{z_j(t) - z_k(t)})} + \bar{F}(z_j(t),t)$, we obtain

$$\ddot{z}_j(t) = -\sum_{k:k\neq j} \frac{\lambda_k i \overline{\dot{z}_j(t) - \dot{z}_k(t)}}{2\pi(\overline{z_j(t) - z_k(t)})^2} + \bar{F}_z(z_j(t),t)\dot{z}_j(t) + \bar{F}_t(z_j(t),t) \tag{2.77}$$

The boundary value of F_z is

$$F_z(z(\alpha,t),t) = \frac{\partial_\alpha F(z(\alpha,t),t)}{z_\alpha} = \frac{\bar{z}_{t\alpha}}{z_\alpha} - \sum_{j=1}^N \frac{\lambda_j i}{2\pi} \frac{1}{(z(\alpha,t) - z_j(t))^2}. \tag{2.78}$$

So

$$|F_z(z(\alpha,t),t)| \leqslant \|z_{t\alpha}\|_\infty \left\| \frac{1}{z_\alpha} \right\|_\infty + N\lambda_{max} d_I(t)^{-2}. \tag{2.79}$$

The boundary value of F_t is

$$\begin{aligned} F_t(z(\alpha,t),t) =& \partial_t F(z(\alpha,t),t) - \frac{\partial_\alpha F(z(\alpha,t),t)}{z_\alpha} z_t \\ =& \partial_t \Big(\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{z(\alpha,t) - z_j(t)} \Big) - \frac{\partial_\alpha (\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{z(\alpha,t)-z_j(t)}}{z_\alpha} z_t \\ =& \bar{z}_{tt} - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{z_t - \dot{z}_j(t)}{(z(\alpha,t) - z_j(t))^2} - \frac{\bar{z}_{t\alpha}}{z_\alpha} z_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{z_t}{(z(\alpha,t) - z_j(t))^2} \end{aligned} \tag{2.80}$$

By maximum principle,

$$\begin{aligned} \|F_t\|_\infty =& \left\| \bar{z}_{tt} - \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{z_t - \dot{z}_j(t)}{(z(\alpha,t) - z_j(t))^2} - \frac{\bar{z}_{t\alpha}}{z_\alpha} z_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{z_t}{(z(\alpha,t) - z_j(t))^2} \right\|_\infty \\ \leqslant& \|z_{tt}\|_\infty + \|z_{t\alpha}\|_\infty \left\| \frac{1}{z_\alpha} \right\|_\infty \|z_t\|_\infty + N\lambda_{max} \|z_t\|_\infty d_I(t)^{-2} + N\lambda_{max} |\dot{z}_j(t)| d_I(t)^{-2}. \end{aligned}$$

By a priori assumption (2.69), for smooth free interface, we have

$$\inf_{\alpha \in \mathbb{R}} |z_\alpha(\alpha,t)| \geqslant \frac{C_1}{2}. \tag{2.81}$$

Substitute the estimate from the previous lemma for $\dot{z}_j(t)$, use Sobolev embedding $\|f\|_{L^\infty} \leqslant \|f\|_{H^1}$, we have

$$\begin{aligned} |\ddot{z}_j(t)| \leqslant& \Big\{ N\lambda_{max} d_P(t)^{-2} (\|z_t\|_{H^1} + N\lambda_{max}(d_P(t)^{-1} + d_I(t)^{-1})) \Big\} + \Big\{ \|z_{tt}\|_{H^1} + \frac{2}{C_1} \|z_t\|_{H^2} \|z_t\|_{H^1} \\ &+ N\lambda_{max} \|z_t\|_{H^1} d_I(t)^{-2} + N\lambda_{max}(\|z_t\|_{H^1} + N\lambda_{max}(d_P(t)^{-1} + d_I(t)^{-1})) d_I(t)^{-2} \Big\} \\ &+ \Big\{ ((\|z_t\|_{H^1} + N\lambda_{max}(d_P(t)^{-1} + d_I(t)^{-1})) \frac{2}{C_1} \|z_t\|_{H^2} + N\lambda_{max} d_I(t)^{-2}) \Big\} \end{aligned} \tag{2.82}$$

(Here, the first bracket is the estimate for $-\sum_{k \neq j} \frac{\lambda_k i \overline{\dot{z}_j(t) - \dot{z}_k(t)}}{2\pi (z_j(t) - z_k(t))^2}$, the second bracket is the estimate for $\bar{F}_z \dot{z}_j(t)$, and the third bracket is the estimate for $\bar{F}_t(z_j(t), t))$.

In abbreviate form, we write the estimate for $\dot{z}_j(t), \ddot{z}_j(t)$ as

$$|\dot{z}_j(t)| + |\ddot{z}_j(t)| \leqslant C(N\lambda_{max}, \|z_t\|_{H^2}, \|z_{tt}\|_{H^1}, d_I(t)^{-1}, d_P(t)^{-1}, C_1). \tag{2.83}$$

$\square$

2.4.2 Quasilinearization.

Take time derivative on both sides of $\bar{z}_{tt} + ia\bar{z}_\alpha = i$, let $u = \bar{z}_t$. We obtain

$$u_{tt} + iau_\alpha = -ia_t\bar{z}_\alpha = -\frac{\bar{z}_{tt} - i}{|z_{tt} + i|} a_t |z_\alpha| := g. \tag{2.84}$$

To show that $a_t\bar{z}_\alpha$ is of lower order, we apply $I - \mathfrak{H}$ on both sides of (2.84). Then we have

$$\begin{aligned} &-i(I - \mathfrak{H})a_t\bar{z}_\alpha = (I - \mathfrak{H})(u_{tt} + iau_\alpha) \\ =&[\partial_t^2 + ia\partial_\alpha, \mathfrak{H}]u + (\partial_t^2 + ia\partial_\alpha)(I - \mathfrak{H})u. \end{aligned} \tag{2.85}$$

By lemma II.12,

$$[\partial_t^2 + ia\partial_\alpha, \mathfrak{H}]u = 2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 \bar{z}_{t\beta} d\beta \tag{2.86}$$

By lemma II.4, we have

$$\begin{aligned} (\partial_t^2 + ia\partial_\alpha)(I - \mathfrak{H})u =& -\frac{i}{\pi}\sum_{j=1}^N (\partial_t^2 + ia\partial_\alpha)\frac{\lambda_j}{z(\alpha,t) - z_j(t)} \\ =&\frac{i}{\pi}\sum_{j=1}^N \lambda_j \Big(\frac{2z_{tt} + i - \ddot{z}_j}{(z(\alpha,t) - z_j(t))^2} - 2\frac{(z_t - \dot{z}_j(t))^2}{(z(\alpha,t) - z_j(t))^3}\Big) \end{aligned} \tag{2.87}$$

By (2.83),

$$|\dot{z}_j(t)| + |\ddot{z}_j(t)| \leqslant C(N\lambda_{max}, \|z_t\|_{H^2}, \|z_{tt}\|_{H^1}, d_I(t)^{-1}, d_P(t)^{-1}, C_1). \tag{2.88}$$

We rewrite $-i(I-\mathfrak{H})a_t\bar{z}_\alpha$ as

$$-i(I-\mathfrak{H})a_t\bar{z}_\alpha = g_1 + g_2, \tag{2.89}$$

where

$$g_1 := 2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} - \frac{1}{\pi i}\int\Big(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\Big)^2\bar{z}_{t\beta}d\beta. \tag{2.90}$$

$$g_2 := \frac{i}{\pi}\sum_{j=1}^N \lambda_j\Big(\frac{2z_{tt}+i-\ddot{z}_j}{(z(\alpha,t)-z_j(t))^2} - 2\frac{(z_t-\dot{z}_j(t))^2}{(z(\alpha,t)-z_j(t))^3}\Big). \tag{2.91}$$

Multiply both sides of (III.11) by $i\frac{z_\alpha}{|z_\alpha|}$ and take real parts, we obtain,

$$(I+\mathfrak{K}^*)a_t|z_\alpha| = Re(\frac{iz_\alpha}{|z_\alpha|}(g_1+g_2)), \tag{2.92}$$

where

$$\mathfrak{K}^*f(\alpha) = p.v.\int Re\Big\{-\frac{1}{\pi i}\frac{z_\alpha}{|z_\alpha|}\frac{|z_\beta(\beta,t)|}{z(\alpha)-z(\beta)}\Big\}f(\beta)d\beta. \tag{2.93}$$

Both g_1 and g_2 are lower order terms.

Assuming the a priori assumptions (2.69) and (2.70), for $0 \leqslant t \leqslant T_0$, $(I+\mathfrak{K}^*)$ is invertible on $L^2(\Sigma)$, so we have

$$a_t|z_\alpha| = (I+\mathfrak{K}^*)^{-1}\Big\{Re(\frac{iz_\alpha}{|z_\alpha|}(g_1+g_2))\Big\}. \tag{2.94}$$

By Lemma III.13, we have

$$\|a_tz_\alpha\|_{H^s} \leqslant C\Big\|\frac{iz_\alpha}{|z_\alpha|}(g_1+g_2)\Big\|_{H^s}, \tag{2.95}$$

for C depends on C_1, C_2, and $\|z_\alpha - 1\|_{H^{s-1}}$.

So (2.84) can be written as

$$u_{tt} + a|z_\alpha|\mathbf{n}\frac{u_\alpha}{z_\alpha} = g, \tag{2.96}$$

where

$$\mathbf{n} = \frac{iz_\alpha}{|z_\alpha|}, \qquad a|z_\alpha| = |z_{tt} + i| = |\bar{u}_t + i|, \tag{2.97}$$

and

$$g := \frac{\bar{z}_{tt} - i}{|z_{tt} + i|}(I + \mathfrak{K}^*)^{-1}\Big\{Re(\frac{iz_\alpha}{|z_\alpha|}(g_1 + g_2))\Big\} \tag{2.98}$$

Denote [2]

$$A := a|z_\alpha|, \qquad D := \frac{\partial_\alpha}{z_\alpha}. \tag{2.99}$$

Let $k \in \mathbb{N}$. Apply ∂_α^k on both sides of (2.96), we have

$$(\partial_\alpha^k u)_{tt} + A\mathbf{n}\partial_\alpha^k(\frac{\partial_\alpha}{z_\alpha}u) = \partial_\alpha^k g - [\partial_\alpha^k, A\mathbf{n}]Du. \tag{2.100}$$

We have

$$[\partial_\alpha^k, A\mathbf{n}]Du = \sum_{m=1}^{k} c_{m,k}\partial_\alpha^m(A\mathbf{n})\partial_\alpha^{k-m}Du, \tag{2.101}$$

where

$$c_{m,k} = \frac{k!}{m!(k-m)!}.$$

So we obtain

$$\begin{cases} \partial_t^2\partial_\alpha^k u + A\mathbf{n}\partial_\alpha^k\frac{\partial_\alpha}{z_\alpha}u = g_k, \\ A = a|z_\alpha| \\ \mathbf{n} = \frac{iz_\alpha}{|z_\alpha|} = \frac{\bar{u}_t + i}{|\bar{u}_t + i|} \\ g_k = \partial_\alpha^k g - \sum_{m=1}^k c_{m,k}\partial_\alpha^m(A\mathbf{n})\partial_\alpha^{k-m}Du. \end{cases} \tag{2.102}$$

[2]This A here is not the same as that in §2.3 and §2.5.

2.4.3 Energy estimates.

Decompose $u = f + p$ as in (1.28). Let $s \in \mathbb{N}$. With quasilinearization (2.102), we define the energy $E(t)$ as

$$E(t) := \sum_{k=0}^{s} \Big\{ \int \frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|} |\partial_\alpha^k u_t|^2 d\alpha + Re \int \mathbf{n}|z_\alpha| D^{k+1} f \overline{D^k f} d\alpha \Big\} \tag{2.103}$$

Note that f is holomorphic in $\Omega(t)$, so is $D^m f$, for any integer $m \geqslant 0$. So we have

$$Re \int \mathbf{n}|z_\alpha| D^{k+1} f \overline{D^k f} d\alpha = Re \int i\partial_\alpha D^k f \overline{D^k f} = \int_{\Omega(t)} |\nabla D^k f|^2 dxdy \geqslant 0.$$

So the energy E is positive.

We can bound u_t by the energy E. Assume the bootstrap assumptions (2.69) and (2.70), since $a|z_\alpha| = |z_{tt} + i|$, we have

$$a|z_\alpha| \leqslant |z_{tt}| + 1 \leqslant \|z_{tt}\|_{L^\infty} + 1 \leqslant \|z_{tt}\|_{H^1} + 1 \leqslant 2\|w_0\|_{H^s} + 1. \tag{2.104}$$

Without loss of generality, we assume $C_2 > 1$. By the definition of E and the definition of $\mathcal{T}$, for $t \in [0, T]$, we have

$$E(t) \geqslant \sum_{k=0}^{s} \int \frac{\inf_{\alpha\in\mathbb{R}} |z_\alpha|^{-2k+1}}{\sup_{\alpha\in\mathbb{R}} a|z_\alpha|} |\partial_\alpha^k u_t(\alpha,t)|^2 d\alpha \tag{2.105}$$

$$\geqslant \sum_{k=0}^{s} \frac{(2C_2)^{-2k+1}}{2\|w_0\|_{H^s} + 1} \int |\partial_\alpha^k u_t(\alpha,t)|^2 d\alpha \tag{2.106}$$

$$\geqslant \frac{(2C_2)^{-2s+1}}{2\|w_0\|_{H^s} + 1} \|z_{tt}(\cdot,t)\|_{H^s}^2. \tag{2.107}$$

So we have

$$\|z_{tt}(\cdot,t)\|_{H^s} \leqslant \frac{(2\|w_0\|_{H^s} + 1)^{1/2}}{(2C_2)^{-s+1/2}} E(t)^{1/2}. \tag{2.108}$$

Let

$$\mathcal{E}(t) := \max_{\tau\in[0,t]} E(\tau). \tag{2.109}$$

Note that

$$\begin{aligned}
&\frac{d}{dt}Re\int \mathbf{n}|z_\alpha|D^{k+1}f\overline{D^kf}d\alpha = \frac{d}{dt}Re\int \frac{iz_\alpha}{|z_\alpha|}|z_\alpha|\frac{\partial_\alpha}{z_\alpha}D^kf\overline{D^kf}d\alpha\\
=&Re\frac{d}{dt}\int i\partial_\alpha D^kf\overline{D^kf}\\
=&Re\frac{d}{dt}\Big\{\int i\partial_\alpha D^ku\overline{D^ku}-\int i\partial_\alpha D^ku\overline{D^kp}-\int i\partial_\alpha D^kp\overline{D^ku}+\int i\partial_\alpha D^kp\overline{D^kp}\Big\}\\
:=&Re\frac{d}{dt}(I_1+I_2+I_3+I_4).
\end{aligned} \tag{2.110}$$

Note that

$$p = -\frac{i}{2\pi}\sum_{j=1}^N \frac{\lambda_j}{z(\alpha,t)-z_j(t)}. \tag{2.111}$$

So we have

$$(p)_t = \frac{i}{2\pi}\sum_{j=1}^N \frac{\lambda_j(z_t-\dot{z}_j(t))}{(z(\alpha,t)-z_j(t))^2}. \tag{2.112}$$

Observe that

$$D^m\frac{1}{z(\alpha,t)-z_j(t)} = \frac{(-1)^m m!}{(z(\alpha,t)-z_j(t))^{m+1}}, \tag{2.113}$$

$$D^m\frac{1}{(z(\alpha,t)-z_j(t))^2} = \frac{(-1)^m (m+1)!}{(z(\alpha,t)-z_j(t))^{m+2}}, \tag{2.114}$$

Therefore, for $k \geqslant 2$, by lemma II.10 and the a priori assumptions (2.69) and (2.70), we have

$$\|D^kp\|_{L^2\cap L^\infty}+\|\partial_\alpha D^kp\|_{L^2\cap L^\infty}+\|\partial_t D^kp\|_{L^2\cap L^\infty}+\|\partial_t\partial_\alpha D^kp\|_{L^2\cap L^\infty}\leqslant \tilde{C}, \tag{2.115}$$

for some

$$\tilde{C} = \tilde{C}(\|z_\beta\|_\infty, \|z_t\|_{L^2}, \|z_{tt}\|_{L^2}, d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{\max}, C_1, C_2).$$

We can take $\tilde{C}$ to be a polynomial with positive coefficients.

We need to estimate $\|D^k u\|_{L^2}$ as well. Note that for $0 \leqslant t \leqslant T_0 \leqslant 1$,

$$\begin{aligned}
\|z_t(t)\|_{H^s} \leqslant & \|v_0\|_{H^s} + \left\| \int_0^t z_{tt}(\cdot,\tau)d\tau \right\|_{H^s} \leqslant \|v_0\|_{H^s} + t \sup_{\tau\in[0,t]} \|z_{tt}(t=\tau)\|_{H^s} \\
\leqslant & \|v_0\|_{H^s} + \frac{(2\|w_0\|_{H^s}+1)^{1/2}}{(2C_2)^{-s+1/2}} \mathcal{E}(t)^{1/2} \\
\leqslant & C(\|v_0\|_{H^s}, \|w_0\|_{H^s}, C_2, \mathcal{E}(t)).
\end{aligned} \tag{2.116}$$

Similarly, we have

$$\|z_\alpha(t) - 1\|_{H^{s-1}} \leqslant C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, C_2, \mathcal{E}(t)). \tag{2.117}$$

Therefore, under the a priori assumption (2.69), using that $D^k = (\frac{\partial_\alpha}{z_\alpha})^k$, we have

$$\|D^k u\|_{L^2} \leqslant C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, C_2, \mathcal{E}(t)), \tag{2.118}$$

for some polynomial C with positive coefficients.

From (2.115), integration by parts if necessary, we see that

$$Re\frac{d}{dt}(I_2 + I_3 + I_4) \leqslant C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(t), d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{max}, C_1, C_2)$$

For I_1,

$$\begin{aligned}
& Re\frac{d}{dt}\int i\partial_\alpha D^k u \overline{D^k u} = Re\ i \int \partial_\alpha \partial_t D^k u \overline{D^k u} + \partial_\alpha D^k u \overline{\partial_t D^k u} \\
= & 2Re \int i\partial_\alpha D^k u \overline{\partial_t D^k u}.
\end{aligned}$$

We have

$$\partial_\alpha D^k u = \frac{1}{z_\alpha^{k-1}} \partial_\alpha^k D u + \frac{1}{z_\alpha^{k-2}} \partial_\alpha^{k-1}(\frac{1}{z_\alpha})\partial_\alpha D u + F_k, \tag{2.119}$$

where F_k consists of lower order terms. We have

$$\|F_k\|_{H^1} \leqslant C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(t), d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{max}, C_1, C_2). \tag{2.120}$$

We have also that

$$\begin{aligned}&\left\|\frac{1}{z_\alpha^{k-2}}\partial_\alpha^{k-1}(\frac{1}{z_\alpha})\partial_\alpha Du\right\|_{L^2}\\ \leqslant& C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(t), d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{max}, C_1, C_2).\end{aligned} \tag{2.121}$$

Similarly, we write

$$\partial_t D^k u = \frac{1}{z_\alpha^k}\partial_t\partial_\alpha^k u + \frac{1}{z_\alpha^{k-1}}\partial_\alpha^{k-1}\partial_t(\frac{1}{z_\alpha})u_\alpha + G_k, \tag{2.122}$$

where G_k consists of lower order terms, and

$$\|G_k\|_{H^1} \leqslant C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(t), d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{max}, C_1, C_2). \tag{2.123}$$

Note that

$$\partial_\alpha^{k-1}\partial_t z_\alpha^{-1} = -\frac{\partial_\alpha^k z_t}{z_\alpha^2} + H_k, \tag{2.124}$$

where H_k consists of lower order terms. So we can obtain

$$\begin{aligned}&\left\|\frac{1}{z_\alpha^{k-1}}\partial_\alpha^{k-1}\partial_t(\frac{1}{z_\alpha})u_\alpha\right\|_{L^2}\\ \leqslant& C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(t), d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{max}, C_1, C_2).\end{aligned} \tag{2.125}$$

Therefore, from the above estimates, we have

$$\begin{aligned}2Re\int i\partial_\alpha D^k u\overline{\partial_t D^k u} =& 2Re\int i\frac{1}{z_\alpha^{k-1}}\partial_\alpha^k Du\frac{1}{\bar z_\alpha^k}\overline{\partial_t\partial_\alpha^k u} + error_k\\ =& 2Re\int \frac{iz_\alpha}{|z_\alpha|}\frac{1}{|z_\alpha|^{2k-1}}\partial_\alpha^k Du\overline{\partial_t\partial_\alpha^k u} + error_k\\ =& 2Re\int \mathbf{n}\frac{1}{|z_\alpha|^{2k-1}}\partial_\alpha^k Du\overline{\partial_t\partial_\alpha^k u} + error_k,\end{aligned}$$

where

$$
\begin{aligned}
error_k =&2Re\int i\partial_\alpha D^k u\overline{\partial_t D^k u}-2Re\int \mathbf{n}\frac{1}{|z_\alpha|^{2k-1}}\partial_\alpha^k Du\overline{\partial_t\partial_\alpha^k u}\\
\leqslant& C(\|\partial_\alpha\xi_0\|_{H^{s-1}},\|v_0\|_{H^s},\|w_0\|_{H^s},\mathcal{E}(t),d_I(t)^{-1},d_P(t)^{-1},N\lambda_{max},C_1,C_2).
\end{aligned}
\tag{2.126}
$$

Observe that

$$
\begin{aligned}
\frac{dE}{dt}=&\sum_{k=0}^{s}\Big\{\int\Big(\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}\Big)_t|\partial_\alpha^k u_t|^2+2Re\int\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}(\partial_\alpha^k u_{tt})\overline{\partial_\alpha^k u_t}+\frac{d}{dt}Re\int\mathbf{n}|z_\alpha|D^{k+1}f\overline{D^k f}d\alpha\\
=&\sum_{k=0}^{s}\Big\{\int\Big(\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}\Big)_t|\partial_\alpha^k u_t|^2+2Re\int\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}(\partial_\alpha^k u_{tt})\overline{\partial_\alpha^k u_t}+2Re\int\mathbf{n}\frac{1}{|z_\alpha|^{2k-1}}\partial_\alpha^k Du\overline{\partial_t\partial_\alpha^k u}\\
&\quad+error_k\Big\}\\
=&\sum_{k=0}^{s}\Big\{\int\Big(\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}\Big)_t|\partial_\alpha^k u_t|^2+2Re\int\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}\Big\{(\partial_\alpha^k u_{tt})+a|z_\alpha|\mathbf{n}\partial_\alpha^k Du\Big\}\overline{\partial_\alpha^k u_t}+error_k\Big\}\\
=&\sum_{k=0}^{s}\Big\{\int\Big(\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}\Big)_t|\partial_\alpha^k u_t|^2+2Re\int\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}g_k\overline{\partial_\alpha^k u_t}+error_k\Big\}\\
\leqslant&\sum_{k=0}^{s}\Big\|\Big(\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}\Big)_t\Big\|_\infty\|\partial_\alpha u_t\|_{L^2}^2+2\Big\|\frac{|z_\alpha|^{-2k+1}}{a|z_\alpha|}\Big\|_\infty\|g_k\|_{L^2}\|\partial_\alpha^k u_t\|_{L^2}+error_k
\end{aligned}
\tag{2.127}
$$

It's easy to obtain the estimate that

$$
\|g_k\|_{L^2}\leqslant C(\|\partial_\alpha\xi_0\|_{H^{s-1}},\|v_0\|_{H^s},\|w_0\|_{H^s},\mathcal{E}(t),d_I(t)^{-1},d_P(t)^{-1},N\lambda_{max},C_1,C_2,\alpha_0),
\tag{2.128}
$$

and

$$
\begin{aligned}
&\Big\|\Big(\frac{|z_\alpha|^{-2k}}{a}\Big)_t\Big\|_\infty+\Big\|\frac{|z_\alpha|^{-2k}}{a}\Big\|_\infty\\
\leqslant& C(\|\partial_\alpha\xi_0\|_{H^{s-1}},\|v_0\|_{H^s},\|w_0\|_{H^s},\mathcal{E}(t),d_I(t)^{-1},d_P(t)^{-1},N\lambda_{max},C_1,C_2,\alpha_0)
\end{aligned}
\tag{2.129}
$$

Then we obtain

$$\frac{dE}{dt} \leqslant C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(t), d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{max}, C_1, C_2, \alpha_0) \tag{2.130}$$

For some polynomial C with positive coefficients. So we obtain

$$\begin{aligned} E(t) \leqslant\ & E(0) \\ & + \int_0^t C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(\tau), d_I(\tau)^{-1}, d_P(\tau)^{-1}, N\lambda_{max}, C_1, C_2, \alpha_0) ds. \end{aligned} \tag{2.131}$$

Note that $E(0) = \mathcal{E}(0)$. Take $sup_{0 \leqslant \tau \leqslant t}$, we obtain

$$\begin{aligned} \mathcal{E}(t) \leqslant\ & \mathcal{E}(0) \\ & + \int_0^t C(\|\partial_\alpha \xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(\tau), d_I(\tau)^{-1}, d_P(\tau)^{-1}, N\lambda_{max}, C_1, C_2, \alpha_0) d\tau. \end{aligned} \tag{2.132}$$

Growth of $d_P(t)^{-1}, d_I(t)^{-1}$. To obtain a closed energy estimate, we need also to control the growth of $d_P(t)^{-1}$ and $d_I(t)^{-1}$. Recall that $d_P(t) = \min_{j \neq k}\{|z_j(t) - z_k(t)|\}$, so we have

$$d_P(t)^{-1} = \max_{1 \leqslant j \neq k \leqslant N} \frac{1}{|z_j(t) - z_k(t)|}. \tag{2.133}$$

Note that

$$\Big|\frac{d}{dt}|z_j(t) - z_k(t)|^{-1}\Big| = \Big|\frac{(z_j(t) - z_k(t)) \cdot (\dot{z}_j(t) - \dot{z}_k(t))}{|z_j(t) - z_k(t)|^3}\Big| \leqslant |\dot{z}_j(t) - \dot{z}_k(t)| d_p(t)^{-2},$$

Use (2.83), and control $\|z_t\|_{H^2}, \|z_{tt}\|_{H^1}$ by $\mathcal{E}$, we obtain

$$|\frac{d}{dt} d_P(t)^{-1}| \leqslant C(N\lambda_{max}, d_P(t)^{-1}, \mathcal{E}, d_I(t)^{-1}, C_1, C_2, \alpha_0). \tag{2.134}$$

To estimate $\frac{d}{dt}d_I(t)^{-1}$, we estimate $\frac{d}{dt}|z_j(t)-z(\alpha,t)|^{-1}$. We have

$$\Big|\frac{d}{dt}|z_j(t)-z(\alpha,t)|^{-1}\Big| = \Big|\frac{(z_j(t)-z(\alpha,t))\cdot(\dot{z}_j(t)-z_t)}{|z_j(t)-z(\alpha,t)|^3}\Big| \leqslant |\dot{z}_j(t)-z_t|d_I(t)^{-2}.$$

Since $|\frac{d}{dt}d_I(t)^{-1}| \leqslant \max_{1\leqslant j\leqslant N}\sup_\alpha |\frac{d}{dt}|z_j(t)-z(\alpha,t)|^{-1}|$, use (2.83), and control $\|z_t\|_{H^2}, \|z_{tt}\|_{H^1}$ by $\mathcal{E}$, we obtain

$$|\frac{d}{dt}d_I(t)^{-1}| \leqslant C(N\lambda_{max}, d_P(t)^{-1}, \mathcal{E}, d_I(t)^{-1}, C_1, C_2, \alpha_0). \tag{2.135}$$

Combine (2.132), (2.134), (2.135), we obtain

$$\frac{d}{dt}\Big(d_P(t)^{-1}+d_I(t)^{-1}+\mathcal{E}(t)\Big) \leqslant C, \tag{2.136}$$

where

$$C = C(\|\partial_\alpha\xi_0\|_{H^{s-1}}, \|v_0\|_{H^s}, \|w_0\|_{H^s}, \mathcal{E}(t), d_I(t)^{-1}, d_P(t)^{-1}, N\lambda_{max}, C_1, C_2, \alpha_0) \tag{2.137}$$

is a polynomial with positive coefficients (the coefficients are absolute constants which do not depend on $N\lambda_{max}$, $d_P(0)^{-1}, d_I(0)^{-1}, \mathcal{E}, C_1, C_2, \alpha_0$, $\|\partial_\alpha\xi_0\|_{H^{s-1}}$, $\|v_0\|_{H^s}$, $\|w_0\|_{H^s}$). So we can use bootstrap argument to obtain closed energy estimates.

Lemma II.28. *Assume the assumptions of Theorem I.24. There exists T_0 depends on $N\lambda_{max}$, $d_P(0)^{-1}$, $d_I(0)^{-1}$, $\|(\partial_\alpha\xi_0, v_0, w_0)\|_{H^{s-1}\times H^s\times H^s}$, $\mathcal{E}(0), C_1, C_2, \alpha_0, s$ such that for all $0\leqslant t\leqslant T_0$,*

$$\begin{cases} \mathcal{E}(t)+d_P(t)^{-1}+d_I(t)^{-1} \leqslant 2(\mathcal{E}(0)+d_P(0)^{-1}+d_I(0)^{-1}),\\ \|z_{tt}(\cdot,t)\|_{H^s} \leqslant 2\|w_0\|_{H^s},\\ \frac{C_1}{2}|\alpha-\beta| \leqslant |z(\alpha,t)-z(\beta,t)| \leqslant 2C_2|\alpha-\beta|\\ \inf_{\alpha\in\mathbb{R}} a(\alpha,t)|z_\alpha| \geqslant \frac{1}{2}\alpha_0. \end{cases} \tag{2.138}$$

Proof. Let T_0 to be determined. Define

$$\mathcal{T} := \{T \in [0, T_0] : \text{ (2.69) } \textit{and} \text{ (2.70) hold for all } 0 \leqslant t \leqslant T\} \tag{2.139}$$

By continuity, $\mathcal{T}$ is closed. Moreover, since $0 \in \mathcal{T}$, we have $\mathcal{T} \neq \emptyset$. Let $T \in \mathcal{T}$. Then (2.136) holds on $[0, T]$. So we have

$$d_P(t)^{-1} + d_I(t)^{-1} + \mathcal{E}(t) \leqslant d_P(0)^{-1} + d_I(0)^{-1} + \mathcal{E}(0) + \int_0^t C d\tau, \tag{2.140}$$

where C is given by (2.137). Since C is a polynomial with positive coefficients, by taking T_0 sufficiently small, T_0 depends only on $N\lambda_{max}$, $d_P(0)^{-1}$, $d_I(0)^{-1}$, $\|(\partial_\alpha \xi_0, v_0, w_0)\|_{H^{s-1}\times H^s \times H^s}$, $\mathcal{E}(0), C_1, C_2, \alpha_0, s$, we have

$$\mathcal{E}(t) + d_P(t)^{-1} + d_I(t)^{-1} \leqslant \frac{3}{2}(\mathcal{E}(0) + d_P(0)^{-1} + d_I(0)^{-1}). \tag{2.141}$$

For $t \in [0, T]$, we have $\mathcal{E}(t) \leqslant 2\mathcal{E}(0)$, so we have by (2.108),

$$\|z_{tt}(\cdot, t)\|_{H^s} \leqslant \sqrt{2}\frac{(2\|w_0\|_{H^s} + 1)^{1/2}}{(2C_2)^{-s+1/2}}\mathcal{E}(0)^{1/2} := M_1. \tag{2.142}$$

Use $z_t(\cdot, t) = z_t(\cdot, 0) + \int_0^t z_{tt}(\cdot, \tau) d\tau$, we have

$$\|z_t(\cdot, t)\|_{H^s} \leqslant \|z_t(\cdot, 0)\|_{H^s} + T_0 M_1 := M_2. \tag{2.143}$$

Use $z_\alpha(\cdot, t) - 1 = z_\alpha(\cdot, 0) - 1 + \int_0^t z_{\tau\alpha}(\cdot, \tau) d\tau$, we obtain

$$\|z_\alpha(\cdot, t) - 1\|_{H^{s-1}} \leqslant \|\partial_\alpha \xi_0\|_{H^{s-1}} + T_0 M_2 := M_3, \tag{2.144}$$

and

$$\|z_\alpha(\cdot, t) - z_\alpha(\cdot, 0)\|_\infty \leqslant \int_0^t \|z_{t\alpha}\|_\infty d\tau \leqslant T_0 M_2. \tag{2.145}$$

By choosing T_0 sufficiently small, we have

$$\sup_{t\in[0,T]} \|z_\alpha(\cdot,t)\|_\infty \leqslant \frac{3}{2}\|z_\alpha(\cdot,0)\|_\infty \leqslant \frac{3}{2}C_2, \tag{2.146}$$

and

$$\inf_{t\in[0,T]}\inf_{\alpha\in\mathbb{R}} |z_\alpha(\alpha,t)| \geqslant \frac{2}{3}|z_\alpha(\alpha,0)| \geqslant \frac{2}{3}C_1. \tag{2.147}$$

Therefore, since $|z(\alpha,t)-z(\beta,t)| = |z_\alpha(\gamma,t)(\alpha-\beta)|$ for some γ between α and β, we have

$$\frac{2}{3}C_1|\alpha-\beta| \leqslant |z(\alpha,t)-z(\beta,t)| \leqslant \frac{3}{2}|\alpha-\beta|, \qquad t\in[0,T]. \tag{2.148}$$

Multiply both sides of the equation $(\partial_t^2 + ia\partial_\alpha)u = -ia_t\bar{z}_\alpha$ by $\bar{u}_t$ and integrate in α, then take real parts, we have

$$\frac{1}{2}\frac{d}{dt}\int |u_t|^2 d\alpha = Re\Big\{-i\int a u_\alpha \bar{u}_t d\alpha - i\int a_t \bar{z}_\alpha \bar{u}_t d\alpha\Big\}. \tag{2.149}$$

For $0\leqslant t\leqslant T$, we have

$$\Big|-i\int a u_\alpha \bar{u}_t d\alpha - i\int a_t\bar{z}_\alpha\bar{u}_t d\alpha\Big| \leqslant \|a|z_\alpha|\|_\infty \Big\|\frac{u_\alpha}{z_\alpha}\Big\|_{L^2}\|u_t\|_{L^2} + \|a_t z_\alpha\|_{L^2}\|u_t\|_{L^2} \tag{2.150}$$

$$\leqslant C(\|w_0\|_{H^s}, M_1, M_2, M_3, C_1, C_2, \alpha_0). \tag{2.151}$$

Similarly,

$$\frac{d}{dt}\|u_t\|_{\dot{H}^1}^2 \leqslant C(\|w_0\|_{H^s}, M_1, M_2, M_3, C_1, C_2, \alpha_0). \tag{2.152}$$

So we have for $0\leqslant t\leqslant T$,

$$\|u_t(\cdot,t)\|_{H^1}^2 = \|u_t(\cdot,0)\|_{H^1}^2 + \int_0^t \frac{d}{d\tau}\|u_t(\cdot,\tau)\|_{H^1}^2 d\tau \leqslant \|u_t(\cdot,0)\|_{H^1}^2 + T_0 C. \tag{2.153}$$

By choosing T_0 sufficiently small, we have for $0 \leqslant t \leqslant T$,

$$\|u_t(\cdot,t)\|_{H^1}^2 \leqslant \frac{3}{2}\|u_t(\cdot,0)\|_{H^1}^2. \tag{2.154}$$

Since $a|z_\alpha| = |z_{tt} + i|$, we have

$$\frac{d}{dt}a|z_\alpha| = \frac{(z_{tt}+i)\cdot z_{ttt}}{|z_{tt}+i|} = \frac{(z_{tt}+i)\cdot(iaz_{t\alpha} + ia_t z_\alpha)}{|z_{tt}+i|}. \tag{2.155}$$

Using (2.155), it's easy to obtain

$$\|a|z_\alpha|(\cdot,t) - a|z_\alpha|(\cdot,0)\|_\infty \leqslant T_0 C. \tag{2.156}$$

By choosing T_0 sufficiently small, we have for $0 \leqslant t \leqslant T$,

$$\|a|z_\alpha|(\cdot,t) - a|z_\alpha|(\cdot,0)\|_\infty \leqslant \frac{1}{3}\alpha_0. \tag{2.157}$$

So we have for

$$\inf_{t\in[0,T]}\inf_{\alpha\in\mathbb{R}} a|z_\alpha|(\alpha,t) \geqslant \frac{2}{3}\alpha_0. \tag{2.158}$$

Combining (2.148), (2.154), and (2.158), together with continuity of these quantities, there must exist $\delta > 0$ such that for $0 \leqslant t < T + \delta$,

$$\begin{aligned} \frac{1}{2}C_1|\alpha-\beta| \leqslant |z(\alpha,t) - z(\beta,t)| &\leqslant 2|\alpha-\beta|, \\ \|u_t(\cdot,t)\|_{H^1}^2 &\leqslant 2\|u_t(\cdot,0)\|_{H^1}^2, \\ \inf_{\alpha\in\mathbb{R}} a|z_\alpha|(\alpha,t) &\geqslant \frac{1}{2}\alpha_0. \end{aligned} \tag{2.159}$$

So $[0, T+\delta) \subset \mathcal{T})$ and therefore $\mathcal{T} = [0, T_0]$, provided that T_0 is sufficiently small, and T_0 depends only on $N\lambda_{max}$, $d_P(0)^{-1}$, $d_I(0)^{-1}$, $\|(\partial_\alpha\xi, v_0, w_0)\|_{H^{s-1}\times H^s\times H^s}$, $\mathcal{E}(0), C_1, C_2, \alpha_0, s$.

□

2.4.4 Proof of Theorem I.24.

Uniqueness is obtained by a similar argument as the energy estimate above. For local existence, one can use iteration method. We refer the readers to S. Wu's works [69][70] for details of this iteration scheme. Moreover, if we let T_0^* be the maximal lifespan, then either $T_0^* = \infty$, or $T_0^* < \infty$, but

$$\lim_{T\to T_0^*-} \|(z_t, z_{tt})\|_{C([0,T];H^s\times H^s)} + \sup_{t\to T_0^*}(d_I(t)^{-1} + d_P(t)^{-1}) = \infty. \tag{2.160}$$

or

$$\lim_{t\to T_0^*-} \inf_{\alpha\in\mathbb{R}} a(\alpha,t)|z_\alpha(\alpha,t)| \leqslant 0, \tag{2.161}$$

or

$$\sup_{\substack{\alpha\neq\beta\\ 0\leqslant t<T_0^*}} \left|\frac{z(\alpha,t)-z(\beta,t)}{\alpha-\beta}\right| + \sup_{\substack{\alpha\neq\beta\\ 0\leqslant t<T_0^*}} \left|\frac{\alpha-\beta}{z(\alpha,t)-z(\beta,t)}\right| = \infty. \tag{2.162}$$

2.5 Long time behavior for small data

In this section we prove Theorem I.10.

2.5.1 Derivation of the cubic structure.

As was explained in the introduction, the main difficulty of studying long time behavior of the system (1.13) is to find a cubic structure for this system. In [71], S. Wu uses $\theta := (I-\mathfrak{H})(z-\bar{z})$ and shows that $(\partial_t^2 - ia\partial_\alpha)\theta$ is cubic for the irrotational case. We use the same θ here. Using lemma II.12,

$$\begin{aligned}
(\partial_t^2 - ia\partial_\alpha)\theta &= (I-\mathfrak{H})(\partial_t^2 - ia\partial_\alpha)(z-\bar{z}) - [\partial_t^2 - ia\partial_\alpha, \mathfrak{H}](z-\bar{z}) \\
&= -2(I-\mathfrak{H})\partial_t \bar{z}_t - 2[z_t, \mathfrak{H}]\frac{\partial_\alpha(z_t - \bar{z}_t)}{z_\alpha} + \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 (z-\bar{z})_\beta d\beta \\
&= -2\partial_t(I-\mathfrak{H})\bar{z}_t - 2[z_t, \mathfrak{H}]\frac{\partial_\alpha z_t}{z_\alpha} + \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 (z-\bar{z})_\beta d\beta.
\end{aligned}$$

Decompose $\bar{z}_t = f + p$ as before, with $p = -\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{1}{z(\alpha,t) - z_j(t)}$. Since $(I-\mathfrak{H})p = 2p$, we have

$$-2\partial_t(I-\mathfrak{H})\bar{z}_t = -2\partial_t(I-\mathfrak{H})p = -4p_t,$$

and

$$-2[z_t,\mathfrak{H}]\frac{\partial_\alpha z_t}{z_\alpha} = -2[\bar{f},\mathfrak{H}]\frac{\partial_\alpha \bar{f}}{z_\alpha} - 2[\bar{p},\mathfrak{H}]\frac{\partial_\alpha \bar{f}}{z_\alpha} - 2[\bar{f},\mathfrak{H}]\frac{\partial_\alpha \bar{p}}{z_\alpha} - 2[\bar{p},\mathfrak{H}]\frac{\partial_\alpha \bar{p}}{z_\alpha}$$

Since f is holomorphic, we have $[f,\mathfrak{H}]\frac{f_\alpha}{z_\alpha} = 0$, and hence $[\bar{f},\bar{\mathfrak{H}}]\frac{\bar{f}_\alpha}{\bar{z}_\alpha} = 0$, so

$$-2[\bar{f},\mathfrak{H}]\frac{\partial_\alpha \bar{f}}{z_\alpha} = -2[\bar{f}, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}]\bar{f}_\alpha, \tag{2.163}$$

which is cubic. So we obtain

$$\begin{aligned}
(\partial_t^2 - ia\partial_\alpha)\theta = &-2[\bar{f}, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}]\bar{f}_\alpha + \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 (z-\bar{z})_\beta d\beta \\
&- 2[\bar{p},\mathfrak{H}]\frac{\partial_\alpha \bar{f}}{z_\alpha} - 2[\bar{f},\mathfrak{H}]\frac{\partial_\alpha \bar{p}}{z_\alpha} - 2[\bar{p},\mathfrak{H}]\frac{\partial_\alpha \bar{p}}{z_\alpha} - 4p_t.
\end{aligned} \tag{2.164}$$

Denote

$$g_c := -2[\bar{f}, \mathfrak{H}\frac{1}{z_\alpha} + \bar{\mathfrak{H}}\frac{1}{\bar{z}_\alpha}]\bar{f}_\alpha + \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 (z-\bar{z})_\beta d\beta. \tag{2.165}$$

$$g_d := -2[\bar{p}, \mathfrak{H}]\frac{\partial_\alpha \bar{f}}{z_\alpha} - 2[\bar{f}, \mathfrak{H}]\frac{\partial_\alpha \bar{p}}{z_\alpha} - 2[\bar{p}, \mathfrak{H}]\frac{\partial_\alpha \bar{p}}{z_\alpha} - 4p_t. \tag{2.166}$$

To control z_{tt}, we consider the quantity

$$\sigma := (I - \mathfrak{H})\partial_t\theta = (I - \mathfrak{H})\partial_t(I - \mathfrak{H})(z - \bar{z}).$$

We have

$$\begin{aligned}(\partial_t^2 - ia\partial_\alpha)\partial_t(I - \mathfrak{H})(z - \bar{z}) =& \partial_t(\partial_t^2 - ia\partial_\alpha)(I - \mathfrak{H})(z - \bar{z}) + ia_t((I - \mathfrak{H})(z - \bar{z}))_\alpha \\ =& \partial_t g + ia_t((I - \mathfrak{H})(z - \bar{z}))_\alpha.\end{aligned} \tag{2.167}$$

Here, $g = g_c + g_d$. Use lemma II.12 ,

$$\begin{aligned}&(\partial_t^2 - ia\partial_\alpha)\sigma = (I - \mathfrak{H})(\partial_t^2 - ia\partial_\alpha)\partial_t(I - \mathfrak{H})(z - \bar{z}) - [\partial_t^2 - ia\partial_\alpha, \mathfrak{H}]\partial_t(I - \mathfrak{H})(z - \bar{z}) \\ =&(I - \mathfrak{H})(\partial_t g + ia_t((I - \mathfrak{H})(z - \bar{z}))_\alpha) - 2[z_t, \mathfrak{H}]\frac{\partial_\alpha\partial_t^2(I - \mathfrak{H})(z - \bar{z})}{z_\alpha} \\ &+ \frac{1}{\pi i}\int\Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2((I - \mathfrak{H})(z - \bar{z}))_{t\beta}d\beta \\ :=&\tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3.\end{aligned} \tag{2.168}$$

Remark II.29. We have

$$\begin{aligned}\tilde{g}_1 =&(I - \mathfrak{H})\partial_t g_c + (I - \mathfrak{H})\partial_t g_d + (I - \mathfrak{H})ia_t(I - \mathfrak{H})(z - \bar{z})_\alpha \\ :=&\tilde{g}_{11} + \tilde{g}_{12} + \tilde{g}_{13}.\end{aligned} \tag{2.169}$$

Note that $\tilde{g}_{11}$ and $\tilde{g}_3$ are obvious cubic or enjoy nice time decay. As one can see later, $\tilde{g}_2$ is cubic as well. Since $a_t\bar{z}_\alpha$ consists of quadratic nonlinearities and terms with sufficiently fast time decay, as long as the point vortices move away from the interface at a speed which has a positive lower bound, so $(\partial_t^2 - ia\partial_\alpha)(I - \mathfrak{H})\partial_t(I - \mathfrak{H})(z - \bar{z})$ consists of cubic or higher order nonlinearities, or nonlinearities with rapid time decay, as long as the point vortices move away from the interface rapidly.

2.5.2 Change of coordinates.

Note that $(a-1)\theta_\alpha$ involves quadratic nonlinearities, which does not directly lead to cubic lifespan. To resolve the problem, we use the diffeomorphism $\kappa : \mathbb{R} \to \mathbb{R}$ such that $\bar{\zeta} - \alpha$ is holomorphic, where $\bar{\zeta} = z \circ \kappa^{-1}$. This κ was used in [71][63] for the irrotational case. Here we need to derive the formulae for b and A for the case with point vortices. Let Ψ be the holomorphic function on $\Omega(t)$ such that

$$\bar{\zeta} - \alpha = \Psi \circ \zeta.$$

We denote

$$D_t\zeta = z_t \circ \kappa^{-1}, \qquad A := (a\kappa_\alpha) \circ \kappa^{-1}, \qquad b = \kappa_t \circ \kappa^{-1}.$$

Then

$$\kappa_t = b \circ \kappa. \tag{2.170}$$

Suppose we know b, then we can recover κ by solving the ODE (2.170).

Recall that in (1.28), we decompose $\bar{z}_t$ as $\bar{z}_t = f + p$. We denote

$$\mathfrak{F} = f \circ \kappa^{-1}, \qquad q = p \circ \kappa^{-1}. \tag{2.171}$$

Since f is the boundary value of the holomorphic function F on $\Omega(t)$, we have

$$\mathfrak{F}(\alpha, t) = F(\zeta(\alpha, t), t), \tag{2.172}$$

In new variables, the water wave system (1.13) can be written as

$$
\begin{cases}
D_t^2\zeta - iA\zeta_\alpha = -i \\
\frac{d}{dt}z_j(t) = (v - \frac{\lambda_j i}{2\pi(z-z_j)})\Big|_{z=z_j} \\
(I-\mathcal{H})(D_t\bar{\zeta} + \sum_{j=1}^N \frac{\lambda_j i}{2\pi(\zeta(\alpha,t)-z_j(t))}) = 0 \\
(I-\mathcal{H})(\bar{\zeta}-\alpha) = 0.
\end{cases}
\tag{2.173}
$$

Here, $\mathcal{H}$ is the Hilbert transform associates with ζ, i.e.

$$
\mathcal{H}f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{\zeta_\beta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)} f(\beta)d\beta. \tag{2.174}
$$

To show that (2.173) is a closed system, we need to derive formula for b and A in terms of the new variable. Once we have shown that this is a closed system, and prove wellposedness for this system, then in turn, this justifies the existence of such change of variable κ^{-1}. Moreover, if we let ϵ_0 be sufficiently small, then in new variables, we have at $t=0$,

$$
\left\| |D|^{1/2}(\zeta(\alpha,0)-\alpha) \right\|_{H^s} + \|\mathfrak{F}(\cdot,0)\|_{H^{s+1/2}} + \|D_t\mathfrak{F}(\cdot,0)\|_{H^s} \leqslant \frac{3}{2}\epsilon. \tag{2.175}
$$

2.5.2.1 Formula for the quantities b and $D_t b$

Note that

$$
D_t\bar{\zeta} = F\circ\zeta - \frac{i}{2\pi}\sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha,t)-z_j(t)}, \quad \lambda_1 = -\lambda_2 = \lambda, \tag{2.176}
$$

Also, $D_t\bar{\zeta}$ can be written as

$$
D_t\bar{\zeta} = D_t(\bar{\zeta}-\alpha) + b = D_t\zeta\Psi_\zeta\circ\zeta + \Psi_t\circ\zeta + b. \tag{2.177}
$$

By (2.176) and (2.177), we have

$$F \circ \zeta - \frac{i}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha,t)-z_j(t)} = D_t\zeta\Psi_\zeta \circ \zeta + \Psi_t \circ \zeta + b. \tag{2.178}$$

Apply $I-\mathcal{H}$ on both sides of the above equation, use the fact that

$$(I-\mathcal{H})\Psi_t \circ \zeta = 0, \qquad (I-\mathcal{H})F \circ \zeta = 0, \qquad \Psi_\zeta \circ \zeta = \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha},$$

we obtain

$$\begin{aligned}(I-\mathcal{H})b =& -(I-\mathcal{H})D_t\zeta\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - (I-\mathcal{H})\frac{i}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha,t)-z_j(t)}\\ =& -[D_t\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - 2\frac{i}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha,t)-z_j(t)}\\ =& -[D_t\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha,t)-z_j(t)},\end{aligned} \tag{2.179}$$

where we've used the fact that $\frac{1}{\zeta(\alpha,t)-z_j(t)}$ is boundary value of a holomorphic function in $\Omega(t)^c$, so

$$(I-\mathcal{H})\frac{1}{\zeta(\alpha,t)-z_j(t)} = \frac{2}{\zeta(\alpha,t)-z_j(t)}. \tag{2.180}$$

So b is quadratic plus terms with sufficient rapid time decay, as long as $z_j(t)$ moves away from the interface rapidly.

We need a formula for $D_t b$ as well. Use $(I-\mathcal{H})b = -[D_t\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha} - \frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha,t)-z_j(t)}$, change of variables, we get

$$(I-\mathfrak{H})b\circ\kappa = -[z_t,\mathfrak{H}]\frac{\bar{z}_\alpha - 1}{z_\alpha} - \frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j}{z(\alpha,t)-z_j(t)}. \tag{2.181}$$

So we have

$$(I-\mathfrak{H})\partial_t b\circ\kappa =[z_t,\mathfrak{H}]\frac{\partial_\alpha b\circ\kappa}{z_\alpha}-[z_{tt},\mathfrak{H}]\frac{\bar{z}_\alpha-1}{z_\alpha}-[z_t,\mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + \frac{1}{\pi i}\int\Big(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\Big)^2(\bar{z}_\beta(\beta,t)-1)d\beta+\frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j(z_t-\dot{z}_j(t))}{(z(\alpha,t)-z_j(t))^2}. \tag{2.182}$$

Changing coordinates by precomposing with κ^{-1}, we obtain

$$(I-\mathcal{H})D_t b =[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha b}{\zeta_\alpha}-[D_t^2\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}-[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha} + \frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\bar{\zeta}_\beta(\beta,t)-1)d\beta+\frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta-\dot{z}_j(t))}{(\zeta(\alpha,t)-z_j(t))^2}. \tag{2.183}$$

So $D_t b$ is quadratic plus terms with sufficient rapid time decay, as long as $z_j(t)$ moves away from the interface rapidly.

2.5.3 The quantity A

Since $\partial_\alpha\mathfrak{F}=\partial_\alpha F(\zeta(\alpha,t),t)=F_\zeta\circ\zeta\zeta_\alpha$, we have

$$F_\zeta\circ\zeta=\frac{\partial_\alpha\mathfrak{F}}{\zeta_\alpha}. \tag{2.184}$$

Use $D_t^2\bar{\zeta}+iA\bar{\zeta}_\alpha=i$. We have

$$\begin{aligned}D_t^2\bar{\zeta} =&D_t(D_t\bar{\zeta})=D_tF\circ\zeta-\frac{i}{2\pi}D_t\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha,t)-z_j(t)}\\ =&D_t\zeta F_\zeta\circ\zeta+\frac{i}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta(\alpha,t)-\dot{z}_j(t))}{(\zeta(\alpha,t)-z_j(t))^2}\\ =&D_t\zeta\frac{\partial_\alpha\mathfrak{F}}{\zeta_\alpha}+\frac{i}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta(\alpha,t)-\dot{z}_j(t))}{(\zeta(\alpha,t)-z_j(t))^2}.\end{aligned} \tag{2.185}$$

Also,

$$iA\bar{\zeta}_\alpha = iA + iA\partial_\alpha(\bar{\zeta} - \alpha) = iA + iA\zeta_\alpha\Psi_\zeta \circ \zeta = iA + (D_t^2\zeta + i)\Psi_\zeta \circ \zeta. \tag{2.186}$$

So we have

$$iA = i - D_t\zeta\frac{\partial_\alpha\mathfrak{F}}{\zeta_\alpha} - \frac{i}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta(\alpha,t) - \dot{z}_j(t))}{(\zeta(\alpha,t) - z_j(t))^2} - (D_t^2\zeta + i)\Psi_\zeta \circ \zeta. \tag{2.187}$$

Apply $I - \mathcal{H}$ on both sides of (2.187), use the fact that $(I - \mathcal{H})\frac{\mathfrak{F}_\alpha}{\zeta_\alpha} = 0$, $(I - \mathcal{H})\Psi_\zeta \circ \zeta = 0$, we obtain

$$i(I - \mathcal{H})A = i - [D_t\zeta, \mathcal{H}]\frac{\partial_\alpha\mathfrak{F}}{\zeta_\alpha} - [D_t^2\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - (I - \mathcal{H})\frac{i}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta(\alpha,t) - \dot{z}_j(t))}{(\zeta(\alpha,t) - z_j(t))^2} \tag{2.188}$$

So we obtain

$$(I - \mathcal{H})A = 1 + i[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha\mathfrak{F}}{\zeta_\alpha} + i[D_t^2\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - (I - \mathcal{H})\frac{1}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta(\alpha,t) - \dot{z}_j(t))}{(\zeta(\alpha,t) - z_j(t))^2}. \tag{2.189}$$

So $A - 1$ is quadratic plus terms with rapid time decay, as long as the point vortices move away from the interface with a speed that has a positive lower bound.

2.5.4 The quantity $\frac{a_t}{a}\circ\kappa^{-1}$

We need a formula for $\frac{a_t}{a}\circ\kappa^{-1}$ as well. Use (III.11) and (2.90), (2.91), then change of variables, we obtain

$$\begin{aligned}&(I-\mathcal{H})\frac{a_t}{a}\circ\kappa^{-1}A\bar{\zeta}_\alpha\\ =&2i[D_t^2\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}+2i[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t^2\bar{\zeta}}{\zeta_\alpha}-\frac{1}{\pi}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(D_t\bar{\zeta})_\beta d\beta\\ &-\frac{1}{\pi}\sum_{j=1}^2\lambda_j\Big(\frac{2D_t^2\zeta+i-\partial_t^2 z_j}{(\zeta(\alpha,t)-z_j(t))^2}-2\frac{(D_t\zeta-\dot{z}_j(t))^2}{(\zeta(\alpha,t)-z_j(t))^3}\Big)\end{aligned} \tag{2.190}$$

So $\frac{a_t}{a}\circ\kappa^{-1}$ is quadratic plus terms with rapid time decay, as long as the point vortices move away from the interface with a speed that has a positive lower bound.

