Quantum K-theory with Level Structure

by

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for my parents

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ABSTRACT

Given a smooth, complex projective variety X, one can associate to it numerical invariants by taking holomorphic Euler characteristics of natural vector bundles on the moduli spaces of stable maps to X. The study of these invariants is called quantum K-theory. Since K-theory is closely related to representation theory, it is natural to revisit quantum K-theory from the representation theoretic point of view. One of the important concepts in representation theory is level. In this thesis, we introduce the notion of level in quasimap theory and refer to it as the level structure.

This thesis consists of two parts. In the first part, we define level structures in quasimap theory as certain determinant line bundles over moduli spaces of quasimaps. By twisting with these determinant line bundles, we define K-theoretic quasimap invariants with level structure. An important case of this construction is quantum K-theory with level structure. We study the basic properties of level structures and show that quantum K-theory with level structure satisfies the same axioms as the ordinary, i.e., Givental-Lee's, quantum K-theory. In the genus-0 case, the invariants are encoded in an important generating series: the J-function. We characterize the values of the J-function in quantum K-theory with level structure. As an application of this characterization, we prove a mirror theorem for toric varieties. One surprising

finding is that the mirrors of some of the simplest examples are Ramanujan's mock theta functions.

In the second part, we study the Verlinde/Grassmannian correspondence, which is a K-theoretic generalization of Witten's result [82]. It relates the Verlinde algebra, a representation theoretic object, with the quantum K-invariants of the Grassmannian with level structure. To prove this correspondence, an important observation is that the Verlinde invariants and quantum K-invariants of the Grassmannian can be defined using the same gauged linear sigma model but with different stability conditions. In this thesis, we study the δ-stability condition, for $\delta \in \mathbb{Q}_+$. In particular, we construct the moduli spaces of δ-stable parabolic N-pairs and prove that they are equipped with canonical perfect obstruction theories. Using virtual structure sheaves, we define Verlinde type invariants over these moduli spaces and prove that they do not change when we vary δ.

CHAPTER I

Introduction

More than a decade ago, quantum K-theory was introduced by Givental and Lee [28, 53] as the K-theoretic analog of quantum cohomology. Its recent revival stems partially from a physical interpretation of quantum K-theory as a 3D-quantum field theory in the 3-manifold of the form $S^1 \times \Sigma$ (see [47, 48]). Because of this mysterious physical connection, the B-model counterpart of quantum K-theory is q-hypergeometric series, itself a classical subject. The above connection was recently confirmed by Givental [34] as the mirror of the so-called *J-function* of the permutation-equivariant quantum K-theory.

Classically, K-theory is more closely related to representation theory, comparing to cohomology theory. It is natural to revisit quantum K-theory from the representation theoretic point of view. In fact, a variant of quantum K-theory was already studied by Aganagic, Okounkov, Smirnov and their collaborators [5, 6, 51, 60, 61, 63, 70] in relation to quantum groups. One of the predominant features of representation theory is the existence of an additional parameter called the level. A natural question is whether it is possible to extend the current version of quantum K-theory to include this notion of level. The primary goal of this thesis is to answer this question affirmatively in the context of quasimap theory. We first explain the motivation for level structures in quantum K-theory. Then we characterize quantum K-theory with level structure in genus zero. In the study of mirror symmetry of this new theory, we see the surprising appearance of Ramanujan's mock theta functions in some of the simplest examples. After that, we give a generalization of Witten's result relating Verlinde numbers and quantum K-invariants of Grassmannians with level structures.

1.1 Motivation

Our motivating example is an old physical result of Witten [82] in the early '90s which relates the quantum cohomology ring of the Grassmannian to the Verlinde algebra. Early explicit physical computations [27, 46, 80] indicate that they are isomorphic as algebras, but have different pairings. In [82], Witten gave a conceptual explanation of the isomorphism, by proposing an equivalence between the quantum field theories which govern the Verlinde algebra and the quantum cohomology of the Grassmannian. His physical derivation of the equivalence naturally leads to a mathematical problem that these two objects are conceptually isomorphic (without referring to the detailed computations). A great deal of work has been done by Agnihotri [7], Marian-Oprea [55, 56, 57], and Belkale [10]. However, to the best of our knowledge, a complete conceptual proof of the equivalence is missing.

Assuming a basic knowledge of quantum K-theory, a key and yet more or less trivial observation is that invariants of Verlinde algebras are *K-theoretic invariants*. To be more precise, let X be a smooth projective variety. Suppose that $\overline{\mathcal{M}}_{g,k}(X,\beta)$ is the moduli space of stable maps to X. Quantum cohomology studies integrals of the form

$$\int_{[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{\mathrm{vir}}} \alpha,$$

where $[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{\mathrm{vir}}$ is the so-called *virtual fundamental cycle* and α is a "tautological" cohomology class. In quantum K-theory, we replace the virtual fundamental cycle by the *virtual structure sheaf* $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)}$. We also replace the integral by the holomorphic Euler characteristic

$$\chi\big(\overline{\mathcal{M}}_{g,k}(X,\beta), E \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)}\big),$$

where E is some natural K-theory class on $\overline{\mathcal{M}}_{g,k}(X,\beta)$. For the Verlinde algebra, the relevant moduli space is the moduli space of semistable parabolic $U(n)$-bundles $U(n,d,\underline{\lambda})$ on a fixed genus g marked curve $(C, p_1, \cdots, p_k)$ with parabolic structure at p_i indexed by λ_{p_i}. Here $\underline{\lambda} = (\lambda_{p,1}, \ldots, \lambda_{p,k})$ is the collection of insertions. Let l be a non-negative integer. The parabolic structures λ_{p_i} correspond to elements $V_{\lambda_{p_i}}$ in the level-l Verlinde algebra $V_l(U(n))$. A new ingredient is a certain determinant line bundle, denoted by det, over the moduli space $U(n,d,\underline{\lambda})$. The level-$l$ Verlinde algebra calculates the holomorphic Euler characteristic

$$\langle V_{\alpha_{\lambda_1}}, \cdots, \alpha_{\lambda_k} \rangle^{l,\mathrm{Verlinde}}_{g,d} = \chi(U(n,d,\underline{\lambda}), \det{}^l).$$

Based on the above description, the Verlinde algebra is clearly a K-theoretic object, and we should compare it with the quantum K-theory of the Grassmannian with an appropriate notion of levels. Let $\mathfrak{B}un_G$ be the moduli stack of principal bundle over curves. Let $\pi : \mathfrak{C}_{\mathfrak{B}un_{g,k}} \to \mathfrak{B}un_G$ be the universal curve and let $\mathfrak{P} \to \mathfrak{C}_{\mathfrak{B}un_{g,k}}$ be the universal principal bundle. Given a finite-dimensional representation R of G, we consider the inverse determinant of cohomology

$$\det{}_R := \left(\det R\pi_*(\mathfrak{P} \times_G R)\right)^{-1}.$$

It is a line bundle over $\mathfrak{B}un_G$. Suppose $X = Z /\!\!/ G$ is a GIT quotient. Let $\mathcal{Q}^\epsilon$ be the moduli stack of ϵ-stable quasimaps to $X = Z /\!\!/ G$ (see Section 2.2). Then there is a natural forgetful morphism $\mu : \mathcal{Q}^\epsilon \to \mathfrak{Bun}_G$. We define the level-$l$ determinant line bundle as

$$\mathcal{D}^{R,l} = \mu^*(\det{}_R)^l.$$

We will often refer to $\mathcal{D}^{R,l}$ as the *level structure*. In general, when X is a smooth complex projective variety, but not a GIT quotient, one can still define determinant line bundles over the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X,\beta)$ as follows. Let $\mathcal{R}$ be a vector bundle over X. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,k}(X,\beta)$ be the universal curve and let $\mathrm{ev} : \mathcal{C} \to X$ be the universal evaluation map. We define the level-l determinant line bundle as

$$\mathcal{D}^l := \left(\det R\pi_*(\mathrm{ev}^*\mathcal{R})\right)^{-l}.$$

This definition agrees with the previous one when X is a GIT quotient (see Definition II.4 and Remark II.5).

With the above definition of the level-l determinant line bundle $\mathcal{D}^{R,l}$, we can define the level-l quantum K-invariants and quasimap invariants by twisting with

$\mathcal{D}^{R,l}$ (see Chapter III). The ordinary quantum K-theory, i.e., Givental-Lee's quantum K-theory, corresponds to the case $l = 0$.

1.2 Mirror theorem and mock theta functions

One of the main results of this thesis is a toric mirror theorem for quantum K-theory with level structure, in the same style as the recent work of Givental [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].

Let X be a smooth complex projective variety and let $\mathcal{R}$ be a vector bundle over X. When $X = Z /\!\!/ G$ is a GIT quotient, we assume $\mathcal{R}$ is of the form $(Z \times R) /\!\!/ G$, with R a finite-dimensional representation of G. By abuse of terminology, we refer to the vector bundle $\mathcal{R}$ as the "representation" R. Let Q be the Novikov variables. We fix a λ-algebra Λ which is equipped with Adams operations Ψ^i, $i = 1, 2, \ldots$. Let $\{\phi_a\}$ be a basis of $K^0(X) \otimes \mathbb{Q}$ and let $\{\phi^a\}$ be the dual basis with respect to the twisted pairing

$$(u, v)^{R,l} := \chi\big(u \otimes v \otimes (\det \mathcal{R})^{-l}\big), \quad \text{where } u, v \in K^0(Z /\!\!/ G) \otimes \mathbb{Q}.$$

Let q be a formal variable and let $\mathbf{t}(q)$ be a Laurent polynomial in q with coefficients in $K^0(X) \otimes \mathbb{Q}$. The *permutation-equivariant* K-theoretic J-function $\mathcal{J}^l_{S_\infty}(\mathbf{t}(q), Q)$ of level l and representation R is defined by

$$\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t}(q), Q) := 1 - q + \mathbf{t}(q) + \sum_a \sum_{\beta \neq 0} Q^\beta \phi^a \left\langle \frac{\phi_a}{1 - qL}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle^{R,l,S_k}_{0,k+1,\beta}.$$

Here $\langle \cdot \rangle^{R,l,S_k}_{0,k+1,\beta}$ denotes the permutation-equivariant quantum K-invariants of level l and L denote the cotangent line bundles. The J-function $\mathcal{J}^{R,l}_{S_\infty}$ can be viewed as elements in the loop space $\mathcal{K}$ defined by

$$\mathcal{K} := [K^0(X) \otimes \mathbb{C}(q)] \otimes \mathbb{C}[[Q]],$$

where $\mathbb{C}(q)$ denotes the field of complex rational functions in q. There is a natural Lagrangian polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where $\mathcal{K}_+$ consists of Laurent polynomials and $\mathcal{K}_-$ consists of reduced rational functions regular at $q = 0$ and vanishing at $q = \infty$. We denote by $\mathcal{L}^{R,l}_{S_\infty}$ the range of $\mathcal{J}^{R,l}_{S_\infty}$, i.e.,

$$\mathcal{L}^{R,l}_{S_\infty} = \cup_{\mathbf{t}(q)\in\mathcal{K}_+} \mathcal{J}^{R,l}_{S_\infty}(\mathbf{t}(q), Q) \subset \mathcal{K}.$$

Due to the stacky structure of the moduli space of stable maps, the J-function $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t}(q), Q)$, as a function in q, has poles at roots of unity. The main technical tool is a generalization of Givental-Tonita's *adelic characterization* of points on the cone $\mathcal{L}^{R,l}_{S_\infty}$ with the presence of level structure, i.e., we describe the Laurent expansion of $\mathcal{J}^{R,l}_{S_\infty}$ at each primitive root of unity in terms of certain twisted *fake* quantum K-theory. The precise statement is rather technical, and we present it in Theorem IV.5. When $l = 0$, the determinant line bundle $\mathcal{D}^{R,l}$ is trivial, and we recover the ordinary quantum K-theory. The adelic characterization of the cone in the ordinary quantum K-theory was introduced in [41], and its generalization to the permutation-equivariant theory is given in [31]. The proofs of all these results are based on application of the virtual Kawasaki's Riemann-Roch formula to moduli spaces of stable maps (see Section 4.1).

Let $\mathcal{L}_{S_\infty}$ denote the range of the permutation-equivariant big J-function in ordinary quantum K-theory (i.e., with trivial level structure). As an application of

Theorem IV.5, we prove that certain "determinantal " modifications of points on $\mathcal{L}_{S_\infty}$ lie on the cone $\mathcal{L}_{S_\infty}^{R,l}$ of quantum K-theory of level l.

Theorem I.1. *If*

$$I = \sum_{\beta \in \mathrm{Eff}(X)} I_\beta Q^\beta$$

lies on $\mathcal{L}_{S_\infty}$, then the point

$$I^{R,l} := \sum_{\beta \in \mathrm{Eff}(X)} I_\beta Q^\beta \prod_i \left(L_i^{-\beta_i} q^{(\beta_i+1)\beta_i/2} \right)^l$$

lies on the cone $\mathcal{L}_{S_\infty}^{R,l}$ of permutation-equivariant quantum K-theory of level l. Here, $\mathrm{Eff}(X)$ *denotes the semigroup of effective curve classes on X, L_i are the K-theoretic Chern roots of $\mathcal{R}$, and $\beta_i := \int_\beta c_1(L_i)$.*

In Theorem IV.17, we give explicit formulas for level-l (torus-equivariant) I-functions of toric varieties. Moreover, we prove the following toric mirror theorem.

Theorem I.2. *Assume that X is a smooth quasi-projective toric variety. The level-l torus-equivariant I-function $(1-q)I^{R,l}$ of X lies on the cone $\mathcal{L}_{S_\infty}^{R,l}$ in the permutation- and torus-equivariant quantum K-theory of level l of X.*

In the study of toric mirror theorems for quantum K-theory with level structure, a remarkable phenomenon is the appearance of Ramanujan's mock theta functions. We first establish some notations. Denote the standard representation of G by St and its dual by $\mathrm{St}^\vee$. When $G = \mathbb{C}^*$, any n-dimensional representation of G is determined by a *charge vector* $(a_1, \ldots, a_n)$ with $a_i \in \mathbb{Z}$: a $\mathbb{C}^*$-action on $\mathbb{C}^n$ can be explicitly described by

$$\lambda \cdot (x_1, \dots, x_n) \to (\lambda^{a_1} x_1, \dots, \lambda^{a_n} x_n), \quad \text{where } \lambda \in \mathbb{C}^*.$$

In the following propositions, we consider GIT quotients $\mathbb{C}^n /\!/ \mathbb{C}^*$, and we refer to the $\mathbb{C}^*$-actions by their associated charge vectors.

Proposition I.3. *Consider $X = \mathbb{C} /\!/ \mathbb{C}^* = [(\mathbb{C}\backslash 0)/\mathbb{C}^*]$, where the $\mathbb{C}^*$-action is the standard action by multiplication. The $\mathbb{C}^*$-equivariant K-ring $K_{\mathbb{C}^*}(X)$ is isomorphic to the representation ring* $\mathrm{Repr}(\mathbb{C}^*)$. *Let $\lambda \in K_{\mathbb{C}^*}(X)$ be the equivariant parameter corresponding to the standard representation. For the $\mathbb{C}^*$-representations* St *and* $\mathrm{St}^\vee$, *we have the following explicit formulas of the equivariant small I-functions*

$$I_X^{\mathrm{St},\, l}(q,Q) = 1 + \sum_{n\geq 1} \frac{q^{\frac{n(n-1)l}{2}}}{(1-\lambda^{-1}q)(1-\lambda^{-1}q^2)\cdots(1-\lambda^{-1}q^n)} Q^n,$$
$$I_X^{\mathrm{St}^\vee,\, l}(q,Q) = 1 + \sum_{n\geq 1} \frac{q^{\frac{n(n+1)l}{2}}}{(1-\lambda^{-1}q)(1-\lambda^{-1}q^2)\cdots(1-\lambda^{-1}q^n)} Q^n,$$

By chosing certain specializations of the parameters, we obtain Ramanujan's mock theta functions of order 3

$$I_X^{\mathrm{St},\, l=1}(q^2,Q)|_{\lambda=-1, Q=q} = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

$$I_X^{\mathrm{St},\, l=1}(q^2,Q)|_{\lambda=q, Q=q} = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})},$$

$$I_X^{\mathrm{St},\, l=1}(q^2,Q)|_{\lambda=-q, Q=1} = 1 + \sum_{n\geq 1} \frac{q^{n(n-1)}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})},$$

and Ramanujan's mock theta functions of order 5

$$I_X^{\mathrm{St},\,l=2}(q,Q)|_{\lambda=-1,Q=q} = 1+\sum_{n\geq 1}\frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)},$$

$$I_X^{\mathrm{St},\,l=2}(q^2,Q)|_{\lambda=q,Q=q^2} = 1+\sum_{n\geq 1}\frac{q^{2n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})},$$

$$I_X^{\mathrm{St}^\vee,\,l=2}(q,Q)|_{\lambda=-1,Q=1} = 1+\sum_{n\geq 1}\frac{q^{n(n+1)}}{(1+q)(1+q^2)\cdots(1+q^n)},$$

$$I_X^{\mathrm{St}^\vee,\,l=2}(q^2,Q)|_{\lambda=q,Q=1} = 1+\sum_{n\geq 1}\frac{q^{2n^2+2n}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}.$$

Proposition I.4. *Let a_1 and a_2 be two positive integers which are coprime. We consider the target $X_{a_1,a_2}=[(\mathbb{C}^2\backslash 0)/\mathbb{C}^*]$ with charge vector (a_1,a_2) and a line bundle $p=[\{(\mathbb{C}^2\backslash 0)\times\mathbb{C}\}/\mathbb{C}^*]$ with charge vector $(a_1,a_2,1)$. Let λ_1 and λ_2 be the equivariant parameters. For the $\mathbb{C}^*$-representations* St *and* $\mathrm{St}^\vee$*, we have the following explicit formulas of the equivariant small I-functions*

$$\begin{aligned}
&I_{X_{a_1,a_2}}^{\mathrm{St},\,l}(q,Q)\\
=&1+\sum_{n\geq 1}\frac{p^{nl}q^{\frac{n(n-1)l}{2}}}{(1-p^{a_1}\lambda_1^{-1}q)\cdots(1-p^{a_1}\lambda_1^{-1}q^{a_1n})(1-p^{a_2}\lambda_2^{-1}q)\cdots(1-p^{a_2}\lambda_2^{-1}q^{a_2n})}Q^n,\\
&I_{X_{a_1,a_2}}^{\mathrm{St}^\vee,\,l}(q,Q)\\
=&1+\sum_{n\geq 1}\frac{p^{nl}q^{\frac{n(n+1)l}{2}}}{(1-p^{a_1}\lambda_1^{-1}q)\cdots(1-p^{a_1}\lambda_1^{-1}q^{a_1n})(1-p^{a_2}\lambda_2^{-1}q)\cdots(1-p^{a_2}\lambda_2^{-1}q^{a_2n})}Q^n.
\end{aligned}$$

By choosing $(a_1,a_2)=(1,1)$ and certain specializations of the parameters, we obtain the following four Ramanujan's mock theta functions of order 3:

$$I_{X_{1,1}}^{\mathrm{St},\,l=2}(q^2,Q)|_{p=1,\lambda_1=\lambda_2=-1,Q=q} = 1+\sum_{n\geq 1}\frac{q^{n^2}}{((1+q)(1+q^2)\cdots(1+q^n))^2},$$

$$I^{\mathrm{St},\,l=2}_{X_{1,1}}(q,Q)|_{p=1,\lambda_1=\frac{1+\sqrt{3}i}{2},\lambda_2=\frac{1-\sqrt{3}i}{2},Q=q} = 1+\sum_{n\geq 1}\frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})},$$

$$\frac{1}{(1-q)^2}I^{\mathrm{St}^\vee,\,l=2}_{X_{1,1}}(q^2,Q)|_{p=1,\lambda_1=\lambda_2=q^{-1},Q=1} = \sum_{n\geq 0}\frac{q^{2n^2+2n}}{((1-q)(1-q^3)\cdots(1-q^{2n+1}))^2},$$

$$\frac{1}{(1+q+q^2)}I^{\mathrm{St}^\vee,\,l=2}_{X_{1,1}}(q^2,Q)|_{p=1,\lambda_1=\frac{-1+\sqrt{3}i}{2}q^{-1},\lambda_2=\frac{-1-\sqrt{3}i}{2}q^{-1},Q=1} = \sum_{n\geq 0}\frac{q^{2n^2+2n}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})}.$$

Proposition I.5. *Let a and b be two positive integers which are coprime. We consider the target $X_{a,-b}=[\{(\mathbb{C}\backslash 0)\times\mathbb{C}\}/\mathbb{C}^*]$ with charge vector $(a,-b)$ and a line bundle $p=[\{(\mathbb{C}\backslash 0)\times\mathbb{C}\times\mathbb{C}\}/\mathbb{C}^*]$ with charge vector $(a,-b,1)$. Let λ and μ be the equivariant parameters of the standard $(\mathbb{C}^*)^2$-action on $X_{a,-b}$. For the $\mathbb{C}^*$-representation* St, *we have the following explicit formula for the equivariant small I-function*

$$\begin{aligned}&I^{\mathrm{St},\,l}_{X_{a,-b}}(q)\\ =&1+\sum_{n\geq 1}(-1)^{bn}\frac{p^{nl-b^2n}q^{\frac{n(n-1)l-bn(bn-1)}{2}}\mu^{-bn}(1-p^b\mu)(1-p^b\mu q)\cdots(1-p^b\mu q^{bn-1})}{(1-p^a\lambda^{-1}q)(1-p^a\lambda^{-1}q^2)\cdots(1-p^a\lambda^{-1}q^{an})}Q^n.\end{aligned}$$

In particular, we have order 7 mock theta functions

$$I^{\mathrm{St},\,l=3}_{X_{2,-1}}(q,Q)|_{p=1,\lambda=1,\mu=q,Q=-q^2} = 1+\sum_{n\geq 1}\frac{q^{n^2}}{(1-q^{n+1})\cdots(1-q^{2n})},$$

$$\frac{q}{1-q}I^{\mathrm{St},\,l=3}_{X_{2,-1}}(q,Q)|_{p=1,\lambda=q^{-1},\mu=q,Q=-q^4} = \sum_{n\geq 1}\frac{q^{n^2}}{(1-q^n)\cdots(1-q^{2n-1})},$$

$$\frac{1}{1-q}I^{\mathrm{St},\,l=3}_{X_{2,-1}}(q,Q)|_{p=1,\lambda=q^{-1},\mu=q,Q=-q^3} = \sum_{n\geq 1}\frac{q^{n^2-n}}{(1-q^n)\cdots(1-q^{2n-1})}.$$

It is interesting that we can recover Ramanujan's mock theta functions using only very simple targets.

One of the attractive features of quantum K-theory is the appearance of q-hypergeometric series as mirrors of K-theoretic J-functions. Recall the definition of the q-Pochhammer symbol

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{for } n>0,$$

and $(a;q)_0 := 1$. A general q-hypergeometric series can be written as

$${}_r\phi_s = \sum_{n\geq 0} \frac{(\alpha_1;q)_n \cdots (\alpha_r;q)_n}{(\beta_1;q)_n \cdots (\beta_s;q)_n} \frac{z^n}{(q;q)_n} [(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r}.$$

For the quantum K-theory of level 0, i.e, Givental-Lee's quantum K-theory, we only see special q-hypergeometric series of the form

$$\sum_{n\geq 0} \frac{(\alpha_1;q)_n \cdots (\alpha_r;q)_n}{(\beta_1;q)_n \cdots (\beta_s;q)_n} \frac{z^n}{(q;q)_n}.$$

The level structure naturally introduces the term

$$[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r}.$$

Proposition I.6. *Consider the target* $X_{\mathbf{1},-\mathbf{1}} := O(-1)^{\oplus r}_{\mathbb{P}^{s-1}} = [\{(\mathbb{C}^s\backslash 0)\times\mathbb{C}^r\}/\mathbb{C}^*]$ *with the charge vector* $(1,1,\cdots,1,-1,-1,\cdots,-1)$. *Let* $p = [\{(\mathbb{C}^s\backslash 0)\times\mathbb{C}^r\times\mathbb{C}\}/\mathbb{C}^*]$ *be a line bundle with charge vector* $(1,\cdots,1,-1,,\cdots,-1,1)$. *Let* $\lambda_1,\cdots,\lambda_s,\mu_1,\cdots,\mu_r$ *be the equivariant parameters of the standard* $(\mathbb{C}^*)^{s+r}$*-action on* $X_{\mathbf{1},-\mathbf{1}}$. *Then the* *equivariant small I-function has the following explicit form*

$$
\begin{aligned}
&I^{\mathrm{St},\,l=1+s}_{X_{\mathbf{1},-\mathbf{1}}}(q)\\
=&1+\sum_{n\geq 1}(-1)^{nr}\prod_{i=1}^{r}(p\mu_i)^{-n}p^{(1+s)n}\frac{(p\mu_1,q)_n\cdots(p\mu_r;q)_n}{(p\lambda_1^{-1}q;q)_n\cdots(p\lambda_s^{-1}q;q)_n}Q^n(q^{\frac{n(n-1)}{2}})^{1+s-r}.
\end{aligned}
$$

Hence we can recover the general q-hypergeometric series by setting $p=1,\lambda_i^{-1}q=\beta_i,\mu_j=\alpha_j,Q=(-1)^{1+s}z\prod_{i=1}^{r}\mu_i$.

Recall that Gromov-Witten theory (of Calabi-Yau varieties) is related to quasi-modular forms. Mock modular forms are another class of modular objects, which are different from the quasi-modular forms. Yet, they share some common properties. The above mirror theorems suggest an exciting possibility that the natural geometric home of mock modular forms is quantum K-theory with non-trivial level structures.

1.3 Verlinde/Grassmannian correspondence

As mentioned in Section 1.1, Verlinde invariants are K-theoretic invariants in the theory of semistable parabolic vector bundles. Hence they should be compared with a version of quantum K-theoretic invariants of Grassmannians, instead of cohomological Gromov-Witten invariants. For this purpose, we introduce the level structure to quantum K-theory mentioned in the previous section.

With the appropriate choice of the level structure in quantum K-theory, we formulate a K-theoretic version of Witten's conjecture. We first introduce some notations. Let l be a non-negative integer. Recall that as a vector space, the Verlinde algebra $V_l(\mathfrak{gl}_n(\mathbb{C}))$ of level l is spanned by a basis $\{V_\lambda\}_{\lambda\in\mathrm{P}_l}$, where $\lambda=(\lambda_1,\ldots,\lambda_n)$

is a partition such that satisfies $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. The set P_l consists of all partitions λ with n parts such that $\lambda_1 \leq l$. There is a geometric construction of the Verlinde numbers. Let C be a smooth curve of genus g, with k marked points $p_1, \dots, p_k$. Let $I = \{p_1, \dots, p_k\}$ be the set of marked points. We assign a partition $\lambda_p = (\lambda_{p,1}, \dots, \lambda_{p,k})$ to each marked point $p \in I$. Let $U(n, d, \underline{\lambda})$ denote the moduli space of S-equivalence classes of parabolic vector bundles of rank n and degree d, with parabolic type determined by the assignment $\underline{\lambda} = (\lambda_p)_{p \in I}$ (see Definition VI.2). There exists an ample line bundle Θ_λ, called the *theta line bundle*, over $U(n, d, \underline{\lambda})$. The *GL Verlinde number* with insertion $\underline{\lambda}$ is defined by

$$\langle V_{\lambda_{p_1}}, \dots, V_{\lambda_{p_k}} \rangle_{g,d}^{l,\mathrm{Verlinde}} := \chi\big(U(n, d, \underline{\lambda}), \Theta_{\underline{\lambda}}\big).$$

We consider a variation of the quantum K-invariants defined in the previous section. We choose one more marked point $x_0 \in C$ which is disjoint from the markings in I. Let $\overline{\mathcal{M}}_C(\mathrm{Gr}(n, N), d)$ denote the *graph space* which is a moduli space parametrizing families of tuples

$$\big((C', x_0', p_1', \dots, p_k'), E, s, \varphi\big),$$

with $(C', x_0', p_1', \dots, p_k')$ a $k+1$-pointed nodal curve of genus g, E a locally free sheaf of degree d on C', s a section of $E \otimes \mathcal{O}_{C'}^N$ satisfying a certain stability condition, and $\varphi : C' \to C$ a morphism such that $\varphi([C']) = [C]$, $\varphi(x_0') = x_0$ and $\varphi(p_i') = p_i$. The stability condition on s ensures there are well-defined evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_C(\mathrm{Gr}(n, N), d) \to \mathrm{Gr}(n, N)$. Let S be the tautological vector bundle over $\mathrm{Gr}(n, N)$

and let $E = S^\vee$ be its dual. A partition $\lambda \in \mathrm{P}_l$ also determines a vector bundle $K_\lambda(S)$ on the Grassmannian $\mathrm{Gr}(n, N)$, for any N. Here, K_λ denotes the *Weyl functor* associated to λ (see [81, §2.1]). By abuse of notation, we denote $K_\lambda(S)$ by V_λ. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_C(\mathrm{Gr}(n, N), d)$ be the projection map. Let $\mathcal{E}$ be the universal vector bundle over the universal curve $\mathcal{C}$. We define the following line bundle:

$$\mathcal{D}^l := \det(\mathcal{E}_{x_0'})^e \otimes \big(\det R\pi_*(\mathcal{E})\big)^{-l},$$

where e is an integer and $\mathcal{E}_{x_0'}$ is the restriction of $\mathcal{E}$ to the distinguished marked point x_0. Let $e = l(1-g) + (ld - |\underline{\lambda}|)/n$, where $|\underline{\lambda}| = \sum_{i,p} \lambda_{p,i}$. If e is an integer, we define the quantum K-theory invariant of $\mathrm{Gr}(n, N)$ with insertions $V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}}$ by

$$\langle \det(E)^e | V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}} \rangle_{C,d}^{l,\mathrm{Gr}(n,n+l)}$$
$$:= \chi\big(\overline{\mathcal{M}}_C(\mathrm{Gr}(n, N), d), \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_C(\mathrm{Gr}(n,N),d)} \otimes \mathcal{D}^l \otimes (\otimes_{i=1}^k \mathrm{ev}_i^* V_{p_i})\big),$$

where $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_C(\mathrm{Gr}(n,N),d)}$ is the virtual structure sheaf. If e is not an integer, the Verlinde invariant is defined as zero.

We propose the following K-theoretic version of Witten's conjecture:

Conjecture I.7 (Verlinde/Grassmannian Correspondence)**.** *Up to an explicit mirror map, the GL Verlinde invariants* $\langle V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}} \rangle_{g,d}^{l,\mathrm{Verlinde}}$ *can be identified with the quantum K-invariants* $\langle \det(E)^e | V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}} \rangle_{C,d}^{l,\mathrm{Gr}(n,n+l)}$ *for $d > n(g-1)$ and* $\lambda_{p_1}, \ldots, \lambda_{p_k} \in \mathrm{P}_l$*.*

In the spirit of Witten's work, we prefer a non-computational proof of the above conjecture. Also, to obtain the mirror map, we need a deep understanding of the geometry of the moduli spaces defining these two invariants.

Remark I.8. Witten's original argument covers quantum cohomology which is a 2D quantum field theory in physics. Quantum K-theory is thought to be a 3D quantum field theory (see a physical derivation in [47]). In this sense, our version of Verlinde/Grassmannian correspondence is new in physics. It would be interesting to give a physical derivation.

Our approach is by following Witten's strategy to lift the problem into the gauged linear sigma model (GLSM) of the Grassmannian. The GLSM of the Grassmannian depends on two stability parameters ϵ and δ (see the precise definitions in Section 5.1 and Section 6.2). The ϵ-stability concerns about the stability of sections of the GLSM, while δ-stability concerns about the stability of bundles. When we vary ϵ or δ, the moduli space undergoes a series of wall-crossings. When ϵ is sufficiently large (denoted by $\epsilon = \infty$), we recover quantum K-theory of the Grassmannian. When δ is sufficiently close to zero (denoted by $= 0+$), we recover Verlinde's theory.

In this thesis, we will only discuss δ-wall-crossings. We leave the study of ϵ-wall-crossings for future research. For technical reasons (see Remark VI.30), we require the partitions to be from a subset $\mathrm{P}_l' \subset \mathrm{P}_l$ consisting of all partitions λ with n parts such that $\lambda_1 < l$. For a generic value of $\delta \in \mathbb{Q}_+$, we denote by $\langle \det(E)^e | V_{\lambda_{p_1}}, \dots, V_{\lambda_{p_k}} \rangle_{C,d}^{l,\delta,\mathrm{Gr}(n,n+l)}$ the δ-stable GLSM invariant with an ordinary insertion $\det(E)^e$ and parabolic insertions $V_{\lambda_{p_1}}, \dots, V_{\lambda_{p_k}}$, where $\lambda_{p_i} \in \mathrm{P}_l'$ (see Definition VI.44). When $\delta = 0+$, the GLSM moduli space admits a morphism to $U(n, d, \underline{\lambda})$.

This morphism is generically a projective bundle, if $d > n(g-1)$ and the open subset $U^s_C(n,d,\underline{\lambda}) \subset U_C(n,d,\underline{\lambda})$ of stable vector bundles is non-empty. Therefore, it allows us to recover the GL Verlinde numbers from ($\delta = 0+$)-stable parabolic GLSM invariants. More precisely, we prove (see Theorem VI.46) the following

Theorem I.9. *Suppose that $d > n(g-1)$, $U^s(n,d,\underline{\lambda}) \neq \emptyset$ and $\lambda_{p_i} \in \mathrm{P}'_l$ for $i = 1, \ldots, k$. Then*

$$\langle V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}} \rangle^{l,\mathrm{Verlinde}}_{g,d} = \langle \det(E)^e | V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}} \rangle^{l,\delta=0^+,\mathrm{Gr}(n,N)}_{C,d}.$$

When $n = 1$, the moduli space of δ-stable parabolic GLSM data does not depend on δ (see Remark VI.8). When $n = 2$, we analyze the geometric wall-crossing of the δ-stable parabolic GLSM moduli spaces. This will allow us to prove the following δ-wall-crossing result.

Theorem I.10. *Assume $n \leq 2$. Suppose that $N \geq n + l$, $d > 2g - 2 + k$, and δ is generic. Then,*

$$\langle \det(E)^e | V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}} \rangle^{l,\delta,\mathrm{Gr}(n,N)}_{C,d}$$

is independent of δ.

The higher rank δ-wall-crossing problem is much more complicated and we leave it for future research.

Remark I.11. The material in this thesis is the result of collaborative work with Yongbin Ruan, and appears also in the preprints [65] and [66].

1.4 Outline

This thesis consists of two parts: in the first part, we develop the general theory of lever structures in quasimap theory, and prove a toric mirror theorem; in the second part, we focus on the case when targets are Grassmannians, and prove some δ-wall-crossing results towards understanding the Verlinde/Grassmannian correspondence.

In Chapter II, we introduce the notion of level in the K-theoretic quasimap theory and establish its main properties. In particular, we show that quantum K-theory with level structure satisfies the same Kontsevich-Manin axioms as Givental-Lee's quantum K-theory.

In Chapter III We define the K-theoretic quasimap invariants with level structure and their permutation-equivariant version. We introduce an important generating series of genus zero invariants: the permuation-equivariant J-function.

In Chapter IV, we focus on the genus zero theory. The range of the permuation-equivariant J-function is a cone in a certain infinite-dimensional symplectic vector space. We characterize this cone of level l. As an application, we compute the K-theoretic toric I-functions of level l, and prove a toric mirror theorem for quantum K-theory with level structure. In the study of mirror symmetry of some of the simplest examples, we see the surprising appearance of Ramanujan's mock theta functions.

In Chapter V, we study the Verlinde/Grassmannian correspondence. We introduce the gauged linear sigma model (GLSM) of Grassmannians, and treat the case of rank 2 δ-wall-crossing in the absence of parabolic structures. We aim to give the reader a general idea of the strategy.

In Chapter VI, we construct the moduli spaces of δ-stable parabolic GLSM/stable pairs. When the degree is small, the moduli spaces are not smooth and this was considered to be a major difficulty in the '90s. With the modern technique of virtual structure sheaves, we can define Verlinde-type invariants using these moduli spaces. When the stability parameter δ is sufficiently small, we recover Verlinde invariants. At the end, we prove the general δ-wall-crossing result in the rank 2 case.

1.5 Notation and conventions

We introduce some basic notations in K-theory. For a Deligne-Mumford stack X, we denote by $K_0(X)$ the Grothendieck group of coherent sheaves on X and by $K^0(X)$ the Grothendieck group of locally free sheaves on X.

For a flat morphism $f : X \to Y$, we have the flat pullback $f^* : K_0(Y) \to K_0(X)$. For a proper morphism $g : X \to Y$, we can define the proper pushforward $f_* : K_0(X) \to K_0(Y)$ by

$$[F] \mapsto \sum_n (-1)^n [R^n f_* F].$$

For a regular embedding $i : X \hookrightarrow Y$ and a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

one can define the *Gysin pullback* $i^! : K_0(Y') \to K_0(X')$ by

$$i^![F] = \sum_i (-1)^! [\mathrm{Tor}_i^Y(F, \mathcal{O}_X)],$$

where $\mathrm{Tor}_i^Y(F, \mathcal{O}_X)$ denotes the Tor sheaf.

Let E be a vector bundle. We define the *K-theoretic Euler class* of E by

$$\lambda_{-1}(E^\vee) := \sum_i (-1)^i \wedge^i E^\vee.$$

Throughout the paper, we consider the rational Grothendieck groups $K_0(X)_{\mathbb{Q}} := K_0(X) \otimes \mathbb{Q}$ and $K^0(X)_{\mathbb{Q}} := K^0(X) \otimes \mathbb{Q}$.

CHAPTER II

Level structure and quasimap theory

2.1 Determinant line bundles

In this section, we briefly review the construction of determinant line bundles.

Let $\mathcal{X}$ be a Deligne-Mumford stack. Let $\mathcal{E}$ be a locally free, finitely generated $\mathcal{O}_{\mathcal{X}}$ module. We define the determinant line bundle of $\mathcal{E}$ as

$$\det(\mathcal{E}) := \wedge^{\mathrm{rank}(\mathcal{E})}\mathcal{E},$$

where $\wedge^i$ denotes the i-th wedge product. In general, let $\mathcal{F}^\bullet$ be a complex of coherent sheaves on $\mathcal{X}$ which has a bounded locally free resolution, i.e., there exists a bounded complex of locally free, finitely generated $\mathcal{O}_{\mathcal{X}}$ modules $\mathcal{G}^\bullet$ and a quasi-isomorphism

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet.$$

We define the determinant line bundle associated to $\mathcal{F}^\bullet$ by

$$\det(\mathcal{F}^\bullet) := \otimes_n \det(\mathcal{G}^n)^{(-1)^n}.$$

We summarize some basic properties of this construction in the following proposition.

Proposition II.1. *Let $\mathcal{F}$ be a complex of coherent sheaves which has a bounded locally free resolution. Then*

1. *The construction of* $\det(\mathcal{F}^\bullet)$ *does not depend on the locally free resolution.*
2. *For every short exact sequence of complexes of sheaves which have bounded locally free resolutions*
$$0 \to \mathcal{F}^\bullet \xrightarrow{\alpha} \mathcal{G}^\bullet \xrightarrow{\beta} \mathcal{H}^\bullet \to 0,$$
we have a functorial isomorphism
$$i(\alpha, \beta) : \det(\mathcal{F}^\bullet) \otimes \det(\mathcal{H}^\bullet) \xrightarrow{\sim} \det(\mathcal{G}^\bullet).$$
3. *The operator* det *commutes with base change. To be more precise, for every (representable) morphism of Deligne-Mumford stacks*
$$g : \mathcal{X} \to \mathcal{Y},$$
we have an isomorphism
$$\det(\mathrm{L}g^*) \xrightarrow{\sim} g^*\det.$$

In the case when $\mathcal{X}$ is a scheme, the above proposition is proved in [50]. Note that these properties are preserved under flat base change. Therefore they hold for stacks as well.

2.2 Level structure in quasimap theory

In this section, we first recall the quasimap theory for nonsingular GIT quotients introduced in [19]. Then we define the level structure in this setting and discuss its generalizations in orbifold quasimap theory.

Let $Z = \mathrm{Spec}(A)$ be a complex affine algebraic variety in $\mathbb{C}^n$ and let G be a reductive group acting on it. Let $\theta : G \to \mathbb{C}^*$ be a character determining a G-equivariant line bundle $L_\theta := Z \times \mathbb{C}$. Let $Z^s(\theta)$ and $Z^{ss}(\theta)$ be the stable and semistable loci, respectively. Throughout the paper, we assume $Z^s(\theta) = Z^{ss}(\theta)$ is nonsingular. Furthermore, we assume that G acts freely on $Z^s(\theta)$. It follows that the GIT quotient $Z /\!\!/_\theta G$ is nonsingular and quasi-projective. For simplicity, we drop θ from the notation of the GIT quotient. The unstable locus is defined as $Z_{\mathrm{us}} := Z - Z^s(\theta)$. Recall that we can identify the G-equivariant Picard group $\mathrm{Pic}^G(Z)$ with the Picard group $\mathrm{Pic}([Z/G])$ of the quotient stack $[Z/G]$ by sending an G-equivariant line bundle L to $[L/G]$. Let $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^G(Z), \mathbb{Z})$.

Definition II.2 ([19])**.** A quasimap is a tuple $(C, p_1, \ldots, p_k, P, s)$ where

- $(C, p_1, \ldots, p_k)$ is a connected, at most nodal, k-pointed projective curve of genus g,
- P is a principal G-bundle on C,
- s is a section of the induced fiber bundle $P \times_G Z$ on C such that (P, s) is of class β, i.e., the homomorphism

$$\mathrm{Pic}^G(Z) \to \mathbb{Z}, \quad L \to \deg_C(s^*(P \times_G L)),$$

  is equal to β.

We require that there are only finitely many base points, i.e., points $p \in C$ such that $s(p) \in Z_{\mathrm{us}}$. An element $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^G(Z), \mathbb{Z})$ is called L_θ-effective if it can be represented as a finite sum of classes of quasimaps. We also refer to β as a curve class. Denote by E the semigroup of L_θ-effective (curve) classes.

A quasimap $(C, p_1, \ldots, p_k, P, s)$ is called *prestable* if the base points are disjoint from the nodes and marked points on C. Given a rational number $\epsilon > 0$, a prestable quasimap is called ϵ-*stable* if it satisfies the following conditions

1. $\omega_{C,\log} \otimes \mathcal{L}_\theta^\epsilon$ is ample, where $\omega_{C,\log} := \omega_C\big(\sum_{i=1}^k p_i\big)$ is the twisted dualizing sheaf of C and

$$\mathcal{L}_\theta := u^*(P \times_G L_\theta) \cong P \times_G \mathbb{C}_\theta.$$

2. $\epsilon l(x) \leq 1$ for every point x in C where

$$l(x) := \text{length}_x\big(\text{coker}(u^*\mathcal{J}) \to \mathcal{O}_C\big).$$

Here $\mathcal{J}$ is the ideal sheaf of the closed subscheme $P \times_G Z_{\text{us}}$ of $P \times_G Z$.

Let $\mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G, \beta) = \{(C, p_1, \ldots, p_k, P, s)\}$ be the moduli stack of ϵ-stable quasimaps. It is shown in [19] that this stack is a separated Deligne-Mumford stack of finite type and it is proper over the affine quotient $Z/_{\text{aff}}\, G := \text{Spec}(A^G)$. When Z has only local complete intersection singularities, the ϵ-stable quasimap space $\mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G, \beta)$ admits a canonical perfect obstruction theory.

Remark II.3. There are two extreme chambers for the stability parameter ϵ.

1. $(\epsilon = \infty)$-stable quasimaps. One can check that when $(g, k) \neq (0, 0)$ and $\epsilon > 1$, the quasimap space $\mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G, \beta)$ is isomorphic to the moduli space of

stable maps $\overline{\mathcal{M}}_{g,k}(Z /\!\!/ G, \beta)$. When $(g,k) = (0,0)$, the same holds with $\epsilon > 2$. Therefore when ϵ is sufficiently large, we denote the ϵ-stable quasimap space by

$$\mathcal{Q}^{\infty}_{g,k}(Z /\!\!/ G, \beta) = \overline{\mathcal{M}}_{g,k}(Z /\!\!/ G, \beta)$$

and refer to it as the $(\epsilon = \infty)$-theory.

2. $(\epsilon = 0+)$-stable quasimaps. Fix $\beta \in E$. For each $\epsilon \in (0, \frac{1}{\beta(L_\theta)}]$, the ϵ-stability is equivalent to the condition that the underlying curve C of a quasimap does not have rational tails and on each rational bridge, the line bundle $\mathcal{L}_\theta$ has strictly positive degree. Since we need to consider different β at the same time, we reformulate the stability condition as

$$\omega_{C,\log} \otimes \mathcal{L}_\theta^{\epsilon} \text{ is ample for all } \epsilon \in \mathbb{Q}_{>0}.$$

Quasimaps which satisfy the above stability condition are referred to as $(\epsilon = 0+)$-stable quasimaps.

To define the level structure, we introduce some notation first. Let $\mathfrak{M}_{g,k}$ be the algebraic stack of pre-stable nodal curves and $\mathfrak{B}un_G$ be the relative moduli stack

$$\mathfrak{B}un_G \xrightarrow{\phi} \mathfrak{M}_{g,k}$$

of principal G-bundles on the fibers of the universal curve $\mathfrak{C}_{g,k} \to \mathfrak{M}_{g,k}$. The morphism ϕ is smooth. There is a forgetful morphism which forgets the section s

$$\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta) \xrightarrow{\mu} \mathfrak{B}un_G$$

Let $\tilde{\pi} : \mathfrak{C}_{\mathfrak{B}un_{g,k}} \to \mathfrak{B}un_{g,k}$ be the universal curve which is the pullback of $\mathfrak{C}_{g,k}$ along ϕ. Let $\tilde{\mathfrak{P}} \to \mathfrak{C}_{\mathfrak{B}un_{g,k}}$ be the universal principal G-bundle. We denote by $\pi : \mathcal{C}_{g,k} \to \mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$ the universal curve on the quasimap space. Let $\mathcal{P} \to \mathcal{C}_{g,k}$ be the universal principal bundle given by the pullback of $\tilde{\mathfrak{P}} \to \mathfrak{C}_{\mathfrak{B}un_{g,k}}$.

Definition II.4. Given a finite-dimensional representation R of G, we define the level-l determinant line bundle over $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$ as

$$\mathcal{D}^{R,l} := \big(\det R\pi_*(\mathcal{P} \times_G R)\big)^{-l}. \tag{2.1}$$[1]

Alternatively, one can define $\mathcal{D}^{R,l}$ to be the pullback via μ of the determinant line bundle $\big(\det R\pi_*(\tilde{\mathfrak{P}} \times_G R)\big)^{-l}$ on $\mathfrak{B}un_{g,k}$.

Remark II.5. The definition mentioned in the introduction is the second one. It is conceptually better in the sense that it does not depend on the different moduli spaces over $\mathfrak{B}un_{g,k}$. In our case, these moduli spaces are the ϵ-stable quasimap spaces $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$ for different ϵ. However, $\mathfrak{B}un_{g,k}$ is an Artin stack, and it is technically more difficult to work with it. Formally, we will use the first definition as the working definition.

Remark II.6. Note that in Definition II.4, the bundle $\mathcal{P} \times_G R$ is the pullback of the vector bundle $[Z \times R/G] \to [Z/G]$ along the evaluation map to the quotient stack $[Z/G]$. Therefore, given a vector bundle $\mathcal{R}$ on X, we can use (2.1) to define a determinant line bundle over the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X, \beta)$, even

[1] According to [19, §6.2], $R\pi_*(\mathcal{P} \times_G R)$ has a two-term locally free resolution. Therefore, we can take the determinant of this complex.

when X is not a GIT quotient. To be more precise, let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,k}(X, \beta)$ be the universal curve and let $\mathrm{ev} : \mathcal{C} \to X$ be the universal evaluation map. We define the level-l determinant line bundle as

$$\mathcal{D}^{R,l} := \left(\det R\pi_*(\mathrm{ev}^*\mathcal{R})\right)^{-l}.$$

We will often abuse the notation by referring to the vector bundle $\mathcal{R}$ as the "representation" R.

The above construction can be easily generalized to orbifold quasimap theory. To be more precise, suppose the target can be written as $[Z^s/G]$. Now, we do not assume G acts freely on the stable locus $Z^s(\theta)$. Therefore $[Z^s/G]$ is in general a Deligne-Mumford stack. The quasimap theory for such orbifold GIT targets is established in [15]. According to [15, Section 2.4.5], we still have universal curves and universal principal G-bundles over moduli spaces of ϵ-stable orbifold quasimaps. Therefore the level-l determinant line bundle can still be defined using (2.1).

2.3 Properties of level structure in quasimap theory

In this section, we study the level-l determinant line bundle in the case $\beta = 0$ and its pullbacks along some natural morphisms between moduli spaces of quasimaps. An important property of the level structure $\mathcal{D}^{R,l}$ is that it splits "correctly" among nodal strata (see Proposition II.9). In the following discussion, we assume $X = Z /\!\!/ G$ is a GIT quotient so that moduli spaces of quasimaps are defined. When X is a smooth projective variety, but not a GIT quotient, the same results hold for determinant line bundles defined in Remark II.6. In fact, the arguments used in the proofs are identical for both cases.

2.3.1 Mapping to a point

Assume that $\beta = 0$. Then any quasimap is a constant map and the morphism

$$\mathcal{Q}^{\epsilon}_{g,k}(X,0) \xrightarrow{\text{stab}\times\text{ev}} \overline{M}_{g,k} \times X$$

is an isomorphism. Here stab : $\mathcal{Q}^{\epsilon}_{g,k}(X,0) \to \overline{M}_{g,k}$ denotes the stabilization morphism of source curves of quasimaps, and ev : $\mathcal{Q}^{\epsilon}_{g,k}(X,0) \to X$ denotes the constant evaluation morphism. Let P be the principal G-bundle $Z^s \to X = Z /\!\!/ G$.

Lemma II.7. *The universal bundle $\mathcal{P}$ over the universal curve $\mathcal{C} = \overline{C}_{g,k} \times X$ is equal to $\pi_2^*(P)$, where $\overline{C}_{g,k}$ is the universal curve over $\overline{M}_{g,k}$ and $\pi_2 : \overline{C}_{g,k} \times X \to X$ is the second projection.*

Proof. In general, there is an evaluation map from the universal curve $\mathcal{C}$ to the quotient stack $[Z/G]$ and $\mathcal{P}$ is the pullback of P along this map. The lemma follows from the observation that the evaluation map is given by the second projection π_2 in this case. □

Corollary II.8. *Let $\mathcal{R} := P \times_G R$ be the associated vector bundle on X and let $\pi : \overline{C}_{g,k} \to \overline{M}_{g,k}$ be the canonical morphism. We have*

$$\mathcal{D}^{R,l} = (\wedge^{\mathrm{rk}(R)g}(R^1\pi_*\mathcal{O}_{\overline{C}_{g,k}} \boxtimes \mathcal{R}))^l \otimes \left(\wedge^{\mathrm{rk}(R)} \mathcal{R}\right)^{-l}.$$

Proof. By Lemma II.7, we have $\mathcal{P} \times_G R = \pi_2^*(\mathcal{R})$. Therefore the pushforward $R\pi_*(\pi_2^*(\mathcal{R}))$ is equal to $\mathcal{O}_{\overline{M}_{g,k}} \boxtimes \mathcal{R} - R^1\pi_*\mathcal{O}_{\overline{C}_{g,k}} \boxtimes \mathcal{R}$ via the projection formula. Note that $\mathrm{rk}(R^1\pi_*\mathcal{O}_{\overline{C}_{g,k}}) = g$. □

2.3.2 Cutting edges

For $i = 1, 2$, we denote by $\mathrm{ev}_{k_i} : \mathcal{Q}^\epsilon_{g,k_{k_i}}(X, \beta_i) \to X$ the evaluation morphism at the last marking. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1) \times_X \mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2) & \xrightarrow{\Phi} & \mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1) \times \mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{k_1} \times \mathrm{ev}_{k_2}} \\ X & \xrightarrow{\Delta} & X \times X, \end{array}$$

where Δ is the diagonal embedding of X. Let $\mathcal{C}$ and $\mathcal{C}'$ denote the universal curves over $\mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1) \times_X \mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2)$ and $\mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1) \times \mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2)$, respectively. Let $\mathcal{P}$ and $\mathcal{P}'$ be the universal principal G-bundles over $\mathcal{C}$ and $\mathcal{C}'$, respectively. We can define level structures $\mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1)\times\mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2)} = \mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1)} \boxtimes \mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2)}$ and $\mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1)\times_X\mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2)}$ using (2.1). The following proposition shows that the level structure splits "correctly" among nodal strata.

Proposition II.9. *Let $x : \mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1) \times_X \mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2) \to \mathcal{C}$ be the section corresponding to the node. Then we have*

$$\Phi^*\big(\mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1)} \boxtimes \mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2)}\big) = \mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1)\times_X\mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2)} \otimes \det\big(x^*(\mathcal{P}\times_G R)\big)^{-l}. \tag{2.2}$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \Phi^*\mathcal{C}' & \xrightarrow{p} & \mathcal{C} \\ & \searrow^{\pi'} & \downarrow{\scriptstyle \pi} \\ & & \mathcal{Q}^\epsilon_{g,k_1}(X,\beta_1) \times_X \mathcal{Q}^\epsilon_{g,k_2}(X,\beta_2). \end{array}$$

Notice that $\mathcal{C}$ is obtained by gluing along two sections of marked points of $\Phi^*\mathcal{C}'$. For any locally free sheaf $\mathcal{F}$ on $\mathcal{C}$, we have a short exact sequence (the normalization exact sequence)

$$0 \to \mathcal{F} \to p_* p^* \mathcal{F} \to x_* x^* \mathcal{F} \to 0.$$

It induces the following natural isomorphism

$$\det(R\pi_*(p_*p^*\mathcal{F}))^{-1} \cong \det(R\pi_*(\mathcal{F}))^{-1} \otimes \det(R\pi_*(x_*x^*\mathcal{F}))^{-1}. \tag{2.3}$$

Note that

$$\det(R\pi_*(p_*p^*\mathcal{F}))^{-1} = \det(R\pi'_*(p^*\mathcal{F}))^{-1} \quad \text{and} \quad \det(R\pi_*(x_*x^*\mathcal{F}))^{-1} = \det(x^*\mathcal{F})^{-1}.$$

Now take $\mathcal{F} = \mathcal{P} \times_G R$ and $\mathcal{F}' = \mathcal{P}' \times_G R$. Finally, the lemma follows from the fact that $p^*\mathcal{F} \cong \Phi^*\mathcal{F}'$, equation (2.3), and cohomology and base change. □

2.3.3 Contractions

Fix g_1, g_2 and k_1, k_2 such that $g = g_1 + g_2$ and $k = k_1 + k_2$. We denote the basic gluing maps by

$$r : \overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1} \to \overline{M}_{g,k},$$

$$q : \overline{M}_{g-1,k+2} \to \overline{M}_{g,k}.$$

Let k' be a non-negative integer. Let $\underline{k}' = (k'_1, \ldots, k'_{m+1})$ and $\underline{\beta} = (\beta_1, \ldots, \beta_{m+1})$ be partitions of k' and β, respectively. For simplicity, we denote by $\mathcal{Q}^{\epsilon}_{m,\underline{k}',\underline{\beta}}$ the fiber product

$$\mathcal{Q}^{\epsilon}_{g_1,k_1+k'_1+1}(X,\beta_1) \times_X \underbrace{\mathcal{Q}^{\epsilon}_{0,2+k'_3}(X,\beta_3) \times_X \ldots \mathcal{Q}^{\epsilon}_{0,2+k'_{m+1}}(X,\beta_{m+1})}_{m-1 \text{ factors}} \times_X \mathcal{Q}^{\epsilon}_{g_2,k_2+k'_2+1}(X,\beta_2)$$

Let st : $\mathcal{Q}^\epsilon_{g,k+k'}(X,\beta) \to \overline{M}_{g,k}$ be the morphism defined by forgetting the (quasi)map and the last k' markings, then stabilizing the source curve. Consider the following commutative diagram.

$$\begin{array}{ccc} \bigsqcup \mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}} & \xrightarrow{\sqcup r_{m,\underline{k}',\underline{\beta}}} & \mathcal{Q}^\epsilon_{g,k+k'}(X,\beta) \\ \downarrow & & \downarrow \mathrm{st} \\ \overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1} & \xrightarrow{r} & \overline{M}_{g,k} \end{array}$$

Here the disjoint union is over partitions of the set of k' marked points of size $\underline{k}'$, and partitions $\underline{\beta}$ of β. The above commutative diagram induces a morphism

$$\Psi_m : \bigsqcup \mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}} \to \left(\overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1}\right) \times_{\overline{M}_{g,k}} \mathcal{Q}^\epsilon_{g,k+k'}(X,\beta).$$

Using the same argument as in the proof of [53, Proposition 11], one can show that the virtual structure sheaves satisfy

$$\sum_m (-1)^{m+1} \Psi_{m*} \sum_{\underline{k}',\underline{\beta}} \mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}}} = r^! \, \mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^\epsilon_{g,k+k'}(X,\beta)}.$$

Let $\mathcal{C}_{g,k+k'}$ and $\mathcal{C}_{m,\underline{k}',\underline{\beta}}$ be the universal curves on $\mathcal{Q}^\epsilon_{g,k+k'}(X,\beta)$ and $\mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}}$, respectively. Let $\mathcal{P}_{g,k+k'}$ and $\mathcal{P}_{m,\underline{k}',\underline{\beta}}$ be the corresponding universal principal G-bundles. The level structures $\mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k+k'}(X,\beta)}$ and $\mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}}}$ can be defined using (2.1). To prove that quantum K-theory with level structure satisfies the same axioms as Givental-Lee's quantum K-theory, we need the following proposition.

Proposition II.10. *We have*

$$\mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{m,\underline{k}',\underline{\beta}}} = (r_{m,\underline{k}',\underline{\beta}})^* \, \mathcal{D}^{R,l}_{\mathcal{Q}^\epsilon_{g,k+k'}(X,\beta)}.$$

Proof. The proposition follows from the following cartesian diagram and cohomology and base change.

$$\begin{array}{ccc} \mathcal{C}_{m,\underline{k}',\underline{\beta}} & \longrightarrow & \mathcal{C}_{g,k+k'} \\ \downarrow & & \downarrow \\ \mathcal{Q}^{\epsilon}_{m,\underline{k}',\underline{\beta}} & \xrightarrow{r_{m,\underline{k}',\underline{\beta}}} & \mathcal{Q}^{\epsilon}_{g,k+k'}(X,\beta) \end{array}$$

□

2.4 K-theoretic field theory

In [53], Lee proved that ordinary quantum K-theory satisfies certain Konservich-Manin axioms. If a theory satisfies those axioms, we refer to it as a *K-theoretic field theory*. We show that quantum K-theory with level structure is also a K-theoretic field theory.

Let $\overline{\mathcal{M}}_{g,k}$ be the moduli stack of k-pointed stable curves of genus g. It is a smooth Deligne-Mumford stack of dimension $3g-3+k$. Moreover, the stack $\overline{\mathcal{M}}_{g,k}$ has resolution property (see, for example, [2]). Therefore, the two K-groups $K^0(\overline{\mathcal{M}}_{g,k})$ and $K_0(\overline{\mathcal{M}}_{g,k})$ are isomorphic. We denote them by $K(\overline{\mathcal{M}}_{g,k})$. Since $\overline{\mathcal{M}}_{g,k}$ has nontrivial stacky structure, its K-group is not isomorphic to its cohomology. Instead, we have

$$K(\overline{\mathcal{M}}_{g,k}) \otimes \mathbb{C} \cong H^*(I\overline{\mathcal{M}}_{g,k}, \mathbb{C}),$$

where $I\overline{\mathcal{M}}_{g,k}$ is the *inertia stack of* $\overline{\mathcal{M}}_{g,k}$. It would be a very interesting problem to study the "tautological K-ring".

Recall that there are several canonical morphisms between different moduli spaces of curves.

Forgetful Morphism: Let

$$\pi : \overline{\mathcal{M}}_{g,k+1} \to \overline{\mathcal{M}}_{g,k}$$

be the morphism forgetting the last marked point. Here, we assume that $2g - 2 + k > 0$. Furthermore, π is isomorphic to the universal curve.

Gluing tree: Let

$$\rho_{\text{tree}} : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,k_1+k_2}$$

be the morphism induced from gluing the last marked point of the first curve and the first marked point of the second curve.

Gluing loop: Let

$$\rho_{\text{loop}} : \overline{M}_{g,k+2} \to \overline{M}_{g+1,k}$$

be the morphism induced from gluing the last two marked points.

Suppose H is a finite-dimensional $\mathbb{Q}$-vector space with a non-degenerate pairing $\langle\ ,\ \rangle : H \times H \to \mathbb{Q}[[Q]]$. Once and for all, we choose a basis $\phi_\alpha, \alpha = 1, \cdots, \dim H$. Denote the metric by $\eta_{\mu\nu} := \langle \phi_\mu, \phi_\nu \rangle$ and its inverse matrix by $\eta^{\mu\nu}$.

Definition II.11. A K-theoretic field theory is a collection of $\mathbb{Q}$-linear maps

$$\Omega_{g,k} : H^{\otimes k} \to K(\overline{M}_{g,k})_{\mathbb{Q}}[[Q]]$$

satisfying the following properties:

C1. The element $\Omega_{g,k}$ is S_k-equivariant, where the action of the symmetric group S_k permutes both the copies of H and the marked points of $\overline{M}_{g,k}$.

C2. Let $g = g_1 + g_2$ and $k = k_1 + k_2$. Then $\Omega_{g,n}$ satisfies the splitting property

$$\rho^*_{\text{tree}}\Omega_{g,k}(a_1, a_2, \ldots, a_k) = \sum_{\mu,\nu} \Omega_{g_1,k_1+1}(a_1, \ldots, a_{k_1}, \phi_\mu)\, \eta^{\mu\nu}\Omega_{g_2,k_2+1}(\phi_\nu, a_{k_1+1}, \ldots, a_k) \tag{2.4}$$

for any $a_1, \ldots, a_k \in H$.

C3. We require

$$\rho^*_{\text{loop}}\, \Omega_{g,k}(a_1, a_2, \ldots, a_k) = \sum_{\mu,\nu} \Omega_{g-1,k+2}\,(a_1, a_2, \ldots, a_n, \phi_\mu, \phi_\nu)\, \eta^{\mu\nu} \tag{2.5}$$

for any $a_1, \ldots, a_k \in H$.

Example II.12. Suppose that X is a smooth projective variety. Denote by E the semigroup of effective curve classes in $H_2(X, \mathbb{Z})$. Let $\overline{\mathcal{M}}_{g,k}(X, \beta)$ denote the moduli space of stable maps to X of degree β. Lee constructed in [53] a virtual structure sheave $\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)}$. Let $H = K(X)_{\mathbb{Q}}$. Define the so-called *quantized metric*

$$\langle a_1, a_2\rangle = \sum_{\beta \in E} Q^\beta \chi\big(\overline{\mathcal{M}}_{0,2}(X, \beta), \mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{0,2}(X,\beta)} \otimes \text{ev}^*_1(a_1) \otimes \text{ev}^*_2(a_2)\big).$$

When $d = 0$, we obtain the Mukai pairing $\chi(X, a_1 \otimes a_2)$. Let

$$\text{st} : \overline{\mathcal{M}}_{g,k}(X, \beta) \to \overline{M}_{g,k}$$

be the *stabilization* morphism defined by forgetting the map and then stabilizing the source curve. Define

$$\Omega_{g,k}(a_1, \cdots, a_k) = \sum_{\beta \in E} Q^\beta \text{st}_*\big(\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)} \otimes \big(\bigotimes_{i=1}^{k} \text{ev}^*_i(a_i)\big)\big)$$

for $a_1, \ldots, a_k \in H$.

According to [53, §4.3], we have the following theorem.

Theorem II.13. $\Omega_{g,k}$ *defines a* K*-theoretic field theory with the quantized metric* $\langle\cdot,\cdot\rangle$.

Similar to the case of cohomological field theories, we can define the shifted K-theoretic field theories. However, there is more than one definition of the shift. Here, we focus on two of them: the ordinary shift and the symmetrized shift. Suppose $t = \sum_i t_i Q^i$ is a formal power series, where $t_i \in H$ for all i. Let

$$\mathrm{ft}^m : \overline{M}_{g,k+m} \to \overline{M}_{g,k}$$

be the forgetful morphism which forgets the last m marked points.

1. We define the ordinary shift of $\Omega_{g,k}$ by

$$\Omega^t_{g,k}(a_1, \cdots, a_k) := \sum_{m\geq 0} \frac{1}{m!} \mathrm{ft}^m_* \Omega_{g,k+m}(a_1, \cdots, a_k, t, \cdots t)$$

for $a_1, \ldots, a_k \in H$. Again, according to [53, §4.3], $\Omega^t_{g,k}$ forms a K-theoretic field theory with the shifted metric

$$\langle a_1, a_2\rangle_t := \sum_{\beta\in E, m\geq 0} Q^d \frac{1}{m!} \chi\big(\overline{\mathcal{M}}_{0,2+m}(X,\beta), \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2+m}(X,\beta)} \otimes \mathrm{ev}_1^*(a_1) \otimes \mathrm{ev}_2^*(a_2) \otimes \bigotimes_{i=3}^{2+m} \mathrm{ev}_i^*(t)\big).$$

2. We define the symmetrized shift of $\Omega_{g,k}$ as follows. Note that

$$\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)} \otimes \bigotimes_{i=1}^{k} \mathrm{ev}_i^*(a_i) \otimes \bigotimes_{i=k+1}^{k+m} \mathrm{ev}_i^*(t)$$

is S_m-equivariant. Therefore, it descends to a K-theory class on the stack $\overline{\mathcal{M}}_{0,k+m}(X,\beta)/S_m$, where S_m acts on $\overline{\mathcal{M}}_{0,k+m}(X,\beta)$ by permuting the last m markings. By abuse of notation, we still denote the stabilization morphism by

$$\mathrm{st} : \overline{\mathcal{M}}_{g,k+m}(X,\beta)/S_m \to \overline{M}_{g,k+m}/S_m.$$

The forgetful morphism ft^m factors through the following morphism:

$$\mathrm{ft}^{S_m} : \overline{M}_{g,k+m}/S_m \to \overline{M}_{g,k}.$$

We define the symmetrized shift of $\Omega_{g,k}$ by

$$\Omega_{g,k}^{S_\infty,t}(a_1,\cdots,a_k) := \sum_{\beta\in E, m\geq 0} Q^d \mathrm{ft}_*^{S_m}\mathrm{st}_*\big(\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k+m}(X,\beta)} \otimes \bigotimes \mathrm{ev}_i^*(a_i) \otimes \bigotimes_{i=k+1}^{k+m} \mathrm{ev}_i^*(t)\big)$$

and the symmetrized paring by

$$\begin{aligned}&\langle a_1, a_2\rangle_t^{S_\infty}\\ := &\sum_{\beta\in E, m\geq 0} Q^d \chi\big(\overline{\mathcal{M}}_{0,2+m}(X,\beta)/S_m, \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2+m}(X,\beta)} \otimes \mathrm{ev}_1^*(a_1)\otimes \mathrm{ev}_2^*(a_2) \otimes \bigotimes_{i=3}^{2+m} \mathrm{ev}_i^*(t)\big).\end{aligned}$$

Proposition II.14. *$\Omega_{g,k}^{S_\infty,t}$ forms a K-theoretic field theory with the pairing $\langle\cdot,\cdot\rangle_t^{S_\infty}$.*

Proof. Axiom **C1** is obviously satisfied. The splitting properties in axioms **C2** and **C3** follow essentially from the properties of virtual structure sheaves proved in [53, §3]. We sketch the proof of the splitting property (2.4). The proof of the splitting property (2.5) is similar.

The key property of virtual structure sheaves that we need to prove (2.4) is discussed in Section 2.3.3. We recall the setup in the quantum K-theory here. Let

$\underline{m} = (m_0, \ldots, m_s)$ and $\underline{\beta} = (\beta_0, \ldots, \beta_s)$ be two $(s+1)$-tuples of non-negative integers such that $\sum_i m_i = m$ and $\sum_i \beta_i = \beta$. Let $\mathcal{M}_{s,\underline{m},\underline{\beta}}$ denote the following stack

$$\overline{\mathcal{M}}_{g_1,k_1+1+m_0}(X,\beta_0) \times_X \underbrace{\overline{\mathcal{M}}_{0,2+m_1}(X,\beta_1) \times_X \ldots \overline{\mathcal{M}}_{0,2+m_{s-1}}(X,\beta_{s-1})}_{m-1 \text{ factors}}$$
$$\times_X \overline{\mathcal{M}}_{g_2,k_2+1+m_s}(X,\beta_s),$$

Consider the following commutative diagram

(2.6)
$$\begin{array}{ccc} \coprod \mathcal{M}_{s,\underline{m},\underline{\beta}} & \longrightarrow & \overline{\mathcal{M}}_{g,k+m}(X,\beta) \\ \downarrow & & \downarrow \\ \overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1} & \xrightarrow{\rho_{\text{tree}}} & \overline{M}_{g,k} \end{array}$$

where the disjoint union is over all $s \geq 1$, partitions of the set of m markings of size $\underline{m}$, and partitions $\underline{\beta}$ of β. The horizontal arrows in (2.6) are given by boundary maps and the vertical arrows are compositions of stabilization morphisms and forgetful morphisms. Let $\mathcal{M}$ denote the fiber product $(\overline{M}_{g_1,k_1+1} \times \overline{M}_{g_2,k_2+1}) \times_{\overline{M}_{g,k}} \overline{\mathcal{M}}_{g,k+m}(X,\beta)$. The commutative diagram (2.6) induces a morphism

$$\Psi_s : \coprod_{s\geq 1} \mathcal{M}_s \to \mathcal{M}.$$

According to [53, Proposition 11], we have

(2.7)
$$\sum_s (-1)^{s+1} \Psi_{s*} \sum_{\underline{m},\underline{\beta}} \mathcal{O}^{\text{vir}}_{\mathcal{M}_{s,\underline{m},\underline{\beta}}} = \rho^!_{\text{tree}}(\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,k+m}(X,\beta)}).$$

Note that the virtual structure sheaves $\mathcal{O}^{\text{vir}}_{\mathcal{M}_{s,\underline{m},\underline{\beta}}}$ and $\mathcal{O}^{\text{vir}}_{\overline{\mathcal{M}}_{g,k+m}(X,\beta)}$ are S_m-equivariant. Hence, (2.7) can be viewed as an equality over $\mathcal{M}/S_m$. Consider the following commutative diagram.

$$
\begin{array}{ccccc}
\left(\coprod \mathcal{M}_{s,\underline{m},\underline{\beta}}\right)/S_m & \xrightarrow{\Phi_s} & \mathcal{M}/S_m & \longrightarrow & \overline{\mathcal{M}}_{g,k+m}(X,\beta)/S_m \\
& \searrow^{\pi''} & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
& & \overline{M}_{g_1,k_1+1}\times\overline{M}_{g_2,k_2+1} & \xrightarrow{\rho_{\mathrm{tree}}} & \overline{M}_{g,k}
\end{array} \tag{2.8}
$$

Then

$$
\begin{aligned}
&\rho^*_{\mathrm{tree}}\pi_*\big(\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k+m}(X,\beta)}\otimes\bigotimes_{i=1}^{k}\mathrm{ev}_i^*(a_i)\otimes\bigotimes_{i=k+1}^{k+m}\mathrm{ev}_i^*(t)\big)\\
=&\pi'_*\rho^!_{\mathrm{tree}}\big(\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k+m}(X,\beta)}\otimes\bigotimes_{i=1}^{k}\mathrm{ev}_i^*(a_i)\otimes\bigotimes_{i=k+1}^{k+m}\mathrm{ev}_i^*(t)\big)\\
=&\sum_s(-1)^{s+1}\pi'_*\Psi_{s*}\big(\sum_{\underline{m},\underline{\beta}}\mathcal{O}^{\mathrm{vir}}_{\mathcal{M}_{s,\underline{m},\underline{\beta}}}\otimes\bigotimes_{i=1}^{k}\mathrm{ev}_i^*(a_i)\otimes\bigotimes_{i=k+1}^{k+m}\mathrm{ev}_i^*(t)\big)\\
=&\sum_s(-1)^{s+1}\pi''_*\big(\sum_{\underline{m},\underline{\beta}}\mathcal{O}^{\mathrm{vir}}_{\mathcal{M}_{s,\underline{m},\underline{\beta}}}\otimes\bigotimes_{i=1}^{k}\mathrm{ev}_i^*(a_i)\otimes\bigotimes_{i=k+1}^{k+m}\mathrm{ev}_i^*(t)\big).
\end{aligned}
$$

It is easy to see that we have the isomorphism:

$$
\left(\coprod \mathcal{M}_{s,\underline{m},\underline{\beta}}\right)/S_m \cong \coprod \left(\mathcal{M}_{s,\underline{m},\underline{\beta}}/(S_{m_0}\times\ldots S_{m_s})\right),
$$

where the disjoint union of the LHS is over all $s\geq 1$, partitions of the *set of m markings* of size $\underline{m}$, and partitions $\underline{\beta}$ of β, while the disjoint union of the RHS is over all $s\geq 1$, partitions $\underline{m}$ of m, and partitions $\underline{\beta}$ of β. The splitting property (2.4) follows from the above computation and the cutting edges axiom of virtual structure sheaves (see [53, Proposition 7]).

□

Example II.15. Suppose that $X = V /\!\!/ G$ is a GIT quotient, where V is an affine variety with at most l.c.i. singularities and G is a complex reductive group. We assume that the stable locus V^s coincides the semistable locus V^{ss}. Let $\mathcal{Q}^{\epsilon}_{g,k}(X,\beta)$ be the moduli space of ϵ-stable quasimaps discussed in Section 2.2. Let $\mathcal{O}^{\text{vir}}$ denote the virtual structure sheaf of $\mathcal{Q}^{\epsilon}_{g,k}(X,\beta)$. Given a finite-dimensional representation R of G and an integer l, one can define the level structure $\mathcal{D}^{R,l}$ by formula (2.1).

Let $\text{st} : \mathcal{Q}^{\epsilon}_{g,k}(X,\beta) \to \overline{M}_{g,k}$ be the stabilization morphism. We define

$$\Omega^{S_\infty,R,l,t,\epsilon}_{g,k}(a_1,\cdots,a_k) := \sum_{\beta\in E, m\geq 0} Q^d \text{ft}^{S_m}_* \text{st}_*\big(\mathcal{O}^{\text{vir}}\otimes\mathcal{D}^{R,l}\otimes\bigotimes_{i=1}^{k}\text{ev}_i^*(a_i)\otimes\bigotimes_{i=k+1}^{k+m}\text{ev}_i^*(t)\big)$$

and the level-l quantized metric by

$$\begin{aligned}&\langle a_1,a_2\rangle^{S_\infty,E,l,\epsilon}_t\\ :=&\sum_{\beta\in E,m\geq 0} Q^d\chi\big(\mathcal{Q}^{\epsilon}_{0,2+m}(X,\beta)/S_m,\mathcal{O}^{\text{vir}}\otimes\mathcal{D}^{R,l}\otimes\text{ev}_1^*(a_1)\otimes\text{ev}_2^*(a_2)\otimes\bigotimes_{i=3}^{2+m}\text{ev}_i^*(t)\big).\end{aligned}$$

We often omit R from the notation if there is no confusion.

The following theorem follows from the same argument as in Proposition II.14 and the splitting property of $\mathcal{D}^{R,l}$ proved in Proposition II.9.

Theorem II.16. $\Omega^{S_\infty,R,l,t,\epsilon}_{g,k}$ *defines a K-theoretic field theory with the pairing* $\langle\cdot,\cdot\rangle^{S_\infty,R,l,\epsilon}_t$.

Remark II.17. Suppose X is a smooth projective variety which is not necessarily a GIT quotient. Given a vector bundle $\mathcal{R}$ over X, we can still define a determinant line bundle $\mathcal{D}^{R,l}$ over the moduli space $\overline{\mathcal{M}}_{g,k}(X,\beta)$ of stable maps by Remark II.6. Then Theorem II.16 still holds for level structures defined in this way.

CHAPTER III

Quantum K-invariants with level structure

3.1 K-theoretic quasimap invariants with level structure

In this section, we first briefly recall Givental-Lee's quantum K-theory. Then we define K-theoretic quasimap invariants with level structure.

The quantum K-theory or K-theoretic Gromov-Witten theory was introduced by Givental-Lee [28, 53]. Let X be a smooth projective variety and let $\overline{\mathcal{M}}_{g,k}(X,\beta)$ be the moduli space of stable maps to X. The moduli space is known to be a proper Deligne-Mumford stack (see for example [9]). In particular, for any coherent sheaf $\mathcal{E}$ on $\overline{\mathcal{M}}_{g,k}(X,\beta)$, we can consider its K-theoretic pushforward to the point $\operatorname{Spec}\mathbb{C}$, i.e., we can take its Euler characteristic

$$\chi(\mathcal{E}) = \sum_i (-1)^i h^i(\mathcal{E}),$$

where $h^i(\mathcal{E}) := \dim_{\mathbb{C}} H^i(\overline{\mathcal{M}}_{g,k}(X,\beta),\mathcal{E})$.

From the perfect obstruction theory, Lee [53] constructed a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}} \in K_0(\overline{\mathcal{M}}_{g,k}(X,\beta))$, where $K_0(\overline{\mathcal{M}}_{g,k}(X,\beta))$ denotes the Grothendieck group of coherent sheaves on $\overline{\mathcal{M}}_{g,k}(X,\beta)$. The virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ has the following properties:

1. If the obstruction sheaf is trivial and hence $\overline{\mathcal{M}}_{g,k}(X,\beta)$ is smooth, then $\mathcal{O}^{\mathrm{vir}}$ is the structure sheaf of $\overline{\mathcal{M}}_{g,k}(X,\beta)$.

2. If the obstruction sheaf Obs is locally free, then $\mathcal{O}^{\mathrm{vir}} = \sum_i (-1)^i \wedge^i \mathrm{Obs}^\vee$. Here $\wedge^i \mathrm{Obs}^\vee$ denotes the i-th wedge product of the dual of the obstruction bundle.

Since we assume X to be smooth, the Grothendieck group of locally free sheaves on X, denoted by $K^0(X)$, is isomorphic to the Grothendieck group of coherent sheaves $K_0(X)$. We denote both of them by $K(X)$. Suppose that E_i are K-theory elements in $K(X)$ and let L_i denote the i-th cotangent line bundles. The *K-theoretic Gromov-Witten invariants* are defined by

$$\left\langle E_1 L_1^{l_1}, \ldots, E_k L_k^{l_k} \right\rangle_{g,k,\beta} = \chi\big(\overline{\mathcal{M}}_{g,k}(X,\beta), \prod_i \mathrm{ev}_i^* E_i \otimes L_i^{l_i} \otimes \mathcal{O}^{\mathrm{vir}}\big),$$

where $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,k}(X,\beta) \to X$ are the evaluation morphisms at the i-th marking. Note that K-theoretic Gromov-Witten invariants, or quantum K-invariants, are closely related to K-theoretic field theories discussed in Section 2.4. In fact, we obtain quantum K-invariants by taking the holomorphic Euler characteristic of the cohomological field theory $\Omega_{g,k}$ discussed in Example III.3.

Let $E \subset H_2(X,\mathbb{Z})$ be the semigroup generated by effective curve classes on X. We define the *quantum K-potential of genus 0* by

$$\mathcal{F}(t,Q) := \frac{1}{2}(t,t) + \sum_{k=0}^{\infty} \sum_{\beta \in E} \frac{Q^\beta}{k!} \langle t, \ldots, t \rangle_{0,k,\beta},$$

where $t \in K(X)_{\mathbb{Q}} := K(X) \otimes \mathbb{Q}$ and $(t,t) := \chi(t \otimes t)$ is the Mukai pairing. Let $\phi_0 = \mathcal{O}_X, \phi_1, \phi_2 \ldots$ be a basis of $K(X)_{\mathbb{Q}}$. One can define the "quantized" pairing on $K(X)_{\mathbb{Q}}$ by

$$((\phi_i, \phi_j)) := \mathcal{F}_{ij} = \partial_{t_i}\partial_{t_j}\mathcal{F}(t, Q).$$

In the following discussion, we assume X can be represented as a GIT quotient $Z /\!\!/ G$. As mentioned before, the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(Z /\!\!/ G, \beta)$ can be identified with $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$ for large ϵ. According to [19], for general ϵ, the ϵ-stable quasimap space $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$ is proper and it admits a two-term perfect obstruction theory, assuming Z has only lci singularities. Hence by the result in [53], one can construct a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ on $\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$.