2.5.5 Cubic structure in new variables

Denote

$$\tilde{\theta}:=(I-\mathcal{H})(\zeta-\bar{\zeta}),\qquad \tilde{\sigma}:=(I-\mathcal{H})D_t\tilde{\theta}. \tag{2.191}$$

We sum up the calculations above, which show that $(D_t^2-iA\partial_\alpha)\tilde{\theta}$ and $(D_t^2-iA\partial_\alpha)\tilde{\sigma}$ consist of cubic terms and terms with rapid time decay, as long as the point vortices move away from the interface rapidly. Recall that $q=p\circ\kappa^{-1}$, so q is given by

$$q=-\sum_{j=1}^2\frac{\lambda_j i}{2\pi}\frac{1}{\zeta(\alpha,t)-z_j(t)}. \tag{2.192}$$

We have

$$\begin{cases}(D_t^2-iA\partial_\alpha)\tilde{\theta}=G\\ (D_t^2-iA\partial_\alpha)\tilde{\sigma}=\tilde{G}\end{cases} \tag{2.193}$$

where $G=G_c+G_d$, with

$$G_c:=-2[\bar{\mathfrak{F}},\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}]\bar{\mathfrak{F}}_\alpha+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta. \tag{2.194}$$

$$G_d := -2[\bar{q}, \mathcal{H}]\frac{\partial_\alpha \bar{\mathfrak{F}}}{\zeta_\alpha} - 2[\bar{\mathfrak{F}}, \mathcal{H}]\frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 2[\bar{q}, \mathcal{H}]\frac{\partial_\alpha \bar{q}}{\zeta_\alpha} - 4D_t q, \tag{2.195}$$

and

$$\begin{aligned}\tilde{G} =&(I-\mathcal{H})(D_t G + i\frac{a_t}{a}\circ\kappa^{-1}A((I-\mathcal{H})(\zeta-\bar{\zeta}))_\alpha) - 2[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha D_t^2(I-\mathcal{H})(\zeta-\bar{\zeta})}{\zeta_\alpha}\\ &+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2\partial_\beta D_t(I-\mathcal{H})(\zeta-\bar{\zeta})d\beta\end{aligned} \tag{2.196}$$

2.5.5.1 Evolution equation for higher order derivatives

Apply ∂_α^k on both sides of (2.358), we have

$$\begin{aligned}(D_t^2 - iA\partial_\alpha)\theta_k &= G_k^\theta\\ (D_t^2 - iA\partial_\alpha)\sigma_k &= G_k^\sigma,\end{aligned} \tag{2.197}$$

where for $0 \leqslant k \leqslant s$,

$$\theta_k = (I-\mathcal{H})\partial_\alpha^k\tilde{\theta}, \qquad \sigma_k = (I-\mathcal{H})\partial_\alpha^k\tilde{\sigma}. \tag{2.198}$$

$$G_k^\theta = (I-\mathcal{H})(\partial_\alpha^k G + [D_t^2 - iA\partial_\alpha, \partial_\alpha^k]\tilde{\theta}) - [D_t^2 - iA\partial_\alpha, \mathcal{H}]\partial_\alpha^k\tilde{\theta}, \tag{2.199}$$

and

$$G_k^\sigma = (I-\mathcal{H})(\partial_\alpha^k \tilde{G} + [D_t^2 - iA\partial_\alpha, \partial_\alpha^k]\tilde{\sigma}) - [D_t^2 - iA\partial_\alpha, \mathcal{H}]\partial_\alpha^k\tilde{\sigma}. \tag{2.200}$$

2.5.6 Energy functional

Define

$$E_k^\theta := \int \frac{1}{A}|D_t\theta_k|^2 + i\theta_k\overline{\partial_\alpha\theta_k}d\alpha. \tag{2.201}$$

By Wu's basic energy lemma (lemma 4.1, [71]), we have

$$\frac{d}{dt}E_k^\theta = \int\frac{2}{A}ReD_t\theta_k\bar{G}_k - \int\frac{1}{A}\frac{a_t}{a}\circ\kappa^{-1}|D_t\theta_k|^2 \tag{2.202}$$

Define

$$E_k^\sigma := \int \frac{1}{A}|D_t\sigma_k|^2 + i\sigma_k\overline{\partial_\alpha\sigma_k}d\alpha. \tag{2.203}$$

Then we have

$$\frac{d}{dt}E_k^\sigma = \int \frac{2}{A}ReD_t\sigma_k\overline{\tilde{G}}_k - \int \frac{1}{A}\frac{a_t}{a}\circ\kappa^{-1}|D_t\sigma_k|^2 \tag{2.204}$$

Define

$$\mathcal{E}_s := \sum_{k=0}^{s}(E_k^\theta + E_k^\sigma). \tag{2.205}$$

2.5.7 The bootstrap assumption and some preliminary estimates

To obtain a priori energy estimates, we make the following bootstrap assumption: Let $T_0 \geqslant 0$, we assume

$$\|\zeta_\alpha - 1\|_{H^s} \leqslant 5\epsilon, \qquad \|\mathfrak{F}\|_{H^{s+1/2}} \leqslant 5\epsilon, \qquad \|D_t\mathfrak{F}\|_{H^s} \leqslant 5\epsilon, \qquad \forall\ t \in [0, T_0]. \tag{2.206}$$

Remark II.30. The assumptions of Theorem I.10 imply that the bootstrap assumption holds at $T_0 = 0$.

As a consequence of (2.206), we have

Lemma II.31 (Chord-arc condition)**.** *Assume the assumptions of Theorem I.10 holds. Assume also the bootstrap assumption (2.206), we have*

$$(1-5\epsilon)|\alpha-\beta| \leqslant |\zeta(\alpha,t) - \zeta(\beta,t)| \leqslant (1+5\epsilon)|\alpha-\beta|, \qquad \forall\ t \in [0, T_0]. \tag{2.207}$$

Proof.

$$|\zeta(\alpha,t) - \zeta(\beta,t)| = |\alpha - \beta + (\zeta(\alpha,t) - \alpha) - (\zeta(\beta,t) - \beta)|. \tag{2.208}$$

Note that

$$|\zeta(\alpha,t) - \alpha - (\zeta(\beta,t) - \beta)| \leqslant \|\zeta_\alpha - 1\|_\infty |\alpha-\beta| \leqslant 5\epsilon|\alpha-\beta|. \tag{2.209}$$

So the conclusion follows by Triangle inequality.

□

Lemma II.32. *Assume the assumptions of Theorem I.10 hold. Assume also the bootstrap assumption (2.206), we have for ϵ sufficiently small,*

$$\|F(\cdot,t)\|_{L^\infty(\Omega(t))} \leqslant 5\epsilon, \tag{2.210}$$

$$\|F_\zeta(\cdot,t)\|_{L^\infty(\Omega(t))} \leqslant 6\epsilon, \tag{2.211}$$

$$|ReF(x+iy,t)| \leqslant 6\epsilon|x|, \tag{2.212}$$

$$\|F_t\|_{L^\infty(\Omega(t))} \leqslant 6\epsilon, \tag{2.213}$$

$$\|F_{\zeta\zeta}\|_{L^\infty(\Omega(t))} \leqslant 10\epsilon, \tag{2.214}$$

$$\|F_{t\zeta}\|_{L^\infty(\Omega(t))} \leqslant 10\epsilon. \tag{2.215}$$

Proof. For (2.210), by maximum principle, we have

$$\|F\|_{L^\infty(\Omega(t))} = \|F\|_{L^\infty(\Sigma(t))} = \|\mathfrak{F}(t)\|_\infty \leqslant 5\epsilon. \tag{2.216}$$

By bootstrap assumption (2.206), for ϵ sufficiently small, we have

$$\|F_\zeta(\zeta(\alpha,t),t)\|_\infty = \left\|\frac{\partial_\alpha \mathfrak{F}(\alpha,t)}{\zeta_\alpha}\right\|_\infty \leqslant \frac{\|\mathfrak{F}\|_{H^s}}{\|\zeta_\alpha\|_\infty} \leqslant \frac{5\epsilon}{1-5\epsilon} \leqslant 6\epsilon. \tag{2.217}$$

By maximum principle, we have

$$\|F_\zeta(\cdot,t)\|_{L^\infty(\Omega(t))} \leqslant \|F_\zeta(\zeta(\alpha,t),t)\|_\infty \leqslant 6\epsilon. \tag{2.218}$$

Note that

$$D_t\mathfrak{F}(\alpha,t) = D_tF(\zeta(\alpha,t),t) = F_t\circ\zeta + D_t\zeta F_\zeta\circ\zeta. \tag{2.219}$$

So we have for ϵ sufficiently small (say, $\epsilon < 1/36$),

$$\|F_t(\zeta(\cdot,t),t)\|_{H^s} \leqslant \|D_t\mathfrak{F}\|_{H^s} + \|D_t\zeta F_\zeta \circ \zeta\|_{H^s} \leqslant 5\epsilon + 6\epsilon(6\epsilon) \leqslant 6\epsilon. \tag{2.220}$$

By Sobolev embedding and maximum principle, we have

$$\|F_t(\cdot,t)\|_{L^\infty(\Omega(t))} \leqslant \|F_t(\zeta(\alpha,t),t)\|_{L^\infty} \leqslant \|F_t \circ \zeta\|_{H^1} \leqslant 6\epsilon. \tag{2.221}$$

Use the fact that ReF is odd, and the estimate $\|F_\zeta\|_{H^1} \leqslant 6\epsilon$, we have

$$|ReF(x+iy,t)| = |ReF(x+iy,t) - ReF(0+iy,t)| \leqslant \|F_\zeta\|_\infty |x| \leqslant 6\epsilon|x|.$$

Note that

$$F_{\zeta\zeta}(\zeta(\alpha,t),t) = (\frac{\partial_\alpha}{\zeta_\alpha})^2 F(\zeta(\alpha,t),t) = \frac{1}{\zeta_\alpha^2}\mathfrak{F}_{\alpha\alpha} - \frac{\zeta_{\alpha\alpha}}{\zeta_\alpha^3}\mathfrak{F}_\alpha. \tag{2.222}$$

For ϵ sufficiently small, by maximum principle, we have

$$\|F_{\zeta\zeta}(\cdot,t)\|_{L^\infty(\Omega(t))} \leqslant \frac{1}{\inf_{\alpha\in\mathbb{R}}\zeta_\alpha^2}\|\mathfrak{F}\|_{H^3} + \frac{\|\zeta_{\alpha\alpha}\|_\infty}{\inf_{\alpha\in\mathbb{R}}\zeta_\alpha^3}\|\mathfrak{F}\|_{H^1} \leqslant \frac{1}{(1-5\epsilon)^2}5\epsilon + \frac{5\epsilon}{(1-5\epsilon)^3}5\epsilon \leqslant 10\epsilon. \tag{2.223}$$

Maximum principle implies $\|F_{t\zeta}\|_{L^\infty(\Omega(t))} \leqslant \|F_{t\zeta}\|_{L^\infty(\partial\Omega(t))}$. Since $F_{t\zeta}(\zeta(\alpha,t),t) = \frac{\partial_\alpha F_t(\zeta(\alpha,t),t)}{\zeta_\alpha}$, by (2.219), we have

$$\begin{aligned}\left\|F_{t\zeta}\circ\zeta\right\|_{L^\infty} =& \left\|\frac{\partial_\alpha F_t(\zeta(\alpha,t),t)}{\zeta_\alpha}\right\|_{L^\infty} \leqslant \|\partial_\alpha F_t(\zeta(\cdot,t),t)\|_\infty \left\|\frac{1}{\zeta_\alpha}\right\|_\infty \\ =& \left\|D_t\mathfrak{F}(\alpha,t) - D_t\zeta F_\zeta\circ\zeta\right\|_{H^s}\left\|\frac{1}{\zeta_\alpha}\right\|_\infty \leqslant 10\epsilon,\end{aligned}$$

□

Assume the bootstrap assumption (2.206), we can obtain control of various characteristics of the point vortices.

Convention. We use K_s to denote a constant that depends on s. We'll use $K_s \sim \frac{((s+12)!)^2}{((s+7)!)^2}$. K_s can be different at different places, up to an absolute multiplicity constant. We also use C to represent an absolute constant.

We'll need the following lemma. Similar versions of this lemma have been appeared in [71].

Lemma II.33. *Assume the bootstrap assumption (2.206), let f, h be real functions. Assume*

$$(I-\mathcal{H})h\bar{\zeta}_\alpha = g \qquad or \qquad (I-\mathcal{H})h = g.$$

Then we have for any $t \in [0, T_0]$,

$$\|h\|_{H^s} \leqslant 2\|g\|_{H^s}. \tag{2.224}$$

We'll use the following estimate a lot.

Lemma II.34. *Assume the assumptions of Theorem I.10 hold. Assume also the bootstrap assumption (2.206), and assume a priori that $d_I(t) \geqslant 1$, $\frac{1}{2} \leqslant \frac{x(t)}{x(0)} \leqslant 2$, $\forall\ t \in [0, T_0]$. Then we have $\forall\ t \in [0, T_0]$,*

$$\|q\|_{H^s} \leqslant K_s^{-1}\epsilon d_I(t)^{-3/2}. \tag{2.225}$$

$$\left\|\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s} \leqslant K_s^{-1}\epsilon d_I(t)^{-5/2}. \tag{2.226}$$

Proof. We prove (2.225). The proof of (2.226) is similar . Let s be a positive integer, we have

$$\|q\|_{H^s}^2 \leqslant \sum_{n=0}^{s}\int_{-\infty}^{\infty}\left|\partial_\alpha^n\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{1}{\zeta(\alpha,t)-z_j(t)}\right|^2 d\alpha$$

Denote $f_j(\alpha,t) := \frac{\lambda_j i}{2\pi}\frac{1}{\alpha - z_j(t)}$, $g := \zeta(\alpha,t)$. Then $\frac{\lambda_j i}{2\pi}\frac{1}{\zeta(\alpha,t)-z_j(t)} = f_j(g(\alpha,t),t)$. By chain rule for composite functions, we have

$$\partial_\alpha^n f_j(g) = \sum_{k=1}^{n} \sum \frac{n!}{(k_1)!...(k_n)!} \partial_\alpha^k f_j(\cdot,t) \circ g \prod_{l=1}^{n} \Big(\frac{\partial_\alpha^l g}{l!}\Big)^{k_l}, \tag{2.227}$$

where the second summation is over all non-negative integers $(k_1, ..., k_n)$ such that

$$\begin{cases} \sum_{l=1}^n k_l = k \\ \sum_{l=1}^n l k_l = n. \end{cases} \tag{2.228}$$

So we have

$$\partial_\alpha^n q = \sum_{k=1}^{n} \sum \frac{n!}{(k_1)!...(k_n)!} \Big(\sum_{j=1}^{2} \partial_\alpha^k f_j(\cdot,t) \circ g\Big) \prod_{l=1}^{n} \Big(\frac{\partial_\alpha^l g}{l!}\Big)^{k_l} \tag{2.229}$$

Note that

$$(\sum_{j=1}^{2} \partial_\alpha^k f_j(\cdot,t)) \circ g = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{(-1)^k k!}{(\zeta(\alpha,t) - z_j(t))^{k+1}} \tag{2.230}$$

$$= \frac{\lambda i (-1)^k k!}{2\pi} \sum_{m=0}^{k} \frac{z_1 - z_2}{(\zeta(\alpha,t) - z_1(t))^{k+1-m} (\zeta(\alpha,t) - z_2(t))^{m+1}} \tag{2.231}$$

use $z_1 - z_2 = 2x(t)$, similar to the proof of lemma II.10, we have

$$\Big\|\sum_{j=1}^{2} \partial_\alpha^k f_j(\cdot,t) \circ g\Big\|_{L^2} \leqslant 100(k+1)! |\lambda x(t)| d_I(t)^{-3/2}. \tag{2.232}$$

Therefore,

$$\begin{aligned} \|\partial_\alpha^n q\|_{L^2} =& \Big\| \sum_{k=1}^{n} \sum \frac{n!}{(k_1)!...(k_n)!} \Big(\sum_{j=1}^{2} \partial_\alpha^k f_j(\cdot,t) \circ g\Big) \prod_{l=1}^{n} \Big(\frac{\partial_\alpha^l g}{l!}\Big)^{k_l} \Big\|_{L^2} \\ \leqslant& \sum_{k=1}^{n} \sum \frac{n!}{(k_1)!...(k_n)!} \prod_{l=1}^{n} \frac{\|\partial_\alpha^l g\|_\infty^{k_l}}{(l!)^{k_l}} \Big\|\sum_{j=1}^{2} \partial_\alpha^k f_j(\cdot,t) \circ g\Big\|_{L^2} \end{aligned}$$

For $l = 1$, we bound $\partial_\alpha^l g$ by $1 + 5\epsilon$. For $l \geqslant 2$, we bound $\|\partial_\alpha^l g\|_\infty$ by 5ϵ. We choose ϵ small so that $(1 + 5\epsilon)^s \leqslant 2$. We bound $(k + 1)!$ by $(n + 1)!$. Use the assumption $x(t) \leqslant 2x(0)$. Use

$$\prod_{l=1}^{n} \|\partial_\alpha^l g\|_\infty^{k_l} \leqslant \prod_{j=1}^{n} (1 + 5\epsilon)^{k_l} \leqslant (1 + 5\epsilon)^s, \tag{2.233}$$

we obtain

$$\|\partial_\alpha^n q\|_{L^2} \leqslant \sum_{k=1}^{n} \sum \frac{n!}{(k_1)!...(k_n)!} \prod_{l=1}^{n} \frac{(1 + 5\epsilon)^{k_l}}{(l!)^{k_l}} \times (100(k + 1)!|\lambda x(t)|d_I(t)^{-3/2}) \tag{2.234}$$

$$\leqslant 400 S(n)|\lambda|x(0)(n + 1)! d_I(t)^{-3/2}, \tag{2.235}$$

where

$$S(n) = \sum_{k=1}^{n} \sum \frac{n!}{(k_1)!...(k_n)!} \prod_{l=1}^{n} \frac{1}{(l!)^{k_l}} \tag{2.236}$$

is called the bell number. We can bound $S(n)$ by

$$S(n) \leqslant n!. \tag{2.237}$$

So we have

$$\|\partial_\alpha^n q\|_{L^2} \leqslant 400|\lambda x(0)| n!(n + 1)! d_I(t)^{-3/2}. \tag{2.238}$$

Therefore,

$$\|q\|_{H^s} \leqslant \Big(\sum_{n=0}^{s} \|\partial_\alpha^n q\|_{L^2}^2 \Big)^{1/2} \tag{2.239}$$

$$\leqslant \Big(\sum_{n=0}^{s} (400|\lambda x(0)| n!(n + 1)! d_I(t)^{-3/2})^2 \Big)^{1/2} \tag{2.240}$$

$$\leqslant 400((s + 2)!)^2 |\lambda x(0)| d_I(t)^{-3/2} \tag{2.241}$$

$$\leqslant K_s^{-1} \epsilon d_I(t)^{-3/2}. \tag{2.242}$$

□

Corollary II.35. *Assume the assumptions of Theorem I.10 hold and assume the bootstrap assumption (2.206), and assume a priori that* $d_I(t) \geqslant 1$, $\frac{1}{2} \leqslant \frac{x(t)}{x(0)} \leqslant 2$, $\forall\ t \in [0, T_0]$. *Then we have*

$$\sup_{t \in [0,T]} \|D_t \zeta\|_{H^s} \leqslant 6\epsilon, \qquad \forall\ t \in [0, T_0]. \tag{2.243}$$

Corollary II.36. *Assume the assumptions of Theorem I.10 hold and assume the bootstrap assumption (2.206), and assume a priori that* $d_I(t) \geqslant 1$, $\frac{1}{2} \leqslant \frac{x(t)}{x(0)} \leqslant 2$, $\forall\ t \in [0, T_0]$. *Then we have*

$$\|b\|_{H^s} \leqslant C\epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-3/2}, \qquad \forall\ t \in [0, T_0]. \tag{2.244}$$

for some absolute constant $C > 0$.

Proof.

$$(I - \mathcal{H})b = -[D_t \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)}.$$

By lemma II.9 and lemma II.34, we have

$$\begin{aligned} \|(I - \mathcal{H})b\|_{H^s} \leqslant & \left\| [D_t \zeta, \mathcal{H}] \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} \right\|_{H^s} + \left\| \frac{i}{\pi} \sum_{j=1}^{2} \frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)} \right\|_{H^s} \\ \leqslant & C\epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-3/2}. \end{aligned}$$

So we have

$$\|b\|_{H^s} \leqslant C\epsilon^2 + K_s^{-1} \epsilon d_I(t)^{-3/2}. \tag{2.245}$$

□

Remark II.37. Again, the a priori assumption $d_I(t) \geqslant 1, \frac{1}{2} \leqslant \frac{x(t)}{x(0)} \leqslant 2$, $\forall\ t \in [0, T_0]$ will be justified by a bootstrap argument.

We need to estimate $\dot{z}_j$ and $\ddot{z}_j$ in a more precise way rather than using the rough estimates in lemma II.27. Let's first derive the estimate for $\dot{z}_j$, then we use this estimate to control $x(t)$ over time. We use the control of $x(t)$ to estimate $\ddot{z}_j$.

Lemma II.38. *Assume the assumptions of Theorem I.10 and the bootstrap assumption (2.206), and assume a priori that* $d_I(t) \geqslant 1$, $\frac{1}{2} \leqslant \frac{x(t)}{x(0)} \leqslant 2$, $\forall\ t \in [0, T_0]$. *Then we have*

$$\sup_{t\in[0,T_0]} |\dot{z}_1(t) - \dot{z}_2(t)| \leqslant 10\epsilon. \tag{2.246}$$

$$\sup_{t\in[0,T_0]} |\dot{z}_j(t) - \frac{\lambda i}{4\pi x(t)}| \leqslant 5\epsilon. \tag{2.247}$$

$$\sup_{t\in[0,T_0]} |\dot{z}_1^2 - \dot{z}_2^2| \leqslant 6|\lambda|\epsilon + 120\epsilon^2 x(t). \tag{2.248}$$

Proof. Note that

$$\dot{z}_1 = \frac{\lambda_2 i}{2\pi(z_1 - z_2)} + \bar{F}(z_1, t) = \frac{\lambda i}{4\pi x(t)} + \bar{F}(z_1, t). \tag{2.249}$$

Similarly,

$$\dot{z}_2 = \frac{\lambda i}{4\pi x(t)} + \bar{F}(z_2, t).$$

By lemma II.32, we have

$$|\dot{z}_1(t) - \dot{z}_2(t)| = |F(z_1, t) - F(z_2, t)| \leqslant 2\|F\|_{L^\infty(\partial\Omega(t))} \leqslant 10\epsilon,$$

and

$$|\dot{z}_j(t) - \frac{\lambda i}{4\pi x(t)}| = |\bar{F}(z_1(t), t)| \leqslant 5\epsilon. \tag{2.250}$$

We have

$$\dot{z}_j^2 = (\frac{\lambda i}{4\pi x(t)})^2 + 2\frac{\lambda i}{4\pi x(t)}\bar{F}(z_j(t), t) + (\bar{F}(z_j(t), t))^2,$$

By mean value theorem, bootstrap assumption (2.206), lemma II.32, we have

$$\begin{aligned} |F(z_1(t), t)^2 - F(z_2(t), t)^2| =& |(F(z_1(t), t) + F(z_2(t), t))F_\zeta(\tilde{x} + iy(t), t)(z_1(t) - z_2(t))| \\ \leqslant& 120\epsilon^2 x(t). \end{aligned} \tag{2.251}$$

Here, $\tilde{x} \in (0, x(t))$.

By (2.212) of lemma II.32, we have

$$\begin{aligned}
|\dot{z}_1^2 - \dot{z}_2^2| =& |\frac{2\lambda i}{4\pi x(t)}(F(z_1(t),t) - F(z_2(t),t)) + F(z_1(t),t)^2 - F(z_2(t),t)^2| \\
=& |\frac{\lambda i}{\pi x(t)} ReF(z_1(t),t) + F(z_1(t),t)^2 - F(z_2(t),t)^2| \\
\leqslant& \frac{|\lambda|}{\pi x(t)} \|F_\zeta\|_\infty x(t) + |F(z_1(t),t)^2 - F(z_2(t),t)^2| \\
\leqslant& 6|\lambda|\epsilon + 120\epsilon^2 x(t).
\end{aligned}$$

□

Another consequence of the bootstrap assumption (2.206) is the following description of the motion of the point vortices, which is the key control of this paper.

Proposition II.39 (key control). *Assume the assumptions of Theorem I.10 and assume the bootstrap assumption (2.206), we have*

$$\frac{1}{2} \leqslant \frac{x(t)}{x(0)} \leqslant 2, \qquad 0 \leqslant t \leqslant T_0. \tag{2.252}$$

Proof. It suffices to prove the case that $x(t)$ is increasing on $0 \leqslant t \leqslant T_0$. The case that $x(t)$ is decreasing follows in a similar way, and other cases are controlled by these two cases.

Denote

$$\mathcal{T} := \Big\{T \in [0, T_0] | \quad \dot{y}(t) \leqslant -\frac{|\lambda|}{20\pi x(0)}, \quad \frac{1}{2} \leqslant \frac{x(t)}{x(0)} \leqslant 2, \quad \widehat{d_I}(t) \geqslant 1 + \frac{|\lambda|}{20\pi x(0)} t, \quad \forall\, t \in [0, T]\Big\}. \tag{2.253}$$

Let's assume $F = F_1 + iF_2$, where F_1, F_2 are real (we remind the readers that F is the holomorphic extension of f. Recall also the notations that $z_1(t) = -x(t) + iy(t)$, $z_2(t) =$

$x(t)+iy(t)$, $x(t)>0, y(t)<0$). From the proof of lemma II.38, we have

$$\begin{cases} \dot{z}_1(t) = \bar{F}(z_1(t),t) + \frac{\lambda_2 i}{2\pi}\frac{1}{\overline{z_1(t)-z_2(t)}} = \bar{F}(z_1(t)) - \frac{|\lambda| i}{4\pi x(t)} \\ \dot{z}_2(t) = \bar{F}(z_2(t),t) + \frac{\lambda_1 i}{2\pi}\frac{1}{\overline{z_2(t)-z_1(t)}} = \bar{F}(z_2(t)) - \frac{|\lambda| i}{4\pi x(t)}. \end{cases} \tag{2.254}$$

So we have

$$\dot{y}(t) = -F_2(z_2(t),t) - \frac{|\lambda|}{4\pi x(t)}. \tag{2.255}$$

By maximum principle and the bootstrap assumption (2.206), we have

$$|F_2(z_2(t),t)| \leqslant |F(z_2(t),t)| \leqslant \|F(\cdot,t)\|_{L^\infty(\Omega(t))} = \|\mathfrak{F}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leqslant 5\epsilon. \tag{2.256}$$

For M relatively large (we take $M=200\pi$), at $t=0$, we have $\frac{|\lambda|}{4\pi x(0)} \geqslant \frac{200\pi\epsilon}{4\pi} = 50\epsilon$. So we have

$$\dot{y}(0) = -F_2(z_2(0),0) - \frac{|\lambda|}{4\pi x(0)} \leqslant -\frac{9|\lambda|}{40\pi x(0)}. \tag{2.257}$$

So $0 \in \mathcal{T}$ and therefore $\mathcal{T} \neq \varnothing$. Clearly, by the definition of $\mathcal{T}$, since $\dot{y}(t), x(t)$ and $\widehat{d}_I(t)$ are continuous, so $\mathcal{T}$ is closed in $[0,T_0]$. To prove $\mathcal{T}=[0,T_0]$, it suffices to prove that if (2.253) holds on $[0,T]$ with $T<T_0$, then there exists $\delta>0$ such that (2.253) holds on $[T,T+\delta)$.

Let $T \in \mathcal{T}$.

By (2.254), we have $\dot{x}(t) = ReF(z_2(t),t)$. Use (2.212) of lemma II.32, use the fact that ReF is odd, by mean value theorem, we have

$$\dot{x}(t) = ReF(z_2(t),t) - ReF(0+iy(t),t) = Re\ F_x(\tilde{x}+iy(t),t)x(t), \tag{2.258}$$

for some $\tilde{x} \in (0, x(t))$.

Since

$$F(z,t) = \frac{1}{2\pi i}\int \frac{\zeta_\beta}{z-\zeta(\beta,t)}\mathfrak{F}(\beta,t)d\beta. \tag{2.259}$$

So we have $\forall\ t \in [0, T_0]$,

$$\partial_x F(z,t) = -\frac{1}{2\pi i}\int \frac{\zeta_\beta}{(z-\zeta(\beta,t))^2}\mathfrak{F}(\beta,t)d\beta. \tag{2.260}$$

By Cauchy-Schwartz inequality and lemma II.10, we have

$$\begin{aligned}|\partial_x F(\tilde{x}+iy(t),t)| \leqslant& \frac{1}{2\pi}\Big(\int \frac{1}{|\tilde{x}+iy(t)-\zeta(\beta,t)|^4}d\beta\Big)^{1/2}\|\zeta_\beta\|_{L^\infty}\|\mathfrak{F}\|_{L^2}\\ \leqslant& C\epsilon\widehat{d}_I(t)^{-3/2}, \qquad \forall\ t \in [0,T_0],\end{aligned} \tag{2.261}$$

for some absolute constant $C > 0$. By direct calculation, we can see that $C \leqslant 2$. So we obtain

$$\dot{x}(t) \leqslant 2\epsilon\widehat{d}_I(t)^{-3/2}x(t). \tag{2.262}$$

So we have

$$\frac{d}{dt}\ln\frac{x(t)}{x(0)} \leqslant 2\widehat{d}_I(t)^{-3/2}\epsilon \leqslant 2(1+\frac{|\lambda|}{20\pi x(0)}t)^{-3/2}\epsilon, \qquad \forall\ t \in [0,T]. \tag{2.263}$$

Then we have for all $t \in [0,T]$,

$$\begin{aligned}x(t) \leqslant& x(0)exp\Big\{2\epsilon\int_0^t(1+\frac{|\lambda|}{20\pi x(0)}\tau)^{-3/2}d\tau\Big\}\\ \leqslant& x(0)exp\Big\{4\epsilon\frac{20\pi x(0)}{|\lambda|}\Big\}\\ \leqslant& x(0)e^{2\epsilon\frac{40\pi}{200\pi\epsilon}} = e^{\frac{2}{5}}x(0) \leqslant \frac{3}{2}x(0).\end{aligned} \tag{2.264}$$

By the continuity of $x(t)$, there exists $\delta > 0$ such that

$$\sup_{t\in[0,T+\delta)}\frac{x(t)}{x(0)} \leqslant 2. \tag{2.265}$$

Next we show that by choosing $\delta > 0$ smaller if necessary, we have

$$\dot{y}(t) \leqslant -\frac{|\lambda|}{20\pi x(0)}, \qquad \widehat{d}_I(t) \geqslant 1 + \frac{|\lambda|}{20\pi x(0)}t, \qquad \forall\ t \in [0, T+\delta). \tag{2.266}$$

Proof of (2.266): By the definition of $\mathcal{T}$, the bootstrap assumption (2.206), and the fact that $\frac{|\lambda|}{x(0)} \geqslant 200\pi\epsilon$, we have

$$\dot{y}(t) \leqslant 5\epsilon - \frac{|\lambda|}{8\pi x(0)} \leqslant -\frac{|\lambda|}{10\pi x(0)}, \qquad \forall\ t \in [0, T]. \tag{2.267}$$

By Fundamental theorem of calculus, we have

$$y(t) = y(0) + \int_0^t \dot{y}(\tau)d\tau. \tag{2.268}$$

We have

$$\begin{aligned}\zeta(\alpha,t) - z_j(t) =&\zeta(\alpha,0) - z_j(0) + \int_0^t \partial_\tau(\zeta(\alpha,\tau) - z_j(\tau))d\tau\\ =&\zeta(\alpha,0) - z_j(0) + \int_0^t D_\tau\zeta(\alpha,\tau)d\tau - \int_0^t b(\alpha,\tau)\partial_\alpha\zeta(\alpha,\tau)d\tau - \int_0^t \dot{z}_j(\tau)d\tau\end{aligned} \tag{2.269}$$

So we have

$$\begin{aligned}Im\{\zeta(\alpha,t) - z_j(t)\} =&Im\Big\{\zeta(\alpha,0) - z_j(0) - \int_0^t \dot{z}_j(\tau)d\tau\Big\} + Im\int_0^t D_\tau\zeta(\alpha,\tau)d\tau\\ &- \int_0^t b(\alpha,\tau)Im\{\partial_\alpha\zeta(\alpha,\tau)\}d\tau.\end{aligned} \tag{2.270}$$

By Sobolev embedding lemma II.1 and the bootstrap assumption, we have

$$\Big|\int_0^t D_\tau\zeta(\alpha,\tau)d\tau\Big| \leqslant 6\epsilon t, \qquad \forall\ t \in [0, T]. \tag{2.271}$$

By Corollary II.36 and Sobolev embedding, we have

$$\Big|\int_0^t b(\alpha,\tau)\partial_\alpha\zeta(\alpha,\tau)d\tau\Big| \leqslant (C\epsilon^2+K_s^{-1}\epsilon d_I(t)^{-3/2})(1+5\epsilon)t \leqslant (C\epsilon^2+K_s^{-1}\epsilon)t. \tag{2.272}$$

Note that $Im\{\zeta(\alpha,0)-z_j(0)\} \geqslant \widehat{d}_I(0) \geqslant 1$, $-Im\{\dot{z}_j(\tau)\} \geqslant \frac{|\lambda|}{10\pi x(0)} > 0$, so we have for all $t \in [0,T]$,

$$\begin{aligned} Im\{\zeta(\alpha,t)-z_j(t)\} \geqslant & \inf_{\alpha\in\mathbb{R}} Im\{\zeta(\alpha,0)-z_j(0)\} + (\frac{|\lambda|}{10\pi x(0)} - 6\epsilon - (C\epsilon^2+K_s^{-1}\epsilon))t \\ \geqslant & 1+\frac{|\lambda|}{18\pi x(0)}t. \end{aligned} \tag{2.273}$$

So we have

$$\widehat{d}_I(t) = \min_{j=1,2}\inf_{\alpha\in\mathbb{R}} Im\{\zeta(\alpha,t)-z_j(t)\} \geqslant 1+\frac{|\lambda|}{18\pi x(0)}t, \qquad \forall\ t\in[0,T]. \tag{2.274}$$

By (2.267), (2.274), the continuity of $\widehat{d}_I(t)$, and the continuity of $y'(t)$, choosing $\delta > 0$ smaller if necessary, we have (2.266).

Since $\mathcal{T}$ is both closed and open as a subspace of $[0,T_0]$, so we must have

$$\mathcal{T} = [0,T_0], \tag{2.275}$$

which concludes the proof of the lemma. □

Because

$$d_I(t) = \min_{j=1,2}\inf_{\alpha\in\mathbb{R}} |\zeta(\alpha,t)-z_j(t)| \geqslant \widehat{d}_I(t),$$

we have the following estimate.

Corollary II.40 (Decay estimate)**.** *Assume the assumptions of Theorem I.10 and assume* *the bootstrap assumption (2.206), we have* $\forall\ t \in [0, T_0]$,

$$d_I(t)^{-1} \leqslant (1 + \frac{|\lambda|}{20\pi x(0)} t)^{-1}. \tag{2.276}$$

We need to estimate $\ddot{z}_j$ and $\ddot{z}_1 - \ddot{z}_2$ as well.

Convention: From now on, if the domain of t is not specified, we assume $t \in [0, T_0]$ by default.

Lemma II.41. *Assume the assumptions of Theorem I.10 and assume the bootstrap assumption (2.206), we have* $\forall\ t \in [0, T_0]$,

$$|\ddot{z}_j(t)| \leqslant 10\epsilon + \frac{6|\lambda|}{x(t)}\epsilon. \tag{2.277}$$

$$|\ddot{z}_1(t) - \ddot{z}_2(t)| \leqslant 220\epsilon^2 x(t) + \epsilon(20x(t) + \frac{5|\lambda|}{\pi}) \tag{2.278}$$

Proof. Take time derivative of (2.249), we have

$$\ddot{z}_j(t) = -\frac{\lambda i x'(t)}{4\pi x(t)^2} + \bar{F}_\zeta(z_j(t), t)\dot{z}_j(t) + \bar{F}_t(z_j, t). \tag{2.279}$$

We have $x'(t) = ReF(z_2(t), t)$. By lemma II.32, we have

$$|F_\zeta(z_j(t), t)| \leqslant \|F_\zeta(\cdot, t)\|_{L^\infty(\Omega(t))} \leqslant \|F_\zeta(\zeta(\alpha, t), t)\|_\infty \leqslant 6\epsilon. \tag{2.280}$$

By Sobolev embedding and lemma II.32, we have

$$\|F_t(\cdot, t)\|_{L^\infty(\Omega(t))} \leqslant \|F_t(\zeta(\alpha, t), t)\|_{L^\infty} \leqslant \|F_t \circ \zeta\|_{H^1} \leqslant 6\epsilon. \tag{2.281}$$

Apply lemma II.32 again, we have

$$|ReF(z_j(t),t)| \leqslant 6\epsilon x(t).$$

So we obtain

$$\begin{aligned}|\ddot{z}_j(t)| \leqslant& \frac{|\lambda||ReF(z_j,t)|}{4\pi x(t)^2} + |F_\zeta(z_j,t)||\dot{z}_j(t)| + |F_t(z_j,t)| \\ \leqslant& \frac{6|\lambda|\epsilon}{4\pi x(t)} + 6\epsilon(\frac{|\lambda|}{4\pi x(t)} + 6\epsilon) + 6\epsilon \\ \leqslant& \frac{6|\lambda|\epsilon}{\pi x(t)} + 10\epsilon.\end{aligned}$$

Here, we assume ϵ^2 sufficiently small such that $36\epsilon^2 \leqslant 4\epsilon$. We have

$$\begin{aligned}&|\ddot{z}_1(t) - \ddot{z}_2(t)| \\ =&|\bar{F}_\zeta(z_1(t),t)\dot{z}_1(t) + \bar{F}_t(z_1(t),t) - \bar{F}_\zeta(z_2(t),t)\dot{z}_2(t) - \bar{F}_t(z_2(t),t)| \\ \leqslant&|F_\zeta(z_1(t),t) - F_\zeta(z_2(t),t)||\dot{z}_1(t)| + |F_\zeta(z_2(t),t)||\dot{z}_1 - \dot{z}_2| + |F_t(z_1,t) - F_t(z_2,t)| \\ \leqslant&\|F_{\zeta\zeta}\|_\infty|z_1 - z_2||\dot{z}_1(t)| + |F_\zeta(z_2(t),t)||F(z_1(t),t) - F(z_2(t),t)| + \|(ReF)_{t\zeta}\|_{L^\infty(\Omega(t))}|z_1 - z_2|.\end{aligned}$$

By lemma II.32,

$$\|F_{\zeta\zeta}\|_\infty \leqslant 10\epsilon, \tag{2.282}$$

and

$$\|F_{t\zeta}\|_{L^\infty(\Omega(t)} \leqslant 10\epsilon. \tag{2.283}$$

Since ReF_t is odd in x and ImF_t is even in x, by mean value theorem, we have

$$\begin{aligned}|F_t(z_1,t) - F_t(z_2,t)| =&2|ReF_t(z_2,t) - ReF_t(0,y,t)|x = 2|ReF_{tx}(\tilde{x},y,t)|x(t) &(2.284)\\ \leqslant&2\|F_{t\zeta}\|_{L^\infty(\Omega(t))}x(t) \leqslant 20\epsilon x(t). &(2.285)\end{aligned}$$

for some $\tilde{x} \in (0, x(t))$.

So we obtain

$$
\begin{aligned}
&|\ddot{z}_1(t) - \ddot{z}_2(t)| \\
\leqslant &10\epsilon(2x(t))(\frac{|\lambda|}{4\pi x(t)} + 5\epsilon) + (6\epsilon)20\epsilon x(t) + (10\epsilon)2x(t) \\
= &220\epsilon^2 x(t) + \epsilon(20x(t) + \frac{5|\lambda|}{\pi}).
\end{aligned}
\tag{2.286}
$$

□

Next, we estimate the quantity $\left\|\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}$, the quantity $\left\|\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{\ddot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}$, and the quantity $\left\|\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{(\dot{z}_j)^2}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}$. These quantities arise from the energy estimates.

Lemma II.42. *Assume the assumptions of Theorem I.10 and assume the bootstrap assumption (2.206). Then we have*

$$
\left\|\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s} \leqslant K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2} + C\epsilon^2. \tag{2.287}
$$

Proof. Replace $\dot{z}_j$ by

$$
\dot{z}_j(t) = \frac{\lambda i}{4\pi x(t)} + \bar{F}(z_j(t), t).
$$

We have

$$
\begin{aligned}
\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2} = &\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{\frac{\lambda i}{4\pi x(t)}}{(\zeta(\alpha,t)-z_j(t))^2} + \bar{F}(z_1(t),t)\sum_{j=1}^2 \frac{\lambda_j i}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_j(t))^2} \\
&+ \frac{\lambda i(\bar{F}(z_1(t),t) - \bar{F}(z_2(t),t))}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_2(t))^2}
\end{aligned}
$$

By lemma II.34 (and use the proof of lemma II.34 to estimate the term $\left\|\frac{1}{(\zeta(\alpha,t)-z_2(t))^2}\right\|_{H^s}$), we have

$$
\begin{aligned}
&\left\|\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}\\
\leqslant&\frac{|\lambda|}{4\pi x(t)}\left\|\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}+\left\|\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}\|F\|_{L^\infty(\Omega(t))}\\
&+\left\|\frac{\lambda i(\bar{F}(z_1(t),t)-\bar{F}(z_2(t),t))}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_2(t))^2}\right\|_{H^s}\\
\leqslant&\frac{|\lambda|}{4\pi x(t)}K_s^{-1}\epsilon d_I(t)^{-5/2}+K_s^{-1}\epsilon d_I(t)^{-5/2}(5\epsilon)+\left|\frac{F(z_1(t),t)-F(z_2(t),t)}{x(t)}\right|\left\|\frac{\lambda x(t)}{(\zeta(\alpha,t)-z_2(t))^2}\right\|_{H^s}\\
\leqslant&K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}+K_s^{-1}\epsilon^2 d_I(t)^{-5/2}+K_s^{-1}\epsilon^2 d_I(t)^{-3/2}.
\end{aligned}
$$

Here, we use lemma II.32 to estimate

$$\left|\frac{F(z_1(t),t)-F(z_2(t),t)}{x(t)}\right|\leqslant 12\epsilon, \tag{2.288}$$

and we use the proof of lemma II.34 to estimate

$$\left\|\frac{\lambda x(t)}{(\zeta(\alpha,t)-z_2(t))^2}\right\|_{H^s}\leqslant K_s^{-1}\epsilon d_I(t)^{-3/2}. \tag{2.289}$$

Since $d_I(t)\geqslant 1$, we simply estimate $\left\|\sum_{j=1}^2\frac{\lambda_j i}{2\pi}\frac{\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}$ by (2.287). □

Lemma II.43. *Assume the assumptions of Theorem I.10 and assume the bootstrap assumption (2.206). Then we have*

$$\left\|\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{(\dot{z}_j)^2}{(\zeta(\alpha,t)-z_j(t))^3}\right\|_{H^s}\leqslant K_s^{-1}\epsilon^2+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}. \tag{2.290}$$

Proof.

$$\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{(\dot{z}_j)^2}{(\zeta(\alpha,t)-z_j(t))^3} = \frac{\lambda i(\dot{z}_1)^2}{2\pi}\left(\frac{1}{(\zeta(\alpha,t)-z_1(t))^3}-\frac{1}{(\zeta(\alpha,t)-z_2(t))^3}\right) + \frac{\lambda i((\dot{z}_1)^2-(\dot{z}_2)^2)}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_2(t))^3} := I + II.$$

We have

$$|I| = \Big|\frac{\lambda i(\dot{z}_1)^2}{2\pi}2x(t)\Big\{\frac{1}{(\zeta(\alpha,t)-z_1(t))^3(\zeta(\alpha,t)-z_2(t))}+\frac{1}{(\zeta(\alpha,t)-z_1(t))^2(\zeta(\alpha,t)-z_2(t)^2)} + \frac{1}{(\zeta(\alpha,t)-z_1(t))(\zeta(\alpha,t)-z_2(t))^3}\Big\}\Big|,$$

and

$$\dot{z}_1^2 = -\frac{\lambda^2}{16\pi^2 x(t)^2}+\frac{\lambda i}{2\pi x(t)}\bar{F}(z_1(t),t)+(\bar{F}(z_1(t),t))^2.$$

Use the proof of lemma II.34, it's easy to see that

$$\left\|\frac{1}{(\zeta(\alpha,t)-z_1(t))^3(\zeta(\alpha,t)-z_2(t))}\right\|_{H^s} \leqslant ((s+6)!)^2 d_I(t)^{-7/2} \tag{2.291}$$

$$\left\|\frac{1}{(\zeta(\alpha,t)-z_1(t))^2(\zeta(\alpha,t)-z_2(t)^2)}\right\|_{H^s} \leqslant ((s+6)!)^2 d_I(t)^{-7/2} \tag{2.292}$$

$$\left\|\frac{1}{(\zeta(\alpha,t)-z_1(t))(\zeta(\alpha,t)-z_2(t)^3)}\right\|_{H^s} \leqslant ((s+6)!)^2 d_I(t)^{-7/2} \tag{2.293}$$

Use the assumption that $\lambda^2+|\lambda x(0)| \leqslant \frac{1}{((s+12)!)^2}\epsilon$ and the fact that $\frac{1}{2}x(0)\leqslant x(t)\leqslant 2x(0)$, we have

$$\begin{aligned}\|I\|_{H^s} \leqslant & \frac{|\lambda|x(t)}{\pi}\left(\frac{\lambda^2}{16\pi^2 x(t)^2}+\frac{|\lambda|}{2\pi x(t)}\times 5\epsilon+25\epsilon^2\right)((s+6)!)^2 d_I(t)^{-7/2}\\ \leqslant & K_s^{-1}\epsilon^2 d_I(t)^{-7/2}+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-7/2}.\end{aligned}$$

By lemma II.38, we have

$$\begin{aligned}\|II\|_{H^s} \leqslant & \frac{|\lambda||\dot{z}_1^2-\dot{z}_2^2|}{2\pi}\left\|\frac{1}{(\zeta(\alpha,t)-z_2(t))^3}\right\|_{H^s}\\ \leqslant & \frac{|\lambda|(6|\lambda|\epsilon+120\epsilon^2 x(t))}{2\pi}((s+6)!)^2 d_I(t)^{-5/2}\\ \leqslant & K_s^{-1}\epsilon^2 d_I(t)^{-5/2}.\end{aligned}$$

Here, we use the assumption

$$\lambda^2+|\lambda x(0)|\leqslant c_0\epsilon, \qquad c_0=\frac{1}{((s+12)!)^2}. \tag{2.294}$$

So we we have

$$\left\|\sum_{j=1}^2\frac{\lambda_j i}{2\pi}\frac{(\dot{z}_j)^2}{(\zeta(\alpha,t)-z_j(t))^3}\right\|_{H^s}\leqslant K_s^{-1}\epsilon^2+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}.$$

□

Lemma II.44. *Assume the assumptions of Theorem I.10 and assume the bootstrap assumption (2.206). Then we have*

$$\left\|\sum_{j=1}^2\frac{\lambda_j i}{2\pi}\frac{\ddot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}\leqslant K_s^{-1}\epsilon^2 d_I(t)^{-5/2}. \tag{2.295}$$

Proof. We have

$$\begin{aligned}&\left\|\sum_{j=1}^2\frac{\lambda_j i}{2\pi}\frac{\ddot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}\\ \leqslant&\left\|\sum_{j=1}^2\frac{\lambda_j i\ddot{z}_1(t)}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}+\left\|\frac{\lambda i(\ddot{z}_1(t)-\ddot{z}_2(t))}{2\pi}\frac{1}{(\zeta(\alpha,t)-z_2(t))^2}\right\|_{H^s}:=I+II.\end{aligned}$$

By the proof of lemma II.34 and by lemma II.41, we have

$$\begin{aligned}
I \leqslant & |\ddot{z}_1(t)| \| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{(\zeta(\alpha,t)-z_j(t))^2} \|_{H^s} \\
\leqslant & (10\epsilon + \frac{6|\lambda|}{x(t)}\epsilon)|\lambda x(0)|((s+6)!)^2 d_I(t)^{-5/2} \\
\leqslant & K_s^{-1}\epsilon^2 d_I(t)^{-5/2}.
\end{aligned}$$

By lemma II.34 and lemma II.41, we have

$$\begin{aligned}
II \leqslant & |\ddot{z}_1(t) - \ddot{z}_2(t)| \left\| \frac{\lambda i}{2\pi} \frac{1}{(\zeta(\alpha,t)-z_2(t))^2} \right\|_{H^s} \\
\leqslant & \Big(220\epsilon^2 x(t) + \epsilon(20x(t) + \frac{5|\lambda|}{\pi})\Big)|\lambda|((s+6)!)^2 d_I(t)^{-5/2} \\
\leqslant & K_s^{-1}\epsilon^2 d_I(t)^{-5/2}.
\end{aligned}$$

Here, we've used the fact that $\lambda^2 + |\lambda x(0)| \leqslant \frac{1}{((s+12)!)^2}\epsilon$. So we obtain

$$\left\| \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{\ddot{z}_j}{(\zeta(\alpha,t)-z_j(t))^2} \right\|_{H^s} \leqslant K_s^{-1}\epsilon^2 d_I(t)^{-5/2}. \tag{2.296}$$

□

2.5.8 Estimates for quantities involved in the energy estimates.

In this subsection, we derive estimates for various quantities that show up in energy estimates.

2.5.8.1 Control $\|\partial_\alpha\tilde{\theta}\|_{H^s}$ by $\|\zeta_\alpha - 1\|_{H^s}$.

Lemma II.45. *Assume the assumptions of Theorem I.10 and assume the assumption (2.206), for* $1 \leqslant k \leqslant s+1$, *we have*

$$\|\partial_\alpha^k\tilde{\theta} - 2\partial_\alpha^{k-1}(\zeta_\alpha - 1)\|_{L^2} \leqslant C\epsilon^2. \tag{2.297}$$

Proof. Since $(I-\mathcal{H})(\bar{\zeta}-\alpha) = 0$, we have $(I+\bar{\mathcal{H}})(\zeta-\alpha) = 2(\zeta-\alpha)$. Therefore,

$$\begin{aligned}
\partial_\alpha^k\tilde{\theta} =& \partial_\alpha^k(I-\mathcal{H})(\zeta-\bar{\zeta}) = \partial_\alpha^k(I-\mathcal{H})(\zeta-\alpha)\\
=& \partial_\alpha^k(I+\bar{\mathcal{H}}-(\bar{\mathcal{H}}+\mathcal{H}))(\zeta-\alpha)\\
=& 2\partial_\alpha^{k-1}(\zeta_\alpha-1) - \partial_\alpha^k(\bar{\mathcal{H}}+\mathcal{H}))(\zeta-\alpha).
\end{aligned}$$

It's easy to obtain that for $1 \leqslant k \leqslant s+1$,

$$\left\|\partial_\alpha^k(\bar{\mathcal{H}}+\mathcal{H})(\zeta-\alpha)\right\|_{L^2} \leqslant C\|\zeta_\alpha-1\|_{H^s}^2 \leqslant C\epsilon^2. \tag{2.298}$$

So we have

$$\left\|\partial_\alpha^k\tilde{\theta} - 2\partial_\alpha^{k-1}(\zeta_\alpha - 1)\right\|_{L^2} \leqslant C\epsilon^2. \tag{2.299}$$

So we obtain (3.376). □

Corollary II.46. *Assume the assumptions of Theorem I.10 and the bootstrap assumption (2.206), we have*

$$\left\|\partial_\alpha\tilde{\theta}\right\|_{H^s} \leqslant 11\epsilon. \tag{2.300}$$

2.5.8.2 Compare $\left\|D_t\tilde{\theta}\right\|_{H^s}$ with $\|D_t\zeta\|_{H^s}$ and $\|D_t\tilde{\sigma}\|_{H^s}$ with $\left\|D_t^2\zeta\right\|_{H^s}$

We need to show that $D_t\tilde{\theta}$ and $D_t\zeta$ are equivalent in certain sense. We have the following:

Lemma II.47. *Assume the assumptions of Theorem I.10 and a priori assumption (2.206), we have*

$$\left\| D_t\tilde{\theta} - 2(\bar{\mathfrak{F}} - q) \right\|_{H^{s+1/2}} \leqslant C\epsilon^2. \tag{2.301}$$

$$\left\| D_t\tilde{\sigma} - 4(D_t\bar{\mathfrak{F}} - D_t q) \right\|_{H^s} \leqslant C\epsilon^2. \tag{2.302}$$

Proof. Recall that $D_t\zeta = \bar{\mathfrak{F}} + \bar{q}$, where $(I - \mathcal{H})\mathfrak{F} = 0$, $(I + \mathcal{H})q = 0$. So we have

$$(I + \bar{\mathcal{H}}))\bar{\mathfrak{F}} = 2\bar{\mathfrak{F}}, \qquad (I + \bar{\mathcal{H}})\bar{q} = 0.$$

We have

$$\begin{aligned} D_t\tilde{\theta} =& D_t(I - \mathcal{H})(\zeta - \bar{\zeta}) = (I - \mathcal{H})(D_t\zeta - D_t\bar{\zeta}) - [D_t\zeta, \mathcal{H}]\frac{\partial_\alpha(\zeta - \bar{\zeta})}{\zeta_\alpha} \\ =& (I - \mathcal{H})(\bar{\mathfrak{F}} + \bar{q} - \mathfrak{F} - q) - [D_t\zeta, \mathcal{H}]\frac{\partial_\alpha(\zeta - \bar{\zeta})}{\zeta_\alpha} \\ =& (I + \bar{\mathcal{H}})\bar{\mathfrak{F}} + (I + \bar{\mathcal{H}})\bar{q} - (\mathcal{H} + \bar{\mathcal{H}})D_t\zeta - 2q - [D_t\zeta, \mathcal{H}]\frac{\partial_\alpha(\zeta - \bar{\zeta})}{\zeta_\alpha} \\ =& 2\bar{\mathfrak{F}} - 2q - (\mathcal{H} + \bar{\mathcal{H}})D_t\zeta - [D_t\zeta, \mathcal{H}]\frac{\partial_\alpha(\zeta - \bar{\zeta})}{\zeta_\alpha}. \end{aligned} \tag{2.303}$$

It's easy to obtain that under a priori assumption (2.206),

$$\left\| -(\mathcal{H} + \bar{\mathcal{H}})D_t\zeta - [D_t\zeta, \mathcal{H}]\frac{\partial_\alpha(\zeta - \bar{\zeta})}{\zeta_\alpha} \right\|_{H^{s+1/2}} \leqslant C\epsilon\|D_t\zeta\|_{H^{s+1/2}} \leqslant C\epsilon^2, \tag{2.304}$$

for some absolute constant $C > 0$.

By triangle inequality,

$$\begin{aligned} \left\| D_t\tilde{\theta} - 2(\bar{\mathfrak{F}} - q) \right\|_{H^{s+1/2}} \leqslant & \left\| -(\mathcal{H} + \bar{\mathcal{H}})D_t\zeta - [D_t\zeta, \mathcal{H}]\frac{\partial_\alpha(\zeta - \bar{\zeta})}{\zeta_\alpha} \right\|_{H^{s+1/2}} \\ \leqslant & C\epsilon^2. \end{aligned} \tag{2.305}$$

So we obtain (2.301).

By (2.303), use

$$(I-\mathcal{H})\bar{\mathfrak{F}} = 2\bar{\mathfrak{F}} - (\bar{\mathcal{H}}+\mathcal{H})\bar{\mathfrak{F}}, \qquad (I-\mathcal{H})q = 2q, \tag{2.306}$$

we have

$$\begin{aligned} D_t\tilde{\sigma} =& D_t(I-\mathcal{H})D_t\tilde{\theta} = D_t(I-\mathcal{H})\Big\{2\bar{\mathfrak{F}} - 2q - (\mathcal{H}+\bar{\mathcal{H}})D_t\zeta - [D_t\zeta,\mathcal{H}]\frac{\partial_\alpha(\zeta-\bar{\zeta})}{\zeta_\alpha}\Big\} \\ =& 4D_t\bar{\mathfrak{F}} - 4D_tq + D_t(I-\mathcal{H})\Big\{-(\mathcal{H}+\bar{\mathcal{H}})D_t\zeta - [D_t\zeta,\mathcal{H}]\frac{\partial_\alpha(\zeta-\bar{\zeta})}{\zeta_\alpha}\Big\} - D_t(\bar{\mathcal{H}}+\mathcal{H})\bar{\mathfrak{F}}. \end{aligned} \tag{2.307}$$

Therefore,

$$\begin{aligned} & \big\|D_t\tilde{\sigma} - 4(D_t\bar{\mathfrak{F}} - D_tq)\big\|_{H^s} && (2.308) \\ =& \left\|D_t(I-\mathcal{H})\Big\{-(\mathcal{H}+\bar{\mathcal{H}})D_t\zeta - [D_t\zeta,\mathcal{H}]\frac{\partial_\alpha(\zeta-\bar{\zeta})}{\zeta_\alpha}\Big\} - D_t(\bar{\mathcal{H}}+\mathcal{H})\bar{\mathfrak{F}}\right\|_{H^s} && (2.309) \\ \leqslant& C\epsilon^2. && (2.310) \end{aligned}$$

□

Corollary II.48. *Assume the bootstrap assumption (2.206), we have*

$$\left\|D_t\tilde{\theta}\right\|_{H^{s+1/2}} \leqslant 11\epsilon, \qquad \|D_t\tilde{\sigma}\|_{H^s} \leqslant 21\epsilon. \tag{2.311}$$

2.5.8.3 Estimate the quantity $\frac{a_t}{a} \circ \kappa^{-1}$.

Recall that

$$\begin{aligned}&(I-\mathcal{H})\frac{a_t}{a}\circ\kappa^{-1}A\bar{\zeta}_\alpha\\ =&2i[D_t^2\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}+2i[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t^2\bar{\zeta}}{\zeta_\alpha}-\frac{1}{\pi}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(D_t\bar{\zeta})_\beta d\beta\\ &-\frac{1}{\pi}\sum_{j=1}^2\lambda_j\Big(\frac{2D_t^2\zeta+i-\partial_t^2 z_j}{(\zeta(\alpha,t)-z_j(t))^2}-2\frac{(D_t\zeta-\dot{z}_j(t))^2}{(\zeta(\alpha,t)-z_j(t))^3}\Big)\end{aligned} \tag{2.312}$$

By lemma II.9, the a priori assumption (2.206), we have

$$\Big\|2i[D_t^2\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}\Big\|_{H^s}\leqslant C\|D_t^2\zeta\|_{H^s}\|D_t\zeta\|_{H^s}\leqslant C\epsilon^2. \tag{2.313}$$

$$\Big\|2i[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t^2\bar{\zeta}}{\zeta_\alpha}\Big\|_{H^s}\leqslant C\|D_t\zeta\|_{H^s}\|D_t^2\zeta\|_{H^s}\leqslant C\epsilon^2. \tag{2.314}$$

$$\Big\|\frac{1}{\pi}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(D_t\bar{\zeta})_\beta d\beta\Big\|_{H^s}\leqslant\|D_t\zeta\|_{H^s}^3\leqslant C(5\epsilon)^3\leqslant C\epsilon^2. \tag{2.315}$$

$$\Big\|\frac{1}{\pi}\sum_{j=1}^2\frac{2\lambda_jD_t^2\zeta}{(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s}\leqslant\|D_t^2\zeta\|_{H^s}\Big\|\frac{2}{\pi}\sum_{j=1}^2\frac{\lambda_j}{(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s} \tag{2.316}$$

By lemma II.34, we have

$$\Big\|\frac{2}{\pi}\sum_{j=1}^2\frac{\lambda_j}{(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s}\leqslant K_s^{-1}\epsilon d_I(t)^{-5/2} \tag{2.317}$$

So we have

$$\Big\|\frac{1}{\pi}\sum_{j=1}^2\frac{2\lambda_jD_t^2\zeta}{(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s}\leqslant\|D_t^2\zeta\|_{H^s}\Big\|\frac{1}{\pi}\sum_{j=1}^2\frac{2\lambda_j}{(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s}\leqslant K_s^{-1}\epsilon^2. \tag{2.318}$$

By lemma II.42, we have

$$\begin{aligned}\left\|\sum_{j=1}^{2}\frac{\lambda_j i}{\pi}\frac{2D_t\zeta\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^3}\right\|_{H^s} \leqslant& 2\|D_t\zeta\|_{H^s}\left\|\sum_{j=1}^{2}\frac{\lambda_j i}{\pi}\frac{\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^3}\right\|_{H^s} \\ \leqslant& 12\epsilon K_s^{-1}\epsilon d_I(t)^{-5/2}\leqslant K_s^{-1}\epsilon^2 d_I(t)^{-5/2}.\end{aligned} \tag{2.319}$$

By lemma II.43 and lemma II.44, we have

$$\begin{aligned}&\left\|\frac{1}{\pi}\sum_{j=1}^{2}\frac{\lambda_j\ddot{z}_j(t)}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}+\left\|\frac{1}{\pi}\sum_{j=1}^{2}\frac{\lambda_j 2(\dot{z}_j(t))^2}{(\zeta(\alpha,t)-z_j(t))^3}\right\|_{H^s} \\ \leqslant& K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}+K_s^{-1}\epsilon^2 d_I(t)^{-5/2}.\end{aligned} \tag{2.320}$$

So we obtain

$$\left\|(I-\mathcal{H})\frac{a_t}{a}\circ\kappa^{-1}A\bar{\zeta}_\alpha\right\|_{H^s}\leqslant C\epsilon^2+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}. \tag{2.321}$$

By lemma II.33 and Sobolev embedding, we have

$$\left\|\frac{a_t}{a}\circ\kappa^{-1}\right\|_{\infty}\leqslant C\epsilon^2+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}. \tag{2.322}$$

2.5.8.4 Estimate the quantity A.

Recall that

$$(I-\mathcal{H})A=1+i[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\mathfrak{F}}{\zeta_\alpha}+i[D_t^2\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}-(I-\mathcal{H})\frac{1}{2\pi}\sum_{j=1}^{2}\frac{\lambda_j(D_t\zeta(\alpha,t)-\dot{z}_j(t))}{(\zeta(\alpha,t)-z_j(t))^2}. \tag{2.323}$$

By lemma II.9, lemma II.34, lemma II.42, we have

$$\begin{aligned}\|(I-\mathcal{H})(A-1)\|_{H^s}\leqslant&\|D_t\zeta\|_{H^s}\|\mathfrak{F}\|_{H^s}+\|D_t^2\zeta\|_{H^s}\|\zeta_\alpha-1\|_{H^s}+\frac{1}{\pi}\|D_t\zeta\|_{H^s}\left\|\sum_{j=1}^2\frac{\lambda_j}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}\\&+\left\|\frac{1}{\pi}\sum_{j=1}^2\frac{\lambda_j\dot{z}_j(t)}{(\zeta(\alpha,t)-z_j(t))^2}\right\|_{H^s}\\\leqslant&C\epsilon^2+K_s^{-1}\epsilon d_I(t)^{-5/2}.\end{aligned}$$

So we have

$$\|A-1\|_{H^s}\leqslant C\epsilon^2+K_s^{-1}\epsilon d_I(t)^{-5/2}. \tag{2.324}$$

Corollary II.49. *Assume the assumptions of Theorem I.10 and assume the bootstrap assumption 2.206. For ϵ sufficiently small, we have*

$$\inf_{\alpha\in\mathbb{R}}A(\alpha,t)\geqslant\frac{9}{10},\qquad\forall t\in[0,T_0]. \tag{2.325}$$

$$\sup_{\alpha\in\mathbb{R}}A(\alpha,t)\leqslant\frac{10}{9},\qquad\forall t\in[0,T_0]. \tag{2.326}$$

2.5.8.5 Estimate the quantity D_tb.

Recall that

$$\begin{aligned}(I-\mathcal{H})D_tb=&[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha b}{\zeta_\alpha}-[D_t^2\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}-[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}\\&+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\bar{\zeta}_\beta(\beta,t)-1)d\beta+\frac{i}{\pi}\sum_{j=1}^2\frac{\lambda_j(D_t\zeta-\dot{z}_j(t))}{(\zeta(\alpha,t)-z_j(t))^2}.\end{aligned} \tag{2.327}$$

By lemma II.9, lemma II.34, lemma II.42, estimate (2.245), we have

$$\begin{aligned}
&\|(I-\mathcal{H})D_t b\|_{H^s}\\
\leqslant&\Big\|[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha b}{\zeta_\alpha}\Big\|_{H^s}+\Big\|[D_t^2\zeta,\mathcal{H}]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}\Big\|_{H^s}+\Big\|[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}\Big\|_{H^s}\\
&+\Big\|\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\bar{\zeta}_\beta(\beta,t)-1)d\beta\Big\|_{H^s}+\Big\|\frac{i}{\pi}\sum_{j=1}^2\frac{\lambda_j(D_t\zeta-\dot{z}_j(t))}{(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s}\\
\leqslant&C\|D_t\zeta\|_{H^s}\|b\|_{H^s}+C\|D_t^2\zeta\|_{H^s}\|\zeta_\alpha-1\|_{H^s}+C\|D_t\zeta\|_{H^s}^2\\
&+\|D_t\zeta\|_{H^s}\Big\|\sum_{j=1}^2\frac{\lambda_j}{\pi(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s}+\Big\|\sum_{j=1}^2\frac{\lambda_j\dot{z}_j}{\pi(\zeta(\alpha,t)-z_j(t))^2}\Big\|_{H^s}+C\|D_t\zeta\|_{H^s}^2\|\zeta_\alpha-1\|_{H^s}\\
\leqslant&C\epsilon(C\epsilon^2+K_s^{-1}\epsilon d_I(t)^{-5/2})+C\epsilon^2+K_s^{-1}\epsilon^2 d_I(t)^{-5/2}+K_s^{-1}\epsilon d_I(t)^{-5/2}+C\epsilon^3\\
\leqslant&C\epsilon^2+K_s^{-1}\epsilon d_I(t)^{-5/2}.
\end{aligned}$$

By lemma II.33, we have

$$\|D_t b\|_{H^s}\leqslant C\epsilon^2+K_s^{-1}\epsilon d_I(t)^{-5/2}. \tag{2.328}$$

2.5.8.6 Estimate $\|G\|_{H^s}$.

Recall that $G=G_c+G_d$, with

$$G_c:=-2[\bar{\mathfrak{F}},\mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}]\bar{\mathfrak{F}}_\alpha+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta:=G_{c1}+G_{c2}. \tag{2.329}$$

$$G_d:=-2[\bar{q},\mathcal{H}]\frac{\partial_\alpha\bar{\mathfrak{F}}}{\zeta_\alpha}-2[\bar{\mathfrak{F}},\mathcal{H}]\frac{\partial_\alpha\bar{q}}{\zeta_\alpha}-2[\bar{q},\mathcal{H}]\frac{\partial_\alpha\bar{q}}{\zeta_\alpha}-4D_tq:=G_{d1}+G_{d2}+G_{d3}+G_{d4}. \tag{2.330}$$

We rewrite G_{c1} as

$$G_{c1}=-\frac{4}{\pi}\int\frac{(D_t\bar{\mathfrak{F}}(\alpha,t)-D_t\bar{\mathfrak{F}}(\beta,t))Im\{\zeta(\alpha,t)-\zeta(\beta,t)\}}{|\zeta(\alpha,t)-\zeta(\beta,t)|^2}\partial_\beta\bar{\mathfrak{F}}(\beta,t)d\beta. \tag{2.331}$$

By lemma II.9, we have

$$\|G_{c1}\|_{H^s} \leqslant C\|\mathfrak{F}\|_{H^s}\|\zeta_\alpha - 1\|_{H^s}\|\mathfrak{F}\|_{H^s} \leqslant C\epsilon^3, \tag{2.332}$$

for some constant C depends on s only. Similarly,

$$\|G_{c2}\|_{H^s} \leqslant C\|D_t\zeta\|_{H^s}^2\|\zeta_\alpha - 1\|_{H^s} \leqslant C\epsilon^3.$$

By lemma II.9, we have

$$\|G_{d1}\|_{H^s} + \|G_{d2}\|_{H^s} \leqslant C\|q\|_{H^s}\|\mathfrak{F}\|_{H^s} \leqslant K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.333}$$

Similarly,

$$\|G_{d3}\|_{H^s} \leqslant C\|q\|_{H^s}^2 \leqslant K_s^{-1}\epsilon^2 d_I(t)^{-3/2}.$$

Use

$$D_t\bar{q} = \sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{D_t\zeta - \dot{z}_j}{(\zeta(\alpha,t) - z_j(t))^2}, \tag{2.334}$$

by lemma II.42, lemma II.34, we have

$$\begin{aligned}\|G_{d4}\|_{H^s} \leqslant& 4\|D_t\zeta\|_{H^s}\left\|\sum_{j=1}^{2}\frac{\lambda_j}{2\pi(\zeta(\alpha,t) - z_j(t))^2}\right\|_{H^s} + 4\left\|\sum_{j=1}^{2}\frac{\lambda_j\dot{z}_j}{2\pi(\zeta(\alpha,t) - z_j(t))^2}\right\|_{H^s} \\ \leqslant& C\epsilon^2 d_I(t)^{-5/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}.\end{aligned} \tag{2.335}$$

So we obtain

$$\|G\|_{H^s} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}. \tag{2.336}$$

As a consequence,

$$\|(I-\mathcal{H})G\|_{H^s} \leqslant 3\|G\|_{H^s} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}. \tag{2.337}$$

2.5.8.7 Estimate $\|(I-\mathcal{H})[D_t^2 - iA\partial_\alpha, \partial_\alpha^k]\tilde{\theta})\|_{L^2}$.

By lemma II.13, we have

$$\begin{aligned}[D_t^2, \partial_\alpha^k]\tilde{\theta} = -\sum_{m=0}^{k-1}\Big[&\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m}\tilde{\theta} + \partial_\alpha^m(b_\alpha\partial_\alpha^{k-m}D_t\tilde{\theta}) + \partial_\alpha^m(b_\alpha[b\partial_\alpha, \partial_\alpha^{k-m}]\tilde{\theta}) + \partial_\alpha^m b_\alpha\partial_\alpha^{k-m}D_t\tilde{\theta}\\ &+ \partial_\alpha^m b_\alpha[b\partial_\alpha, \partial_\alpha]\partial_\alpha^{k-m-1}\tilde{\theta}\Big]\end{aligned}$$

<u>The quantity $\|\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m}\tilde{\theta}\|_{L^2}$.</u> For $0 \leqslant m \leqslant k-1, k \leqslant s$, we have

$$\|\partial_\alpha^m(D_t b_\alpha)\partial_\alpha^{k-m}\tilde{\theta}\|_{L^2} \leqslant \|D_t b_\alpha\|_{H^m}\|\partial_\alpha^{k-m}\tilde{\theta}\|_{H^m} \tag{2.338}$$

Since $D_t b_\alpha = \partial_\alpha D_t b + b b_\alpha$, we have

$$\begin{aligned}\|D_t b_\alpha\|_{H^m} \leqslant& \|\partial_\alpha D_t b\|_{H^m} + \|b_\alpha b\|_{H^m} \leqslant \|D_t b\|_{H^s} + \|b\|_{H^s}^2\\ \leqslant& C\epsilon^2 + K_s^{-1}\epsilon d_I(t)^{-5/2} + (C\epsilon^2 + K_s^{-1}\epsilon d_I(t)^{-3/2})^2\\ \leqslant& C\epsilon^2 + K_s^{-1}\epsilon d_I(t)^{-5/2}.\end{aligned}$$

and since $k-m \geqslant 1$, by Corollary II.46, we have

$$\|\partial_\alpha^{k-m}\tilde{\theta}\|_{H^s} \leqslant 11\epsilon, \tag{2.339}$$

we obtain

$$\|\partial_\alpha^m(D_tb_\alpha)\partial_\alpha^{k-m}\tilde{\theta}\|_{L^2}\leqslant\|D_tb_\alpha\|_{H^s}\|\partial_\alpha\tilde{\theta}\|_{H^s}\leqslant C\epsilon^3+K_s^{-1}\epsilon^2d_I(t)^{-3/2}. \tag{2.340}$$

<u>The quantity $\partial_\alpha^m(b_\alpha\partial_\alpha^{k-m}D_t\tilde{\theta})$.</u> Similar to the previous case, we have for $0\leqslant m\leqslant k-1, k\leqslant s$, and assume bootstrap assumption (2.206),

$$\|\partial_\alpha^m(b_\alpha\partial_\alpha^{k-m}D_t\tilde{\theta})\|_{L^2}\leqslant\|b_\alpha\|_{H^{k-1}}\|D_t\tilde{\theta}\|_{H^k}\leqslant C\epsilon^3+K_s^{-1}\epsilon^2d_I(t)^{-3/2}. \tag{2.341}$$

<u>The quantity $\partial_\alpha^m(b_\alpha[b\partial_\alpha,\partial_\alpha^{k-m}]\tilde{\theta})$.</u> We have for $0\leqslant m\leqslant k-1, k\leqslant s$, and assume bootstrap assumption (2.206),

$$\|\partial_\alpha^m(b_\alpha[b\partial_\alpha,\partial_\alpha^{k-m}]\tilde{\theta})\|_{L^2}\leqslant C\|b_\alpha\|_{H^{s-1}}\|b\|_{H^s}\|\theta_\alpha\|_{H^{s-1}}\leqslant C\epsilon^3. \tag{2.342}$$

<u>The quantity $\partial_\alpha^m b_\alpha\partial_\alpha^{k-m}D_t\tilde{\theta}$.</u> We have for $0\leqslant m\leqslant k-1, k\leqslant s$,

$$\|\partial_\alpha^m b_\alpha\partial_\alpha^{k-m}D_t\tilde{\theta}\|_{L^2}\leqslant C\|b_\alpha\|_{H^{k-1}}\|D_t\tilde{\theta}\|_{H^k}\leqslant C\epsilon^3+K_s^{-1}\epsilon^2d_I(t)^{-3/2}. \tag{2.343}$$

<u>The quantity $\partial_\alpha^m b_\alpha[b\partial_\alpha,\partial_\alpha]\partial_\alpha^{k-m-1}\tilde{\theta}$.</u> We have for $0\leqslant m\leqslant k-1, k\leqslant s$,

$$\|\partial_\alpha^m b_\alpha[b\partial_\alpha,\partial_\alpha]\partial_\alpha^{k-m-1}\tilde{\theta}\|_{L^2}\leqslant C\|b\|_{H^s}^2\|D_t\theta\|_{H^s}\leqslant C\epsilon^3. \tag{2.344}$$

So we obtain

$$\|[D_t^2,\partial_\alpha^k]\tilde{\theta}\|_{L^2}\leqslant C\epsilon^3+K_s^{-1}C\epsilon^2d_I(t)^{-3/2}. \tag{2.345}$$

<u>The quantity $\|[iA\partial_\alpha,\partial_\alpha^k]\tilde{\theta}\|_{L^2}$</u> Use similar argument, we obtain

$$\|[iA\partial_\alpha,\partial_\alpha^k]\theta\|_{L^2}\leqslant C\epsilon^3+K_s^{-1}\epsilon^2d_I(t)^{-3/2}. \tag{2.346}$$

So we obtain

$$\|(I-\mathcal{H})[D_t^2 - iA\partial_\alpha, \partial_\alpha^k]\tilde{\theta})\|_{L^2} \leqslant C\epsilon^3 + K_s^{-1}C\epsilon^2 d_I(t)^{-3/2}. \tag{2.347}$$

2.5.8.8 Estimate $\|[D_t^2 - iA\partial_\alpha, \mathcal{H}]\partial_\alpha^k\tilde{\theta}\|_{L^2}$

Note that by identity (3.384),

$$[D_t^2 - iA\partial_\alpha, \mathcal{H}]\partial_\alpha^k\tilde{\theta} = 2[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha} - \frac{1}{\pi i}\int \Big(\frac{\zeta(\alpha,t)-\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2 \partial_\beta\partial_\beta^k\tilde{\theta}d\beta \tag{2.348}$$

Clearly, for $k \leqslant s$, and assume (2.206), we have

$$\Big\|\frac{1}{\pi i}\int \Big(\frac{\zeta(\alpha,t)-\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2 \partial_\beta\partial_\beta^k\tilde{\theta}d\beta\Big\|_{L^2} \leqslant C\epsilon^3. \tag{2.349}$$

- Estimate $\|[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}\|_{L^2}$.

$[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}$ is not obvious cubic. However, since $\partial_\alpha^k\tilde{\theta}$ is almost anti-holomorphic, and $D_t\zeta = \bar{\mathfrak{F}} + \bar{q}$, with $\bar{\mathfrak{F}}$ anti-holomorphic and $\bar{q}$ decays rapidly in time as long as the point vortices move away from the free interface rapidly, we expect this quantity consists of cubic terms and quadratic terms which decay rapidly. To see this, decompose

$$\partial_\alpha^k\tilde{\theta} := \frac{1}{2}(I-\mathcal{H})\partial_\alpha^k\tilde{\theta} + \frac{1}{2}(I+\mathcal{H})\partial_\alpha^k\tilde{\theta}.$$

Note that for $k \geqslant 1$,

$$\begin{aligned}(I+\mathcal{H})\partial_\alpha^k\tilde{\theta} =& (I+\mathcal{H})\partial_\alpha^k(I-\mathcal{H})(\zeta-\bar{\zeta}) = -[\partial_\alpha^k, \mathcal{H}]\tilde{\theta}\\ =& -\sum_{m=0}^{k-1}\partial_\alpha^m[\zeta_\alpha - 1, \mathcal{H}]\frac{\partial_\alpha\partial_\alpha^{k-m-1}\tilde{\theta}}{\zeta_\alpha}.\end{aligned}$$

By lemma II.9 and lemma II.45,

$$\|(I+\mathcal{H})\partial_\alpha^k\tilde{\theta}\|_{L^2}\leqslant C\|\zeta_\alpha-1\|_{H^k}\|\partial_\alpha\tilde{\theta}\|_{H^{s-1}}\leqslant C\epsilon^2. \tag{2.350}$$

Therefore, by lemma II.9, we have

$$\left\|[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I+\mathcal{H})\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}\right\|_{L^2}\leqslant C\epsilon^3. \tag{2.351}$$

We rewrite $[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I-\mathcal{H})\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}$ as

$$\begin{aligned}&[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I-\mathcal{H})\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}\\=&[\frac{1}{2}(I+\mathcal{H})D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I-\mathcal{H})\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}+[\frac{1}{2}(I-\mathcal{H})D_t\theta,\mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I-\mathcal{H})\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}:=I+II.\end{aligned}$$

Clearly, $II=0$. Since

$$\frac{1}{2}(I+\mathcal{H})D_t\zeta=\frac{1}{2}(I+\mathcal{H})\bar{q}+\frac{1}{2}(\mathcal{H}+\bar{\mathcal{H}})\bar{\mathfrak{F}}, \tag{2.352}$$

Use lemma II.9, lemma II.34, and similar to the estimate of $\|G_{d1}\|_{H^s}$ in §2.5.8.6, we have

$$\left\|[\frac{1}{2}(I+\mathcal{H})\bar{q},\mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I-\mathcal{H})\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}\right\|_{L^2}\leqslant C\|q\|_{H^k}\|\partial_\alpha\theta\|_{H^{k-1}}\leqslant K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.353}$$

It's easy to obtain

$$\left\|(\mathcal{H}+\bar{\mathcal{H}})\mathfrak{F}\right\|_{H^s}\leqslant C\epsilon^2. \tag{2.354}$$

So we obtain

$$\left\|[(\mathcal{H}+\bar{\mathcal{H}})\mathfrak{F},\mathcal{H}]\frac{\partial_\alpha\frac{1}{2}(I-\mathcal{H})\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}\right\|_{H^s}\leqslant C\epsilon^3. \tag{2.355}$$

Therefore,

$$\left\|(I-\mathcal{H})[D_t^2-iA\partial_\alpha,\mathcal{H}]\partial_\alpha^k\tilde{\theta}\right\|_{L^2}\leqslant 3\left\|[D_t^2-iA\partial_\alpha,\mathcal{H}]\partial_\alpha^k\tilde{\theta}\right\|_{L^2}\leqslant C\epsilon^3+K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.356}$$

2.5.8.9 Estimate for $\|G_k^\theta\|_{L^s}$.

Collect the estimates from (2.337), (2.347), (2.356), we obtain

$$\left\|G_k^\theta\right\|_{L^2}\leqslant C\epsilon^3+K_s^{-1}\epsilon^2 d_I(t)^{-3/2}+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}. \tag{2.357}$$

2.5.9 Estimate $\left\|(I-\mathcal{H})\partial_\alpha^k\tilde{G}\right\|_{L^2}$

Recall that

$$\begin{aligned}\tilde{G}=&(I-\mathcal{H})(D_tG+i\frac{a_t}{a}\circ\kappa^{-1}A((I-\mathcal{H})(\zeta-\bar{\zeta}))_\alpha)-2[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t^2(I-\mathcal{H})(\zeta-\bar{\zeta})}{\zeta_\alpha}\\&+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(D_t(I-\mathcal{H})(\zeta-\bar{\zeta}))_\beta d\beta\end{aligned} \tag{2.358}$$

2.5.9.1 Estimate $\|D_tG\|_{H^k}$

D_tG is given by

$$D_tG=(\partial_t g)\circ\kappa^{-1}.$$

$g = g_c + g_d$, and

$$\begin{aligned}\partial_t g_c =&\partial_t\Big\{-2[\bar f, \mathfrak{H}\frac{1}{z_\alpha}+\bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}]\bar f_\alpha + \frac{1}{\pi i}\int\Big(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\Big)^2(z-\bar z)_\beta d\beta\Big\}\\ =&-2[\bar f_t, \mathfrak{H}\frac{1}{z_\alpha}+\bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}]\bar f_\alpha - 2[\bar f, \mathfrak{H}\frac{1}{z_\alpha}+\bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}]\bar f_{t\alpha}\\ &-\frac{4}{\pi}\int(\bar f(\alpha,t)-\bar f(\beta,t))\Big(\partial_t Im\Big\{\frac{z_t(\alpha,t)-z_t(\beta,t)}{(z(\alpha,t)-z(\beta,t))^2}\Big\}\Big)\partial_\beta \bar f(\beta,t)d\beta\\ &+\frac{2}{\pi i}\int\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\Big\{\frac{z_{tt}(\alpha,t)-z_{tt}(\beta,t)}{z(\alpha,t)-z(\beta,t)}-\frac{(z_t(\alpha,t)-z_t(\beta,t))^2}{(z(\alpha,t)-z(\beta,t))^2}\Big\}(z-\bar z)_\beta d\beta\\ &+\frac{1}{\pi i}\int\Big(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)}\Big)^2(z_t-\bar z_t)_\beta d\beta\end{aligned}$$

So we have

$$\begin{aligned}D_tG_c =&-2[D_t\bar{\mathfrak{F}}, \mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}]\bar{\mathfrak{F}}_\alpha - 2[\bar{\mathfrak{F}}, \mathcal{H}\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}\frac{1}{\bar\zeta_\alpha}]\partial_\alpha D_t\bar{\mathfrak{F}}\\ &-\frac{4}{\pi}\int(\bar{\mathfrak{F}}(\alpha,t)-\bar{\mathfrak{F}}(\beta,t))\Big(D_t Im\Big\{\frac{\zeta(\alpha,t)-\zeta(\beta,t)}{(\zeta(\alpha,t)-\zeta(\beta,t))^2}\Big\}\Big)\partial_\beta\bar{\mathfrak{F}}(\beta,t)d\beta\\ &+\frac{2}{\pi i}\int\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big\{\frac{D_t^2\zeta(\alpha,t)-D_t^2\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}-\frac{(D_t\zeta(\alpha,t)-D_t\zeta(\beta,t))^2}{(\zeta(\alpha,t)-\zeta(\beta,t))^2}\Big\}(\zeta-\bar\zeta)d\beta\\ &+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(D_t\zeta-D_t\bar\zeta)_\beta d\beta\end{aligned}$$

Recall that

$$g_d := -2[\bar p, \mathfrak{H}]\frac{\partial_\alpha \bar f}{z_\alpha} - 2[\bar f, \mathfrak{H}]\frac{\partial_\alpha\bar p}{z_\alpha} - 2[\bar p, \mathfrak{H}]\frac{\partial_\alpha \bar p}{z_\alpha} - 4p_t. \tag{2.359}$$

So

$$\begin{aligned}\partial_t g_d =&-2[\bar p, \mathfrak{H}]\frac{\partial_{\alpha t}\bar f}{z_\alpha} - \frac{2}{\pi i}\int\Big(\frac{\bar p(\alpha,t)-\bar p(\beta,t)}{z(\alpha,t)-z(\beta,t))}\Big)_t\partial_\beta\bar f(\beta,t)d\beta\\ &-2[\bar f, \mathfrak{H}]\frac{\partial_\alpha\bar p_t}{z_\alpha} - \frac{2}{\pi i}\int\Big(\frac{\bar f(\alpha,t)-\bar f(\beta,t)}{z(\alpha,t)-z(\beta,t)}\Big)_t\partial_\beta\bar p(\beta,t)d\beta\\ &-2[\bar p, \mathfrak{H}]\frac{\partial_\alpha\bar p_t}{z_\alpha} - \frac{2}{\pi i}\int\Big(\frac{\bar p(\alpha,t)-\bar p(\beta,t)}{z(\alpha,t)-z(\beta,t)}\Big)_t\partial_\beta\bar p(\beta,t)d\beta\\ &-4p_{tt}.\end{aligned}$$

So we have

$$
\begin{aligned}
D_tG_d = &-2[\bar{q},\mathcal{H}]\frac{\partial_\alpha D_t\bar{\mathfrak{F}}}{\zeta_\alpha}-\frac{2}{\pi i}\int\Big(D_t\frac{\bar{q}(\alpha,t)-\bar{q}(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)\partial_\beta\bar{\mathfrak{F}}(\beta,t)d\beta\\
&-2[\bar{\mathfrak{F}},\mathcal{H}]\frac{\partial_\alpha D_t\bar{q}}{\zeta_\alpha}-\frac{2}{\pi i}\int\Big(D_t\frac{\bar{\mathfrak{F}}(\alpha,t)-\bar{\mathfrak{F}}(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)\partial_\beta\bar{q}(\beta,t)d\beta\\
&-2[\bar{q},\mathcal{H}]\frac{\partial_\alpha D_t\bar{q}}{\zeta_\alpha}-\frac{2}{\pi i}\int\Big(D_t\frac{\bar{q}(\alpha,t)-\bar{q}(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)\partial_\beta\bar{q}(\beta,t)d\beta\\
&-4D_t^2q.
\end{aligned}
\tag{2.360}
$$

D_tG_c is cubic, we have

$$
\begin{aligned}
\|D_tG_c\|_{H^k}\leqslant &C_s(\|D_t\mathfrak{F}\|_{H^k}\|\zeta_\alpha-1\|_{H^k}\|\mathfrak{F}\|_{H^k}+\|\mathfrak{F}\|_{H^k}^2\|D_t\zeta\|_{H^k}+\|D_t\zeta\|_{H^k}\left\|D_t^2\zeta\right\|_{H^k}\|\zeta_\alpha-1\|_{H^k}\\
&+\|D_t\zeta\|_{H^k}^3\|\zeta_\alpha-1\|_{H^k}+\|D_t\zeta\|_{H^k}^3)\\
\leqslant &C\epsilon^3.
\end{aligned}
$$

D_tG_d consists of cubic terms or terms with rapid time decay. By (2.335), we have

$$
\|D_tq\|_{H^k}\leqslant C\epsilon^2 d_I(t)^{-5/2}+K_s^{-1}\epsilon d_I(t)^{-5/2}. \tag{2.361}
$$

Note that $D_tG_d+4D_t^2q$ is at least quadratic. Use lemma II.9, lemma II.34, and similar to the estimate of $\|G_{d1}\|_{H^s}$ in §2.5.8.6, we obtain

$$
\left\|D_tG_d+4D_t^2q\right\|_{H^k}\leqslant C\epsilon^3+K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.362}
$$

Note that

$$
D_t^2q=\sum_{j=1}^{2}\frac{\lambda_j i}{2\pi}\frac{D_t^2\zeta-\ddot{z}_j(t)}{(\zeta(\alpha,t)-z_j(t))^2}-\sum_{j=1}^{2}\frac{\lambda_j i}{\pi}\frac{(D_t\zeta)^2-2D_t\zeta\dot{z}_j}{(\zeta(\alpha,t)-z_j(t))^3}-\sum_{j=1}^{2}\frac{\lambda_j i}{\pi}\frac{\dot{z}_j^2}{(\zeta(\alpha,t)-z_j(t))^3}. \tag{2.363}
$$

Use lemma II.42, lemma II.43, lemma II.44, we have

$$\left\|4D_t^2 q\right\|_{H^k} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-3/2}. \tag{2.364}$$

Then we have

$$\|D_t G\|_{H^k} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-3/2}. \tag{2.365}$$

Therefore,

$$\|(I-\mathcal{H})D_t G\|_{H^k} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-3/2}. \tag{2.366}$$

2.5.9.2 Estimate $\left\|2[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t^2(I-\mathcal{H})(\zeta-\bar{\zeta})}{\zeta_\alpha}\right\|_{H^k}$

The way that we estimate for this quantity is the same as that for $[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}$. We obtain

$$\left\|2[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t^2(I-\mathcal{H})(\zeta-\bar{\zeta})}{\zeta_\alpha}\right\|_{H^k} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.367}$$

So we obtain

$$\left\|(I-\mathcal{H})\partial_\alpha^k\tilde{G}\right\|_{L^2} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-3/2}. \tag{2.368}$$

2.5.9.3 Estimate $[D_t^2 - iA\partial_\alpha,\mathcal{H}]\partial_\alpha^k\tilde{\sigma}$

Use

$$\begin{aligned}[D_t^2 - iA\partial_\alpha,\mathcal{H}]\partial_\alpha^k\tilde{\sigma} =& 2[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\partial_\alpha^k\tilde{\sigma}}{\zeta_\alpha} - \frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2\partial_\beta^{k+1}\tilde{\sigma}(\beta,t)d\beta\\ :=& I_1 + I_2.\end{aligned}$$

Clearly,

$$\|I_2\|_{L^2} \leqslant C\|D_t\zeta\|_{H^k}^2\|\tilde{\sigma}\|_{H^k} \leqslant C\epsilon^3. \tag{2.369}$$

Note that

$$[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha D_t\partial_\alpha^k\tilde{\sigma}}{\zeta_\alpha} = [D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\partial_\alpha^k D_t\tilde{\sigma}}{\zeta_\alpha} + [D_t\zeta,\mathcal{H}]\frac{\partial_\alpha[D_t,\partial_\alpha^k]\tilde{\sigma}}{\zeta_\alpha}$$

The second term $[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha[D_t,\partial_\alpha^k]\tilde{\sigma}}{\zeta_\alpha}$ is cubic, it's easy to obtain

$$\left\|[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha[D_t,\partial_\alpha^k]\tilde{\sigma}}{\zeta_\alpha}\right\|_{L^2} \leqslant C\epsilon^3. \tag{2.370}$$

The way that we estimate for this quantity is the same as that for $[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\partial_\alpha^k\tilde{\theta}}{\zeta_\alpha}$. We obtain

$$\left\|[D_t\zeta,\mathcal{H}]\frac{\partial_\alpha\partial_\alpha^k D_t\tilde{\sigma}}{\zeta_\alpha}\right\|_{H^k} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.371}$$

So we obtain

$$\left\|[D_t^2 - iA\partial_\alpha,\mathcal{H}]\partial_\alpha^k\tilde{\sigma}\right\|_{L^2} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.372}$$

2.5.9.4 Estimate $\left\|(I-\mathcal{H})[D_t^2 - iA\partial_\alpha,\partial_\alpha^k]\tilde{\sigma}\right\|_{L^2}$

The way that we estimate this quantity is the same as that for $\left\|(I-\mathcal{H})[D_t^2 - iA\partial_\alpha,\partial_\alpha^k]\tilde{\theta})\right\|_{L^2}$. We obtain

$$\left\|(I-\mathcal{H})[D_t^2 - iA\partial_\alpha,\partial_\alpha^k]\tilde{\sigma}\right\|_{L^2} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2}. \tag{2.373}$$

2.5.9.5 Estimate for $\left\|G_k^\sigma\right\|_{L^2}$.

Collect the estimates from (2.366), (2.368), (2.372), (2.373), we obtain

$$\|G_k^\sigma\|_{L^2} \leqslant C\epsilon^3 + K_s^{-1}\epsilon^2 d_I(t)^{-3/2} + K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-3/2}. \tag{2.374}$$

2.5.10 A priori energy estimates

We derive energy estimates in this subsection. We'll prove the following.

Proposition II.50. *Assume the assumptions of Theorem I.10, assume the bootstrap assumption (2.206), we have for all* $t \in [0, T_0]$,

$$\frac{d}{dt}\mathcal{E}_s(t) \leqslant C\epsilon^4 + K_s^{-1}\epsilon^3 d_I(t)^{-3/2} + K_s^{-1}\epsilon^2 \frac{|\lambda|}{x(0)} d_I(t)^{-5/2}. \tag{2.375}$$

Proof. From (2.202) and (2.204), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_s(t) &= \frac{d}{dt}\sum_{k=0}^{s}(E_k^\theta + E_k^\sigma) \\ &= \sum_{k=0}^{s}\Big(\int \frac{2}{A} Re D_t\theta_k \overline{G_k^\theta} - \int \frac{1}{A}\frac{a_t}{a}\circ\kappa^{-1}|D_t\theta_k|^2 + \int \frac{2}{A} Re D_t\sigma_k \overline{G_k^\sigma} - \int \frac{1}{A}\frac{a_t}{a}\circ\kappa^{-1}|D_t\sigma_k|^2\Big) \end{aligned} \tag{2.376}$$

By Corollary II.48, we have

$$\|D_t\tilde{\theta}\|_{H^s} \leqslant 11\epsilon, \qquad \|D_t\tilde{\sigma}\|_{H^s} \leqslant 21\epsilon. \tag{2.377}$$

By Corollary II.49, (2.322), (2.357), (2.374), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_s(t) \leqslant & \sum_{k=0}^{s}\Big(2\Big\|\frac{1}{A}\Big\|_{\infty}\|D_t\theta_k\|_{L^2}\Big\|G_k^{\theta}\Big\|_{L^2}+\Big\|\frac{1}{A}\Big\|_{\infty}\Big\|\frac{a_t}{a}\circ\kappa^{-1}\Big\|_{\infty}\|D_t\theta_k\|_{L^2}^2\\ & +2\Big\|\frac{1}{A}\Big\|_{\infty}\|D_t\sigma_k\|_{L^2}\|G_k^{\sigma}\|_{L^2}+\Big\|\frac{1}{A}\Big\|_{\infty}\Big\|\frac{a_t}{a}\circ\kappa^{-1}\Big\|_{\infty}\|D_t\sigma_k\|_{L^2}^2\Big)\\ \leqslant & \sum_{k=0}^{s}\Big(4\times 11\epsilon(C\epsilon^3+K_s^{-1}\epsilon^2 d_I(t)^{-3/2}+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2})\\ & +4\times(C\epsilon^2+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2})\times(11\epsilon)^2\\ & +4\times 21\epsilon(C\epsilon^3+K_s^{-1}\epsilon^2 d_I(t)^{-3/2}+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-3/2})\\ & +4\times(C\epsilon^2+K_s^{-1}\epsilon\frac{|\lambda|}{x(0)}d_I(t)^{-5/2})\times(21\epsilon)^2\Big)\\ \leqslant & C\epsilon^4+K_s^{-1}\epsilon^3 d_I(t)^{-3/2}+K_s^{-1}\epsilon^2\frac{|\lambda|}{x(0)}d_I(t)^{-5/2}.\end{aligned}$$

Here, we simply bound $\|\frac{1}{A}\|_{\infty}$ by $\frac{1}{2}$. □

Before we use the bootstrap argument to complete the proof of Theorem I.10, we need to show that the energy $\mathcal{E}_s$ is equivalent to $4(\Big\|D_t\tilde{\theta}\Big\|_{H^s}^2+\|D_t\tilde{\sigma}\|_{H^s}^2+\Big\||D|^{1/2}\tilde{\theta}\Big\|_{H^s}^2+\Big\||D|^{1/2}\tilde{\sigma}\Big\|_{H^s}^2)$.

Lemma II.51. *Assume the assumptions of Theorem I.10, and assume the bootstrap assumption (2.206). Then we have*

$$\Big|\mathcal{E}_s-4(\Big\|D_t\tilde{\theta}\Big\|_{H^s}^2+\|D_t\tilde{\sigma}\|_{H^s}^2+\Big\||D|^{1/2}\tilde{\theta}\Big\|_{H^s}^2+\Big\||D|^{1/2}\tilde{\sigma}\Big\|_{H^s}^2)\Big|\leqslant C\epsilon^3. \tag{2.378}$$

Proof. Recall that

$$\mathcal{E}_s=\sum_{k=0}^{s}\Big\{\int\frac{1}{A}|D_t\theta_k|^2+i\theta_k\overline{\partial_\alpha\theta_k}d\alpha+\int\frac{1}{A}|D_t\sigma_k|^2+i\sigma_k\overline{\partial_\alpha\sigma_k}d\alpha\Big\}, \tag{2.379}$$

where

$$\theta_k=(I-\mathcal{H})\partial_\alpha^k\tilde{\theta},\quad \sigma_k=(I-\mathcal{H})\partial_\alpha^k\tilde{\sigma},\quad \tilde{\theta}:=(I-\mathcal{H})(\zeta-\bar{\zeta}),\quad \tilde{\sigma}:=(I-\mathcal{H})D_t\tilde{\theta}. \tag{2.380}$$

It's easy to obtain that

$$\|A-1\|_{H^s} \leqslant C\epsilon. \tag{2.381}$$

So

$$\mathcal{E}_s = \sum_{k=0}^{s} \Big\{ \int |D_t\theta_k|^2 + i\theta_k \overline{\partial_\alpha \theta_k} d\alpha + \int |D_t\sigma_k|^2 + i\sigma_k \overline{\partial_\alpha \sigma_k} d\alpha \Big\} + O(\epsilon^3). \tag{2.382}$$

We have

$$\theta_k = \partial_\alpha^k (I-\mathcal{H})\tilde{\theta} + [\partial_\alpha^k, \mathcal{H}]\tilde{\theta} = 2\partial_\alpha^k \tilde{\theta} + [\partial_\alpha^k, \mathcal{H}]\tilde{\theta}. \tag{2.383}$$

So we have

$$\Big\| D_t\theta_k - 2\partial_\alpha^k D_t\tilde{\theta} \Big\|_{L^2} \leqslant \Big\| D_t[\partial_\alpha^k, \mathcal{H}]\tilde{\theta} \Big\|_{L^2} + 2\Big\| [D_t, \partial_\alpha^k]\tilde{\theta} \Big\|_{L^2} \leqslant C\epsilon^2. \tag{2.384}$$

Similarly, we have

$$\Big\| D_t\sigma_k - 2\partial_\alpha^k D_t\tilde{\sigma} \Big\|_{L^2} \leqslant C\epsilon^2. \tag{2.385}$$

Therefore,

$$\Big| \sum_{k=0}^{s} \int (|D_t\theta_k|^2 + |D_t\sigma_k|^2) d\alpha - 4(\Big\| \partial_\alpha^k D_t\tilde{\theta} \Big\|_{L^2}^2 + \Big\| \partial_\alpha^k D_t\tilde{\sigma} \Big\|_{L^2}^2) \Big| \leqslant C\epsilon^3. \tag{2.386}$$

Decompose $\tilde{\theta}$ as

$$\tilde{\theta} = \frac{1}{2}(I+\mathbb{H})\tilde{\theta} + \frac{1}{2}(I-\mathbb{H})\tilde{\theta} \tag{2.387}$$

Note that since $\tilde{\theta} = (I-\mathcal{H})(\zeta - \bar{\zeta})$, it's easy to obtain

$$\Big\| |D|^{1/2} \frac{1}{2}(I+\mathbb{H})\tilde{\theta} \Big\|_{H^s} \leqslant C\epsilon^2. \tag{2.388}$$

Then we have

$$\Big| \Big\| |D|^{1/2}\tilde{\theta} \Big\|_{H^s}^2 - \Big\| |D|^{1/2} \frac{1}{2}(I-\mathbb{H})\tilde{\theta} \Big\|_{H^s}^2 \Big| \leqslant C\epsilon^3. \tag{2.389}$$

Note that

$$\left\| |D|^{1/2}\frac{1}{2}(I-\mathbb{H})\tilde{\theta}\right\|_{H^s}^2 = i\sum_{k=0}^{s}\int \partial_\alpha^k\frac{1}{2}(I-\mathbb{H})\tilde{\theta}\overline{\partial_\alpha^{k+1}\frac{1}{2}(I-\mathbb{H})\tilde{\theta}}d\alpha \tag{2.390}$$

Use the fact that

$$(I-\mathbb{H})\tilde{\theta} = 2\tilde{\theta} + (\mathcal{H}-\mathbb{H})\tilde{\theta}, \tag{2.391}$$

and use

$$\left\|\partial_\alpha^k |D|^{1/2}(\mathcal{H}-\mathbb{H})\tilde{\theta}\right\|_{L^2} \leqslant C\epsilon^2, \tag{2.392}$$

we obtain

$$\Big|\int i\theta_k\overline{\partial_\alpha\theta_k}d\alpha - 4\|\partial_\alpha^k|D|^{1/2}\tilde{\theta}\|_{L^2}^2\Big| \leqslant C\epsilon^3. \tag{2.393}$$

Similarly,

$$\Big|\int i\sigma_k\overline{\partial_\alpha\sigma_k}d\alpha - 4\left\|\partial_\alpha^k|D|^{1/2}\tilde{\sigma}\right\|_{L^2}^2\Big| \leqslant C\epsilon^3. \tag{2.394}$$

By (2.384), (2.385), (2.393), and (2.394), we obtain

$$\Big|\mathcal{E}_s - 4\sum_{k=0}^{s}\Big\{\left\|\partial_\alpha^k D_t\tilde{\theta}\right\|_{L^2}^2 + \left\|\partial_\alpha^k D_t\tilde{\sigma}\right\|_{L^2}^2 + \left\|\partial_\alpha^k|D|^{1/2}\tilde{\theta}\right\|_{L^2}^2 + \left\|\partial_\alpha^k|D|^{1/2}\tilde{\sigma}\right\|_{L^2}^2\Big\}\Big| \leqslant C\epsilon^3. \tag{2.395}$$

□

Corollary II.52. *Assume the assumptions of Theorem I.10, then*

$$\mathcal{E}_s(0) \leqslant 17\epsilon^2. \tag{2.396}$$

Proposition II.53. *Assume the assumptions of Theorem I.10, there exists $\delta > 0$ such that*

$$\|\zeta_\alpha - 1\|_{H^s} \leqslant 5\epsilon, \qquad \|\mathfrak{F}\|_{H^{s+1/2}} \leqslant 5\epsilon, \qquad \|D_t\mathfrak{F}\|_{H^s} \leqslant 5\epsilon \qquad t \in [0, \delta\epsilon^{-2}] \tag{2.397}$$

Indeed, we can choose δ to be an absolute constant.

Proof. Let $\delta > 0$ to be determined. Let

$$\mathcal{T} := \Big\{ T \in [0, \delta\epsilon^{-2}] : \quad \|\zeta_\alpha - 1\|_{H^s} \leqslant 5\epsilon,\ \|\mathfrak{F}\|_{H^{s+1/2}} \leqslant 5\epsilon,\ \|D_t\mathfrak{F}\| H^s \leqslant 5\epsilon, \quad \forall\ t \in [0, T] \Big\} \tag{2.398}$$

At $t = 0$, we have

$$\|\mathfrak{F}\|_{H^{s+1/2}} + \|D_t\mathfrak{F}\| H^s \leqslant \frac{3}{2}\epsilon. \tag{2.399}$$

To obtain estimate of $\|\zeta_\alpha - 1\|_{H^s}$, use $D_t^2\zeta - iA\zeta_\alpha = -i$, we have

$$\zeta_\alpha - 1 = \frac{D_t^2\zeta - i(A-1)}{iA}. \tag{2.400}$$

We have $D_t^2\zeta = D_t\bar{\mathfrak{F}} + D_t\bar{q}$, and

$$D_t q = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{D_t\zeta - \dot{z}_j}{(\zeta(\alpha, t) - z_j(t))^2}.$$

We have

$$\|D_t q\|_{H^s} \leqslant C\epsilon^2 + K_s^{-1}\epsilon. \tag{2.401}$$

Use (2.324), we obtain

$$\|\zeta_\alpha(\cdot, 0) - 1\|_{H^s} \leqslant \|D_t\mathfrak{F}(\cdot, 0)\|_{H^s} + C\epsilon^2 + K_s^{-1}\epsilon \leqslant 2\epsilon. \tag{2.402}$$

Therefore, $0 \in \mathcal{T}$, so $\mathcal{T} \neq \emptyset$. Since $\|\zeta_\alpha - 1\|_{H^s}, \|\mathfrak{F}\|_{H^{s+1/2}}, \|D_t\mathfrak{F}\|_{H^s}$ are continuous in t, we have $\mathcal{T}$ is closed. To prove $\mathcal{T} = [0, \delta\epsilon^{-2}]$, it suffices to prove that if $T_0 < \delta\epsilon^{-2}$, then there exists $c > 0$ such that $[0, T_0 + c) \subset \mathcal{T}$.

Assume $T_0 \in \mathcal{T}$ and assume $T_0 < \delta\epsilon^{-2}$. By Proposition II.50, we have for any $t_0 \leqslant T_0$,

$$\begin{aligned}\mathcal{E}_s(t_0) =&\mathcal{E}_s(0) + \int_0^{t_0} \frac{d}{dt}\mathcal{E}_s(t)dt\\ \leqslant&17\epsilon^2 + \int_0^{t_0}(C\epsilon^4 + K_s^{-1}\epsilon^3 d_I(t)^{-3/2} + K_s^{-1}\epsilon^2\frac{|\lambda|}{x(0)}d_I(t)^{-5/2})dt\\ \leqslant&17\epsilon^2 + C\epsilon^4 T_0 + K_s^{-1}\epsilon^3\int_0^{t_0}((1+\frac{|\lambda|}{20\pi x(0)}t)^{-1})^{3/2}dt\\ &+ K_s^{-1}\epsilon^2\frac{|\lambda|}{x(0)}\int_0^{t_0}((1+\frac{|\lambda|}{20\pi x(0)}t)^{-1})^{3/2}dt\\ \leqslant&17\epsilon^2 + C\epsilon^4 T_0 + K_s^{-1}\epsilon^3\frac{x(0)}{|\lambda|} + K_s^{-1}\epsilon^2.\end{aligned}$$

Since $\frac{|\lambda|}{x(0)} \geqslant M\epsilon$, we have

$$K_s^{-1}\epsilon^3\frac{x(0)}{|\lambda|} \leqslant K_s^{-1}\epsilon^3(M)^{-1}\epsilon^{-1} = K_s^{-1}M^{-1}\epsilon^2 \leqslant \frac{1}{2}\epsilon^2.$$

Since $T_0 \leqslant \delta\epsilon^{-2}$, if we choose $\delta \leqslant C^{-1}$, then

$$C\epsilon^4 T_0 \leqslant \epsilon^2.$$

Therefore we have

$$\sup_{t\in[0,T_0]}\mathcal{E}_s(t) \leqslant 19\epsilon^2.$$

By lemma II.51, we obtain

$$4\sum_{k=0}^{s}\left\{\|\partial_\alpha^k D_t\tilde{\theta}\|^2 + \|\partial_\alpha^k D_t\tilde{\sigma}\|^2 + \|\partial_\alpha^k |D|^{1/2}\tilde{\theta}\|_{L^2}^2 + \|\partial_\alpha^k |D|^{1/2}\tilde{\sigma}\|_{L^2}^2\right\} \leqslant \mathcal{E}_s + C\epsilon^3 \leqslant 20\epsilon^2. \quad (2.403)$$

So we have

$$\|D_t\tilde{\theta}\|_{H^{s+1/2}} + \|D_t\tilde{\sigma}\|_{H^s} + \||D|^{1/2}\tilde{\theta}\|_{H^s} \leqslant 5\epsilon, \quad (2.404)$$

By lemma II.47, we obtain

$$\|\mathfrak{F}\|_{H^{s+1/2}} \leqslant K_s^{-1}\epsilon + \frac{1}{2}\|D_t\tilde{\theta}\|_{H^{s+1/2}} \leqslant 3\epsilon. \tag{2.405}$$

$$\|D_t\mathfrak{F}\|_{H^s} \leqslant \frac{1}{4}\|D_t\tilde{\sigma}\|_{H^s} + K_s^{-1}\epsilon \leqslant 2\epsilon. \tag{2.406}$$

Since $\bar{\zeta} - \alpha$ is holomorphic, we have

$$\tilde{\theta} = (I - \mathcal{H})(\bar{\zeta} - \zeta) = (I - \mathcal{H})(\zeta - \alpha) = 2(\zeta - \alpha) - (\mathcal{H} + \bar{\mathcal{H}})(\zeta - \alpha). \tag{2.407}$$

It's easy to obtain

$$\left\| |D|^{1/2}(\left(\tilde{\theta} - 2(\zeta - \alpha)\right)\right\|_{H^s} = \||D|^{1/2}(\mathcal{H} + \bar{\mathcal{H}})(\zeta - \alpha)\|_{H^s} \leqslant C\epsilon^2. \tag{2.408}$$

By (2.404), we obtain

$$2\||D|^{1/2}(\zeta - \alpha)\|_{H^s} \leqslant \||D|^{1/2}\tilde{\theta}\|_{H^s} + C\epsilon^2 \leqslant 6\epsilon. \tag{2.409}$$

So we have

$$\left\| |D|^{1/2}(\zeta - \alpha)\right\|_{H^s} \leqslant 3\epsilon. \tag{2.410}$$

To obtain control of $\|\zeta_\alpha - 1\|_{H^s}$, again we use

$$\zeta_\alpha - 1 = \frac{D_t^2\zeta - i(A-1)}{iA}. \tag{2.411}$$

It's easy to obtain

$$\|\zeta_\alpha - 1\|_{H^s} \leqslant \|D_t\mathfrak{F}\|_{H^s} + C\epsilon^2 + K_s^{-1}\epsilon \leqslant 3\epsilon. \tag{2.412}$$

By continuity, we can choose $c > 0$ sufficiently small such that

$$\|\zeta_\alpha - 1\|_{H^s} \leqslant 5\epsilon,\ \|\mathfrak{F}\|_{H^{s+1/2}} \leqslant 5\epsilon,\ \|D_t\mathfrak{F}\|H^s \leqslant 5\epsilon, \quad \forall\ t \in [0, T_0 + c) \tag{2.413}$$

So we must have $\mathcal{T} = [0, \delta\epsilon^{-2}]$, for some absolute constant $\delta > 0$. □

2.5.11 Change of variables back to lagrangian coordinates

Next, we need to change of variables back to system (1.13). So we need to control κ on time interval $[0, \delta\epsilon^{-2}]$. We have

$$\kappa_t = b \circ \kappa \tag{2.414}$$

So we have

$$\partial_t \kappa_\alpha = b_\alpha \circ \kappa \kappa_\alpha. \tag{2.415}$$

Recall that

$$(I - \mathcal{H})b = -[D_t\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - \frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j}{\zeta(\alpha, t) - z_j(t)}. \tag{2.416}$$

So we have

$$(I - \mathcal{H})b_\alpha = [\zeta_\alpha, \mathcal{H}]b - \partial_\alpha[D_t\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} + \frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j\zeta_\alpha}{(\zeta(\alpha, t) - z_j(t))^2}. \tag{2.417}$$

Clearly,

$$\left\|[\zeta_\alpha, \mathcal{H}]b - \partial_\alpha[D_t\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}\right\|_{H^1} \leqslant C\epsilon^2, \tag{2.418}$$

and

$$\left\|\frac{i}{\pi}\sum_{j=1}^{2}\frac{\lambda_j\zeta_\alpha}{(\zeta(\alpha, t) - z_j(t))^2}\right\|_{H^1} \leqslant K_s^{-1}d_I(t)^{-5/2}\epsilon, \tag{2.419}$$

for some absolute constant $C > 0$. By lemma II.33, we have

$$\|b_\alpha\|_{H^1} \leqslant C\epsilon^2 + K_s^{-1}d_I(t)^{-5/2}\epsilon. \tag{2.420}$$

By Sobolev embedding, we have

$$\|b_\alpha \circ \kappa\|_\infty = \|b_\alpha\|_\infty \leqslant \|b_\alpha\|_{H^1} \leqslant C\epsilon^2 + K_s^{-1}d_I(t)^{-5/2}\epsilon. \tag{2.421}$$

It's easy to obtain that

$$\|\kappa_\alpha(\cdot,0)-1\|_\infty \leqslant C\epsilon. \tag{2.422}$$

So we obtain

$$\kappa_\alpha(\alpha,t)-\kappa_\alpha(\alpha,0) = \int_0^t \kappa_{\alpha\tau}(\alpha,\tau)d\tau \tag{2.423}$$

$$= \int_0^t b_\alpha \circ \kappa(\alpha,\tau)\kappa_\alpha(\alpha,\tau)d\tau. \tag{2.424}$$

Let $\delta_1 > 0$ be a constant to be determined.

$$\mathcal{T}_1 := \Big\{T \in [0,\delta_1\epsilon^{-2}] : \sup_{t\in[0,T]} \|\kappa_\alpha(\cdot,t)-\kappa_\alpha(\cdot,0)\|_\infty \leqslant \frac{1}{10}\Big\} \tag{2.425}$$

In particular, if $t \in \mathcal{T}_1$, then for ϵ sufficiently small, we have $\frac{4}{5} \leqslant \kappa_\alpha \leqslant \frac{6}{5}$. Also, $\mathcal{T}_1$ is closed. For $T \in \mathcal{T}_1$, we have for any $t \in [0,T]$,

$$\Big|\kappa_\alpha(\alpha,t)-\kappa_\alpha(\alpha,0)\Big| \leqslant \int_0^t \Big(C\epsilon^2 + K_s^{-1}d_I(\tau)^{-5/2}\epsilon\Big)d\tau \tag{2.426}$$

$$\leqslant \int_0^t (C\epsilon^2 + K_s^{-1}(1+\frac{|\lambda|}{20\pi x(0)})^{-5/2}\epsilon)d\tau \tag{2.427}$$

$$\leqslant C\epsilon^2 t + K_s^{-1}\frac{20\pi x(0)}{|\lambda|}\frac{2}{3}\epsilon \tag{2.428}$$

$$\leqslant C\epsilon^2 t + \frac{1}{15K_s}. \tag{2.429}$$

Here we've used the assumption $\frac{|\lambda|}{x(0)} \geqslant 200\pi\epsilon$. Choose $\delta_1 = \frac{1}{30C}$. Then we have

$$\sup_{t\in[0,T]} \|\kappa_\alpha(\cdot,t)-\kappa_\alpha(\cdot,0)\|_\infty \leqslant \frac{1}{20}. \tag{2.430}$$

Therefore, $\mathcal{T}_1$ is open in $[0,\delta_1\epsilon^{-2}]$, we must have $\mathcal{T}_1 = [0,\delta_1\epsilon^{-2}]$.