Definition III.1. The *K-theoretic quasimap invariants of level l* are defined by

$$\big\langle E_1 L_1^{l_1}, \ldots, E_k L_k^{l_k} \big\rangle_{g,k,\beta}^{Z/\!\!/G,R,l,\epsilon} = \chi\big(\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta), \prod_i \mathrm{ev}_i^* E_i \otimes L_i^{l_i} \otimes \mathcal{O}^{\mathrm{vir}} \otimes \mathcal{D}^{R,l}\big) \in \mathbb{Z},$$

where $E_i \in K(Z /\!\!/ G) \otimes \mathbb{Q}$.

We shall usually suppress $Z /\!\!/ G$ from the notation if there is no confusion. Note that these invariants are all integers.

3.2 Quasimap graph space and $\mathcal{J}^{R,l,\epsilon}$-function

In this section, we first recall the definition and properties of the ϵ-stable quasimap graph space. Then we define an important generating series $\mathcal{J}^{R,l,\epsilon}$ of K-theoretic quasimap invariants of level l.

Given a rational number $\epsilon > 0$, the *quasimap graph space*, denoted by $\mathcal{QG}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$, is introduced in [19]. It is the moduli space of the tuples

$$((C, x_1, \dots x_k), P, u, \varphi),$$

where $((C, x_1, \dots x_k), P, u)$ is a prestable quasimap, satisfying $\epsilon l(x) < 1$ for every point x on C, and the new data φ is a degree 1 morphism from C to $\mathbb{P}^1$. The curve C has a unique rational component C_0 such that $\varphi|_{C_0} : C_0 \to \mathbb{P}^1$ is an isomorphism and the complement C/C_0 is contracted by φ. The ampleness condition imposed on the tuples is modified to:

$$\omega_{\overline{C \setminus C_0}}(\sum x_i + \sum y_j) \otimes \mathcal{L}^{\epsilon}_{\theta} \text{ is ample,}$$

where x_i are marked points on $\overline{C \setminus C_0}$ and y_i are the nodes $\overline{C \setminus C_0} \cap C_0$. It is shown in [19] that the quasimap graph space is also a separated Deligne-Mumford stack which is proper over the affine quotient. Moreover, when Z has only lci singularities, the canonical obstruction theory on the graph space is perfect. Similarly, we can define the level-l determinant line bundle $\mathcal{D}^{R,l}_{\mathcal{QG}}$ on $\mathcal{QG}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)$ using the universal principal G-bundles over its universal curve.

There is a natural $\mathbb{C}^*$-action on the graph spaces. Let $[x_0, x_1]$ be homogeneous coordinates on $\mathbb{P}^1$, and set $0 := [1, 0]$ and $\infty := [0, 1]$. We consider the standard $\mathbb{C}^*$-action on $\mathbb{P}^1$:

$$t \cdot [x_0, x_1] = [tx_0, x_1], \quad \forall t \in \mathbb{C}^*. \tag{3.1}$$

It induces an action on the ϵ-stable quasimap graph space $\mathcal{QG}^{\epsilon}_{g,k}(Z/\!\!/G,\beta)$ by rescaling the parametrized rational component. According to [18, §4.1], the $\mathbb{C}^*$-fixed locus can be described as

$$(\mathcal{QG}^{\epsilon}_{g,k}(Z /\!\!/ G,\beta))^{\mathbb{C}^*} = \coprod F^{g_1,k_1,\beta_1}_{g_2,k_2,\beta_2},$$

where the disjoint union is over all possible splittings

$$g = g_1 + g_2, \quad k = k_1 + k_2, \quad \beta = \beta_1 + \beta_2,$$

with $g_i, k_i \geq 0$ and β_i effective. In the stable cases, an ϵ-stable parametrized quasimap $((C, x_1, \ldots, x_k), P, u, \varphi) \in F^{g_1,k_1,\beta_1}_{g_2,k_2,\beta_2}$ is obtained by gluing two ϵ-stable quasimaps of types (g_1, k_1, β_1) and (g_2, k_2, β_2) to a constant map $\mathbb{P}^1 \to p \in Z /\!\!/ G$ at 0 and ∞, respectively. Therefore, the component $F^{g_1,k_1,\beta_1}_{g_2,k_2,\beta_2}$ is isomorphic to the fiber product

$$\mathcal{Q}^{\epsilon}_{g_1,k_1+\bullet}(Z /\!\!/ G,\beta_1) \times_{Z/\!\!/G} \mathcal{Q}^{\epsilon}_{g_2,k_2+\bullet}(Z /\!\!/ G,\beta_2)$$

over the evaluation maps at the special marked points $\bullet$. When one of the components at 0 or ∞ is unstable, we use the following conventions.

1. For the unstable cases $(g_1, k_1, \beta_1) = (0, 0, 0)$ or $(0, 1, 0)$ (and likewise for (g_2, k_2, β_2)), we define

$$\mathcal{Q}^{\epsilon}_{0,0+\bullet}(Z /\!\!/ G, 0) := Z /\!\!/ G, \quad \mathcal{Q}^{\epsilon}_{0,1+\bullet}(Z /\!\!/ G, 0) := Z /\!\!/ G, \quad \mathrm{ev}_{\bullet} = \mathrm{Id}_{Z/\!\!/G}.$$

2. For the unstable cases $(g_1, k_1, \beta_1) = (0, 0, \beta_1)$ with $\beta_1 \neq 0$ and $\epsilon \leq \frac{1}{\beta_1(L_\theta)}$, we denote by

$$\mathcal{Q}_{0,0+\bullet}(Z /\!\!/ G,\beta_1)_0$$

the moduli space of quasimaps $(C = \mathbb{P}^1, P, u)$ such that $u(x) \in P \times_G Z^s$ for $x \neq 0 \in \mathbb{P}^1$ and $0 \in \mathbb{P}^1$ is a base point of length $\beta_1(L_\theta)$. Similarly, we define $\mathcal{Q}_{0,0+\bullet}(Z /\!\!/ G, \beta_2)_\infty$ to be the moduli space of quasimaps whose only base point is of length $\beta_2(L_\theta)$ and located at ∞. Using these definitions we have

$$F^{g,k,\beta_1}_{0,0,\beta_2} \cong \mathcal{Q}^\epsilon_{g,k+\bullet}(Z /\!\!/ G, \beta_1) \times_{Z/\!\!/G} \mathcal{Q}_{0,0+\bullet}(Z /\!\!/ G, \beta_2)_\infty$$

for $k \geq 1$ and $\epsilon \leq \frac{1}{\beta_2(L_\theta)}$. Similarly we have

$$F^{0,0,\beta_1}_{g,k,\beta_2} \cong \mathcal{Q}_{0,0+\bullet}(Z /\!\!/ G, \beta_1)_0 \times_{Z/\!\!/G} \mathcal{Q}^\epsilon_{g,k+\bullet}(Z /\!\!/ G, \beta_2)$$

for $k \geq 1$ and $\epsilon \leq \frac{1}{\beta_1(L_\theta)}$. When $g = k = 0$ and $\epsilon \leq \min\{\frac{1}{\beta_1(L_\theta)}, \frac{1}{\beta_2(L_\theta)}\}$, we have

$$F^{0,0,\beta_1}_{0,0,\beta_2} \cong \mathcal{Q}_{0,0+\bullet}(Z /\!\!/ G, \beta_1)_0 \times_{Z/\!\!/G} \mathcal{Q}_{0,0+\bullet}(Z /\!\!/ G, \beta_2)_\infty.$$

We denote by $\mathcal{R}$ the vector bundle $Z \times_G R \to [Z/G]$ and its restriction to $Z /\!\!/ G$. We define the twisted pairing on $K(Z /\!\!/ G)_{\mathbb{Q}}$ by

$$(u, v)^{R,l} := \chi\big(u \otimes v \otimes (\det \mathcal{R})^{-l}\big), \quad \text{where } u, v \in K(Z /\!\!/ G)_{\mathbb{Q}}. \tag{3.2}$$

Let $\{\phi_a\}$ be a basis of $K(Z /\!\!/ G)_{\mathbb{Q}}$ and let $\{\phi^a\}$ be the dual basis with respect to the above twisted pairing $(\cdot, \cdot)^{R,l}$. Let $t = \sum_i t^i \phi_i \in K(Z /\!\!/ G)_{\mathbb{Q}}$. We define the $\mathcal{J}^{R,l,\epsilon}$*-function of level* l to be

$$\mathcal{J}^{R,l,\epsilon}(t, Q) = 1 - q + t + \sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} \frac{Q^\beta}{k!} \phi^a \left\langle \frac{\phi_a}{1 - qL}, t, \ldots, t \right\rangle^{R,l,\epsilon}_{0,k+1,\beta}. \tag{3.3}$$ [1]

[1] In Ciocan-Fontanine-Kim's convention, the J-function starts at 1. The definition given here agrees with Givental's convention in which the J-function starts at the dilaton shift $1 - q$.

In the above summation, the quasimap moduli spaces are empty when $k = 0, \beta \neq 0, \beta(L_\theta) \leq 1/\epsilon$, and the unstable terms are defined by $\mathbb{C}^*$-localization on the graph space $\mathcal{QG}^\epsilon_{0,0}(Z /\!\!/ G, \beta)$. To be more precise, we consider the fixed point locus $F_{0,\beta} := \mathcal{Q}_{0,0+\bullet}(Z /\!\!/ G, \beta)_0$ of the $\mathbb{C}^*$-action. The unstable terms in (3.3) are defined to be

$$(1-q)\sum_a \sum_{\beta\neq 0,\, \beta(L_\theta)\leq 1/\epsilon} Q^\beta \chi\Big(F_{0,\beta}, \mathcal{O}^{\mathrm{vir}}_{F_{0,\beta}} \otimes \mathrm{ev}^*(\phi_a) \otimes \Big(\frac{\mathrm{tr}_{\mathbb{C}^*}\mathcal{D}^{R,l}}{\mathrm{tr}_{\mathbb{C}^*} \wedge^* N^\vee_{F_{0,\beta}}}\Big)\Big)\phi^a.$$

where $N_{F_{0,\beta}}$ is the *virtual* normal bundle of the fixed locus $F_{0,\beta}$ in $\mathcal{QG}^\epsilon_{0,0}(Z /\!\!/ G, \beta)$ and $\wedge^* N^\vee_{F_{0,\beta}} := \sum_i (-1)^i \wedge^i N^\vee_{F_0}$ is the K-theoretic Euler class of $N_{F_{0,\beta}}$. Here the trace of a $\mathbb{C}^*$-equivariant bundle V, when restricted to the fixed point locus, is a virtual bundle defined by the eigenspace decomposition with respect to the $\mathbb{C}^*$-action, i.e., we have

$$\mathrm{tr}_{\mathbb{C}^*}(V) := \sum_i q^i\, V(i),$$

where $t \in \mathbb{C}^*$ acts on $V(i)$ as multiplication by t^i.

For $1 < \epsilon \leq \infty$, i.e., the $(\epsilon = \infty)$-theory, (3.3) defines $\mathcal{J}$-function in the quantum K-theory of level l. In this case, we use $\langle \cdot \rangle^{R,l,\infty}$ or simply $\langle \cdot \rangle^{R,l}$ to denote quantum K-invariants of level l. Following Givental-Tonita [41], we introduce the *symplectic loop space formalism.* Recall that E denotes the semigroup of L_θ-effective curve classes on $Z /\!\!/_\theta G$. The *Novikov ring* $\mathbb{C}[[Q]]$ is defined as

$$\mathbb{C}[[Q]] := \overline{\{\sum_{\beta\in E} c_\beta Q^\beta | c_\beta \in \mathbb{C}\}}.$$

Here the completion is taken with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ denotes the maximal ideal generated by nonzero elements of E. We define the *loop space* as

$$\mathcal{K} := [K(Z /\!\!/ G) \otimes \mathbb{C}(q)] \otimes \mathbb{C}[[Q]],$$

where $\mathbb{C}(q)$ is the field of complex rational functions in q. By viewing the elements in $\mathbb{C}(q) \otimes \mathbb{C}[[Q]]$ as the coefficients, we extend the twisted pairing $(\cdot,\cdot)^{R,l}$ to $\mathcal{K}$ via linearity. There is a natural symplectic form Ω on $\mathcal{K}$ defined by

$$\Omega(f,g) := [\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}](f(q), g(q^{-1}))^{R,l}\frac{dq}{q}, \quad \text{where } f, q \in \mathcal{K}. \tag{3.4}$$

With respect to Ω, there is a *Lagrangian polarization* $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where

$$\mathcal{K}_+ = [K(Z /\!\!/ G) \otimes \mathbb{C}[q, q^{-1}]] \otimes \mathbb{C}[[Q]] \quad \text{and} \quad \mathcal{K}_- = \{f \in \mathcal{K} | f(0) \neq \infty, f(\infty) = 0\}.$$

As before, let $\{\phi_a\}$ be a basis of $K(Z /\!\!/ G)_{\mathbb{Q}}$ and let $\{\phi^a\}$ be the dual basis with respect to the twisted pairing $(\cdot,\cdot)^{R,l}$. Let $\mathbf{t}(q) = \sum_{i,j} t^i_j \phi_i q^j \in \mathcal{K}_+$ be an arbitrary Laurent polynomial. We define the *big $\mathcal{J}$-function of level l* to be the function $\mathcal{J}^{R,l}(\mathbf{t}(q), Q) : \mathcal{K}_+ \to \mathcal{K}$ given by

$$\mathcal{J}^{R,l}(\mathbf{t}(q), Q) = 1 - q + \mathbf{t}(q) + \sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} \frac{Q^\beta}{k!} \phi^a \left\langle \frac{\phi_a}{1-qL}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle^{R,l,\infty}_{0,k+1,\beta}.$$

Define the *genus-0 K-theoretic descendant potential of level l* by

$$\mathcal{F}^{R,l}(\mathbf{t}, Q) := \sum_{k,\beta} \frac{Q^\beta}{k!} \langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \rangle^{R,l,\infty}_{0,k,\beta}. \tag{3.5}$$

We can identify the cotangent bundle $T^*\mathcal{K}_+$ with the symplectic loop space $\mathcal{K}$ via the Lagrangian polarization and the *dilaton shift* $f \to f + (1-q)$. Then $\mathcal{J}^{R,l}$ coincides with the differential of the descendant potential up to the dilaton shift, i.e., we have

$$\mathcal{J}^{R,l} = 1 - q + \mathbf{t}(q) + d_t \mathcal{F}^{R,l}(\mathbf{t}, Q).$$

In the case $l = 0$, the above fact is proved in [41, §2]. The same argument works for arbitrary l.

For $(\epsilon = 0+)$-stable quasimap theory, the definition (3.3) gives the $\mathcal{I}$-function of level l of $Z /\!\!/ G$:

$$\begin{aligned}\mathcal{I}^{R,l}(t,Q) &:= \mathcal{J}^{R,l,0+}(t,Q)/(1-q)\\ &= 1 + \frac{t}{1-q} + \sum_a \sum_{\beta \neq 0} Q^\beta \chi\left(F_{0,\beta}, \mathcal{O}^{\mathrm{vir}}_{F_{0,\beta}} \otimes \mathrm{ev}^*(\phi_a) \otimes \left(\frac{\mathrm{tr}_{\mathbb{C}^*} \mathcal{D}^{R,l}}{\mathrm{tr}_{\mathbb{C}^*} \wedge^* N^\vee_{F_{0,\beta}}}\right)\right)\phi^a\\ &+ \sum_a \sum_{k\geq 1, (k,\beta)\neq(1,0)} \frac{Q^\beta}{k!}\phi^a \left\langle \frac{\phi_a}{(1-q)(1-qL)}, t, \ldots, t\right\rangle^{R,l,\epsilon=0+}_{0,k+1,\beta},\end{aligned}$$

where $t \in K(Z /\!\!/ G)_{\mathbb{Q}}$.

3.3 Permutation-equivariant quasimap K-theory with level structure

Givental [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] introduced the *permutation-equivariant* quantum K-theory, which takes into account the S_n-action on the moduli spaces of stable maps by permuting the marked points. The definition can be easily generalized to incorporate the level structure.

Let Λ be a λ-algebra, i.e. an algebra over $\mathbb{Q}$ equipped with abstract Adams operations $\Psi^k, k = 1, 2, \ldots$. Here $\Psi^k : \Lambda \to \Lambda$ are ring homomorphisms which satisfy $\Psi^r\Psi^s = \Psi^{rs}$ and $\Psi^1 = \mathrm{id}$. We often assume that Λ includes the Novikov variables, the algebra of symmetric polynomials in a given number of variables, and the torus equivariant K-ring of a point. We also assume that Λ has a maximal ideal Λ_+ and is equipped with the Λ_+-adic topology. For example, we can choose

$$\Lambda = \mathbb{Q}[[N_1, N_2, \dots]][[Q]][\Lambda_0^{\pm}, \dots, \Lambda_N^{\pm}],$$

where N_i are the Newton polynomials (in infinitely or finitely many variables) and Q denotes the Novikov variable(s). The parameters Λ_i denote the torus-equivariant parameters. The Adams operations Ψ^r act on N_m and Q by $\Psi^r(N_m) = N_{rm}$ and $\Psi^r(Q^\beta) = Q^{r\beta}$, respectively. We assume their actions on the torus-equivariant parameters are trivial.

Similar to the "ordinary" quasimap K-theory with level structure, we define the loop space by

$$\mathcal{K} := [K(Z /\!\!/ G) \otimes \Lambda] \otimes \mathbb{C}(q).$$

As before, it is equipped with a symplectic form defined by (3.4), and it has a Lagrangian polarization

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-,$$

where $\mathcal{K}_+$ is the subspace of Laurent polynomials in q and $\mathcal{K}_-$ is the subspace of reduced rational functions which are regular at $q = 0$ and vanish at $q = \infty$.

Consider the natural S_k-action on the quasimap moduli space $\mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G, \beta)$ by permuting the k marked points. Notice that the virtual structure sheaf $\mathcal{O}_{\mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G,\beta)}$ and the determinant line bundle $\mathcal{D}^{R,l}$ are invariant under this action. Therefore we have the following S_k-module

$$\big[\mathbf{t}(L), \dots, \mathbf{t}(L)\big]_{g,k,\beta} := \sum_m (-1)^m H^m\big(\mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^\epsilon_{g,k}(Z /\!\!/ G,\beta)} \otimes \mathcal{D}^{R,l} \otimes_{i=1}^k \mathbf{t}(L_i)\big),$$

for $\mathbf{t}(q) \in \mathcal{K}_+$.

Definition III.2. The correlators of the permutation-equivariant quasimap K-theory of level l are defined by

$$\big\langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \big\rangle_{g,k,\beta}^{R,l,\epsilon,S_k} := \pi_*\big(\mathcal{O}^{\mathrm{vir}}_{\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G,\beta)} \otimes \mathcal{D}^{R,l} \otimes_{i=1}^{k} \mathbf{t}(L_i)\big),$$

where π_* is the K-theoretic pushforward along the projection

$$\pi : \big[\mathcal{Q}^{\epsilon}_{g,k}(Z /\!\!/ G, \beta)/S_k\big] \to [\mathrm{pt}].$$

Remark III.3. When the λ-algebra Λ is chosen to be $\mathbb{Z}[[Q]]$, we refer to the invariants as the *symmetrized* invariants. The pushforward map in Definition III.2 carries information only about the dimensions of the S_k-invariant parts of sheaf cohomology $\big[\mathbf{t}(L), \ldots, \mathbf{t}(L)\big]_{g,k,\beta}$. We refer to [37, Example 4] for more details.

For the permutation-equivariant quasimap K-theory, we also consider the J^{ϵ}-function, and define the cone $\mathcal{L}_{S_\infty}$ to be the range of the J^{∞}-function.

Definition III.4. The permutation-equivariant K-theoretic J^{ϵ}-function of $Z /\!\!/ G$ of level l is defined by

(3.6)
$$\mathcal{J}^{R,l,\epsilon}_{S_\infty}(\mathbf{t}(q), Q) := 1-q+\mathbf{t}(q)+\sum_{a} \sum_{(k,\beta)\neq(0,0),(1,0)} Q^{\beta} \left\langle \frac{\phi_a}{1-qL}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle_{0,k+1,\beta}^{R,l,\epsilon,S_k} \phi^a,$$

where the unstable terms in the summation are the same as those in (3.3).

Note that in the above definition of permutation-equivariant J-function, we do not need to divide each term by $k!$.

Definition III.5. We define the Givental's cone $\mathcal{L}^{R,l}_{S_\infty}$ as the range of $\mathcal{J}^{R,l,\infty}_{S_\infty}$, i.e.,

$$\mathcal{L}^{R,l}_{S_\infty} := \bigcup_{\mathbf{t}(q)\in\mathcal{K}_+} \mathcal{J}^{R,l,\infty}_{S_\infty}(\mathbf{t}(q), Q) \subset \mathcal{K}.$$

Remark III.6. In the ordinary, i.e. permutation-*non*-equivariant, quantum K-theory, the range of the $\mathcal{J}$-function is a cone which coincides with the differential of the descendant potential (up to the dilaton shift). Therefore the range of the ordinary K-theoretic $\mathcal{J}$-function is a Lagrangian cone in the loop space $\mathcal{K}$. However, in the permutation-equivariant theory, it is explained in [35] that the cone $\mathcal{L}^{R,l}_{S_\infty}$ is not Lagrangian.

3.4 The level structure in equivariant quasimap theory and orbifold quasimap theory

When $Z /\!\!/ G$ is not proper, one can still define the *equivariant* quasimap invariants if $Z /\!\!/ G$ has an additional torus action such that the the fixed point loci in the quasimap moduli spaces are proper. It is explained in [19, §6.3] how to define cohomological quasimap invariants via virtual localization. Similarly, one can define equivariant K-theoretic quasimap invariants (with level structure) for noncompact GIT targets using the K-theoretic virtual localization formula (see [64, §3.2]). With this understood, we define the $\mathcal{J}^{R,l,\epsilon}$-function of equivariant K-theoretic quasimap invariants of level l using (3.3). Its permutation-equivariant generalization is straightforward.

If we do not assume G acts freely on the stable locus Z^s, then the target $X := [Z^s/G]$ is naturally an orbifold. For such orbifold GIT targets, a quasimap is a tuple

$((C, x_1, \ldots, x_k), [u])$ where $(C, x_1, \ldots, x_k)$ is a k-pointed, genus g twisted curve (see [4, §4]) and $[u]$ is a representable morphism from $(C, x_1, \ldots, x_k)$ to X. We refer the reader to [15, §2.3] for the details of the ϵ-stability imposed on those tuples. We denote by $\mathcal{Q}^{\epsilon}_{g,k}(X, \beta)$ the moduli stack of ϵ-stable quasimaps to the orbifold X. It is shown in [15, Theorem 2.7] that this moduli stack is Deligne-Mumford and proper over the affine quotient. Furthermore if Z only has l.c.i. singularties, then $\mathcal{Q}^{\epsilon}_{g,k}(X, \beta)$ has a canonical perfect obstruction theory. Let $\mathcal{C} \to \mathcal{Q}^{\epsilon}_{g,k}(X, \beta)$ be the universal curve. The universal principal G-bundle $\mathcal{P} \to \mathcal{C}$ is defined as the pullback of the principal G-bundle $Z \to [Z/G]$ via the universal morphism $[u] : \mathcal{C} \to [Z/G]$. In the orbifold setting, we can still define the level-l determinant line bundle $\mathcal{D}^{R,l}$ using (2.1).

According to [15, §2.5.1], there are natural evaluation morphisms

$$\mathrm{ev}_i : \mathcal{Q}^{\epsilon}_{g,k}(X, \beta) \to \bar{I}_{\mu}X, \ \big((C, x_1, \ldots, x_k), [u]\big) \mapsto [u]_{|x_i}, \quad \text{for } i = 1, \ldots, k.$$

Here $\bar{I}_{\mu}X$ denotes the rigidified cyclotomic inertia stack of X which parameterizes representable maps from gerbes banded by finite cyclic groups to X. Let L_i be the universal cotangent line bundle whose fiber at $((C, x_1 \ldots, x_k), [u])$ is the cotangent space of the coarse curve $\underline{C}$ of C at the i-th marked point $\underline{x_i}$. For non-negative integers l_i and classes $E_i \in K^0(\bar{I}_{\mu}X) \otimes \mathbb{Q}$, we define the K-theoretic quasimap invariants of level l as

$$\big\langle E_1 L_1^{l_1}, \ldots, E_k L_k^{l_k} \big\rangle^{X,R,l,\epsilon}_{g,k,\beta} = \chi\big(\mathcal{Q}^{\epsilon}_{g,k}(X, \beta), \prod_i \mathrm{ev}_i^* E_i \otimes L_i^{l_i} \otimes \mathcal{O}^{\mathrm{vir}} \otimes \mathcal{D}^{R,l}\big).$$

When $\epsilon = \infty$ and $l = 0$, this definition recovers the K-theoretic Gromov-Witten invariants of X defined in [78].

In the orbifold setting, one can still define the quasimap graph space $\mathcal{QG}^{\epsilon}_{g,k}(X,\beta)$ (see [15, §2.5.3]). The definition of the determinant line bundle $\mathcal{D}^{R,l}$ over the graph space is straightforward. We choose a basis $\{\phi_a\}$ of $K^0(\bar{I}_\mu X) \otimes \mathbb{Q}$. Let $\{\phi^a\}$ be the dual basis with respect to the twisted pairing $(\cdot,\cdot)^{R,l}$ on $K^0(\bar{I}_\mu X) \otimes \mathbb{Q}$ given by

$$(u, v)^{R,l} := \chi\big(\bar{I}_\mu X, u \otimes \bar{\iota}^* v \otimes \det{}^{-l}(\bar{I}_\mu \mathcal{R})\big).$$

Here $\bar{\iota}$ is the involution induced by $(x, g) \mapsto (x, g^{-1})$ and $\bar{I}_\mu \mathcal{R}$ is a vector bundle over $\bar{I}_\mu X$ such that the fiber over (x, H), with $H \subset \mathrm{Aut}\, x$, is the H-fixed subspace of $\mathcal{R}_x$. With all the notations understood, it is straightforward to adapt the definition of cohomological orbifold quasimap $\mathcal{J}^{\epsilon}$-function [15, Definition 3.1] to the (permutation-equivariant) K-theoretic setting.

CHAPTER IV

Characterization of genus-0 theory, mirror theorem, and mock theta functions

4.1 Adelic Characterization in quantum K-theory with level structure

In this section, we focus on quantum K-theory, i.e., $(\epsilon = \infty)$-quasimap theory. We first recall the (virtual) Lefschetz-Kawasaki's Riemann-Roch formula. This is the main tool in analyzing the poles of the J-function. We give an adelic characterization of points on the cone $\mathcal{L}^{R,l}_{S_\infty}$ in Theorem IV.5. As an application of the adelic characterization, we prove that certain "determinantal " modifications of points on the cone $\mathcal{L}_{S_\infty}$ of level 0 lie on the cone $\mathcal{L}^{R,l}_{S_\infty}$ of level l. This result will be used in the proof of the toric mirror theorem in Section 4.2. We assume in this section that X is a smooth projective variety which is not necessarily a GIT quotient. The determinant line bundle $\mathcal{D}^{R,l}$ is defined as in Remark II.6.

4.1.1 Virtual Lefschetz-Kawasaki's Riemann-Roch formula

To understand the poles of the generating series of the permutation-equivariant quantum K-invariants, we recall the *Lefschetz-Kawasaki's Riemann-Roch formula* in

[38].

Let h be a finite order automorphism of a holomorphic orbibundle E over a compact smooth orbifold $\mathcal{M}$. The (super)trace of h on the sheaf cohomology $H^*(\mathcal{M}, E)$ can be computed as an integral over the h-fixed point locus $I\mathcal{M}^h$ in the inertia orbifold $I\mathcal{M}$:

(4.1)

$$\operatorname{tr}_h H^*(\mathcal{M}, E) = \chi^{fake}\left(I\mathcal{M}^h, \frac{\operatorname{tr}_{\tilde{h}} E}{\operatorname{tr}_{\tilde{h}} \wedge^* N^\vee_{I\mathcal{M}^h}}\right) := \int_{[I\mathcal{M}^h]} \operatorname{td}(T_{I\mathcal{M}^h}) \operatorname{ch}\left(\frac{\operatorname{tr}_{\tilde{h}} E}{\operatorname{tr}_{\tilde{h}} \wedge^* N^\vee_{I\mathcal{M}^h}}\right).$$

We explain the ingredients of this formula as follows. By definition, we can choose an atlas of local charts $U \to U/G(x)$ of $\mathcal{M}$. The local description of the inertia orbifold $I\mathcal{M}$ near $x \in \mathcal{M}$ is given by $[\coprod_{g\in G(x)} U^g/G(x)]$, where $U^g \subset U$ denotes the fixed point locus of g. The automorphism h can be lifted to an automorphism $\tilde{h}$ of the chart U^g. We denote by $(U^g)^{\tilde{h}}$ the fixed point locus of $\tilde{h}$ in U^g. Then the local description of the orbifold $I\mathcal{M}^h$ is given by $[\coprod_g (U^g)^{\tilde{h}}/G(x)]$. We refer to the connected components of $I\mathcal{M}^h$ as *Kawasaki strata*. Near a point $(x, [g]) \in I\mathcal{M}^h$, the tangent and normal orbiford bundles $T_{I\mathcal{M}^h}$ and $N_{I\mathcal{M}^h}$ are identified with the tangent bundle and normal bundle to $(U^g)^{\tilde{h}}$ in U, respectively. In the denominator of the right side of (4.1), $\wedge^* N^\vee_{I\mathcal{M}^h} := \sum_{i\geq 0}(-1)^i \wedge^i N^\vee_{I\mathcal{M}^h}$ is the K-theoretic Euler class of the normal bundle $N_{I\mathcal{M}^h}$. The trace bundle $\operatorname{tr}_{\tilde{h}} F$ is the virtual orbifold bundle:

$$\operatorname{tr}_{\tilde{h}} F := \sum_\lambda \lambda F_\lambda,$$

where F_λ are the eigen-bundles of h corresponding to the eigenvalues λ. Finally, td and ch denote the *Todd class* and *Chern character*.

By choosing h to be the identity map, we obtain the *Kawasaki's Riemann-Roch formula* [49] from (4.1):

$$\chi(\mathcal{M}, E) = \chi^{fake}\left(I\mathcal{M}, \frac{\mathrm{tr}_g\, E}{\mathrm{tr}_g\ \wedge^* N^{\vee}_{I\mathcal{M}}}\right). \tag{4.2}$$

When $\mathcal{M}$ is no longer smooth, Tonita [76] proved a *virtual* Kawasaki's formula: under the assumption that $\mathcal{M}$ has a perfect obstruction theory and admits an embedding into a smooth orbifold which has the resolution property, Kawasaki's formula still holds true if we replace the structure sheaves, tangent, and normal bundles in the formula by their virtual counterparts. According to [2], the moduli stacks of stable maps to smooth projective varieties satisfy the assumptions of Tonita's theorem. In the next subsection, we apply the virtual Kawasaki's Riemann-Roch (KRR) formula to $\overline{\mathcal{M}}_{g,k}(X, \beta)/S_k$ to study the poles of the J-function.

4.1.2 Adelic characterization

In this subsection, we first recall the adelic characterization [31] of the cone $\mathcal{L}_{S_\infty}$ in the level-0 permutation-equivariant quantum K-theory. Then we generalize it to describe points on the cone $\mathcal{L}^{R,l}_{S_\infty}$ of level l.

The level-0 case

In the level-0 case, i.e., Givental-Lee's quantum K-theory, the permutation-equivariant invariants are defined as

$$\langle \mathbf{t}(L), \ldots, \mathbf{t}(L)\rangle^{S_k}_{g,k,\beta} := \chi\big(\overline{\mathcal{M}}_{g,k}(X, \beta)/S_k, \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,k}(X,\beta)} \otimes_{i=1}^{k} \mathbf{t}(L_i)\big),$$

where $\mathbf{t}(q)$ is a Laurent polynomial in q with coefficients in $K^0(X)\otimes\mathbb{Q}$. This is a special case of Definition III.2 with ϵ sufficiently large and $l=0$. The J-function is defined as

$$\mathcal{J}_{S_\infty}(\mathbf{t}(q),Q) := 1-q+\mathbf{t}(q)+\sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} Q^\beta \left\langle \frac{\phi_a}{1-qL}, \mathbf{t}(L),\ldots,\mathbf{t}(L)\right\rangle^{S_k}_{0,k+1,\beta}\phi^a.$$

Here $\{\phi_a\}$ and $\{\phi^a\}$ are basis of $K^0(X)\otimes\mathbb{Q}$ dual with respect to the Mukai pairing

$$(\phi_a,\phi_b) := \chi(\phi_a\otimes\phi_b).$$

Recall the definition of the loop space $\mathcal{K}$ from Section 3.2:

$$\mathcal{K} := [K^0(X)\otimes\mathbb{C}(q)]\otimes\mathbb{C}[[Q]].$$

With respect to the symplectic form (3.4) with $l=0$, there is a Lagrangian polarization $\mathcal{K}=\mathcal{K}_+\oplus\mathcal{K}_-$, where $\mathcal{K}_+$ consists of Laurent polynomials in q and $\mathcal{K}_-$ consists of reduced rational functions in q. The Givental's cone $\mathcal{L}_{S_\infty}\subset\mathcal{K}$ is defined as the image of $\mathcal{J}_{S_\infty}:\mathcal{K}_+\to\mathcal{K}$.

To study the poles of the series $\mathcal{J}_{S_\infty}(q)$, we apply the virtual KRR formula to the stack $\overline{\mathcal{M}}_{g,k+1}(X,\beta)/S_k$, where the symmetric group acts on the last k markings. By the virtual KRR formula, each term in the J-function can be written as a summations of fake Euler characteristics over the Kawasaki strata. Note that the Kawasaki strata parametrize stable maps with prescribed automorphisms, i.e., equivalence classes of pairs (C,f,h), where (C,f) is a stable map to X and h is an automorphism of the map. Here, h is allowed to permute the last k markings, but it has to preserve the first marking (with the insertion $1/(1-qL)$). Denote by η the eigenvalue of h on the *cotangent* line to the curve at the first marking.

There are two types of Kawasaki strata. Over the Kawasaki strata with $\eta = 1$, the input $\mathrm{tr}_h(1/(1-qL))$ in the fake Euler characteristics becomes $1/(1-q\bar{L})$, where $1-\bar{L}$ is nilpotent. From the finite expansion

$$\frac{1}{1-q\bar{L}} = \sum_{i\geq 0} \frac{q^i(\bar{L}-1)^i}{(1-q)^{i+1}},$$

we see that the contributions to the J-function from the Kawasaki strata with $\eta = 1$ have poles at $q = 1$.

Over the Kawasaki strata where $\eta \neq 1$ is a primitive m-th root of unity, the insertion $\mathrm{tr}_h(1/(1-qL))$ in the fake Euler characteristics becomes $1/(1-q\eta\bar{L})$, where $1-\bar{L}$ is nilpotent. By considering the finite expansion

$$\frac{1}{1-q\eta\bar{L}} = \sum_{i\geq 0} \frac{(q\eta)^i(\bar{L}-1)^i}{(1-q\eta)^{i+1}},$$

we see that they contribute terms with possible poles at the root of unity η^{-1} to the J-function. We refer the reader to [31] for a nice diagram cataloging all the strata.

For each primitive m-th root of unity η, we denote by $\mathcal{J}_{S_\infty}(\mathbf{t})_{(\eta)}$ the Laurent expansion of the J-function in $1-q\eta$ and regard it as an element in the loop space of power Q-series with vector Laurent series in $1-q\eta$ as coefficients:

$$\mathcal{K}^\eta := K^0(X)\left[\left[\frac{1}{1-q\eta}, 1-q\eta\right]\right] \otimes \mathbb{C}[[Q]].$$

The contributions from the untwisted sector of $\overline{\mathcal{M}}_{g,k}(X,\beta)/S_k$ in the virtual KRR formula are called the *fake* K-theoretic Gromov-Witten (GW) invariants. More precisely, they are defined by

$$\langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \rangle^{fake}_{0,k,\beta} := \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{k} \mathrm{ch}(\mathbf{t}(L_i))\, \mathrm{Td}(T^{\mathrm{vir}}),$$

where $[\overline{\mathcal{M}}_{g,k}(X,\beta)]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli space and T^{vir} is the virtual tangent bundle of $\overline{\mathcal{M}}_{g,k}(X,\beta)$. We define the J-function in the fake quantum K-theory by

$$\mathcal{J}_{fake} : \mathcal{K}^1_+ \to \mathcal{K}^1,$$

$$\mathcal{J}_{fake}(\mathbf{t}(q), Q) := 1 - q + \mathbf{t}(q) + \sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} \frac{Q^\beta}{k!} \phi^a \left\langle \frac{\phi_a}{1-qL}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle^{fake}_{0,k+1,\beta}.$$

Here the input $\mathbf{t}(q)$ belongs to

$$\mathcal{K}^1_+ := K^0(X)[[1-q]] \otimes \mathbb{C}[[Q]].$$

Denote by $\mathcal{L}_{fake} \subset \mathcal{K}^1$ the range of the series $\mathcal{J}_{fake}$. The negative space $\mathcal{K}^1_-$ of the polarization is spanned by $\phi^a q^k/(1-q)^{k+1}$, $a = 1, \ldots, \dim K^0(X)_{\mathbb{Q}}$, $k = 0, 1, \ldots$. The input $\mathbf{t}(q)$ of $\mathcal{J}_{fake}$ can be obtained from the projection of $\mathcal{J}_{fake}$ to $\mathcal{K}^1_+$ along $\mathcal{K}^1_-$.

In [31], Givental gives the following adelic characterization of the values of $\mathcal{J}_{S_\infty}(\mathbf{t})$:

Theorem IV.1 ([31])**.** *The values of $\mathcal{J}_{S_\infty}(\mathbf{t})$ are characterized by the following requirements:*

1. *$\mathcal{J}_{S_\infty}(\mathbf{t})$ has possible poles only at 0, ∞, and roots of unity;*
2. *$\mathcal{J}_{S_\infty}(\mathbf{t})_{(1)}$ lies on $\mathcal{L}_{fake}$;*

3. for every primitive root of unity η of order $m \neq 1$,

$$\mathcal{J}_{S_\infty}(\mathbf{t})_{(\eta)}(q^{1/m}/\eta) \in \sqrt{\frac{\lambda_{-1}(T_X^\vee)}{\lambda_{-1}(\Psi^m T_X^\vee)}} \times \exp \sum_{i \geq 1} \left(\frac{\Psi^i T_X^\vee}{i(1-\eta^{-i}q^{i/m})} - \frac{\Psi^{im} T_X^\vee}{i(1-q^{im})} \right) \mathcal{T}_m\big(\mathcal{J}_{S_\infty}(\mathbf{t})_{(1)}\big),$$

where Ψ^m is the m-th Adams operation on $K^0(X)_{\mathbb{Q}}$ which acts on line bundles as $L \mapsto L^m$, $\lambda_{-1}(E^\vee) = \sum_i (-1)^i \wedge^i E^\vee$ is the K-theoretic Euler class of a vector bundle E, and $\mathcal{T}_m(\mathbf{f})$ is the space described in Definition IV.2 below.

We recall the following definition from [77].

Definition IV.2. Let $\mathbf{f}$ be a point on $\mathcal{L}_{fake}$ and let $T(\mathbf{f})$ be the tangent space to $\mathcal{L}_{fake}$ at $\mathbf{f}$, considered as the image of a map $S(q,Q) : \mathcal{K}^1_+ \to \mathcal{K}$. We extend the Adams operations from $K^0(X)_{\mathbb{Q}}$ to $\mathcal{K}^1$ by $\Psi^m(q) = q^m$ and $\Psi^m(Q) = Q^m$. Let $\Psi^{\frac{1}{m}}$ be the inverse of Ψ^m, acting as $q \mapsto q^{1/m}$ and $Q \to Q^{1/m}$. We define the space $\mathcal{T}_m(\mathbf{f})$ to be the image of the conjugate of $S(q,Q)$:

$$\Psi^m \circ S(q,Q) \circ \Psi^{\frac{1}{m}} : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Remark IV.3. As explained in [77, Remark 5.6], the explicit operator in condition (3) of Theorem IV.1 can be written as a composition $\Box_\eta \Box_m^{-1}$. The definitions of the operators $\Box_\eta$ and $\Box_m$ are given in Proposition IV.10.