Let $\delta_0 := \min\{\delta,\delta_1\}$. Since $\kappa_\alpha \geqslant \frac{3}{5}$ on $[0,\delta_0\epsilon^{-2}]$, we can change of variables back to lagrangian coordinates and conclude the proof of Theorem I.10.

CHAPTER III

Justification of the Peregrine soliton from the full water waves

3.1 Notation and convention

Assume f a function on boundary of $\Omega(t)$. By saying f holomorpihc, we mean f is boundary value of a holomorhpic function in $\Omega(t)$. Let $h \in L^2_{loc}(\mathbb{R})$, if h is neither periodic nor vanishing at spatial infinity, then we say that h is nonvanishing.

We use $C(X_1, X_2, ..., X_k)$ to denote a positive constant C depends continuous on the parameters $X_1, ..., X_k$. Throughout this paper, such constant $C(X_1, ..., X_k)$ could be different even we use the same letter C. The commutator $[A, B] = AB - BA$. Given a function $g(\cdot, t) : \mathbb{R} \to \mathbb{R}$, the composition $f(\cdot, t) \circ g := f(g(\cdot, t), t)$. We identify the $\mathbb{R}^2$ with the complex plane. A point (x, y) is identified as $x + iy$. For a point $z = x + iy$, $\bar{z}$ represents the complex conjugate of z.

3.2 Preliminaries

In this section, we define the class of holomorphic functions that are considered in this paper, and define the function spaces, norms involved. Also, we collect the preliminary analytical tools such as double layer potential theory, commutator estimates and some basic identities.

3.2.1 Two classes of holomorphic functions

We define two classes of holomorphic functions.

(1) Bounded holomorphic functions which decays nontangentially,

(2) Periodic holomorphic functions which approaches 0 as $y \to -\infty$.

Periodic holomorphic functions are used to explore the periodic water waves system, while bounded holomorphic functions which decays non-tangentially is a good setting for water waves with initial data of the form X^s. For convenience, we introduce the following notation.

Definition III.1. Denote

$$\mathcal{H}ol_{\mathcal{N}}(\Omega(t)) : = \Big\{F(\cdot, t) : \Omega(t) \to \mathbb{C} \text{ bounded holomorphic, decays nontangentially in } \Omega(t).\Big\} \tag{3.1}$$

Denote

$$\begin{aligned}\mathcal{H}ol_{\mathcal{P}}(\Omega^0(t)) := \Big\{\phi(\cdot, t) :&\Omega^0(t) \to \mathbb{C} \ \text{ bounded, holomorphic, } 2\pi \text{ periodic in } \Omega^0(t),\\ &\lim_{\operatorname{Im} z \to -\infty} \phi(z, t) = 0\Big\}.\end{aligned} \tag{3.2}$$

Remark III.2. Let $f \in L^2_{loc}(\mathbb{R})$. If $f = \Phi \circ \zeta$ for some $\Phi \in \mathcal{H}ol_{\mathcal{N}}(\Omega(t))$, then we say $f \in \mathcal{H}ol_{\mathcal{N}}(\Omega(t))$.

3.2.2 Fourier transform

In this subsection we define the Fourier transform on $\mathbb{R}$ and on $\mathbb{T} := [-\pi, \pi]$.

Definition III.3. Let $f \in L^2(\mathbb{R})$, then we Fourier transform of f as

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Let $g \in L^2(\mathbb{T})$. Then define Fourier transform of f on $\mathbb{T}$, still denoted by $\widehat{g}$:

$$\widehat{g}(\xi) := \frac{1}{2\pi} \int_{\mathbb{T}} g(x) e^{-ix\xi} dx.$$

3.2.3 Function spaces

In this subsection, we define some function spaces that we'll use in this paper.

Definition III.4. (1) Let $s \geqslant 0$, we define

$$H^s(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} (1 + |2\pi\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty\},$$

and we define the norm $\|\cdot\|_{H^s}$ by

$$\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} (1 + |2\pi\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi.$$

(2) We define

$$H^s(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : \sum_{m \in \mathbb{Z}} (1 + m^2)^s |\widehat{f}(m)|^2 < \infty\},$$

and we define the norm

$$\|f\|_{H^s(\mathbb{T})}^2 := \sum_{m \in \mathbb{Z}} (1 + m^2)^s |\widehat{f}(m)|^2.$$

(3) Let $J = \mathbb{R}$ or $\mathbb{T}$. Without loss of generality, assume $s \geqslant 0$ is an integer. Define

$$W^{s,\infty}(J) := \{f \in L^\infty(J) : \sum_{m=0}^{s} \|\partial_\alpha^m f\|_{L^\infty(J)} < \infty\}.$$

Define the norm

$$\|f\|_{W^{s,\infty}(J)} := \sum_{m=0}^{s} \|\partial_\alpha^m f\|_{L^\infty(J)}.$$

We'll use the following Sobolev embedding a lot.

Lemma III.5. *(1) If $s > 1/2$, and $f \in H^s(\mathbb{R})$, then $f \in L^\infty$, and*

$$\|f\|_{L^\infty(\mathbb{R})} \leqslant C(s)\|f\|_{H^s(\mathbb{R})}.$$

(2) If $s > 1/2$, and $f \in H^s(\mathbb{T})$, then $f \in L^\infty$, and

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant C(s)\|f\|_{H^s(\mathbb{T})}.$$

Definition III.6. Let $s \geqslant 0$. Let $s_0 > 3/2$ be fixed. Define

$$X^s := \{f = f_0 + f_1 : f_1 \in H^s(\mathbb{R}), f_0 \in H^{s+s_0}(\mathbb{T})\}. \tag{3.3}$$

Associate X^s with the norm

$$\|f\|_{X^s} = \|f_0\|_{H^{s+s_0}(\mathbb{T})} + \|f_1\|_{H^s(\mathbb{R})}. \tag{3.4}$$

Lemma III.7. *Let $s \geqslant 0$. Then X^s is a Banach space.*

Remark III.8. Let $f \in X^s$. The decomposition $f = f_0 + f_1$ for $f_0 \in H^{s+s_0}(\mathbb{T})$ and $f_1 \in H^s(\mathbb{R})$ is unique.

3.2.4 Hilbert transform and double layer potential

Let $\zeta = \zeta(\alpha, t)$ be a chord-arc for every fixed time t, we denote the Hilbert transform associated with ζ by $\mathcal{H}_\zeta$, i.e.,

$$\mathcal{H}_\zeta f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{\zeta_\beta(\beta)}{\zeta(\alpha, t) - \zeta(\beta, t)} f(\beta) d\beta. \tag{3.5}$$

Remark III.9. We'll also use the notation $\mathcal{H}_\omega, \mathcal{H}_{\tilde{\zeta}}, \mathcal{H}_{\tilde{\omega}}$ in this paper, represents Hilbert transform associated with $\omega, \tilde{\zeta}, \tilde{\omega}$, respectively. We denote $\mathbb{H}$ the Hilbert transform associated with $\zeta(\alpha) = \alpha$.

The double layer potential operator $\mathcal{K}$ associated with ζ is given by

$$\mathcal{K}_\zeta f(\alpha) := p.v. \int_{-\infty}^{\infty} \mathrm{Re}\{\frac{1}{\pi i}\frac{\zeta_\beta}{\zeta(\alpha,t)-\zeta(\beta,t)}\} f(\beta)d\beta. \tag{3.6}$$

The adjoint of the double layer potential operator $\mathcal{K}_\zeta^*$ associated with ζ is defined by

$$\mathcal{K}_\zeta^* f(\alpha) := p.v. \int_{-\infty}^{\infty} \mathrm{Re}\{-\frac{1}{\pi i}\frac{\zeta_\alpha}{|\zeta_\alpha|}\frac{|\zeta_\beta|}{\zeta(\alpha,t)-\zeta(\beta,t)}\} f(\beta)d\beta. \tag{3.7}$$

For periodic functions, we use also the following version of Hilbert transform. Let $\Gamma(t)$ be a chord-arc, $\Gamma(t) = \{\omega(\alpha,t) : \alpha \in \mathbb{R}\}$, where $\omega - \alpha$ is periodic. Let $\Omega^0(t)$ be the region below the curve $\Gamma(t)$. Define the periodic Hilbert transform associated with Γ as

$$\mathcal{H}_p f(\alpha) := \frac{1}{2\pi i} p.v. \int_{\mathbb{T}} \omega_\beta(\beta)\cot(\frac{1}{2}(\omega(\alpha,t)-\omega(\beta,t))f(\beta)d\beta. \tag{3.8}$$

Remark III.10. Please note the difference between $\mathcal{H}_\omega$ ad $\mathcal{H}_p$.

The corresponding double layer potential operator $\mathcal{K}_p$ is given by

$$\mathcal{K}_p f(\alpha) := p.v. \int_{\mathbb{T}} \mathrm{Re}\{\frac{1}{2\pi i}\omega_\beta(\beta)\cot(\frac{1}{2}(\omega(\alpha,t)-\omega(\beta,t)))\} f(\beta)d\beta. \tag{3.9}$$

The corresponding adjoint $\mathcal{K}_p^*$ of $\mathcal{K}_p$ is given by

$$\mathcal{K}_p^* f(\alpha) := p.v. \int_{\mathbb{T}} \mathrm{Re}\{-\frac{1}{2\pi i}\frac{\omega_\alpha}{|\omega_\alpha|}|\omega_\beta(\beta)|\cot(\frac{1}{2}(\omega(\alpha,t)-\omega(\beta,t)))\} f(\beta)d\beta. \tag{3.10}$$

3.2.4.1 Characterization of holomorphic functions

For holomorphic functions which decay nontangentially, we have the following description.

Lemma III.11. *Let $f \in \mathcal{H}ol_{\mathcal{N}}(\Omega(t))$. Then $\mathcal{H}_\zeta f$ is defined and*

$$(I - \mathcal{H}_\zeta)f = 0.$$

Proof. This is a consequence of Cauchy's theorem. □

We have the following well-known characterization of periodic holomorphic functions.

Lemma III.12. *Assume $f \in L^2(\mathbb{T})$. Then $f \in \mathcal{H}ol_{\mathcal{P}}(\Omega^0(t))$ if and only if*

$$(I - \mathcal{H}_p)f = 0.$$

We'll use the following boundedness of Hilbert transform and double layer potential operators. Suppose that ζ, ω exist on $[0, T_0]$ for some constant $T_0 > 0$, and satisfy the following chord-arc condition: There exist constants $\alpha_0, \beta_0, \alpha_0', \beta_0'$ such that for all $t \in [0, T_0]$,

$$\alpha_0|\alpha - \beta| \leqslant |\zeta(\alpha, t) - \zeta(\beta, t)| \leqslant \beta_0|\alpha - \beta|, \tag{3.11}$$

and

$$\alpha_0'|\alpha - \beta| \leqslant |\omega(\alpha, t) - \omega(\beta, t)| \leqslant \beta_0'|\alpha - \beta|, \tag{3.12}$$

Lemma III.13. *Assume $\zeta(\alpha, t), \omega(\alpha, t)$ satisfy (3.11) and (3.12), respectively. Then there exist constants $C_1 = C_1(\alpha_0, \beta_0)$ and $C_2 = C_2(\alpha_0', \beta_0')$ such that*

$$\left\|\mathcal{H}_\zeta f\right\|_{L^2(\mathbb{R})} \leqslant C_1\|f\|_{L^2(\mathbb{R})}. \tag{3.13}$$

$$\left\|\mathcal{H}_p f\right\|_{L^2(\mathbb{T})} \leqslant C_2\|f\|_{L^2(\mathbb{T})}. \tag{3.14}$$

$$\left\|(I - \mathcal{K}_\zeta)^{-1} f\right\|_{L^2(\mathbb{R})} \leqslant C_1\|f\|_{L^2(\mathbb{R})}. \tag{3.15}$$

$$\left\|(I - \mathcal{K}_p)^{-1} f\right\|_{L^2(\mathbb{T})} \leqslant C_2\|f\|_{L^2(\mathbb{T})}. \tag{3.16}$$

$$\left\|(I - \mathcal{K}_\zeta^*)^{-1} f\right\|_{L^2(\mathbb{R})} \leqslant C_1\|f\|_{L^2(\mathbb{R})}. \tag{3.17}$$

$$||(I - \mathcal{K}_p^*)^{-1} f||_{L^2(\mathbb{T})} \leqslant C_2||f||_{L^2(\mathbb{T})}. \tag{3.18}$$

Proof. See for example Chapter 4 of [59] for the case on $L^2(\mathbb{R})$. The case on $L^2(\mathbb{T})$ can be proved in a similar way. □

Remark III.14. Because we consider smooth and small solution, indeed we have for real function f, for ω such that $\omega - \alpha$ small, an easy calculation gives

$$\left\|\mathcal{K}_p f\right\|_{H^s(\mathbb{T})} \leqslant C\epsilon \|f\|_{H^s(\mathbb{T})}.$$

From this, the boundedness of $(I - \mathcal{K}_p)^{-1}$ follows immediately.

3.2.5 Some basic identities

For convenience, we record a variant of Lemma II.12 here. The differences between Lemma II.12 and Lemma III.15 are: First, Lemma II.12 is in lagrangian coordinates, while Lemma III.15 is in another coordinates. Second, Lemma III.15 holds not only for functions vanishing at ∞, but also for functions in X^s. Nevertheless, the proof of the two lemmas are the same.

Lemma III.15. *Let $T_0 > 0$ be fixed. Assume $D_t\zeta, \zeta_\alpha - 1 \in C^1([0, T_0]; X^1)$, $f \in X^2$. We have*

$$[D_t, \mathcal{H}_\zeta]f = [D_t\zeta, \mathcal{H}_\zeta]\frac{\partial_\alpha f}{\zeta_\alpha} \tag{3.19}$$

$$[D_t^2, \mathcal{H}_\zeta]f = [D_t^2\zeta, \mathfrak{H}]\frac{f_\alpha}{\zeta_\alpha} + 2[D_t\zeta, \mathcal{H}_\zeta]\frac{\partial_\alpha D_t f}{\zeta_\alpha} \tag{3.20}$$

$$- \frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{D_t\zeta(\alpha,t) - D_t\zeta(\beta,t)}{\zeta(\alpha,t) - \zeta(\beta,t)}\right)^2 f_\beta(\beta,t)d\beta \tag{3.21}$$

$$[A\partial_\alpha, \mathcal{H}_\zeta]f = [A\zeta_\alpha, \mathcal{H}_\zeta]\frac{f_\alpha}{\zeta_\alpha}, \qquad \partial_\alpha\mathcal{H}_\zeta f = \zeta_\alpha\mathcal{H}_\zeta\frac{f_\alpha}{\zeta_\alpha} \tag{3.22}$$

$$[D_t^2 - iA\partial_\alpha, \mathcal{H}_\zeta]f = 2[D_t\zeta, \mathcal{H}_\zeta]\frac{\partial_\alpha D_t f}{\zeta_\alpha} - \frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{D_t\zeta(\alpha,t) - D_t\zeta(\beta,t)}{\zeta(\alpha,t) - \zeta(\beta,t)}\right)^2 f_\beta(\beta,t)d\beta \tag{3.23}$$

For proof, see [71].

We also need some commutator identities for periodic functions.

Lemma III.16. *Let $T_0 > 0$ be fixed. Assume that $f \in C^2_{t,x}([0,T_0]\times\mathbb{T})$. We have*

$$[\partial_\alpha, \mathcal{H}_p]f = [\omega_\alpha, \mathcal{H}_p]\frac{f_\alpha}{\omega_\alpha}. \tag{3.24}$$

$$[g\partial_\alpha, \mathcal{H}_p]f = [g\omega_\alpha, \mathcal{H}_p]\frac{f_\alpha}{\omega_\alpha}, \qquad \forall\ g \in L^\infty(\mathbb{T}). \tag{3.25}$$

$$[\partial_t, \mathcal{H}_p]f = [\omega_t, \mathcal{H}_p]\frac{f_\alpha}{\omega_\alpha}. \tag{3.26}$$

$$[D_t^0, \mathcal{H}_p]f = [D_t^0\omega, \mathcal{H}_p]\frac{f_\alpha}{\omega_\alpha}. \tag{3.27}$$

$$\begin{aligned}[(D_t^0)^2, \mathcal{H}_p]f =&[(D_t^0)^2\omega, \mathcal{H}_p]\frac{f_\alpha}{\omega_\alpha} + 2[D_t^0\omega, \mathcal{H}_p]\frac{\partial_\alpha D_t^0 f}{\omega_\alpha}\\ &- \frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha) - D_t^0\omega(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2 f_\beta d\beta.\end{aligned} \tag{3.28}$$

$$[(D_t^0)^2 - iA_0\partial_\alpha, \mathcal{H}_p]f =2[D_t^0\omega, \mathcal{H}_p]\frac{\partial_\alpha D_t^0 f}{\omega_\alpha} - \frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha) - D_t^0\omega(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2 f_\beta d\beta. \tag{3.29}$$

Proof. Note that

$$\begin{aligned}\mathcal{H}_p f(\alpha) =& -\frac{1}{\pi i}\int_{\mathbb{T}}\partial_\beta \log\sin(\frac{1}{2}(\omega(\alpha)-\omega(\beta)))f(\beta)d\beta\\ =&\frac{1}{\pi i}\int_{\mathbb{T}}\log\sin(\frac{1}{2}(\omega(\alpha)-\omega(\beta)))f_\beta(\beta)d\beta.\end{aligned}$$

Using this, we obtain (3.24). (3.25) is proved exactly the same way. (3.26) is prove similarly. (3.27) is a direct consequence of (3.25) and (3.26).

To prove (3.28), by changing of variable, it suffices to prove

$$\begin{aligned}&[\partial_t^2, \mathcal{H}_p]f\\ =&[\partial_t^2\omega, \mathcal{H}_p]\frac{f_\alpha}{\omega_\alpha} + 2[\omega_t, \mathcal{H}_p]\frac{f_{t\alpha}}{\omega_\alpha} - \frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{\omega_t(\alpha)-\omega_t(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2 f_\beta d\beta.\end{aligned} \tag{3.30}$$

(3.30) is a direct consequence of (3.26) and the following identity:

$$[\partial_t^2, \mathcal{H}_p]f = \partial_t[\partial_t, \mathcal{H}_p]f + [\partial_t, \mathcal{H}_p]\partial_t f.$$

□

Remark III.17. The identities in lemma III.15 and lemma III.16 hold true in BMO sense.

3.2.6 Basic commutator estimates

We'll need to use the commutator estimates which we've used in Chapter 2. For the reader's convenience, we record them in the following. Let $m \geqslant 1$ be an integer. Define

$$S_1(A, f) = \int_{\mathbb{R}} \prod_{j=1}^{m} \frac{A_j(\alpha) - A_j(\beta)}{\gamma_j(\alpha) - \gamma_j(\beta)} \frac{f(\beta)}{\gamma_0(\alpha) - \gamma_0(\beta)} d\beta. \tag{3.31}$$

$$S_2(A, f) = \int_{\mathbb{R}} \prod_{j=1}^{m} \frac{A_j(\alpha) - A_j(\beta)}{\gamma_j(\alpha) - \gamma_j(\beta)} f_\beta(\beta) d\beta. \tag{3.32}$$

We have the following comutator estimates, which can be found in [63], [71].

Proposition III.18. *(1) Assume each γ_j satisfies the chord-arc condition*

$$C_{0,j}|\alpha - \beta| \leqslant |\gamma_j(\alpha) - \gamma_j(\beta)| \leqslant C_{1,j}|\alpha - \beta|, \tag{3.33}$$

where $C_{0,j}, C_{1,j}$ are positive constants and $C_{0,j} \leqslant C_{1,j}, 1 \leqslant j \leqslant m$. Then both $\|S_1(A, f)\|_{L^2}$ and $\|S_2(A, f)\|_{L^2}$ are bounded by

$$C \prod_{j=1}^{m} \left\| A_j' \right\|_{X_j} \|f\|_{X_0},$$

where one of the $X_0, X_1, ...X_m$ is equal to L^2 and the rest are L^∞. The constant C depends on $\left\| \gamma_j' \right\|_{L^\infty}^{-1}, j = 1, .., m$.

(2) Let $s \geqslant 3$ be given, and suppose chord-arc condition (3.33) holds for each γ_j, assume $\gamma_j - 1 \in H^{s-1}$.then

$$\|S_2(A,f)\|_{H^s} \leqslant C \prod_{j=1}^{m} \left\| A_j' \right\|_{Y_j} \|f\|_Z ,$$

where for all $j = 1, ..., m$, $Y_j = H^{s-1}$ or $W^{s-2,\infty}$ and $Z = H^s$ or $W^{s-1,\infty}$. The constant C depends on $\left\| \gamma_j' - 1 \right\|_{H^{s-1}}, j = 1, ..., m$.

Let $m \geqslant 1$ be integer. Define

$$\tilde{S}_1(f,g) := [g, \mathcal{H}_p] \frac{f_\alpha}{\omega_\alpha}. \tag{3.34}$$

$$\tilde{S}_2(A,f) = \int_{\mathbb{T}} \prod_{j=1}^{m} \frac{A_j(\alpha) - A_j(\beta)}{\sin(\frac{1}{2}(\omega(\alpha,t) - \omega(\beta,t))} f_\beta(\beta) d\beta. \tag{3.35}$$

We have

Proposition III.19. *Assume ω satisfies the chord-arc condition (3.12). Then*

$$\left\| \tilde{S}_1(f,g) \right\|_{H^s(\mathbb{T})} \leqslant C \|f\|_{H^s(\mathbb{T})} \|g\|_{H^s(\mathbb{T})} , \tag{3.36}$$

$$\left\| \tilde{S}_2(A,f) \right\|_{H^s} \leqslant C \prod_{j=1}^{m} \left\| A_j' \right\|_{H^{s-1}(\mathbb{T})} \|f\|_{H^s} , \tag{3.37}$$

where the constant C depends on $\|\omega_\alpha\|_{H^{s-1}(\mathbb{T})}, j = 1, ..., m$.

Proof. This can be derived from Proposition III.18. □

3.3 Water wave system in periodic setting

In this section, we use S. Wu's method (see ([71], [72]) to give a sketch of proof of long time existence of water wave system in periodic setting. Long time existence of periodic water waves is not new, the methods in [71][39][1][32] all imply cubic lifespan for 2d gravity water waves with small initial data. We use S. Wu's method to sketch the proof here because we need to bound the quantities such as $b_0, A_0, \omega-\alpha, D_t^0\omega, (D_t^0)^2\omega$ on time scale $O(\epsilon^{-2})$ in later sections, and also because we need to use this method to prove the remainder term r_0 remains small for sufficiently long times. Solution of this periodic water waves system has the same boundary values at spatial infinity as a water waves system whose initial data is assumed to be in X^s.

3.3.1 Notation

Recall that $D_t^0=\partial_t+b_0\partial_\alpha$, for some function b_0. $D_t=\partial_t+b\partial_\alpha$. $\Omega^0(t)$ be the region bounded above by the graph ω.

3.3.2 Set up of the periodic water waves system

Consider the periodic water waves system

$$\begin{cases}((D_t^0)^2-iA_0\partial_\alpha)\omega=-i\\ \bar{\omega}-\alpha, D_t^0\bar{\omega}\in \mathcal{H}ol_{\mathcal{P}}(\Omega^0(t)).\end{cases} \tag{3.38}$$

Let $\kappa_0:\mathbb{R}\to\mathbb{R}$ be a diffeomorphism given by

$$(\kappa_0)_t=b_0\circ\kappa_0. \tag{3.39}$$

Then the change of coordinates $\alpha\mapsto\kappa_0(\alpha)$ brings the system (3.38) back to Lagrangian coordinates, namely, with $(a_0\partial_\alpha\kappa_0)\circ\kappa_0^{-1}=A_0$, we have

$$\begin{cases}(\omega\circ\kappa_0)_{tt}-ia_0\partial_\alpha\omega\circ\kappa_0=-i\\ (\overline{\omega\circ\kappa_0})_t\in \mathcal{H}ol_{\mathcal{P}}(\Omega^0(t)).\end{cases} \tag{3.40}$$

Take ∂_t on both sides of the above equation, we get :

$$(\partial_t^2 - ia_0\partial_\alpha)(\omega\circ\kappa_0)_t = i(a_0)_t\partial_\alpha\omega\circ\kappa_0 = \frac{(a_0)_t}{a_0} ia_0\omega_\alpha\circ\kappa. \tag{3.41}$$

Precomposing with κ_0^{-1} on both sides of the above equation, we obtain,

$$((D_t^0)^2 - iA_0\partial_\alpha)D_t^0\omega = i\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}A_0\omega_\alpha. \tag{3.42}$$

Similar to the derivation of formula for b, A, $\frac{a_t}{a}\circ\kappa^{-1}$ in [71], we can derive formula for $b_0, A_0, \frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}$. We give the details for the derivation of formula for b_0. Formula for $A_0, \frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}$ follow in a similar way.

3.3.3 Formula for b_0

We have

$$(I-\mathcal{H}_p)b_0 = -[D_t^0\omega, \mathcal{H}_p]\frac{\bar{\omega}_\alpha - 1}{\omega_\alpha}. \tag{3.43}$$

Proof. By assumption, $\bar{\omega}-\alpha = \Phi_0(\omega(\alpha,t),t), D_t\bar{\omega} = \Psi_0(\omega(\alpha,t),t)$, where $\Phi_0, \Psi_0 \in \mathcal{H}ol_{\mathcal{P}}$. We have

$$\begin{aligned} D_t^0\bar{\omega} =&(\partial_t + b_0\partial_\alpha)(\bar{\omega}-\alpha) + b_0\\ =&D_t^0\omega(\Phi_0)_\omega\circ\omega + (\Phi_0)_t\circ\omega + b_0. \end{aligned} \tag{3.44}$$

Note that $D_t^0\bar{\omega}, (\Phi_0)_t\circ\omega \in \mathcal{H}ol_{\mathcal{P}}$, we have

$$(I-\mathcal{H}_p)D_t\bar{\omega} = 0, \qquad (I-\mathcal{H}_p)(\Phi_0)_t\circ\omega = 0.$$

Also note that

$$(\Phi_0)_\omega\circ\omega = \frac{\bar{\omega}_\alpha - 1}{\omega_\alpha}.$$

Apply $I - \mathcal{H}_p$ on both sides of (3.44), we have

$$(I-\mathcal{H}_p)b_0 = (I-\mathcal{H}_p)D_t^0\omega\frac{\bar{\omega}_\alpha - 1}{\omega_\alpha} = -[D_t^0\omega, \mathcal{H}_p]\frac{\bar{\omega}_\alpha - 1}{\omega}.$$

□

3.3.4 Formula for A_0

$$(I-\mathcal{H}_p)(A_0-1) = i[(D_t^0)^2\omega, \mathcal{H}_p]\frac{\bar{\omega}_\alpha - 1}{\omega_\alpha} + i[(D_t^0)\omega, \mathcal{H}_p]\frac{\partial_\alpha(D_t^0)\bar{\omega}}{\omega_\alpha}. \tag{3.45}$$

3.3.5 Formula for $\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}$

We have

$$\begin{aligned}-i(I-\mathcal{H}_p)\Big(A_0\bar{\omega}_\alpha(\frac{(a_0)_t}{a_0})\circ\kappa_0^{-1}\Big) =& 2[(D_t^0)^2\omega, \mathcal{H}_p]\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha} + 2[D_t^0\omega, \mathcal{H}_p]\frac{\partial_\alpha(D_t^0)^2\bar{\omega}}{\omega_\alpha} \\ &- \frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha) - D_t^0\omega(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2\partial_\beta D_t^0\bar{\omega}d\beta.\end{aligned} \tag{3.46}$$

3.3.6 Local well-posedness

(3.38) is a fully nonlinear system. To prove local well-posedness, one way is to quasilinearize this system. In [69], S. Wu showed that for water waves which vanish at infinity, one can quasilinearize (3.38) by just taking one derivative in time. For periodic case, we have quasilinearization

$$\begin{cases}((D_t^0)^2 - iA_0\partial_\alpha)D_t^0\omega = i\frac{(a_0)_t}{a_0}\circ\kappa^{-1}\omega_\alpha, \\ D_t^0\bar{\omega}\in\mathcal{H}ol_{\mathcal{P}}(\Omega^0(t)).\end{cases} \tag{3.47}$$

By formula (3.43), (3.45) together with Proposition III.19, the system (3.47) is a quasilinear system. So the local well-posedness can be obtained similar to the work of S. Wu[69]. We omit the details and state the result as follows:

Theorem III.1 (local well-posedness)**.** *Let $s \geqslant 4$. Assume that $\omega(0), D_t^0\omega(0), (D_t^0)^2\omega(0)$ satify the compatiability condition, i.e., $\bar{\omega}(0)-\alpha, D_t^0\bar{\omega}(0) \in \mathcal{H}ol_{\mathcal{P}}(\Omega^0(0))$, and*

$$(I-\mathcal{H}_{\mathcal{P}})(A_0(0)-1) = i[(D_t^0)^2\omega(0), \mathcal{H}_{\mathcal{P}}]\frac{\partial_\alpha\bar{\omega}(0)-\alpha}{\partial_\alpha\omega(0)} + i[D_t^0\omega(0), \mathcal{H}_{\mathcal{P}}]\frac{\partial_\alpha D_t^0\bar{\omega}(0)}{\partial_\alpha\omega(0)}.$$

Assume $(\partial_\alpha\omega(0)-1, D_t^0\omega(0), (D_t^0)^2\omega(0)) \in H^s(\mathbb{T})\times H^{s+1/2}(\mathbb{T})\times H^s(\mathbb{T})$. Then there is $T>0$ depending on the norm of the initial data such that the water waves system (3.38) has a unique solution $\omega = \omega(\alpha,t)$ for $t\in[0,T]$, satisfying

$$(\omega_\alpha(t)-1, D_t^0\omega(t), (D_t^0)^2\omega(t)) \in C([0,T]; H^s(\mathbb{T})\times H^{s+1/2}(\mathbb{T})\times H^s(\mathbb{T})). \tag{3.48}$$

Moreover, if T_{max} is the supremum over all such times T, then either $T_{max}=\infty$, or $T_{max}<\infty$, but

$$\lim_{t\uparrow T_{max}} \left\|(D_t^0\omega(t), (D_t^0)^2\omega(t))\right\|_{H^s(\mathbb{T})\times H^s(\mathbb{T})} = \infty, \tag{3.49}$$

or

$$\sup_{\alpha\neq\beta}\left|\frac{\omega(\alpha,t)-\omega(\beta,t)}{\alpha-\beta}\right| + \sup_{\alpha\neq\beta}\left|\frac{\alpha-\beta}{\omega(\alpha,t)-\omega(\beta,t)}\right| = \infty, \tag{3.50}$$

or

$$\lim_{t\uparrow T_{max}} \inf_{\alpha\in\mathbb{R}} A_0(\alpha,t)|\omega_\alpha(\alpha,t)| \leqslant 0. \tag{3.51}$$

3.3.7 Long time behavior

We use S. Wu's method ([71]) to study the long time behavior of periodic water waves with small initial data. Consider the quantity $\tilde{\theta}_0 := (I-\mathcal{H}_{\mathcal{P}})(\omega-\bar{\omega})$ and $\tilde{\sigma}_0 := D_t^0(I-\mathcal{H}_{\mathcal{P}})(\omega-\bar{\omega})$. One can show that

$$\left\|\partial_\alpha\tilde{\theta}_0\right\|_{H^{s'}(\mathbb{T})} \approx \|\partial_\alpha\omega-1\|_{H^{s'}(\mathbb{T})}, \qquad \|\tilde{\sigma}_0\|_{H^{s'+1/2}(\mathbb{T})} \approx \left\|D_t^0\omega\right\|_{H^{s'+1/2}(\mathbb{T})}. \tag{3.52}$$

Apply lemma III.16, we obtain

$$
\begin{aligned}
&((D_t^0)^2 - iA_0)\tilde{\theta}_0\\
=& -2[D_t^0\omega, \mathcal{H}_p\frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_p\frac{1}{\bar{\omega}_\alpha}]\frac{\partial_\alpha D_t^0\omega}{\omega_\alpha} + \frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha) - D_t^0\omega(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2\partial_\beta(\omega-\bar{\omega})d\beta \quad (3.53)\\
:=& G_0,
\end{aligned}
$$

and

$$((D_t^0)^2 - iA_0\partial_\alpha)\tilde{\sigma}_0 = D_t^0 G_0 + [(D_t^0)^2 - iA_0\partial_\alpha, D_t^0]\tilde{\theta}_0. \tag{3.54}$$

Note that

$$[(D_t^0)^2 - iA_0\partial_\alpha, D_t^0]\tilde{\theta}_0 = i(\frac{(a_0)_t}{a_0})\circ\kappa_0^{-1}A_0\partial_\alpha\tilde{\theta}_0, \tag{3.55}$$

which is cubic. So we can prove long time existence. We state the result as follows.

Theorem III.2 (Long time existence)**.** *Let $s' \geqslant 6$. Let $\omega(0), D_t^0\omega(0), (D_t^0)^2\omega(0)$ satisfy the compatibility condition as in Theorem III.1. There exists $\epsilon_0 = \epsilon_0(s') > 0$ such that for all $\epsilon < \epsilon_0$, if*

$$\big\|(\partial_\alpha\omega(0) - 1, D_t^0\omega(0), (D_t^0)^2\omega(0))\big\|_{H^{s'}(\mathbb{T})\times H^{s'+1/2}(\mathbb{T})\times H^{s'}(\mathbb{T})} \leqslant \epsilon,$$

then there exists a positive constant $C_0 = C_0(s')$ such that the solution to (3.38) exists on $[0, C_0\epsilon^{-2}]$, and

$$\sup_{t\in[0,C_0\epsilon^{-2}]}\big\|(\partial_\alpha\omega(t) - 1, D_t^0\omega(t), (D_t^0)^2\omega(t))\big\|_{H^{s'}(\mathbb{T})\times H^{s'+1/2}(\mathbb{T})\times H^{s'}(\mathbb{T})} \leqslant 2\epsilon.$$

As a consequence, use formula (3.43), (3.45), (3.46), and use lemma III.13, we obtain bounds for $b_0, A_0, \frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}$.

Corollary III.20. *With the assumptions in Theorem III.2, there exists $C > 0$ independent of ϵ and ω such that for all $t \in [0, C_0\epsilon^{-2}]$,*

$$\|b_0(t)\|_{H^{s'}(\mathbb{T})} + \|A_0(t) - 1\|_{H^{s'}(\mathbb{T})} + \Big\|\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}\Big\|_{H^{s'}(\mathbb{T})} \leqslant C\epsilon^2. \tag{3.56}$$

In particular, by Sobolev embedding, we have

$$\|b_0(t)\|_{W^{s'-1,\infty}(\mathbb{T})} + \|A_0(t)-1\|_{W^{s'-1,\infty}(\mathbb{T})} + \left\|\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}\right\|_{W^{s'-1,\infty}(\mathbb{T})} \leqslant C\epsilon^2. \tag{3.57}$$

Proof. For (3.56), take real part of equations (3.43), (3.45), and (3.46), respectively, then use (3.18). For (3.57), use lemma III.5 and (3.56). □

Another consequence is the following.

Corollary III.21. *The quantities* $\mathcal{H}_\omega(\bar{\omega}-\alpha)$, $\mathcal{H}_\omega D_t^0\bar{\omega}$, $\mathcal{H}_\omega\frac{\bar{\omega}_\alpha-1}{\omega_\alpha}$ *and* $\mathcal{H}_\omega\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha}$ *are well-defined, and*

$$(I-\mathcal{H}_\omega)(\bar{\omega}-\alpha)=0. \tag{3.58}$$

$$(I-\mathcal{H}_\omega)D_t^0\bar{\omega}=0. \tag{3.59}$$

$$(I-\mathcal{H}_\omega)\frac{\bar{\omega}_\alpha-1}{\omega_\alpha}=0. \tag{3.60}$$

$$(I-\mathcal{H}_\omega)\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha}=0. \tag{3.61}$$

Proof. The functions $\bar{\omega}-\alpha$, $\frac{\bar{\omega}_\alpha-1}{\omega_\alpha}$, $D_t^0\bar{\omega}$, $\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha}$ are in $\mathcal{H}ol_{\mathcal{N}}(\Omega^0(t))$. Then the corollary follows from lemma III.11. □

Remark III.22. Note that in the above corollary, the Hilbert transform is defined by

$$\mathcal{H}_\omega f(\alpha)=\frac{1}{\pi i}p.v.\int_{-\infty}^{\infty}\frac{\omega_\beta}{\omega(\alpha)-\omega(\beta)}f(\beta)d\beta.$$

For a bounded smooth function f, $\mathcal{H}_\omega f$ does not always define an L^∞ function. In such cases, $\mathcal{H}_\omega f$ is interpreted in BMO sense.

Notation Denote

$$\mathcal{P}:=D_t^2-iA\partial_\alpha, \qquad \mathcal{P}_0:=(D_t^0)^2-iA_0\partial_\alpha.$$

3.4 Water waves system with data in X^s

We consider long time existence of non-vanishing water waves system with data of the form $X^s := H^s(\mathbb{R}) + H^{s'}(\mathbb{T})$. This is a natural generalization of the current known long time existence results for water waves. Moreover, if we restrict ourselves to smooth water waves, then this class of water waves has included many physically relevant situations.

Let $(\omega, D_t^0\omega, (D_t^0)^2\omega)$ be the solution to the periodic water wave system in the previous section. We consider the class of solutions of water wave system with boundary values $\omega(\alpha, t)$ at $\alpha = \pm\infty$, i.e., we consider

$$\begin{cases} (D_t^2 - iA\partial_\alpha)\zeta = -i \\ \bar{\zeta} - \alpha, \ D_t\bar{\zeta} \in \mathcal{H}ol_{\mathcal{N}}(\Omega(t)) \\ \lim_{\alpha\to\pm\infty}(\zeta - \omega) = 0 \\ \lim_{\alpha\to\pm\infty}(D_t\zeta - D_t^0\omega) = 0. \end{cases} \tag{3.62}$$

Recall that $D_t = \partial_t + b\partial_\alpha$. b and A cannot be arbitrary. Instead, they are determined by the water wave system and the constraint that $\bar{\zeta} - \alpha, D_t\bar{\zeta} \in \mathcal{H}ol_{\mathcal{N}}(\Omega(t))$. Denote

$$\bar{\zeta} - \alpha = \Phi(\zeta(\alpha, t), t), \qquad D_t\bar{\zeta} = \Psi(\zeta(\alpha, t), t),$$

where $\Phi, \Psi \in \mathcal{H}ol_{\mathcal{N}}(\Omega(t))$.

In the following subsections, we derive formula for b, A and $\frac{a_t}{a} \circ \kappa^{-1}$, etc. The derivation is almost the same as that in [71], except that we are dealing with functions which are not necessarily vanishing at ∞, and that the formula might hold only in BMO sense.

3.4.1 Formula for b and b_0

$$(I - \mathcal{H}_\zeta)b = -[D_t\zeta, \mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}, \tag{3.63}$$

and

$$(I-\mathcal{H}_\omega)b_0 = -[D_t^0 b_0, \mathcal{H}_\omega]\frac{\partial_\alpha\bar{\omega}-1}{\omega_\alpha}. \tag{3.64}$$

Proof. We have

$$D_t\bar{\zeta} = b + \Phi_t\circ\zeta + D_t\zeta\Phi_\zeta\circ\zeta = b+\Phi_t\circ\zeta + D_t\zeta\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}.$$

Since $D_t\bar{\zeta}, \Phi_t, \Phi_\zeta \in \mathcal{H}_{\mathcal{N}}(\Omega(t))$, if we apply $(I-\mathcal{H}_\zeta)$ on both sides of the above equation and use lemma III.11, we have

$$\begin{aligned}0 =&(I-\mathcal{H}_\zeta)b + (I-\mathcal{H}_\zeta)D_t\zeta\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}\\ =&(I-\mathcal{H}_\zeta)b + [D_t\zeta,\mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}.\end{aligned}$$

So we have[1]

$$(I-\mathcal{H}_\zeta)b = -[D_t\zeta,\mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}. \tag{3.65}$$

So we obtain (3.63). Use completely the same proof, we have

$$(I-\mathcal{H}_\omega)b_0 = -[D_t^0 b_0, \mathcal{H}_\omega]\frac{\partial_\alpha\bar{\omega}-1}{\omega_\alpha}. \tag{3.66}$$

So we obtain (3.64). □

3.4.2 Formula for A

Use the water wave system, we have $D_t^2\bar{\zeta} + iA\bar{\zeta}_\alpha = i$. Note that

$$D_t^2\bar{\zeta} = D_tD_t\bar{\zeta} = D_t\Psi\circ\zeta = \Psi_t\circ\zeta + D_t\zeta\Psi_\zeta\circ\zeta \qquad \Psi_\zeta = \frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}. \tag{3.67}$$

[1] Note that $\mathcal{H}_\zeta b$ and $\mathcal{H}_\omega b_0$ are defined as BMO functions. Moreover, $\mathcal{H}_\zeta b - \mathcal{H}_\omega b_0 \in H^s(\mathbb{R})$. Similar properties hold for other quantities such as A, A_0.

$$iA\bar{\zeta}_\alpha = iA + iA\partial_\alpha(\bar{\zeta} - \alpha) = iA + iA\zeta_\alpha \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} = iA + (D_t^2\zeta + i)\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}. \tag{3.68}$$

Since $\Psi_t, \frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} \in \mathcal{H}ol_{\mathcal{N}}$, by lemma III.11, we have

$$(I - \mathcal{H}_\zeta)\Psi_\zeta \circ \zeta = 0, \qquad (I - \mathcal{H}_\zeta)\Psi_t \circ \zeta = 0, \qquad (I - \mathcal{H}_\zeta)\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} = 0. \tag{3.69}$$

Apply $(I - \mathcal{H}_\zeta)$ on both sides of $D_t^2\bar{\zeta} + iA\bar{\zeta}_\alpha = i$, use (3.67), (3.68), and (3.69), we have

$$(I - \mathcal{H}_\zeta)D_t\zeta\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha} + (I - \mathcal{H}_\zeta)i(A - 1) + (I - \mathcal{H}_\zeta)D_t^2\zeta\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} = 0. \tag{3.70}$$

Use (3.69) again, we can write (3.70) in commutator form:

$$(I - \mathcal{H}_\zeta)(A - 1) = i[D_t\zeta, \mathcal{H}_\zeta]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha} + i[D_t^2\zeta, \mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}. \tag{3.71}$$

Use completely the same argument, we get a nonlocal version of formula for A_0:

$$(I - \mathcal{H}_\omega)(A_0 - 1) = i[D_t^0\omega, \mathcal{H}_\omega]\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha} + i[(D_t^0)^2\omega, \mathcal{H}_\omega]\frac{\bar{\omega}_\alpha - 1}{\omega_\alpha}. \tag{3.72}$$

Remark III.23. Formula (3.71) implies that $A - 1$ is quadratic.

3.4.3 Formula for $\frac{a_t}{a} \circ \kappa^{-1}$ and $\frac{(a_0)_t}{a_0} \circ \kappa_0^{-1}$

Apply D_t on both sides of $(D_t^2 + iA\partial_\alpha)\bar{\zeta} = i$, we have

$$(D_t^2 + iA\partial_\alpha)D_t\bar{\zeta} = -i\frac{a_t}{a} \circ \kappa^{-1}A\bar{\zeta}_\alpha. \tag{3.73}$$

Apply $(I-\mathcal{H}_\zeta)$ on both sides of (3.73), use $(I-\mathcal{H}_\zeta)D_t\bar{\zeta}=0$, we have

$$\begin{aligned}
&-i(I-\mathcal{H}_\zeta)\frac{a_t}{a}\circ\kappa^{-1}A\bar{\zeta}_\alpha\\
=&(I-\mathcal{H}_\zeta)(D_t^2+iA\partial_\alpha)D_t\bar{\zeta}\\
=&[D_t^2+iA\partial_\alpha,\mathcal{H}_\zeta]D_t\bar{\zeta}\\
=&2[D_t^2\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}+2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t^2\bar{\zeta}}{\zeta_\alpha}-\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta D_t\bar{\zeta}d\beta.
\end{aligned}\tag{3.74}$$

So we have

$$\begin{aligned}
&(I-\mathcal{H}_\zeta)(\frac{a_t}{a})\circ\kappa^{-1}A\bar{\zeta}_\alpha\\
=&2i[D_t^2\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}+2i[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t^2\bar{\zeta}}{\zeta_\alpha}-\frac{1}{\pi}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta D_t\bar{\zeta}d\beta.
\end{aligned}\tag{3.75}$$

Use the same argument, we get nonlocal version of formula for $\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}$:

$$\begin{aligned}
(I-\mathcal{H}_\omega)(\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}A_0\bar{\omega}_\alpha)=&2i[(D_t^0)^2\omega,\mathcal{H}_\omega]\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha}+2i[D_t^0\omega,\mathcal{H}_\omega]\frac{\partial_\alpha(D_t^0)^2\bar{\omega}}{\omega_\alpha}\\
&-\frac{1}{\pi}\int\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\omega(\alpha)-\omega(\beta)}\Big)^2\partial_\beta D_t^0\bar{\omega}d\beta
\end{aligned}\tag{3.76}$$

3.4.4 Formula for $b-b_0$

Write $b=b_0+b_1$. By (3.63), (3.64), we have

$$(I-\mathcal{H}_\zeta)b-(I-\mathcal{H}_\omega)b_0=-[D_t\zeta,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}](\bar{\zeta}_\alpha-1)+[D_t^0\omega,\mathcal{H}_\omega\frac{1}{\omega_\alpha}](\bar{\omega}_\alpha-1)$$

So we obtain

$$\begin{aligned}
(I-\mathcal{H}_\zeta)(b-b_0)=&-[D_t\zeta-D_t^0\omega,\mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}-[D_t^0\omega,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}-\mathcal{H}_\omega\frac{1}{\omega_\alpha}](\bar{\zeta}_\alpha-1)\\
&-[D_t^0\omega,\mathcal{H}_\omega\frac{1}{\omega_\alpha}](\bar{\xi}_1)\alpha-(\mathcal{H}_\zeta-\mathcal{H}_\omega)b_0.
\end{aligned}\tag{3.77}$$

3.4.5 Formula for $\frac{a_t}{a}\circ\kappa^{-1}-\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}$

We have

$$\begin{aligned}&(I-\mathcal{H}_\zeta)\Big\{A\bar{\zeta}_\alpha\Big[\frac{a_t}{a}\circ\kappa^{-1}-\frac{(a_0)_t}{a_0}\circ\kappa_0^{-1}\Big]\Big\}\\=&(I-\mathcal{H}_\zeta)\Big(A\bar{\zeta}_\alpha\Big(\frac{a_t}{a}\Big)\circ\kappa^{-1}\Big)-(I-\mathcal{H}_\omega)\Big(A_0\bar{\omega}_\alpha\Big(\frac{(a_0)_t}{a_0}\Big)\circ\kappa_0^{-1}\Big)\\&+(\mathcal{H}_\zeta-\mathcal{H}_\omega)\Big(A_0\bar{\omega}_\alpha\Big(\frac{(a_0)_t}{a_0}\Big)\circ\kappa_0^{-1}\Big).\end{aligned}\tag{3.78}$$

3.4.6 Formula for D_tb_1

The idea of deriving formula for D_tb_1 is the same as that for b_1: find formula for D_tb and $D_t^0b_0$ and then consider their difference. The derivation of formula for D_tb and $D_t^0b_0$ are similar to that in S. Wu's paper (See Proposition 2.7 of [71]). We record the formula as follows.

$$\begin{aligned}(I-\mathcal{H}_\zeta)D_tb=&[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha(2b-D_t\bar{\zeta})}{\zeta_\alpha}-[D_t^2\zeta,\mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}\\&+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2(\bar{\zeta}_\beta(\beta)-1)d\beta.\end{aligned}\tag{3.79}$$

For the periodic part, we have

$$\begin{aligned}(I-\mathcal{H}_\omega)D_t^0b_0=&[D_t^0\omega,\mathcal{H}_\omega]\frac{\partial_\alpha(2b_0-D_t^0\bar{\omega})}{\zeta_\alpha}-[(D_t^0)^2\omega,\mathcal{H}_\omega]\frac{\bar{\omega}_\alpha-1}{\omega_\alpha}\\&+\frac{1}{\pi i}\int\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\omega(\alpha)-\omega(\beta)}\Big)^2(\bar{\omega}_\beta(\beta)-1)d\beta.\end{aligned}\tag{3.80}$$

Subtract (3.80) from (3.79) , we have

$$\begin{aligned}
&(I-\mathcal{H}_\zeta)D_t b_1\\
=&(I-\mathcal{H}_\zeta)D_t b-(I-\mathcal{H}_\omega)D_t^0 b_0+(\mathcal{H}_\zeta-\mathcal{H}_\omega)D_t^0 b_0-(I-\mathcal{H}_\zeta)b_1\partial_\alpha b_0\\
=&[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha(2b-D_t\bar{\zeta})}{\zeta_\alpha}-[D_t^0\omega,\mathcal{H}_\omega]\frac{\partial_\alpha(2b_0-D_t^0\bar{\omega})}{\zeta_\alpha}\\
&-[D_t^2\zeta,\mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}+[(D_t^0)^2\omega,\mathcal{H}_\omega]\frac{\bar{\omega}_\alpha-1}{\omega_\alpha}\\
&+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2(\bar{\zeta}_\beta(\beta)-1)d\beta-\frac{1}{\pi i}\int\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\omega(\alpha)-\omega(\beta)}\Big)^2(\bar{\omega}_\beta(\beta)-1)d\beta\\
&+(\mathcal{H}_\zeta-\mathcal{H}_\omega)D_t^0 b_0\\
&-(I-\mathcal{H}_\zeta)b_1\partial_\alpha b_0.
\end{aligned} \tag{3.81}$$

3.4.7 Formula for $A - A_0$

By (3.71) and (3.72), we have

$$\begin{aligned}
(I-\mathcal{H}_\zeta)(A-A_0)=&i[D_t^2\zeta,\mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}-i[(D_t^0)^2\omega,\mathcal{H}_\omega]\frac{\bar{\omega}_\alpha-1}{\omega_\alpha}\\
&+i[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha}-i[D_t^0\omega,\mathcal{H}_\omega]\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha}\\
&+(\mathcal{H}_\omega-\mathcal{H}_\zeta)(A_0-1).
\end{aligned} \tag{3.82}$$

Now we have formula for $b_1, D_t b_1, A_1$. So that we have a quasilinear system. It's not difficult to obtain local well-posedness of this quasilinear system. We omit the details and focus on the long time existence.

3.4.8 A discussion on long time existence

In order to prove long time well-posedness, one idea is to find some quantity ζ with $D_t\theta \approx D_t\zeta$, such that

$$\mathcal{P}\theta = cubic.$$

In [71], S. Wu take $\theta = (I - \mathcal{H}_\zeta)(\zeta - \alpha)$ and show that $\mathcal{P}\theta$ consists of cubic and higher order terms. For water waves that is neither periodic nor vanishing at spatial infinity, if we take $\theta = (I - \mathcal{H}_\zeta)(\zeta - \alpha)$, then $\mathcal{P}\theta$ is still cubic, at least in BMO sense. As was explained in the introduction, since θ is not in any $L^2(\mathbb{R})$ based spaces, it's difficult to associate $\mathcal{P}\theta = cubic$ with an appropriate energy which still preserves this cubic structure.

Note however that, given any compatible initial data $(\zeta(\alpha, 0), D_t\zeta(\alpha, 0), D_t^2\zeta(\alpha, 0))$ be such that

$$(\partial_\alpha\zeta(\alpha, 0) - 1, D_t\zeta(\alpha, 0), D_t^2\zeta(\alpha, 0)) \in X^s \times X^{s+1/2} \times X^s,$$

by Theorem III.2, $\omega(\alpha, t), D_t^0\omega(\alpha, t), (D_t^0)^2\omega(\alpha, t)$ exists on time scale $O(\epsilon^{-2})$, so we need only to consider long time existence for $\xi_1 := \zeta - \omega$ and $D_t\xi_1$: and it's advantageous to do so, because $\xi_1(\alpha, t)$ and $D_t\xi_1(\alpha, t)$ vanish as $\alpha \to \infty$, while $\zeta - \alpha$ and $D_t\zeta$ oscillate at ∞. It turns out that $\mathcal{P}(I - \mathcal{H}_\zeta)\xi_1$ consists of cubic and higher order nonlinearities. So we are able to prove long time existence in our situation.

3.4.9 Governing equation for ξ_1

3.4.9.1 $\mathcal{P}(I - \mathcal{H}_\zeta)(\zeta - \alpha)$

In [71], the key ingredients that S. Wu derived $\mathcal{P}(I - \mathcal{H}_\zeta)(\zeta - \alpha)$ are:

$$(I - \mathcal{H}_\zeta)D_t\bar{\zeta} = 0, \quad (I - \mathcal{H}_\zeta)(\bar{\zeta} - \alpha) = 0, \quad (I - \mathcal{H}_\zeta)\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} = 0, \quad \mathcal{P}\zeta = -i. \tag{3.83}$$

In our situation, (3.83) is still true, despite that we have non-vanishing water waves at ∞. Use the same derivation as in [71], we have

$$\begin{aligned} &\mathcal{P}(I - \mathcal{H}_\zeta)(\zeta - \alpha) \\ =& -2[D_t\zeta, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta + \frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha) - D_t\zeta(\beta)}{\zeta(\alpha) - \zeta(\beta)}\Big)^2\partial_\beta(\zeta - \bar{\zeta})d\beta \\ :=& G. \end{aligned} \tag{3.84}$$

Similarly,

$$\begin{aligned}
&\mathcal{P}_0(I-\mathcal{H}_\omega)(\omega-\alpha)\\
=&-2[D_t^0\omega, \mathcal{H}_\omega\frac{1}{\omega_\alpha}+\bar{\mathcal{H}}_\omega\frac{1}{\bar{\omega}_\alpha}]\partial_\beta D_t^0\omega+\frac{1}{\pi i}\int_{-\infty}^{\infty}\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\omega(\alpha)-\omega(\beta)}\Big)^2\partial_\beta(\omega-\bar{\omega})d\beta \qquad (3.85)\\
:=&G_0.
\end{aligned}$$

3.4.9.2 An equivalent quantity of ξ_1

Denote

$$\lambda := (I-\mathcal{H}_\zeta)(\zeta-\alpha)-(I-\mathcal{H}_\omega)(\omega-\alpha). \qquad (3.86)$$

The reason we consider this quantity is that at least formally, we have known $\mathcal{P}(I-\mathcal{H}_\zeta)(\zeta-\bar{\zeta})$ and $\mathcal{P}_0(I-\mathcal{H}_\omega)(\omega-\bar{\omega})$ consist of cubic and higher order terms. So the quantity $\mathcal{P}(I-\mathcal{H}_\zeta)(\zeta-\bar{\zeta})-\mathcal{P}_0(I-\mathcal{H}_\omega)(\omega-\bar{\omega})$ is at least cubic. Moreover, $\mathcal{P}-\mathcal{P}_0$ is quadratic. So $\mathcal{P}\lambda$ is cubic.

Note that λ might not be holomorphic in $\Omega(t)$. To avoid loss of derivatives, we consider the quantity

$$\theta := (I-\mathcal{H}_\zeta)\lambda. \qquad (3.87)$$

First we derive water waves equation for $\mathcal{P}\lambda$, and then we derive $\mathcal{P}(I-\mathcal{H}_\zeta)\lambda$. Direct calculation gives

$$\begin{aligned}
D_t^2-(D_t^0)^2 =& D_t^2-D_tD_t^0+D_tD_t^0-(D_t^0)^2\\
=& D_t(b_1\partial_\alpha)+b_1\partial_\alpha D_t^0 \qquad (3.88)\\
=& (D_tb_1)\partial_\alpha+b_1D_t\partial_\alpha+b_1\partial_\alpha D_t^0.
\end{aligned}$$

Then we have

$$\begin{aligned}\mathcal{P}\lambda =&\mathcal{P}(I-\mathcal{H}_\zeta)(\zeta-\alpha)-\mathcal{P}_0(I-\mathcal{H}_\omega)(\omega-\alpha)+(\mathcal{P}-\mathcal{P}_0)(I-\mathcal{H}_\omega)(\omega-\alpha)\\ =&-2[D_t\zeta,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta+2[D_t^0\omega,\mathcal{H}_\omega\frac{1}{\omega_\alpha}+\bar{\mathcal{H}}_\omega\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega\\ &+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta\\ &-\frac{1}{\pi i}\int\Big(\frac{D_t^0\omega(\alpha,t)-D_t^0\omega(\beta,t)}{\omega(\alpha,t)-\omega(\beta,t)}\Big)^2(\omega-\bar{\omega})_\beta d\beta\\ &+D_tb_1\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha)+b_1D_t\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha)\\ &+b_1\partial_\alpha D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)-iA_1\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha).\end{aligned}\tag{3.89}$$

So $\mathcal{P}\lambda$ consists of cubic and higher order nonlinearities. Note that

$$\begin{aligned}\mathcal{P}\theta =&\mathcal{P}(I-\mathcal{H}_\zeta)\lambda\\ =&(I-\mathcal{H}_\zeta)\mathcal{P}\lambda-[\mathcal{P},\mathcal{H}_\zeta]\lambda.\end{aligned}\tag{3.90}$$

We have

$$[\mathcal{P},\mathcal{H}_\zeta]\lambda =2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\lambda}{\zeta_\alpha}-\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta\lambda d\beta.\tag{3.91}$$

Note that

$$\begin{aligned}\lambda =&(I-\mathcal{H}_\zeta)(\zeta-\omega)+(\mathcal{H}_\omega-\mathcal{H}_\zeta)(\omega-\alpha)\\ =&(I-\mathcal{H}_\zeta)\xi_1+(\mathcal{H}_\omega-\mathcal{H}_\zeta)(\omega-\alpha)\\ :=&\lambda_1+\lambda_2.\end{aligned}\tag{3.92}$$

λ_2 is quadratic, so $[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\lambda_2}{\zeta_\alpha}$ is cubic. λ_1 is holomorphic in $\Omega(t)^c$, so $[D_t\zeta,\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\lambda_1$ is cubic. So

$$[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\lambda_1}{\zeta_\alpha} = -[D_t\zeta,\bar{\mathcal{H}}_\zeta]\frac{\partial_\alpha D_t\lambda_1}{\bar{\zeta}_\alpha}+[D_t\zeta,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\lambda_1$$

is cubic.

3.4.10 Governing equation for $D_t\theta$.

The nonlinearities $\mathcal{P}(I-\mathcal{H}_\zeta)D_t\theta$ contains a term of the form $D_t^2 b_1$, which loses derivatives in energy estimates. So we consider the quantity

$$\sigma = (I-\mathcal{H}_\zeta)[D_t(I-\mathcal{H}_\zeta)(\zeta-\alpha) - D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)].$$

Denote

$$\chi := D_t(I-\mathcal{H}_\zeta)(\zeta-\alpha) - D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha).$$

Remark III.24. If we replace σ by χ, then we'll lose one derivative in the energy estimates. The advantage of $(I-\mathcal{H}_\zeta)$ acting on χ is that $(I+\mathcal{H}_\zeta)\sigma = 0$, so $(I+\mathcal{H}_\zeta)\partial_\alpha\sigma = -[\partial_\alpha, \mathcal{H}_\zeta]\sigma$, which prevents losing one derivative.

We have

$$\begin{aligned} &\mathcal{P}D_t(I-\mathcal{H}_\zeta)(\zeta-\alpha)\\ =&D_t\mathcal{P}(I-\mathcal{H}_\zeta)(\zeta-\alpha) + [\mathcal{P}, D_t](I-\mathcal{H}_\zeta)(\zeta-\alpha)\\ =&D_tG + [\mathcal{P}, D_t](I-\mathcal{H}_\zeta)(\zeta-\alpha). \end{aligned}$$

And we have

$$\begin{aligned} &\mathcal{P}_0 D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)\\ =&D_t^0\mathcal{P}_0(I-\mathcal{H}_\omega) + [\mathcal{P}_0, D_t^0](I-\mathcal{H}_\omega)(\omega-\alpha)\\ =&D_t^0 G_0 + [\mathcal{P}_0, D_t^0](I-\mathcal{H}_\omega)(\omega-\alpha). \end{aligned}$$

So we have

$$
\begin{aligned}
\mathcal{P}\chi =&\mathcal{P}D_t(I-\mathcal{H}_\zeta)(\zeta-\alpha)-\mathcal{P}_0D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)\\
&+(\mathcal{P}-\mathcal{P}_0)D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)\\
=&D_tG-D_t^0G_0+(\mathcal{P}-\mathcal{P}_0)D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)\\
&+[\mathcal{P},D_t](I-\mathcal{H}_\zeta)(\zeta-\alpha)-[\mathcal{P}_0,D_t^0](I-\mathcal{H}_\omega)(\omega-\alpha)
\end{aligned}
\tag{3.93}
$$

We have

$$
\mathcal{P}\sigma=(I-\mathcal{H})\mathcal{P}\chi-[\mathcal{P},\mathcal{H}_\zeta]\chi
$$

$$
[\mathcal{P},\mathcal{H}]\chi=2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\chi}{\zeta_\alpha}-\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta\chi d\beta. \tag{3.94}
$$

Use the same argument as we did for $2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\lambda}{\zeta_\alpha}$, we can show that $2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t\chi}{\zeta_\alpha}$ is indeed cubic.

3.4.11 Long time existence

With previous preparations, use standard energy method (similar to those in [71], [63]), we can complete the proof of Theorem I.24. A minor modification of the argument in §3.7 -§3.9 also gives a proof of Theorem I.24. We omit the details of the proof here.

Remark III.25. In our set up, we need the periodic solution ω to have $\frac{3}{2}+$ more derivatives than the decaying part ξ_1. This requirement is of course not optimal. However, it's enough for us to justify the Peregrine soliton from the full water waves.

Remark III.26. Theorem I.24 can be interpreted as: Periodic water wave system is stable under Sobolev perturbation (note that this perturbation is indeed not small relative to the periodic part).

3.5 Multiscale analysis and the derivation of NLS from full water waves equation

Our goal of this section is to formally derive the NLS from full water waves, which is similar to that in ([63]), except that the water waves we are considering do not vanish at infinity. The method we use to derive the NLS is the multiscale analysis. Let

$$\alpha_0 := \alpha, \quad \alpha_1 := \epsilon\alpha, \quad t_0 := t, \quad t_1 := \epsilon t, \quad t_2 := \epsilon^2 t. \tag{3.95}$$

Assume that $\bar{\zeta} - \alpha, D_t\bar{\zeta} \in \mathcal{H}ol(\Omega(t))$. Assume ζ can be expanded as a power series of ϵ, i.e.,

$$\zeta = \alpha + \sum_{n \geqslant 1} \epsilon^n \zeta^{(n)}. \tag{3.96}$$

We assume $\zeta^{(1)}$ is wave packet like, i.e., $\zeta^{(1)} = B(\alpha_1, t_0, t_1, t_2)e^{i\phi}$, where $\phi = k\alpha + \gamma t$ for some constants $k, \gamma > 0$. We don't assume $B \in H^s(\mathbb{R})$. Instead, we assume $B = B_0 + B_1$, with $B_0 = B_0(t)$ independent of α, and $B_1 \in H^s(\mathbb{R})$.

Because $\bar{\zeta} - \alpha$ is holomorphic, the leading order of $\bar{\zeta} - \alpha$ must be close to a holomorphic function. If $B_0 \equiv 0$, we the following result:

Lemma III.27 (Propositioin 3.1 in [63]). *Let $f = g(\epsilon\alpha)e^{-ik\alpha}$, with $g \in H^{s+m}(\mathbb{R})$, $k \neq 0$ and $s, m \geqslant 0$ be given, assume $\epsilon \leqslant 1$ and $g \in H^{s+m}(\mathbb{R})$. Then*

$$\|(I - sgn(k))\mathbb{H})f\|_{H^s(\mathbb{R})} \leqslant C\frac{\epsilon^{m-\frac{1}{2}}}{k^m}\|g\|_{H^{s+m}(\mathbb{R})},$$

for some constant $C = C(s)$.

If f is oscillating at ∞, we have the following.

Lemma III.28. *Let c be a constant and assume $f = ce^{-ik\alpha} + g(\epsilon\alpha)e^{-ik\alpha}$, with $g \in H^{s+m}(\mathbb{R})$,*

$k \neq 0$ and $s, m \geqslant 0$ be given. Assume $\epsilon \leqslant 1$. Then

$$\|(I - sgn(k)\mathbb{H})f\|_{H^s(\mathbb{R})} \leqslant C\frac{\epsilon^{m-\frac{1}{2}}}{k^m}\|g\|_{H^{s+m}(\mathbb{R})},$$

for some constant $C = C(s)$.

Proof. For $k \neq 0$, we have

$$(I - sgn(k)\mathbb{H})e^{-ik\alpha} = 0. \tag{3.97}$$

Therefore, by lemma III.27, we have

$$\|(I - \mathbb{H})f\|_{H^s(\mathbb{R})} = \left\|(I - \mathbb{H})g(\epsilon\alpha)e^{-ik\alpha}\right\|_{H^s(\mathbb{R})} \leqslant C\frac{\epsilon^{m-\frac{1}{2}}}{k^m}\|g\|_{H^{s+m}(\mathbb{R})}. \tag{3.98}$$

□

Now we are ready to carry out asymptotic expansion and derive the focusing NLS. As in [63], we use the following equation to perform multiscale analysis.

$$\begin{aligned}(D_t^2 - iA\partial_\alpha)(I - \mathcal{H}_\zeta)(\zeta - \bar{\zeta}) = &-2[D_t\zeta, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\bar{\zeta}\\ &+ \frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha) - D_t\zeta(\beta)}{\zeta(\alpha) - \zeta(\beta)}\Big)^2\partial_\beta(\zeta - \bar{\zeta})d\beta := G.\end{aligned} \tag{3.99}$$

So we need to expand every quantity/operator as asymptotic series in ϵ. These have been done in ([63]). We expand b, A, G as

$$b = \sum_{n\geqslant 0}\epsilon^n b^{(n)}, \quad A = \sum_{n\geqslant 0}\epsilon^n A^{(n)}, \quad G = \sum_{n\geqslant 0}\epsilon^n G_n. \tag{3.100}$$

Since b and $A - 1$ are quadratic and G is cubic, we have

$$b^{(0)} = b^{(1)} = A^{(1)} = G_1 = G_2 = 0, \quad A^{(0)} = 1. \tag{3.101}$$

Expand $\mathcal{H}_\zeta = H_0 + \sum_{n\geqslant 1} \epsilon^n H_n$. Then

$$\begin{aligned}
H_0 f(\alpha) =& \mathbb{H} f(\alpha), \\
H_1 f(\alpha) =& [\zeta^{(1)}, H_0]\partial_{\alpha_0} f, \\
H_2 f(\alpha) =& [\zeta^{(1)}, H_0] f_{\alpha_1} + [\zeta^{(2)}, H_0] f_{\alpha_0} - [\zeta^{(1)}, H_0]\zeta^{(1)}_{\alpha_0} f_{\alpha_0} + \frac{1}{2}[\zeta^{(1)}, [\zeta^{(1)}, H_0]]\partial^2_{\alpha_0} f.
\end{aligned} \tag{3.102}$$

See §3.1 in [63] for the derivation of H_0, H_1 and H_2.

3.5.1 $O(1)$ hierarchy

This simply gives

$$A^{(0)} = 1.$$

3.5.2 ϵ hierarchy

We have

$$(\partial_{t_0}^2 - i\partial_{\alpha_0})(I - H_0)\zeta^{(1)} = 0.$$

Since $\zeta^{(1)} = B(\alpha_1, t_0, t_1, t_2)e^{i\phi}$, by Lemma III.28, we have

$$(I - H_0)\zeta^{(1)} = 2\zeta^{(1)} + O(\epsilon^4). \tag{3.103}$$

So we have

$$(\partial_{t_0}^2 - i\partial_{\alpha_0})\zeta^{(1)} = O(\epsilon^4).$$

Then we get $\zeta^{(1)} = B(\alpha_1, t_1, t_2)e^{i(k\alpha+\gamma t)}$, with $\gamma^2 = k$. We simply choose $\gamma = \sqrt{k}$, as what we expected.

3.5.3 ϵ^2 level

We need

$$(\partial_{t_0}^2 - i\partial_{\alpha_0})(I - H_0)\zeta^{(2)} = -(2\partial_{t_0}\partial_{t_1} - i\partial_{\alpha_1})(I - H_0)\zeta^{(1)} + (\partial_{t_0}^2 - i\partial_{\alpha_0})H_1\zeta^{(1)}.$$

Note that

$$\begin{aligned}(\partial_{t_0}^2 - i\partial_{\alpha_0})H_1\zeta^{(1)} =&(\partial_{t_0}^2 - i\partial_{\alpha_0})[\zeta^{(1)}, H_0]\partial_{\alpha_0}Be^{i\phi}\\ =&ik(\partial_{t_0}^2 - i\partial_{\alpha_0})[\zeta^{(1)}, I + H_0]Be^{i\phi}\\ =&O(\epsilon^4).\end{aligned}$$

To avoid secular terms, we choose $\zeta^{(1)}$ such that

$$-(2\partial_{t_0}\partial_{t_1} - i\partial_{\alpha_1})(I - H_0)\zeta^{(1)} = 0.$$

This is equivalent to

$$B_{t_1} - \frac{1}{2\gamma}B_{\alpha_1} = 0.$$

So we choose $B = B(X,T)$, with $X = \alpha_1 + \frac{1}{2\gamma}t_1 = \epsilon(\alpha + \frac{1}{2\gamma}t)$, $T = t_2 = \epsilon^2 t$. Note that $\frac{1}{2\gamma} = \frac{\partial\gamma}{\partial k}$, so B travels at the group velocity.

To choose $\zeta^{(2)}$, we use $(I - \mathcal{H})(\bar{\zeta} - \alpha) = 0$.

$$\begin{aligned}(I - H_0)\bar{\zeta}^{(2)} =&H_1\bar{\zeta}^{(1)} = [\zeta^{(1)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(1)}\\ =&-ik[\zeta^{(1)}, H_0]\bar{B}e^{-i\phi}\\ =&ik[\zeta^{(1)}, I - H_0]\bar{B}e^{-i\phi}\\ =&-ik(I - H_0)|B|^2.\end{aligned}$$

We choose

$$\zeta^{(2)} = \frac{ik}{2}(I + H_0)|B|^2 + \frac{ik}{2}|B_0|^2 = \frac{ik}{2}(I + H_0)(|B|^2 - |B_0|^2) + ik|B_0|^2.$$

Note that

$$(I - H_0)\bar{\zeta}^{(2)} = -\frac{ik}{2}(I - H_0)(I - H_0)(|B|^2 - |B_0|^2) - ik(I - H_0)|B_0|^2 = -ik(I - H_0)|B|^2.$$

3.5.4 ϵ^3 level

First, we need to expand $b = \sum_{n\geqslant 0} \epsilon^n b^{(n)}$. Since $(I - \mathcal{H})b = -[D_t\zeta, \mathcal{H}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}$ is quadratic, we have $b^{(0)} = b^{(1)} = 0$. For b_2, we have

$$\begin{aligned}(I - H_0)b^{(2)} =& - [\partial_{t_0}\zeta^{(1)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(1)} \\ =& - \gamma k[\zeta^{(1)}, H_0]\bar{\zeta}^{(1)} = \gamma k[\zeta^{(1)}, I - H_0]\bar{\zeta}^{(1)} = -\gamma k(I - H_0)|B|^2.\end{aligned} \tag{3.104}$$

Since $b^{(2)}$ is real, we have

$$b^{(2)} = -\gamma k|B|^2.$$

We need also to expand $A = \sum_{n\geqslant 0} \epsilon^n A^{(n)}$. Clearly, $A^{(0)} = 1$, and $A^{(1)} = 0$. We have

$$(I - H_0)A^{(2)} = i[\partial_{t_0}^2\zeta^{(1)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(1)} + i[\partial_{t_0}\zeta^{(1)}, H_0]\partial_{\alpha_0}\partial_{t_0}\bar{\zeta}^{(1)} = 0.$$

Since $A^{(2)}$ is real, we have $A^{(2)} = 0$.

Use exactly the same calculation as in ([63]), we obtain

$$G_3 = 2k^3 B|B|^2 e^{i\phi}.$$

Then for $O(\epsilon^3)$ terms, we have

$$\begin{aligned}(\partial_{t_0}^2 - i\partial_{\alpha_0})(I - H_0)\zeta^{(3)} =& - (\partial_{t_0}^2 - i\partial_{\alpha_0})(-H^{(1)})\zeta^{(2)} - (\partial_{t_0}^2 - i\partial_{\alpha_0})(-H^{(2)})\zeta^{(1)} \\ & - (2\partial_{t_0}\partial_{t_1} - i\partial_{\alpha_1})(I - H_0)\zeta^{(2)} - (2\partial_{t_0\partial_{t_1}} - i\partial_{\alpha_1})(-H^{(1)})\zeta^{(1)} \\ & - (2\partial_{t_0t_2} + \partial_{t_1}^2 + 2b_2\partial_{t_0}\partial_{\alpha_0})(I - H_0)\zeta^{(1)} + G_3 \\ =& - (2\partial_{t_0t_2} + \partial_{t_1}^2 + 2b_2\partial_{t_0}\partial_{\alpha_0})(I - H_0)\zeta^{(1)} + 2k^3 B|B|^2 e^{i\phi} \\ =& - 2\gamma(2iB_T - \gamma^{''} B_{XX} + k^2\gamma B|B|^2)e^{i\phi},\end{aligned}$$

where $\gamma^{''} = \frac{d^2\gamma}{dk^2} = -\frac{1}{4k^{3/2}}$. To avoid secular growth, we choose B such that

$$2iB_T - \gamma^{''} B_{XX} + k^2\gamma B|B|^2 = 0.$$

So B solves the focusing cubic NLS. So we have

$$(\partial_{t_0}^2 - i\partial_{\alpha_0})(I - H_0)\zeta^{(3)} = 0.$$

From $(I - \mathcal{H}_\zeta)(\bar{\zeta} - \alpha) = 0$, we have

$$\begin{aligned}(I - H_0)\bar{\zeta}^{(3)} =& H^{(1)}\bar{\zeta}^{(2)} + H^{(2)}\bar{\zeta}^{(1)} \\ =& [\zeta^{(1)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(2)} + [\zeta^{(2)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(1)} + [\zeta^{(1)}, H_0]\partial_{\alpha_1}\zeta^{(1)} \\ & - [\zeta^{(1)}, H_0]\overline{\partial_{\alpha_0}\zeta^{(1)}}\partial_{\alpha_0}\zeta^{(1)} + \frac{1}{2}[\zeta^{(1)}, [\zeta^{(1)}, H_0]]\partial_{\alpha_0}^2\bar{\zeta}^{(1)} \\ =& (I - H_0)B\bar{B}_X - k^2 B e^{i\phi}(I + H_0)|B|^2 + k^2 B e^{i\phi} H_0 |B|^2 \\ =& - k^2 B|B|^2 e^{i\phi} + (I - H_0)B\bar{B}_X.\end{aligned}$$

We choose

$$\zeta^{(3)} = -\frac{1}{2}k^2\bar{B}|B|^2 e^{-i\phi} + \frac{1}{2}(I + H_0)(\bar{B}B_X).$$

So we have an approximated solution

$$\begin{aligned}\tilde{\zeta} =&\alpha + \epsilon B(X,T)e^{i\phi} + \epsilon^2\Big\{\frac{ik}{2}(I+H_0)(|B|^2 - |B_0|^2) + ik|B_0|^2\Big\}\\ &+ \epsilon^3\Big\{-\frac{1}{2}k^2\bar{B}|B|^2e^{-i\phi} + \frac{1}{2}(I+H_0)(\bar{B}B_X)\Big\}.\end{aligned} \tag{3.105}$$

To find b_3, we have

$$\begin{aligned}&(I-H_0)b_3\\ =&-[\partial_{t_0}\zeta^{(2)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(1)} - [\partial_{t_1}\zeta^{(1)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(1)} - [\partial_{t_0}\zeta^{(1)}, H_1]\partial_{\alpha_0}\bar{\zeta}^{(1)}\\ &-[\partial_{t_0}\zeta^{(1)}, H_0]\partial_{\alpha_1}\bar{\zeta}^{(1)} - [\partial_{t_0}\zeta^{(1)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(2)} - [\partial_{t_0}\zeta^{(2)}, H_0]\partial_{\alpha_0}\bar{\zeta}^{(1)}(-\partial_{\alpha_0}\zeta^{(1)})\\ =&i\gamma(I+H_0)(B\bar{B}_X - \frac{1}{2}\bar{B}B_X) - 2ik^2\bar{B}|B|^2e^{-i\phi}.\end{aligned}$$

So we have

$$b_3 = \mathrm{Re}\{i\gamma(I+H_0)(B\bar{B}_X - \frac{1}{2}\bar{B}B_X) - 2ik^2\bar{B}|B|^2e^{-i\phi}\}. \tag{3.106}$$

We decompose $\tilde{\zeta}$ into vanishing part and periodic part:

$$\tilde{\zeta} := \alpha + \tilde{\xi}_0 + \tilde{\xi}_1, \qquad \tilde{\omega} := \alpha + \tilde{\xi}_0. \tag{3.107}$$

where

$$\begin{aligned}\tilde{\xi}_1 =&\epsilon B_1 e^{i\phi} + \epsilon^2\frac{ik}{2}(I+H_0)(|B|^2 - |B_0|^2)\\ &+ \epsilon^3\Big\{-\frac{1}{2}k^2(\bar{B}|B|^2 - \bar{B}_0|B_0|^2)e^{-i\phi} + \frac{1}{2}(I+H_0)(\bar{B}B_X)\Big\}.\end{aligned} \tag{3.108}$$

$$\tilde{\xi}_0 =\tilde{\zeta} - \tilde{\xi}_1 - \alpha = \epsilon B_0 e^{i\phi} + ik|B_0|^2\epsilon^2 - \frac{1}{2}\epsilon^3k^2\bar{B}_0|B_0|^2e^{-i\phi}. \tag{3.109}$$

So $\tilde{\omega}-\alpha$ is periodic. Also, we decompose $\tilde{b}$ as $\tilde{b}_1+\tilde{b}_0$, where

$$\begin{aligned}\tilde{b}_1 =& -\epsilon^2\omega k(|B|^2-|B_0|^2)+\epsilon^3\Big\{\operatorname{Re}\{i\omega(I+H_0)(B\bar{B}_X-\frac{1}{2}\bar{B}B_X)\\ &-2ik^2(\bar{B}|B|^2-\bar{B}_0|B_0|^2)e^{-i\phi}\Big\}.\end{aligned} \tag{3.110}$$

$$\tilde{b}_0=-\epsilon^2|B_0|^2+\epsilon^3\operatorname{Re}\{-2ik^2\bar{B}_0|B_0|^2e^{-i\phi}\}. \tag{3.111}$$

So $\tilde{b}_1\in H^s$, and $\tilde{b}_0$ is periodic. We choose $\tilde{A}$ as

$$\tilde{A}:=A^{(0)}+\epsilon A^{(1)}+\epsilon^2A^{(2)}=1.$$

Notation. Denote

$$\begin{cases}\xi=\zeta-\alpha,\\ \xi_0=\omega-\alpha,\quad \xi_1=\xi-\xi_0,\\ r_0=\omega-\tilde{\omega},\\ r_1=\xi_1-\tilde{\xi}_1, \tilde{D}_t:=\partial_t+\tilde{b}\partial_\alpha,\\ \tilde{D}_t^0=\partial_t+\tilde{b}_0\partial_\alpha,\\ \tilde{\mathcal{P}}:=\tilde{D}_t^2-i\tilde{A}\partial_\alpha,\\ \tilde{\mathcal{P}}_0:=(\tilde{D}_t^0)^2-i\tilde{A}_0\partial_\alpha.\end{cases} \tag{3.112}$$

And recall that the leading order $\zeta^{(1)}$ is

$$\zeta^{(1)}=Be^{i\phi}:=\zeta_0^{(1)}+\zeta_1^{(1)},$$

where $B=B_0+B_1$ solves NLS, B_0 is a constant, and $B_1\in H^{s+7}(\mathbb{R})$.

$$\zeta=Be^{i\phi},\quad \zeta_0^{(1)}=B_0e^{i\phi},\quad \zeta_1^{(1)}=B_1e^{i\phi}, \tag{3.113}$$

Because B scales like ϵ^2 in t, in order to observe the modulation of the amplitute, the solution ζ must exist on time interval whose length scales like ϵ^{-2}, i.e., we must have long time existence for the water waves system.

3.5.5 Well posedness of NLS

Theorem III.3. *Let $s \geqslant 1$. There exists $e_0 = e_0(\|B_1(0)\|_{H^s}) > 0$ such that the cauchy problem*

$$\begin{cases} iB_t + B_{xx} = -2|B|^2 B, \\ B(0) = B_0(0) + B_1(0) = 1 + f, \quad B_1(0) \in H^s(\mathbb{R}). \end{cases} \tag{3.114}$$

is locally well-posed on $[0, e_0]$, and satisfies

$$\|B\|_{L_t^\infty H_x^s(\mathbb{R})} \leqslant C(\|B_1(0)\|_{H^s(\mathbb{R})}). \tag{3.115}$$

For a proof, see for example [28].

Remark III.29. The global well-posedness of (3.114) for $B_1(0) \in H^s(\mathbb{R})$ is still open, due to the lack of coercive conservation law. In [9], D. Bilman and P.D. Miller presented a more robust version of the inverse scattering transform that admits the Peregrine solution in particular and all $L^1(\mathbb{R})$ perturbations of the background more generally. Therefore, for a large class of initial data, (3.114) admits global solutions.

In the following sections, we obtain energy estimates for the remainder terms r_0, r_1, respectively. With good energy estimates on the remainder terms, we are able to prove existence of solutions to full water wave equations whose leading term modulated according to the NLS.

3.6 Energy estimate I: r_0

Because formally, $\zeta - \tilde{\zeta} = O(\epsilon^4)$, $\lim_{\alpha\to\pm\infty}(\zeta - \omega) = 0$, and $\omega - \alpha$ periodic, we have $(\tilde{\omega}, \tilde{D}_t^0\tilde{\omega}, \tilde{b}_0, \tilde{A}_0)$ approximate (3.38) with error $O(\epsilon^4)$, at least formally.

In this section, we obtain a priori energy estimates for the remainder r_0. The idea is the same as that in [63]: use the facts that $\mathcal{P}_0(I-\mathcal{H}_p)\xi_0 = cubic$ and $\tilde{\omega}, \tilde{D}_t^0\tilde{\omega}, \tilde{b}_0, \tilde{A}_0$ approximates $\omega, D_t^0\omega, b_0, A_0$ up to ϵ^4, respectively, we derive water wave equations for a quantity which is equivalent to r_0. With these equations, we can then obtain energy estimates for r_0 on time scale ϵ^{-2}.

Remark III.30. As before, we use the periodic Hilbert transform. The nonlocal Hilbert transform $\mathcal{H}_\omega$ is used when we estimate the error term r_1.

First, let's derive water wave equation for r_0.

3.6.1 Governing equation for r_0

We have

$$\begin{aligned}
&((D_t^0)^2 - iA_0\partial_\alpha)(I-\mathcal{H}_p)r_0\\
=&((D_t^0)^2 - iA_0\partial_\alpha)(I-\mathcal{H}_p)\tilde{\omega} - 2[D_t^0\omega, \mathcal{H}_p\frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_p\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega\\
&+\frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha,t) - D_t^0\omega(\beta,t)}{\sin(\frac{\pi}{2}(\omega(\alpha,t)-\omega(\beta,t)))}\Big)^2(\omega-\bar{\omega})(\beta)d\beta\\
:=&\mathcal{G}.
\end{aligned}$$

We split $\mathcal{G}$ as $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4$, where

$$\mathcal{G}_1 := ((D_t^0)^2 - iA_0\partial_\alpha)(I-\mathcal{H}_p)\tilde{\omega} - ((\tilde{D}_t^0)^2 - i\tilde{A}_0\partial_\alpha)(I-\tilde{\mathcal{H}}_p)\tilde{\omega}. \tag{3.116}$$

$$\mathcal{G}_2 := -2[D_t^0\omega, \mathcal{H}_p\frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_p\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega + 2[\tilde{D}_t^0\tilde{\omega}, \tilde{\mathcal{H}}_p\frac{1}{\tilde{\omega}_\alpha} + \bar{\tilde{\mathcal{H}}}_p\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha\tilde{D}_t^0\tilde{\omega}, \tag{3.117}$$

and

$$
\begin{aligned}
\mathcal{G}_3 :=& \frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha,t)-D_t^0\omega(\beta,t)}{\sin(\frac{\pi}{2}(\omega(\alpha,t)-\omega(\beta,t)))}\Big)^2(\omega-\bar{\omega})(\beta)d\beta\\
&-\frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha,t)-\tilde{D}_t^0\tilde{\omega}(\beta,t)}{\sin(\frac{\pi}{2}(\tilde{\omega}(\alpha,t)-\tilde{\omega}(\beta,t)))}\Big)^2(\tilde{\omega}-\bar{\tilde{\omega}})(\beta)d\beta,
\end{aligned}
\tag{3.118}
$$

and

$$
\begin{aligned}
\mathcal{G}_4 :=& (\tilde{D}_t^0)^2-i\tilde{A}_0\partial_\alpha)(I-\mathcal{H}_p)\tilde{\omega}-2[\tilde{D}_t^0\tilde{\omega},\tilde{\mathcal{H}}_p\frac{1}{\tilde{\omega}_\alpha}+\bar{\tilde{\mathcal{H}}}_p\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha\tilde{D}_t^0\tilde{\omega}\\
&+\frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha,t)-\tilde{D}_t^0\tilde{\omega}(\beta,t)}{\sin(\frac{\pi}{2}(\tilde{\omega}(\alpha,t)-\tilde{\omega}(\beta,t)))}\Big)^2(\tilde{\omega}-\bar{\tilde{\omega}})(\beta)d\beta.
\end{aligned}
\tag{3.119}
$$

3.6.2 Governing equation for $D_t^0(I-\mathcal{H}_p)r_0$

We need to derive an equation to control $D_t^0 r_0$ as well. We consider instead the quantity

$$
\sigma_0 := (I-\mathcal{H}_p)\Big\{D_t^0(I-\mathcal{H}_p)(\omega-\alpha)-\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)\Big\}.
$$

We have

$$
\begin{aligned}
&((D_t^0)^2-iA_0\partial_\alpha)(I-\mathcal{H}_p)D_t^0(I-\mathcal{H}_p)(\omega-\alpha)\\
=&-[(D_t^0)^2-iA_0\partial_\alpha,\mathcal{H}_p]D_t^0(I-\mathcal{H}_p)(\omega-\alpha)\\
&+(I-\mathcal{H}_p)((D_t^0)^2-iA_0\partial_\alpha)D_t^0(I-\mathcal{H}_p)(\omega-\alpha)\\
=&-2[D_t^0\zeta,\mathcal{H}_p]\frac{\partial_\alpha(D_t^0)^2(I-\mathcal{H}_p)(\omega-\alpha)}{\omega_\alpha}\\
&+\frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2\partial_\beta D_t^0(I-\mathcal{H}_p)(\omega(\beta)-\beta)d\beta\\
&+(I-\mathcal{H}_p)[(D_t^0)^2-iA_0\partial_\alpha,D_t^0](I-\mathcal{H}_p)(\omega-\alpha)+(I-\mathcal{H}_p)D_t^0\mathcal{G}.
\end{aligned}
\tag{3.120}
$$

And we have

$$
\begin{aligned}
&((D_t^0)^2 - iA_0\partial_\alpha)(I-\mathcal{H}_p)\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)\\
=&-[(D_t^0)^2 - iA_0\partial_\alpha, \mathcal{H}_p]\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)\\
&+(I-\mathcal{H}_p)((D_t^0)^2 - iA_0\partial_\alpha)\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)\\
=&-2[D_t^0\omega, \mathcal{H}_p]\frac{\partial_\alpha D_t^0\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)}{\omega_\alpha}\\
&+\frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2\partial_\beta\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\beta)(\beta)d\beta\\
&+(I-\mathcal{H}_p)((D_t^0)^2 - iA_0\partial_\alpha)\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)\\
=&-2[D_t^0\omega, \mathcal{H}_p]\frac{\partial_\alpha D_t^0\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)}{\omega_\alpha}\\
&+\frac{1}{4\pi i}\int_{\mathbb{T}}\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\sin(\frac{\pi}{2}(\omega(\alpha)-\omega(\beta)))}\Big)^2\partial_\beta\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\beta)(\beta)d\beta\\
&+(I-\mathcal{H}_p)((\tilde{D}_t^0)^2 - i\tilde{A}_0\partial_\alpha)\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha)\\
&+(I-\mathcal{H}_p)((D_t^0)^2 - iA_0\partial_\alpha-(\tilde{D}_t^0)^2+i\tilde{A}_0\partial_\alpha)\tilde{D}_t^0(I-\tilde{\mathcal{H}}_p)(\tilde{\omega}-\alpha).
\end{aligned}
\tag{3.121}
$$

Use (3.120) and (3.121), we have

$$((D_t^0)^2 - iA_0\partial_\alpha)\sigma_0 = \text{fourth order}. \tag{3.122}$$

Use alsmost the derivation and the estimates as that in N. Totz and S. Wu's work ([63]), we obtain the following theorem.

Theorem III.4. *Let $s'\geqslant 6$. Let $\mathcal{T}$ be given as in Theorem I.16. Let $\tilde{\omega}, \tilde{b}_0, \tilde{A}_0$ be given as in Section 5. There is compatible initial data*

$$(\omega(0), D_t^0\omega(0), (D_t^0)^2\omega(0))$$

to water waves system (3.38) such that

$$\left\|(\omega_\alpha(0)-1, D_t^0\omega(0), (D_t^0)^2\omega(0)) - (\tilde{\omega}_\alpha - 1, \tilde{D}_t^0\tilde{\omega}(t), (\tilde{D}_t^0)^2\tilde{\omega}(t))\right\|_{\mathcal{H}^{s'}(\mathbb{T})} \leqslant C\epsilon^2, \tag{3.123}$$

where $\mathcal{H}^s(\mathbb{T}) := H^{s'}(\mathbb{T}) \times H^{s'+1/2}(\mathbb{T}) \times H^{s'}(\mathbb{T})$. *Moreover, for all such initial data, there is a a unique solution* $(\omega, D_t^0\omega, b_0, A_0)$ *to (3.38) on time interval* $[0, \mathcal{T}\epsilon^{-2}]$ *such that*

$$\sup_{t\in[0,\mathcal{T}\epsilon^{-2}]}\left\|(\omega_\alpha - 1, D_t^0\omega(t), (D_t^0)^2\omega(t)) - (\tilde{\omega}_\alpha - 1, \tilde{D}_t^0\tilde{\omega}(t), (\tilde{D}_t^0)^2\tilde{\omega}(t))\right\|_{\mathcal{H}^{s'}(\mathbb{T})} \leqslant C\epsilon^{3/2}. \tag{3.124}$$

Also, there is some constant $C = C(s')$ *such that*

$$\sup_{t\in[0,\mathcal{T}\epsilon^{-2}]}\left(\left\|b_0 - \tilde{b}_0\right\|_{H^{s'}(\mathbb{T})} + \left\|A_0 - \tilde{A}_0\right\|_{H^{s'}(\mathbb{T})} + \left\|D_t^0 b_0 - \tilde{D}_t^0\tilde{b}_0\right\|_{H^{s'}(\mathbb{T})}\right) \leqslant C\epsilon^{5/2},. \tag{3.125}$$

$$\sup_{t\in[0,\mathcal{T}\epsilon^{-2}]}\left(\left\|b_0 - \tilde{b}_0\right\|_{W^{s'-1,\infty}} + \left\|A_0 - \tilde{A}_0\right\|_{W^{s'-1,\infty}} + \left\|D_t^0 b_0 - \tilde{D}_t^0\tilde{b}_0\right\|_{W^{s'-1,\infty}}\right) \leqslant C\epsilon^3. \tag{3.126}$$

3.7 Governing equation for r_1

In this section, we derive a governing equation for the remainder term r_1. Because we need to obtain long time energy estimates for the error term (we'll prove that the error term r_1 has norm $O(\epsilon^{3/2})$ in Sobolev space), the nonlinearities of the equations governing r_1 has to be at least of fourth order. Since $(D_t^2 - iA\partial_\alpha)r_1$ is not obviously fourth order, we find some equivalent quantity of r_1 and consider its water wave equation.

3.7.1 Governing equation for r_1

Recall that

$$\lambda := (I - \mathcal{H}_\zeta)\xi - (I - \mathcal{H}_\omega)\xi_0.$$

We have shown that $\mathcal{P}\lambda$ consists of cubic and higher order nonlinearities. Define

$$\tilde{\lambda} := (I - \mathcal{H}_{\tilde{\zeta}})\tilde{\xi} - (I - \mathcal{H}_{\tilde{\omega}})\tilde{\xi}_0.$$

Because $\tilde{\lambda}$ approximates λ to $O(\epsilon^4)$, we expect

$$\mathcal{P}\tilde{\lambda} = O(\epsilon^3), \quad and \quad \mathcal{P}(\lambda - \tilde{\lambda}) = O(\epsilon^4).$$

Also, as we can see later, $\lambda - \tilde{\lambda}$ is equivalent to r_1 in appropriate sense. So it's natural to consider $\mathcal{P}(\lambda - \tilde{\lambda})$. However, $\lambda - \tilde{\lambda}$ is not the boundary value of a holomorhpic function in $\Omega(t)^c$. There is the trouble of losing derivatives in energy estimates if we use $\lambda - \tilde{\lambda}$. To resolve this problem, we consider the quantity ρ_1 define by

$$\rho_1 := (I - \mathcal{H}_{\zeta})(\lambda - \tilde{\lambda}). \tag{3.127}$$

Then ρ_1 is holomorphic in $\Omega(t)^c$.

We show that $\mathcal{P}\rho_1$ consists of fourth and higher order terms. The idea is to take advantage of the facts that $\mathcal{P}[(I - \mathcal{H}_{\zeta})(\zeta - \alpha) - (I - \mathcal{H}_{\omega})(\omega - \alpha)]$ is approximated by $\tilde{\mathcal{P}}[(I - \mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta} - \alpha) - (I - \mathcal{H}_{\tilde{\omega}})(\tilde{\omega} - \alpha)]$ to $O(\epsilon^4)$, so their difference would be of order $O(\epsilon^4)$.

To be precise, because $\tilde{\zeta}, \tilde{b}$ approximate ζ, b to the order of $O(\epsilon^4)$, we have $(\tilde{\zeta}, \tilde{D}_t\tilde{\zeta}, \tilde{b}, \tilde{A})$

satisfy (3.89) to the order of $O(\epsilon^4)$, i.e.,

$$\begin{aligned}
&(\tilde{D}_t^2 - i\tilde{A}\partial_\alpha)[(I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha) - (I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)]\\
=&-2[\tilde{D}_t\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha} + \bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha \tilde{D}_t\tilde{\zeta} + 2[\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha} + \bar{\mathcal{H}}_{\tilde{\omega}}\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha \tilde{D}_t^0\tilde{\omega}\\
&+\frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t\tilde{\zeta}(\alpha,t) - \tilde{D}_t\tilde{\zeta}(\beta,t)}{\tilde{\zeta}(\alpha,t)-\tilde{\zeta}(\beta,t)}\Big)^2(\tilde{\zeta}-\bar{\tilde{\zeta}})_\beta d\beta\\
&-\frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha,t) - \tilde{D}_t^0\tilde{\omega}(\beta,t)}{\tilde{\omega}(\alpha,t)-\tilde{\omega}(\beta,t)}\Big)^2(\tilde{\omega}-\bar{\tilde{\omega}})_\beta d\beta\\
&+\tilde{D}_t\tilde{b}_1\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha) + \tilde{b}_1\tilde{D}_t\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)\\
&+\tilde{b}_1\partial_\alpha\tilde{D}_t^0(I-\mathcal{H}_\omega)(\tilde{\omega}-\alpha) - i\tilde{A}_1\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha) + \epsilon^4\mathcal{R},
\end{aligned}$$

where $\epsilon^4\mathcal{R}$ is a known function (in terms of $B(X,T)$) which satisfies

$$\left\|\epsilon^4\mathcal{R}\right\|_{H^s} \leqslant C\epsilon^{7/2}.$$

Remark III.31. Throughout this paper, we'll use the notation $\epsilon^4\mathcal{R}$ frequently (and sometimes $\epsilon^4\mathcal{R}_1, \epsilon^4\mathcal{R}_2$). It might represent different quantities. However, it always represents a quantity which is in terms of $B(X,T)$ and ϵ, and satisfies the estimate

$$\left\|\epsilon^4\mathcal{R}\right\|_{H^{s+7}} \leqslant C\epsilon^{7/2}.$$

So we have

$$\begin{aligned}
&(D_t^2 - iA\partial_\alpha)(\lambda-\tilde{\lambda}) = \mathcal{P}[(I-\mathcal{H}_\zeta)(\zeta-\alpha) - (I-\mathcal{H}_\omega)(\omega-\alpha)]\\
&-\tilde{\mathcal{P}}[(I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha) - (I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)]\\
&+(\mathcal{P}-\tilde{\mathcal{P}})[(I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha) - (I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)]\\
:=&\sum_{m=1}^{5}\mathcal{R}_{1m}.
\end{aligned} \tag{3.128}$$

where

$$\begin{aligned}\mathcal{R}_{11} =& -2[D_t\zeta, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta + 2[D_t^0\omega, \mathcal{H}_\omega\frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega \\ & -2[\tilde{D}_t\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha} + \bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha \tilde{D}_t\tilde{\zeta} + 2[\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha} + \bar{\mathcal{H}}_{\tilde{\omega}}\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha \tilde{D}_t^0\tilde{\omega}.\end{aligned} \tag{3.129}$$

$$\begin{aligned}\mathcal{R}_{12} =&\Big\{\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta \\ & - \frac{1}{\pi i}\int\Big(\frac{D_t^0\omega(\alpha,t)-D_t^0\omega(\beta,t)}{\omega(\alpha,t)-\omega(\beta,t)}\Big)^2(\omega-\bar{\omega})_\beta d\beta\Big\} \\ & - \Big\{\frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t\tilde{\zeta}(\alpha,t)-\tilde{D}_t\tilde{\zeta}(\beta,t)}{\tilde{\zeta}(\alpha,t)-\tilde{\zeta}(\beta,t)}\Big)^2(\tilde{\zeta}-\bar{\tilde{\zeta}})_\beta d\beta \\ & - \frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha,t)-\tilde{D}_t^0\tilde{\omega}(\beta,t)}{\tilde{\omega}(\alpha,t)-\tilde{\omega}(\beta,t)}\Big)^2(\tilde{\omega}-\bar{\tilde{\omega}})_\beta d\beta\Big\},\end{aligned} \tag{3.130}$$

and

$$\begin{aligned}\mathcal{R}_{13} =& D_t b_1\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha) + b_1 D_t\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha) \\ & - \Big\{\tilde{D}_t\tilde{b}_1\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha) + \tilde{b}_1\tilde{D}_t\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)\Big\},\end{aligned} \tag{3.131}$$

and

$$\begin{aligned}\mathcal{R}_{14} =& b_1\partial_\alpha D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha) - iA_1\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha) \\ & - \Big\{\tilde{b}_1\partial_\alpha\tilde{D}_t^0(I-\mathcal{H}_\omega)(\tilde{\omega}-\alpha) - i\tilde{A}_1\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)\Big\},\end{aligned} \tag{3.132}$$

and

$$\mathcal{R}_{15} = \epsilon^4\mathcal{R}. \tag{3.133}$$

Denote $\mathcal{R}_{16}$ and $\mathcal{R}_{17}$ as follows:

$$\mathcal{R}_{16} = -2[D_t\zeta, \mathcal{H}_\zeta]\frac{\partial_\alpha D_t(\lambda-\tilde{\lambda})}{\zeta_\alpha}, \tag{3.134}$$

and

$$\mathcal{R}_{17} = \frac{1}{\pi i}\int \Big(\frac{D_t\zeta(\alpha) - D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2 \partial_\beta(\lambda - \tilde{\lambda})d\beta. \tag{3.135}$$

Then by (3.128), lemma III.15, we have

$$\begin{aligned}\mathcal{P}\rho_1 =&\mathcal{P}(I-\mathcal{H}_\zeta)(\lambda-\tilde{\lambda}) = (I-\mathcal{H}_\zeta)\mathcal{P}(\lambda-\tilde{\lambda}) - [\mathcal{P},\mathcal{H}_\zeta](\lambda-\tilde{\lambda})\\ =&(I-\mathcal{H}_\zeta)\sum_{m=1}^{5}\mathcal{R}_{1m} + \mathcal{R}_{16}+\mathcal{R}_{17}.\end{aligned} \tag{3.136}$$

Note that $\mathcal{R}_{16}$ and $\mathcal{R}_{17}$ are not obvious fourth order, so we need to explore the cancellations hidden behind when we estimate these terms.

3.7.2 Governing equation for time evolution of r_1

We need to control $D_t r_1$ as well. Denote

$$\delta := D_t(I-\mathcal{H}_\zeta)(\zeta-\alpha) - D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha).$$

We know that $\mathcal{P}\delta$ consists of cubic and higher order terms. Denote

$$\tilde{\delta} := \tilde{D}_t(I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha) - \tilde{D}_t^0(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha).$$

Then because $\tilde{\mathcal{P}}\tilde{\delta}$ approximates $\mathcal{P}\delta$ to $O(\epsilon^4)$, and $\mathcal{P}-\tilde{\mathcal{P}} = O(\epsilon^3)$, we expect $\mathcal{P}(\delta-\tilde{\delta}) = O(\epsilon^4)$. However, $\delta-\tilde{\delta}$ is not holomorphic in $\Omega(t)^c$, which would lose derivatives in energy estimates. So we consider the quantity $\sigma_1 := (I-\mathcal{H})_\zeta)(\delta-\tilde{\delta})$.

By direct calculation, we have

$$\begin{aligned}\mathcal{P}\delta =& D_tG - D_t^0G_0 + (\mathcal{P}-\mathcal{P}_0)D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)\\ &+ [\mathcal{P},D_t](I-\mathcal{H}_\zeta)(\zeta-\alpha) - [\mathcal{P}_0,D_t^0](I-\mathcal{H}_\omega)(\omega-\alpha).\end{aligned} \tag{3.137}$$

and

$$\begin{aligned}\tilde{\mathcal{P}}\tilde{\delta} =&\tilde{D}_t\tilde{G}-\tilde{D}_t^0\tilde{G}_0+(\tilde{\mathcal{P}}-\tilde{\mathcal{P}}_0)\tilde{D}_t^0(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)+\epsilon^4\mathcal{R}\\ &+[\tilde{\mathcal{P}},\tilde{D}_t](I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha)-[\tilde{\mathcal{P}}_0,\tilde{D}_t^0](I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)\end{aligned}\tag{3.138}$$

So we have

$$\begin{aligned}\mathcal{P}(\delta-\tilde{\delta}) =&\mathcal{P}\delta-\tilde{\mathcal{P}}\tilde{\delta}+(\mathcal{P}-\tilde{\mathcal{P}})\tilde{\delta}+\epsilon^4\mathcal{R}\\ =&D_tG-D_t^0G_0-\tilde{D}_t\tilde{G}+\tilde{D}_t^0\tilde{G}_0+(\mathcal{P}-\mathcal{P}_0)D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)\\ &-(\tilde{\mathcal{P}}-\tilde{\mathcal{P}}_0)\tilde{D}_t^0(I-\tilde{\omega})(\tilde{\omega}-\alpha)+[\mathcal{P},D_t](I-\mathcal{H}_\zeta)(\zeta-\alpha)\\ &-[\mathcal{P}_0,D_t^0](I-\mathcal{H}_\omega)(\omega-\alpha)-[\tilde{\mathcal{P}},\tilde{D}_t](I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha)\\ &+[\tilde{\mathcal{P}}_0,\tilde{D}_t^0](I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)+\epsilon^4\mathcal{R}\\ :=&\mathcal{S}_1.\end{aligned}\tag{3.139}$$

Then $\mathcal{S}_1$ is fourth order. So we have

$$\begin{aligned}\mathcal{P}\sigma_1 =&\mathcal{P}(I-\mathcal{H}_\zeta)(\delta-\tilde{\delta})\\ =&(I-\mathcal{H}_\zeta)\mathcal{P}(\delta-\tilde{\delta})-[\mathcal{P},\mathcal{H}](\delta-\tilde{\delta})\\ =&(I-\mathcal{H}_\zeta)\mathcal{S}_1-2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t(\delta-\tilde{\delta})}{\zeta_\alpha}\\ &+\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta(\delta-\tilde{\delta})d\beta.\end{aligned}\tag{3.140}$$

We use equations (3.136) and (3.140) to study the evolution of r_1. A first step is to construct an appropriate energy which controls certain norm of r_1, and then show that this control exists for a sufficiently long time.

3.7.3 Construction of energy

In this subsection, we construct energy for the water wave equations (3.136) and (3.140). The energy is essentially the same as the energy used by S. Wu in [71] and the energy by N.

Totz and S. Wu in [63].

First, let's recall again the basic energy estimates by S. Wu([71]):

Lemma III.32 (Basic lemma, Lemma 4.1 in [71])**.** *Let Θ satisfies the equation*

$$(D_t^2 - iA\partial_\alpha)\Theta = G$$

and Θ is smooth and decays fast at infinity. Let

$$E_0(t) := \int \frac{1}{A}|D_t\Theta(\alpha,t)|^2 + i\Theta(\alpha,t)\partial_\alpha\bar{\Theta}(\alpha,t)d\alpha. \tag{3.141}$$

Then

$$\frac{dE_0}{dt} = \int \frac{2}{A}\operatorname{Re}(D_t\Theta\bar{G}) - \frac{1}{A}\frac{a_t}{a}\circ\kappa^{-1}|D_t\Theta|^2 d\alpha. \tag{3.142}$$

Moreover, if Θ is the boundary value of a holomorphic function in $\Omega(t)^c$, then

$$\int i\Theta\partial_\alpha\bar{\Theta}d\alpha = -\int i\bar{\Theta}\partial_\alpha\Theta d\alpha \geqslant 0. \tag{3.143}$$

Notations: Denote

$$\rho_1^{(n)} := \partial_\alpha^n\rho_1, \qquad \sigma_1^{(n)} := \partial_\alpha^n\sigma_1. \tag{3.144}$$

Because $\rho_1^{(n)}$ and $\sigma_1^{(n)}$ are not necessarily holomorphic in $\Omega(t)^c$, if we decompose them as

$$\begin{aligned} \rho_1^n =& \frac{1}{2}(I-\mathcal{H}_\zeta)\rho_1^n + \frac{1}{2}(I+\mathcal{H}_\zeta)\rho_1^{(n)} := \phi_1^{(n)} + \mathcal{R}_1^{(n)} \\ \sigma_1^{(n)} =& \frac{1}{2}(I-\mathcal{H}_\zeta)\sigma_1^{(n)} + \frac{1}{2}(I+\mathcal{H}_\zeta)\sigma_1^n := \Psi_1^{(n)} + \mathcal{S}_1^{(n)}. \end{aligned} \tag{3.145}$$

and define

$$\mathcal{E}_n(t) := \int \frac{1}{A}|D_t\rho_1^{(n)}|^2 + i\phi_1^{(n)}\partial_\alpha\bar{\phi}_1^{(n)}d\alpha. \tag{3.146}$$

$$\mathcal{F}_n(t) := \int \frac{1}{A}|D_t\sigma_1^{(n)}|^2 + i\sigma_1^{(n)}\partial_\alpha\bar{\sigma}_1^{(n)}d\alpha. \tag{3.147}$$

Define the energy as

$$\mathcal{E}(t) := \sum_{n=0}^{s} (\mathcal{E}_n(t) + \mathcal{F}_n(t)). \tag{3.148}$$

By lemma III.32, each $\mathcal{E}_n$ is positive. σ_1^n might not be holomorphic in $\Omega(t)^c$. However, we'll show that it is still essentially positive. We'll show that this energy controls $\|D_t r_1\|_{H^s} + \|(r_1)_\alpha\|_{H^s}$.