The general case of level l

Let R be a vector bundle over X. According to Remark II.6, the level structure $\mathcal{D}^{R,l}$ is defined as

$$\mathcal{D}^{R,l} = \left(\det \mathcal{R}_{k,\beta}\right)^{-l},$$

where $\mathcal{R}_{k,\beta} := R\pi_*(\mathrm{ev}^*\mathcal{R})$ is the *index bundle*, $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,k}(X,\beta)$ is the universal curve, and $\mathrm{ev} : \mathcal{C} \to X$ is the universal evaluation morphism. To state the adelic characterization theorem of the cone $\mathcal{L}^{R,l}_{S_\infty}$ in the permutation-equivariant quantum K-theory with level structure, we introduce the fake invariants of level l

$$\langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \rangle^{fake,R,l}_{0,k,\beta} := \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{k} \mathrm{ch}(\mathbf{t}(L_i))\, \mathrm{Td}(T^{\mathrm{vir}})\, \mathrm{ch}(\mathcal{D}^{R,l}),$$

and the fake J-function of level l

$$\mathcal{J}^{R,l}_{fake}(\mathbf{t}(q), Q) := 1-q+\mathbf{t}(q)+\sum_a \sum_{(k,\beta)\neq(0,0),(1,0)} \frac{Q^\beta}{k!} \phi^a \left\langle \frac{\phi_a}{1-qL}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle^{fake,R,l}_{0,k+1,\beta}.$$

Here $\{\phi^a\}$ is the dual basis of $\{\phi_a\}$ with respect to the twisted pairing

$$(u,v)^{R,l} = \chi(u \otimes v \otimes (\det \mathcal{R})^{-l}).$$

Denote by $\mathcal{L}^{R,l}_{fake} \subset \mathcal{K}^1$ the range of the series $\mathcal{J}^{R,l}_{fake}$. Based on the relationship [29] between gravitational descendants and ancestors of fake quantum K-theory, one can show that $\mathcal{L}^{R,l}_{fake}$ is an overruled Lagrangian cone. We refer the reader to [41, §3] for more details.

Convention IV.4. We will consider various twisted theories, in which the pairings are usually different. In particular, the dual bases $\{\phi^a\}$ which appear in the definitions of various J-functions may not be the same. To relate J-functions in different theories, we need to regard them as elements of the same loop space. This is achieved by rescaling the elements in loop spaces. For example, there is a rescaling map

$$\big(\mathcal{K}^1,\ (\,,\,)^{R,l}\big) \to \big(\mathcal{K}^1,\ (\,,\,)\big),$$
$$E \mapsto E \otimes (\det \mathcal{R})^{-l/2},$$

which identifies the loop space in fake quantum K-theory of level l with that in fake quantum K-theory of level 0.

One of our main results is the following adelic characterization of values of the big J-function in quantum K-theory of level l, generalizing Theorem IV.1.

Theorem IV.5. *The values of $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})$ are characterized by the following requirements:*

1. *$\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})$ has possible poles only at 0, ∞, and roots of unity;*
2. *$\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(1)}$ lies on $\mathcal{L}^{R,l}_{fake}$;*
3. *for every primitive root of unity η of order $m \neq 1$,*

$$\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(\eta)}(q^{1/m}/\eta) \in \sqrt{\frac{\lambda_{-1}(T_X^\vee)}{\lambda_{-1}(\Psi^m T_X^\vee)}} \times \exp \sum_{i\geq 1} \left(\frac{\Psi^i T_X^\vee}{i(1-\eta^{-i} q^{i/m})} - \frac{\Psi^{im} T_X^\vee}{i(1-q^{im})} \right) \mathcal{T}^{R,l}_m\big(\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(1)}\big),$$

where the space $\mathcal{T}^{R,l}_m(\mathbf{f})$ is defined as in Definition IV.2 but starting with a point $\mathbf{f} \in \mathcal{L}^{R,l}_{fake}$.

The proof of Theorem IV.5

We follow the proofs of [41] and [77]. The first requirement in Theorem IV.5 is obviously satisfied. For the second condition, we apply the virtual KRR formula to the J-function. Let $\widetilde{\mathbf{T}}(q)$ be the sum of $\mathbf{t}(q)$ and all contributions in $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})$ which are regular at $q = 1$. According to Proposition II.9, the level structure $\mathcal{D}^{R,l}$ splits "correctly" over nodal strata. With this understood, the following proposition follows from an argument identical to the one given in [77, Proposition 5.2].

Proposition IV.6. *We have*

$$\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t}(q))_{(1)} = \mathcal{J}^{R,l}_{fake}(\widetilde{\mathbf{T}}(q))$$

as elements in $\mathcal{K}^1$. In particular, it shows that $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(1)}$ lies on the cone $\mathcal{L}^{R,l}_{fake}$.

Before we move on to prove the third condition in Theorem IV.5, we characterize the cone $\mathcal{L}^{R,l}_{fake}$ of fake quantum K-theory of level l in terms of the cone $\mathcal{L}_{fake}$ of level 0. This characterization will be needed later in the proof of Theorem IV.5.

Note that the fake quantum K-theory is a version of twisted cohomological Gromov-Witten theory. The machinery of twisted cohomological Gromov-Witten invariants was introduced in [22], and generalized in various directions in [75, 79]. Let $\mathcal{H}$ be the loop space of the cohomological GW theory of X

$$\mathcal{H} := H^{even}(X, \mathbb{C})[z^{-1}, z]][[Q]].$$

It is equipped with a natural symplectic form Ω_H on $\mathcal{H}$ given by

$$\Omega_H\big(\mathbf{f}, \mathbf{g}\big) = \mathrm{Res}_{z=0}\big(\mathbf{f}(-z), \mathbf{g}(z)\big)dz,$$

where the pairing $(\,,)$ is the Poincaré pairing. With respect to Ω_H, there is a Lagrangian polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_+ := H^{even}(X, \mathbb{C})[[z]][[Q]], \quad \mathcal{H}_- := \frac{1}{z}H^{even}(X, \mathbb{C})[z^{-1}][[Q]].$$

Inside $\mathcal{H}$, one can define an overruled Lagrangian cone $\mathcal{L}_H$ by the image of the cohomological big J-function. Since we will not use the explicit description of $\mathcal{L}_H$ in this paper, we refer the reader to [22] for the basic definitions in the cohomological GW theory.

Convention IV.7. Throughout this subsection, we identify $\mathcal{K}^1$ with $\mathcal{H}$ via the Chern character

$$\mathrm{qch} : \mathcal{K}^1 \to \mathcal{H},$$

$$E \mapsto \mathrm{ch}(E),\ q \mapsto e^z.$$

Hence K-theoretic insertions (e.g., ϕ_a and L) in the correlators of twisted cohomological theories should be understood as their Chern characters (e.g., $\mathrm{ch}\,\phi_a$ and $\mathrm{ch}\,L$).

By definition, the fake quantum K-invariants are obtained from the cohomological invariants by inserting the Todd class $\mathrm{Td}(T^{\mathrm{vir}})$ of the virtual tangent bundle T^{vir}. According to [20], the virtual tangent bundle can be written as

$$T^{\mathrm{vir}} = \pi_*(\mathrm{ev}^* T_X - 1) - \pi_*(L_{k+1}^\vee - 1) - (\pi_* i_* \mathcal{O}_{\mathcal{Z}})^\vee \tag{4.3}$$

in $K^0(\overline{\mathcal{M}}_{0,k}(X,\beta))$. Here, $i : \mathcal{Z} \to \mathcal{C}$ is the embedding of the nodal locus. The three parts correspond respectively to: (i) deformations of maps to X of a fixed source curve, (ii) deformations of complex structure and configuration of markings, and (iii) smoothing the nodes.

It is proved in [21] that the cone $\mathcal{L}_{fake}$ (of level 0) is given explicitly in terms of $\mathcal{L}_H$:

$$\mathrm{qch}(\mathcal{L}_{fake}) = \Delta \mathcal{L}_H,$$

where the *loop group transformation* Δ is the Euler-Maclaurin asymptotics of the infinite product

$$\Delta \sim \prod_{\text{Chern roots } x \text{ of } T_X} \prod_{r=1}^{\infty} \frac{x - rz}{1 - e^{-x+rz}}.$$

Here, Δ acts on $\mathcal{H}$ by the pointwise multiplication, and it is determined only by the Todd class of the first summand in the expression of T^{vir}. The second and third summands in (4.3) are respectively responsible for the changes of the dilaton shifts and the polarizations between $\mathcal{H}$ and $\mathcal{K}^1$. We refer to [20] for the details.

The fake quantum K-invariants of level l are obtained from those of level 0 by inserting one more class $\mathrm{ch}(\mathcal{D}^{R,l})$. Its effect on the Lagrangian cone is described in the following proposition.

Proposition IV.8. *Under the identification $z = \log q$, we have the following identity*

$$\mathcal{L}^{R,l}_{fake} = \exp\Big(- l \Big(\frac{\mathrm{ch}_2 \mathcal{R}}{z} \Big) \Big) \mathcal{L}_{fake}$$

in the loop space $\big(\mathcal{K}^1, (\ ,\ )\big)$.

Proof. Recall that the level structure $\mathcal{D}^{R,l}$ is defined as a certain power of the determinant of the index bundle $\mathcal{R}_{k,\beta} = R\pi_*(\mathrm{ev}^*\mathcal{R})$. Note that

$$\mathrm{ch}(\mathcal{D}^{R,l}) = \exp\big(- l \cdot \mathrm{ch}_1(\mathcal{R}_{k,\beta})\big). \tag{4.4}$$

According to [22], the cone of a theory twisted by a general multiplicative characteristic class of the form

$$\exp\Big(\sum_{i\geq 0} s_i \,\mathrm{ch}_i(\mathcal{R}_{k,\beta})\Big)$$

is obtained from the cone of the untwisted theory by applying the operator

$$\exp\left(\sum_{m,i\geq 0} s_{2m-1+i}\frac{B_{2m}}{(2m)!}\mathrm{ch}_i(\mathcal{R})\cdot z^{2m-1}\right).$$

Here the Bernoulli numbers B_{2m} are defined by

$$\frac{t}{1-e^{-t}} = 1+\frac{t}{2}+\sum_{m\geq 1}\frac{B_{2m}}{(2m)!}t^{2m},$$

and the operator acts on $\mathcal{H}$ by the pointwise multiplication. For the twisting class (4.4), we have $s_1 = -l$ and $s_i = 0$ if $i \neq 1$. By applying the above result, we obtain the corresponding loop group transformation:

$$\exp\left(-l\left(\frac{\mathrm{ch}_2(\mathcal{R})}{z}+\frac{\mathrm{ch}_0(\mathcal{R})\cdot z}{12}\right)\right).$$

Note that the cone $\mathcal{L}_{fake}$, being overruled, is invariant under multiplication by functions of z. Therefore, we can ignore the second summand in the exponent of the above operator. □

Now let us prove the third condition in Theorem IV.5. Let $\eta \neq 1$ be a primitive root of unity of order m. The Kawasaki strata in $\overline{\mathcal{M}}_{g,k+1}(X,\beta)/S_k$ which contribute terms with poles at $q = \eta^{-1}$ to the J-function are called the *stem spaces* in [41]. We give a brief description of stem spaces here, and we refer the reader to [41, §8] for more details. Let (C', f, h) be a point in these strata. Consider the unique maximal subcurve $C_+ \subset C'$ containing the first marking where the m-th power h^m acts as the identity. Here we also require that the nodes between components in C_+ are *balanced*, i.e., we require the eigenvalues of h on the two branches of a node in C_+ are inverse to each other. Hence the subcurve C_+ is a chain of $\mathbb{P}^1$, on which h acts as multiplication by η. There are only two smooth points on C_+ which are fixed by h: the first marking on the first component, and one more on the last component. The second point is called the *butt* in [41]. The butt can be a regular point, a marking, or a node in C'. The automorphism h acts on the cotangent space at the butt by η^{-1}. The other marked points and unbalanced nodes on C_+ are cyclically permuted by h. We denote by C the quotient of C_+ by the $\mathbb{Z}_m$-symmetry generated by h. The quotient curve C together with the induced quotient stable map is called a *stem* in [41]. Note that a stem curve can carry unramified marked points, coming from symmetric configurations of m-tuples of markings on the cover, or nodes, coming from m-tuples of symmetric nodes on the cover, where further components of C', cyclically permuted by h, are attached.

One of the key observations in [41] is that the data (C_+, C, f) also represents a stable map to the orbifold $X/\mathbb{Z}_m = X \times B\mathbb{Z}_m$ in the sense of [14] and [3]. Therefore, the contributions with poles at $q = \eta^{-1}$ in the KRR formula for the J-function can be expressed as cohomological integrals over the moduli space of stable maps to $X \times B\mathbb{Z}_m$, twisted by the Todd classes of the traces of the virtual tangent and normal bundles of the Kawasaki strata, and the Chern class of the trace of the level structure

$\mathcal{D}^{R,l}$. To be more precise, we introduce some notations first. Let $\overline{\mathcal{M}}^{X,\beta}_{0,k+2}(\eta)$ denote the stem space. It parametrizes stems of degree β, which are quotient maps by the $\mathbb{Z}_m$-symmetry generated by g. Here g acts by η and η^{-1} on the cotangent lines at the first and last markings of the covering curve, respectively. The only markings on the covering curve fixed by h are the first and last markings. Note that the stem space is a Kawasaki stratum in $\overline{\mathcal{M}}_{0,mk+2}(X, m\beta)$. According to [41, Proposition 5], the stem space $\overline{\mathcal{M}}^{X,\beta}_{0,k+2}(\eta)$ is isomorphic to the moduli space $\overline{\mathcal{M}}^{X/\mathbb{Z}_m,\beta}_{0,k+2}(g, 1, \ldots, 1, g^{-1})$ of stable maps to the orbifold $X/\mathbb{Z}_m$. Here, the sequence $(g, 1, \ldots, 1, g^{-1})$ indicates the sectors where the evaluation maps land. We also consider the stem space $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\eta)$ parametrizing stems whose butts are regular points. Similarly, we have an isomorphism between $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\eta)$ and $\overline{\mathcal{M}}^{X/\mathbb{Z}_m,\beta}_{0,k+1}(g, 1, \ldots, 1)$.

For simplicity, we denote the stem space $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\eta)$ by $\overline{\mathcal{M}}$. Modelling on the contributions in the virtual KRR formula applied to the stack $\overline{\mathcal{M}}_{g,k+1}(X, \beta)/S_k$, we define the correlators in the stem theory of level l by

$$\left\langle \frac{\phi}{1 - qL^{1/m}}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle^{stem,R,l}_{0,k+1,\beta} := \int_{[\overline{\mathcal{M}}]^{\mathrm{vir}}} \mathrm{td}(T_{\overline{\mathcal{M}}})\mathrm{ch}\left(\frac{\mathrm{ev}_1^*\,\phi \cdot \prod_{i=2}^k \mathrm{ev}_i^*\,\mathbf{t}(L_i) \cdot \mathrm{tr}_g\,\mathcal{D}^{R,l}}{\left(1 - qL_1^{1/m}\right) \cdot \mathrm{tr}_g\left(\wedge^* N^{\vee}_{\overline{\mathcal{M}}}\right)} \right).$$

Here $[\overline{\mathcal{M}}]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli space $\overline{\mathcal{M}}^{X/\mathbb{Z}_m,\beta}_{0,k+1}(g, 1, \ldots, 1)$ of stable maps to $X/\mathbb{Z}_m$, and $T_{\overline{\mathcal{M}}}$ and $N_{\overline{\mathcal{M}}}$ are, respectively, the virtual tangent and normal bundles to $\overline{\mathcal{M}}$, considered as a Kawasaki stratum in $\overline{\mathcal{M}}_{0,mk+1}(X, m\beta)$. The line bundle L_1 is formed by the cotangent spaces of stem curves at the first markings, while $L_1^{1/m}$ corresponds to the cotangent line bundle of the covering curves (see [41, §7] for the explanation). From the definition, we see that the stem theory of level l

is a type of twisted cohomological GW theory of $X \times B\mathbb{Z}_m$.

Before we investigate the stem theory further, let us recall some basic facts about the GW theory of the orbifold $X \times B\mathbb{Z}_m$. In this case, the Lagrangian cone of the cohomological GW theory of $X \times B\mathbb{Z}_m$ is the product of m copies of the Lagrangian cone of the GW theory of X. It lies inside the product of m copies of the Fock space $\mathcal{H}$. We refer to each copy of the Lagrangian cone as a *sector*. These sectors are labeled by elements of $\mathbb{Z}_m = \{1, g, \dots, g^{m-1}\}$.

The following proposition relates the Laurent expansion of $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})$ at $q = \eta^{-1}$ to generating series in stem theory.

Proposition IV.9. *Let $\delta\mathbf{t}(q)$ be the contributions in $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})$ which are regular at $q = \eta^{-1}$, i.e., $\delta\mathbf{t}(q) = 1 - q + \mathbf{t}(q) + \tilde{\mathbf{t}}(q)$, where $\tilde{\mathbf{t}}(q)$ is the sum of all the contributions from Kawasaki strata $\overline{\mathcal{M}}^{X,\beta}_{0,k+1}(\xi)$ with $\xi \neq \eta$. Then*

$$\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(\eta)} = \delta\mathbf{t}(q) + \sum_a \sum_{(k,\beta)\neq(0,0)} \frac{Q^{m\beta}}{k!}\phi^a \left\langle \frac{\phi_a}{1 - q\eta L^{1/m}}, \mathbf{T}(L), \dots, \mathbf{T}(L), \delta\mathbf{t}(L^{1/m}/\eta) \right\rangle^{stem,R,l}_{0,k+2,\beta},$$

where

1. *the evaluation morphisms at the marked points land in the twisted sector of $B\mathbb{Z}_m$ labeled by the sequence $(g, 1, \dots, 1, g^{-1})$,*
2. *$\mathbf{T}(L) = \Psi^m \widetilde{\mathbf{T}}(L)$, where Ψ^m acts on cotangent line bundles $L \mapsto L^m$, elements of $K^0(X)_{\mathbb{Q}}$, and Novikov variables $Q^\beta \mapsto Q^{m\beta}$,*
3. *$\widetilde{\mathbf{T}}(q)$ is the input point of $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(1)}$, i.e., it is determined by*

$$1 - q + \widetilde{\mathbf{T}}(q) = \left(\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(1)}\right)_+.$$

where $(\cdots)_+$ *denotes the projection along* $\mathcal{K}^1_-$ *to* $\mathcal{K}^1_+$.

Proof. Proposition II.9 shows that the determinant line bundle $\mathcal{D}^{R,l}$ factorizes "nicely" over nodal strata. With this in mind, the argument of [41, Proposition 2] applies here with one slight change: when determining $\widetilde{\mathbf{T}}(q)$, we do not impose the same condition $\mathbf{t}(q) = 0$ as in [41, Proposition 2]. This is because in the permutation-equivariant theory, we are allowed to permute marked points. □

Proposition IV.9 shows that the Laurent expansion $\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(\eta)}$ of the J-function around $q = \eta^{-1}$ can be identified with a tangent vector to the cone of stem theory of level l:

$$\delta\mathcal{J}^{st,R,l}(\delta\mathbf{t}, \mathbf{T}') := \delta\mathbf{t}(q^{1/m}) + \sum_a \sum_{(k,\beta)\neq(0,0)} \frac{Q^\beta}{k!}\phi^a \left\langle \frac{\phi_a}{1 - q^{1/m}L^{1/m}}, \mathbf{T}'(L), \ldots, \mathbf{T}'(L), \delta\mathbf{t}(L^{1/m}) \right\rangle^{stem,R,l}_{0,k+2,\beta},$$

after replacing $q\eta$ with $q^{1/m}$ and Q^β with $Q^{m\beta}$ (but not in $\delta\mathbf{t}$). Here the input point $\mathbf{T}'(q)$ is obtained from $\mathbf{T}(q)$ by replacing Q^β with $Q^{\beta/m}$, and it belongs to the sector labeled by 1. The tangent vector $\delta\mathcal{J}^{st,R,l}(\delta\mathbf{t}, \mathbf{T})$ belongs to the sector labeled by g^{-1}.

Now let us study the stem theory of level l using the formalism of twisted cohomological GW theory of $X \times B\mathbb{Z}_m$. It follows from the definition that the stem theory of level l is obtained from the untwisted cohomological Gromov-Witten theory of $X/B\mathbb{Z}_m$ by twisting the following classes:

$$\mathrm{td}(T_{\overline{\mathcal{M}}})/\mathrm{ch}\big(\mathrm{tr}_g\big(\wedge^* N^\vee_{\overline{\mathcal{M}}}\big)\big) \tag{4.5}$$

and

$$\mathrm{ch}\big(\mathrm{tr}_g\,\mathcal{D}^{R,l}\big). \tag{4.6}$$

The trace in the first twisting class (4.5) is computed in [41, §8], and the effects of this twisting class on the Lagrangian cone and the genus zero potential of the untwisted theory are also studied in [41, §8]. We summarize them in the following proposition.

Proposition IV.10 ([41])**.** *The effects of the twisting class (4.5) on the Lagrangian cone and the genus zero potential of the untwisted theory are described as follows:*

(i) The sectors labeled by 1 and g^{-1} are rotated by the operators $\square_m$ and $\square_\eta$, respectively. These two operators are the Euler-Maclaurin asymptotics of the infinite products

$$\square_m \sim \prod_i \left(\sqrt{\frac{x_i}{1-e^{-mx_i}}}\prod_{r=1}^{\infty}\frac{x_i - rz}{1-e^{-mx_i+rmz}}\right),$$
$$\square_\eta \sim \prod_i \left(\sqrt{\frac{x_i}{1-e^{-x_i}}}\prod_{r=1}^{\infty}\frac{x_i - rz}{1-\eta^{-r}e^{-x_i+rz/m}}\right),$$

where x_i are the Chern roots of the tangent bundle T_X.

(ii) The dilaton shift changes from $-z$ to $1-q^m$.

(iii) There are changes of polarizations of symplectic loop spaces. More precisely, in the sector labeled by 1, the negative space of the polarization is spanned by

$$\phi^a\Psi^m(q^k/(1-q)^{k+1}),$$

whereas in the sector labeled by g^{-1}, it is spanned by

$$\phi^a q^{k/m}/(1-q^{1/m})^{k+1}.$$

We compute the second twisting class (4.6), and describe its effect on the Lagrangian cone. Let p be the universal family of stem curves. By abuse of notation, we still use ev to denote the universal evaluation morphism from the universal family of quotient curves to $X/\mathbb{Z}_m$. Let $\mathbb{C}_{\eta^i}$ be the topologically trivial line bundle on $X/\mathbb{Z}_m$ on which g acts as multiplication by η^i. According to a simple argument in [77], the trace of the index bundle $R\pi_*(\mathrm{ev}^*\mathcal{R})$ can be expressed as

$$\mathrm{tr}_g\big(R\pi_*(\mathrm{ev}^*\mathcal{R})\big) = \sum_{i=0}^{m-1} \eta^i \, Rp_*(\mathrm{ev}^*\,\mathcal{R}\otimes\mathbb{C}_{\eta^i}).$$

For simplicity, we denote $Rp_*(\mathrm{ev}^*\,\mathcal{R}\otimes\mathbb{C}_{\eta^i})$ by $\bar{\mathcal{R}}_i$. Then we have

$$\begin{aligned}
\mathrm{ch}\,\mathrm{tr}_g\,\mathcal{D}^{R,l} &= \mathrm{ch}\left(\det \sum_{i=0}^{m-1} \eta^i\,\bar{\mathcal{R}}_i\right)^{-l} \\
&= \mathrm{ch}\left(\prod_{i=0}^{m-1} (\eta^i)^{\mathrm{ch}_0\,\bar{\mathcal{R}}_i} \prod_{i=0}^{m-1} \det\bar{\mathcal{R}}_i\right)^{-l} \\
&= \prod_{i=0}^{m-1} \exp\Bigg(-l\Big(i\log(\eta)\,\mathrm{ch}_0\,\bar{\mathcal{R}}_i + \mathrm{ch}_1\,\bar{\mathcal{R}}_i\Big)\Bigg) \qquad (4.7)
\end{aligned}$$

Proposition IV.11. *Twisting by the class (4.6) rotates the sector labeled by the identity of the Lagrangian cone of $X\times B\mathbb{Z}_m$ by*

$$D_m := \exp\Bigg(-ml\bigg(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\bigg)\Bigg)$$

The sector labeled by g^{-1} is rotated by the same operator D_m.

Proof. The proof is based on the orbifold quantum Riemann-Roch theorem developed in [79]. Let E be an orbifold vector bundle over $X \times B\mathbb{Z}_m$. Consider a general twisting class

$$\exp\Big(\sum_{j\geq 0} s_j \operatorname{ch}_j p_*\big(\operatorname{ev}^* E\big)\Big).$$

According to [79, Theorem 1], it corresponds to the rotation by the following operator

$$\exp\Big(\sum_{j\geq 0} s_j \Big(\sum_{n\geq 0} \frac{(A_n)_{j+1-n} z^{n-1}}{n!} + \frac{\operatorname{ch}_j E^{(0)}}{2}\Big)\Big).$$

Here A_n is an operator which acts on all sectors. The restriction $(A_n)|_{X,g^i}$ of A_n to the sector labeled by g^i is defined by

$$(A_n)|_{(X,g^i)} = \sum_{r=0}^{m-1} B_n\big(\frac{r}{m}\big) \operatorname{ch} E_i^{(r)},$$

where $E_i^{(r)}$ (respectively $E^{(0)}$) is the subbundle of the restriction of E to (X, g^i) on which g^i acts with eigenvalue $e^{2\pi i r/m}$ (respectively 1). The notation $(A_n)_j$ denotes the degree j component of the operator A_n. The Bernoulli polynomials are defined by

$$\sum_{n\geq 0} B_n(x) \frac{t^n}{n!} = \frac{te^{tx}}{e^t - 1}.$$

In our case, the twisting class is given by (4.7). Let $\tilde{D}_{\eta,i}$ denote the symplectic transformations corresponding to the i-th factor of the twisting class (4.7), restricted to (X, g^{-1}). For each $i \in \{0, \ldots, m-1\}$, the operator A_n in the definition of $\tilde{D}_{\eta,i}$ is given by

$$(A_n)|_{(X,g^{-1})} = B_n\big(\frac{i}{k}\big) \operatorname{ch} \mathcal{R}.$$

By the orbifold quantum Riemann-Roch theorem, the operator $\tilde{D}_{\eta,i}$ equals

$$\exp\Big(-l\log(\eta)\Big(\frac{\mathrm{ch}_1\mathcal{R}}{z}+B_1(\frac{i}{m})\,\mathrm{ch}_0\mathcal{R}\Big)-l\Big(\frac{\mathrm{ch}_2\mathcal{R}}{z}+B_1(\frac{i}{m})\,\mathrm{ch}_1\mathcal{R}+\frac{B_2(i/m)\,\mathrm{ch}_0\mathcal{R}}{2}z\Big)-l\frac{\mathrm{ch}_1\mathcal{R}}{2}\Big)$$

Let $\tilde{D}_\eta = \prod_{i=0}^{m-1}\tilde{D}_{\eta,i}$. To simplify the expression of $\tilde{D}_\eta$, we use the fact that $n\log(\eta) = 0$ if n is an integer divisible by m. Keeping this in mind, we obtain

$$\tilde{D}_\eta = \exp\Big(-l\Big(\frac{m\,\mathrm{ch}_2\mathcal{R}}{z}+\frac{\mathrm{ch}_0\mathcal{R}}{12\,m}z+\frac{\mathrm{ch}_0\mathcal{R}}{6}\log(\eta)\Big)\Big).$$

Note that the factor $\exp(-l(z\,\mathrm{ch}_0\mathcal{R}/(12\,m)+\log(\eta)\,\mathrm{ch}_0\mathcal{R}/6))$ in $\tilde{D}_\eta$ is a scalar z-series and thus it preserves the overruled Lagrangian cone. We can drop it and obtain the operator D_m.

For the sector labeled by the identity, we denote by $\tilde{D}_{m,i}$ the restriction of the operator corresponding to the i-th factor of the twisting class (4.7). It is easy to check that

$$(A_n)|_{(X,1)} = B_n(0)\,\mathrm{ch}\,\mathcal{R},$$

and the operator $\tilde{D}_{m,i}$ equals

$$\exp\Big(-il\log(\eta)\Big(\frac{\mathrm{ch}_1\mathcal{R}}{z}\Big)-ml\Big(\frac{\mathrm{ch}_2\mathcal{R}}{z}+\frac{\mathrm{ch}_0\mathcal{R}}{12}z\Big)\Big).$$

Let $\tilde{D}_m = \prod_{i=0}^{m-1}\tilde{D}_{m,i}$. Again by using the fact that $n\log(\eta) = 0$ if $m|n$, we can simplify the operator $\tilde{D}_m$ to

$$\exp\Big(-ml\Big(\frac{\mathrm{ch}_2\mathcal{R}}{z}+\frac{\mathrm{ch}_0\mathcal{R}}{12}z\Big)\Big).$$

We can drop the second term in the exponent because it is a constant z-series.

□

The above discussion can be summarized in the following proposition.

Proposition IV.12. $\mathrm{qch}\,\delta\mathcal{J}^{st,R,l}(\delta\mathbf{t},\mathbf{T})$ *lies in the tangent space*

$$\Box_\eta\,\Box_m^{-1}\big(\mathcal{T}_{\mathcal{I}^{tw}}\,\Box_m\,D_m\,\mathcal{L}_H\big)$$

to the cone of the stem theory of level l at a certain point $\mathcal{I}^{tw}$. The input $\mathbf{T}$ satisfies

$$\mathrm{qch}\big(1-q^m+\mathbf{T}(q)\big)=\big[\mathcal{I}^{tw}\big]_+,$$

where $[\cdots]_+$ denotes the projection along the negative space of the polarization of the sector labeled by 1.

Proof. The argument of [77, Proposition 5.8] applies here. We briefly explain the relation between the application point $\mathcal{I}^{tw}$ and the input $\mathbf{T}$. Note that the application point $\mathcal{I}^{tw}$ lies on the Lagrangian of the stem theory of level l in the sector labeled by 1. According to Proposition IV.10 (ii), the new dilaton shift is $1-q^m$. This explains the equality in the proposition. □

To prove the third condition in Theorem IV.5, we need to identify $\mathcal{T}_{\mathcal{I}^{tw}}\Box_m D_m\mathcal{L}_H$ with $\mathcal{T}_m\big(\mathcal{J}_{S_\infty}(\mathbf{t})_{(1)}\big)$. We first show that

Proposition IV.13.

$$\mathrm{qch}^{-1}\big(\Box_m\,D_m\,\mathcal{L}_H\big)=\widetilde{\Psi}^m\mathcal{L}^{R,l}_{fake}.$$

Here, the Adams operation $\widetilde{\Psi}^m$ acts on K-theory classes of X and q by $\Psi^m(q)=q^m$, but not on the Novikov variables Q.

Proof. It is proved in [41, Proposition 9] that $\mathrm{qch}^{-1}(\square_m \mathcal{L}_H) = \Psi^m \mathcal{L}_{fake}$. In that proof, one needs to extend the action of the Adams operator Ψ^m on cohomology classes via the Chern isomorphism:

$$\mathrm{ch}\big(\Psi^m(\mathrm{ch}^{-1} a)\big) = m^{\deg(a)/2} a.$$

By Proposition IV.8, we have $\mathcal{L}^{R,l}_{fake} = D_1 \mathcal{L}_{fake}$. We conclude the proof by noticing that $\Psi^m(D_1) = D_m$. □

Let $\widetilde{\mathbf{T}}(q)$ be the input point of $\mathcal{J}^{R,l}_{fake}(\widetilde{\mathbf{T}}(q))$ determined by

$$1 - q + \widetilde{\mathbf{T}}(q) = \big(\mathcal{J}^{R,l}_{S_\infty}(\mathbf{t})_{(1)}\big)_+.$$

where $(\cdots)_+$ denotes the projection along $\mathcal{K}^1_-$ to $\mathcal{K}^1_+$. Let $\mathbf{T}'(q)$ be the input point of $\mathcal{I}^{tw}$ such that

$$\widetilde{\Psi}^m\big(\mathcal{J}^{R,l}_{fake}(\widetilde{\mathbf{T}}(q))\big) = \mathcal{I}^{tw}\big(\mathbf{T}'(q)\big). \tag{4.8}$$

We claim that $\widetilde{\Psi}^m(\widetilde{\mathbf{T}}(q)) = \mathbf{T}'(q)$. This equality holds because according to Proposition IV.10 and Proposition IV.13, the operation $\widetilde{\Psi}^m : \mathcal{K}^1 \to \mathcal{K}^1$ identifies the cone $\mathcal{L}^{R,l}_{fake}$ with the cone $\square_m D_m \mathcal{L}_H$, the polarization of the fake quantum K-theory with the polarization in the sector labeled by 1 of the stem theory, and the old dilaton shift $1 - q$ with the new one $1 - q^m$. Therefore $\widetilde{\Psi}^m$ must also map the input point $\widetilde{\mathbf{T}}(q)$ of the fake J-function to the input point $\mathbf{T}'(q)$ of $\mathcal{I}^{tw}$. Recall from Proposition IV.9 that we have $\Psi^m(\widetilde{\mathbf{T}}(q)) = \mathbf{T}(q)$. Then it follows from the definitions of Ψ^m and $\widetilde{\Psi}^m$ that $\mathbf{T}'(q)$ is obtained from $\mathbf{T}(q)$ by replacing Q^β with $Q^{\beta/m}$.

By differentiate the relation (4.8), we get

$$\widetilde{\Psi}^m\left(\mathbf{f}(q)+\sum\frac{Q^\beta}{k!}\phi^a\left\langle\frac{\phi_a}{1-qL},\widetilde{\mathbf{T}}(L),\ldots,\widetilde{\mathbf{T}}(L),\mathbf{f}(L)\right\rangle_{0,k+2,\beta}^{fake,R,l}\right)$$
$$=\widetilde{\Psi}^m\mathbf{f}(q)+\sum\frac{Q^\beta}{k!}\widetilde{\Psi}^m\phi^a\left\langle\frac{\widetilde{\Psi}^m\phi_a}{1-q^mL^m},\mathbf{T}'(L),\ldots,\mathbf{T}'(L),\widetilde{\Psi}^m\mathbf{f}(L)\right\rangle_{0,k+2,\beta}^{stem,R,l}.$$

The RHS is a tangent vector in $\mathcal{T}_{\mathcal{I}^{tw}}\square_m D_m\mathcal{L}_H$ along the direction of $\delta\mathbf{t}':=\widetilde{\Psi}^m\mathbf{f}(q)$. The LHS becomes $\Psi^m\circ S(q,Q)\circ\Psi^{1/m}(\delta\mathbf{t}')$ after we replace Q^β with $Q^{m\beta}$ (including such a change in $\widetilde{\mathbf{T}}$ but excluding it in $\mathbf{f}(q)$). This concludes the proof of Theorem IV.5.

Remark IV.14. When the target X is an orbifold, the adelic characterization of points on the cone $\mathcal{L}$ of the ordinary, i.e., permutation-*non*-equivariant, quantum K-theory is developed in [78]. In this case, the Lagrangian cone $\mathcal{L}$ has different sectors, and each sector corresponds to a connected component of the rigidified inertia stack $\bar{I}_\mu X$. Let $f: C\to X$ be an orbifold stable map. Here C is an orbifold curve with possible orbifold structures at the marked points and nodes. Let $\underline{f}:\underline{C}\to\underline{X}$ be the map between coarse moduli spaces. There is a short exact sequence

$$1\to K\to \mathrm{Aut}(f)\to\mathrm{Aut}(\underline{\mathrm{f}})\to 1.$$

The kernel K consists of automorphisms of $C\to X$ that fix $\underline{C}\to\underline{X}$. These automorphisms are referred to as "ghost automorphisms" in [1], and they arise from the stacky nodes of the source curve.

To analyze poles of K-theoretic J-functions in the orbifold setting, we still apply the virtual KRR formula to the moduli space of orbifold stable maps. In this case, there are extra contributions from twisted sectors corresponding to ghost automorphisms. The key observation in [78] is that once we add the appropriate contributions from ghost automorphisms in the definition of fake K-theoretic GW invariants, the formalism of adelic characterizations carries over to the orbifold setting. We refer the reader to [78, Definition 3.1] for the precise definition of fake K-theoretic invariants and [78, Theorem 4.1] for the adelic characterization in the orbifold and permutation-non-equivariant setting. We only mention that if we restrict to the untwisted sector of the cone $\mathcal{L}$, the main theorem in [78] specializes to Theorem IV.1.

The generalization of [78, Theorem 4.1] to the permutation-equivariant setting is straightforward: we only need to change the application point of the tangent space in [78, Definition 4.3] from $\mathcal{J}_1(0)$ to the Laurent expansion $(\mathcal{J}_{S\infty}(\mathbf{t}))_1$ of the J-function at $q = 1$. Since the determinant line bundle splits "correctly" among nodal strata, we can also generalize [78, Theorem 4.1] to permutation-equivariant quantum K-theory with level structure. In this paper, we focus on recovering examples of mock theta functions. For this purpose, we only need to consider the untwisted sector of Lagrangian cones of orbifold targets. Once we make this restriction, the statement of the adelic characterization is the same as in Theorem IV.5.

4.1.3 Determinantal modification

In this subsection, we use the adelic characterization to prove Theorem I.1 which gives us a way to obtain points on $\mathcal{L}^{R,l}_{S_\infty}$ by making certain "determinantal" modifications to points on the level-0 cone $\mathcal{L}_{S_\infty}$.

Let us restate Theorem I.1.

Theorem IV.15. *If*

$$I = \sum_{\beta \in \mathrm{Eff}(X)} I_\beta Q^\beta$$

lies on $\mathcal{L}_{S_\infty}$*, then the point*

$$I^{R,l} := \sum_{\beta \in \mathrm{Eff}(X)} I_\beta Q^\beta \prod_i \left(L_i^{-\beta_i} q^{(\beta_i+1)\beta_i/2} \right)^l$$

lies on the cone $\mathcal{L}^{R,l}_{S_\infty}$ *of permutation-equivariant quantum* K*-theory of level* l*. Here,* $\mathrm{Eff}(X)$ *denotes the semigroup of effective curve classes on* X*,* L_i *are the* K*-theoretic Chern roots of* $\mathcal{R}$*, and* $\beta_i := \int_\beta c_1(L_i)$*.*

Proof. Suppose $I = \sum_{\beta \in \mathrm{Eff}(X)} I_\beta Q^\beta$ is a point on $\mathcal{L}_{S_\infty}$. Let $I^{R,l}$ be its "determinantal" modification $\sum_\beta I_\beta Q^\beta \prod_i \left(L_i^{-\beta_i} q^{\beta_i(\beta_i-1)/2} \right)^l$. According to Convention IV.4, we compare different cones in the same loop spare $\mathcal{K}^1$. In particular, $\mathcal{L}^{R,l}_{fake}$ and the tangent space in Theorem IV.5 are viewed as subspaces of $\mathcal{K}^1$. Therefore, to show $I^{R,l}$ lies on $\mathcal{L}^{R,l}_\infty$, we need to work with the series after the rescaling

$$\widetilde{I}^{R,l} := (\det \mathcal{R})^{-l/2} I^{R,l}.$$

We denote by $I_{(\eta)}$ and $\widetilde{I}^{R,l}_{(\eta)}$ the Laurent expansions of I and $\widetilde{I}^{R,l}$ in $1 - q\eta$, respectively. Let $Q_1, \ldots, Q_n$ be the Novikov variables. Let p_i be the degree 2 cohomology classes corresponding to Q_i, and let $P_i = e^{-p_i} \in K^0(X)$. We denote by L_i the K-theoretic Chern roots of $\mathcal{R}$. In other words, we have $\mathrm{ch}\, L_i = e^{l_i}$, where l_i are cohomological Chern roots of $\mathcal{R}$. We write l_i as a linear function $f_i(p_1, \ldots, p_n)$ in terms of the basis $p_1, \ldots, p_n$.