3.7.4 Evolution of $\mathcal{E}_n$ and $\mathcal{F}_n$

To show that r_1 remains small (in the sense of some appropriate norm), we need to show that the energy $\mathcal{E}_s$ remains small for a long time. So we need to analyze the evolution of $\mathcal{E}_n$ and $\mathcal{F}_n$. Note that

$$\begin{aligned}(D_t^2 - iA\partial_\alpha)\rho_1^{(n)} =& \partial_\alpha^n (D_t^2 - iA\partial_\alpha)\rho_1 + [D_t^2 - iA\partial_\alpha, \partial_\alpha^n]\rho_1 \\ =& \partial_\alpha^n (I - \mathcal{H}_\zeta) \sum_{m=1}^{5} \mathcal{R}_{1m} + \partial_\alpha^n (\mathcal{R}_{16} + \mathcal{R}_{17}) + [D_t^2 - iA\partial_\alpha, \partial_\alpha^n]\rho_1 \\ :=& \mathcal{C}_{1,n}.\end{aligned} \tag{3.149}$$

Similarly, we derive governing equation for $\sigma_1^{(n)} = \partial_\alpha^n \sigma_1$. We have

$$\begin{aligned}(D_t^2 - iA\partial_\alpha)\sigma_1^{(n)} =& \partial_\alpha^n (D_t^2 - iA\partial_\alpha)\sigma_1 + [D_t^2 - iA\partial_\alpha, \partial_\alpha^n]\sigma_1 \\ :=& C_{2,n}.\end{aligned} \tag{3.150}$$

By basic lemma III.32, equations (3.149) and (3.150), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_n(t) =& \int \frac{2}{A} \operatorname{Re}(D_t \rho_1^{(n)} \bar{\mathcal{C}}_{1,n}) - \frac{1}{A}\frac{a_t}{a} \circ \kappa^{-1} |D_t \rho_{1,n}^{(n)}|^2 d\alpha \\ &+ 2 \operatorname{Im} \int \partial_t \mathcal{R}_1^{(n)} \partial_\alpha \bar{\phi}_1^{(n)} + \partial_t \phi_1^{(n)} \partial_\alpha \bar{\mathcal{R}}_{1,n}^{(n)} + \partial_t \mathcal{R}_1^{(n)} \partial_\alpha \bar{\mathcal{R}}_1^{(n)}\end{aligned} \tag{3.151}$$

And

$$\frac{d}{dt}\mathcal{F}_n^{\sigma_1}(t)=\int\frac{2}{A}\operatorname{Re}(D_t\sigma_1^{(n)}\bar{\mathcal{C}}_{2,n})-\frac{1}{A}\frac{a_t}{a}\circ\kappa^{-1}|D_t\sigma_1^{(n)}|^2d\alpha. \tag{3.152}$$

3.8 Bound for some quantities

In this section, we obtain bounds for the quantities which will be used in the energy estimates in next section. We bound these quantities in terms of an auxiliary quantity E_s, which is essentially equivalent to the energy $\mathcal{E}_s$.

3.8.1 An auxiliary quantity for the energy functional and an a priori assumption

The energy functional $\mathcal{E}_s$ is not very convenient in the energy estimates, so we introduce the quantity

$$E_s^{1/2}:=\|D_tr_1\|_{H^s}+\|(r_1)_\alpha\|_{H^s}+\left\|D_t^2r_1\right\|_{H^s}. \tag{3.153}$$

Let $T_0>0$, we make the following a priori assumption

$$\sup_{t\in[0,T_0]}E_s(t)^{1/2}\leqslant\epsilon. \tag{3.154}$$

Remark III.33. We'll eventually show that on time scale ϵ^{-2}, $\mathcal{E}_s\lesssim\epsilon^3$ and

$$E_s(t)\leqslant C(\mathcal{E}+\epsilon^{5/2}), \tag{3.155}$$

therefore,

$$\sup_{t\in[0,T_0]}E_s(t)\lesssim\epsilon^{3/2}, \tag{3.156}$$

which is much better than (3.154). Since this a priori assumption is easy to justify by a bootstrap argument, we won't provide the details for this justification.

We'll control $\frac{d\mathcal{E}}{dt}$ in terms of E_s and ϵ, then we can obtain energy estimates on a lifespan of lengh $O(\epsilon^{-2})$. For this purpose, we control the quantities appear in the energy estimates in terms of E_s and ϵ.

Convention. In this and the next section, if not specified, then

$$0 \leqslant t \leqslant \min\{T_0, \mathcal{T}\epsilon^{-2}\}$$

and the bootstrap assumption (3.154) holds. Here, $\mathcal{T}$ is the same as that in Theorem I.16.

3.8.1.1 Consequence of the a priori assumption

Lemma III.34. *Assume the bootstrap assumption (3.154). We have*

$$\|\zeta_\alpha - 1\|_{W^{s-1,\infty}} \leqslant C\epsilon. \tag{3.157}$$

Proof. We have

$$\zeta_\alpha - 1 = (r_1 + r_0 + \tilde{\zeta})_\alpha - 1 = (r_1)_\alpha + (\tilde{\zeta}_\alpha - 1 + (r_0)_\alpha).$$

So we have

$$\|\zeta_\alpha - 1\|_{W^{s-1,\infty}} \leqslant \|(r_1)_\alpha\|_{W^{s-1,\infty}} + \left\|r_0 + \tilde{\zeta}_\alpha - 1\right\|_{W^{s-1,\infty}} \leqslant C\epsilon. \tag{3.158}$$

□

We'll need to use Lemma II.33. For convenience, we record it as follows.

Lemma III.35. *Assume the bootstrap assumption (3.154), let f, h be real functions. Assume*

$$(I - \mathcal{H}_\zeta)h\bar{\zeta}_\alpha = g \qquad or \qquad (I - \mathcal{H}_\zeta)h = g.$$

Then we have for any $t \in [0, T_0]$,

$$\|h\|_{H^s} \leqslant 2\|g\|_{H^s}. \tag{3.159}$$

3.8.1.2 The equivalence of ρ_1 and r_1

Lemma III.36. *Assume the a priori assumption (3.154). We have*

$$\|\partial_\alpha(\rho_1 - 2r_1)\|_{H^s} \leqslant C(\epsilon E_s^{1/2} + \epsilon^{5/2}), \qquad \|D_t(\rho_1 - 2r_1)\|_{H^{s+1/2}} \leqslant C(\epsilon E_s^{1/2} + \epsilon^{5/2}). \quad (3.160)$$

Proof. We have $\lambda := (I - \mathcal{H}_\zeta)(\zeta - \alpha) - (I - \mathcal{H}_\omega)(\omega - \alpha)$, and $\tilde{\lambda} := (I - \mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta} - \alpha) - (I - \mathcal{H}_{\tilde{\omega}})(\tilde{\omega} - \alpha)$. Recall that $\rho_1 := (I - \mathcal{H}_\zeta)(\lambda - \tilde{\lambda})$. So we have

$$\partial_\alpha \rho_1 = \partial_\alpha (I - \mathcal{H}_\zeta)(\lambda - \tilde{\lambda}).$$

We have

$$\lambda - \tilde{\lambda} = (I - \mathcal{H}_\zeta) r_1 + (\mathcal{H}_\omega - \mathcal{H}_\zeta)(\omega - \alpha) + (\mathcal{H}_{\tilde{\zeta}} - \mathcal{H}_\zeta)(\tilde{\zeta} - \alpha) - (\mathcal{H}_{\tilde{\omega}} - \mathcal{H}_\zeta)(\tilde{\omega} - \alpha). \quad (3.161)$$

Denote

$$\tilde{\gamma} := (\mathcal{H}_\omega - \mathcal{H}_\zeta)(\omega - \alpha) + (\mathcal{H}_{\tilde{\zeta}} - \mathcal{H}_\zeta)(\tilde{\zeta} - \alpha) - (\mathcal{H}_{\tilde{\omega}} - \mathcal{H}_\zeta)(\tilde{\omega} - \alpha). \quad (3.162)$$

So we have

$$\partial_\alpha \rho_1 = \partial_\alpha (I - \mathcal{H}_\zeta)(\lambda - \tilde{\lambda}) \quad (3.163)$$

$$= \partial_\alpha (I - \mathcal{H}_\zeta)(I - \mathcal{H}_\zeta) r_1 + \partial_\alpha (I - \mathcal{H}_\zeta)\tilde{\gamma} \quad (3.164)$$

$$= 2\partial_\alpha (I - \mathcal{H}_\zeta) r_1 + \partial_\alpha (I - \mathcal{H}_\zeta)\tilde{\gamma} \quad (3.165)$$

$$= 2\partial_\alpha r_1 - \partial_\alpha (I + \mathcal{H}_\zeta) r_1 + \partial_\alpha (I - \mathcal{H}_\zeta)\tilde{\gamma}. \quad (3.166)$$

We are aiming to prove that

$$\left\| -\partial_\alpha (I + \mathcal{H}_\zeta) r_1 + \partial_\alpha (I - \mathcal{H}_\zeta)\tilde{\gamma} \right\|_{H^s} \leqslant C\epsilon E_s^{1/2} + C\epsilon^{5/2}. \quad (3.167)$$

For $\left\|\partial_\alpha (I+\mathcal{H}_\zeta) r_1\right\|_{H^s}$, we have

$$(I+\mathcal{H}_\zeta) r_1 = (\overline{\mathcal{H}_\zeta}+\mathcal{H}_\zeta) r_1 + (I-\overline{\mathcal{H}_\zeta}) r_1. \tag{3.168}$$

The kernal of $\overline{\mathcal{H}_\zeta}+\mathcal{H}_\zeta$ is of order one, so it's easy to obtain that

$$\left\|\partial_\alpha (\overline{\mathcal{H}_\zeta}+\mathcal{H}_\zeta) r_1\right\|_{H^s} \leqslant C\epsilon E_s^{1/2}. \tag{3.169}$$

Decompose

$$r_1 = (\zeta-\alpha) - (\tilde{\zeta}-\alpha) - (\omega-\alpha) + (\tilde{\omega}-\alpha).$$

Use $(I-\overline{\mathcal{H}_\zeta})(\zeta-\alpha)=0$, $(I-\overline{\mathcal{H}_\omega})(\omega-\alpha)=0$, we have

$$(I-\overline{\mathcal{H}_\zeta}) r_1 = -(I-\overline{\mathcal{H}_\zeta})(\tilde{\zeta}-\alpha) + (I-\overline{\mathcal{H}_\zeta})(\tilde{\omega}-\alpha) + (\overline{\mathcal{H}_\zeta}-\overline{\mathcal{H}_\omega})(\omega-\alpha) \tag{3.170}$$

$$= -(I-\overline{\mathcal{H}_\zeta})(\tilde{\zeta}-\alpha) + (I-\overline{\mathcal{H}_\omega})(\tilde{\omega}-\alpha) + (\overline{\mathcal{H}_\zeta}-\overline{\mathcal{H}_\omega})(\omega-\tilde{\omega}) \tag{3.171}$$

By the construction of $\tilde{\zeta}$ and $\tilde{\omega}$, we have

$$(I-\overline{\mathcal{H}_\zeta})(\tilde{\zeta}-\alpha) = O(\epsilon^4), \qquad (I-\overline{\mathcal{H}_\omega})(\tilde{\omega}-\alpha) = O(\epsilon^4). \tag{3.172}$$

Therefore,

$$\left\|\partial_\alpha (I-\overline{\mathcal{H}_\zeta})(\tilde{\zeta}-\alpha) - \partial_\alpha (I-\overline{\mathcal{H}_\omega})(\tilde{\omega}-\alpha)\right\|_{H^s} \leqslant C\epsilon^{7/2}. \tag{3.173}$$

Since

$$\|\omega-\tilde{\omega}\|_{W^{s,\infty}} = \|r_0\|_{W^{s,\infty}} \leqslant C\epsilon^2, \tag{3.174}$$

and

$$\|\partial_\alpha(\zeta-\omega)\|_{H^s} \leqslant C\epsilon^{1/2}, \tag{3.175}$$

we have

$$\left\|(\overline{\mathcal{H}_\zeta} - \overline{\mathcal{H}_\omega})(\omega - \tilde{\omega})\right\|_{H^s} \leqslant C\epsilon^{5/2}. \tag{3.176}$$

So we obtain

$$\|\partial_\alpha(\rho_1 - 2r_1)\|_{H^s} \leqslant C(\epsilon E_s^{1/2} + \epsilon^{5/2}). \tag{3.177}$$

Use similar argument, we have

$$\|D_t(\rho_1 - 2r_1)\|_{H^{s+1/2}} \leqslant C(\epsilon E_s^{1/2} + \epsilon^{5/2}). \tag{3.178}$$

□

Corollary III.37. *Assume the a priori assumption (3.154), we have*

$$\|\partial_\alpha \rho_1\|_{H^s} \leqslant C\epsilon, \qquad \|D_t\rho_1\|_{H^{s+1/2}} \leqslant C\epsilon. \tag{3.179}$$

3.8.2 Bound $\tilde{b}$, $\tilde{b}_1$, b_1 and $b_1 - \tilde{b}_1$

From the definition of $\tilde{b}$ and $\tilde{b}_1$, we have

$$\left\|\tilde{b}\right\|_{W^{s,\infty}} \leqslant C\epsilon^2, \tag{3.180}$$

and

$$\left\|\tilde{b}_1\right\|_{H^s} \leqslant C\epsilon^{3/2}. \tag{3.181}$$

It's not difficult to bound $\|b_1\|_{H^s}$ by $C\epsilon^{3/2}$, and therefore $\left\|b_1 - \tilde{b}_1\right\|_{H^s} \leqslant C\epsilon^{3/2}$. However, it turns out that we need a bound better than $C\epsilon^{3/2}$ for the quantity $\left\|b_1 - \tilde{b}_1\right\|_{H^s}$.

Because $\tilde{b}_1$ approximates b_1 to the order of $O(\epsilon^4)$, we have

$$\begin{aligned}(I - \mathcal{H}_{\tilde{\zeta}})\tilde{b}_1 = &- [\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\zeta}}]\frac{\bar{\tilde{\zeta}}_\alpha - 1}{\tilde{\zeta}_\alpha} - [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha} - \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}](\bar{\tilde{\zeta}}_\alpha - 1)\\ &- [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}](\bar{\tilde{\xi}}_1)_\alpha - (\mathcal{H}_{\tilde{\zeta}} - \mathcal{H}_{\tilde{\omega}})\tilde{b}_0 + \epsilon^4\mathcal{R},\end{aligned} \tag{3.182}$$

where $\epsilon^4\mathcal{R}$ is a function of $\tilde{b}_1$ such that $||\epsilon^4\mathcal{B}||_{H^{s+7}} \lesssim \epsilon^{7/2}$.

To derive a formula for $b_1 - \tilde{b}_1$, we subtract (3.77) from (3.182). In order to explore the cancellation relations and obtain good estimates, we group the similar terms together (the terminology 'similar' should be clear in the context). We obtain the following

$$
\begin{aligned}
&(I-\mathcal{H}_\zeta)(b_1-\tilde{b}_1) = (I-\mathcal{H}_\zeta)b_1 - (I-\mathcal{H}_{\tilde{\zeta}})\tilde{b}_1 - (\mathcal{H}_{\tilde{\zeta}}-\mathcal{H}_\zeta)\tilde{b}_1\\
=&-[D_t\zeta - D_t^0\omega, \mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} + [\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\zeta}}]\frac{\bar{\tilde{\zeta}}_\alpha - 1}{\tilde{\zeta}_\alpha} && := B_1\\
&-[D_t^0\omega, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} - \mathcal{H}_\omega\frac{1}{\omega_\alpha}](\bar{\zeta}_\alpha - 1) + [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha} - \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}](\bar{\tilde{\zeta}}_\alpha - 1) && := B_2\\
&-[D_t^0\omega, \mathcal{H}_\omega\frac{1}{\omega_\alpha}](\bar{\xi}_1)_\alpha + [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}](\bar{\tilde{\xi}}_1)_\alpha && := B_3\\
&-(\mathcal{H}_\zeta - \mathcal{H}_\omega)b_0 + (\mathcal{H}_{\tilde{\zeta}} - \mathcal{H}_{\tilde{\omega}})\tilde{b}_0 && := B_4\\
&+\epsilon^4\mathcal{R} && := B_5\\
&-(\mathcal{H}_{\tilde{\zeta}} - \mathcal{H}_\zeta)\tilde{b}_1 && := B_6.
\end{aligned}
$$

To estimate B_1, we write B_1 as

$$
\begin{aligned}
B_1 =& -[D_t\zeta - D_t^0\omega - \tilde{D}_t\tilde{\zeta} + \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}\\
&+[\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha} - \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}](\bar{\zeta}_\alpha - 1)\\
&-[\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}]\bar{r}_\alpha\\
:=& B_{11} + B_{12} + B_{13}.
\end{aligned}
$$

The terms consist of B_1 are 'similar'. The advantages of writing B_1 in this form are:

- Each $B_{1j}(j=1,2,3)$ contains a factor which is in H^s (therefore L^2 estimate is possible).
- Each B_{1j} contains a factor which explores the cancellation relations between the exact solution and the approximation.

3.8.2.1 Estimate $\|B_{11}\|_{H^s}$

To estimate B_{11}, we rewrite the quantity

$$\begin{aligned}
&D_t\zeta - D_t^0\omega - \tilde{D}_t\tilde{\zeta} + \tilde{D}_t^0\tilde{\omega}\\
=&(D_t - \tilde{D}_t)\zeta + \tilde{D}_t(\zeta-\tilde{\zeta}) - (D_t^0 - \tilde{D}_t^0)\omega - \tilde{D}_t^0(\omega-\tilde{\omega})\\
=&(b-\tilde{b})\zeta_\alpha + \tilde{D}_t r - (b_0-\tilde{b}_0)\omega_\alpha - \tilde{D}_t^0 r_0\\
=&(b_1-\tilde{b}_1)\zeta_\alpha + (b_0-\tilde{b}_0)(\zeta_\alpha - \omega_\alpha) + (\tilde{D}_t - \tilde{D}_t^0)r_0 + \tilde{D}_t r_1\\
=&(b_1-\tilde{b}_1)\zeta_\alpha + (b_0-\tilde{b}_0)\partial_\alpha\xi_1 + \tilde{b}_1\partial_\alpha r_0 + (\tilde{D}_t - D_t)r_1 + D_t r_1\\
=&(b_1-\tilde{b}_1)\zeta_\alpha + (b_0-\tilde{b}_0)\partial_\alpha\xi_1 + \tilde{b}_1\partial_\alpha r_0 - (b-\tilde{b})\partial_\alpha r_1 + D_t r_1\\
=&(b_1-\tilde{b}_1)(\zeta_\alpha - (r_1)_\alpha) + (b_0-\tilde{b}_0)\partial_\alpha\tilde{\xi}_1 + \tilde{b}_1(r_0)_\alpha + D_t r_1\\
=&(b_1-\tilde{b}_1)(\tilde{\zeta}_\alpha + (r_0)_\alpha) + (b_0-\tilde{b}_0)\partial_\alpha\tilde{\xi}_1 + \tilde{b}_1(r_0)_\alpha + D_t r_1.
\end{aligned} \tag{3.183}$$

Use proposition III.18 and estimate (3.158) for $\zeta_\alpha - 1$, we have

$$\begin{aligned}
&\|B_{11}\|_{H^s}\\
\leqslant& C\left\|(b_1-\tilde{b}_1)(\tilde{\zeta}_\alpha + (r_0)_\alpha) + (b_0-\tilde{b}_0)\partial_\alpha\tilde{\xi}_1 + \tilde{b}_1(r_0)_\alpha + D_t r_1\right\|_{H^s}\|\zeta_\alpha - 1\|_{W^{s-1,\infty}}\\
\leqslant& C\Big(\left\|b_1-\tilde{b}_1\right\|_{H^s}\left\|\tilde{\zeta}_\alpha + (r_0)_\alpha\right\|_{W^{s,\infty}} + \left\|\tilde{b}_1\right\|_{H^s}\|(r_0)_\alpha\|_{W^{s,\infty}} + \left\|b_0-\tilde{b}_0\right\|_{H^s}\left\|\partial_\alpha\tilde{\xi}_1\right\|_{H^s}\\
&+\|D_t r_1\|_{H^s}\Big)\|\zeta_\alpha - 1\|_{W^{s-1,\infty}}\\
\leqslant& C\Big(\left\|b_1-\tilde{b}_1\right\|_{H^s} + \epsilon^{3/2} + E_s^{1/2}\Big)\epsilon.
\end{aligned}$$

Here, we have used (see Theorem III.4)

$$\left\|\tilde{b}_1\right\|_{H^s} \leqslant C\epsilon^{3/2}, \quad \|(r_0)_\alpha\|_{W^{s-1,\infty}} \leqslant C\epsilon^2, \quad \|b_0-\tilde{b}_0\|_{W^{s,\infty}} \leqslant C\epsilon^2.$$

3.8.2.2 Estimate $\|B_{12}\|_{H^s}$.

For B_{12}, we write

$$\tilde{D}_t\tilde{\zeta}-\tilde{D}_t^0\tilde{\omega}=(\tilde{D}_t-\tilde{D}_t^0)\tilde{\zeta}+\tilde{D}_t^0(\tilde{\zeta}-\tilde{\omega})=\tilde{b}_1\partial_\alpha\tilde{\zeta}+\tilde{D}_t^0\tilde{\xi}_1.$$

Then by proposition III.18, we have

$$\begin{aligned}\|B_{12}\|_{H^s}=&\left\|\tilde{D}_t\tilde{\zeta}-\tilde{D}_t^0\tilde{\omega}\right\|_{H^s}\|\zeta_\alpha-1\|_{W^{s-1,\infty}}\left\|\tilde{\zeta}-\tilde{\omega}\right\|_{W^{s-1,\infty}}\\ \leqslant&||\tilde{b}_1\partial_\alpha\tilde{\zeta}+\tilde{D}_t^0\tilde{\xi}_1||_{H^s}||\zeta_\alpha-1||_{W^{s-1,\infty}}||\tilde{\zeta}-\tilde{\omega}||_{W^{s-1,\infty}}\\ \leqslant&C\epsilon^{3/2}||\zeta_\alpha-1||_{W^{s-1,\infty}}\\ \leqslant&C\epsilon^{5/2}.\end{aligned}\tag{3.184}$$

3.8.2.3 Estimate $\|B_{13}\|_{H^s}$.

Use proposition III.18, we have

$$\begin{aligned}\|B_{13}\|_{H^s}=&\left\|[\tilde{b}_1\tilde{\zeta}_\alpha+\tilde{D}_t^0\tilde{\xi}_1,\mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}]\bar{r}_\alpha\right\|_{H^s}\\ \leqslant&C\left\|\tilde{b}_1\tilde{\zeta}_\alpha+\tilde{D}_t^0\tilde{\xi}_1\right\|_{W^{s,\infty}}\|\partial_\alpha r_1\|_{H^s}+C\left\|\tilde{b}_1\tilde{\zeta}_\alpha+\tilde{D}_t^0\tilde{\xi}_1\right\|_{H^s}\|\partial_\alpha r_0\|_{W^{s-1,\infty}}\Big)\\ \leqslant&C\epsilon E_s^{1/2}+\epsilon^{5/2}.\end{aligned}$$

Here, we've used the estimates

$$\|\tilde{b}_1\|_{H^s}\leqslant C\epsilon^{3/2},\quad\|\tilde{D}_t^0\tilde{\xi}_1\|_{W^{s,\infty}}\leqslant C\epsilon,\quad\|\partial_\alpha r_0\|_{W^{s-1,\infty}}\leqslant C\epsilon^2,\quad\|\tilde{D}_t^0\tilde{\xi}_1\|_{H^s}\leqslant C\epsilon^{1/2}.\tag{3.185}$$

So we have

$$\|B_1\|_{H^s}\leqslant C\epsilon\left\|b_1-\tilde{b}_1\right\|_{H^s}+C\epsilon E_s^{1/2}+C\epsilon^{5/2}.\tag{3.186}$$

Use the same argument, we show that

$$\|B_2\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.187}$$

$$\|B_3\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.188}$$

$$\|B_4\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.189}$$

For B_5, it's trivially that

$$\|B_5\|_{H^s} \leqslant C\epsilon^{5/2}. \tag{3.190}$$

And

$$\|B_6\|_{H^s} \leqslant C\epsilon^{5/2} + CE_s^{1/2}. \tag{3.191}$$

By lemma III.35, we have

Lemma III.38. *Assume the a priori assumption (3.154), then we have*

$$\left\|b_1 - \tilde{b}_1\right\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.192}$$

Proof. From the estimates for $B_j, j = 1, ..., m$, we have

$$\left\|b_1 - \tilde{b}_1\right\|_{H^s} \leqslant C\epsilon\left\|b_1 - \tilde{b}_1\right\|_{H^s} + C\epsilon^{5/2} + C\epsilon E_s^{1/2}.$$

For ϵ small such that $C\epsilon < 1$, we have

$$\left\|b_1 - \tilde{b}_1\right\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.193}$$

□

Corollary III.39. *Under the assumptions of lemma III.38, we have*

$$\|b_1\|_{H^s} \leqslant C\epsilon^{3/2}, \qquad \|b\|_{W^{s-1,\infty}} \leqslant C\epsilon^2. \tag{3.194}$$

Corollary III.40. *Under the assumptions of lemma III.38, we have*

$$\|D_t\zeta\|_{W^{s-1,\infty}} \leqslant C\epsilon. \tag{3.195}$$

Proof. We write $D_t\zeta$ as

$$\begin{aligned} D_t\zeta =& D_t r_1 + D_t r_0 + D_t\tilde{\zeta} \\ =& D_t r_1 + D_t^0 r_0 + (b - b_0)\partial_\alpha r_0 + \tilde{D}_t\tilde{\zeta} + (b - \tilde{b})\partial_\alpha\tilde{\zeta}. \end{aligned} \tag{3.196}$$

So we have

$$\|D_t\zeta\|_{W^{s-1,\infty}} \leqslant C\epsilon + CE_s^{1/2} \leqslant C\epsilon. \tag{3.197}$$

□

3.8.3 Bound $D_t b_1$ and $D_t(b_1 - \tilde{b}_1)$

From the definition of $\tilde{D}_t\tilde{b}_1$, it's easy to obtain that

$$\left\|\tilde{D}_t\tilde{b}_1\right\|_{H^s} \leqslant C\epsilon^{3/2}, \qquad \left\|\tilde{D}_t\tilde{b}_1\right\|_{W^{s,\infty}} \leqslant C\epsilon^2. \tag{3.198}$$

To estimate $D_t(b_1 - \tilde{b}_1)$, we need to derive a formula for $D_t(b_1 - \tilde{b}_1)$. Since $D_t(b_1 - \tilde{b}_1)$ is real, it suffices to estimate $(I - \mathcal{H}_\zeta)D_t(b_1 - \tilde{b}_1)$. We have

$$\begin{aligned} (I - \mathcal{H}_\zeta)D_t(b_1 - \tilde{b}_1) =& \Big((I - \mathcal{H}_\zeta)D_t b_1 - (I - \mathcal{H}_{\tilde{\zeta}})\tilde{D}_t\tilde{b}_1\Big) \\ &+ (\mathcal{H}_\zeta - \mathcal{H}_{\tilde{\zeta}})\tilde{D}_t\tilde{b}_1 - (I - \mathcal{H}_\zeta)(b_1 - \tilde{b}_1)\partial_\alpha\tilde{b}_1. \end{aligned}$$

For $(I - \mathcal{H}_\zeta)D_t b_1$, we have a formula given by (3.81). Since $\tilde{b}_1, \tilde{b}_0, \tilde{\zeta}, \tilde{\omega}$ approximate b_1, b_0, ζ, ω

to the order $O(\epsilon^4)$, respectively, we have the following formula for $(I-\mathcal{H}_{\tilde{\zeta}})\tilde{D}_t\tilde{b}_1$:

$$
\begin{aligned}
&(I-\mathcal{H}_{\tilde{\zeta}})\tilde{D}_t\tilde{b}\\
=&[\tilde{D}_t\tilde{\zeta},\mathcal{H}_{\tilde{\zeta}}]\frac{\partial_\alpha(2\tilde{b}-\tilde{D}_t\bar{\tilde{\zeta}})}{\tilde{\zeta}_\alpha}-[\tilde{D}_t^0\tilde{\omega},\mathcal{H}_{\tilde{\omega}}]\frac{\partial_\alpha(2\tilde{b}_0-\tilde{D}_t^0\bar{\tilde{\omega}})}{\tilde{\zeta}_\alpha}-[\tilde{D}_t^2\tilde{\zeta},\mathcal{H}_{\tilde{\zeta}}]\frac{\bar{\tilde{\zeta}}_\alpha-1}{\tilde{\zeta}_\alpha}+[(\tilde{D}_t^0)^2\tilde{\omega},\mathcal{H}_{\tilde{\omega}}]\frac{\bar{\tilde{\omega}}_\alpha-1}{\tilde{\omega}_\alpha}\\
&+\frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t\tilde{\zeta}(\alpha)-\tilde{D}_t\tilde{\zeta}(\beta)}{\tilde{\zeta}(\alpha)-\tilde{\zeta}(\beta)}\Big)^2(\bar{\tilde{\zeta}}_\beta(\beta)-1)d\beta-\frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha)-\tilde{D}_t^0\tilde{\omega}(\beta)}{\tilde{\omega}(\alpha)-\tilde{\omega}(\beta)}\Big)^2(\bar{\tilde{\omega}}_\beta(\beta)-1)d\beta\\
&+(\mathcal{H}_{\tilde{\zeta}}-\mathcal{H}_{\tilde{\omega}})\tilde{D}_t^0\tilde{b}_0-(I-\mathcal{H}_{\tilde{\zeta}})\tilde{b}_1\partial_\alpha\tilde{b}_0+\epsilon^4\mathcal{R},
\end{aligned}
\tag{3.199}
$$

where $\epsilon^4\mathcal{R}$ satisfies $||\epsilon^4\mathcal{R}||_{H^{s+7}}\leqslant C\epsilon^{7/2}$. So we have

$$(I-\mathcal{H}_\zeta)D_t(b_1-\tilde{b}_1):=\sum_{m=1}^{6}F_m.$$

where each F_m are given as follows:

$$
\begin{aligned}
F_1=&[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha(2b-D_t\bar{\zeta})}{\zeta_\alpha}-[\tilde{D}_t\tilde{\zeta},\mathcal{H}_{\tilde{\zeta}}]\frac{\partial_\alpha(2\tilde{b}-\tilde{D}_t\bar{\tilde{\zeta}})}{\tilde{\zeta}_\alpha}\\
&-[D_t^0\omega,\mathcal{H}_\omega]\frac{\partial_\alpha(2b_0-D_t^0\bar{\omega})}{\zeta_\alpha}+[\tilde{D}_t^0\tilde{\omega},\mathcal{H}_{\tilde{\omega}}]\frac{\partial_\alpha(2\tilde{b}_0-\tilde{D}_t^0\bar{\tilde{\omega}})}{\tilde{\zeta}_\alpha}.
\end{aligned}
\tag{3.200}
$$

$$F_2=-[D_t^2\zeta,\mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha-1}{\zeta_\alpha}+[\tilde{D}_t^2\tilde{\zeta},\mathcal{H}_{\tilde{\zeta}}]\frac{\bar{\tilde{\zeta}}_\alpha-1}{\tilde{\zeta}_\alpha}+[(D_t^0)^2\omega,\mathcal{H}_\omega]\frac{\bar{\omega}_\alpha-1}{\omega_\alpha}-[(\tilde{D}_t^0)^2\tilde{\omega},\mathcal{H}_{\tilde{\omega}}]\frac{\bar{\tilde{\omega}}_\alpha-1}{\tilde{\omega}_\alpha}. \tag{3.201}$$

$$F_3 = \frac{1}{\pi i}\int \Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2(\bar{\zeta}_\beta(\beta)-1)d\beta - \frac{1}{\pi i}\int \Big(\frac{\tilde{D}_t\tilde{\zeta}(\alpha)-\tilde{D}_t\tilde{\zeta}(\beta)}{\tilde{\zeta}(\alpha)-\tilde{\zeta}(\beta)}\Big)^2(\bar{\tilde{\zeta}}_\beta(\beta)-1)d\beta$$
$$-\frac{1}{\pi i}\int \Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\omega(\alpha)-\omega(\beta)}\Big)^2(\bar{\omega}_\beta(\beta)-1)d\beta + \frac{1}{\pi i}\int \Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha)-\tilde{D}_t^0\tilde{\omega}(\beta)}{\tilde{\omega}(\alpha)-\tilde{\omega}(\beta)}\Big)^2(\bar{\tilde{\omega}}_\beta(\beta)-1)d\beta \tag{3.202}$$

$$F_4 = (\mathcal{H}_\zeta - \mathcal{H}_\omega)D_t^0 b_0 - (\mathcal{H}_{\tilde{\zeta}} - \mathcal{H}_{\tilde{\omega}})\tilde{D}_t^0\tilde{b}_0. \tag{3.203}$$

$$F_5 = -(I-\mathcal{H}_\zeta)b_1\partial_\alpha b_0 + (I-\mathcal{H}_{\tilde{\zeta}})\tilde{b}_1\partial_\alpha\tilde{b}_0. \tag{3.204}$$

$$F_6 = \epsilon^4\mathcal{R} \tag{3.205}$$

3.8.3.1 Estimate $\|F_1\|_{H^s}$.

We write F_1 as

$$\begin{aligned}
F_1 =& [D_t\zeta - \tilde{D}_t\tilde{\zeta}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}]\partial_\alpha(2b - D_t\bar{\zeta}) + [\tilde{D}_t\tilde{\zeta}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} - \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}}]\partial_\alpha(2b - D_t\bar{\zeta})\\
&+ [\tilde{D}_t\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}]\partial_\alpha(2(b-\tilde{b}) - (D_t\bar{\zeta} - \tilde{D}_t\bar{\tilde{\zeta}}))\\
&- \Big\{[D_t^0\omega - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\omega\frac{1}{\omega_\alpha}]\partial_\alpha(2b_0 - D_t^0\bar{\omega}) + [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\omega\frac{1}{\omega_\alpha} - \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}}]\partial_\alpha(2b_0 - D_t^0\bar{\omega})\\
&+ [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}]\partial_\alpha(2(b_0-\tilde{b}_0) - (D_t^0\bar{\omega} - \tilde{D}_t^0\bar{\tilde{\omega}}))\Big\}\\
=& \Big\{[D_t\zeta - \tilde{D}_t\tilde{\zeta}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}]\partial_\alpha(2b - D_t\bar{\zeta}) - [D_t^0\omega - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\omega\frac{1}{\omega_\alpha}]\partial_\alpha(2b_0 - D_t^0\bar{\omega})\Big\}\\
&+ \Big\{[\tilde{D}_t\tilde{\zeta}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} - \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}}]\partial_\alpha(2b - D_t\bar{\zeta}) - [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\omega\frac{1}{\omega_\alpha} - \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}}]\partial_\alpha(2b_0 - D_t^0\bar{\omega})\Big\}\\
&+ \Big\{[\tilde{D}_t\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}]\partial_\alpha(2(b-\tilde{b}) - (D_t\bar{\zeta} - \tilde{D}_t\bar{\tilde{\zeta}})) - [\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}]\partial_\alpha(2(b_0-\tilde{b}_0) - (D_t^0\bar{\omega} - \tilde{D}_t^0\bar{\tilde{\omega}}))\Big\}\\
:=& F_{11} + F_{12} + F_{13}.
\end{aligned}$$

The estimates for F_{11}, F_{12} and F_{13} are similar, so we give the details of F_{11} only. We rewrite F_{11} as

$$\begin{aligned}F_{11} =&[D_t\zeta - \tilde{D}_t\tilde{\zeta} - (D_t^0\omega - \tilde{D}_t^0\tilde{\omega}), \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}]\partial_\alpha(2b - D_t\bar{\zeta})\\ &+ [D_t^0\omega - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} - \mathcal{H}_\omega\frac{1}{\omega_\alpha}]\partial_\alpha(2b - D_t\bar{\zeta})\\ &+ [D_t^0\omega - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\omega\frac{1}{\omega_\alpha}]\partial_\alpha(2(b - b_0) - (D_t\bar{\zeta} - D_t^0\bar{\omega}))\\ :=&F_{111} + F_{112} + F_{113}.\end{aligned} \tag{3.206}$$

For F_{111}, use (3.183),

$$D_t\zeta - \tilde{D}_t\tilde{\zeta} - (D_t^0\omega - \tilde{D}_t^0\tilde{\omega}) = (b_1 - \tilde{b}_1)(\tilde{\zeta}_\alpha + (r_0)_\alpha) + \tilde{b}_1(r_0)_\alpha + D_t r_1. \tag{3.207}$$

Use (3.197), (3.192), and Corollary III.39, we have

$$\begin{aligned}\|F_{111}\|_{H^s} \leqslant& C\Big\|D_t\zeta - \tilde{D}_t\tilde{\zeta} - (D_t^0\omega - \tilde{D}_t^0\tilde{\omega})\Big\|_{H^s}\|2b - D_t\bar{\zeta}\|_{W^{s-1,\infty}}\\ \leqslant&\Big\|(b_1 - \tilde{b}_1)(\tilde{\zeta}_\alpha + (r_0)_\alpha) + \tilde{b}_1(r_0)_\alpha + D_t r_1\Big\|_{H^s}\|2b - D_t\bar{\zeta}\|_{W^{s-1,\infty}}\\ \leqslant& C\Big((\epsilon^{5/2} + \epsilon E_s^{1/2}) + \epsilon^{5/2} + E_s^{1/2}\Big)\epsilon\\ \leqslant& C\epsilon^{5/2} + C\epsilon E_s^{1/2}.\end{aligned} \tag{3.208}$$

For F_{112} and F_{113}, use proposition III.18, (3.192), (III.39), (3.197), it's easy to obtain the estimate

$$\|F_{112}\|_{H^s} + \|F_{113}\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.209}$$

So we obtain the estimate

$$\|F_{11}\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.210}$$

Estimates for F_{12}, F_{13} are similar to that of F_{11}, we obtain

$$\|F_{12}\|_{H^s} + \|F_{13}\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.211}$$

So we have

$$\|F_1\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.212}$$

3.8.3.2 Estimate $\|F_2\|_{H^s}$.

We rewrite F_2 as

$$\begin{aligned} F_2 =& -[D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_\zeta \frac{1}{\zeta_\alpha}](\bar{\zeta}_\alpha - 1) - [\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_\zeta \frac{1}{\zeta_\alpha} - \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}](\bar{\zeta}_\alpha - 1) - [\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}]\partial_\alpha(\bar{\zeta} - \bar{\tilde{\zeta}}) \\ &+ [(D_t^0)^2\omega - (\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\omega \frac{1}{\omega_\alpha}](\bar{\omega}_\alpha - 1) + [(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\omega\frac{1}{\omega_\alpha} - \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}}](\bar{\omega}_\alpha - 1) \\ &+ [(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}](\bar{\omega}_\alpha - \bar{\tilde{\omega}}_\alpha) \\ :=& \sum_{m=1}^{3} F_{2m}, \end{aligned} \tag{3.213}$$

where

$$F_{21} = -[D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}](\bar{\zeta}_\alpha - 1) + [(D_t^0)^2\tilde{\omega} - (\tilde{D}_t^0)^2\omega, \mathcal{H}_\omega\frac{1}{\omega_\alpha}](\bar{\omega}_\alpha - 1). \tag{3.214}$$

$$F_{22} = -[\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} - \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}](\bar{\zeta}_\alpha - 1) + [(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\omega\frac{1}{\omega_\alpha} - \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}}](\bar{\omega}_\alpha - 1) \tag{3.215}$$

$$F_{23} = -[\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}]\partial_\alpha(\bar{\zeta} - \bar{\tilde{\zeta}}) + [(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}](\bar{\omega}_\alpha - \bar{\tilde{\omega}}_\alpha). \tag{3.216}$$

The estimates for F_{21}, F_{22}, F_{23} are similar, so we give the details of estimates of F_{21} only. We rewrite F_{21} as

$$F_{21} := \sum_{m=1}^{3} F_{21m}, \tag{3.217}$$

where

$$F_{211} = -[D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta} - (D_t^0)^2\omega + (\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}](\bar{\zeta}_\alpha - 1). \tag{3.218}$$

$$F_{212} = -[(D_t^0)^2\omega - (\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} - \mathcal{H}_\omega\frac{1}{\omega_\alpha}](\bar{\zeta}_\alpha - 1). \tag{3.219}$$

$$F_{213} = -[(D_t^0)^2\omega - (\tilde{D}_t^0)^2\omega, \mathcal{H}_\omega \frac{1}{\omega_\alpha}](\bar{\zeta}_\alpha - \bar{\omega}_\alpha). \tag{3.220}$$

To estimate F_{211}, we rewrite $D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta} - (D_t^0)^2\omega + (\tilde{D}_t^0)^2\tilde{\omega}$ as

$$\begin{aligned}
&D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta} - (D_t^0)^2\omega + (\tilde{D}_t^0)^2\tilde{\omega}\\
=&D_t^2(\xi_1+\omega) - \tilde{D}_t^2(\tilde{\xi}_1+\tilde{\omega}) - (D_t^0)^2\omega + (\tilde{D}_t^0)^2\tilde{\omega}\\
=&D_t^2\xi_1 + (D_t^2 - (D_t^0)^2)\omega - \tilde{D}_t^2\tilde{\xi}_1 - ((D_t^0)^2 - (\tilde{D}_t^0)^2)\tilde{\omega}\\
=&D_t^2 r_1 + (D_t^2 - \tilde{D}_t^2)\tilde{\xi}_1 + (D_t^2 - (D_t^0)^2)\omega - ((D_t^0)^2 - (\tilde{D}_t^0)^2)\tilde{\omega}
\end{aligned} \tag{3.221}$$

Lemma III.41. *Assume the a priori assumption (3.154). We have*

$$\begin{aligned}
&\left\|(D_t^2 - \tilde{D}_t^2)\tilde{\xi}_1\right\|_{H^s} + \left\|(D_t^2 - (D_t^0)^2)\omega\right\|_{H^s} + \left\|((D_t^0)^2 - (\tilde{D}_t^0)^2)\tilde{\omega}\right\|_{H^s}\\
\leqslant& C\epsilon^{5/2} + C\epsilon\left\|D_t(b_1 - \tilde{b}_1)\right\|_{H^s}.
\end{aligned} \tag{3.222}$$

where

$$C = C(\left\|D_t^0\omega\right\|_{H^{s'}(\mathbb{T})}, \left\|\tilde{D}\tilde{\xi}_1\right\|_{H^s(\mathbb{R})})$$

for $s' > s + \frac{3}{2}$.

Proof. We have

$$\begin{aligned}
&D_t^2 - (D_t^0)^2 = D_t(D_t - D_t^0) + (D_t - D_t^0)D_t^0\\
=&D_t(b - b_0)\partial_\alpha + (b - b_0)\partial_\alpha D_t^0\\
=&(D_t b_1)\partial_\alpha + b_1 D_t\partial_\alpha + b_1\partial_\alpha D_t^0.
\end{aligned}$$

So we have

$$\begin{aligned}
&\left\|(D_t^2-(D_t^0)^2)\omega\right\|_{H^s}\\
\leqslant&\|(D_tb_1)\partial_\alpha\omega\|_{H^s}+\|b_1D_t\partial_\alpha\omega\|_{H^s}+\left\|b_1\partial_\alpha D_t^0\omega\right\|_{H^s}\\
\leqslant&\|D_tb_1\|_{H^s}\|\omega_\alpha\|_{W^{s,\infty}}+\|b_1\|_{H^s}\|\omega_\alpha\|_{W^{s,\infty}}+\|b_1\|_{H^s}\left\|\partial_\alpha D_t^0\omega\right\|_{W^{s,\infty}}\\
\leqslant&\left\|D_t(b_1-\tilde b_1)\right\|_{H^s}\|\omega_\alpha\|_{W^{s,\infty}}+\left\|D_t\tilde b_1\right\|_{H^s}\|\omega_\alpha\|_{W^{s,\infty}}+\|b_1\|_{H^s}\|\omega_\alpha\|_{W^{s,\infty}}+\|b_1\|_{H^s}\left\|\partial_\alpha D_t^0\omega\right\|_{W^{s,\infty}}\\
\leqslant&C\epsilon^{5/2}+C\epsilon\left\|D_t(b_1-\tilde b_1)\right\|_{H^s}.
\end{aligned}$$

We decompose $D_t^2-(D_t^0)^2$ and $(D_t^0)^2-(\tilde D_t^0)^2$ in a similar way. With these decompositions, the lemma follows easily. □

By (3.221), lemma III.41, and proposition III.18, we have

$$\|F_{211}\|_{H^s}\leqslant C\epsilon(\epsilon^{5/2}+\epsilon\left\|D_t(b_1-\tilde b_1)\right\|_{H^s})\leqslant C\epsilon^{7/2}+C\epsilon^2\left\|D_t(b_1-\tilde b_1)\right\|_{H^s}. \tag{3.223}$$

The estimates for F_{212}, F_{213} are the same and we obtain

$$\|F_{212}\|_{H^s}+\|F_{213}\|_{H^s}\leqslant C\epsilon^{5/2}+C\epsilon^2\left\|D_t(b_1-\tilde b_1)\right\|_{H^s}. \tag{3.224}$$

So we obtain

$$\|F_{21}\|_{H^s}\leqslant C\epsilon^{5/2}+C\epsilon^2\left\|D_t(b_1-\tilde b_1)\right\|_{H^s}. \tag{3.225}$$

Similarly, we have

$$\|F_{22}\|_{H^s}\leqslant C\epsilon^{5/2}+C\epsilon E_s^{1/2}+C\epsilon^2\left\|D_t(b_1-\tilde b_1)\right\|_{H^s}. \tag{3.226}$$

$$\|F_{23}\|_{H^s}\leqslant C\epsilon^{5/2}+C\epsilon E_s^{1/2}+C\epsilon^2\left\|D_t(b_1-\tilde b_1)\right\|_{H^s}. \tag{3.227}$$

So we obtain

$$\|F_2\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2} + C\epsilon^2 \left\| D_t(b_1 - \tilde{b}_1) \right\|_{H^s}. \tag{3.228}$$

Similar to the estimates for F_1 and F_2, we obtain

$$\|F_3\|_{H^s} + \|F_4\|_{H^s} + \|F_5\|_{H^s} + \|F_6\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2} + C\epsilon^2 \left\| D_t(b_1 - \tilde{b}_1) \right\|_{H^s}. \tag{3.229}$$

So we obtain

$$\left\| D_t(b_1 - \tilde{b}_1) \right\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2} + C\epsilon^2 \left\| D_t(b_1 - \tilde{b}_1) \right\|_{H^s}. \tag{3.230}$$

Therefore,

$$\left\| D_t(b_1 - \tilde{b}_1) \right\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2} \leqslant C\epsilon^2. \tag{3.231}$$

Since

$$\left\| D_t\tilde{b}_1 \right\|_{H^s} = \left\| \tilde{D}_t\tilde{b}_1 + b_1\partial_\alpha\tilde{b}_1 \right\|_{H^s} \leqslant \left\| \tilde{D}_t\tilde{b}_1 \right\|_{H^s} + \|b_1\|_{H^s} \left\| \partial_\alpha\tilde{b}_1 \right\|_{H^s} \leqslant C\epsilon^{3/2}, \tag{3.232}$$

we obtain

$$\|D_t b_1\|_{H^s} \leqslant C\epsilon^{3/2}. \tag{3.233}$$

3.8.4 Bound $A_1 - \tilde{A}_1$

Indeed, recall that $\tilde{A}_1 = 0$. We show that $\left\| A_1 - \tilde{A}_1 \right\|_{H^s}$ can be bounded by $C\epsilon^{5/2} + C\epsilon E_s^{1/2}$. Since $\tilde{A}_1$ satisfies the formula (3.82) for A_1 up to $O(\epsilon^3)$, we have

$$\begin{aligned}(I - \mathcal{H}_{\tilde{\zeta}})(\tilde{A} - \tilde{A}_0) =& i[\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}]\frac{\bar{\tilde{\zeta}}_\alpha - 1}{\tilde{\zeta}_\alpha} - i[(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}]\frac{\bar{\tilde{\omega}}_\alpha - 1}{\tilde{\omega}_\alpha} \\ &+ i[\tilde{D}_t\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}]\frac{\partial_\alpha\tilde{D}_t\bar{\tilde{\zeta}}}{\tilde{\zeta}_\alpha} - i[\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}]\frac{\partial_\alpha\tilde{D}_t^0\bar{\tilde{\omega}}}{\tilde{\omega}_\alpha} \\ &+ (\mathcal{H}_{\tilde{\omega}} - \mathcal{H}_{\tilde{\zeta}})(\tilde{A}_0 - 1) + \epsilon^3\mathcal{R}\end{aligned} \tag{3.234}$$

Subtract (3.234) from (3.82), we obtain the following formula for $A_1 - \tilde{A}_1$. We group similar terms together and write it in the following manner:

$$
\begin{aligned}
&(I - \mathcal{H}_\zeta)(A_1 - \tilde{A}_1) = (I - \mathcal{H}_\zeta)A_1 - (I - \mathcal{H}_{\tilde{\zeta}})\tilde{A}_1 + (\mathcal{H}_\zeta - \mathcal{H}_{\tilde{\zeta}})\tilde{A}_1 \\
=&\Big\{ i[D_t^2\zeta, \mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} - i[\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}]\frac{\bar{\tilde{\zeta}}_\alpha - 1}{\tilde{\zeta}_\alpha} - i[(D_t^0)^2\omega, \mathcal{H}_\omega]\frac{\bar{\omega}_\alpha - 1}{\omega_\alpha} + i[(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}]\frac{\bar{\tilde{\omega}}_\alpha - 1}{\tilde{\omega}_\alpha}\Big\} \\
&+ \Big\{ i[D_t\zeta, \mathcal{H}_\zeta]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha} - i[\tilde{D}_t\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}}]\frac{\partial_\alpha \tilde{D}_t\bar{\tilde{\zeta}}}{\tilde{\zeta}_\alpha} - i[D_t^0\omega, \mathcal{H}_\omega]\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha} + i[\tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\omega}}]\frac{\partial_\alpha \tilde{D}_t^0\bar{\tilde{\omega}}}{\tilde{\omega}_\alpha}\Big\} \\
&+ \Big\{ (\mathcal{H}_\omega - \mathcal{H}_\zeta)(A_0 - 1) - (\mathcal{H}_{\tilde{\omega}} - \mathcal{H}_{\tilde{\zeta}})(\tilde{A}_0 - 1)\Big\} \\
&+ (\mathcal{H}_\zeta - \mathcal{H}_{\tilde{\zeta}})\tilde{A}_1 \\
&+ \epsilon^4\mathcal{R} \\
:=& \sum_{m=1}^{5} K_m.
\end{aligned}
$$

Note that we can estimate K_1 in exactly the same way as we did for the quantity F_{211}, and we obtain estimate

$$\|K_1\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.235}$$

K_2 can be estimated the same way as we did for the quantity F_1, and we obtain

$$\|K_2\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.236}$$

Estimates for K_3, K_4 and K_5 are straight forward , we have

$$\|K_3\|_{H^s} + \|K_4\|_{H^s} + \|K_5\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.237}$$

So we obtain

$$\Big\|(I - \mathcal{H}_\zeta)(A_1 - \tilde{A}_1)\Big\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.238}$$

Therefore,

$$\left\|A_1 - \tilde{A}_1\right\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.239}$$

Corollary III.42. *We have*

$$\|A_1\|_{H^s} \leqslant C\epsilon^{3/2}. \tag{3.240}$$

Corollary III.43. *Assume the a priori assumption (3.154), then*

$$\inf_{t\in[0,T_0]} \inf_{\alpha\in\mathbb{R}} A(\alpha,t) \geqslant \frac{1}{2}, \qquad \sup_{t\in[0,T_0]} \sup_{\alpha\in\mathbb{R}} A(\alpha,t) \leqslant 2. \tag{3.241}$$

Proof. We have $A = A_0 + A_1$. By Theorem III.2, $\|A_0 - 1\|_\infty \leqslant C\epsilon^2$. By Corollary III.42, $\|A_1\|_\infty \leqslant C\epsilon^{3/2}$. Therefore, for ϵ sufficiently small, we have (3.241). □

Definition III.44. Denote L by the quantity

$$\begin{aligned} L =&[\mathcal{P}, D_t](I-\mathcal{H}_\zeta)(\zeta-\alpha) - [\mathcal{P}_0, D_t^0](I-\mathcal{H}_\omega)(\omega-\alpha)\\ &-[\tilde{\mathcal{P}}, \tilde{D}_t](I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha) + [\tilde{\mathcal{P}}_0, \tilde{D}_t^0](I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha) \end{aligned} \tag{3.242}$$

This quantity arises in the energy estimates in the next section, so we need also to bound it in terms of E_s and ϵ.