It is clear that $\widetilde{I}^{R,l}$ satisfies the first condition in Theorem IV.5. Now, we check the second condition. It follows from the Lemma in the proof of [22, Theorem 2] that the operator

$$\Phi := \prod_{i=1}^{n} \exp\Big(l \Big(\frac{\big(f_i(p_j - zQ_j\partial_{Q_j})\big)^2}{2z} + \frac{f_i\big(p_j - zQ_j\partial_{Q_j}\big)}{2} \Big)\Big)$$

preserves $\mathcal{L}_{fake}$. Define $d_j = \langle c_1(p_j), \beta \rangle$ to be the components of the degree β. By a simple computation, one can show that

$$\begin{aligned}
\Phi(Q^\beta) &= \prod_{i=1}^{n} \exp\Big(l \Big(\frac{\big(f_i(p_j - zd_j)\big)^2}{2z} + \frac{f_i\big(p_j - zd_j\big)}{2} \Big)\Big) Q^\beta \\
&= \exp\Big(l \Big(\frac{\operatorname{ch}_2 \mathcal{R}}{2z} + \frac{\operatorname{ch}_1 \mathcal{R}}{2} \Big)\Big)\, Q^\beta \\
&\cdot \prod_{i=1}^{n} \exp\Big(l \Big(\frac{\big(f_i(p_j - zd_j)\big)^2}{2z} + \frac{f_i\big(p_j - zd_j\big)}{2} \Big)\Big) \exp\Big(- l \Big(\frac{\operatorname{ch}_2 \mathcal{R}}{2z} + \frac{\operatorname{ch}_1 \mathcal{R}}{2} \Big)\Big) \\
&= \exp\Big(l \Big(\frac{\operatorname{ch}_2 \mathcal{R}}{z} + \frac{\operatorname{ch}_1 \mathcal{R}}{2} \Big)\Big)\, Q^\beta \prod_i \big(L_i^{-\beta_i} q^{\beta_i(\beta_i - 1)/2} \big)^l,
\end{aligned}$$

It follows that

$$\Phi\big(\overline{Q} \cdot I_{(1)}\big)/\overline{Q} = \exp\big(l \big(\operatorname{ch}_2 \mathcal{R}/z \big)\big) \widetilde{I}^{R,l}_{(1)}, \tag{4.9}$$

where $\overline{Q} := \prod Q_i$. Since the LHS lies on $\mathcal{L}_{fake}$, we conclude that the Laurent expansion of $\widetilde{I}^{R,l}$ at $q = 1$ lies on the cone $\mathcal{L}^{R,l}_{fake} = \exp\big(- l \big(\operatorname{ch}_2 \mathcal{R}/z \big)\big) \mathcal{L}_{fake}$.

Now we check the third condition. Suppose the tangent space to $\mathcal{L}_{fake}$ at $I_{(1)}$ is given as the image of a map

$$S(q, Q) : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Then by (4.9), the tangent space to $\mathcal{L}^{R,l}_{fake}$ at $\widetilde{I}^{R,l}_{(1)}$ is given as the image of a map

$$S'(q,Q) = \exp\Big(-l\Big(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\Big)\Big)\Phi \circ S(q,Q) : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Here we use the fact that the Novikov variables are contained in the λ-algebra and hence they preserve tangent spaces.

Recall from Definition IV.2 that the space $\mathcal{T}_m(I_{(1)})$ is defined as the image of a map

$$\Psi^m \circ S(q,Q) \circ \Psi^{1/m} : \mathcal{K}^1_+ \to \mathcal{K}^1.$$

Then $\mathcal{T}^{R,l}_m(\widetilde{I}^{R,l}_{(1)})$ is given as the image of

$$\begin{aligned}
\Psi^m \circ S'(q,Q) \circ \Psi^{1/m} &= \Psi^m \circ \exp\Big(-l\Big(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\Big)\Big)\Phi \circ S(q,Q) \circ \Psi^{1/m} \\
&= \exp\Big(-ml\Big(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\Big)\Big)\Psi^m \circ \Phi \circ S(q,Q) \circ \Psi^{1/m} \\
&= D_m\, \Phi^m \left(\Psi^m \circ S(q,Q) \circ \Psi^{1/m}\right),
\end{aligned}$$

where $\Phi^m := \Psi^m(\Phi)$ is given as follows

$$\Psi^m(\Phi) = \prod_{i=1}^n \exp\bigg(l\bigg(\frac{\big(f_i(mp_j - zQ_j\partial_{Q_j})\big)^2}{2mz} + \frac{f_i\big(mp_j - zQ_j\partial_{Q_j}\big)}{2}\bigg)\bigg).$$

Here we use the fact that the Adams operation Ψ^m acts on the degree two classes z and p_j as multiplication by m, and its action on the differential operator $zQ_j\partial Q_j$ is trivial[1]. This shows that $\mathcal{T}^{R,l}_m(\widetilde{I}^{R,l}_{(1)}) = D_m\Phi^m\big(\mathcal{T}_m(I_{(1)})\big)$.

By the assumption, we have

$$I_{(\eta)}(q^{1/m}/\eta) \in \Box_\eta \Box_m^{-1} \mathcal{T}_m(I_{(1)}).$$

[1]This is because $\Psi^m(zQ_j\partial Q_j) = mzQ_j^m \partial Q_j^m = zQ_j\partial Q_j$.

Then

$$\widetilde{I}^{R,l}_{(\eta)}(q^{1/m}/\eta) = \sum_{\beta}(I_\beta)_{(\eta)}(q^{1/m}/\eta)Q^\beta \prod_i \left(L_i^{-\beta_i} q^{(\beta_i+1)\beta_i/(2m)}\eta^{-(\beta_i+1)\beta_i/2}\right)^l$$

$$(4.10)\qquad = \exp\left(- ml\left(\frac{\mathrm{ch}_2\,\mathcal{R}}{z}\right)\right)\big(\Phi(\overline{Q}\cdot I_{(\eta)})\big)(q^{1/m}/\eta)/\overline{Q}.$$

By an elementary computation using the fact that $m \cdot \log(\eta) = 0$, one can show that for any series $\mathbf{f}$ in q and Q, we have

$$(4.11)\qquad \big(\Phi(\mathbf{f})\big)\big(q^{1/m}/\eta\big) = \Phi^m \mathcal{D}_\eta \mathbf{f}(q^{1/m}/\eta),$$

where the operator $\mathcal{D}_\eta$ is defined by

$$\mathcal{D}_\eta = \prod_{i=1}^n \exp\left(l\left(-f_i\big(\log(\eta)(Q_j\partial Q_j)^2\big)+\frac{f_i\big(\log(\eta)Q_j\partial Q_j\big)}{2}-\frac{m-1}{m}\frac{f_i\big(mp_j - zQ_j\partial_{Q_j}\big)}{2}\right)\right).$$

Here the substitution $q \mapsto q^{1/m}/\eta$ corresponds to the change $z \mapsto z/m - \log(\eta)$ in the expression of Φ.

It follows from (4.11) that (4.10) equals

$$(4.12)\qquad \big(D_m \Phi^m \mathcal{D}_\eta\big(\overline{Q}\cdot I_\eta(q^{1/m}/\eta)\big)\big)/\overline{Q} \in D_m \Phi^m \mathcal{D}_\eta \Box_\eta \Box_m^{-1} \mathcal{T}_m(I_{(1)}).$$

Since all the operators above have constant coefficients (i.e. independent of Q), they commute. We claim that $\mathcal{D}_\eta$ preserves $\mathcal{T}_m(I_{(1)})$. This is because by definition, we have

$$\mathcal{D}_\eta \mathcal{T}_m(I_{(1)}) = \mathcal{D}_\eta \Psi^m \circ S(q,Q) \circ \Psi^{1/m}\mathcal{K}^1_+.$$

Let

$$\mathcal{D}_{\eta,1} = \prod_{i=1}^n \exp\big(l\big(- f_i\big(\log(\eta)(Q_j\partial Q_j)^2\big) + f_i\big(\log(\eta)Q_j\partial Q_j\big)/2\big)\big)$$

and

$$\mathcal{D}_{\eta,2} = \prod_{i=1}^{n} \exp\big(l\big(-(m-1) f_i \big(mp_j - zQ_j \partial_{Q_j}\big)/(2m)\big)\big)$$

be the two factors of $\mathcal{D}_\eta$. Then it is easy to check that the first factor $\mathcal{D}_{\eta,1}$ commutes with $\Psi^m \circ S(q,Q) \circ \Psi^{1/m}$, and hence preserves $\mathcal{T}_m(I_{(1)})$. The second factor $\mathcal{D}_{\eta,2}$ satisfies the commutation relation:

$$\mathcal{D}_{\eta,2}\Psi^m = \Psi^m \prod_{i=1}^{n} \exp\big(l\big(-(m-1) f_i \big(p_j - zQ_j \partial_{Q_j}\big)/(2m)\big)\big).$$

According to [41, Corollary 1], the second operator on the RHS preserves the tangent space $S(q,Q) \circ \Psi^{1/m}\mathcal{K}_+^1$. Therefore we have shown that the space $\mathcal{T}_m(I_{(1)})$ is $\mathcal{D}_\eta$-invariant.

We can further simplify the space on the RHS of (4.12) as follows

$$\begin{aligned}
D_m\Phi^m\mathcal{D}_\eta\Box_\eta\Box_m^{-1}\mathcal{T}_m(I_{(1)}) &= D_m\Phi^m\Box_\eta\Box_m^{-1}\mathcal{D}_\eta\mathcal{T}_m(I_{(1)}) \\
&= D_m\Phi^m\Box_\eta\Box_m^{-1}\mathcal{T}_m(I_{(1)}) \\
&= D_m\Phi^m\Box_\eta\Box_m^{-1}(\Phi^m)^{-1}D_m^{-1}(\mathcal{T}_m^{R,l}(\widetilde{I}_{(1)}^{R,l})) \\
&= \Box_\eta\Box_m^{-1}(\mathcal{T}_m^{R,l}(\widetilde{I}_{(1)}^{R,l})).
\end{aligned}$$

This concludes the proof.

□

Remark IV.16. Suppose the target is an orbifold. As explained in Remark IV.14, the adelic characterization of points on the untwisted sector of the Lagrangian cone of X is the same as the one given in Theorem IV.5. Using the same proof as above, we can show that if I is a point on the untwisted sector of $\mathcal{L}_{S_\infty}$, then the determinantal modification $I^{R,l}$ lies on the untwisted sector of $\mathcal{L}^{R,l}_{S_\infty}$.

4.2 Toric mirror theorem and mock theta functions

In this subsection, we first explicitly compute the (torus-equivariant) small I-functions with level structures for toric varieties, using quasimap graph spaces. Then we use torus localization to prove a toric mirror theorem (Theorem I.2), following Givental [33]. In the study of quantum K-theory with non-trivial level structures, a remarkable phenomenon is the appearance of Ramanujan's mock theta functions.

Let $M \cong \mathbb{Z}^n$ be a n-dimensional lattice and let N be its dual lattice. For every complete nonsingular fan $\Sigma \subset N_{\mathbb{R}}$, we can associate a n-dimensional smooth projective variety X_Σ. We denote by $\Sigma(1)$ the set of 1-dimensional cones in Σ. Let $m = |\Sigma(1)|$. Each $\rho \in \Sigma(1)$ determines a Weil divisor D_ρ on X_Σ and the Picard group of X_Σ is determined by the following short exact sequence:

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X_\Sigma) \to 0. \tag{4.13}$$

Here the inclusion is defined by $m \mapsto \sum_\rho \langle m, \rho\rangle D_\rho$. Now let us describe the quotient construction of X_Σ. Since $\mathrm{Pic}(X_\Sigma)$ is torsion free, we choose an integral basis $\{L_1, \ldots, L_s\}$ of it, where $s = m - n$. Then the inclusion map in (4.13) is given by an integral $s \times n$ matrix $Q = (Q_{a\rho})$ which is called the charge matrix of X_Σ. Applying $\mathrm{Hom}(-, \mathbb{C}^*)$ to the exact sequence (4.13), we get an exact sequence.

$$1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to N \otimes \mathbb{C}^* \to 1,$$

where $G := \mathrm{Hom}(\mathrm{Pic}(X_\Sigma), \mathbb{C}^*) \cong (\mathbb{C}^*)^s$. The first map in the above short exact sequence defines the following G-action on $\mathbb{C}^{\Sigma(1)}$

$$\mathbf{t} \cdot (z_{\rho_1}, \ldots, z_{\rho_m}) = \left(\prod_{a=1}^{s} t_a^{Q_{a\rho_1}} z_{\rho_1}, \ldots, \prod_{a=1}^{s} t_a^{Q_{a\rho_m}} z_{\rho_m} \right), \tag{4.14}$$

where $\mathbf{t} = (t_1, \ldots, t_s) \in (\mathbb{C}^*)^s$. By choosing an appropriate linearization of the trivial line bundle on $\mathbb{C}^{\Sigma(1)}$ (see e.g., [23], Chapter 12), the semistable and stable loci are equal. We denote this linearized trivial line bundle by L_Σ and the stable loci by $U(\Sigma)$. Let z_ρ be the coordinates in $\mathbb{C}^{\Sigma(1)}$. We define a subvarity

$$Z(\Sigma) = \{(z_\rho) \in \mathbb{C}^{\Sigma(1)} | \prod_{\rho \not\subset \sigma} z_\rho = 0, \sigma \in \Sigma\}.$$

Then we have

$$U(\Sigma) = \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma).$$

The toric variety X_Σ is the geometric quotient $U(\Sigma)/G$. Let P be the principal G-bundle $\mathbb{C}^{\Sigma(1)} \to [\mathbb{C}^{\Sigma(1)}/G]$. Let $\pi_i : G \to \mathbb{C}^*$ be the projection to the i-th component and let R_j be the characters given by $\mathbf{t} = (t_1, \ldots, t_s) \to \prod_{a=1}^{s} t_a^{Q_{a\rho_j}}$ for $1 \leq j \leq m$. Then the line bundles L_i and $\mathcal{O}(-D_{\rho_j})$ are the restrictions of the associated line bundles of P with the characters π_i and R_j, respectively, to X_Σ.

Note that X_Σ admits a $T^m := (\mathbb{C}^*)^{\Sigma(1)}$-action. We denote by P_i and U_ρ the T^m-equivariant line bundles corresponding to L_i and $\mathcal{O}(D_\rho)$, respectively. In the T^m-equivariant K-group $K^0_{T^m}(X_\Sigma) \otimes \mathbb{Q}$, we have the following multiplicative relation:

$$U_\rho = \prod_{i=1}^{s} P_i^{\otimes Q_{i\rho}} \Lambda_\rho^{-1},$$

where Λ_ρ are the generators of $\mathrm{Repr}(T^m)$ corresponding to the projection to the component labeled by ρ.

Now let us compute the (T^m-equivariant) small I-function of X_Σ with level structures, using the quasimap graph space. Let $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^G(\mathbb{C}^{\Sigma(1)}), \mathbb{Z})$ be an L_Σ-effective class. According to [17, Lemma 3.1.8], a point in the quasimap graph space $QG^{\epsilon=0+}_{0,k}(X_\Sigma, \beta)$ is specified by the following data

$$((C, p_1, \ldots, p_k), \{\mathcal{P}_i | i = 1, \ldots, s\}, \{u_\rho\}_{\rho\in\Sigma(1)}, \varphi),$$

where

- $(C, p_1, \ldots, p_s)$ is a connected, at most nodal, curve of genus 0 and p_i are distinct nonsingular points of C,
- $\mathcal{P}_i$ are line bundles on C of degree $f_i := \beta(L_i)$,
- $u_\rho \in \Gamma(C, \mathcal{L}_\rho)$, where $\mathcal{L}_\rho$ is defined by

$$\mathcal{L}_\rho := \otimes_{i=1}^{s} \mathcal{P}_i^{\otimes Q_{i\rho}},$$

- $\varphi : C \to \mathbb{P}^1$ is a regular map such that $\varphi_*[C] = [\mathbb{P}^1]$.

The stability conditions are discussed in Section 3.2. In the case when $(g, k) = (0, 0)$, we have $C \cong \mathbb{P}^1$ and $\mathcal{P}_i \cong \mathcal{O}_{\mathbb{P}^1}(f_i)$. The line bundles $\mathcal{L}_\rho$ are isomorphic to $\mathcal{O}_{\mathbb{P}^1}(\sum_{i=1}^{s} f_i Q_{i\rho}) = \mathcal{O}_{\mathbb{P}^1}(\beta_\rho)$, where $\beta_\rho := \beta(\mathcal{O}(D_\rho))$. Therefore, a point on $QG^{\epsilon=0+}_{0,0}(X_\Sigma, \beta)$ is specified by sections $\{u_\rho \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\beta_\rho)) | \rho \in \Sigma(1)\}$. We choose coordinates $[x_0, x_1]$ on $\mathbb{P}^1$ and consider the standard action $\mathbb{C}^*$-action defined by (3.1).

Let F_0 be the distinguished fixed point locus parametrizing quasimaps whose degrees are concentrated only at 0. According to [17, §7.2], we have the identification

$$F_0 \cong \bigcap_{\{\rho | \beta_\rho < 0\}} D_\rho \subset X_\Sigma$$

$$(z_\rho \, x_0^{\beta_\rho}) \to (z_\rho), \tag{4.15}$$

where (z_ρ) are the coordinates on X_Σ.

Let R be a character of $G = (\mathbb{C}^*)^s$ defined by $\mathbf{t} \cdot z = \prod_{i=1}^s t_i^{r_i} z$ where $r_i \in \mathbb{Z}$. Recall that the small I-function of X_Σ of level l and representation R is defined by

$$I^{R,l}(q) = 1 + \sum_a \sum_{\beta \neq 0} Q^\beta \chi\Bigg(F_0, \mathrm{ev}^*(\phi_a) \otimes \Bigg(\frac{\mathrm{tr}_{\mathbb{C}^*} \mathcal{D}^{R,l}}{\mathrm{tr}_{\mathbb{C}^*} \wedge^* \left(N^{\mathrm{vir}}_{F_0/QG}\right)^\vee}\Bigg)\Bigg) \phi^a,$$

It is not difficult to check that under the identification (4.15), we can identify the virtual normal bundle $N^{\mathrm{vir}}_{F_0/QG}$ in $K^0(F_0)$ with

$$N^{\mathrm{vir}}_{F_0/QG} = \sum_{\{\rho | \, \beta_\rho > 0\}} \sum_{i=1}^{\beta_\rho} \mathcal{O}(D_\rho)|_{F_0} \otimes \mathbb{C}_{-i} - \sum_{\{\rho | \, \beta_\rho < 0\}} \sum_{i=1}^{\beta_\rho - 1} \mathcal{O}(D_\rho)|_{F_0} \otimes \mathbb{C}_i, \tag{4.16}$$

where $\mathbb{C}_a$ denotes the representation of $\mathbb{C}^*$ on $\mathbb{C}$ with weight $a \in \mathbb{Z}$. Let $\mathcal{P}$ be the universal principal G-bundle on $F_0 \times \mathbb{P}^1 \subset X_\Sigma \times \mathbb{P}^1$. Then the associated line bundle $\mathcal{P} \times_G R$ can be identified with $\otimes_{i=1}^s L_i^{r_i} \otimes \mathcal{O}_{\mathbb{P}^1}(\beta_R)$, where $\beta_R := \sum_{i=1}^s r_i f_i$. We denote the line bundle $\otimes_{i=1}^s L_i^{r_i}$ by $\mathcal{R}$. Let $\pi : F_0 \times \mathbb{P}^1 \to F_0$ be the projection. When $\beta_R \geq 0$, we have

$$\begin{aligned} \mathcal{D}^{R,l} &= \det{}^{-l} R\pi_*(\mathcal{R} \otimes \mathcal{O}_{\mathbb{P}^1}(\beta_R)) \\ &= \det{}^{-l}(\mathcal{R} \otimes R^0\pi_*(\mathcal{O}_{\mathbb{P}^1}(\beta_R))) \\ &= \mathcal{R}^{-l(\beta_R+1)} \otimes \mathbb{C}_{l\beta_R(\beta_R+1)/2}. \end{aligned} \tag{4.17}$$

When $\beta_R < 0$, a similar calculation shows that we have the same formula $\mathcal{D}^{R,l} = \mathcal{R}^{-l(\beta_R+1)} \otimes \mathbb{C}_{l\beta_R(\beta_R+1)/2}$.

We give the explicit formulas of the (torus-equivariant) small I-functions of toric varieties in the following proposition.

Proposition IV.17. *The small I-function of a toric variety X_Σ of level l and character R is given by*

$$I^{R,l}(q) = 1 + \sum_{\beta \in \mathrm{Eff}(X)} Q^\beta \, \mathcal{R}^{-l\beta_R} \, q^{l\beta_R(\beta_R+1)/2} \prod_{\rho \in \Sigma(1)} \frac{\prod_{j=-\infty}^{0}(1 - \mathcal{O}(-D_\rho)q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1 - \mathcal{O}(-D_\rho)q^j)},$$

and its equivariant version is given by

$$I^{R,l,eq}(q) = 1 + \sum_{\beta \in \mathrm{Eff}(X)} Q^\beta \, \tilde{\mathcal{R}}^{-l\beta_R} \, q^{l\beta_R(\beta_R+1)/2} \prod_{\rho \in \Sigma(1)} \frac{\prod_{j=-\infty}^{0}(1 - U_\rho \, q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1 - U_\rho \, q^j)}.$$

Here $\mathcal{R} := \otimes_{i=1}^{s} L_i^{r_i}$ is the line bundle associated to the character R, and $\tilde{\mathcal{R}} = \otimes_{i=1}^{s} P_i^{r_i}$ and U_ρ are the equivariant line bundles corresponding to $\mathcal{R}$ and $\mathcal{O}(-D_\rho)$, respectively.

Proof. The proposition follows easily from (4.16) and (4.17). Note that one factor $\mathcal{R}^{-l}$ in (4.17) disappears due to the change of pairings (see Convention IV.4). □

By extending Givental's localization argument in [33] to the setting with level structure, we prove a toric mirror theorem. Now let us restate Theorem I.2.

Theorem IV.18. *Assume that X_Σ is a smooth quasi-projective toric variety. Let $I^{R,l,eq}(q)$ be the level-l torus-equivariant small I-function given in Proposition IV.17. Then the series $(1-q)I^{R,l,eq}(q)$ lies on the cone $\mathcal{L}_{S_\infty}^{R,l,eq}$ in the symmetrized torus-equivariant quantum K-theory of level l of X_Σ.*

Proof. For simplicity, we denote by I the I-function $I^{R,l,eq}$ of X_Σ. Let $\{\phi_\alpha\}_{\alpha\in X_\Sigma^{T^m}}$ be the fixed point basis of $K^0_{T^m}(X_\Sigma)$ and let $\{\phi^\alpha\}$ be the dual basis with respect to the pairing (3.2). For each fixed point α, we denote by $J(\alpha)\subset\Sigma(1)$ the cardinality-s subset such that α equals the intersection $\cap_{\rho\notin J(\alpha)}D_\rho$. Write $I=\sum_\alpha I^{(\alpha)}\phi_\alpha$. We denote by $U_\rho(\alpha)$ and $\mathcal{R}(\alpha)$ the restrictions of U_ρ and $\mathcal{R}$ to the fixed point α, respectively. For $\rho\in J(\alpha)$, we have $U_\rho(\alpha)=1$. Hence $I^{(\alpha)}$ can be explicitly written as

$$I^{(\alpha)}(q)=1+\sum_{\beta\in\mathrm{Eff}'(X_\Sigma)}Q^\beta\frac{\mathcal{R}(\alpha)^{-l\beta_R}\,q^{l\beta_R(\beta_R+1)/2}}{\prod_{\rho\in J(\alpha)}\prod_{j=1}^{\beta_\rho}(1-q^j)}\prod_{\rho\notin J(\alpha)}\frac{\prod_{j=-\infty}^{0}(1-U_\rho(\alpha)\,q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1-U_\rho(\alpha)\,q^j)}.$$

Here $\mathrm{Eff}'(X_\Sigma)$ denotes the semigroup of effective curve classes β such that $\beta_\rho\geq 0$. The terms with $\beta_\rho<0$ disappear because there is a factor $(1-q^0)$ in the numerators.

We first observe that for the point target, the cone $\mathcal{L}^{pt,R,l}_{S_\infty}$ of level l coincides with the cone $\mathcal{L}^{pt}_\infty$ of level 0. This is because in the case of the point target, the determinant line bundle $\mathcal{D}^{R,l}$ is always topologically trivial (with possible equivariant weights). Combining this observation with the fact that the level structure $\mathcal{D}^{R,l}$ splits "nicely" among nodal strata, we can extend the argument in [30] to prove that a point $f=\sum_\alpha f^{(\alpha)}\phi_\alpha$ lies on $\mathcal{L}^{R,l,eq}_{S_\infty}$ if and only iff the following are satisfied:

1. When expanded as meromorphic functions with poles only at roots of unity, $f^{(\alpha)}$ lie on the cone $\mathcal{L}^{pt}_{S_\infty}$ in the permutation-equivariant quantum K-theory of the point target space.
2. Away from $q=0$, ∞, and roots of unity, $f^{(\alpha)}$ may have at most simple poles at $q=U_\rho(\alpha)^{-1/m}$, $\rho\notin J(\alpha)$, $m=1,2,\ldots$, for generic values of $\Lambda_1,\ldots,\Lambda_m$. The

residues satisfy the following recursion relations

$$\mathrm{Res}_{q=U_\rho(\alpha)^{-1/m}} f^{(\alpha)}(q)\frac{dq}{q} = -\frac{\phi^\alpha Q^{md_{\alpha\rho}}}{C_{\alpha\rho}(m)} f^{(\rho)}\big(U_\rho(\alpha)^{-1/m}\big).$$

Here $C_{\alpha\rho}(m) = \lambda_{-1}(T_p\,\overline{\mathcal{M}}_{0,2}(X_\Sigma, md_{\alpha\rho})) \cdot \big(\mathcal{R}(\alpha)^{-md^R_{\alpha\rho}}\, q^{md^R_{\alpha\rho}(md^R_{\alpha\rho}+1)/2}\big)^l$, where

(a) T_p denotes the virtual tangent space to the moduli space at the point p represented by the m-multiple cover of the one-dimensional orbit connecting α and ρ. The explicit formula of the equivariant weights of the K-theoretic Euler class $\lambda_{-1}(T_p)$ is given in [33].

(b) $d_{\alpha\rho}$ denotes the degree of the one-dimensional orbit connecting α and ρ.

(c) $d^R_{\alpha\rho} := \langle d_{\alpha\rho}, c_1(\mathcal{R})\rangle$.

We want to show that $(1-q)I$ satisfies (1) and (2). It is proved in the main theorem[2] of [33] that the series

$$\widetilde{I}^{(\alpha)} = 1 + \sum_{\beta\in \mathrm{Eff}'(X_\Sigma)} \frac{Q^\beta}{\prod_{\rho\in J(\alpha)}\prod_{j=1}^{\beta_\rho}(1-q^j)} \prod_{\rho\notin J(\alpha)} \frac{\prod_{j=-\infty}^{0}(1-U_\rho(\alpha)\,q^j)}{\prod_{j=-\infty}^{\beta_\rho}(1-U_\rho(\alpha)\,q^j)}$$

represents a value of $\mathcal{J}^{pt}_{S_\infty}(\mathbf{t}(q), Q)/(1-q)$, i.e., $(1-q)\widetilde{I}^{(\alpha)}$ lies on the cone $\mathcal{L}^{pt}_{S_\infty}$. Note that $I^{(\alpha)}$ is obtained from $\widetilde{I}^{(\alpha)}$ by a "determinantal" modification. Therefore, it follows from Theorem I.1 that $(1-q)I^{(\alpha)}$ lies on $\mathcal{L}^{pt,R,l}_{S_\infty} = \mathcal{L}^{pt}_{S_\infty}$.

To prove $(1-q)I^{(\alpha)}$ satisfies the second condition, we rewrite $\mathcal{R}(\alpha)^{-l\beta_R}\, q^{l\beta_R(\beta_R+1)/2}$ as

$$\frac{\prod_{j=-\infty}^{\beta_R}\big(\mathcal{R}(\alpha)^{-1}\,q^j\big)^l}{\prod_{j=-\infty}^{0}\big(\mathcal{R}(\alpha)^{-1}\,q^j\big)^l}. \tag{4.18}$$

[2]In [33], the I-function is defined to sum over all $\beta \in \mathbb{Z}^m$. However, the same argument works if we restrict the summation to curve classes in the semigroup $\mathrm{Eff}(X_\Sigma)$.

Note that for all j, we have $\mathcal{R}(\alpha) = \mathcal{R}(\beta)\lambda^{-d^R_{\alpha\rho}}$, where $\lambda = U_\rho(\alpha)$. Hence at $q = \lambda^{-1/m}$,

$$\mathcal{R}(\alpha)^{-1}q^j = \mathcal{R}(\beta)^{-1}\, q^{j-md^R_{\alpha\rho}}.$$

The formula (4.18) is equivalent to

$$\mathcal{R}(\alpha)^{-mld^R_{\alpha\rho}}\, q^{mld^R_{\alpha\rho}(md^R_{\alpha\rho}+1)/2} \frac{\prod_{j=-\infty}^{\beta_R - md^R_{\alpha\rho}} \left(\mathcal{R}(\beta)^{-1}\, q^j\right)^l}{\prod_{j=-\infty}^{0} \left(\mathcal{R}(\beta)^{-1}\, q^j\right)^l}$$

at $q = \lambda^{-1/m}$. Combing the above equivalence with the result on page 10 of [33], we obtain

$$(1-q)I^{(\alpha)}(q) = \frac{Q^{md_{\alpha\rho}}}{1-q^m\lambda}\, \frac{\phi^\alpha}{C_{\alpha\rho}(m)}(1-q)I^{(\rho)}(q),$$

which is equivalent to the residue formula in Condition (2).

Since I is defined over the λ-algebra $\mathbb{Z}[\Lambda_1^\pm, \ldots, \Lambda_m^\pm][[Q]]$, it takes value in the symmetrized theory (see Remark III.3). □

When X_Σ is projective, we may pass to the non-equivariant limit in Theorem IV.18 to obtain

Corollary IV.19. *When X_Σ is a smooth projective toric variety, the level-l small I-function $I^{R,l}$ given in Proposition IV.17 lies on the cone $\mathcal{L}^{R,l}_{S_\infty}$ in the symmetrized quantum K-theory of level l.*

We denote by St and $\mathrm{St}^\vee$ the standard representation and its dual representation of $\mathbb{C}^*$. As corollaries of Theorem IV.17, we give proofs for Proposition I.3-I.5 and I.6.

Proof of Proposition I.3. Let the target be $X = (\mathbb{C}\backslash 0)/\mathbb{C}^*$ where the action is the standard action. Then the Proposition follows directly from the Theorem IV.17. $\square$

Proof of Proposition I.4. Let the target be $X_{a_1,a_2} = (\mathbb{C}^2\backslash\{(0,0)\})/\mathbb{C}^*$ with charge vector (a_1, a_2). As mentioned in Remark IV.14 and Remark IV.16, we only consider the untwisted component of the orbifold I-function and its formula is given by Theorem IV.17. $\square$

Proof of Proposition I.5. For positive integers a, b, we consider $X_{a,-b} = \{(\mathbb{C} - 0) \times \mathbb{C}\}/\mathbb{C}^*$ with charge vector $(a, -b)$. Let λ, μ be the generators of $\mathrm{Repr}((\mathbb{C}^*)^2)$ corresponding to the first and second projections of $(\mathbb{C}^*)^2$ onto its factors. From Theorem IV.17, the untwisted component of the orbifold I-function is given by

$$
\begin{aligned}
&I^{\mathrm{St},\,l}_{X_{a,-b}}(q)\\
=&1 + \sum_{n\geq 1} \frac{p^{nl} q^{\frac{n(n-1)l}{2}} (1-p^{-b}\mu^{-1})(1-p^{-b}\mu^{-1}q^{-1})\cdots(1-p^{-b}\mu^{-1}q^{1-bn})}{(1-p^{a}\lambda^{-1}q)(1-p^{a}\lambda^{-1}q^{2})\cdots(1-p^{a}\lambda^{-1}q^{an})} Q^n\\
=&1 + \sum_{n\geq 1} (-1)^{bn} \frac{p^{nl-b^2 n} q^{\frac{n(n-1)l-bn(bn-1)}{2}} \mu^{-bn}(1-p^{b}\mu)(1-p^{b}\mu q^{1})\cdots(1-p^{b}\mu q^{bn-1})}{(1-p^{a}\lambda^{-1}q)(1-p^{a}\lambda^{-1}q^{2})\cdots(1-p^{a}\lambda^{-1}q^{an})} Q^n.
\end{aligned}
$$

$\square$

Proof of Proposition I.6. We consider the target $O(-1)^{\oplus r}_{\mathbb{P}^{s-1}} = X_{\mathbf{1},-\mathbf{1}} = \{(\mathbb{C}^s - 0) \times \mathbb{C}^r\}/\mathbb{C}^*$ with the charge vector $(1, 1, \cdots, 1, -1, -1, \cdots, -1)$. It follows from Theorem

IV.17 that

$$
\begin{aligned}
&I^{\mathrm{St},\, l=1+s}_{X_{\mathbf{1},\mathbf{-1}}}(q)\\
=&1+\sum_{n\geq 1} Q^n p^{nl} q^{\frac{n(n-1)l}{2}} \frac{(p^{-1}\mu_1^{-1},q)_n\cdots(p^{-1}\mu_r^{-1};q)_n}{(p\lambda_1^{-1}q;q)_n\cdots(p\lambda_s^{-1}q;q)_n}\\
=&1+\sum_{n\geq 1}(-1)^{nr}\prod_{i=1}^{r}(p\mu_i)^{-n}p^{(1+s)n}\frac{(p\mu_1,q)_n\cdots(p\mu_r;q)_n}{(p\lambda_1^{-1}q;q)_n\cdots(p\lambda_s^{-1}q;q)_n}Q^n(q^{\frac{n(n-1)}{2}})^{1+s-r}.
\end{aligned}
$$

□

CHAPTER V

The GLSM of Grassmannians and wall-crossing

The Grassmannian can be expressed as a geometric invariant theory (GIT) quotient $M_{n\times N} /\!\!/ \mathrm{GL}_n(\mathbb{C})$, where $M_{n\times N}$ denotes the vector space of $n \times N$ complex matrices. For any GIT quotient, we can construct a gauged linear sigma model (GLSM) which recovers the nonlinear sigma model (physical counterpart of GW-theory) at one of its limit. In Witten's physical argument, he obtained the gauged WZW model (physical counterpart of Verlinde's theory) at another limit. A mathematical theory of the GLSM has been constructed by Fan-Jarvis-Ruan [25] where the parameter in the GLSM is interpreted as stability parameter ϵ. Recently, Choi-Kiem [16] introduces several more stability parameters for abelian gauge group. To simplify the notation, we postpone the introduction of parabolic structures to the next chapter. Throughout the rest of the chapter, we fix a smooth curve C of genus $g \geq 2$ and a marked point $x_0 \in C$.

5.1 The GLSM of Grassmannians and its stability conditions

The GIT description of the Grassmannian gives rise to a moduli problem of the GLSM data

$$\big(C', x_0', E, s \in H^0(E' \otimes \mathcal{O}_{C'}^N), \varphi\big)$$

where C' is a genus g (possibly) nodal curve, E is a vector bundle of rank n and degree d on C', and $\varphi : C' \to C$ is a morphism of degree one (i.e., $\varphi([C']) = [C]$) such that $\varphi(x_0') = x_0$. A point $x \in C'$ is called a *base point* if the N sections s do not span the fiber of E at x.

To obtain proper Deligne-Mumford stacks, we need to impose certain stability conditions on the GLSM data. There are several choices and we focus on two of them: ϵ-stability and δ-stability. Roughly speaking, ϵ-stability condition is imposed on the N sections s and δ-stability is imposed on the bundle E.

5.1.1 ϵ-stability

In general, one can impose the ϵ-stability condition on the GLSM data for any $\epsilon \in \mathbb{Q}_+$. In this paper we are only interested in two cases: $\epsilon = \infty$ and $\epsilon = 0^+$.

Definition V.1. (1) The data $(C', x_0', E, s, \varphi)$ is called $(\epsilon = \infty)$-stable if s defines a stable map into $\mathrm{Gr}(n, N) = M_{n\times N} /\!\!/ \mathrm{GL}_n(\mathbb{C})$. Namely, s has no base point.

(2) The data (C', E, s, φ) is called $(\epsilon = 0^+)$-stable if s defines a stable quotient into $\mathrm{Gr}(n, N) = M_{n\times N} /\!\!/ \mathrm{GL}_n(\mathbb{C})$. Namely, (i) the base point of s is different from nodes and the marked point x_0; (ii) there is no rational tail (an irreducible component

which is of genus zero and has exactly one node on them); (iii) the degrees of E on rational bridges (irreducible components which are rational and carries exactly two special points, markings or nodes) are positive.

For $\epsilon = 0+$ or ∞, we denote by $\overline{\mathcal{M}}^{\epsilon}_{C}(\mathrm{Gr}(n,N),d)$ the moduli space of ϵ-stable GLSM data $(C', x'_0, E, s, \varphi)$. According to [67], the moduli stack $\overline{\mathcal{M}}^{\epsilon}_{C}(\mathrm{Gr}(n,N),d)$ is a proper Deligne-Mumford stack with a canonical perfect obstruction theory. Hence, it admits a virtual structure sheave $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}^{\epsilon}_{C}(\mathrm{Gr}(n,N),d)} \in K_0(\overline{\mathcal{M}}^{\epsilon}_{C}(\mathrm{Gr}(n,N),d))$.

In this thesis, we focus on δ-wall-crossings, which will be introduced in the next subsection. The study of the relation between $(\delta = \infty)$-invariants and $(\epsilon = 0+)$-invariants, and the relation between $(\epsilon = 0+)$-invariants and $(\epsilon = \infty)$-invariants is work in progress [67].

5.1.2 δ-stability

It turns out that the GLSM with δ-stability has been studied much early under the name of stable pairs. Its moduli space is constructed using geometric invariant theory (GIT). In the following discussion, we only consider the fixed marked curve (C, x_0).

Suppose that F is a vector bundle on C. The rank and degree of F are denoted by $r(F)$ and $d(F)$, respectively. We define the slope of F as $\mu(F) := d(F)/r(F)$. Recall the definition of Bradlow N-pairs and its stability conditions.

Definition V.2. [11] A Bradlow N-pair (E, s) consists of a vector bundle E of rank

n and degree d over C, together with N sections $s \neq 0 \in H^0(E \otimes \mathcal{O}_C^N)$. A sub-pair

$$(E', s') \subset (E, s),$$

consists of a subbundle $\iota : E' \hookrightarrow E$ and N sections $s' : \mathcal{O}_C^N \to E'$ such that

$$\iota \circ s' = s \quad s \in \mathrm{H}^0(E' \otimes \mathcal{O}_C^N), \quad \text{and}$$
$$s' = 0 \quad s \notin \mathrm{H}^0(E' \otimes \mathcal{O}_C^N).$$

A quotient pair (E'', s'') consists of a quotient bundle $q : E \to E''$ with $s'' = q \circ s$.

We will focus on the case $N \geq n$. The slope of an N-pair (E, s) is defined by

$$\mu(E, s) = \mu(E) + \frac{\theta(s)\delta}{r(E)},$$

where $\theta(s) = 1$ if $s \neq 0$ and 0 otherwise.

Definition V.3. Let $\delta \in \mathbb{Q}_+$. A Bradlow N-pair of degree d is δ-semistable if for all nonzero sub-pairs $(E', s') \subsetneq (E, s)$, we have

$$\mu(E', s') \leq \mu(E, s).$$

An N-pair (E, s) is δ-stable if the above inequality is strict.

Lemma V.4. *Suppose $\phi : (E_1, s_1) \to (E_2, s_2)$ is a nonzero morphism of δ-semistable pairs. Then $\mu(E_1, s_1) \leq \mu(E_2, s_2)$. Furthermore, if (E_1, s_1) and (E_2, s_2) are δ-stable pairs with the same slope, then ϕ is an isomorphism. In particular, for a δ-stable pair (E, s) with $s \neq 0$, there are no endomorphisms of E preserving s except the identity, and no endomorphisms of E annihilating s except 0.*

Proof. The proof is standard (cf. [54, Lemma 7]), and we omit the details. □

Lemma V.5. *Let (E, s) be a δ-semistable parabolic N-pair of rank n and degree d. Assume that $\mu(E, s) > 2g - 1 + \delta$. Then $H^1(E) = 0$ and E is globally generated, i.e., the morphism*

$$H^0(E) \otimes \mathcal{O}_C \to E$$

is surjective.

Proof. It suffices to show that $H^1(E(-p)) = 0$ for any point $p \in E$. Indeed, if $H^1(E(-p)) = 0$, the lemma follows from the long exact sequence of cohomology groups for the short exact sequence:

$$0 \to E(-p) \to E \to E_p \to 0.$$

Now suppose $H^1(E(-p)) \neq 0$. By Serre duality, we have $H^1(E(-p)) = (H^0(E^\vee \otimes \omega_C(p)))^\vee$, where ω_C is the cotangent sheaf of C. Therefore a nonzero element in $H^1(E(-p))$ induces a nonzero morphism $\phi : E \to \omega_C(p)$. Let L be the image sheaf of ϕ. Since L is a subsheaf of $\omega_C(p)$, we have $d(L) \leq 2g - 1$. Let s'' be the induced N sections of L. It follows that $\mu(E, s) > 2g - 1 + \delta \geq d(L) + \theta(s'')\delta$, contradicting the δ-semistability of (E, s). □

The stability parameter δ is called *generic* if there is no strictly δ-semistable N-pair. Otherwise, δ is called *critical*. We also refer to the critical values of δ as *walls*. An N-pair (E, s) is called *non-degenerate* if $s \neq 0$. For a generic δ, the moduli space of non-degenerate δ-stable N pairs $\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d)$ can be constructed using GIT (see [74, §8] and [54]). Furthermore, there exists a universal N-pair

$$S : \mathcal{O}^N_{\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d)\times C} \to \mathcal{E}$$

over the universal curve $\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d)\times C$.

Example V.6. According to [11, Proposition 3.14][1], if $\delta > (n-1)d$, all δ-semistable pairs (E,s) are δ-stable and the stability condition is equivalent to having the N sections s generically generating E. In other words, the moduli space of δ-stable pairs is the Grothendieck's Quot scheme when δ is sufficiently large. In this case, we denote it by $\overline{\mathcal{M}}_C^{\infty}(\mathrm{Gr}(n,N),d)$. The Grothendieck's Quot scheme $\overline{\mathcal{M}}_C^{\infty}(\mathrm{Gr}(n,N),d)$ is a fine moduli space for the functor that assigns to each scheme T the set of equivalent morphisms $S : \mathcal{O}^N_{C\times T} \to \tilde{E}$ such that $\tilde{E}$ is locally free, for every closed point x of T, the restriction $\tilde{E}|_{C\times\{x\}}$ has rank n and degree d, and the restriction of the morphism $S|_{C\times\{x\}}$ is surjective at all but a finite number of points.

A standard argument in deformation theory (cf. [54, §5]) shows that the Zariski tangent space of $\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d)$ is isomorphic to the hypercohomology

$$\mathbb{H}^1(\mathrm{End}(E) \to E\otimes \mathcal{O}_C^N).$$

For simplicity, we denote the i-th hypercohomogy of the complex $\mathrm{End}(E) \to E\otimes\mathcal{O}_C^N$

[1]The stability parameter τ in [11] is related to δ by $d+\delta = n\tau$.

by $\mathbb{H}^{i-1}$, for $i = 0, 1, 2$. We have the following long exact sequence:

$$0 \to \mathbb{H}^{-1} \to H^0(\mathrm{End}(E)) \to (H^0(E))^N \to \mathbb{H}^0$$

$$\to H^1(\mathrm{End}(E)) \to (H^1(E))^N \to \mathbb{H}^1 \to 0.$$

If (E, s) is δ-stable, then by Lemma V.4, the map $H^0(\mathrm{End}(E)) \to (H^0(E))^N$ is injective. Therefore $\mathbb{H}^{-1} = 0$. In general, the hypercohomology group $\mathbb{H}^1$ is not zero, and hence the moduli space is not smooth. Nevertheless, we can still show that it is virtually smooth. The following proposition is a special case of Proposition VI.35.

Proposition V.7. *For a generic value of $\delta \in \mathbb{Q}_+$, the moduli space of non-degenerate δ-stable N-pairs $\overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n, N), d)$ has a perfect obstruction theory.*

The following corollary follows from Proposition V.7 and the construction in [53, §2.3].

Corollary V.8. *There exists a virtual structure sheaf*

$$\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n,N),d)} \in K_0(\overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n, N), d)).$$

for the moduli space of δ-stable N-pairs $\overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n, N), d)$

When there is no confusion, we will simply denote the virtual structure sheaf of $\overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n, N), d)$ by $\mathcal{O}^{\mathrm{vir}}$.

Let $\pi : \overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n, N), d) \times C \to \overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n, N), d)$ be the projection map and let $\mathcal{E}$ be the universal bundle over $\overline{\mathcal{M}}_C^\delta(\mathrm{Gr}(n, N), d) \times C$. Consider the derived pushforward $R\pi_*(\mathcal{E}) = [R^0\pi_*(\mathcal{E}) \to R^1\pi_*(\mathcal{E})]$. A two-term locally free resolution of $R\pi_*(\mathcal{E})$ can be easily obtained as follows. Let $O(1)$ be an ample line bundle on C. Since the family of δ-stable N-pairs is bounded, there exists a surjection

$$B \to \mathcal{E}(m) \to 0,$$

for $m \gg 0$. Here B is a trivial vector bundle. The kernel, denoted by A, is also a vector bundle on $\overline{\mathcal{M}}^{\delta}_{C}(\mathrm{Gr}(n,N),d) \times C$, and we have a short exact sequence

$$0 \to A(-m) \to B(-m) \to \mathcal{E} \to 0.$$

Note that $R^0\pi_*(A(-m)) = R^0\pi_*(B(-m)) = 0$. Therefore, the following two-term complex of vector bundles

$$R^1\pi_*(A(-m)) \to R^1\pi_*(B(-m))$$

is a resolution of $R\pi_*(\mathcal{E})$.

Denote the rank of $R^1\pi_*(A(-m))$ and $R^1\pi_*(B(-m))$ by r_A and r_B, respectively. Recall from Section 2.1 that we define the inverse determinant line bundle of cohomology by

$$\big(\det R\pi_*(\mathcal{E})\big)^{-1} := \bigwedge^{r_B} R^1\pi_*(B(-m)) \otimes \big(\bigwedge^{r_A} R^1\pi_*(A(-m))\big)^{-1}.$$

This line bundle does not depend on the choice of the locally free resolutions of $R\pi_*(\mathcal{E})$. As before, we refer to it as the level structure.

Let E denote the dual of the tautological bundle on $\mathrm{Gr}(n,N)$. The following definition is motivated by Corollary VI.46.

Definition V.9. Let $e = l(1-g) + ld/n$. If e is an integer, we define the level-l K-theoretic δ-stable N-pair invariant by

$$\langle \det(E)^e \rangle_{C,d}^{l,\delta,\mathrm{Gr}(n,N)} = \chi\big(\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d), \mathcal{O}^{\mathrm{vir}} \otimes \det(\mathcal{E}_{x_0})^e \otimes \big(\det R\pi_*(\mathcal{E})\big)^{-l}\big),$$

where $\mathcal{E}_{x_0} = \mathcal{E}|_{\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d)\times\{x_0\}}$ denotes the restriction. If e is not an integer, we define $\langle \det(E)^e \rangle_{C,d}^{l,\delta,\mathrm{Gr}(n,N)}$ to be zero.

5.2 Rank-two δ-wall-crossing in the absence of parabolic structures

In this section, we prove Theorem V.11, which is the special case of Theorem I.10 in the absence of parabolic structures. The stability parameter δ is a critical value if $(d-\delta)/2 \in \mathbb{N}_+$. We study how the moduli space $\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d)$ changes when the stability parameter δ crosses a wall and prove a wall-crossing theorem.

Let i be a half-integer such that $\delta = 2i$ is a critical value. Note that $i \in (0, d/2)$. A δ-semistable vector bundle must split $E = L \oplus M$ where L, M are line bundles of degrees $d/2 - i$ and $d/2 + i$, respectively, and $s \in H^0(L \otimes \mathcal{O}_C^N)$. Let $\nu > 0$ be a small real number such that $2i$ is the only critical value in $(2i-\nu, 2i+\nu)$. For simplicity, we denote by $\mathcal{M}_{i,d}^{\pm}$ the moduli spaces $\overline{\mathcal{M}}_C^{2i\pm\nu}(\mathrm{Gr}(2,N),d)$. Let $\mathcal{W}_{i,d}^{+}$ be the subscheme of $\mathcal{M}_{i,d}^{+}$ parametrizing $(2i+\nu)$-pairs which are not $(2i-\nu)$-stable. Similarly, we denote by $\mathcal{W}_{i,d}^{-}$ the subscheme of $\mathcal{M}_{i,d}^{-}$ which parametrizes $(2i-\nu)$-pairs which are not $(2i+\nu)$-stable. The subschemes $\mathcal{W}_{i,d}^{\pm}$ are called the *flip loci*.

Let (E, s) be an N-pair in $\mathcal{W}_{i,d}^{-}$. It follows from the definition that there exists a short exact sequence

$$0 \to L \to E \to M \to 0,$$

where L, M are line bundles of degree $d/2 - i$ and $d/2 + i$, respectively, and $s \in H^0(L \otimes \mathcal{O}_C^N)$ (cf. [74, §8]). Notice that L and M are unique since L is the saturated subsheaf of E containing s. Similarly, for a pair (E, s) in $\mathcal{W}_{i,d}^+$. There exists a unique subline bundle M of E of degree $d/2 + i$ which fits into a short exact sequence:

$$0 \to M \to E \to L \to 0.$$

Let $\tilde{\mathcal{L}}$ be a Poincaré bundle over $\mathrm{Pic}^{d/2-i}C \times C$ and let $p : \mathrm{Pic}^{d/2-i}C \times C \to \mathrm{Pic}^{d/2-i}C$ be the projection. If $d/2 - i > 2g - 1$, then we have $R^1 p_* \tilde{\mathcal{L}} = 0$. Hence $U := (R^0 p_* \tilde{\mathcal{L}})^N$ is a vector bundle. We define $Z_{i,d} := \mathbb{P}U \times \mathrm{Pic}^{d/2+i}C$. Let $\mathcal{M}$ be a Poincaré bundle over $\mathrm{Pic}^{d/2+i}C \times C$. Notice that $H^0(\mathrm{Pic}^{d/2-i}C, \mathrm{End}\, U) = H^0(\mathrm{Pic}^{d/2-i}C \times C, U^\vee \otimes \tilde{\mathcal{L}} \otimes \mathcal{O}^N) = H^0(\mathbb{P}U \times C, \mathcal{O}_{\mathbb{P}U}(1) \otimes \tilde{\mathcal{L}} \otimes \mathcal{O}^N)$. Therefore there exists a tautological section of $\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}U}^N$, where $\mathcal{L} := \mathcal{O}_{\mathbb{P}U}(1) \otimes \tilde{\mathcal{L}}$. This tautological section induces an injection $\alpha : \mathcal{M}\mathcal{L}^{-1} \to \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}U}^N$. We denote by $\mathcal{F}$ the cokernel of α. By abuse of notation, we use the same notations $\mathcal{M}$ and $\mathcal{L}$ to denote the pullbacks of the corresponding universal line bundles to $Z_{i,d} \times C$. Let $\pi : Z_{i,d} \times C \to Z_{i,d}$ be the projection. The flip loci $\mathcal{W}_{i,d}^\pm$ are characterized by the following proposition.

Proposition V.10 ([11, 74]). *Assume $d/2 - i > 2g - 1$. Let $\mathcal{V}_{i,d}^+ = R^0\pi_*(\mathcal{F})$ and $\mathcal{V}_{i,d}^- = R^1\pi_*(\mathcal{M}^{-1}\mathcal{L})$. Then we have*

$$\mathcal{W}_{i,d}^\pm \cong \mathbb{P}(\mathcal{V}_{i,d}^\pm).$$

Let $q_\pm : \mathcal{W}^\pm_{i,d} \to Z_{i,d}$ *be the projective bundle maps. Then the morphisms* $\mathcal{W}^\pm_{i,d} \to \mathcal{M}^\pm_{i,d}$ *are regular embeddings with normal bundles* $q^*_\pm \mathcal{V}^\mp_{i,d} \otimes \mathcal{O}_{\mathcal{W}^\pm_{i,d}}(-1)$*. Moreover, we have the following two short exact sequences of universal bundles:*

$$0 \to \tilde{q}^*_- \mathcal{L} \to \mathcal{E}^-_i|_{\mathcal{W}^-_{i,d}\times C} \to \tilde{q}^*_- \mathcal{M} \otimes \mathcal{O}_{\mathcal{W}^-_{i,d}}(-1) \to 0, \tag{5.1}$$

$$0 \to \tilde{q}^*_+ \mathcal{M} \otimes \mathcal{O}_{\mathcal{W}^+_{i,d}}(1) \to \mathcal{E}^+_i|_{\mathcal{W}^+_{i,d}\times C} \to \tilde{q}^*_+ \mathcal{L} \to 0, \tag{5.2}$$

where $\mathcal{E}^\pm_i$ *are the universal bundles over* $\mathcal{M}^\pm_{i,d}$ *and* $\tilde{q}_\pm : \mathcal{W}^\pm_{i,d} \times C \to Z_{i,d} \times C$ *are the projective bundle maps.*

Theorem V.11. *Suppose that* $N \geq 2 + l$, $d > 2(g-1)$ *and* δ *is generic. Then* $\langle \det(E)^e \rangle^{l,\delta,\mathrm{Gr}(2,N)}_{C,d}$ *is independent of* δ.

By abuse of notation, we denote by π the projection maps $\mathcal{M}^\pm_{i,d} \times C \to \mathcal{M}^\pm_{i,d}$. To prove Theorem V.11, we need the following lemma.

Lemma V.12. *Let* $\mathcal{D}_{i,\pm} = \det(\mathcal{E}^\pm_{i,x_0})^e \otimes \left(\det R\pi_*(\mathcal{E}^\pm_i)\right)^{-l}$, *where* $\mathcal{E}^\pm_{i,x_0} = \mathcal{E}^\pm_i|_{\mathcal{M}^\pm_{i,d}\times\{x_0\}}$. *Then*

1. *the restriction of* $\mathcal{D}_{i,-}$ *to a fiber of* $\mathbb{P}(\mathcal{V}^-_{i,d})$ *is* $\mathcal{O}(il)$, *and*
2. *the restriction of* $\mathcal{D}_{i,+}$ *to a fiber of* $\mathbb{P}(\mathcal{V}^+_{i,d})$ *is* $\mathcal{O}(-il)$.

Proof. The lemma follows easily from the short exact sequences (5.1) and (5.2). □

Proof of Theorem V.11. We prove the claim by showing that the invariant does not change when δ crosses a critical value $2i$. The proof is divided into two cases:

Case 1. Assume that $d/2 - i > 2g - 1$. Then $\mathcal{M}_{i,d}^{\pm}$ are smooth. According to Theorem 3.44 of [11], we have the following diagram.

$$\begin{array}{ccc} & \widetilde{\mathcal{M}}_{i,d} & \\ {}^{p_-}\swarrow & & \searrow^{p_+} \\ \mathcal{M}_{i,d}^{-} & & \mathcal{M}_{i,d}^{+} \end{array}$$

where $p_{\pm}$ are blow-down maps onto the smooth subvarieties $\mathcal{W}_{i,d}^{\pm} \cong \mathbb{P}\big(\mathcal{V}_{i,d}^{\pm}\big)$, and the exceptional divisor $A_{i,d} \subset \widetilde{\mathcal{M}}_{i,d}$ is isomorphic to the fiber product $A_{i,d} \cong \mathbb{P}\big(\mathcal{V}_{i,d}^{-}\big) \times_{Z_{i,d}} \times \mathbb{P}\big(\mathcal{V}_{i,d}^{+}\big)$.

Since $p_{\pm}$ are blow-ups with smooth centers, we have $(q_{\pm})_*\big([\mathcal{O}_{\widetilde{\mathcal{M}}_{i,d}}]\big) = [\mathcal{O}_{\mathcal{M}_{i,d}^{\pm}}]$. Let $\mathcal{D}_{i,\pm}$ be the line bundles defined in Lemma V.12. It follows from the projection formula that

$$\chi(\mathcal{M}_{i,d}^{\pm}, \mathcal{D}_{i,\pm}) = \chi(\widetilde{\mathcal{M}}_{i,d}, p_{\pm}^*(\mathcal{D}_{i,\pm})). \tag{5.3}$$

We only need to compare $p_{\pm}^*(\mathcal{D}_{i,\pm})$ over $\widetilde{\mathcal{M}}_{i,d}$. Notice that the restriction of $\mathcal{O}_{A_{i,d}}(A_{i,d})$ to $A_{i,d}$ is $\mathcal{O}_{\mathbb{P}(\mathcal{V}_{i,d}^{+})}(-1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_{i,d}^{-})}(-1)$. Therefore, by Lemma V.12, we have

$$p_-^*(\mathcal{D}_{i,-}) = p_+^*(\mathcal{D}_{i,+})(-ilA_{i,d}).$$

For $1 \le j \le il$, we consider the following short exact sequence:

(5.4)

$$0 \to p_+^*(\mathcal{D}_{i,+})(-jA_{i,d}) \to p_+^*(\mathcal{D}_{i,+})(-(j-1)A_{i,d}) \to p_+^*(\mathcal{D}_{i,+}) \otimes \mathcal{O}_{A_{i,d}}(-(j-1)A_{i,d}) \to 0.$$

Define $\mathcal{L}_{i,d} := (\mathcal{M}_{x_0}^e \otimes (\det R\pi_*\mathcal{M})^{-l}) \otimes (\mathcal{L}_{x_0}^e \otimes (\det R\pi_*\mathcal{L})^{-l})$, where $\mathcal{M}_{x_0}$ and $\mathcal{L}_{x_0}$ denote the restrictions of $\mathcal{M}$ and $\mathcal{L}$ to $Z_{i,d} \times \{x_0\}$, respectively. Then by Lemma V.12, the restriction of $\mathcal{D}_{i,+}$ to $A_{i,d}$ is $\mathcal{L}_i \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{i,d})}(-li)$. By taking the Euler characteristic of (5.4), we obtain

$$\begin{aligned}&\chi\big(\widetilde{\mathcal{M}}_{i,d}, p_+^*(\mathcal{D}_{i,+})(-(j-1)A_{i,d})\big) - \chi\big(\widetilde{\mathcal{M}}_{i,d}, p_+^*(\mathcal{D}_{i,+})(-jA_{i,d})\big)\\ =&\chi\left(A_{i,d}, \mathcal{L}_{i,d} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{i,d})}(-li+j-1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^-_{i,d})}(j-1)\right) \quad \text{for } 1 \le j \le il.\end{aligned}$$

Let $n_+ = N(d/2+i+1-g)-2i-1+g$ be the rank of $\mathcal{V}^+_{i,d}$. A simple calculation shows that $n_+ > li$ when $l \le N-2$. Hence every term in the Leray spectral sequence of the fibration $\mathbb{P}^{n_+-1} \to A_{i,d} \to \mathbb{P}(\mathcal{V}^-_{i,d})$ vanishes, which implies that $\chi(\widetilde{\mathcal{M}}_{i,d}, p_-^*(\mathcal{D}_{i,-})) = \chi(\widetilde{\mathcal{M}}_{i,d}, p_+^*(\mathcal{D}_{i,+}))$ when $d/2-i > 2g-1$. This concludes the proof of the first case.

Case 2. When $d/2 - i \le 2g-1$, the moduli spaces $\mathcal{M}^{\pm}_{i,d}$ are singular. We can choose a divisor $D = x_1 + \cdots + x_k$ where $x_1, \ldots, x_k$ are distinct points on C, disjoint from $I \cup \{x_0\}$. Assume k is sufficiently large such that $d/2-i+k > 2g-1$. By Lemma VI.39, there are embeddings $\iota_D : \mathcal{M}^{\pm}_{i,d} \hookrightarrow \mathcal{M}^{\pm}_{i,d+2k}$. Let $\mathcal{E}_{\pm}$ and $\mathcal{E}'_{\pm}$ be the universal vector bundles on $\mathcal{M}^{\pm}_{i,d} \times C$ and $\mathcal{M}^{\pm}_{i,d+2k} \times C$, respectively. According to Proposition VI.40, we have $\iota_{D*}\big(\mathcal{O}^{\mathrm{vir}}_{\mathcal{M}^{\pm}_{i,d}}\big) = \lambda_{-1}(((\mathcal{E}'^{\vee}_{\pm})_D)^N)$, where $(\mathcal{E}'^{\vee}_{\pm})_D$ denote the restrictions of the dual of $\mathcal{E}'_{\pm}$ to $\mathcal{M}^{\pm}_{i,d+2k} \times D$. Let $\mathcal{D}'_{i,\pm} = \det((\mathcal{E}'_{\pm})_{x_0})^e \otimes \det(R\pi_*(\mathcal{E}'_{\pm}))^{-l}$ be the determinant line bundle on $\mathcal{M}^{\pm}_{i,d+2k}$. According to Corollary VI.49, to show that $\chi\big(\mathcal{M}^-_{i,d}, \mathcal{D}_{i,-} \otimes \mathcal{O}^{\mathrm{vir}}_{\mathcal{M}^-_{i,d}}\big) = \chi\big(\mathcal{M}^+_{i,d}, \mathcal{D}_{i,+} \otimes \mathcal{O}^{\mathrm{vir}}_{\mathcal{M}^+_{i,d}}\big)$, it suffices to show that

$$\chi\big(\mathcal{M}^-_{i,d+2k}, \mathcal{D}'_{i,-} \otimes \lambda_{-1}(((\mathcal{E}'^{\vee}_{-})_D)^N)\big) = \chi\big(\mathcal{M}^+_{i,d+2k}, \mathcal{D}'_{i,+} \otimes \lambda_{-1}(((\mathcal{E}'^{\vee}_{+})_D)^N)\big).$$

By abuse of notation, we denote by $p_{\pm}$ the blow-down maps from $\widetilde{\mathcal{M}}_{i,d+2k}$ to $\mathcal{M}^{\pm}_{i,d+2k}$. Let $\tilde{p}_{\pm} : \widetilde{\mathcal{M}}_{i,d+2k} \times C \to \mathcal{M}^{\pm}_{i,d+2k} \times C$ be the base change of $p_{\pm}$. By a straightforward modification of the proof of [73, Proposition 3.17], one can show that $p_-^*(\mathcal{E}'_-)$ is an *elementary modification* of $p_-^*(\mathcal{E}_+)$ along the divisor $A_{i,d+2k}$. More precisely, we have the following short exact sequence

$$0 \to p_-^*(\mathcal{E}'_-) \to p_+^*(\mathcal{E}'_+) \otimes \mathcal{O}(A_{i,d+2k}) \to \iota_*\big(\mathcal{L} \otimes \iota^*(\mathcal{O}(A_{i,d+2k}))\big) \to 0, \tag{5.5}$$

over $\widetilde{\mathcal{M}}_{i,d+2k} \times C$. Here $\iota : A_{i,d+2k} \hookrightarrow \widetilde{\mathcal{M}}_{i,d+2k}$ is the embedding. Applying the functor $\mathcal{H}om(-,\mathcal{O})$ to (5.5), we obtain

$$0 \to p_+^*(\mathcal{E}'^\vee_+) \otimes \mathcal{O}(-A_{i,d+2k}) \to p_-^*(\mathcal{E}'^\vee_-) \to \iota_*\big(\mathcal{L}^\vee\big) \to 0. \tag{5.6}$$

Recall that $(\mathcal{E}'^\vee_\pm)_D = \bigoplus_{i=1}^k (\mathcal{E}'^\vee_\pm)_{x_i}$. Then it follows from (5.6) that

$$\begin{aligned}
p_-^*\big(\lambda_{-1}((\mathcal{E}'^\vee_-)_{x_i})\big) &= 1 - p_-^*\big((\mathcal{E}'^\vee_-)_{x_i}\big) + p_-^*\big(\det(\mathcal{E}'^\vee_-)_{x_i}\big) \\
&= 1 - p_+^*\big((\mathcal{E}'^\vee_+)_{x_i}\big) \otimes \mathcal{O}(-A_{i,d+2k}) - \iota_*\big(\mathcal{L}^\vee_{x_i}\big) \\
&+ p_+^*\big(\det(\mathcal{E}'^\vee_+)_{x_i}\big) \otimes \mathcal{O}(-A_{i,d+2k}) \\
&= 1 - \mathcal{O}(-A_{i,d+2k}) - \iota_*\big(\mathcal{L}^\vee_{x_i}\big) \\
&+ p_+^*\big(\lambda_{-1}((\mathcal{E}'^\vee_+)_{x_i})\big) \otimes \mathcal{O}(-A_{i,d+2k}) \\
&= \iota_*\big(1 - \mathcal{L}^\vee_{x_i}\big) + p_+^*\big(\lambda_{-1}((\mathcal{E}'^\vee_+)_{x_i})\big) \otimes \mathcal{O}(-A_{i,d+2k})
\end{aligned}$$

in $K^0(\widetilde{\mathcal{M}}_{i,d+2k})$. Notice that

$$p_+^*\big(\lambda_{-1}((\mathcal{E}'^\vee_+)_{x_i})\big) \otimes \mathcal{O}(-A_{i,d+2k}) = p_+^*\big(\lambda_{-1}((\mathcal{E}'^\vee_+)_{x_i})\big) - \iota_*\big(\iota^*(p_+^*(\lambda_{-1}((\mathcal{E}'^\vee_+)_{x_i})))\big). \tag{5.7}$$

Using the short exact sequence (5.2), we obtain the following equality in $K^0(\widetilde{\mathcal{M}}_{i,d+2k})$:

(5.8)

$$\iota^*\big(p_+^*(\lambda_{-1}((\mathcal{E}'^\vee_+)_{x_i}))\big) = 1 - \mathcal{M}^\vee_{x_i} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{i,d+2k})}(-1) - \mathcal{L}^\vee_{x_i} + \mathcal{M}^\vee_{x_i}\mathcal{L}^\vee_{x_i} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{i,d+2k})}(-1).$$

By combining (5.6), (5.7) and (5.8), we obtain

$$(5.9)\quad p_-^*\big(\lambda_{-1}((\mathcal{E}_-'^\vee)_{x_i})\big) = p_+^*\big(\lambda_{-1}((\mathcal{E}_+'^\vee)_{x_i})\big) + \iota_*\big(\mathcal{M}_{x_i}^\vee(1-\mathcal{L}_{x_i}^\vee)\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V}_{i,d+2k}^+)}(-1)\big).$$

By taking the N-th power of both sides of (5.9) and then taking the product of all $1 \leq i \leq k$, we get

$$(5.10)\qquad p_-^*\big(\lambda_{-1}((\mathcal{E}_-'^\vee)_D)^N\big) = p_+^*\big(\lambda_{-1}((\mathcal{E}_+'^\vee)_D)^N\big) + \iota_*(\alpha).$$

Here α is an explicit K-theory class of the form

$$\alpha = \sum_{m=1}^{kN} \alpha_m \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_{i,d+2k}^+)}(-m),$$

where α_m are explicit combinations of vector bundles whose restrictions to a fiber of $\mathbb{P}(\mathcal{V}_{i,d+2k}^+)$ are trivial. To obtain (5.10), one needs to use the excess intersection formula

$$\iota^*\iota_* F = F \otimes \big(1 - \mathcal{O}_{A_{i,d+2k}}(-A_{i,d+2k})\big) \qquad \text{for } F \in K^0(A_{i,d+2k}).$$

By Lemma V.12, we have $p_-^*(\mathcal{D}_{i,-}') = p_+^*((\mathcal{D}_{i,+}')(-ilA_{i,d+2k})$. Then it follows from the exact sequence (5.4) that

$$(5.11)\qquad p_-^*\big(\mathcal{D}_{i,-}'\big) = p_+^*\big(\mathcal{D}_{i,+}'\big) + \sum_{j=1}^{il} \iota_*\big(\beta_j \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_{i,d+2k}^+)}(-j)\big) \quad \text{in } K^0(\widetilde{\mathcal{M}}_{i,d+2k}).$$

Here $\beta_j = \mathcal{L}_{i,d+2k} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_{i,d}^-)}(il-j)$, whose restriction to a fiber of $\mathbb{P}(\mathcal{V}_{i,d+2k}^+)$ is trivial. By combining (5.10) and (5.11), we get

$$p_-^*\big(\mathcal{D}_{i,-}' \otimes \lambda_{-1}((\mathcal{E}_-'^\vee)_D)^N\big) = p_+^*\big(\mathcal{D}_{i,+}' \otimes \lambda_{-1}((\mathcal{E}_+'^\vee)_D)^N\big) + \sum_{j=1}^{kN+il} \iota_*\big(\gamma_j \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}_{i,d+2k}^+)}(-j)\big),$$

where the restrictions of $\gamma_j \in K^0(A_{i,d+2k})$ to a fiber of $\mathbb{P}(\mathcal{V}^+_{i,d+2k})$ are trivial.

The rest of the argument is similar to the one given in the proof of the first case. Let $n_+ = N(d/2+k+i+1-g)-2i-1+g$ be the rank of $\mathcal{V}^+_{i,d+2k}$. A simple calculation shows that $n_+ > il+kN$ when $l \leq N-2$ and $d > 2(g-1)$. For $1 \leq j \leq kN+il$, we have $\chi\big(\gamma_j \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{i,d+2k})}(-j)\big) = 0$ because every term in the Leray spectral sequence of the fibration $\mathbb{P}^{n_+-1} \to A_{i,d+2k} \to \mathbb{P}(\mathcal{V}^-_{i,d+2k})$ vanishes. This concludes the proof of the second case.

$\square$

CHAPTER VI

Parabolic structure and the general δ-wall-crossing

In this chapter, we introduce the parabolic structure to the GLSM. In this new setting, the parabolic structure can be viewed as K-theoretic insertions. An interesting aspect of this construction is that the parabolic structure intertwines with the stability condition.

6.1 Irreducible representations of $\mathfrak{gl}_n(\mathbb{C})$

In this section, we recall some basic facts about the representations of $\mathfrak{gl}_n(\mathbb{C})$.

Let $\mathfrak{gl}_n(\mathbb{C})$ be the *general linear Lie algebra* of all $n \times n$ complex matrices, with $[X, Y] = XY - YX$. We have the triangular decomposition

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-,$$

where $\mathfrak{h}$ is the Cartan subalgebra consisting of all diagonal matrices and $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) is the subalgebra of upper triangular (resp., lower triangular) matrices. Let $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ and let $\mathfrak{h}_0^*$ be the real subspace of $\mathfrak{h}^*$ generated by the roots of

$\mathfrak{gl}_n(\mathbb{C})$. We fix an isomorphism $\mathfrak{h}_0^* \cong \mathbb{R}^n$ such that the simple roots α_i can be expressed as

$$\alpha_i = e_i - e_{i+1}, \qquad \text{for } 1 \leq i \leq n-1.$$

Here $\{e_i\}$ is the standard basis of $\mathbb{R}^n$. The fundamental weights $\omega_i \in \mathfrak{h}_0^*$ are given by

$$\omega_i = e_1 + \cdots + e_i, \qquad \text{for } 1 \leq i \leq n.$$

Consider the set

$$\mathrm{P}_+ = \big\{\lambda = \sum_{i=1}^{n-1} m_i\omega_i + m_n\omega_n \mid m_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq n-1 \text{ and } m_n \in \mathbb{Z}\big\}.$$

An element λ in P_+ is called a *dominant weight*. A dominant weight λ can also be expressed in term of the standard basis $\{e_i\}$ as follows:

$$\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n,$$

where $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \cdots \geq \lambda_n$. In the following discussion, we will denote a dominant weight λ by the *partition* $(\lambda_1, \ldots, \lambda_n)$. If a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfies $\lambda_n \geq 0$, one can identify it with its *Young diagram*, i.e., a left-justified shape of n rows of boxes of length $\lambda_1, \ldots, \lambda_n$.

There is a bijection between the set P_+ of dominant weights and the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{gl}_n(\mathbb{C})$-modules. More precisely, for each dominant weight λ, one can assign a unique finite-dimensional irreducible $\mathfrak{gl}_n(\mathbb{C})$-modules V_λ. Here V_λ is generated by a unique vector v_λ (up to a scalar) with the properties $\mathfrak{n}^+.v_\lambda = 0$ and $H.v_\lambda = \lambda(H)v_\lambda$ for all $H \in \mathfrak{h}$. The $\mathfrak{gl}_n(\mathbb{C})$-module V_λ

is called the *highest weight module* with *highest weight* λ and the vector v_λ is called the *highest weight vector*. Given a $\mathfrak{gl}_n(\mathbb{C})$-module V, we denote its dual by $V^\vee$.

Fix a non-negative integer l. We denote by P_l the set of dominant weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that

$$l \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

To a partition $\lambda \in \mathrm{P}_l$, we associate the complement partition λ^* in P_l:

$$\lambda^* : l \geq l - \lambda_n \geq \cdots \geq l - \lambda_1 \geq 0.$$

Given a partition λ, we define $|\lambda| = \sum_{i=1}^n \lambda_i$, which is the total number of boxes in its Young diagram.

Now we recall a geometric construction of the highest weight $\mathfrak{gl}_N(\mathbb{C})$-modules. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ in P_l. Let $(r_1, \ldots, r_k)$ be the sequence of jumping indices of λ (i.e. $l \geq \lambda_1 = \cdots = \lambda_{r_1} > \lambda_{r_1+1} = \cdots = \lambda_{r_2} > \ldots$). We define a sequence of non-negative integers $a = (a_1, \ldots, a_k)$, where $a_j = l - \lambda_{r_j}$ for $1 \leq j \leq k$. For $1 \leq j \leq k$, we introduce positive integers

$$d_j = a_{j+1} - a_j.$$

Here a_{k+1} is defined to be l. Define a sequence $m = (m_1, \ldots, m_k)$, where $m_i = r_i - r_{i-1}$. We denote by Fl_m the flag variety which parametrizes all sequences

$$\mathbb{C}^n = V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_k \supsetneq V_{k+1} = 0,$$

where V_j are complex linear subspaces of $\mathbb{C}^n$ and $m_j = \dim V_j - \dim V_{j+1}$, for all $1 \leq j \leq k$. The k-tuple m is referred to as the *type* of the flag variety Fl_m. Let Q_j be the universal quotient bundle over Fl_m of rank $r_j = \sum_{i=1}^{j} m_i$, for $1 \le j \le k$. Notice that Q_k is the trivial bundle of rank n over Fl_m. We define the *Borel-Weil-Bott line bundle* L_λ of type λ by

$$L_\lambda = \bigotimes_{j=1}^{k} (\det Q_j)^{d_j}.$$

Lemma VI.1. *If λ is a dominant weight, then the following holds:*

1. $H^i(\mathrm{Fl}_m, L_\lambda) = 0$, *if $i > 0$.*
2. *The $\mathfrak{gl}_n(\mathbb{C})$-module $H^0(\mathrm{Fl}_m, L_\lambda)$ is isomorphic to $V_\lambda^\vee$.*

Proof. The proof is similar to that of [62, Proposition 6.3] and we briefly recall it here. We denote by Fl the complete flag variety parametrizing complete flags in $\mathbb{C}^n$. For $i = 1, \ldots, n$, we define $\tilde{d}_{m_j} = d_j$, and $\tilde{d}_i = 0$ if $i \ne m_1, \ldots, m_k$. Let $\tilde{Q}_i$ be the universal quotient bundle over Fl of rank i, for $1 \le i \le n$. According to the Borel-Weil-Bott theorem for $\mathfrak{gl}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$ (see, for example, [81, Chapter 4]), we have $H^i(\mathrm{Fl}, \otimes_{i=1}^{n} (\det \tilde{Q}_i)^{\tilde{d}_i}) = 0$, if $i > 0$, and the $\mathfrak{gl}_n(\mathbb{C})$-module $H^0(\mathrm{Fl}, \otimes_{i=1}^{n} (\det \tilde{Q}_i)^{\tilde{d}_i})$ is the dual of the highest weight representation V_λ. Consider the surjective flat morphism

$$h : \mathrm{Fl} \to \mathrm{Fl}_m.$$

For any point $x \in \mathrm{Fl}_m$, the fiber $h^{-1}(x)$ is a product of flag varieties. In particular, the fibers are smooth and connected. By [43, III 12.9], we have $h_*(\mathcal{O}_{\mathrm{Fl}}) = \mathcal{O}_{\mathrm{Fl}_m}$. Notice that the anticanonical line bundle of a product of flag varieties is ample. By the Kodaira vanishing theorem, we have

$$H^i(h^{-1}(x), \mathcal{O}_{h^{-1}(x)}) = 0, \quad \text{for any } x \in \mathrm{Fl}_m, \text{ and } i > 1.$$

The Grauert's theorem [43, III 12.9] implies that $R^i h_*(\mathcal{O}_{\mathrm{Fl}}) = 0$ for $i > 0$. The lemma follows from the projection formula and the following relation:

$$h^*\left(\bigotimes_{i=1}^{n}(\det \tilde{Q}_i)^{\tilde{d}_i}\right) = \bigotimes_{i=1}^{k}(\det Q_i^{d_i}).$$

□

6.2 Parabolic N-pairs and δ-stability

In this section, we generalize the notion of Bradlow N-pairs to parabolic Bradlow N-pairs, which can be viewed as parabolic GLSM data to the Grassmannian. We define the stability condition for parabolic N-pairs and it intertwines with parabolic structures. We fix a fixed smooth curve C of genus g, with one distinguished marked point x_0 and k distinct ordinary marked points $p_1, \ldots, p_k$. Let $I = \{p_1, \ldots, p_k\}$ be the set of ordinary marked points. Throughout the discussion, we assume $g > 1$. This assumption is not essential and the case $g \leq 1$ will be discussed in Remark VI.29.

We first give a brief review on parabolic vector bundles.

Definition VI.2. A *parabolic vector bundle* on C is a collection of data $(E, \{f_p\}_{p\in I}, \underline{a})$ where

- E is a vector bundle of rank n and degree d on C.
- For each marked point $p \in I$, f_p denotes a filtration in the fiber $E_p := E|_p$

$$E_p = E_{1,p} \supsetneq E_{2,p} \supsetneq \cdots \supsetneq E_{l_p,p} \supsetneq E_{l_p+1,p} = 0.$$

- The vector $\underline{a} = (a_p)_{p \in I}$ is a collection of integers such that

$$a_p = (a_{1,p}, \dots, a_{l_p,p}),\ 0 \le a_{1,p} < a_{2,p} < \cdots < a_{l_p,p} < l.$$

For $p \in I$ and $1 \le i \le l_p$, the integers $a_{i,p}$ are called the *parabolic weights* and $m_{i,p} := \dim E_{i,p} - \dim E_{i+1,p}$ are called the *multiplicities* of $a_{i,p}$. Let $m_p = (m_{1,p}, \dots, m_{l_p,p})$ and let $\underline{m} = (m_p)_{p \in I}$. The pair $(\underline{a}, \underline{m})$ is referred to as the *parabolic type* of the parabolic vector bundle E. The data f_p can be viewed as an element in the flag variety $\mathrm{Fl}_{m_p}(E_p)$ of type m_p. Define $r_{i,p} := \sum_{j=1}^{i} m_{j,p} = \dim E_p/E_{i+1,p}$ for $1 \le i \le l_p$. Denote $|a_p| := \sum_{i=1}^{l_p} m_{i,p}\, a_{i,p}$ and $|\underline{a}| := \sum_{p \in I} |a_p|$. We define the *parabolic degree* of E by

$$d_{\mathrm{par}}(E) = d + \frac{|\underline{a}|}{l},$$

and the *parabolic slope* by

$$\mu_{\mathrm{par}}(E) = \frac{d_{\mathrm{par}}(E)}{r(E)},$$

where $r(E) = \operatorname{rank} E$.

Suppose F is a subbundle of E and Q is the corresponding quotient bundle. Then F and Q inherit canonical parabolic structures from E. More precisely, given a marked point p, there is an induced filtration $\{F_{i,p}\}_i$ of the fiber F_p, which consists of distinct terms in the collection $\{F \cap E_{i,p}\}_i$. The parabolic weights $a'_{i,p}$ of F are defined such that if j is the largest integer satisfying $F_{i,p} \subset E_{j,p}$, then define $a'_{i,p} = a_{j,p}$. If F is a locally free subsheaf of E but not a subbundle, one can define the induced parabolic structure on F in the same way. For the quotient bundle $q : E \to Q$, we define a filtration $\{Q_{i,p}\}_i$ of Q_p by choosing distinct terms in the collection $\{q(E_{i,p})\}_i$. The parabolic weights $a''_{i,p}$ of Q are defined such that if j is the largest integer satisfying $q(E_{j,p}) = Q_{i,p}$, then define $a''_{i,p} = a_{j,p}$. We call $0 \to F \to E \to Q \to 0$ an *exact sequence of parabolic vector bundles* if it is an exact sequence of vector bundles, and F and G have the induced parabolic structures from E. One can check that the parabolic degree is additive on exact sequences, i.e., $d_{\text{par}}(E) = d_{\text{par}}(F) + d_{\text{par}}(Q)$.