3.8.5 Bound L

We know that:

$$[\mathcal{P}, D_t](I-\mathcal{H}_\zeta)(\zeta-\alpha) = \left(\frac{a_t}{a}\right)\circ\kappa^{-1} iA\partial_\alpha(I-\mathcal{H}_\zeta)(\zeta-\alpha), \tag{3.243}$$

where $\left(\frac{a_t}{a}\right)\circ\kappa^{-1}$ is given by

$$\begin{aligned} (I-\mathcal{H}_\zeta)\Big(A\bar{\zeta}_\alpha\Big(\frac{a_t}{a}\Big)\circ\kappa^{-1}\Big) =&2i[D_t^2\zeta, \mathcal{H}]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha} + 2i[D_t\zeta, \mathcal{H}]\frac{\partial_\alpha D_t^2\bar{\zeta}}{\zeta_\alpha}\\ &-\frac{1}{\pi}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta D_t\bar{\zeta}(\beta)d\beta. \end{aligned} \tag{3.244}$$

Similarly, we have

$$[\mathcal{P}_0, D_t^0](I-\mathcal{H}_\omega)(\omega-\alpha) = \Big(\frac{(a_0)_t}{a_0}\Big)\circ\kappa_0^{-1} i A_0\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha), \tag{3.245}$$

where $\Big(\frac{(a_0)_t}{a_0}\Big)\circ\kappa_0^{-1}$ is given by

$$\begin{aligned}(I-\mathcal{H}_\omega)\Big(A_0\bar{\omega}_\alpha\Big(\frac{(a_0)_t}{a_0}\Big)\circ\kappa_0^{-1}\Big) =& 2i[(D_t^0)^2\omega,\mathcal{H}]\frac{\partial_\alpha D_t^0\bar{\omega}}{\omega_\alpha} + 2i[D_t^0\omega,\mathcal{H}]\frac{\partial_\alpha D_t^2\bar{\zeta}}{\zeta_\alpha}\\ &-\frac{1}{\pi}\int\Big(\frac{D_t^0\omega(\alpha)-D_t^0\omega(\beta)}{\omega(\alpha)-\omega(\beta)}\Big)^2\partial_\beta D_t^0\bar{\omega}(\beta)d\beta.\end{aligned} \tag{3.246}$$

For brevity, denote

$$\psi = \Big(\frac{a_t}{a}\Big)\circ\kappa^{-1}, \quad \psi_0 = \Big(\frac{(a_0)_t}{a_0}\Big)\circ\kappa_0^{-1}.$$

Let $\tilde{\psi}$ be the approximation of ψ to the order $O(\epsilon^4)$, and $\tilde{\psi}_0$ the approximation of ψ_0 to the order $O(\epsilon^4)$. We have formula for $(I-\mathcal{H}_{\tilde{\zeta}})\tilde{\psi}$ and $(I-\mathcal{H}_{\tilde{\omega}})\tilde{\psi}_0$:

$$\begin{aligned}(I-\mathcal{H}_{\tilde{\zeta}})\Big(\tilde{A}\bar{\tilde{\zeta}}_\alpha\tilde{\psi}\Big) =& 2i[\tilde{D}_t^2\tilde{\zeta},\mathcal{H}]\frac{\partial_\alpha\tilde{D}_t\bar{\tilde{\zeta}}}{\tilde{\zeta}_\alpha} + 2i[\tilde{D}_t\tilde{\zeta},\mathcal{H}]\frac{\partial_\alpha\tilde{D}_t^2\bar{\tilde{\zeta}}}{\tilde{\zeta}_\alpha}\\ &-\frac{1}{\pi}\int\Big(\frac{\tilde{D}_t\tilde{\zeta}(\alpha)-\tilde{D}_t\tilde{\zeta}(\beta)}{\tilde{\zeta}(\alpha)-\tilde{\zeta}(\beta)}\Big)^2\partial_\beta\tilde{D}_t\bar{\tilde{\zeta}}(\beta)d\beta + \epsilon^4\mathcal{R}_1,\end{aligned} \tag{3.247}$$

and

$$\begin{aligned}(I-\mathcal{H}_{\tilde{\omega}})\Big(\tilde{A}_0\bar{\tilde{\omega}}_\alpha\tilde{\psi}_0\Big) =& 2i[(\tilde{D}_t^0)^2\tilde{\omega},\mathcal{H}]\frac{\partial_\alpha\tilde{D}_t^0\bar{\tilde{\omega}}}{\tilde{\omega}_\alpha} + 2i[\tilde{D}_t^0\tilde{\omega},\mathcal{H}]\frac{\partial_\alpha(\tilde{D}_t^0)^2\bar{\tilde{\omega}}}{\tilde{\zeta}_\alpha}\\ &-\frac{1}{\pi}\int\Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha)-\tilde{D}_t^0\tilde{\omega}(\beta)}{\tilde{\omega}(\alpha)-\tilde{\omega}(\beta)}\Big)^2\partial_\beta\tilde{D}_t^0\bar{\tilde{\omega}}(\beta)d\beta + \epsilon^4\mathcal{R}_2,\end{aligned} \tag{3.248}$$

where

$$\big\|\epsilon^4\mathcal{R}_1-\epsilon^4\mathcal{R}_2\big\|_{H^s}\leqslant\epsilon^{7/2}.$$

Denote

$$\theta=(I-\mathcal{H}_\zeta)(\zeta-\alpha), \quad \theta_0=(I-\mathcal{H}_\omega)(\omega-\alpha), \quad \tilde{\theta}=(I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha), \quad \tilde{\theta}_0=(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha).$$

Then

$$L = \psi\partial_\alpha\theta - \tilde{\psi}\partial_\alpha\tilde{\theta} - \psi_0\partial_\alpha\theta_0 + \tilde{\psi}_0\partial_\alpha\tilde{\theta}_0.$$

We rewrite L in the following form:

$$\begin{aligned}
L =&(\psi - \psi_0)\partial_\alpha\theta + \psi_0\partial_\alpha(\theta - \theta_0) - (\tilde{\psi} - \tilde{\psi}_0)\partial_\alpha\tilde{\theta} - \tilde{\psi}_0\partial_\alpha(\tilde{\theta} - \tilde{\theta}_0)\\
=&(\psi - \psi_0)\partial_\alpha(\theta - \tilde{\theta}) + (\psi - \psi_0 - (\tilde{\psi} - \tilde{\psi}_0))\partial_\alpha\tilde{\theta}\\
&+ (\psi_0 - \tilde{\psi}_0)\partial_\alpha(\theta - \theta_0) + \tilde{\psi}_0\partial_\alpha(\theta - \theta_0 - (\tilde{\theta} - \tilde{\theta}_0))\\
:=&L_1 + L_2 + L_3 + L_4.
\end{aligned}$$

The advantage of writing L in this form is that, each L_i can be written in the form $L_i = yz$, where $y \in H^s$, and $z = z_1 + z_2$, where $z_1 \in H^s$, and $z_2 \in W^{s,\infty}$. Note that we cannot estimate z directly in $W^{s,\infty}$, because z_1 might lose one derivative.

3.8.5.1 Estimate L_1.

First we estimate $\theta - \tilde{\theta}$. We have

$$\theta - \tilde{\theta} =(I - \mathcal{H}_\zeta)r + (\mathcal{H}_{\tilde{\zeta}} - \mathcal{H}_\zeta)(\tilde{\zeta} - \alpha).$$

Let $\mathcal{H}_\zeta^*$ be the adjoint of $\mathcal{H}_\zeta$, i.e.,

$$\mathcal{H}_\zeta^* f = -\zeta_\alpha\mathcal{H}\frac{f}{\zeta_\alpha}.$$

Then

$$
\begin{aligned}
\partial_\alpha(\theta-\tilde\theta) =&(I-\mathcal{H}_\zeta^*)r_\alpha+(\mathcal{H}_{\tilde\zeta}^*-\mathcal{H}_\zeta^*)(\tilde\zeta_\alpha-1)\\
=&(I-\mathcal{H}_\zeta^*)(r_1)_\alpha+(I-\mathcal{H}_\zeta^*)(r_0)_\alpha\\
&+\frac{1}{\pi i}\int\frac{\zeta_\alpha(\tilde\zeta(\alpha)-\tilde\zeta(\beta))-\tilde\zeta_\alpha(\zeta(\alpha)-\zeta(\beta))}{(\zeta(\alpha)-\zeta(\beta))(\tilde\zeta(\alpha)-\tilde\zeta(\beta))}(\tilde\zeta_\beta-1)d\beta\\
=&(I-\mathcal{H}_\zeta^*)(r_1)_\alpha+(I-\mathcal{H}_\zeta^*)(r_0)_\alpha && := Z_1+Z_2\\
&+\frac{1}{\pi i}\int\frac{(r_1)_\alpha(\tilde\zeta(\alpha)-\tilde\zeta(\beta))-\tilde\zeta_\alpha((r_1)_\alpha-(r_1)_\beta)}{(\zeta(\alpha)-\zeta(\beta))(\tilde\zeta(\alpha)-\tilde\zeta(\beta))}(\tilde\zeta_\beta-1)d\beta && := Z_3\\
&+\frac{1}{\pi i}\int\frac{(r_0)_\alpha(\tilde\zeta(\alpha)-\tilde\zeta(\beta))-\tilde\zeta_\alpha((r_0)_\alpha-(r_0)_\beta)}{(\zeta(\alpha)-\zeta(\beta))(\tilde\zeta(\alpha)-\tilde\zeta(\beta))}(\tilde\zeta_\beta-1)d\beta && := Z_4\\
:=&Z_1+Z_2+Z_3+Z_4.
\end{aligned}
$$

We have $Z_1 := (I-\mathcal{H}_\zeta^*)(r_1)_\alpha$, so

$$\|Z_1\|_{H^s}\leqslant CE_s^{1/2}.$$

For Z_3, use proposition III.18, we obtain

$$\|Z_3\|_{H^s}\leqslant C\epsilon E_s^{1/2}.$$

Use the fact that $||\partial_\alpha r_0||_{W^{s,\infty}}\leqslant C\epsilon^2$, it's straightforward to prove that

$$\|Z_2+Z_4\|_{W^{s,\infty}}\leqslant C\epsilon^2.$$

Next we estiamte $\psi - \psi_0$, we consider $(I-\mathcal{H}_\zeta)(\psi-\psi_0)A\bar{\zeta}_\alpha$. Note that

$$\begin{aligned}
&(I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha\psi - (I-\mathcal{H}_\omega)A_0\bar{\omega}_\alpha\psi_0\\
=&(I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha\psi - (I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha\psi_0 + (I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha\psi_0 - (I-\mathcal{H}_\zeta)A\bar{\omega}_\alpha\psi_0\\
&+(I-\mathcal{H}_\zeta)A\bar{\omega}_\alpha\psi_0 - (I-\mathcal{H}_\zeta)A_0\bar{\bar{\omega}}_\alpha\psi_0 + (I-\mathcal{H}_\zeta)A_0\bar{\bar{\omega}}\psi - (I-\mathcal{H}_\omega)A_0\bar{\bar{\omega}}\psi_0\\
=&(I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha(\psi-\psi_0) + (I-\mathcal{H}_\zeta)A(\bar{\xi}_1)_\alpha\tilde{\psi}\\
&+(I-\mathcal{H}_\zeta)A_1\bar{\omega}_\alpha\psi + (\mathcal{H}_\omega-\mathcal{H}_\zeta)A_0\bar{\omega}_\alpha\psi_0.
\end{aligned}$$

So we have

$$\begin{aligned}
(I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha(\psi-\psi_0) =& (I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha\psi - (I-\mathcal{H}_\omega)A_0\bar{\omega}_\alpha\psi_0\\
&-\Big\{(I-\mathcal{H}_\zeta)A(\bar{\xi}_1)_\alpha\tilde{\psi} + (I-\mathcal{H}_\zeta)A_1\bar{\omega}_\alpha\psi + (\mathcal{H}_\omega-\mathcal{H}_\zeta)A_0\bar{\omega}_\alpha\psi_0\Big\}
\end{aligned}$$

For $(I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha\psi - (I-\mathcal{H}_\omega)A_0\bar{\omega}_\alpha\psi_0$, subtract (3.246) from (3.244), and then group similar terms. We have estimated terms of these kinds before, so we omitt the details. We have

$$\Big\|(I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha\psi - (I-\mathcal{H}_\omega)\tilde{A}\bar{\tilde{\zeta}}_\alpha\tilde{\psi}\Big\|_{H^s} \leqslant C(\epsilon^{5/2}+\epsilon E_s^{1/2}). \tag{3.249}$$

Since $\tilde{\psi}$, ψ_0 and A_1 are quadratic, it's easy to show that

$$\Big\|(I-\mathcal{H}_\zeta)A(\bar{\xi}_1)_\alpha\tilde{\psi} + (I-\mathcal{H}_\zeta)A_1\bar{\omega}_\alpha\psi + (\mathcal{H}_\omega-\mathcal{H}_\zeta)A_0\bar{\omega}_\alpha\psi_0\Big\|_{H^s} \leqslant C\epsilon^{5/2}. \tag{3.250}$$

Combined the above estimates, we obtain

$$\begin{aligned}
||\psi-\psi_0||_{H^s} \leqslant& C||(I-\mathcal{H}_\zeta)A\bar{\zeta}_\alpha(\psi-\psi_0)||_{H^s}\\
\leqslant& C(\epsilon E_s^{1/2}+\epsilon^{5/2}).
\end{aligned} \tag{3.251}$$

So we have

$$\|L_1\|_{H^s} \leqslant C(\epsilon E_s^{1/2} + \epsilon^{5/2})(E_s^{1/2} + \epsilon^2) \leqslant C(\epsilon^{7/2} + \epsilon E_s). \tag{3.252}$$

The quantities L_2, L_3 and L_4 can be estimated in similar manner, and obtain

$$\|L_2\|_{H^s} + \|L_3\|_{H^s} + \|L_4\|_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon E_s). \tag{3.253}$$

So we have

$$||L||_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon E_s). \tag{3.254}$$

3.9 Energy estimates

In Section 3.7, we derive equations governing the evolution of r_1 and $D_t r_1$, respectively, and define energy for these quantities. In Section 3.8, we obtain aproiri bounds for some quantities which will be used in energy estimates. In this section, we will obtain bounds for the energy $\mathcal{E}_s$. For this purpose, we estimate the quantity appear in $\frac{d\mathcal{E}_s}{dt}$, and bound $\frac{d\mathcal{E}_s}{dt}$ in terms of E_s and ϵ. Then we obtain

$$\frac{d\mathcal{E}_s}{dt} \leqslant C(E_s^2 + \epsilon E_s^{3/2} + \epsilon^2 E_s + \epsilon^{7/2} E_s^{1/2} + \epsilon^5). \tag{3.255}$$

Then we show that E_s is essentially controlled by $\mathcal{E}_s$:

$$E_s^{1/2} \leqslant C(\mathcal{E}_s + \epsilon^{5/2}). \tag{3.256}$$

(3.255) and (3.256) will together give the bound

$$\mathcal{E}_s \leqslant C\epsilon^3 \tag{3.257}$$

on time scale ϵ^{-2}.

3.9.1 Estimate $(I-\mathcal{H}_\zeta)\mathcal{R}_{11}$

It suffices to estimate $\mathcal{R}_{11}$. Recall that

$$\begin{aligned}\mathcal{R}_{11}=&-2[D_t\zeta,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta+2[D_t^0\omega,\mathcal{H}_\omega\frac{1}{\omega_\alpha}+\bar{\mathcal{H}}_\omega\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega\\&-2[\tilde{D}_t\tilde{\zeta},\mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}+\bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha\tilde{D}_t\tilde{\zeta}+2[\tilde{D}_t^0\tilde{\omega},\mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}+\bar{\mathcal{H}}_{\tilde{\omega}}\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha\tilde{D}_t^0\tilde{\omega}.\end{aligned}\tag{3.258}$$

First, we rewrite $I:=-2[D_t\zeta,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta+2[D_t^0\omega,\mathcal{H}_\omega\frac{1}{\omega_\alpha}+\bar{\mathcal{H}}_\omega\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega$ as

$$\begin{aligned}I=&-2[D_t\zeta-D_t^0\omega,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta\\&-2[D_t^0\omega,\mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}}-\mathcal{H}_\omega\frac{1}{\omega_\alpha}-\bar{\mathcal{H}}_\omega\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t\zeta\\&-2[D_t^0\omega,\mathcal{H}_\omega\frac{1}{\omega_\alpha}+\bar{\mathcal{H}}_\omega\frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\xi_1\\:=&I_1+I_2+I_3.\end{aligned}\tag{3.259}$$

And we rewrite $II:=-2[\tilde{D}_t\tilde{\zeta},\mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}+\bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha\tilde{D}_t\tilde{\zeta}+2[\tilde{D}_t^0\tilde{\omega},\mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}+\bar{\mathcal{H}}_{\tilde{\omega}}\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha\tilde{D}_t^0\tilde{\omega}$ as

$$\begin{aligned}II=&-2[\tilde{D}_t\tilde{\zeta}-\tilde{D}_t^0\tilde{\omega},\mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}+\bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha\tilde{D}_t\tilde{\zeta}\\&-2[\tilde{D}_t^0\tilde{\omega},\mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha}+\bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}}-\mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}-\bar{\mathcal{H}}_{\tilde{\omega}}\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha\tilde{D}_t\tilde{\zeta}\\&-2[\tilde{D}_t^0\tilde{\omega},\mathcal{H}_{\tilde{\omega}}\frac{1}{\tilde{\omega}_\alpha}+\bar{\mathcal{H}}_{\tilde{\omega}}\frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha\tilde{D}_t^0\tilde{\xi}_1\\:=&II_1+II_2+II_3.\end{aligned}\tag{3.260}$$

3.9.1.1 Estimate $\|I_1 + II_1\|_{H^s}$

We have

$$\begin{aligned} I_1 + II_1 =& -2[D_t\zeta - D_t^0\omega - (\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}), \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta \\ & -2[\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha} - \Big(\mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha} + \bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}_\alpha}\Big)]\partial_\alpha D_t\zeta \\ & -2[\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}, \mathcal{H}_{\tilde{\zeta}}\frac{1}{\tilde{\zeta}_\alpha} + \bar{\mathcal{H}}_{\tilde{\zeta}}\frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha(D_t\zeta - \tilde{D}_t\tilde{\zeta}) \\ :=& \Lambda_1 + \Lambda_2 + \Lambda_3. \end{aligned}$$

Denote

$$h := D_t\zeta - D_t^0\omega - (\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}).$$

We have

$$\begin{aligned} & [h, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta \\ =& -\frac{2}{\pi i}\int \frac{\mathrm{Im}\{\zeta(\alpha)-\zeta(\beta)\}(h(\alpha)-h(\beta))}{|\zeta(\alpha)-\zeta(\beta)|^2}\partial_\beta D_t\zeta(\beta)d\beta. \end{aligned} \tag{3.261}$$

Use (3.183),

$$D_t\zeta - \tilde{D}_t\tilde{\zeta} - (D_t^0\omega - \tilde{D}_t^0\tilde{\omega}) = (b_1 - \tilde{b}_1)(\tilde{\zeta}_\alpha + (r_0)_\alpha) + \tilde{b}_1(r_0)_\alpha + D_t r_1, \tag{3.262}$$

and proposition III.18, lemmma III.38, corollary III.39, lemma III.34, corollary III.40, we have

$$\begin{aligned} \Big\|[h, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta\Big\|_{H^s} \leqslant& \|h\|_{H^s}\|\mathrm{Im}\,\zeta_\alpha\|_{X^{s-1,\infty}}\|D_t\zeta\|_{W^{n-1,\infty}} \\ \leqslant& C(\epsilon^{3/2} + E_s^{1/2})\epsilon^2 \\ \leqslant& C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \end{aligned} \tag{3.263}$$

So we obtain

$$\|\Lambda_1\|_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2} + \epsilon E_s + E_s^{3/2}). \tag{3.264}$$

Λ_2 is a singular integral of the form $S_2(A, f)$, whose kernel is at least of order two. Note that

$$\left\|\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}\right\|_{H^s} = \left\|\tilde{D}_t\tilde{\xi}_1 + \tilde{b}_1\tilde{\omega}_\alpha\right\|_{H^s} \leqslant \left\|\tilde{D}_t\tilde{\xi}_1\right\|_{H^s} + \left\|\tilde{b}_1\tilde{\omega}_\alpha\right\|_{H^s} \leqslant C\epsilon^{1/2}. \tag{3.265}$$

By proposition III.18, lemma III.34, corollary III.40, we have

$$\begin{aligned} \|\Lambda_2\|_{H^s} \leqslant & C\left\|\tilde{D}_t\tilde{\zeta} - \tilde{D}_t^0\tilde{\omega}\right\|_{H^s}\left\|\mathrm{Im}(\zeta - \tilde{\zeta})_\alpha\right\|_{W^{s-1,\infty}}\left\|\zeta_\alpha - \tilde{\zeta}_\alpha\right\|_{W^{s-1,\infty}}\|D_t\zeta\|_{W^{s-1,\infty}} \\ \leqslant & C\epsilon^{7/2}. \end{aligned} \tag{3.266}$$

The same argument gives

$$||\Lambda_3||_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.267}$$

So we obtain

$$||I_1 + II_1||_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.268}$$

We estimate $I_2 + II_2$ and $I_3 + II_3$ in the same way as we did for $I_1 + II_1$, and obtain

$$\|I_2 + II_2\|_{H^s} + \|I_3 + II_3\|_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.269}$$

So we obtain

$$\|I + II\|_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.270}$$

So we obtain

$$||(I - \mathcal{H}_\zeta)\mathcal{R}_{11}||_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.271}$$

3.9.2 Estimate $(I-\mathcal{H}_\zeta)\mathcal{R}_{12}$

The way we estimate $\mathcal{R}_{12}$ is similar to that of $\mathcal{R}_{11}$. Recall that

$$\begin{aligned}\mathcal{R}_{12}=&\Big\{\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha,t)-D_t\zeta(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta\\&-\frac{1}{\pi i}\int\Big(\frac{D_t^0\omega(\alpha,t)-D_t^0\omega(\beta,t)}{\omega(\alpha,t)-\omega(\beta,t)}\Big)^2(\omega-\bar{\omega})_\beta d\beta\Big\}\\&-\Big\{\frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t\tilde{\zeta}(\alpha,t)-\tilde{D}_t\tilde{\zeta}(\beta,t)}{\tilde{\zeta}(\alpha,t)-\tilde{\zeta}(\beta,t)}\Big)^2(\tilde{\zeta}-\bar{\tilde{\zeta}})_\beta d\beta\\&-\frac{1}{\pi i}\int\Big(\frac{\tilde{D}_t^0\tilde{\omega}(\alpha,t)-\tilde{D}_t^0\tilde{\omega}(\beta,t)}{\tilde{\omega}(\alpha,t)-\tilde{\omega}(\beta,t)}\Big)^2(\tilde{\omega}-\bar{\tilde{\omega}})_\beta d\beta\Big\}\end{aligned}\tag{3.272}$$

The idea is again to decompose $\zeta=\omega+\tilde{\xi}_1+r_1$ and $D_t=D_t^0+b_1\partial_\alpha$ to explore the cancellations, and then use Proposition III.18 to obtain appropriate estimates. For example, in the decomposition we'll obtain terms like

$$\mathcal{R}_{121}:=\frac{1}{\pi i}\int\Big(\frac{D_tr_1(\alpha,t)-D_tr_1(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)}\Big)^2(\zeta-\bar{\zeta})_\beta d\beta.$$

Then we have

$$||\mathcal{R}_{121}||_{H^s}\leqslant C\|D_tr_1\|_{H^s}\|D_tr_1\|_{W^{s-1,\infty}}\|\operatorname{Im}\zeta_\alpha\|_{W^{s-1,\infty}}\leqslant C\epsilon^2E_s^{1/2}.\tag{3.273}$$

Other terms can be estimated in a similar way, and we obtain

$$\|\mathcal{R}_{12}\|_{H^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}).\tag{3.274}$$

3.9.3 Estimate $\mathcal{R}_{13}$

Recall that

$$\begin{aligned}\mathcal{R}_{13}=&D_tb_1\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha)+b_1D_t\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha)\\&-\Big\{\tilde{D}_t\tilde{b}_1\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)+\tilde{b}_1\tilde{D}_t\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)\Big\}\end{aligned}\tag{3.275}$$

We provide the detail for the estimate of

$$D_t b_1 \partial_\alpha (I-\mathcal{H}_\omega)(\omega-\alpha) - \tilde{D}_t \tilde{b}_1 \partial_\alpha (I-\mathcal{H}_{\tilde{\omega}})(\omega-\alpha).$$

The estimate for $b_1 D_t \partial_\alpha (I-\mathcal{H}_\omega)(\omega-\alpha) - \tilde{b}_1 \tilde{D}_t \partial_\alpha (I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)$ can be obtained in the same way. We have

$$\begin{aligned}
& D_t b_1 \partial_\alpha (I-\mathcal{H}_\omega)(\omega-\alpha) - \tilde{D}_t \tilde{b}_1 \partial_\alpha (I-\mathcal{H}_{\tilde{\omega}})(\omega-\alpha)\\
=& D_t (b_1-\tilde{b}_1)\partial_\alpha (I-\mathcal{H}_\omega)(\omega-\alpha) + (b_1-\tilde{b}_1)\partial_\alpha \tilde{b}_1 \partial_\alpha (I-\mathcal{H}_\omega)(\omega-\alpha)\\
&+ \tilde{D}_t \tilde{b}_1 \partial_\alpha (\mathcal{H}_{\tilde{\omega}}-\mathcal{H}_\omega)(\omega-\alpha) + \tilde{D}_t \tilde{b}_1 \partial_\alpha (I-\mathcal{H}_{\tilde{\omega}})(\omega-\tilde{\omega})\\
:=& J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.276}$$

By Theorem III.2, Sobolev embedding, and (3.231), we have

$$\begin{aligned}
\|J_1\|_{H^s} =& \left\| D_t (b_1-\tilde{b}_1)\partial_\alpha (I-\mathcal{H}_\omega)(\omega-\alpha) \right\|_{H^s}\\
\leqslant & \left\| D_t (b_1-\tilde{b}_1) \right\|_{H^s} \|\omega_\alpha - 1\|_{W^{s,\infty}}\\
\leqslant & C(\epsilon^{5/2} + \epsilon E_s^{1/2})\epsilon\\
\leqslant & C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}).
\end{aligned}$$

By Theorem III.2, Sobolev embedding, and corollary III.39, we have

$$\begin{aligned}
\|J_2\|_{H^s} \leqslant & \left\| b_1-\tilde{b}_1 \right\|_{H^s} \left\| \partial_\alpha \tilde{b}_1 \right\|_{W^{s,\infty}} \|\partial_\alpha (I-\mathcal{H}_\omega)(\omega-\alpha)\|_{W^{s,\infty}}\\
\leqslant & C\epsilon^{7/2}.
\end{aligned}$$

Estimates for J_3, J_4 are similar. So we obtain So we have

$$\left\| (I-\mathcal{H}_\zeta)\mathcal{R}_{13} \right\|_{H^s(\mathbb{R})} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.277}$$

3.9.4 Estimate $(I-\mathcal{H}_\zeta)\mathcal{R}_{14}$

Write $\mathcal{R}_{14}$ as

$$\mathcal{R}_{14}=\mathcal{R}_{141}+\mathcal{R}_{142},$$

where

$$\mathcal{R}_{141}=b_1\partial_\alpha D_t^0(I-\mathcal{H}_\omega)(\omega-\alpha)-\tilde{b}_1\partial_\alpha\tilde{D}_t^0(I-\mathcal{H}_\omega)(\tilde{\omega}-\alpha), \tag{3.278}$$

and

$$\mathcal{R}_{142}=-iA_1\partial_\alpha(I-\mathcal{H}_\omega)(\omega-\alpha)+i\tilde{A}_1\partial_\alpha(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha). \tag{3.279}$$

Estimates for these two terms are straightforward, we obtain

$$\Big\|(I-\mathcal{H}_\zeta)\mathcal{R}_{14}\Big\|_{H^s(\mathbb{R})}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}). \tag{3.280}$$

3.9.5 Estimate $\mathcal{R}_{16}$

Recall that

$$\mathcal{R}_{16}=-2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t(\lambda-\tilde{\lambda})}{\zeta_\alpha}. \tag{3.281}$$

To obtain better estimates, we explore the fact that $\frac{\partial_\alpha D_t(\lambda-\tilde{\lambda})}{\zeta_\alpha}$ is almot holomorphic in $\Omega(t)^c$. Write $\mathcal{R}_{16}$ as

$$\begin{aligned}\mathcal{R}_{16}=&-2[D_t\zeta,\mathcal{H}_\zeta+\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}]\frac{\partial_\alpha D_t(\lambda-\tilde{\lambda})}{\zeta_\alpha}+2[D_t\zeta,\bar{\mathcal{H}}\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t(\lambda-\tilde{\lambda})\\ :=&\mathcal{R}_{161}+\mathcal{R}_{162}.\end{aligned}$$

It's easy to see that

$$\|\mathcal{R}_{161}\|_{H^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}). \tag{3.282}$$

To estimate $\mathcal{R}_{162}$, we write

$$\begin{aligned}\mathcal{R}_{162} =& -2D_t\zeta(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha D_t(\lambda-\tilde{\lambda})}{\bar{\zeta}_\alpha}+2(I-\bar{\mathcal{H}}_\zeta)D_t\zeta\frac{\partial_\alpha D_t(\lambda-\tilde{\lambda})}{\bar{\zeta}_\alpha}\\ :=&\mathcal{R}_{1621}+\mathcal{R}_{1622}.\end{aligned}$$

Note that

$$\begin{aligned}\lambda-\tilde{\lambda}=&(I-\mathcal{H}_\zeta)(\zeta-\alpha)-(I-\mathcal{H}_\omega)(\omega-\alpha)-((I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha)-(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha))\\ =&(I-\mathcal{H}_\zeta)r_1+(\mathcal{H}_\omega-\mathcal{H}_\zeta)(\omega-\alpha)+(\mathcal{H}_{\tilde{\zeta}}-\mathcal{H}_\zeta)(\tilde{\zeta}-\alpha)-(\mathcal{H}_{\tilde{\omega}}-\mathcal{H}_\zeta)(\tilde{\omega}-\alpha).\end{aligned}$$

The last three terms are quadratic, and it's quite easy to see that they are bounded in H^s by

$$\epsilon E_s^{1/2}+\epsilon^{5/2}.$$

So to bound $\mathcal{R}_{1621}$, it suffices to bound $-2D_t\zeta(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha D_t(I-\mathcal{H}_\zeta)r_1}{\bar{\zeta}_\alpha}$. We have

$$\begin{aligned}&(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha D_t(I-\mathcal{H}_\zeta)r_1}{\bar{\zeta}_\alpha}\\ =&(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha(I-\mathcal{H}_\zeta)D_tr_1}{\bar{\zeta}_\alpha}-(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha r_1}{\zeta_\alpha}}{\bar{\zeta}_\alpha}.\end{aligned}$$

$(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha r_1}{\zeta_\alpha}}{\zeta_\alpha}$ satisfies the estimate

$$\Big\|(I-\bar{\mathcal{H}}_{\bar{\zeta}_\alpha})\frac{\partial_\alpha[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha r_1}{\bar{\zeta}_\alpha}}{\zeta_\alpha}\Big\|_{H^s}\leqslant C\epsilon E_s^{1/2}.$$

And

$$\begin{aligned}&(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha(I-\mathcal{H}_\zeta)D_tr_1}{\bar{\zeta}_\alpha}\\ =&(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha(I+\bar{\mathcal{H}}_{\bar{\zeta}})D_tr_1}{\bar{\zeta}_\alpha}-(I-\bar{\mathcal{H}}_\zeta)\frac{\partial_\alpha(\mathcal{H}_\zeta+\bar{\mathcal{H}}_{\bar{\zeta}})D_tr_1}{\bar{\zeta}_\alpha}.\end{aligned}$$

Note that the first term is zero, while the second term satisfies desired estimates. So we have

$$\left\| (I - \bar{\mathcal{H}}_\zeta) \frac{\partial_\alpha (I - \mathcal{H}_\zeta) D_t r_1}{\bar{\zeta}_\alpha} \right\|_{H^s} \leqslant C(\epsilon E_s^{1/2} + \epsilon^{5/2}).$$

We can estimate $\|\mathcal{R}_{1622}\|_{H^s}$ in a similar way. So we obtain

$$\|\mathcal{R}_{16}\|_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.283}$$

3.9.6 Estimate $\mathcal{R}_{17}$

It's easy to obtain estimate

$$\|\mathcal{R}_{17}\|_{H^2} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.284}$$

Since $\|\mathcal{R}_{15}\|_{H^s} \leqslant \epsilon^{7/2}$, we obtain

$$\left\| (I - \mathcal{H}_\zeta) \sum_{j=1}^{5} \mathcal{R}_j + \mathcal{R}_{16} + \mathcal{R}_{17} \right\|_{H^s} \leqslant C(\epsilon^{7/2} + \epsilon E_s^{1/2}). \tag{3.285}$$

3.9.7 Estimate $[D_t^2 - iA\partial_\alpha, \partial_\alpha^n]\rho_1$

We have

$$\begin{aligned}
&[D_t^2 - iA\partial_\alpha, \partial_\alpha^n]\rho_1 \\
=& \sum_{m=1}^{n} \partial_\alpha^{n-m} [D_t^2 - iA\partial_\alpha, \partial_\alpha] \partial_\alpha^{m-1} \rho_1 \\
=& \sum_{m=1}^{n} \partial_\alpha^{n-m} (D_t[D_t, \partial_\alpha] + [D_t, \partial_\alpha] D_t + iA_\alpha \partial_\alpha) \partial_\alpha^{m-1} \rho_1 \\
=& - \sum_{m=1}^{n} \partial_\alpha^{n-m} D_t(b_\alpha \partial_\alpha) \partial_\alpha^{m-1} \rho_1 - \sum_{m=1}^{n} \partial_\alpha^{n-m} b_\alpha D_t \partial_\alpha^{m-1} \rho_1 + i \sum_{m=1}^{n} \partial_\alpha^{n-m} A_\alpha \partial_\alpha^m \rho_1 \\
:=& K_1 + K_2 + K_3.
\end{aligned} \tag{3.286}$$

To estimate K_1, we use

$$\begin{aligned}&\partial_\alpha^{n-m}D_t(b_\alpha\partial_\alpha)\partial_\alpha^{m-1}\rho_1=\partial_\alpha^{n-m}(D_tb_\alpha\partial_\alpha^m\rho_1+b_\alpha D_t\partial_\alpha^m\rho_1)\\ =&\sum_{j=1}^{n-m}C_{n-m,j}(\partial_\alpha^{n-m-j}D_tb_\alpha\partial_\alpha^{m+j}\rho_1+\partial_\alpha^{n-m-j}b_\alpha\partial_\alpha^jD_t\partial_\alpha^m\rho_1),\end{aligned}\tag{3.287}$$

where

$$C_{n-m,j}=\frac{(n-m)!}{j!(n-m-j)!}.\tag{3.288}$$

Write $D_tb_\alpha=\partial_\alpha D_tb-(b_\alpha)^2$. If $m>1$, then use (3.231), corollary III.39, we have

$$\begin{aligned}\left\|\partial_\alpha^{n-m-j}D_tb_\alpha\partial_\alpha^{m+j}\rho_1\right\|_{L^2}\leqslant&\left\|\partial_\alpha^{n-m-j}D_tb_\alpha\right\|_\infty\partial_\alpha^{m+j}\rho_1||_{L^2}\\ \leqslant&(\|b_\alpha\|_{W^{n-1,\infty}}^2+\left\|\partial_\alpha^{n-m-j+1}D_tb\right\|_\infty)E_s^{1/2}\\ \leqslant&C\epsilon^2E_s^{1/2}.\end{aligned}\tag{3.289}$$

Similarly, we have

$$\left\|\partial_\alpha^{n-m-j}b_\alpha\partial_\alpha^jD_t\partial_\alpha^m\rho_1)\right\|_{L^2}\leqslant C\epsilon^2E_s^{1/2}.\tag{3.290}$$

If $m=1$ and $j=0$, then we cannot simply estimate $\partial_\alpha^{n-1}D_tb_\alpha$ in L^∞, because if $n=s$, we'll lose derivatives. To avoid loss of derivatives, we decompose

$$D_tb=D_tb_1+b_1\partial_\alpha b_0+D_t^0b_0=D_t(b_1-\tilde{b}_1)+\tilde{D}_t\tilde{b}_1+b_1\partial_\alpha(b_0+\tilde{b}_1)+D_t^0b_0.$$

Then for $n\geqslant 3$, by corollary III.37, Theorem III.2, corollary III.39, we have

$$\begin{aligned}&\|\partial_\alpha\rho_1\partial_\alpha^nD_tb\|_{L^2}\\ =&\left\|\partial_\alpha\rho_1(\partial_\alpha^nD_t(b_1-\tilde{b}_1)+\partial_\alpha^n\tilde{D}_t\tilde{b}_1+\partial_\alpha^n(b_1\partial_\alpha(\tilde{b}_1+b_0)+\partial_\alpha^n(D_t^0b_0))\right\|_{L^2}\\ \leqslant&\left\|D_t(b_1-\tilde{b}_1)\right\|_{H^n}\|\partial_\alpha\rho_1\|_{L^\infty}+\|b_1\|_{H^n}\left\|\tilde{b}_1+b_0\right\|_{W^{n+1,\infty}}\|\partial_\alpha\rho_1\|_{L^\infty}+\|\partial_\alpha\rho_1\|_{L^2}\left\|D_t^0b_0\right\|_{W^{n,\infty}}\\ \leqslant&C\epsilon^2E_s^{1/2}.\end{aligned}\tag{3.291}$$

The quantity $\partial_\alpha^{n-1}(b_\alpha)^2\partial_\alpha\rho_1$ can be estimated similarly, and we obtain

$$\left\|\partial_\alpha^{n-1}(b_\alpha)^2\partial_\alpha\rho_1\right\|_{L^2} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.292}$$

So we have

$$\left\|\partial_\alpha^{n-m-j}D_t b_\alpha \partial_\alpha^{m+j}\rho_1\right\|_{L^2} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.293}$$

We use same argument to obtain

$$\left\|\partial_\alpha^{n-m-j}b_\alpha\partial_\alpha^j D_t\partial_\alpha^m\rho_1)\right\|_{L^2} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.294}$$

So we obtain

$$\|K_1\|_{L^2} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.295}$$

Estimate K_2, K_3 in a similar way, we obtain

$$\|K_2 + K_3\|_{L^2} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.296}$$

So we have

$$\left\|[D_t^2 - iA\partial_\alpha, \partial_\alpha^n]\rho_1\right\|_{L^2} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.297}$$

So we obtain estimate for $\mathcal{C}_{1,n}$: for $0 \leqslant n \leqslant s$,

$$\left\|\mathcal{C}_{1,n}\right\|_{L^2} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.298}$$

3.9.8 Estimate $\partial_\alpha^n(I - \mathcal{H}_\zeta)\mathcal{S}_1$

In this subsection we obtain estimate for the quantity $\partial_\alpha^n(I - \mathcal{H}_\zeta)\mathcal{S}_1$. Since

$$\partial_\alpha^n(I - \mathcal{H}_\zeta)\mathcal{S}_1 = (I - \mathcal{H}_\zeta)\partial_\alpha^n\mathcal{S}_1 - [\partial_\alpha^n, \mathcal{H}_\zeta]\mathcal{S}_1,$$

it suffices to estimate $\partial_\alpha^n \mathcal{S}_1$.

Recall that

$$\begin{aligned}\mathcal{S}_1 =& D_tG - D_t^0G_0 - \tilde{D}_t\tilde{G} + \tilde{D}_t^0\tilde{G}_0 + (\mathcal{P}-\mathcal{P}_0)\tilde{D}_t(I-\mathcal{H}_\omega)(\omega-\alpha)\\ &-(\tilde{\mathcal{P}}-\tilde{\mathcal{P}}_0)\tilde{D}_t^0(I-\tilde{\omega})(\tilde{\omega}-\alpha)+\epsilon^4\mathcal{R}.\end{aligned}$$

3.9.8.1 Estimate $\left\|D_tG - D_t^0G_0 - \tilde{D}_t\tilde{G} + \tilde{D}_t^0\tilde{G}_0\right\|_{H^s}$

We have

$$\begin{aligned}D_tG =& -2[D_t^2\zeta, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta - 2[D_t\zeta, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha}+\bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t^2\zeta\\ &+\frac{2}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta D_t\zeta(\beta)d\beta\\ &-\frac{2}{\pi i}\int\frac{|D_t\zeta(\alpha)-D_t\zeta(\beta)|^2}{(\bar{\zeta}(\alpha)-\bar{\zeta}(\beta))^2}\partial_\beta D_t\zeta(\beta)d\beta\\ &+\frac{4}{\pi}\int\frac{(D_t\zeta(\alpha)-D_t\zeta(\beta))(D_t^2\zeta(\alpha)-D_t^2\zeta(\beta))}{(\zeta(\alpha)-\zeta(\beta))^2}\partial_\beta \operatorname{Im}\zeta(\beta)d\beta\\ &+\frac{2}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta \operatorname{Im} D_t\zeta(\beta)d\beta\\ &-\frac{4}{\pi}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^3\partial_\beta \operatorname{Im}\zeta(\beta)d\beta.\end{aligned} \tag{3.299}$$

And

$$\begin{aligned}
D_t^0 G_0 =& -2[(D_t^0)^2\omega, \mathcal{H}_\omega \frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega \frac{1}{\omega_\alpha}]\partial_\alpha D_t^0 \omega \\
& -2[D_t^0\omega, \mathcal{H}_\omega \frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega \frac{1}{\omega_\alpha}]\partial_\alpha (D_t^0)\omega \\
& + \frac{2}{\pi i}\int \Big(\frac{D_t^0\omega(\alpha,t) - D_t^0\omega(\beta,t)}{\omega(\alpha,t)-\omega(\beta,t)}\Big)^2 \partial_\beta D_t^0\omega(\beta,t)d\beta \\
& \frac{4}{\pi}\int \frac{(D_t^0\omega(\alpha) - D_t^0\omega(\beta))((D_t^0)^2\omega(\alpha) - (D_t^0)^2\omega(\beta))}{(\omega(\alpha)-\omega(\beta))^2}\partial_\beta \operatorname{Im}\{\omega(\beta)\}d\beta \qquad (3.300) \\
& - \frac{2}{\pi i}\int \frac{|D_t^0\omega(\alpha,t) - D_t^0\omega(\beta,t)|^2}{(\bar{\omega}(\alpha,t)-\bar{\omega}(\beta,t))^2}\partial_\beta D_t^0\omega(\beta,t)d\beta \\
& + \frac{2}{\pi}\int \Big(\frac{D_t^0\omega(\alpha,t) - D_t^0\omega(\beta,t)}{\omega(\alpha,t)-\omega(\beta,t)}\Big)^2 \partial_\beta \operatorname{Im}\{D_t^0\omega(\beta,t)\}d\beta \\
& - \frac{4}{\pi}\int \Big(\frac{D_t^0\omega(\alpha,t) - D_t^0\omega(\beta,t)}{\omega(\alpha,t)-\omega(\beta,t)}\Big)^2 \partial_\beta \operatorname{Im}\{\omega(\beta,t)\}d\beta
\end{aligned}$$

$\tilde{D}_t\tilde{G}$ and $\tilde{D}_t^0\tilde{G}_0$ are given similarly: For $\tilde{D}_t\tilde{G}$, replace D_t by $\tilde{D}_t$ and replance ζ by $\tilde{\zeta}$ in (3.299). For $\tilde{D}_t^0\tilde{G}_0$, replace D_t^0 by $\tilde{D}_t^0$, and replace ω by $\tilde{\omega}$ in (3.300).

- Estimate the quantity

$$\begin{aligned}
\mathcal{S}_{11} :=& -2[D_t^2\zeta, \mathcal{H}_\zeta \frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta \frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta + 2[\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}} \frac{1}{\tilde{\zeta}_\alpha} + \bar{\mathcal{H}}_{\tilde{\zeta}} \frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha \tilde{D}_t\tilde{\zeta} \\
& + 2[(D_t^0)^2\omega, \mathcal{H}_\omega \frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega \frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega - 2[(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_{\tilde{\omega}} \frac{1}{\tilde{\omega}_\alpha} + \bar{\mathcal{H}}_{\tilde{\omega}} \frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha \tilde{D}_t^0\tilde{\omega}
\end{aligned}$$

Rewrite $\mathcal{S}_{11}$ as

$$\begin{aligned}
\mathcal{S}_{11} =& -2[D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_\zeta \frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta \frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta && := S_{111}\\
& -2[\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_\zeta \frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta \frac{1}{\bar{\zeta}_\alpha} - \Big(\mathcal{H}_{\tilde{\zeta}} \frac{1}{\tilde{\zeta}_\alpha} + \bar{\mathcal{H}}_{\tilde{\zeta}} \frac{1}{\bar{\tilde{\zeta}}_\alpha}\Big)]\partial_\alpha D_t\zeta && := S_{112}\\
& -2[\tilde{D}_t^2\tilde{\zeta}, \mathcal{H}_{\tilde{\zeta}} \frac{1}{\tilde{\zeta}_\alpha} + \bar{\mathcal{H}}_{\tilde{\zeta}} \frac{1}{\bar{\tilde{\zeta}}_\alpha}]\partial_\alpha (D_t\zeta - \tilde{D}_t\tilde{\zeta}) && := S_{113}\\
& +2[(D_t^0)^2\omega - (\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\omega \frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega \frac{1}{\bar{\omega}_\alpha}]\partial_\alpha D_t^0\omega && := S_{114}\\
& +2[(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\omega \frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega \frac{1}{\bar{\omega}_\alpha} - \Big(\mathcal{H}_{\tilde{\omega}} \frac{1}{\tilde{\omega}_\alpha} + \bar{\mathcal{H}}_{\tilde{\omega}} \frac{1}{\bar{\tilde{\omega}}_\alpha}\Big)]\partial_\alpha D_t^0\omega && := S_{115}\\
& +2[(\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_{\tilde{\omega}} \frac{1}{\tilde{\omega}_\alpha} + \bar{\mathcal{H}}_{\tilde{\omega}} \frac{1}{\bar{\tilde{\omega}}_\alpha}]\partial_\alpha (D_t^0\omega - \tilde{D}_t^0\tilde{\omega}) && := S_{116}.
\end{aligned}$$

We give details of estimate of $S_{111} + S_{114}$ only, the estimates for $S_{112} + S_{115}$ and $S_{113} + S_{116}$ are similar. Write $S_{111} + S_{114}$ as

$$\begin{aligned}
S_{111} + S_{114} =& -2[D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta} - ((D_t^0)^2\omega - (\tilde{D}_t^0)^2\tilde{\omega}), \mathcal{H}_\zeta \frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta \frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha D_t\zeta\\
& -2[(D_t^0)^2\omega - (\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\zeta \frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta \frac{1}{\bar{\zeta}_\alpha} - \Big(\mathcal{H}_\omega \frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega \frac{1}{\bar{\omega}_\alpha}\Big)]\partial_\alpha D_t\zeta\\
& -2[(D_t^0)^2\omega - (\tilde{D}_t^0)^2\tilde{\omega}, \mathcal{H}_\omega \frac{1}{\omega_\alpha} + \bar{\mathcal{H}}_\omega \frac{1}{\bar{\omega}_\alpha}]\partial_\alpha (D_t\zeta - D_t^0\omega)\\
:=& M_1 + M_2 + M_3.
\end{aligned}$$

To estimate M_1, we use (3.221)

$$\begin{aligned}
& D_t^2\zeta - \tilde{D}_t^2\tilde{\zeta} - (D_t^0)^2\omega + (\tilde{D}_t^0)^2\tilde{\omega}\\
=& D_t^2 r_1 + (D_t^2 - \tilde{D}_t^2)\tilde{\xi}_1 + (D_t^2 - (D_t^0)^2)\omega - ((D_t^0)^2 - (\tilde{D}_t^0)^2)\omega
\end{aligned} \tag{3.301}$$

Denote

$$q := (D_t^2 - \tilde{D}_t^2)\tilde{\xi}_1 + (D_t^2 - (D_t^0)^2)\omega - ((D_t^0)^2 - (\tilde{D}_t^0)^2)\omega.$$

We have the rough estimate

$$\|q\|_{H^s} \leqslant C(E_s^{1/2} + \epsilon^{3/2}). \tag{3.302}$$

Decompose $D_t\zeta$ as

$$D_t\zeta = D_t r_1 + D_t^0\omega + b_1\omega_\alpha + \tilde{D}_t\tilde{\xi}_1 + (b - \tilde{b})\partial_\alpha\tilde{\xi}_1.$$

Then M_1 can be written as

$$\begin{aligned}
M_1 =& -2[D_t^2 r_1, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha(D_t r_1 + b_1\omega_\alpha + (b - \tilde{b})\partial_\alpha\tilde{\xi}_1) && (3.303)\\
& -2[D_t^2 r_1, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha(D_t^0\omega + \tilde{D}_t\tilde{\xi}_1) && (3.304)\\
& -2[q, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha(D_t r_1 + b_1\omega_\alpha + (b - \tilde{b})\partial_\alpha\tilde{\xi}_1) && (3.305)\\
& -2[q, \mathcal{H}_\zeta\frac{1}{\zeta_\alpha} + \bar{\mathcal{H}}_\zeta\frac{1}{\bar{\zeta}_\alpha}]\partial_\alpha(D_t^0\omega + \tilde{D}_t\tilde{\xi}_1) && (3.306)\\
:=& M_{11} + M_{12} + M_{13} + M_{14}. && (3.307)
\end{aligned}$$

For M_{11}, we have

$$\begin{aligned}
\|M_{11}\|_{H^s} \leqslant& C\left\|D_t^2 r_1\right\|_{H^s}\|Im\zeta_\alpha\|_{W^{s-1,\infty}}\left\|D_t r_1 + b_1\omega_\alpha + (b - \tilde{b})\partial_\alpha\tilde{\xi}_1\right\|_{H^s} && (3.308)\\
\leqslant& C\epsilon E_s^{1/2}(E_s^{1/2} + \epsilon^{3/2}) && (3.309)\\
\leqslant& C\epsilon^2 E_s^{1/2}. && (3.310)
\end{aligned}$$

For M_{12}, we have

$$\|M_{12}\|_{H^s} \leqslant C\left\|D_t^2 r_1\right\|_{H^s}\|Im\zeta_\alpha\|_{W^{s-1,\infty}}\left\|D_t^0\omega + \tilde{D}_t\tilde{\xi}_1\right\|_{W^{s,\infty}} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.311}$$

For M_{13}, we have

$$\|M_{13}\|_{H^s} \leqslant C\|q\|_{H^s}\|Im\zeta_\alpha\|_{W^{s-1,\infty}}\left\|D_t r_1 + b_1\omega_\alpha + (b-\tilde{b})\partial_\alpha\tilde{\xi}_1\right\|_{H^s} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.312}$$

For M_{14}, we have

$$\|M_{14}\|_{H^s} \leqslant C\|q\|_{H^s}\|Im\zeta_\alpha\|_{W^{s-1,\infty}}\left\|D_t^0\omega + \tilde{D}_t\tilde{\xi}_1\right\|_{W^{s,\infty}} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.313}$$

So we have

$$\|M_1\|_{H^s} \leqslant C\epsilon^2 E_s^{1/2}. \tag{3.314}$$

M_2 and M_3 can be estimated similarly, we obtain

$$\|M_2\|_{H^s} + \|M_3\|_{H^s} \leqslant C\epsilon^2 E_s^{1/2} + \epsilon^{7/2}. \tag{3.315}$$

So we have

$$\|S_{111} + S_{114}\|_{H^s} \leqslant C(\epsilon^2 E_s^{1/2} + \epsilon^{7/2}). \tag{3.316}$$

Similarly,

$$\|S_{112} + S_{115}\|_{H^s} \leqslant C(\epsilon^2 E_s^{1/2} + \epsilon^{7/2}). \tag{3.317}$$

$$\|S_{113} + S_{116}\|_{H^s} \leqslant C(\epsilon^2 E_s^{1/2} + \epsilon^{7/2}). \tag{3.318}$$

So we have

$$\|\mathcal{S}_{11}\|_{H^s} \leqslant C(\epsilon^2 E_s^{1/2} + \epsilon^{7/2}). \tag{3.319}$$

We can estimate other quantities in $D_tG - D_t^0G_0 - \tilde{D}_t\tilde{G} + \tilde{D}_t^0\tilde{G}_0$ similarly, and obtain

$$\left\|D_tG - D_t^0G_0 - \tilde{D}_t\tilde{G} + \tilde{D}_t^0\tilde{G}_0\right\|_{H^s} \leqslant C(\epsilon^2 E_s^{1/2} + \epsilon^{7/2}). \tag{3.320}$$

3.9.8.2 Estimate $\left\|(\mathcal{P}-\mathcal{P}_0)\tilde{D}_t(I-\mathcal{H}_\omega)(\omega-\alpha)-(\tilde{\mathcal{P}}-\tilde{\mathcal{P}}_0)\tilde{D}_t^0(I-\tilde{\omega})(\tilde{\omega}-\alpha)\right\|_{H^s}$

We have

$$\begin{aligned}&(\mathcal{P}-\mathcal{P}_0)\tilde{D}_t(I-\mathcal{H}_\omega)(\omega-\alpha)-(\tilde{\mathcal{P}}-\tilde{\mathcal{P}}_0)\tilde{D}_t^0(I-\tilde{\omega})(\tilde{\omega}-\alpha)\\=&(\mathcal{P}-\mathcal{P}_0-\tilde{\mathcal{P}}+\tilde{\mathcal{P}}_0)\tilde{D}_t(I-\mathcal{H}_\omega)(\omega-\alpha)+(\tilde{\mathcal{P}}-\tilde{\mathcal{P}}_0)[\tilde{D}_t(I-\mathcal{H}_\omega)(\omega-\alpha)-\tilde{D}_t^0(I-\tilde{\omega})(\tilde{\omega}-\alpha)]\\:=&U_1+U_2.\end{aligned}\tag{3.321}$$

U_2 is a known function, it's easy to obtain that

$$\|U_2\|_{H^s}\leqslant C\epsilon^{7/2}.\tag{3.322}$$

The operator

$$\begin{aligned}\mathcal{P}-\mathcal{P}_0-\tilde{\mathcal{P}}+\tilde{\mathcal{P}}_0=&D_tb_1\partial_\alpha+b_1\partial_\alpha D_t^0-\tilde{D}_t\tilde{b}_1\partial_\alpha-\tilde{b}_1\partial_\alpha\tilde{D}_t^0\\=&(b_1-\tilde{b}_1)\partial_\alpha+\tilde{D}_t(b_1-\tilde{b}_1)\partial_\alpha+(b_1-\tilde{b}_1)\partial_\alpha D_t^0+\tilde{b}_1\partial_\alpha(b_0-\tilde{b}_0)\partial_\alpha.\end{aligned}$$

Then it's easy to obtain

$$\|U_1\|_{H^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}).\tag{3.323}$$

So we have

$$\begin{aligned}&\left\|(\mathcal{P}-\mathcal{P}_0)\tilde{D}_t(I-\mathcal{H}_\omega)(\omega-\alpha)-(\tilde{\mathcal{P}}-\tilde{\mathcal{P}}_0)\tilde{D}_t^0(I-\tilde{\omega})(\tilde{\omega}-\alpha)\right\|_{H^s}\\\leqslant&C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}).\end{aligned}\tag{3.324}$$

3.9.8.3 Estimate L

We have obtained estimate for the quantity L in the previous section, we have

$$\begin{aligned}&(I-\mathcal{H}_\zeta)(\zeta-\alpha)-(I-\mathcal{H}_\omega)(\omega-\alpha)-(I-\mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta}-\alpha)+(I-\mathcal{H}_{\tilde{\omega}})(\tilde{\omega}-\alpha)\\ \leqslant&C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}).\end{aligned} \tag{3.325}$$

Therefore

$$\|\partial_\alpha^n\mathcal{S}_1\|_{L^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}). \tag{3.326}$$

So we have

$$\left\|\partial_\alpha^n(I-\mathcal{H}_\zeta)\mathcal{S}_1\right\|_{L^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}). \tag{3.327}$$

3.9.9 Estimate $[D_t^2-iA\partial_\alpha,\partial_\alpha^n]\sigma_1$

The way we estimate this term is similar to that of $[D_t^2-iA\partial_\alpha,\partial_\alpha^n]\rho_1$. We omitt the details. We have

$$\left\|[D_t^2-iA\partial_\alpha,\partial_\alpha^n]\sigma_1\right\|_{H^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}). \tag{3.328}$$

3.9.10 Estimate $\left\|-2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t(\delta-\tilde{\delta})}{\zeta_\alpha}\right\|_{H^s}$

Estimates of this quantity similar to that of $\mathcal{R}_{16}$, we have

$$\left\|-2[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha D_t(\delta-\tilde{\delta})}{\zeta_\alpha}\right\|_{H^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}). \tag{3.329}$$

3.9.11 Estimate $\left\|\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta(\delta-\tilde{\delta})d\beta\right\|_{H^s}$

Estimates of this quantity similar to that of $\mathcal{R}_{17}$, we have

$$\left\|\frac{1}{\pi i}\int\Big(\frac{D_t\zeta(\alpha)-D_t\zeta(\beta)}{\zeta(\alpha)-\zeta(\beta)}\Big)^2\partial_\beta(\delta-\tilde{\delta})d\beta\right\|_{H^s}\leqslant C(\epsilon^{7/2}+\epsilon^2E_s^{1/2}). \tag{3.330}$$

Sum up these estimates

Lemma III.45. *We have*

$$\left\|\mathcal{C}_{2,n}\right\|_{L^2} = \left\|\mathcal{P}\sigma_1^{(n)}\right\|_{L^2} \leqslant C(\epsilon^{7/2} + \epsilon^2 E_s^{1/2}). \tag{3.331}$$