Definition VI.3. Let $(E, \{f_p\}_{p\in I}, \underline{a})$ and $(E', \{f'_p\}_{p\in I}, \underline{a}')$ be two parabolic vector bundles. A morphism $\phi : E \to E'$ of vector bundles is said to be parabolic if the restrictions ϕ_p satisfy $\phi_p(E_{i,p}) \subset E'_{j+1,p}$ whenever $a_i > a'_j$, and strongly parabolic if $\phi_p(E_{i,p}) \subset E'_{j+1,p}$ whenever $a_i \geq a'_j$

Suppose $0 \to F \xrightarrow{i} E \xrightarrow{\pi} Q \to 0$ is an exact sequence of parabolic bundles. Then by definition i and π are parabolic homomorphisms. We denote by $\mathcal{P}ar\mathcal{H}om(E, E')$ and $\mathcal{SP}ar\mathcal{H}om(E, E')$ the subsheaves of $\mathcal{H}om(E, E')$ consisting of parabolic and strongly parabolic homomorphisms, respectively. The spaces of their global sections are denoted by $\text{ParHom}(E, E')$ and $\text{SParHom}(E, E')$, respectively. There are two natural skyscraper sheaves $K_{E,E'}$ and $SK_{E,E'}$ supported on the set of marked points I such that

$$0 \to \mathcal{P}ar\mathcal{H}om(E, E') \to \mathcal{H}om(E, E') \to K_{E,E'} \to 0,$$

$$0 \to \mathcal{SP}ar\mathcal{H}om(E, E') \to \mathcal{H}om(E, E') \to SK_{E,E'} \to 0.$$

Let $m_{i,p}$ and $m'_{i,p}$ be the multiplicities of the weights $a_{i,p}$ and $a'_{i,p}$, respectively. According to [12, Lemma 2.4], we have

$$\chi(K_{E,E'}) = \sum_{\substack{p\in I \\ (i,j)\in T_p}} m_{i,p}\, m_{j,p} \tag{6.1}$$

where $T_p = \{(i,j)|a_{i,p} > a'_{j,p}\}$. Using a similar argument to that of [12, Lemma 2.4], one can show that

$$\chi(SK_{E,E'}) = \sum_{\substack{p\in I \\ (i,j)\in ST_p}} m_{i,p}\, m_{j,p}$$

where $ST_p = \{(i,j)|a_{i,p} \geq a'_{j,p}\}$. When $E = E'$, we denote by $\mathcal{P}ar\mathcal{E}nd(E)$ the subsheaf of parabolic endomorphisms.

In [83], Yokogawa introduced an abelian category of parabolic $\mathcal{O}_C$-modules which has enough injective objects. It contains the category of parabolic vector bundles as a full (not abelian) subcategory. Hence, one can define the right derived functor $\mathrm{Ext}^i(E,-)$ of $\mathrm{ParHom}(E,-)$. The following lemmas show that the functors Ext^i for parabolic bundles behave similarly to the ordinary Ext functors for locally free sheaves.

Lemma VI.4. *[83, Lemma 3.6, Proposition 3.7] If E and E' are parabolic vector bundles, then there are canonical isomorphisms*

1. $\mathrm{Ext}^i(E,E') \cong \mathrm{H}^i(\mathcal{P}ar\mathcal{H}om(E,E'))$.
2. *Serre duality:* $\mathrm{Ext}^i(E, E'\otimes\omega_C(D)) \cong \mathrm{H}^{1-i}(\mathcal{SP}ar\mathcal{H}om(E',E))^\vee$, *where ω_C is the cotangent sheaf of C and $D = \sum_{p\in I} p$.*

Lemma VI.5. *[83, Lemma 1.4] The group $\mathrm{Ext}^1(E'',E')$ parametrizes isomorphism classes of extensions of $(E'', \{f''_p\}_{p\in I}, \underline{a}'')$ by $(E', \{f'_p\}_{p\in I}, \underline{a}')$.*

Now let us define *parabolic Bradlow N-pairs*.

Definition VI.6. A parabolic Bradlow N-pair $(E,\{E_{i,p}\},\underline{a},s)$ consists of a parabolic vector bundle $(E,\{f_p\}_{p\in I},\underline{a})$ of rank n and degree d, together with N sections $s \in H^0(E\otimes\mathcal{O}_C^N)$. A parabolic sub-pair

$$(E',s')\subset(E,s),$$

consists of a parabolic subbundle $\iota : E' \hookrightarrow E$ and N sections $s' : \mathcal{O}^N \to E'$ such that

$$\begin{aligned} \iota\circ s' = s \quad & s\in \mathrm{H}^0(E'\otimes\mathcal{O}^N), \quad \text{and} \\ s' = 0 \quad & s\notin \mathrm{H}^0(E'\otimes\mathcal{O}^N). \end{aligned}$$

A quotient pair (E'',s'') consists of a quotient parabolic bundle $q : E \to E''$ with $s'' = q\circ s$.

We shall abbreviate the parabolic N-pair $(E,\{E_{i,p}\},\underline{a},s)$ as (E,s) when there is no confusion. We define the parabolic slope of a parabolic N-pair by

$$\mu_{\mathrm{par}}(E,s) = \mu_{\mathrm{par}}(E) + \frac{\delta\theta(s)}{r(E)},$$

where $\theta(s)=1$ if $s\neq 0$ and 0 otherwise.

Definition VI.7. Let $\delta\in\mathbb{Q}_+$. A parabolic N-pair of degree d is δ-semistable if for all sub-pairs $(E',s')\subset(E,s)$, we have

$$\mu_{\mathrm{par}}(E',s')\leq\mu_{\mathrm{par}}(E,s).$$

A parabolic N-pair (E,s) is δ-stable if the above inequality is strict.

Remark VI.8. Suppose that the rank n is 1. Then according to Definition VI.7, any parabolic N-pair is stable with respect to all values of δ.

Remark VI.9. Note that a parabolic N-pair $(E, 0)$ is (semi-)stable if E is a (semi-)stable parabolic vector bundle. We will focus on *non-degenerate* parabolic pairs, i.e., pairs (E, s) with $s \neq 0$.

In the following, we list some basic properties of δ-stable and semistable parabolic N-pairs, parallel to the corresponding results for N-pairs without parabolic structures.

Lemma VI.10. *Suppose $\phi : (E_1, s_1) \to (E_2, s_2)$ is a nonzero parabolic morphism of δ-semistable pairs. Then $\mu_{\mathrm{par}}(E_1, s_1) \leq \mu_{\mathrm{par}}(E_2, s_2)$. Furthermore, if (E_1, s_1) and (E_2, s_2) are δ-stable parabolic pairs with the same parabolic slope, then ϕ is an isomorphism. In particular, for a non-degenerate δ-stable parabolic pair N-pair (E, s), there are no parabolic endomorphisms of E preserving s except the identity, and no parabolic endomorphisms of E annihilating s except 0.*

Lemma VI.11 (Harder-Narasimhan Filtration)**.** *Let (E, s) be a parabolic N-pair. There exists a canonical filtration by sub-pairs*

$$0 \subsetneq (F_1, s_1) \subsetneq (F_2, s_2) \subsetneq \cdots \subsetneq (F_m, s_m) = (E, s)$$

such that for all i we have

1. $(\mathrm{gr}_i, \bar{s}_i) := (F_i, s_i)/(F_{i-1}, s_{i-1})$ *are δ-semistable.*
2. $\mu_{\mathrm{par}}(\mathrm{gr}_i, \bar{s}_i) > \mu_{\mathrm{par}}(\mathrm{gr}_{i+1}, \bar{s}_{i+1})$.

Proof. Notice that the parabolic slope μ_{par} is additive on short exact sequences of parabolic N-pairs. The proof is the same as the proof of the existence and uniqueness of Harder-Narasimhan filtration of a pure sheaf (see for example the proof of [45, Theorem 1.3.4]). $\square$

Lemma VI.12 (Jordan-Hölder Filtration)**.** *Let (E, s) be a δ-semistable parabolic N-pair. A Jordan-Hölder filtration of (E, s) is a filtration*

$$0 \subsetneq (G_1, s_1) \subsetneq (G_2, s_2) \subsetneq \cdots \subsetneq (G_m, s_m) = (E, s)$$

such that the factors $(\mathrm{gr}_i, \bar{s}_i) := (F_i, s_i)/(F_{i-1}, s_{i-1})$ are δ-stable with slope $\mu_{\text{par}}(E, s)$. Moreover, the graded object $\mathrm{gr}(E, s) := \oplus \mathrm{gr}_i$ does not depend on the filtration.

Proof. The proof is standard. See for example the proof of [45, Proposition 1.5.] in the case of semistable sheaves. $\square$

For δ-semistable parabolic N-pairs of rank n and degree d, we have the following boundedness result.

Lemma VI.13. *Let (E, s) be a δ-semistable parabolic N-pair. Suppose that*

$$\mu_{\text{par}}(E, s) > 2g - 1 + |I| + \delta.$$

Then $H^1(E) = 0$ and E is globally generated, i.e., the morphism

$$H^0(E) \otimes \mathcal{O}_C \to E$$

is surjective.

Proof. The proof is similar to that of Lemma V.5. It suffices to show that $H^1(E(-p)) = 0$ for any point $p \in E$. Suppose $H^1(E(-p)) \neq 0$. By Serre duality, we have $H^1(E(-p)) = (H^0(E^\vee \otimes \omega_C(p))^\vee$, where ω_C is the dualizing sheaf of C. Therefore a nonzero element in $H^1(E(-p))$ induces a nonzero morphism $\phi : E \to \omega_C(p)$. Let L be the image sheaf of ϕ. Since L is a subsheaf of $\omega_C(p)$, we have $\deg(L) \leq 2g - 1$. Let s'' be the induced N sections of L. It follows that $\mu_{\mathrm{par}}(E, s) > 2g - 1 + |I| + \delta \geq d_{\mathrm{par}}(L) + \theta(s'')\delta$, which contradicts the δ-semistability of (E, s). □

Corollary VI.14. *Fix the rank n, degree d and the parabolic type $(\underline{a}, \underline{m})$. The family of vector bundles underlying δ-semistable parabolic N-pairs of rank n, degree d and parabolic type $(\underline{a}, \underline{m})$ on a smooth curve C is bounded.*

Proof. Let $\mathcal{O}(1)$ be a locally free sheaf of degree one on C. By Lemma VI.13, we have $H^1(E(m)) = 0$ if

$$m + \mu_{\mathrm{par}}(E, s) > 2g - 1 + |I| + \delta.$$

The boundedness of δ-semistable pairs follows from [45, Lemma 1]. □

The following lemma shows that for a bounded family of parabolic N-pairs, the family of the factors of their Harder-Narasimhan filtrations is also bounded.

Lemma VI.15. *Let T be a scheme of finite type. Suppose $S : \mathcal{O}^N_{T\times C} \to \mathcal{E}$ is a flat family of parabolic N-pairs over $T \times C$. For any closed point $t \in T$, we denote by $\{(\mathrm{gr}^t_i, s^t_i)\}_i$ the Harder-Narasimhan factors of $(\mathcal{E}_t, S_t)$, where $\mathcal{E}_t = \mathcal{E}|_{\mathrm{Spec}\, k(t)\times C}$ and S_t*

is the restriction of the N sections to the fiber over t. Then the family $\{(\mathrm{gr}_i^t, s_i^t)\}_{i,t\in T}$ is bounded.

Proof. The proof is identical to that of Lemma 9 in [54] for N-pairs without parabolic structures. $\square$

6.3 GIT construction of the moduli stack of δ-stable parabolic N-pairs

In this section, we show that the moduli stack $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})$ of parabolic N-pairs is an Artin stack, locally of finite type. For a generic value of $\delta \in \mathbb{Q}_+$ (see Definition VI.19), we prove that the substack $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ parametrizing non-degenerate δ-stable parabolic N-pairs is a projective variety. In fact, we will construct it using geometric invariant theory (GIT). Throughout the discussion, we fix the degree d, rank n, parabolic weights $\underline{a} = (a_{i,p})$ and their multiplicities $\underline{m} = (m_p)_{p\in I}$, where $m_p = (m_{i,p})$.

Definition VI.16. Let T be a scheme. A family of parabolic N-pairs $(\mathcal{E}, \{f_p\}, S)$ over T is a locally free sheaf $\mathcal{E}$, flat over T, together with a morphism of sheaves $\mathcal{O}_{T\times C}^N \to \mathcal{E}$ on $T\times C$ and a section f_p of the relative flag variety $\mathrm{Fl}_{m_p}(\mathcal{E}|_{T\times\{p\}})$ of type m_p for each $p \in I$.

An isomorphism $(\mathcal{E}, \{f_p\}, S) \to (\mathcal{E}', \{f'_p\}, S')$ of families of parabolic N-pairs over T is given by a parabolic isomorphism $\Phi : \mathcal{E} \to \mathcal{E}'$ such that $\Phi(S) = S'$.

Let $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})$ be the groupoid of parabolic N-pairs of rank n, degree d and type $(\underline{a},\underline{m})$. Let $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ be the groupoid of parabolic vector bundles with the same numerical data. It is easy to see that $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ is a fiber product of flag bundles over the moduli stack of vector bundles $\mathfrak{Bun}_C(d,n)$. The moduli stack $\mathfrak{Bun}_C(d,n)$ is a smooth Artin stack (see, for example, [44]). Therefore, $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ is also a smooth Artin stack. There is a representable forgetful morphism $q : \mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \to \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$. Let $\mathfrak{E}$ be the universal vector bundle over $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a}) \times C$ and let $\pi : \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a}) \times C \to \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ be the projection. Let ω be the relative dualizing sheaf of π, which is just the pullback of the cotangent sheaf ω_C of C along the second projection to C.

Proposition VI.17. *There is a natural isomorphism of* $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$*-stacks*

$$\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \to \operatorname{Spec}\operatorname{Sym}\big(R^1\pi_*\big((\mathfrak{E}^\vee)^N \otimes \omega\big)\big).$$

In particular, $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})$ *is an abelian cone over* $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$*.*

Proof. The same arguments given in the proof of [68, Proposition 1.8] apply here. □

Corollary VI.18. *The moduli stack of parabolic N-pairs* $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})$ *is an Artin stack and the forgetful morphism* $q : \mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \to \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ *is strongly representable.*

Definition VI.19. A value of $\delta \in \mathbb{Q}_+$ is called generic if there is no strictly δ-semistable N-pairs. Otherwise, δ is called critical. A critical value of δ is also called a wall.

Let $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ be the substack of $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})$ which parametrizes non-degenerate δ-stable N-pairs (E,s). In the following, we will use GIT to give an alternate construction of $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$, modeled on the construction of moduli spaces of (semi)stable pairs given in [54].

The semistability condition of parabolic N-pairs can be described in terms of dimensions of global sections. We fix an ample line bundle $\mathcal{O}(1)$ on C of degree one. For any locally free sheaf E on C, we define $E(m) := E \otimes \mathcal{O}(1)^{\otimes m}$. If E is a parabolic vector bundle, there is a natural parabolic structure on $E(m)$. Given a non-degenerate parabolic N-pair (E,s) of degree d, rank n and parabolic type $(\underline{a},\underline{m})$, we define

$$\mu_{\mathrm{par}}^{\delta}(m) := \mu_{\mathrm{par}}(E(m)) + \frac{\delta}{r(E)} = \frac{d+nm}{n} + \frac{|\underline{a}|}{nl} + \frac{\delta}{n}$$

Before we describe the GIT construction, we recall the special cases for curves of the Le Potier-Simpson estimate and a boundedness result due to Grothendieck. The Le Potier-Simpson estimate allows us to give uniform bounds for the dimension of global sections of a vector bundle in terms of its slope. We refer the reader to [45, Theorem 3.3.1] and [69, Corollary 1.7] for the general theorem in higher dimensions. Suppose the Harder-Narasimhan filtration of a vector bundle E with respect to the ordinary slope μ is given by

$$0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = E.$$

Define $\mu_{\max}(E) = \mu(E_1)$ and $\mu_{\min}(E) = \mu(E_k/E_{k-1})$. Denote $[t]_+ := \max\{0,t\}$ for any real number t.

Lemma VI.20 (Le Potier-Simpson)**.** *Let C be a smooth curve. For any locally free* *sheaf F on C, we have*

$$\frac{h^0(F)}{r(F)} \leq \left[\mu_{\max}(F) + c\right]_+,$$

where $r(F) = \operatorname{rank} F$ and the constant $c := r(F)(r(F)+1)/2 - 1$.

The following lemma is on the boundedness of subsheaves. We refer the reader to [45, Lemma 1.7.9] for the general results

Lemma VI.21 (Grothendieck)**.** *Let C be a smooth curve and let F be a locally free sheaf on C. Then the family of subsheaves $F' \subset F$ with slopes bounded below, such that the quotient F/F' is locally free, is bounded.*

Let (E, s) be a non-degenerate parabolic N-pair. In the following discussion, we will always denote a sub-pair of (E, s) by (E', s'), with the induced parabolic type $\underline{a}'$. Similarly, we will always denote a quotient pair of (E, s) by (E'', s''), with the induced parabolic type $\underline{a}''$.

Lemma VI.22. *There exists an integer m_0 such that for any integer $m \geq m_0$, the following assertions are equivalent.*

(1) The parabolic N-pair (E, s) is stable.

(2) For any nontrivial proper sub-pair (E', s'),

$$\frac{h^0(E'(m)) + \theta(s')\delta}{r(E')} + \frac{|\underline{a}'|}{r(E')l} < \mu^{\delta}_{\mathrm{par}}(m) + 1 - g.$$

(3) For any proper quotient pair (E'', s'') with $r(E'') > 0$,

$$\frac{h^0(E''(m)) + \theta(s'')\delta}{r(E'')} + \frac{|\underline{a}''|}{r(E'')l} > \mu^{\delta}_{\mathrm{par}}(m) + 1 - g.$$

δ-semistability can be characterized similarly by replacing $<$ by $\leq$ in (ii) *and* (iii).

Proof. (1) $\Rightarrow$ (2): By Lemma VI.14 and Lemma VI.15, there exists a constants μ such that $\mu_{\max}(E) \leq \mu$. Let (E', s') be a proper nontrivial sub-pair and let $\nu = \mu_{\min}(E')$. It follows from Lemma VI.20 that there exists a constant c depending only on n such that

$$\frac{h^0(E'(m))}{r(E')} \leq (1 - \frac{1}{n})[\mu + m + c]_+ + \frac{1}{n}[\nu + m + c]_+. \tag{6.2}$$

Let $A > 0$ be a constant satisfying $d + n(1-g) + nm \geq n(m - A)$. Since there are only finite many choices for $\theta(s')\delta/r(E')$ and $|\underline{a}'|/r(E')l$, it is possible to choose an integer ν_0 such that

$$(1 - \frac{1}{n})\mu + \frac{1}{n}\nu_0 + c + \frac{\theta(s')\delta}{r(E')} + \frac{|\underline{a}'|}{r(E')l} < -A + \frac{\delta}{n} + \frac{|\underline{a}|}{nl}. \tag{6.3}$$

Enlarging m_0 if necessary, we can assume that $\mu + m + c$ and $\nu + m + c$ are positive. Therefore

$$(1 - \frac{1}{n})[\mu + m + c]_+ + \frac{1}{n}[\nu + m + c]_+ = (1 - \frac{1}{n})\mu + \frac{1}{n}\nu + m + c. \tag{6.4}$$

If $\nu \leq \nu_0$, then it follows from (6.2), (6.4) and (6.3) that

$$\begin{aligned}
\frac{h^0(E'(m)) + \theta(s')\delta}{r(E')} + \frac{|\underline{a}'|}{r(E')l} &< m - A + \frac{\delta}{n} + \frac{|\underline{a}|}{nl} \\
&\leq \frac{d + n(1-g) + nm}{n} + \frac{\delta}{n} + \frac{|\underline{a}|}{nl} \\
&= \mu^{\delta}_{\mathrm{par}}(m) + 1 - g.
\end{aligned}$$

If $\nu > \nu_0$, then by Grothendieck's Lemma VI.21, the family of such E' is bounded. Enlarging m_0 if necessary, we have

$$h^0(E'(m)) = \chi(E'(m)) = d(E') + r(E')m + r(E')(1-g)$$

for all $m \geq m_0$. By the δ-stability of (E, s), we have

$$\frac{h^0(E'(m)) + \theta(s')\delta}{r(E')} + \frac{|\underline{a}'|}{r(E')l} = \mu_{\text{par}}(E', s') + m + 1 - g < \mu^{\delta}_{\text{par}}(m) + 1 - g.$$

(2) $\Rightarrow$ (3): Consider the short exact sequence

$$0 \to E' \to E \to E'' \to 0.$$

There exists an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, we have $H^1(E(m)) = 0$. It follows that $H^1(E''(m)) = 0$. Suppose $(h^0(E'(m)) + \theta(s')\delta)/r(E') + |\underline{a}'|/(r(E')l) < \mu^{\delta}_{\text{par}}(m) + 1 - g$. Since $\mu(E'(m)) \leq h^0(E'(m))/r(E')$, we have $\mu_{\text{par}}(E', s') < \mu_{\text{par}}(E, s)$. It follows from the additivity of the parabolic δ-slope of pairs that

$$\begin{aligned} \mu^{\delta}_{\text{par}}(m) + 1 - g &= \mu_{\text{par}}(E, s) + m + 1 - g \\ &< \mu_{\text{par}}(E'', s'') + m + 1 - g \\ &= \frac{h^0(E''(m)) + \theta(s'')\delta}{r(E'')} + \frac{|\underline{a}''|}{r(E'')l}. \end{aligned}$$

(3) $\Rightarrow$ (1): Suppose that (E, s) is not stable. Let $(E'', \{f''_p\}, s'')$ be a quotient pair of (E, s) such that

$$\mu_{\text{par}}(E'', s'') \leq \mu_{\text{par}}(E, s)$$

There exists an $m_0 \in \mathbb{N}$ satisfying for all $m \geq m_0$, $H^1(E(m)) = 0$. Let E' be the kernel of the quotient morphism $E \to E''$. Then by the long exact sequence of cohomology groups associated to $0 \to E' \to E \to E'' \to 0$, we have $H^1(E''(m)) = 0$ and hence $h^0(E''(m)) = d(E'') + r(E'')(1-g)$. It follows that

$$\begin{aligned}\frac{h^0(E''(m)) + \theta(s'')\delta}{r(E'')} + \frac{|\underline{a}''|}{r(E'')l} &= \mu_{\mathrm{par}}(E'', s'') + m + 1 - g \\ &\leq \mu_{\mathrm{par}}(E, s) + m + 1 - g \\ &= \mu^{\delta}_{\mathrm{par}}(m) + 1 - g,\end{aligned}$$

which contradicts the hypothesis. Therefore, (E, s) is δ-stable.

The equivalence of three assertions for δ-semistability can be proved similarly. □

By Lemma VI.14 and Lemma VI.15, there exists an $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ and any δ-stable parabolic N-pair (E, s), the following conditions are satisfied.

1. $E(m)$ is globally generated and has no higher cohomology. Similar results hold for their Harder-Narasimhan factors.
2. The three assertions in Lemma VI.22 are equivalent.

We fix such an m. Let (E, s) be a δ-semistable N-pairs. Then the vector bundle E can be realized as a quotient

$$q : H^0(E(m)) \otimes \mathcal{O}_C(-m) \twoheadrightarrow E$$

and the section s induces a linear map

$$\phi : H^0(\mathcal{O}_C(m))^N \to H^0(E(m)).$$

Let V be a fixed complex vector space of dimension $\dim(V) = P(m)$ where $P(m) := \chi(E(m)) = d + mn + n(1-g)$.

After fixing an isomorphism between $H^0(E(m))$ and V, we have the following diagram.

$$\begin{array}{ccccc} K & \stackrel{\iota}{\hookrightarrow} & H^0(\mathcal{O}_C(m))^N \otimes \mathcal{O}_C(-m) & \xrightarrow{\text{ev}} & \mathcal{O}_C^N \\ & & \downarrow \phi & & \downarrow s \\ & & V \otimes \mathcal{O}_C(-m) & \xrightarrow{q} & E \end{array}$$

Here K denotes the kernel of the evaluation map $\text{ev} : H^0(\mathcal{O}_C(m))^N \otimes \mathcal{O}_C(-m) \to \mathcal{O}_C^N$. Let

$$\mathbb{P} = \mathbb{P}(\text{Hom}(H^0(\mathcal{O}_C(m))^N, V))$$

and let

$$Q = \text{Quot}_C^{n,d}(V \otimes \mathcal{O}_C(-m)).$$

be the Grothendieck's Quot scheme which parametrizes coherent quotients of $V \otimes \mathcal{O}_C(-m)$ over C of rank n and degree d. Notice that the spaces P and Q are fine moduli spaces with universal families

$$H^0(\mathcal{O}_C(m))^N \otimes \mathcal{O}_{\mathbb{P}} \to V \otimes \mathcal{O}_{\mathbb{P}}(1) \tag{6.5}$$

and

$$V \otimes \mathcal{O}_C(-m) \twoheadrightarrow \widetilde{\mathcal{E}}. \tag{6.6}$$

Here $\mathcal{O}_{\mathbb{P}}(1)$ denotes the anti-tautological line bundle on $\mathbb{P}$. By abuse of notation, we will still denote by $\mathcal{O}_{\mathbb{P}}(1)$ and $\widetilde{\mathcal{E}}$ the pullbacks of the corresponding universal sheaves to $Q \times \mathbb{P} \times C$.

We consider the locally closed subscheme

$$Z \subset Q \times \mathbb{P}$$

consisting of points $([q], [\phi])$ which satisfy the following properties:

- E is a locally free.
- $q \circ \phi \circ \iota = 0$.
- The quotient q induces an isomorphism $V \to H^0(E(m))$.

Let $p \in I$ be a marked point. We denote by Fl_{m_p} the relative flag varitey of locally-free quotients of $\widetilde{\mathcal{E}}_p := \widetilde{\mathcal{E}}|_{Z\times\{p\}}$ of type $m_p = (m_{i,p})$ (cf. [42, §2]). Let $\pi_p : \mathrm{Fl}_{m_p} \to Z$ be the projection. There exists a universal filtration of $\pi_p^*(\widetilde{\mathcal{E}}_p)$ by coherence subsheaves

$$\pi_p^*(\widetilde{\mathcal{E}}_p) = \mathcal{F}_{1,p} \supsetneq \cdots \supsetneq \mathcal{F}_{l_p,p} \supsetneq \mathcal{F}_{l_p+1,p} = 0$$

such that the universal quotient bundles $\mathcal{Q}_{i,p} := \pi_p^*(\widetilde{\mathcal{E}}_p)/\mathcal{F}_{i+1,p}$ are locally free of rank $r_{i,p} = \sum_{j=1}^{i} m_{j,p}$.

Let R be the fiber product

$$R := \mathrm{Fl}_{m_{p_1}} \times_Z \cdots \times_Z \mathrm{Fl}_{m_{p_k}},$$

where $p_1, \ldots, p_k$ are the ordinary marked points. By abuse of notation, we still denote by $\mathcal{Q}_{i,p}$ the pullback of $\mathcal{Q}_{i,p}$ to R. A δ-semistable parabolic N-pair (E, s) can be represented by a point $([q], \{[\tilde{f}_p]\}, [\phi])$ in R. There is a natural right $\mathrm{SL}(V)$-action on $Q \times \mathbb{P}$ given by

$$([q], [\phi])g = ([q \circ g], [g^{-1} \circ \phi])$$

for $g \in \mathrm{SL}(V)$ and $([q],[\phi]) \in Q \times \mathbb{P}$. It is easy to see that Z is invariant under this $\mathrm{SL}(V)$-action. Notice that the natural right $\mathrm{SL}(V)$-action on $V \otimes \mathcal{O}_C(-m)$ induces a right $\mathrm{SL}(V)$-action on $\widetilde{\mathcal{E}}$ via the universal quotient morphism $V \otimes \mathcal{O}_C(-m) \twoheadrightarrow \widetilde{\mathcal{E}}$. Therefore, $\mathrm{SL}(V)$ also acts on the relative flag variety Fl_{m_p} for $p \in I$ and the universal quotient bundles $\mathcal{Q}_{i,p}$ have natural $\mathrm{SL}(V)$-linearizations.

Pick a sufficiently large integer t such that $t > m$ and we have the following embedding

$$\begin{aligned} Q = \mathrm{Quot}_C^{n,d}(V \otimes \mathcal{O}_C(-m)) &\hookrightarrow \mathrm{Gr}(V \otimes H^0(\mathcal{O}_C(t-m)), \chi_t), \\ [q : V \times \mathcal{O}_C(-m) \twoheadrightarrow E] &\to [H^0(q(t)) : V \otimes H^0(\mathcal{O}_C(t-m)) \twoheadrightarrow H^0(E(t))]. \end{aligned}$$

For such a t, there is a $\mathrm{SL}(V)$-equivariant embedding

$$\begin{aligned} T : R \hookrightarrow & \mathrm{Gr}(V \otimes H^0(\mathcal{O}_C(t-m)), \chi_t) \\ & \times \prod_{p \in I} \{\mathrm{Gr}(V, r_{1,p}) \times \cdots \times \mathrm{Gr}(V, r_{l_p-1,p})\} \times \mathbb{P}, \\ ([q], \{[\tilde{f}_p]\}, [\phi]) \mapsto & ([H^0(q(t))], \{E_p/E_{2,p}, \ldots, E_p/E_{l_p,p}\}, [\phi]), \end{aligned}$$

where $\chi_t = \chi(E(t))$ and $r_{i,p} = \sum_{j=1}^{i} m_{j,p} = \dim E_p/E_{i+1,p}$. For simplicity, we denote $\mathrm{Gr}(V \otimes H^0(\mathcal{O}_C(t-m)), \chi_t)$ by $\mathbb{G}_t$ and $\mathrm{Gr}(V, r_{j,p})$ by $\mathbb{G}_{j,p}$ for $1 \leq j \leq l_p - 1$.

Let $\bar{R}$ be the closure of $T(R)$ in $\mathbb{G}_t \times \prod_{p \in I} \{\mathbb{G}_{1,p} \times \cdots \times \mathbb{G}_{l_p-1,p}\} \times \mathbb{P}$. Let $\mathcal{O}_{\mathbb{G}_t}(1)$ and $\mathcal{O}_{\mathbb{G}_{i,p}}(1)$ be the canonical ample generators of the Grassmannians. Let $\mathcal{O}_{\mathbb{P}}(1)$ be the anti-canonical line bundle on $\mathbb{P}$. Notice that the ample line bundles $\mathcal{O}_{\mathbb{G}_t}(1)$, $\mathcal{O}_{\mathbb{G}_{i,p}}(1)$ and $\mathcal{O}_{\mathbb{P}}(1)$ all have standard $\mathrm{SL}(V)$-linearizations. For positive integers a_1, a_2 and

$b_{j,p}$ for $p \in I, 1 \leq j \leq l_p - 1$, we consider the $\mathrm{SL}(V)$-linearized line bundle

$$L = \mathcal{O}_{\mathbb{G}_t}(a_2) \boxtimes \{\boxtimes_{p\in I,\, j} \mathcal{O}_{\mathbb{G}_{j,p}}(b_{j,p})\} \boxtimes \mathcal{O}_{\mathbb{P}}(a_1).$$

We study the GIT stability condition of $\mathbb{G}_t \times \prod_{p\in I}\{\mathbb{G}_{1,p} \times \cdots \times \mathbb{G}_{l_p-1,p}\} \times \mathbb{P}$ with respect to L. Let $\lambda : \mathbb{C}^* \to \mathrm{SL}(V)$ be a one parameter subgroup. For any closed point $z \in \bar{R}$, we denote by $o_z : \mathrm{SL}(V) \times \{z\} \to \bar{R}$ the orbit map. The morphism $o_z \circ \lambda$ extends to a morphism $g : \mathbb{A}^1 \to \bar{R}$. Notice that $g(0)$ is a fixed point of the $\mathbb{C}^*$-action. Suppose any element $x \in \mathbb{C}^*$ acts on the fiber $L|_{g(0)}$ by multiplying x^w for some $w \in \mathbb{Z}$. Then we define the *Hilbert-Mumford weight*

$$\mu^L(z, \lambda) = -w.$$

By the Hilbert-Mumford criterion, a closed point $z \in \bar{R}$ is stable (semistable) with respect to L if and only if $\mu^L(z,\lambda) > 0$ (respectively $\mu^L(z,\lambda) \geq 0$) for all one parameter subgroups of $\mathrm{SL}(V)$. Now let us compute $\mu^L(z,\lambda)$ for a point $z = ([q], \{[\tilde{f}_p]\}, [\phi]) \in \bar{R}$. A one parameter subgroup λ induces a $\mathbb{C}^*$-action on V. Let $w_1 < w_2 < \cdots < w_s$ be the weights of this $\mathbb{C}^*$-action. Then there exists a filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_s = V,$$

such that V_i/V_{i-1} is the isotypic component of weight $w_i \in \mathbb{Z}$. We denote by $i(\phi)$ the smallest i such that $\operatorname{im}\phi \subset V_i$. Define $w(\phi) = w_{i(\phi)}$. Consider the ascending filtration of E by

$$F_i = q(V_i \otimes \mathcal{O}_C(-m)).$$

Note that $F_s = E$. Let $\mathrm{gr}_i = F_i/F_{i-1}$. Notice that the family of subsheaves $E' \subset E$ of the form $q(V' \otimes \mathcal{O}_C(-m))$ for some subspace $V' \subset V$ is bounded. We can pick large enough t such that we also have

$$H^1(F_i(t)) = 0 \quad \text{and} \quad H^1(\mathrm{gr}_i(t)) = 0, \quad \text{for } 1 \leq i \leq s. \tag{6.7}$$

Denote $Q_{j,p} := E_p/E_{j+1,p}$ for $1 \leq j \leq l_p - 1$. Let $q_{j,p} : V \twoheadrightarrow Q_{j,p}$ be the surjective maps induced by $V \otimes \mathcal{O}_C(-m) \twoheadrightarrow E$. We consider the ascending filtrations of $Q_{j,p}$ by

$$Q^i_{j,p} = q_{j,p}(V_i), \quad \text{for } 1 \leq i \leq s.$$

Define $Q^0_{j,p} = 0$. Let $r^i_{j,p} = \dim Q^i_{j,p}$. Note that $r^s_{j,p} = r_{j,p}$.

Suppose F is a coherent sheaf on C. Then its Hilbert polynomial is defined as the polynomial $P_F(t) := \chi(F(t)) = r(F)t + d(F) + r(F)(1-g)$ in t. An explicit formula for $\mu^L(z,\lambda)$ is given in the following lemma.

Lemma VI.23.

$$\begin{aligned} \mu^L(z,\lambda) = a_1 w(\phi) - a_2 \sum_{1\leq i\leq s} w_i\big(P_{F_i}(t) - P_{F_{i-1}}(t)\big) \\ - \sum_{p\in I} \sum_{\substack{1\leq i\leq s \\ 1\leq j\leq l_p-1}} b_{j,p} w_i\big(r^i_{j,p} - r^{i-1}_{j,p}\big). \end{aligned}$$

Proof. The Hilbert-Mumford weight satisfies that

$$\mu^{L_1 \boxtimes L_2} = \mu^{L_1} + \mu^{L_2}.$$

Hence we can compute $\mu^{\mathcal{O}_{\mathbb{G}_t}(a_1)}(z,\lambda), \mu^{\mathcal{O}_{\mathbb{G}_{j,p}}(b_{j,p})}(z,\lambda)$, and $\mu^{\mathcal{O}_{\mathbb{P}}(a_2)}(z,\lambda)$ separately.

First, we calculate the contribution from $\mathcal{O}_{\mathbb{P}}(a_1)$ to $\mu^L(z,\lambda)$. Let $\{e^i_\nu\}_\nu$ be a basis of V_i. Then we can write ϕ as

$$\phi = \bigoplus_{i,\nu} \phi_i^\nu \otimes e^i_\nu \in (H^0(\mathcal{O}_C(m)^N))^\vee \otimes V,$$

where $\phi_i^\nu \in (H^0(\mathcal{O}_C(m)^N))^\vee$. By definition, $i(\phi)$ is the largest i such that $\phi_i^\nu \neq 0$ for some ν. Since an element in $\mathrm{SL}(V)$ acts on V as its inverse, the contribution from $\mathcal{O}_{\mathbb{P}}(a_1)$ to $\mu^L(z,\lambda)$ is

$$a_1 w(\phi).$$

Second, we consider $\mathcal{O}_{\mathbb{G}_t}(a_2)$. According to [45, Lemma 4.4.3], we have

$$\lim_{x\to 0}\, [q]\cdot\lambda(x) = \bigoplus_{i=1}^{s} H^0(\mathrm{gr}_i(t)) \ \in \mathbb{G}_t.$$

The fiber of $\mathcal{O}_{\mathbb{G}_t}(1)$ at the limiting point is

$$\bigwedge^{\chi(E(t))} \bigoplus_{i=1}^{s} H^0(\mathrm{gr}_i(t)).$$

The weight of $\mathbb{C}^*$-action is

$$\sum_{i=1}^{s} w_i h^0(\mathrm{gr}_i(t)) = \sum_{i=1}^{s} w_i\big(P_{F_i}(t) - P_{F_{i-1}}(t)\big).$$

Therefore, the contribution from $\mathcal{O}_{\mathbb{G}_t}(a_2)$ to $\mu^L(z,\lambda)$ is

$$-a_2 \sum_{i=1}^{s} w_i\big(P_{F_i}(t) - P_{F_{i-1}}(t)\big).$$

Finally, it follows easily from the computations of [59, Chapter 4, §4] that the contribution to $\mu^L(z,\lambda)$ from $\mathcal{O}_{\mathbb{G}_{j,p}}(b_{j,p})$ is

$$-\sum_{1\leq i\leq s} b_{j,p} w_i\big(\dim Q^i_{j,p} - \dim Q^{i-1}_{j,p}\big).$$

□

Lemma VI.24. *Let* $z = ([q], \{[\tilde{f_p}]\}, [\phi]) \in \bar{R}$ *be a point with the associated parabolic* N*-pair* (E, s)*. For sufficiently large* t *such that (6.7) holds, then the following two conditions are equivalent.*

(1) z *is GIT-stable with respect to* L*.*

(2) For any nontrivial proper subspace $W \subset V$*, let* $F = q(W \otimes \mathcal{O}(-m))$*. Then*

$$P_F(t) > \frac{a_1}{a_2}\left(\theta_W(\phi) - \frac{\dim W}{\dim V}\right) + P(t)\frac{\dim W}{\dim V} + \sum_{p \in I} \sum_{1 \le j \le l_p - 1} \frac{b_{j,p}}{a_2}\left(r_{j,p}\frac{\dim W}{\dim V} - r^W_{j,p}\right), \tag{6.8}$$

where $r^W_{j,p} = \dim q_{j,p}(W)$ *and* $\theta_W(\phi) = 1$ *if* $\operatorname{im}\phi \subset W$ *and 0 otherwise.*

GIT-semistablity can be also characterized by replacing $>$ *by* $\geq$ *in equation (6.8).*

Proof. (1) $\Rightarrow$ (2): Suppose z is GIT-stable with respect to L. Let $h = \dim W$. We consider the one parameter subgroup give by

$$\lambda(x) = \begin{pmatrix} x^{h-P(m)}\mathrm{id}_h & \\ & x^h \mathrm{id}_{P(m)-h} \end{pmatrix},$$

where $\lambda(x)$ acts on W by multiplying $x^{h-P(m)}$ and its compliment by multiplying x^h. If $\operatorname{im}\phi \subset W$, then by the Hilbert-Mumford criterion and Lemma VI.23, we have

$$0 < \mu^L(z, \lambda) = a_1(h - P(m)) + a_2 P(m) P_F(t) - a_2 h P(t) + \sum_{p \in I} \sum_{1 \le j \le l_p - 1} b_{j,p}\big(r^W_{j,p} P(m) - r_{j,p} h\big).$$

Since $\dim V = P(m)$, the above inequality is equivalent to

$$P_F(t) > \frac{a_1}{a_2}\left(1 - \frac{\dim W}{\dim V}\right) + P(t)\frac{\dim W}{\dim V} + \sum_{p\in I}\sum_{1\le j\le l_p-1}\frac{b_{j,p}}{a_2}\left(r_{j,p}\frac{\dim W}{\dim V} - r^W_{j,p}\right).$$

If $\operatorname{im}\phi \not\subset W$, then

$$0 < \mu^L(z,\lambda) = a_1 h + a_2 P(m)P_F(t) - a_2 h P(t) + \sum_{p\in I}\sum_{1\le j\le l_p-1} b_{j,p}\big(r^W_{j,p}P(m) - r_{j,p}h\big),$$

which is equivalent to

$$P_F(t) > -\frac{a_1}{a_2}\left(\frac{\dim W}{\dim V}\right) + P(t)\frac{\dim W}{\dim V} + \sum_{p\in I}\sum_{1\le j\le l_p-1}\frac{b_{j,p}}{a_2}\left(r_{j,p}\frac{\dim W}{\dim V} - r^W_{j,p}\right).$$

(2) $\Rightarrow$ (1): It follows from inequality (6.8) that

$$\begin{aligned}\mu^L(z,\lambda) > a_1 w_s - a_2 w_s P(t) + \left(\frac{a_2 P(t)}{\dim V} - \frac{a_1}{\dim V}\right)\sum_{i=1}^{s-1}(w_{i+1}-w_i)\dim V_i \\ + \sum_{p\in I}\sum_{1\le j\le l_p-1}\frac{b_{j,p}r_{j,p}}{\dim V}\left(\sum_{i=1}^{s-1}(w_{i+1}-w_i)\dim V_i\right) \\ - \sum_{p\in I}\sum_{1\le j\le l_p-1} b_{j,p}\left(\sum_{i=1}^{s-1}(w_{i+1}-w_i)\, r^i_{j,p}\right) \\ - \sum_{p\in I}\sum_{\substack{1\le i\le s\\ 1\le j\le l_p-1}} b_{j,p}w_i\big(r^i_{j,p} - r^{i-1}_{j,p}\big) \\ = 0.\end{aligned}$$

Here we use the fact that

$$\sum_{i=1}^{s-1}(w_{i+1}-w_i)\dim V_i = w_s \dim V$$

since λ is a one parameter subgroup of $\mathrm{SL}(V)$. Therefore z is GIT-stable. $\square$

Let I denote the number of ordinary marked points. To relate GIT-(semi)stability with δ-(semi)stability, we make the following choice:

$$a_1 = nl(t-m)\delta, a_2 = P(m)l + |\underline{a}| + \delta l - n\sum_{p\in I} a_{l_p,p},$$

and

$$b_{j,p} = (a_{j+1,p} - a_{j,p})n(t-m) \quad \text{for } 1 \leq j \leq l_p - 1.$$

Let

$$L = \mathcal{O}_{\mathbb{G}_t}(a_2) \boxtimes \{\boxtimes_{p\in I,\, j} \mathcal{O}_{\mathbb{G}_{j,p}}(b_{j,p})\} \boxtimes \mathcal{O}_{\mathbb{P}}(a_1)$$

be the polarization.