3.9.12 Estimate the quantity $\frac{a_t}{a} \circ \kappa^{-1}$

By (3.75), lemma III.35, corollary III.43, it's easy to obtain

$$\left\|\frac{a_t}{a} \circ \kappa^{-1}\right\|_{L^\infty} \leqslant \epsilon^{3/2} + \epsilon E_s^{1/2}. \tag{3.332}$$

3.9.13 Estimate $\mathcal{R}_1^{(n)}$

Recall that

$$\mathcal{R}_1^{(n)} = \frac{1}{2}(I + \mathcal{H}_\zeta)\rho_1^{(n)} = \frac{1}{2}(I + \mathcal{H}_\zeta)\partial_\alpha^n \rho_1.$$

So we have

$$\begin{aligned}
\mathcal{R}_1^{(n)} =& \frac{1}{2}\partial_\alpha^n (I + \mathcal{H}_\zeta)\rho_1 - \frac{1}{2}[\partial_\alpha^n, \mathcal{H}_\zeta]\rho_1 \\
=& \frac{1}{2}\partial_\alpha^n (I + \mathcal{H}_\zeta)(I - \mathcal{H}_\zeta)(\lambda - \tilde{\lambda}) - \frac{1}{2}\sum_{m=1}^n \partial_\alpha^{n-m}[\zeta_\alpha - 1, \mathcal{H}_\zeta]\frac{\partial_\alpha^m \rho_1}{\zeta_\alpha} \\
=& -\frac{1}{2}\sum_{m=1}^n \partial_\alpha^{n-m}[\zeta_\alpha - 1, \mathcal{H}_\zeta]\frac{\partial_\alpha^m \rho_1}{\zeta_\alpha}.
\end{aligned}$$

Decompose $\zeta_\alpha - 1 = (r_1)_\alpha + (r_0)_\alpha + \tilde{\zeta}_\alpha - 1$. We estimate $(r_1)_\alpha$ in $H^s(\mathbb{R})$ while estimate $(r_0)_\alpha + \tilde{\zeta}_\alpha - 1$ in $W^{s,\infty}$. By Proposition III.18,

$$\begin{aligned}
\left\|\partial_\alpha \mathcal{R}_1^{(n)}\right\|_{L^2} \leqslant & \sum_{m=1}^n \left\|\partial_\alpha^{n-m+1}[\zeta_\alpha - 1, \mathcal{H}_\zeta]\frac{\partial_\alpha^m \rho_1}{\zeta_\alpha}\right\|_{L^2} \\
\leqslant & C\Big(\|(r_1)_\alpha\|_{H^s} + \left\|(r_0)_\alpha + \tilde{\zeta}_\alpha - 1\right\|_{W^{s,\infty}}\Big)\|\partial_\alpha \rho_1\|_{H^s} \\
\leqslant & C(E_s^{1/2} + \epsilon)E_s^{1/2}.
\end{aligned} \tag{3.333}$$

We need also to bound $\partial_t\mathcal{R}_1^{(n)}$. We have

$$\begin{aligned}\partial_t\mathcal{R}_1^{(n)} =& -\frac{1}{2}\sum_{m=1}^{n}\partial_\alpha^{n-m}\partial_t[\zeta_\alpha-1,\mathcal{H}_\zeta]\frac{\partial_\alpha^m\rho_1}{\zeta_\alpha}\\ =& -\frac{1}{2}\sum_{m=1}^{n}\partial_\alpha^{n-m}\Big\{[\partial_\alpha\zeta_t,\mathcal{H}_\zeta]\frac{\partial_\alpha^m\rho_1}{\zeta_\alpha}+[\zeta_\alpha-1,\mathcal{H}_\zeta]\frac{\partial_\alpha^m\partial_t\rho_1}{\zeta_\alpha}\\ &-\frac{1}{\pi i}\int\frac{(\zeta_\alpha(\alpha)-\zeta_\beta(\beta))(\zeta_t(\alpha)-\zeta_t(\beta))}{(\zeta(\alpha)-\zeta(\beta))^2}\partial_\beta^m\rho_1(\beta)d\beta\Big\}\end{aligned}$$

Write $\partial_t = D_t - b\partial_\alpha$, then use Proposition III.18 to obtain desired estimates. For example, the term $\partial_\alpha^{n-m}[\partial_\alpha D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha^m\rho_1}{\zeta_\alpha}$ can be estimated as follows

$$\begin{aligned}&\Big\|\partial_\alpha^{n-m}[\partial_\alpha D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha^m\rho_1}{\zeta_\alpha}\Big\|_{L^2}\\ \leqslant&\big\|\partial_\alpha(D_t\zeta-D_t^0\omega)\big\|_{H^{n-1}}+\big\|\partial_\alpha D_t^0\omega\big\|_{W^{n-1,\infty}}\|\rho_1\|_{H^n}\\ \leqslant&C(E_s+\epsilon E_s^{1/2}+\epsilon^{5/2}).\end{aligned}$$

Similar argument gives

$$\Big\|\partial_t\mathcal{R}_1^{(n)}\Big\|_{L^2}\leqslant C(E_s+\epsilon E_s^{1/2}+\epsilon^{5/2}). \tag{3.334}$$

So we have

$$\int\partial_t\mathcal{R}_1^{(n)}\partial_\alpha\bar{\mathcal{R}}_1^{(n)}d\alpha\leqslant C(E_s^2+\epsilon E_s^{3/2}+\epsilon^2E_s+\epsilon^{7/2}E_s^{1/2}+\epsilon^5). \tag{3.335}$$

3.9.14 Estimate $\phi_1^{(n)}$

Recall that

$$\phi_1^{(n)}=\frac{1}{2}(I-\mathcal{H}_\zeta)\partial_\alpha^n\rho_1.$$

We use the rough estimate for $\phi_1^{(n)}$: for $n \leqslant s$, apply Lemma III.36, we have

$$\begin{aligned}\left\|\partial_\alpha \phi_1^{(n)}\right\|_{L^2} =&\frac{1}{2}\left\|(I-\mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 - [\zeta_\alpha - 1, \mathcal{H}_\zeta]\frac{\rho_1^{n+1}}{\zeta_\alpha}\right\|_{L^2}\\ \leqslant&\|\partial_\alpha \rho_1\|_{H^n}\\ \leqslant&C(E_s^{1/2}+\epsilon^{3/2}).\end{aligned} \tag{3.336}$$

3.9.15 Estimate $\int \partial_\alpha \phi_1^{(n)} \overline{\partial_t \mathcal{R}_1^{(n)}}$

If we use (3.334) and (3.336), then

$$\int \partial_\alpha \phi_1^{(n)} \overline{\partial_t \mathcal{R}_1^{(n)}} \leqslant \left\|\phi_1^{(n)}\right\|_{L^2}\left\|\partial_t \mathcal{R}_1^{(n)}\right\|_{L^2} \tag{3.337}$$

$$\leqslant C(E_s^{1/2}+\epsilon^{3/2})(E_s + \epsilon E_s^{1/2} + \epsilon^{5/2}) \tag{3.338}$$

$$\leqslant C(E_s^{3/2}+\epsilon^4), \tag{3.339}$$

which is not good enough. So we need to explore the cancellation between $\phi_1^{(n)}$ and $\partial_t \mathcal{R}_1^{(n)}$. We have

$$\begin{aligned}\partial_t \mathcal{R}_1^{(n)} =&\frac{1}{2}\partial_t (I+\mathcal{H}_\zeta)\mathcal{R}_1^{(n)}\\ =&\frac{1}{2}(I+\mathcal{H}_\zeta)\partial_t \mathcal{R}_1^{(n)} + [\zeta_t, \mathcal{H}]\frac{\partial_\alpha \mathcal{R}_1^{(n)}}{\zeta_\alpha}.\end{aligned}$$

For the second term, we have

$$\begin{aligned}\int \partial_\alpha \phi_1^{(n)} \overline{[\zeta_t, \mathcal{H}]\frac{\partial_\alpha \mathcal{R}_1^{(n)}}{\zeta_\alpha}} d\alpha \leqslant&(E_s^{1/2}+\epsilon^{3/2})((E_s^{1/2}+\epsilon)E_s^{1/2}\epsilon\\ \leqslant&C(\epsilon E_s^{3/2} + \epsilon^2 E_s + \epsilon^{7/2}E_s^{1/2}).\end{aligned}$$

For the first term, note that

$$\partial_\alpha \phi_1^{(n)} = \frac{1}{2}(I - \mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 - \frac{1}{2}[\zeta_\alpha - 1, \mathcal{H}_\zeta]\partial_\alpha^{n+1}\rho_1. \tag{3.340}$$

We have

$$\begin{aligned}
\int [\zeta_\alpha - 1, \mathcal{H}_\zeta]\partial_\alpha^{n+1}\rho_1 \overline{(I + \mathcal{H}_\zeta)\partial_t \mathcal{R}_1^{(n)}} d\alpha \leqslant & C\|\zeta_\alpha - 1\|_{W^{s-1,\infty}} \|\partial_\alpha \rho_1\|_{H^s} \left\|\partial_t \mathcal{R}_1^{(n)}\right\|_{L^2} && (3.341)\\
\leqslant & C\epsilon(E_s^{1/2} + \epsilon^{3/2})(E_s + \epsilon E_s^{1/2} + \epsilon^{5/2}) && (3.342)\\
\leqslant & C(\epsilon E_s^{3/2} + \epsilon^2 E_s + \epsilon^{7/2} E_s^{1/2} + \epsilon^5). && (3.343)
\end{aligned}$$

We have

$$\begin{aligned}
& \int (I - \mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 \overline{(I + \mathcal{H}_\zeta)\partial_t \mathcal{R}_1^{(n)}} d\alpha && (3.344)\\
= & \int (I - \mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 (I - \mathcal{H}_\zeta)\overline{\partial_t \mathcal{R}_1^{(n)}} d\alpha + \int (I - \mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 (\overline{\mathcal{H}_\zeta} + \mathcal{H}_\zeta)\overline{\partial_t \mathcal{R}_1^{(n)}} d\alpha && (3.345)
\end{aligned}$$

We have

$$\begin{aligned}
& \int (I - \mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 (\overline{\mathcal{H}_\zeta} + \mathcal{H}_\zeta)\overline{\partial_t \mathcal{R}_1^{(n)}} d\alpha && (3.346)\\
\leqslant & \left\|\partial_\alpha^{n+1}\rho_1\right\|_{L^2} \|Im\zeta_\alpha\|_{W^{s-1,\infty}} \left\|\partial_t \mathcal{R}_1^{(n)}\right\|_{L^2} && (3.347)\\
\leqslant & C\epsilon(E_s^{1/2} + \epsilon^{3/2})(E_s + \epsilon E_s^{1/2} + \epsilon^{5/2}) && (3.348)\\
\leqslant & C(\epsilon E_s^{3/2} + \epsilon^2 E_s + \epsilon^{7/2} E_s^{1/2} + \epsilon^5). && (3.349)
\end{aligned}$$

By Cauchy integral formula, we have

$$\int (I - \mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 (I - \mathcal{H}_\zeta)\overline{\partial_t \mathcal{R}_1^{(n)}} \zeta_\alpha d\alpha = 0$$

So we have

$$\int (I-\mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 (I-\mathcal{H}_\zeta)\overline{\partial_t \mathcal{R}_1^{(n)}}d\alpha = \int (I-\mathcal{H}_\zeta)\partial_\alpha^{n+1}\rho_1 (I-\mathcal{H}_\zeta)\overline{\partial_t \mathcal{R}_1^{(n)}}(1-\zeta_\alpha)d\alpha \tag{3.350}$$

$$\leqslant \left\|\partial_\alpha^{n+1}\rho_1\right\|_{L^2}\|\zeta_\alpha - 1\|_{W^{s-1,\infty}}\left\|\partial_t \mathcal{R}_1^{(n)}\right\|_{L^2} \tag{3.351}$$

$$\leqslant C\epsilon(E_s^{1/2}+\epsilon^{3/2})(E_s+\epsilon E_s^{1/2}+\epsilon^{5/2}) \tag{3.352}$$

$$\leqslant C(\epsilon E_s^{3/2}+\epsilon^2 E_s+\epsilon^{7/2}E_s^{1/2}+\epsilon^5). \tag{3.353}$$

So we obtain

$$\frac{d\mathcal{E}_n}{dt} \leqslant C(E_s^2+\epsilon E_s^{3/2}+\epsilon^2 E_s+\epsilon^{7/2}E_s^{1/2}+\epsilon^5). \tag{3.354}$$

The estimate of $\frac{d\mathcal{F}_n}{dt}$ is almost the same, and we obtain

$$\frac{d\mathcal{F}_n}{dt} \leqslant C(\epsilon E_s^{3/2}+\epsilon^2 E_s+\epsilon^{7/2}E_s^{1/2}+\epsilon^5). \tag{3.355}$$

Combine (3.354) and (3.355), we obtain

$$\frac{d\mathcal{E}_s}{dt} \leqslant C(\epsilon E_s^{3/2}+\epsilon^2 E_s+\epsilon^{7/2}E_s^{1/2}+\epsilon^5). \tag{3.356}$$

3.9.16 Control E_s in terms of $\mathcal{E}_s$

To obtain bound on the energy $\mathcal{E}_s$, it's remaining to bound E_s in terms of $\mathcal{E}_s$. First, we make the following reamrk.

Lemma III.46. *We have for $n \leqslant s$,*

$$\int \sigma_1^{(n)}\overline{\partial_\alpha \sigma_1^{(n)}}d\alpha \geqslant -C(\epsilon^5+\epsilon^{5/2}E_s^{1/2}+\epsilon E_s+E_s^{3/2}).$$

Proof. We decompose $\sigma_1^{(n)} = \Psi_1^{(n)} + \mathcal{S}_1^{(n)}$ as in (3.145). So we have

$$\int \sigma_1^{(n)} \overline{\partial_\alpha \sigma_1^{(n)}} d\alpha = \int \Psi_1^{(n)} \overline{\partial_\alpha \Psi_1^{(n)}} d\alpha + \int \mathcal{S}_1^{(n)} \overline{\partial_\alpha \mathcal{S}_1^{(n)}} d\alpha + \int \Psi_1^{(n)} \overline{\partial_\alpha \mathcal{S}_1^{(n)}} d\alpha + \int \mathcal{S}_1^{(n)} \overline{\partial_\alpha \Psi_1^{(n)}} d\alpha$$
$$:= W_1 + W_2 + W_3 + W_4.$$

By lemma III.32, we have $W_1 \geqslant 0$. Similar to the estimate for $\mathcal{R}_1^{(n)}$ and $\phi_1^{(n)}$, we have

$$\left\| \partial_\alpha \mathcal{S}_1^{(n)} \right\|_{H^s} \leqslant C(E_s + \epsilon E_s^{1/2} + \epsilon^{5/2}). \tag{3.357}$$

$$\left\| \Psi_1^{(n)} \right\|_{H^s} \leqslant C(E_s^{1/2} + \epsilon^{5/2}). \tag{3.358}$$

The lemma follows directly from the above three equations. □

3.9.16.1 Bound $\left\|D_t^2 r_1\right\|_{H^s}$ by $\|D_t r_1\|_{H^s}$ and $\|\partial_\alpha r_1\|_{H^s}$

First we derive equation governing $D_t^2 r_1$. We have by water waves equation

$$D_t^2 \zeta = iA\zeta_\alpha - i, \qquad (D_t^0)^2 \omega = iA_0 \omega_\alpha - i. \tag{3.359}$$

$$\tilde{D}_t^2 \tilde{\zeta} = i\tilde{A}\tilde{\zeta}_\alpha - i + \epsilon^4 \mathcal{R}_1, \qquad (\tilde{D}_t^0)^2 \tilde{\omega} = i\tilde{A}_0 \partial_\alpha \tilde{\omega} - i + \epsilon^4 \mathcal{R}_2. \tag{3.360}$$

where

$$\left\| \epsilon^4 \mathcal{R}_1 - \epsilon^4 \mathcal{R}_2 \right\|_{H^{s+7}} \leqslant C\epsilon^{7/2}.$$

So we have

$$D_t^2 \zeta - (D_t^0)^2 \omega - (\tilde{D}_t^2 \tilde{\zeta} - (\tilde{D}_t^0)^2 \tilde{\omega}) = iA\zeta_\alpha - iA_0 \omega_\alpha - (i\tilde{A}\tilde{\zeta}_\alpha - i\tilde{A}_0 \tilde{\omega}_\alpha) + error, \tag{3.361}$$

with

$$\|error\|_{H^s} \leqslant C\epsilon^{7/2}.$$

We write left hand side of (3.361) as

$$\begin{aligned}\text{LHS of (3.361)} =&D_t^2\xi_1 + (D_t^2 - (D_t^0)^2)\omega - (\tilde{D}_t^2\tilde{\xi}_1 + (\tilde{D}_t^2 - (\tilde{D}_t^0)^2)\tilde{\omega})\\ =&D_t^2 r_1 + (D_t^2 - \tilde{D}_t^2)\tilde{\xi}_1 + (D_t^2 - (D_t^0)^2)\omega + (\tilde{D}_t^2 - (\tilde{D}_t^0)^2)\tilde{\omega}.\end{aligned} \tag{3.362}$$

Split $A = A_0 + A_1$, $\tilde{A} = \tilde{A}_0 + \tilde{A}_1$. We write the right hand side of (3.361) as (omitt the ϵ^4 term)

$$\begin{aligned}\text{RHS of (3.361)} =&iA_1\zeta_\alpha + iA_0\partial_\alpha\xi_1 - (i\tilde{A}_1\tilde{\zeta}_\alpha + i\tilde{A}_0\partial_\alpha\tilde{\xi}_1)\\ =&i(A_1 - \tilde{A}_1)\zeta_\alpha + i\tilde{A}_1\partial_\alpha r + i(A_0 - \tilde{A}_0)\partial_\alpha\xi_1 + i\tilde{A}_0\partial_\alpha r_1.\end{aligned} \tag{3.363}$$

By (3.361), (3.362), and (3.363), we obtain

$$\begin{aligned}D_t^2 r_1 =&i(A_1 - \tilde{A}_1)\zeta_\alpha + i\tilde{A}_1\partial_\alpha r + i(A_0 - \tilde{A}_0)\partial_\alpha\xi_1 + i\tilde{A}_0\partial_\alpha r_1\\ &- \Big\{(D_t^2 - \tilde{D}_t^2)\tilde{\xi}_1 + (D_t^2 - (D_t^0)^2)\omega + (\tilde{D}_t^2 - (\tilde{D}_t^0)^2)\tilde{\omega}\Big\}\end{aligned} \tag{3.364}$$

By (3.239), decompose $\zeta_\alpha = (r_1)_\alpha + (r_0)_\alpha + \tilde{\zeta}_\alpha$, it's easy to obtain

$$\Big\|i(A_1 - \tilde{A}_1)\zeta_\alpha\Big\|_{H^s} \leqslant C\Big\|A_1 - \tilde{A}_1\Big\|_{H^s} \leqslant C\epsilon^{5/2} + C\epsilon E_s^{1/2}. \tag{3.365}$$

By corollary III.42, we have

$$\|iA_1\partial_\alpha\xi_1\|_{H^s} \leqslant \|A_1\|_{H^s}\|\partial_\alpha\xi_1\|_{H^s} \leqslant \epsilon^{5/2}. \tag{3.366}$$

By (3.126) and Sobolev embedding, we have

$$\Big\|i(A_0 - \tilde{A}_0)\partial_\alpha\xi_1\Big\|_{H^s} \leqslant \Big\|A_0 - \tilde{A}_0\Big\|_{W^{s,\infty}}\|\partial_\alpha\xi_1\|_{H^s} \leqslant C\epsilon^{7/2}. \tag{3.367}$$

Obviously,

$$\left\|i\tilde{A}_0\partial_\alpha r_1\right\|_{H^s} \leqslant \left\|\tilde{A}_0\right\|_{W^{s,\infty}}\|\partial_\alpha r_1\|_{H^s} \leqslant C\|\partial_\alpha r_1\|_{H^s}. \tag{3.368}$$

By lemma III.41 and (3.231), we have

$$\left\|(D_t^2 - \tilde{D}_t^2)\tilde{\xi}_1 + (D_t^2 - (D_t^0)^2)\omega + (\tilde{D}_t^2 - (\tilde{D}_t^0)^2)\tilde{\omega}\right\|_{H^s} \leqslant C\epsilon^{5/2}. \tag{3.369}$$

Recall that $\tilde{A}_1 = 0$. So $i\tilde{A}_1\partial_\alpha r = 0$.

Use (3.364), together with (3.365)-(3.369), and recall that $E_s^{1/2} := \|\partial_\alpha r_1\|_{H^s} + \|D_t r_1\|_{H^s} + \left\|D_t^2 r_1\right\|_{H^s}$, we obtain

$$\begin{aligned}
\left\|D_t^2 r_1\right\|_{H^s} \leqslant & C\|\partial_\alpha r_1\|_{H^s} + C\epsilon E_s^{1/2} + \epsilon^{5/2} && (3.370)\\
\leqslant & C(\|\partial_\alpha r_1\|_{H^s} + \|D_t r_1\|_{H^s}) + C\epsilon\left\|D_t^2 r_1\right\|_{H^s} + C\epsilon^{5/2}. && (3.371)
\end{aligned}$$

Therefore, for ϵ sufficiently small, we have

$$\left\|D_t^2 r\right\|_{H^s} \leqslant C(\|\partial_\alpha r_1\|_{H^s} + \|D_t r_1\|_{H^s} + \epsilon^{5/2}). \tag{3.372}$$

Now we are ready to bound E_s by $\mathcal{E}_s$.

Lemma III.47. *We have*

$$\|\partial_\alpha r_1\|_{H^s} + \|D_t r_1\|_{H^s} \leqslant C(\mathcal{E}^{1/2} + \epsilon^{5/2}). \tag{3.373}$$

Proof. **Step 1.** Show that

$$\|D_t r_1\|_{H^s} \leqslant C(\epsilon E_s^{1/2} + E_s + \epsilon^{5/2} + \|D_t\rho_1\|_{H^s}). \tag{3.374}$$

See the proof of lemma III.36.

Step 2. Show that

$$\|(r_1)_\alpha\|_{H^s} \leqslant C(\|D_t\sigma_1\|_{H^s} + \epsilon E_s^{1/2} + \epsilon^{5/2}).$$

Use the fact that $\bar{r}_1$ is almost holomorphic, similar to the argument in step 1, we have

$$\|(r_1)_\alpha\|_{H^s} \leqslant C(\left\|(I-\mathcal{H}_\zeta)(r_1)_\alpha\right\|_{H^s} + \epsilon E_s^{1/2} + \epsilon^{5/2}). \tag{3.375}$$

To bound $\left\|(I-\mathcal{H}_\zeta)(r_1)_\alpha\right\|_{H^s}$ in terms of $\|D_t\sigma_1\|_{H^s}$ plus an error term, consider

$$U := iA\zeta_\alpha - iA_0\omega_\alpha - i\tilde{A}\tilde{\zeta}_\alpha + i\tilde{A}_0\tilde{\omega}_\alpha.$$

Use the fact that $\tilde{A} = \tilde{A}_0 = 1$, we have on one hand,

$$\begin{aligned} U =& i(\zeta_\alpha - \omega_\alpha - \tilde{\zeta}_\alpha + \tilde{\omega}_\alpha) + i(A-1)\zeta_\alpha - i(A_0-1)\omega_\alpha - i(\tilde{A}-1)\tilde{\zeta}_\alpha + i(\tilde{A}_0-1)\tilde{\omega}_\alpha \\ =& i(r_1)_\alpha + i(A-A_0)\omega_\alpha + i(A-1)\xi_\alpha. \end{aligned}$$

So

$$\left\|(I-\mathcal{H}_\zeta)(r_1)_\alpha\right\|_{H^s} \leqslant \left\|(I-\mathcal{H}_\zeta)U\right\|_{H^s} + C(\epsilon E_s^{1/2} + E_s + \epsilon^{5/2}). \tag{3.376}$$

On the other hand, use water wave equations

$$(D_t^2 - iA\partial_\alpha)\zeta = -i, \qquad ((D_t^0)^2 - iA_0\partial_\alpha)\tilde{\omega} = -i,$$

and the fact that $\tilde{\zeta}, \tilde{\omega}$ are good approximations:

$$(\tilde{D}_t^2 - i\tilde{A}\partial_\alpha)\tilde{\zeta} = -i + \epsilon^4\tilde{\mathcal{R}}, \qquad ((\tilde{D}_t^0)^2 - i\tilde{A}_0\partial_\alpha)\tilde{\omega} = -i + \epsilon^4\tilde{\mathcal{R}}_0,$$

with

$$\left\|\epsilon^4\tilde{\mathcal{R}}-\epsilon^4\tilde{\mathcal{R}}_0\right\|_{H^s}\leqslant \epsilon^{7/2}.$$

Denote $\epsilon^4\mathcal{R}:=\epsilon^4\tilde{\mathcal{R}}-\epsilon^4\tilde{\mathcal{R}}_0$. So we have

$$\begin{aligned}U=&D_t^2\zeta-(D_t^0)^2\omega-\tilde{D}_t^2\tilde{\zeta}+(\tilde{D}_t^0)^2\tilde{\omega}+\epsilon^4\mathcal{R}\\=&D_t(D_t\zeta-D_t^0\omega-\tilde{D}_t\tilde{\zeta}+\tilde{D}_t^0\tilde{\omega})+(b-b_0)\partial_\alpha D_t^0\omega+(b-\tilde{b}_0)\partial_\alpha\tilde{D}_t\tilde{\zeta}\\&-(b-\tilde{b}_0)\partial_\alpha\tilde{D}_t^0\tilde{\omega}+\epsilon^4\mathcal{R}.\end{aligned}$$

We have

$$\begin{aligned}&\left\|(b-b_0)\partial_\alpha D_t^0\omega+(b-\tilde{b}_0)\partial_\alpha\tilde{D}_t\tilde{\zeta}-(b-\tilde{b}_0)\partial_\alpha\tilde{D}_t^0\tilde{\omega}\right\|_{H^s}\\\leqslant&C(\epsilon E_s^{1/2}+\epsilon^{5/2}).\end{aligned}\tag{3.377}$$

Denote

$$U_{error}:=(b-b_0)\partial_\alpha D_t^0\omega+(b-\tilde{b}_0)\partial_\alpha\tilde{D}_t\tilde{\zeta}-(b-\tilde{b}_0)\partial_\alpha\tilde{D}_t^0\tilde{\omega}+\epsilon^4\mathcal{R}.$$

We have

$$\begin{aligned}(I-\mathcal{H}_\zeta)U=&(I-\mathcal{H}_\zeta)D_t(D_t\zeta-D_t^0\omega-\tilde{D}_t\tilde{\zeta}+\tilde{D}_t^0\tilde{\omega})+(I-\mathcal{H}_\zeta)U_{error}\\=&D_t(I-\mathcal{H}_\zeta)(D_t\zeta-D_t^0\omega-\tilde{D}_t\tilde{\zeta}+\tilde{D}_t^0\tilde{\omega})\\&+[D_t\zeta,\mathcal{H}_\zeta]\frac{\partial_\alpha(D_t\zeta-D_t^0\omega-\tilde{D}_t\tilde{\zeta}+\tilde{D}_t^0\tilde{\omega})}{\zeta_\alpha}+(I-\mathcal{H}_\zeta)U_{error}.\end{aligned}\tag{3.378}$$

Because we want to get estiamtes in terms of $D_t\sigma_1$, we rewrite $(I-\mathcal{H}_\zeta)(D_t\zeta-D_t^0\omega-\tilde{D}_t\tilde{\zeta}+\tilde{D}_t^0\tilde{\omega})$

as

$$
\begin{aligned}
&(I-\mathcal{H}_\zeta)(D_t\zeta - D_t^0\omega - \tilde{D}_t\tilde{\zeta} + \tilde{D}_t^0\tilde{\omega})\\
=&(I-\mathcal{H}_\zeta)(D_t\xi - D_t^0\xi_0 - \tilde{D}_t\tilde{\xi} + \tilde{D}_t^0\tilde{\xi}_0 + b - b_0 - \tilde{b} - \tilde{b}_0)\\
=&(I-\mathcal{H}_\zeta)D_t\xi - (I-\mathcal{H}_\omega)D_t^0\xi_0 - (I-\mathcal{H}_{\tilde{\zeta}})\tilde{D}_t\tilde{\xi} + (I-\mathcal{H}_{\tilde{\omega}})\tilde{D}_t^0\tilde{\xi}_0 \qquad (3.379)\\
&+ (\mathcal{H}_\zeta - \mathcal{H}_\omega)D_t^0\xi_0 + (\mathcal{H}_\zeta - \mathcal{H}_{\tilde{\zeta}})\tilde{\xi} - (\mathcal{H}_\zeta - \mathcal{H}_{\tilde{\omega}})\tilde{D}_t^0\tilde{\xi}_0 + (I-\mathcal{H}_\zeta)(b - b_0 - \tilde{b} - \tilde{b}_0)\\
:=&(\delta - \tilde{\delta}) + V_1 + V_2,
\end{aligned}
$$

where

$$V_1 := (\mathcal{H}_\zeta - \mathcal{H}_\omega)D_t^0\xi_0 + (\mathcal{H}_\zeta - \mathcal{H}_{\tilde{\zeta}})\tilde{\xi} - (\mathcal{H}_\zeta - \mathcal{H}_{\tilde{\omega}})\tilde{D}_t^0\tilde{\xi}_0,$$

and

$$V_2 := (I-\mathcal{H}_\zeta)(b - b_0 - \tilde{b} - \tilde{b}_0).$$

We have

$$\|D_tV_1\|_{H^s} + \|D_tV_2\|_{H^s} \leqslant C(\epsilon E_s^{1/2} + E_s + \epsilon^{5/2}). \qquad (3.380)$$

Combine (3.378), (3.379), and (3.380), use the fact that

$$\delta - \tilde{\delta} = \sigma_1 + error,$$

where the H^s norm of the error is controlled by $C(\epsilon E_s + E_s + \epsilon^{5/2}$. We obtain

$$\big\|(I-\mathcal{H}_\zeta)U\big\|_{H^s} = \|D_t\sigma_1\|_{H^s} + C(\epsilon E_s^{1/2} + \epsilon^{5/2}) \qquad (3.381)$$

Combine (3.375), (3.376) and (3.381), we have

$$\|(r_1)_\alpha\|_{H^s} \leqslant C(\|D_t\sigma_1\|_{H^s} + \epsilon E_s^{1/2} + \epsilon^{5/2}). \qquad (3.382)$$

Step 3. Control $\|D_t\sigma_1\|_{H^s}$ and $\|D_t\rho_1\|_{H^s}$ by $\mathcal{E}_s$.

By corollary III.43 and lemma III.46, we have

$$\|D_t\sigma_1\|_{H^s} + \|D_t\rho_1\|_{H^s} \leqslant 2\mathcal{E}_s + C(\epsilon^5 + \epsilon^{5/2}E_s^{1/2} + \epsilon E_s + E_s^{3/2}). \tag{3.383}$$

Combine (3.374) and (3.382), we obtain

$$E_s^{1/2} \leqslant C(\mathcal{E}_s + \epsilon E_s^{1/2} + \epsilon^{5/2}).$$

So we obtian

$$E_s^{1/2} \leqslant C(\mathcal{E}_s + \epsilon^{5/2}). \tag{3.384}$$

□

Combine (3.356) and (3.384), we obtain

$$\frac{d\mathcal{E}_s}{dt} \leqslant C(\epsilon^{7/2}\mathcal{E}_s^{1/2} + \epsilon^2\mathcal{E}_s + \epsilon\mathcal{E}_s^{3/2} + \epsilon^5). \tag{3.385}$$

By bootstrap argument, we obtain

Proposition III.48. *Let s, k, $B(0), B(X,T)$, M_0, $\zeta^{(1)}$ and $\mathcal{T}$ be given as in Theorem I.16. Let ϵ_0 be given. Suppose $E_s(0) = M_0\epsilon^3$. Then there exists a probably smaller ϵ_0 depends on k, s, $\mathcal{T}, M_0, \|B(0)-1\|_{H^{s+7}}$ so that for all $0 < \epsilon < \epsilon_0$ and $0 \leqslant t \leqslant \min\{T_0, \mathcal{T}\epsilon^{-2}\}$, we have $\mathcal{E}(t) \leqslant C\epsilon^3$, for some constant $C = C(k, s, M_0, \mathcal{T}, \|B(0)-1\|_{H^{s+7}})$.*

3.10 Justification of the NLS from full water waves

In this section, we show that for non-vanishing wave packet-like data, the solution to the water wave system exists on the $O(\epsilon^{-2})$ time scale, and is well-approximated by the wave packet whose modulation evolves according the the 1d focusing NLS. Let's summarize what we have done so far:

1. Use the NLS to construct approximate solutions to the full water waves. Let B be a solution to 1d focusing NLS. We show that there is an approximate solution $\tilde{\zeta} = \alpha + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)}$ to the water waves system on time scale $O(\epsilon^{-2})$ such that $\zeta^{(1)} = Be^{i\phi}$.

2. A priori energy estimate of error term. We show that if $(\zeta, D_t\zeta, D_t^2\zeta)$ is a solution to water waves system, with approximation $(\tilde{\zeta}, \tilde{D}_t\tilde{\zeta}, \tilde{D}^2\tilde{\zeta})$, then the remainder term $r := \zeta - \tilde{\zeta}$ satisfies some good energy estimates on time scale $O(\epsilon^{-2})$.

In the next subsection, we show that there exists initial data (ζ_0, v_0, w_0) such that $\bar{\zeta}_0 - \alpha$ and $\bar{v}_0$ are holomorphic, and $(\zeta_0, v_0, w_0) - (\tilde{\zeta}(0), \tilde{D}_t\tilde{\zeta}(0), \tilde{D}_t^2\tilde{\zeta}(0)) = O(\epsilon^{3/2})$ in appropriate sense.

3.10.1 Construction of appropriate initial data

In this subsection, we construct initial data to the water waves system which is close to the approximation $(\tilde{\zeta}, \tilde{D}_t\tilde{\zeta}, \tilde{D}_t^2\tilde{\zeta})$. To be precise, we need

(I-1) $\bar{\zeta}_0 - \alpha, D_t\bar{\zeta}_0 \in \mathcal{H}ol_{\mathcal{N}}(\Omega(0))$, which is equivalent to

$$(I - \mathcal{H}_{\zeta_0})(\bar{\zeta}_0 - \alpha) = 0, \qquad (I - \mathcal{H}_{\zeta_0})D_t\bar{\zeta}(0) = 0.$$

(I-2) $\zeta_0 = \omega(0) + \xi_1(0)$, such that

$$(I - \mathcal{H}_p)(\bar{\omega}(0) - \alpha) = 0, \qquad (I - \mathcal{H}_p)D_t^0\omega(0) = 0.$$

(I-3) The distance between $\omega(0)$ and $\tilde{\omega}(0)$ is small:

$$\|\omega(0) - \tilde{\omega}(0)\|_{W^{s'+1,\infty}} \leqslant C\epsilon^2.$$

$$\left\|D_t^0\omega(0) - \tilde{D}_t^0\tilde{\omega}(0)\right\|_{W^{s',\infty}} \leqslant C\epsilon^2.$$

And the distance between ξ_1 and $\tilde{\xi}_1$ is also small:

$$\left\|\partial_\alpha(\xi_1(0) - \tilde{\xi}_1(0))\right\|_{H^s(\mathbb{R})} \leqslant C\epsilon^{3/2}.$$

$$\left\|D_t\xi_1(0) - \tilde{D}_t\tilde{\xi}_1(0)\right\|_{H^{s+1/2}(\mathbb{R})} \leqslant C\epsilon^{3/2}.$$

(I-4) $A_0(0)$, $(D_t^0)^2\omega(0)$, $D_t^0\omega(0)$ satisfy the following compatibility condition

$$(I-\mathcal{H}_p)(A_0(0)-1) = i[(D_t^0)^2\omega(0), \mathcal{H}_p]\frac{\partial_\alpha\bar{\omega}(0)-1}{\partial_\alpha\omega(0)} + i[D_t^0\omega(0), \mathcal{H}_p]\frac{\partial_\alpha D_t^0\bar{\omega}(0)}{\partial_\alpha\omega(0)}. \quad (3.386)$$

(I-5) $A(0)$, $D_t^2\zeta(0)$, $D_t\zeta(0)$ satisfy the following compatibility condition

$$(I-\mathcal{H}_{\zeta_0})(A(0)-1) = i[D_t^2\zeta(0), \mathcal{H}_{\zeta_0}]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha} + i[D_t\zeta(0), \mathcal{H}_{\zeta(0)}]\frac{\partial_\alpha D_t\bar{\zeta}(0)}{\zeta_\alpha} \qquad (3.387)$$

In the following lemma, we show that initial data satisfy (I-1)-(I-5) exist.

Lemma III.49. *For sufficiently small $\epsilon_0 > 0$, there exist $\omega(0)$, $\zeta(0) = \zeta_0$ such that for all $\epsilon < \epsilon_0$, (I1)-(I5) hold.*

To prove lemma III.49, we first prove the following.

Lemma III.50. *Let $\tilde{\omega}_0 = \alpha + \epsilon\omega^{(1)} + \epsilon^2\omega^{(2)} + \epsilon^3\omega^{(3)}$ be such that the $\omega^{(1)} = c_0e^{ik\alpha}$ for some constant c_0, and $\omega^{(2)}, \omega^{(3)} \in W^{s'+1,\infty}(\mathbb{T})$. Then there exists $\omega(0)$ be such that $\omega(0) - \alpha$ is periodic with period 2π,*

$$\|\omega(0) - \tilde{\omega}\|_{W^{s'+1,\infty}(\mathbb{T})} \leqslant C\epsilon^2,$$

and $\bar{\omega}(0) - \alpha = \Phi_0(\omega(\alpha, 0))$, where Φ_0 is holomorphic in the domain bounded above by the *curve* $\omega(0)$, *satisfying*

$$\lim_{\operatorname{Im} z \to -\infty} \Phi_0(z) = 0.$$

In the appendix, we show that if we define $\Gamma := \{\alpha + ce^{ik\alpha} : \alpha \in \mathbb{R}\}$ and Ω_-^0 the domain which is below Γ, then $ce^{-ik\alpha}$ is not holomorphic in Ω_-^0. So we cannot simply take $\omega(0) = \alpha + ce^{ik\alpha}$.

Lemma III.50 is a direct consequence of the following lemma.

Lemma III.51. *Let $s' \geqslant 0$ and $\epsilon \geqslant 0$ be fixed. There exists $\epsilon_0 = \epsilon_0(s') > 0$ be sufficiently small such that for all ϵ with $0 \leqslant \epsilon \leqslant \epsilon_0$, there exists ω such that $\omega - \alpha \in W^{s'+1,\infty}(\mathbb{T})$, satisfying*

$$\bar{\omega} - \alpha = ce^{-i\omega}, \tag{3.388}$$

and

$$\left\| \omega - \alpha - \bar{c}e^{i\alpha} \right\|_{W^{s'+1,\infty}(\mathbb{T})} \leqslant C\epsilon^2, \tag{3.389}$$

for some constant $C = C(s') > 0$.

Proof. We prove the lemma by iteration. It suffices to prove the case that $s' = 0$. Let $\omega_0 := \alpha$ and define

$$\bar{\omega}_{n+1} = \alpha + ce^{-i\omega_n}.$$

Then $\omega_1 = \alpha + \bar{c}e^{i\alpha}$. We have

$$\|\omega_1 - \omega_0\|_\infty = |c| = \epsilon.$$

Then we have

$$
\begin{aligned}
\|\omega_2-\omega_1\|_\infty =&|c|\left\|e^{-i\omega_0}\right\|_\infty\left\|e^{-i(\omega_1-\omega_0)}-1\right\|_\infty\\
=&\epsilon\left\|\sum_{k\geqslant 1}\frac{(-i(\omega_1-\omega_0))^k}{k!}\right\|_\infty\\
\leqslant&\epsilon\|\omega_1-\omega_0\|_\infty\sum_{n\geqslant 1}\frac{\|\omega_1-\omega_0\|_\infty^{k-1}}{k!}\\
\leqslant&\epsilon\|\omega_1-\omega_0\|_\infty\sum_{n\geqslant 1}\frac{\epsilon^{k-1}}{k!}\leqslant\epsilon\|\omega_1-\omega_0\|_\infty\sum_{n\geqslant 1}\frac{\epsilon^{k-1}}{(k-1)!}\\
\leqslant&2\epsilon\|\omega_1-\omega_0\|_\infty.
\end{aligned}
$$

From this, we have also that

$$
\|\omega_2-\alpha\|_\infty\leqslant\|\omega_2-\omega_1\|_\infty+\|\omega_1-\alpha\|_\infty\leqslant\epsilon+2\epsilon^2.
$$

We claim that

$$
\|\omega_n-\alpha\|_\infty\leqslant\epsilon+\sum_{k=2}^{n}(2\epsilon)^k, \tag{3.390}
$$

and

$$
\|\omega_n-\omega_{n-1}\|_\infty\leqslant 2\epsilon\|\omega_{n-1}-\omega_{n-2}\|_\infty. \tag{3.391}
$$

Indeed,

$$
\begin{aligned}
\|\omega_{n+1}-\omega_n\|_\infty =&|c|\left\|e^{-i\alpha}e^{-i(\omega_{n-1}-\alpha)}\right\|_\infty\left\|e^{-i(\omega_n-\omega_{n-1})}-1\right\|_\infty\\
\leqslant&\epsilon e^{2\epsilon}\|\omega_n-\omega_{n-1}\|_\infty\sum_{k\geqslant 1}\frac{\|\omega_n-\omega_{n-1}\|_\infty^{k-1}}{k!}\\
\leqslant&\epsilon e^{2\epsilon}\|\omega_n-\omega_{n-1}\|_\infty\sum_{k\geqslant 1}\frac{(2\epsilon)^{k-1}}{(k-1)!}\quad\leqslant\epsilon e^{2\epsilon}\|\omega_n-\omega_{n-1}\|_\infty e^{2\epsilon}\\
\leqslant&2\epsilon\|\omega_n-\omega_{n-1}\|_\infty.
\end{aligned}
$$

So the induction hypothesis is verified, and the claim follows. So

$$\|\omega_n - \omega_{n-1}\|_\infty \leqslant (2\epsilon)^n.$$

There exists $\xi_0 \in L^\infty$ such that

$$\omega_n - \alpha \to \xi_0.$$

It's easy to show that $\xi_0 \in W^{s'+1}(\mathbb{T})$ and $\omega_n - \alpha \to \xi_0$ in $W^{s'+1,\infty}(\mathbb{T})$. Moreover, if we denote $\omega := \xi_0 + \alpha$, we have

$$\left\|\omega - \alpha - \bar{c}e^{ik\alpha}\right\|_{W^{s'+1,\infty}} \leqslant C\epsilon^2.$$

So the proof of Lemma III.51 and hence Lemma III.50 is completed. □

With lemma III.50, we can prove lemma III.49.

Proof of lemma III.49. Given $\tilde{\zeta}$ and $\tilde{\omega}$ given by (3.107)-(3.109), by Lemma III.50, there exists ω_0 with $\omega_0 - \alpha \in W^{s'+1,\infty}(\mathbb{T})$ such that $\bar{\omega}_0 - \alpha = B_0(0)e^{-i\omega_0}$ and $\left\|\omega_0 - \alpha - B_0(0)\epsilon e^{i\alpha}\right\|_{W^{s'+1,\infty}} \leqslant C(s)\epsilon^2$. By Lemma III.11, we have

$$(I - \mathcal{H}_{\omega_0})(\bar{\omega}_0 - \alpha) = B_0(0)(I - \mathcal{H}_{\omega_0})e^{-i\omega_0} = 0. \tag{3.392}$$

We want to find ζ_0 such that

$$(I - \mathcal{H}_{\zeta_0})(\bar{\zeta}_0 - \alpha) = 0. \tag{3.393}$$

Write

$$\zeta_0 = \omega_0 + \xi_1(0).$$

We want

$$\xi_1(0) - \tilde{\xi}_1(0) = O(\epsilon^{3/2}). \tag{3.394}$$

We simply write $\xi_1(0)$ as ξ_1 and $\tilde{\xi}_1(0)$ as $\tilde{\xi}_1$. By (3.393), we expect that

$$\bar{\zeta}_0 - \alpha = (I + \mathcal{H}_{\zeta_0})f, \tag{3.395}$$

for some function f. By (3.394), we expect that f should be closed to $\tilde{\xi}$. It's easy to see that we can take

$$f = \frac{1}{2}(\overline{\tilde{\xi}_1 + \omega_0 - \alpha}).$$

Denote $\xi_0 := \omega_0 - \alpha$. Recall that $(I - \mathcal{H}_{\omega_0})\bar{\omega}_0 = 0$, so we have

$$\frac{1}{2}(I + \mathcal{H}_{\omega_0})\bar{\xi}_0 = \bar{\xi}_0.$$

So we have

$$\begin{aligned}(I + \mathcal{H}_{\zeta_0})f =& \frac{1}{2}(I + \mathcal{H}_{\zeta_0})\bar{\tilde{\xi}}_1 + \frac{1}{2}(I + \mathcal{H}_{\zeta_0})\bar{\xi}_0\\ =& \frac{1}{2}(I + \mathcal{H}_{\zeta_0})\bar{\tilde{\xi}}_1 + \frac{1}{2}(I + \mathcal{H}_{\omega_0})\bar{\xi}_0 + (\mathcal{H}_{\zeta_0} - \mathcal{H}_{\omega_0})\bar{\xi}_0\\ =& \frac{1}{2}(I + \mathcal{H}_{\zeta_0})\bar{\tilde{\xi}}_1 + \bar{\xi}_0 + (\mathcal{H}_{\zeta_0} - \mathcal{H}_{\omega_0})\bar{\xi}_0\end{aligned}$$

So (3.395) is equivalent to

$$\bar{\xi}_1 = \frac{1}{2}(I + \mathcal{H}_{\zeta_0})\bar{\tilde{\xi}}_1 + (\mathcal{H}_{\zeta_0} - \mathcal{H}_{\omega_0})\bar{\xi}_0. \tag{3.396}$$

(3.396) can be solved by iteration: let $g_0 = 0$, $z_0 = \alpha$. Assume g_n has been constructed, define $z_n = g_n + \omega_0$. Then define g_{n+1} by

$$\bar{g}_{n+1} = \frac{1}{2}(I + \mathcal{H}_{z_n})\bar{\tilde{\xi}}_1 + (\mathcal{H}_{z_n} - \mathcal{H}_{\omega_0})\bar{\xi}_0.$$

Then it's easy to prove that $\{g_n\}$ defines a Cauchy sequence in $H^{s+7}(\mathbb{R})$, given that $\tilde{\xi}_1 \in H^{s+7}(\mathbb{R})$. See lemma 5.1 of [63] for example.

Use the same argument, we can show that $D_t\bar{\zeta}(0) \in \mathcal{H}ol_{\mathcal{N}}$, and (1), (2), (3) hold. (4) and (5) can be proved similarly.

$\square$

3.10.2 Long time well-posedness

By energy estimates in the previous section and the initial data constructed above, we can prove the following theorem.

Theorem III.5. *Let M_0, s, k, $B(0)$, $\mathcal{T}$ be given as in Theorem I.16. Denote the $B(X,T)$ the solution of (3.114) with initial data $B(0)$, and let $\zeta_1^{(1)}$ be defined as in (3.113). Then there is $\epsilon_0 = \epsilon_0(k, s, M_0, \|B_1(0)\|_{H^{s+7}(\mathbb{R})}, \mathcal{T}) > 0$ so that for $\epsilon < \epsilon_0$, there exists compatible initial data (ζ_0, v_0, w_0) to water waves system such that*

$$(\zeta_0, v_0, w_0) = (\omega_0 + \xi_1(0), D_t^0\omega_0 + (v_1)_0, (D_t^0)^2\omega_0 + (w_1)_0),$$

where $(\omega_0, D_t^0\omega_0, (D_t^0)^2\omega_0)$ is a compatible initial data for periodic water waves sytem (3.38), satisfying

$$\begin{aligned}&\Big\|(\partial_\alpha\omega_0, D_t^0\omega_0, (D_t^0)^2\omega_0) - \epsilon(\partial_\alpha\zeta_0^{(1)}(0), \partial_t\zeta_0^{(1)}(0), \partial_t^2\zeta_0^{(1)}(0))\Big\|_{H^{s'}(\mathbb{T})\times H^{s'+1/2}(\mathbb{T})\times H^{s'}(\mathbb{T})}\\&\leqslant M_0\epsilon^2,\end{aligned} \tag{3.397}$$

$$\begin{aligned}&\Big\|(\partial_\alpha\xi_1(0), (v_1)_0, (w_1)_0) - \epsilon(\partial_\alpha\zeta_1^{(1)}(0), \partial_t\zeta_1^{(1)}(0), \partial_t^2\zeta_1^{(1)}(0)\Big\|_{H^s\times H^{s+1/2}\times H^s}\\&\leqslant M_0\epsilon^{3/2},\end{aligned} \tag{3.398}$$

and for all such initial data, there exists a possibly smaller $\epsilon_0 > 0$ such that the water waves system has a unique solution $\zeta(\alpha,t)$ with $(\partial_\alpha(\zeta-\alpha), D_t\zeta, D_t^2\zeta) \in C([0, \mathcal{T}\epsilon^{-2}]; X^s \times X^{s+1/2} \times X^s)$ satisfying

$$\sup_{0\leqslant t\leqslant \mathcal{T}\epsilon^{-2}}\Big\|(\zeta_\alpha(t)-1, D_t\zeta(t), D_t^2\zeta(t) - \epsilon(\zeta_\alpha^{(1)}(t), \zeta_t^{(1)}, \zeta_{tt}^{(1)})\Big\|_{X^s\times X^{s+1/2}\times X^s} \leqslant C\epsilon^{3/2}, \tag{3.399}$$

for some constant $C = C(k, s, M_0, \mathcal{T}, \|B(0) - 1\|_{H^{s+7}})$.

In particular, if we take B to be the Peregrine soliton, then we justify the Peregrine soliton from the full water waves.

3.10.3 Rigorous justification of the Peregrine soliton in Lagrangian coordinates

Let's change of variables back the our more familiar lagrangian coordinates. We have $\kappa_t = b(\kappa)$. This gives a smooth function $\kappa : \mathbb{R} \to \mathbb{R}$. Taking k_0 smaller if necessary, it's easy to show that

$$\|\kappa_\alpha\|_{W^{s-1,\infty}} \geqslant 1/2, \quad \forall\ t \in [0, \mathcal{T}\epsilon^{-2}]. \tag{3.400}$$

So κ is a diffeomorphism. Let $z = \zeta \circ \kappa$, a be such that $(a\kappa_\alpha) \circ \kappa^{-1} = A$, we obtain water waves equation (1.81), which is in Lagrangian coordinates. We can then obtain estimates for the remainder term in Lagrangian coordinates.

Remark III.52. In Lagrangian coordinates, $\zeta_\alpha - 1$ becomes $z_\alpha - \kappa_\alpha$. So we have

$$z_\alpha - 1 = z_\alpha - \kappa_\alpha + (\kappa_\alpha - 1). \tag{3.401}$$

We have

$$\sup_{t \in [0, \mathcal{T}\epsilon^{-2}]} \|z_\alpha - \kappa_\alpha\|_{X^{s-1/2}} \lesssim \epsilon^{3/2}. \tag{3.402}$$

However, it seems that $\|\kappa_\alpha - 1\|_{X^{s-1/2}}$ can be as large as $\epsilon^{1/2}$ on time scale $O(\epsilon^{-2})$. So we are unable to rigorously justify the modulation approximation for $Re\{z_\alpha - 1\}$. Please see [63] for more details.

APPENDIX

APPENDIX A

Holomorphicity of plane waves

Let $\zeta(\alpha) = \alpha + ce^{ik\alpha}, \alpha \in \mathbb{R}$, c is a small constant, $k > 0$, for simplicity, assume k is an integer. Let

$$\Gamma := \{\zeta(\alpha) : \alpha \in \mathbb{R}\}.$$

Then Γ is a graph. Let Ω_+ be the region above Γ, and Ω_- the region below Γ. On one hand, it' easy to prove that

Lemma A.1. *α, $e^{-ik\alpha}$ and $e^{ik\alpha}$ are holomorphic in Ω_+.*

On the other hand, we'll show that $e^{ik\alpha}$ cannot be boundary value of a bounded holomorphic function in Ω_-.

Lemma A.2. *If $c \neq 0$, then $e^{ik\alpha}$ cannot be boundary value of a holomorphic function in Ω_-.*

Proof. If $e^{ik\alpha}$ is boundary value of a holomorphic function in Ω_-, then $e^{ik\alpha}$ is entire, and so α is entire. Assume $\alpha = \Phi(\zeta(\alpha))$, Φ entire. Let $\Psi(\zeta) = \zeta + ce^{ik\zeta}$. Then Ψ is entire, and $\Psi(\alpha) = \zeta(\alpha)$. So we have

$$\begin{cases} \Psi \circ \Phi(\zeta(\alpha)) = \zeta(\alpha) \\ \Phi \circ \Psi(\alpha) = \alpha. \end{cases}$$

$\Psi \circ \Phi$ and $\Phi \circ \Psi$ are entire, we must have $\Psi \circ \Phi(z) \equiv z, \Phi \circ \Psi(z) \equiv z$. So Ψ and Φ are inverse of each other.

If $c \neq 0$, then the function $z + ce^{ikz}$ has an essential singularity at ∞ because ce^{ikz} does. By Picard's theorem, $z + ce^{ikz}$ attains all values in $\mathbb{C}$ infinitely many times with at most one exception. Suppose z_0 is this exception, i.e., $z + ce^{ikz} = z_0$ has finitely many solutions (possibly none). But then $z + ce^{ikz} = z_0 + 2\pi$ has infinitely many solutions. Then

$$z - 2\pi + ce^{ikz} = z_0 \quad \Rightarrow \quad z - 2\pi + ce^{ik(z-2\pi)} = z_0.$$

So $z + ce^{ikz} = z_0$ has infinitely many solutions, a contradiction.

In particular, $z + ce^{ikz} = 0$ has infinitely many solutions. So Ψ is not invertible, contradiction. □

Lemma A.3. *If $e^{-ik\alpha}$ is boundary value of a holomorphic function in Ω_-, then $e^{ik\alpha}$ is also holomorphic in Ω_-.*

Proof. Let $e^{-ik\alpha} = G(\zeta(\alpha))$, where G is holomorphic in Ω_-. Then the zeros of G is a discrete set, which we denote by S. We'll show that $S = \varnothing$. Since $\zeta = \alpha + ce^{ik\alpha}$, we have

$$\alpha = \zeta(\alpha) - \frac{c}{e^{-ik\alpha}} = \zeta(\alpha) - \frac{c}{G(\zeta(\alpha))}.$$

Define$H(\zeta) := \zeta - \frac{c}{G(\zeta)}$, $\zeta \in \Omega_-$. Then H has boundary value α. So α is boundary value of a meromorphic function in Ω_-, with poles at S.

Note that $e^{-ikH(\zeta(\alpha))}$ has boundary values $e^{-ik\alpha}$, and $e^{-ikH(\zeta)}$ is holomorphic in $\Omega_-\backslash S$, by uniqueness extension of holomorphic functions, we must have $e^{-ikH(\zeta)} = G(\zeta)$ on $\Omega_-\backslash S$.

If $S \neq \varnothing$, then take $z_0 \in S$. Then since $G(z_0)$ is defined, z_0 must be a removable singularity of $e^{-ikH(\zeta)}$. However, since z_0 is a pole of $H(\zeta)$, so z_0 is an essential singularity of $e^{-ikH(\zeta)}$, a contradiction. So $S = \varnothing$.

So we conclude that α is holomorphic in Ω_-, and so $e^{ik\alpha}$ is holomorphic in Ω_-. □

Corollary A.4. *$e^{-ik\alpha}$ cannot be the boundary value of a holomorphic function in Ω_- if $c \neq 0$.*

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