We fix a sufficiently large t such that

1. (6.7) holds;

2. (6.8) holds if and only if it holds as an inequality of polynomials in t.

Corollary VI.25. *If* $([q], \{[f_p]\}, [\phi]) \in \bar{R}$ *is GIT-semistable, then*

$$H^0(q(m)) : V \to H^0(E(m))$$

is injective and E *is torsion free.*

Proof. It is straightforward to check that the coefficient of t on the RHS of the inequality (6.8) equals

$$n \cdot \frac{l \dim W + \theta_W(\phi)\delta l - \sum_{p\in I}\sum_{1\le j\le l_p-1}(a_{j+1,p}-a_{j,p})r^W_{j,p}}{l\dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}}.$$

Let W be the kernel of $H^0(q(m)) : V \to H^0(E(m))$. Then $G = q(W\otimes\mathcal{O}(-m)) = 0$. The LHS of the inequality (6.8) is zero, while the coefficient of t on the RHS of the inequality is greater than or equal to

$$\frac{nl\dim W}{l\dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}}.$$

It follows that $W = 0$.

Let T be the torsion subsheaf of E. Since $V \to E(m)$ is surjective, it is easy to show that $H^0(T(m)) \subset V$ as subspaces in $H^0(T(m))$. Let $W = H^0(T(m))$. Suppose $W \neq 0$. Then the coefficient of t on the RHS of the inequality (6.8) is positive because

$$\sum_{p\in I}\sum_{1\le j\le l_p-1}(a_{j+1,p}-a_{j,p})r^W_{j,p} \le \sum_{p\in I}(a_{l_p,p}-a_{1,p})r^W_{l_p-1,p} < l\dim W.$$

Here we use the fact that $a_{l_p,p} - a_{1,p} < l$. We get a contradiction because the LHS of (6.8) is a constant. Therefore we must have $W = 0$ and $F = 0$. □

Proposition VI.26. *Let $([q], \{[\tilde{f}_p]\}, [\phi])$ be a point in $\bar{R}$ and let (E, s) be the corresponding parabolic N-pair. Then the following are equivalent.*

(1) $([q], \{[\tilde{f}_p]\}, [\phi])$ is GIT-(semi)stable with respect to L.

(2) (E, s) is δ-(semi)stable and q induces an isomorphism $V \xrightarrow{\sim} H^0(E(m))$.

Proof. (1) ⇒ (2): Let $z = ([q], \{[\tilde{f}_p]\}, [\phi])$ be a GIT-semistable point in $\bar{R}$, where $q : V \otimes \mathcal{O}(-m) \to E$ is a quotient. According to Corollary VI.25, E is locally free and q induces an injection $V \hookrightarrow H^0(E(m))$. Let $\pi : E \twoheadrightarrow E''$ be a quotient bundle. Denote by K the kernel of π. We have an exact sequence $0 \to K \to E \xrightarrow{\pi\circ\alpha} E'' \to 0$. Let $W = V \cap H^0(K(m))$. Then

$$h^0(E''(m)) \geq h^0(E(m)) - h^0(K(m)) \geq \dim V - \dim W. \tag{6.9}$$

Let $F = q(W \otimes \mathcal{O}(-m))$. Since F is a subsheaf of K, we have $r(F) \leq r(K) = r(E) - r(E'')$. By comparing the coefficients of t on both sides of inequality (6.8), we have

(6.10)

$$\begin{aligned} n - r(E'') &\geq r(F) \\ &\geq \frac{n \dim W}{\dim V} \cdot \frac{l \dim V + |\underline{a}| - n\sum_{p\in I} a_{l_p,p}}{l \dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}} \\ &+ \theta_W(\phi)\frac{nl\delta}{l \dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}} \\ &+ \sum_{p\in I}\sum_{1\leq j\leq l_p-1} n\frac{a_{j+1,p} - a_{j,p}}{l \dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}}\left(r_{j,p}\frac{\dim W}{\dim V} - r^W_{j,p}\right). \end{aligned}$$

Let $\underline{a}'' = (a''_p)_{p\in I}$ and $\underline{a}' = (a'_p)_{p\in I}$ be the induced parabolic weights of E'' and the kernel K, respectively. It is not difficult to show that the following hold:

$$\sum_{1\leq j\leq l_p-1} (a_{j+1,p} - a_{j,p}) r_{j,p} = n a_{l_p,p} - |a_p|, \text{ and}$$

$$\sum_{1\leq j\leq l_p-1} (a_{j+1,p} - a_{j,p}) r^W_{j,p} \leq (n - r(E'')) a_{l_p,p} - |a'_p|.$$

Then it follows from inequality (6.10) that

$$n - r(E'') \geq \frac{nl \dim W}{l \dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}} + \theta_W(\phi)\frac{nl\delta}{l \dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}} + \frac{n|\underline{a}'| - n(n - r(E''))\sum_{p\in I} a_{l_p,p}}{l \dim V + |\underline{a}| + l\delta - n\sum_{p\in I} a_{l_p,p}}. \tag{6.11}$$

Notice that $|\underline{a}| = |\underline{a}'| + |\underline{a}''|$. Then we can rewrite inequality (6.11) as

$$\frac{\dim V - \dim W + (1 - \theta_W(\phi))\delta}{r(E'')} + \frac{|\underline{a}''|}{r(E'')l} \geq \frac{\dim V + \delta}{n} + \frac{|\underline{a}|}{nl}. \tag{6.12}$$

Note that if $\theta(s'') = \pi \circ s = 0$, then $\operatorname{im}\phi \subset W$ and hence $1 - \theta_W(\phi) = 0$. Therefore $\theta(s'') \geq 1 - \theta_W(\phi)$. Combining (6.12) and (6.9), we have

$$\frac{h^0(E''(m)) + \theta(s'')\delta}{r(E'')} + \frac{|\underline{a}''|}{r(E'')l} \geq \frac{P(m) + \delta}{n} + \frac{|\underline{a}|}{nl}.$$

According to Lemma VI.22, the pair (E, s) is semistable.

Let $z = ([q], \{[\tilde{f}_p]\}, [\phi])$ be a GIT-stable point. Suppose (E, s) is not stable. Then by the previous discussion, (E, s) is strictly semistable. Then there exists a destabilizing sub-pair (E', s'). Let $W = H^0(E'(m)) \subset H^0(E(m)) \cong V$. It is clear that $\theta(s') = \theta_W(\phi)$. We have

$$\frac{h^0(E'(m)) + \theta(s')\delta}{r(E')} + \frac{|\underline{a}'|}{r(E')l} = \frac{P(m) + \delta}{n} + \frac{|\underline{a}|}{nl}.$$

By an elementary calculation, one can show that the RHS of the inequality (6.8) is equal to $P_{E'}(t)$. It contradicts with the fact that $z = ([q], \{[\tilde{f}_p]\}, [\phi])$ is GIT-stable.

(2) $\Rightarrow$ (1): If (E, s) is stable and $q(m)$ induces an isomorphism between global

sections. For any nontrivial subspace $W \subsetneq V$, let $F = q(W \otimes \mathcal{O}(-m))$ and let (F, s') be the corresponding sub-pair. If $(F, s') = (E, s)$, then inequality (6.8) obviously holds. Thus we assume that (F, s') is a proper sub-pair. By Lemma VI.22, we have

$$\frac{h^0(F(m)) + \theta(s')\delta}{r(F)} + \frac{|\underline{a}'|}{r(F)l} < \frac{h^0(E(m)) + \delta}{n} + \frac{|\underline{a}|}{nl}.$$

The above inequality is equivalent to

$$r(F) > n\frac{|\underline{a}'| + l\,h^0(F(m)) + \theta(s')\delta l - r(F)\sum_{p\in I} a_{l_p,p}}{l\dim V + \delta l + |\underline{a}| - n\sum_{p\in I} a_{l_p,p}}. \tag{6.13}$$

Notice that $\dim W \le h^0(F(m))$, which follows from the following commutative diagram.

$$\begin{array}{ccc} W & \longrightarrow & H^0(F(m)) \\ \big\downarrow & & \big\downarrow \\ V & \xrightarrow{\ \cong\ } & H^0(E(m)) \end{array}$$

By combining the inequality (6.13), $\dim W \le h^0(F(m))$ and

$$\sum_{p\in I}\sum_{1\le j\le l_p-1}(a_{j+1,p} - a_{j,p})r^W_{j,p} = -|\underline{a}'| + r(F)\sum_{p\in I} a_{l_p,p},$$

we obtain

$$r(F) > n\frac{l\dim W + \theta(s')\delta l - \sum_{p\in I}\sum_{1\le j\le l_p-1}(a_{j+1,p} - a_{j,p})r^W_{j,p}}{l\dim V + \delta l + |\underline{a}| - n\sum_{p\in I} a_{l_p,p}}.$$

It implies that the leading coefficient of $P_F(t)$ is great than the leading coefficient of the polynomial on the right hand side of (6.8). Therefore, $([q], \{[\tilde{f}_p]\}, [\phi])$ is GIT-stable.

Assume (E, s) is strictly δ-semistable. We need to show that the corresponding point $([q], \{[\tilde{f}_p]\}, [\phi])$ is GIT-semistable. Choose any nontrivial subspace $W \subsetneq V$. Let

$F = q(W \otimes \mathcal{O}(-m))$ and let (F, s') be the corresponding sub-pair. Since all these F are in a bounded family, we can assume $h^0(F(m)) = \chi(F(m))$. As discussed in the previous case, if $(F, s') = (E, s)$ or (F, s') is not a destabilizing sub-pair, we are done. Therefore, we assume (F, s') is a destabilizing sub-pair such that $\dim W = h^0(F(m))$ and

$$r(F) = n\frac{|\underline{a}'| + l\dim W + \theta(s')\delta l - r(F)\sum_{p\in I} a_{l_p,p}}{l\dim V + \delta l + |\underline{a}| - n\sum_{p\in I} a_{l_p,p}}.$$

This shows that the coefficients of t on both sides of the inequality (6.8) are equal. A tedious but elementary computation shows that the constant terms of the left hand side of (6.8) is also equal to the constant term on the right hand side. This concludes the proof. We leave the details to the reader.

□

Recall that a value of $\delta \in \mathbb{Q}_+$ is called critical, or a wall if there are strictly δ-semistable N-pairs.

Lemma VI.27. *For fixed d, n and parabolic type $(\underline{a}, \underline{m})$, there are only finitely many critical values of δ.*

Proof. It suffices to show that the destabilizing sub-pairs form a bounded family. The same arguments used in [54, Proposition 6] work here. □

Theorem VI.28. *If δ is generic, the moduli groupoid $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N), d, \underline{a})$ of non-degenerate δ-stable parabolic N-pairs is isomorphic to $\bar{R} /\!\!/_L \mathrm{SL}(V)$. In particular, it is a projective variety.*

Proof. The proof is standard. See for example the proof of [54, Theorem 1]. □

Remark VI.29. The GIT construction of the moduli space $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ also works in the case $g \leq 1$. However, for some choices of the parabolic type $(\underline{a},\underline{m})$, the moduli space $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ is empty when the stability parameter δ is sufficiently close to zero. This is because by definition, when δ is sufficiently close to zero, the underline parabolic vector bundle E of a δ-stable pair (E,s) is parabolic semistable (see Section 6.5), and the moduli space $U(n,d,\underline{a},\underline{m})$ of S-equivalence classes of semistable parabolic vector bundles may be empty for some parabolic types $(\underline{a},\underline{m})$ (cf. [12, §5]). In this paper, we only consider parabolic types $(\underline{a},\underline{m})$ such that $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ is nonempty for all generic δ.

Remark VI.30. In the definition of the parabolic data, we assume that the last parabolic weight $a_{l_p,p}$ is less than l. In the case $a_{l_p,p}=l$, the (coarse) moduli space of S-equivalence classes of semistable *parabolic sheaves* is constructed in [72]. However, there are some differences in this new case. According to [72, Remark 2.4], when $a_{l_p,p}=l$, a strictly semistable parabolic sheaf can have torsion supported on the marked point $\{p\}$. In the GIT construction of the moduli space in the case $a_{l_p,p}=l$, a point corresponding to a stable parabolic sheaf is strictly GIT semistable (see [72, Proposition 2.12]). In the setting of parabolic N-pairs, if $a_{l_p,p}=l$, all values of δ are critical values and the GIT construction discussed in this section does not produce a fine moduli space. Therefore we only consider the case $a_{l_p,p}<l$ in this paper.

The universal parabolic N-pair $S:\mathcal{O}^N\to\mathcal{E}$ over $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})\times C$ can be constructed using GIT. To be more precise, we have a morphism

$$H^0(\mathcal{O}_C(m))^N \otimes \mathcal{O} \otimes \mathcal{O}_C(-m) \to \widetilde{\mathcal{E}} \otimes \mathcal{O}_{\mathbb{P}}(1)$$

over $\bar{R} \times C$, induced by the universal families (6.5) and (6.6). By the definition of $\bar{R}$, the morphism above induces N sections

$$\widetilde{S} : H^0(\mathcal{O}_C)^N \otimes \mathcal{O} = \mathcal{O}_{\bar{R}}^N \to \widetilde{\mathcal{E}} \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Let z be a point in $\bar{R}$. By Lemma VI.10 and Lemma 4.3.2 in [45], the only stabilizers in $\mathrm{SL}(V)$ of z are the χ_m-root of unity, where $\chi_m = \dim(V)$. They act oppositely on $\widetilde{\mathcal{E}}$ and $\mathcal{O}_{\mathbb{P}}(1)$. Therefore, by Kempf's descent lemma (c.f. [24, Théorèm 2.3]), $\widetilde{\mathcal{E}} \otimes \mathcal{O}_{\mathbb{P}}(1)$ descends to a bundle $\mathcal{E}$ on $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N), d, \underline{a}) \times C$, with N sections $S \in H^0(\mathcal{E} \otimes \mathcal{O}^N)$ induced by $\widetilde{S}$. Moreover, the tautological flags of $\widetilde{\mathcal{E}}|_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})\times\{p\}} \otimes \mathcal{O}_{\mathbb{P}}(1)$ descend to the universal flags of $\mathcal{E}|_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})\times\{p\}}$, for $p \in I$. We denote by $(\mathcal{E}, \{f_p\}, S)$ the universal parabolic N-pair over $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N), d, \underline{a}) \times C$.

Example VI.31. When δ is sufficiently large, the stability condition stabilizes. More precisely, we have the following lemma.

Lemma VI.32. *Let $d_{\mathrm{par}} = d + |\underline{a}|/l$. Suppose $\delta > (n-1)d_{\mathrm{par}}$. Then there is no strictly δ-semistable parabolic N-pair. Furthermore, a parabolic N-pair (E, s) is δ-stable if and only if the N sections generically generate the fiber of E on C.*

Proof. The proof is a direct generalization of the proof of Proposition 3.14 in [11] for N-pairs without parabolic structures. We first show that if (E, s) is δ-semistable, then $s : \mathcal{O}_C^N \to E$ generically generate the fiber of E on C. Suppose s does not generically generates the fiber of E, then it spans a proper subbundle $E' \subsetneq E$. Denote the induced quotient pair by $(E'', \{f_p''\}, s'')$, where $E'' = E/E'$ and $s'' = 0$. Then

$$\mu_{\text{par}}(E'', s'') = \mu_{\text{par}}(E'') \leq d_{\text{par}}(E'') \leq d_{\text{par}}(E) < \frac{d_{\text{par}}(E) + \delta}{n} = \mu_{\text{par}}(E, s)$$

which contradicts with the δ-semistability of (E, s).

We conclude the proof by showing that if $s : \mathcal{O}_C^N \to E$ generically generates the fiber of E, then (E, s) is δ-stable. Let E' be a proper subbundle (equivalently, a saturated subsheaf) of E. Then $s \notin H^0(E' \otimes \mathcal{O}_C^N)$ because s generically generates the fiber of E. Hence, $\mu_{\text{par}}(E', s') = \mu_{\text{par}}(E')$. We only need to show that $d_{\text{par}}(E') \leq d_{\text{par}}(E)$. If this holds, we have

$$\mu_{\text{par}}(E', s') = \mu_{\text{par}}(E') \leq d_{\text{par}}(E') \leq d_{\text{par}}(E) < \frac{d_{\text{par}}(E) + \delta}{n} = \mu_{\text{par}}(E, s),$$

and it implies that (E, s) is δ-stable. To prove $d_{\text{par}}(E') \leq d_{\text{par}}(E)$, we consider the underlying parabolic bundle $(E, \{f_p\})$ of the parabolic N-pair. Suppose the Harder-Narasimhan filtration of $(E, \{f_p\})$ with respect to the parabolic slope of parabolic bundles is given by

$$0 \subsetneq (E_1, \{f_p^1\}) \subsetneq (E_2, \{f_p^2\}) \subsetneq \cdots \subsetneq (E_k, \{f_p^k\}) = (E, \{f_p\}).$$

Here $(E_1, \{f_p^1\})$ is the maximal destabilizing parabolic subbundle of $(E, \{f_p\})$. For all subbundle $E' \subset E$, one has $\mu_{\text{par}}(E') \leq \mu_{\text{par}}(E_1)$. Hence we only need to show that $d_{\mathrm{par}}(E_1) \leq d_{\mathrm{par}}(E)$. Consider the exact sequence

$$0 \to E_{k-1} \to E \to E/E_{k-1} \to 0.$$

Since N sections generically generate the fiber of E, the bundle E/E_{k-1} has non-trivial sections. Thus $d_{\mathrm{par}}(E/E_{k-1}) \geq 0$. By the properties of the Harder-Narasimhan filtration, we obtain $\mu_{\mathrm{par}}(E_i/E_{i-1}) > \mu_{\mathrm{par}}(E/E_{k-1}) \geq 0$ for $i < k$. By induction, we assume that $d_{\mathrm{par}}(E/E_i) \geq 0$ for $i < m$. Then from the exact sequence

$$0 \to E_m/E_{m-1} \to E/E_m \to E/E_{m-1} \to 0,$$

it follows that $d_{\mathrm{par}}(E/E_m) = d_{\mathrm{par}}(E_m/E_{m-1}) + d_{\mathrm{par}}(E/E_{m-1}) \geq 0$. In particular, it shows that $d_{\mathrm{par}}(E/E_1) \geq 0$ and hence $d_{\mathrm{par}}(E_1) = d_{\mathrm{par}}(E) - d_{\mathrm{par}}(E/E_1) \leq d_{\mathrm{par}}(E)$. $\square$

Let $\overline{\mathcal{M}}_Q(d,n,N)$ be the Grothendieck's Quot scheme which parametrizes quotients $\mathcal{O}_C^N \to Q \to 0$, where Q is a coherent sheaf on C of rank n and degree d. Let $0 \to \mathcal{F} \to \mathcal{O}^N \to \mathcal{Q} \to 0$ be the tautological exact sequence of universal bundles over $\overline{\mathcal{M}}_Q(d,n,N) \times C$. We denote by $\mathcal{E} = \mathcal{F}^\vee$. Let $I = \{p_1, \ldots, p_k\}$ be the set of marked points and let $\mathrm{Fl}_{m_p}(\mathcal{E}_{p_i})$ be the relative flag variety of type m_{p_i}, where $\mathcal{E}_{p_i} = \mathcal{E}|_{\overline{\mathcal{M}}_Q(d,n,N)\times\{p_i\}}$. We define

$$\mathrm{Fl}_{\mathrm{Quot}} = \mathrm{Fl}_{m_1}(\mathcal{E}_{p_1}) \times_{\overline{\mathcal{M}}_Q(d,n,k)} \cdots \times_{\overline{\mathcal{M}}_Q(d,n,k)} \times \mathrm{Fl}_{m_k}(\mathcal{E}_{p_k}).$$

By Lemma VI.32, the moduli space of δ-stable parabolic N-pairs is isomorphic to $\mathrm{Fl}_{\mathrm{Quot}}$ when $\delta > (n-1)d_{\mathrm{par}}$.

6.4 Perfect obstruction theory

In this section, we show that for a generic value of δ, the moduli space of δ-stable parabolic N-pairs $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ has a canonical perfect obstruction. We construct a virtual structure sheaf on the moduli space and discuss its basic properties.

The following proposition follows from Proposition VI.17 and the same argument as in [68, Proposition 1.12].

Proposition VI.33. *The morphism q locally factorizes as the composition of a closed embedding followed by a smooth morphism.*

Let $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \xrightarrow{\iota} M \xrightarrow{p} \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ be a local factorization of the forgetful morphism $q : \mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \to \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$. Denote by $\mathcal{I}$ the ideal sheaf of $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \to M$. Let Ω be the relative cotangent sheaf of $M \to \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ and let L_q be the cotangent complex of the morphism q. Then the truncated cotangent complex $\tau_{\geq -1}L_q$ is isomorphic to $[\mathcal{I}|_{\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})} \to \Omega|_{|\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})}]$.

Let $\bar{\mathcal{E}}$ be the universal bundle over $\mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \times C$. Let

$$\bar{\pi} : \mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a}) \times C \to \mathfrak{M}_C^{\mathrm{par}}(\mathrm{Gr}(n,N),d,\underline{a})$$

be the projection and let $\bar{\omega}$ be the relative dualizing sheaf of $\bar{\pi}$.

Proposition VI.34. *There is a canonical morphism*

$$E^{\bullet} := R\bar{\pi}_*((\bar{\mathcal{E}}^{\vee})^N \otimes \bar{\omega}[1]) \to L_q$$

which induces a relative perfect obstruction theory for $q : \mathfrak{M}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a}) \to \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$.

Proof. The same arguments used in [68, Proposition 2.4, Proposition 2.6] work here.

□

Proposition VI.35. *The relative perfect obstruction theory* $E^\bullet \to \tau_{\geq 1} L_q$ *induces an absolute perfect obstruction theory on* $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$.

Proof. For simplicity, we denote the smooth stack $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ by $\mathbf{B}$. Consider

$$\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a}) \xrightarrow{q} \mathbf{B} \to \operatorname{Spec}\mathbb{C}.$$

We have a distinguished triangle of cotangent complexes

$$\mathbb{L}q^* L_{\mathbf{B}} \to L_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})} \to L_q \to \mathbb{L}q^* L_{\mathbf{B}}[1].$$

By Proposition VI.34, we have a canonical morphism $g : E^\bullet \to L_q$ which induces the relative perfect obstruction theory for q. We define $F^\bullet$ to be the shifted mapping cone $C(f)[-1]$ of the composite morphism:

$$f : E^\bullet \to L_q \to \mathbb{L}q^* L_{\mathbf{B}}[1].$$

By the axioms of triangulated categories, we have a morphism $F^\bullet \to L_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})}$.

The moduli stack $\mathbf{B} = \mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ is a fiber product of flag bundles over the moduli stack of vector bundles $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$. Therefore, $\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a})$ is smooth and the cotangent complex $L_{\mathbf{B}}$ is isomorphic to a two-term complex concentrated at [0,1]. Also note that $H^1(L_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})}) = 0$ because the moduli space $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ is a scheme. Then it is straightforward to check that this induces a perfect obstruction theory on $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$. □

Remark VI.36. Let $T_{\mathbf{B}}$ be the tangent complex of $\mathbf{B}$, dual to $L_{\mathbf{B}}$. By the definition of $F^\bullet$, we have a distinguished triangle

$$F^\bullet \to E^\bullet \to \mathbb{L}_q^* L_{\mathbf{B}}[1] \to F^\bullet[1].$$

By taking its dual, we have

$$\mathbb{L}_q^* T_{\mathbf{B}}[-1] \to (E^\bullet)^\vee \to (F^\bullet)^\vee \to \mathbb{L}_q^* T_{\mathbf{B}}.$$

It induces a long exact sequence of cohomology sheaves

$$\begin{aligned} 0 \to H^0(\mathbb{L}_q^* T_{\mathbf{B}}[-1]) \to H^0\big((E^\bullet)^\vee\big) \to H^0\big((F^\bullet)^\vee\big) \to \\ \to H^1(\mathbb{L}_q^* T_{\mathbf{B}}[-1]) \to H^1\big((E^\bullet)^\vee\big) \to H^1\big((F^\bullet)^\vee\big) \to 0. \end{aligned} \tag{6.14}$$

Let $z = (E,s) \in \overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ be a closed point and let $t = [E] \in \mathbf{B}$. The fiber of the locally free sheaf $H^1(\mathbb{L}_q^* T_{\mathbf{B}}[-1])$ at z is isomorphic to $\mathrm{Ext}^1(E,E)$ and the fiber $H^0(\mathbb{L}_q^* T_{\mathbf{B}}[-1])|_z$ is isomorphic to the infinitesimal automorphism group $\mathrm{Ext}^0(E,E)$. The fiber $H^i\big((E^\bullet)^\vee\big)|_z$ can be identified with $\big(\mathbb{H}^i(E)\big)^N$ for $i = 0,1$. Let $\mathbb{H}^i = \mathbb{H}^{i+1}(\mathcal{P}ar\mathcal{E}nd(E) \to E \otimes \mathcal{O}_C^N)$ be the hypercohomology groups, for $i = 0,1$. We have the following long exact sequence of hypercohomology groups

$$0 \to H^0(\mathcal{P}ar\mathcal{E}nd(E)) \to (H^0(E))^N \to \mathbb{H}^0 \to H^1(\mathcal{P}ar\mathcal{E}nd(E)) \to (H^1(E))^N \to \mathbb{H}^1 \to 0. \tag{6.15}$$

Comparing with (6.14), we can identify the stalks $H^i\big((F^\bullet)^\vee\big)|_z$ with the hypercohomology groups $\mathbb{H}^i$, for $i=0,1$.

Corollary VI.37. *If the degree d is sufficiently large such that $\mu_{\mathrm{par}}(E,s) > 2g-1+|I|+\delta$. Then the moduli space $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ is smooth.*

Proof. If $\mu_{par}(E,s) > 2g-1+|I|+\delta$, then by Lemma VI.13, we have $H^1(E)=0$. It follows from the long exact sequence (6.15) that the obstruction space $\mathbb{H}^1$ vanishes. Therefore, the moduli space is smooth. □

Corollary VI.38. *For generic δ, the moduli space of δ-stable parabolic N-pairs $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ has a virtual structure sheaf*

$$\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})} \in K_0(\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})).$$

Proof. The corollary follows from Proposition VI.35 and the construction in [53, §2.3]. We describe an equivalent construction here. By Proposition VI.35 and Definition 2.2 in [64], one can define a virtual pullback:

$$q^! : K_0(\mathbf{B}) \to K_0(\overline{\mathcal{M}}_C^{\delta}(\mathrm{Gr}(n,N),d)).$$

The virtual structure sheaf is defined as

$$\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})} = q^!(\mathcal{O}_{\mathbf{B}}),$$

where $\mathcal{O}_{\mathbf{B}}$ is the structure sheaf of $\mathbf{B}$. □

The following lemma shows that there are natural embeddings between moduli spaces of stable parabolic N-pairs of different degrees.

Lemma VI.39. *Let D be an effective divisor whose support is disjoint from the set I of ordinary markings. If (E, s) is a δ-(semi)stable parabolic N-pair, then so is $(E(D), s(D))$. Here $s(D)$ is defined as the composition $\mathcal{O}_C^N \hookrightarrow \mathcal{O}_C^N(D) \xrightarrow{s} E(D)$. Conversely, if ϕ vanishes on D and (E, s) is δ-(semi)stable, then so is $(E(-D), s(-D))$.*

Proof. The lemma follows easily from the the fact that for any vector bundle F, we have

$$\mu_{\text{par}}(F(D)) = \deg D + \mu_{\text{par}}(F).$$

□

Let d_D be the degree of D. Lemma VI.39 shows that there is an embedding

$$\iota_D : \overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n, N), d, \underline{a}) \hookrightarrow \overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n, N), d + nd_D, \underline{a}). \tag{6.16}$$

In fact, we can choose D such that $\overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n, N), d, \underline{a})$ is the zero locus of a section of a vector bundle on $\overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n, N), d + nd_D, \underline{a})$. Suppose that D is the sum of d_D distinct points $x_1, \ldots, x_{d_D}$ on C. Let $\mathcal{E}'$ and $\mathcal{E}$ be the universal bundles over $\overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n, N), d, \underline{a}) \times C$ and $\overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n, N), d + nd_D, \underline{a}) \times C$, respectively. We define a vector bundle $\mathcal{E}_D$ on $\overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n, N), d + nd_D, \underline{a})$ by

$$\mathcal{E}_D = \bigoplus_{i=1}^{d_D} \mathcal{E}_{x_i},$$

where $\mathcal{E}_{x_i} := \mathcal{E}|_{\overline{\mathcal{M}}_C^{\text{par},\delta}(\text{Gr}(n,N), d+nd_D, \underline{a}) \times \{x_i\}}$ denotes the restriction of the universal bundle $\mathcal{E}$ to the point x_i.

Proposition VI.40. *There is a canonical section* $S_D \in H^0(\mathcal{E}_D^{\oplus N})$*, induced by the universal* N*-pair* $S : \mathcal{O}^N \to \mathcal{E}$*, and the image of* ι_D *is the scheme-theoretic zero locus of* S_D*. Moreover, we have the following relation between the virtual structure sheaves:*

$$\iota_{D*}\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})} = \lambda_{-1}((\mathcal{E}_D^{\vee})^{\oplus N}) \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d+nd_D,\underline{a})}$$

in $K_0(\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d+nd_D,\underline{a}))$.

Proof. The canonical section S_D is defined by the restrictions of the universal N-pair to $\{x_i\}$, i.e.,

$$S_D = \big(S|_{\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d+nd_D,\underline{a})\times\{x_i\}}\big)_{i\in I}.$$

Suppose Z is the zero-scheme of S_D. Then the restriction of the universal N-pair to $Z\times C$ factors:

$$\mathcal{O}^N_{Z\times C} \xrightarrow{S'} \mathcal{E}|_{Z\times C}(-D) \hookrightarrow \mathcal{E}|_{Z\times C}.$$

By Lemma VI.39, the section S' defines a family of stable parabolic N-pairs over $Z\times C$. Hence S' induces a morphism $Z \to \overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ which is inverse to ι_D. This proves the first part of the proposition.

For simplicity, we denote by $\overline{\mathcal{M}}_d$ and $\overline{\mathcal{M}}_{d+nd_D}$ the moduli spaces $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d,\underline{a})$ and $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N),d+nd_D,\underline{a})$, respectively. Consider the following commutative diagram.

$$\begin{array}{ccc}
\overline{\mathcal{M}}_d & \xhookrightarrow{\quad\iota_D\quad} & \overline{\mathcal{M}}_{d+nd_D} \\
\big\downarrow q' & & \big\downarrow q \\
\mathfrak{Bun}_C^{\mathrm{par}}(d,n,\underline{a}) & \xrightarrow{\ t\ } & \mathfrak{Bun}_C^{\mathrm{par}}(d+nd_D,n,\underline{a})
\end{array}$$

The morphism t is defined by mapping a parabolic bundle E to $E(D)$. Notice that it induces an isomorphism between $\mathfrak{Bun}_C^{\mathrm{par}}(d, n, \underline{a})$ and $\mathfrak{Bun}_C^{\mathrm{par}}(d + nd_D, n, \underline{a})$. Therefore, we can identify these two moduli stacks of parabolic vector bundles by t and use $\mathfrak{B}$ to denote both of them. Consider the morphisms

$$\overline{\mathcal{M}}_d \xrightarrow{\iota_D} \overline{\mathcal{M}}_{d+nd_D} \xrightarrow{q} \mathfrak{B}.$$

Let $q' = q \circ \iota_D$. Let $\pi : C \times \overline{\mathcal{M}}_{d+nd_D} \to \overline{\mathcal{M}}_{d+nd_D}$ and $\pi' : C \times \overline{\mathcal{M}}_d \to \overline{\mathcal{M}}_d$ be the projection maps. By abuse of notation, we denote by ι_D the embedding of $C \times \overline{\mathcal{M}}_d$ into $C \times \overline{\mathcal{M}}_{d+nd_D}$. By Proposition VI.34, we have two relative perfect obstruction theories

$$E^\bullet := R\pi_*((\mathcal{E}^\vee)^N \otimes \omega[1]) \to L_q$$

and

$$E'^\bullet := R\pi'_*((\mathcal{E}'^\vee)^N \otimes \omega[1]) \to L_{q'}.$$

Here ω is the pullback of the dualizing sheaf of C to the universal curve via the projection map. Consider the following short exact sequence

$$0 \to \mathcal{E}' \to \iota_D^*\mathcal{E} \to \mathcal{E}_D \to 0,$$

where $\mathcal{E}_D = \bigoplus_{i=1}^{d_D} \mathcal{E}_{x_i}$. It induces a distinguished triangle

$$\big(R\pi'_*(\iota_D^*\mathcal{E}^N)\big)^\vee \to \big(R\pi'_*((\mathcal{E}')^N)\big)^\vee \to (\mathcal{E}_D^N)^\vee[1] \to \big(R\pi'_*(\iota^*\mathcal{E}^N)\big)^\vee[1].$$

By Grothendieck duality and cohomology and base change, we have $\big(R\pi'_*(\iota_D^*\mathcal{E}^N)\big)^\vee = \mathbb{L}\iota_D^* E^\bullet$ and $\big(R\pi'_*((\mathcal{E}')^N)\big)^\vee = E'^\bullet$. By the axioms of triangulated categories, we obtain a morphism

$$(\mathcal{E}_D^N)^\vee[1] \to L_{\iota_D},$$

and the following morphism of distinguished triangles.

$$\begin{array}{ccccccc}
\mathbb{L}\iota_D^* E^\bullet & \longrightarrow & E'^\bullet & \longrightarrow & (\mathcal{E}_D^N)^\vee[1] & \longrightarrow & \mathbb{L}\iota_D^* E^\bullet[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{L}\iota_D^* L_q & \longrightarrow & L_{q'} & \longrightarrow & L_{\iota_D} & \longrightarrow & \mathbb{L}\iota_D^* L_q[1]
\end{array}$$

over $\overline{\mathcal{M}}_d$. It follows from the long exact sequences in cohomology that $(\mathcal{E}_D^N)^\vee[1] \to L_{\iota_D}$ is a perfect obstruction theory for ι_D. Recall that $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{d+nd_D}} = q^! \mathcal{O}_{\mathfrak{B}}$ and $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_d} = (q')^! \mathcal{O}_{\mathfrak{B}}$. By the functoriality property of virtual pullbacks proved in [64, Proposition 2.11], we have

$$\iota_D^! \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{d+nd_D}} = \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_d}.$$

Let $0_{\mathcal{E}_D} : \overline{\mathcal{M}}_{d+nd_D} \to \mathcal{E}_D^{\oplus N}$ be the zero section embedding. Consider the following Cartesian diagram.

$$\begin{array}{ccc}
\overline{\mathcal{M}}_d & \xrightarrow{\iota_D} & \overline{\mathcal{M}}_{d+nd_D} \\
\iota_D \downarrow & & \downarrow 0_{\mathcal{E}_D} \\
\overline{\mathcal{M}}_{d+nd_D} & \xrightarrow{S_D} & \mathcal{E}_D^{\oplus N}
\end{array}$$

Using the fact that virtual pullbacks commute with push-forward, we obtain

$$\iota_{D*} \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_d} = 0^!_{\mathcal{E}_D} S_{D*} \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{d+nd_D}}.$$

Note that $S_{D*} = 0_{\mathcal{E}_D *}$, since the two sections are homotopic. The proposition follows from the excess intersection formula in K-theory (c.f. [26, Chapter VI]). $\square$

6.5 $(\delta = 0^+)$-chamber and Verlinde type invariants

When δ is sufficiently close to 0, the stability condition stabilizes. We refer to it as the $(\delta = 0^+)$-chamber. The theory of the GLSM at $(\delta = 0^+)$-chamber is related to the theory of semistable bundles in an explicit way. We describe this connection in this section.

We first consider the case without parabolic structures. We assume the genus of C is greater than 1, i.e., $g > 1$. Let δ_+ be the smallest critical value. For $\delta \in (0, \delta_+)$, we denote the moduli space of δ-stable parabolic N-pairs by $\overline{\mathcal{M}}_C^{0+}(\mathrm{Gr}(n,N), d)$. It is not difficult to check for $0 < \delta < \delta_+$,

- If (E, s) is a δ-stable pair then E is a semistable bundle.
- Conversely, if E is stable, then (E, s) is δ-stable for any choice of nonzero $s \in H^0(E \otimes \mathcal{O}_C^N)$.

Let $U_C(n, d)$ be the moduli space of S-equivalence classes of semistable vector bundles of rank n and degree d (cf. [52]). From the analysis above, we have a forgetful morphism

$$q : \overline{\mathcal{M}}_C^{0+}(\mathrm{Gr}(n,N), d) \to U_C(n, d),$$

which forgets N sections. Let $[E] \in U_C(n, d)$ be a closed point where E is a stable bundle. Then (E, s) is δ-stable for any nonzero N sections s. Hence the fibre of q over $[E]$ is $\mathbb{P}H^0(E)$. If $d > n(g-1)$, then any bundle E must have non-zero sections by Riemann-Roch. Therefore the image of the forgetful morphism q contains the non-empty open subset $U_C^s(n, d) \subset U_C(n, d)$ parametrizing isomorphism classes of stable vector bundles. Note that $\overline{\mathcal{M}}_C^{0+}(\mathrm{Gr}(n,N),d)$ is proper and $U_C(n,d)$ is irreducible. Hence we have shown that q is surjective if $d > n(g-1)$.

In the case $(n,d) = 1$, there are no strictly semistable vector bundles and the moduli space $U(n,d)$ is smooth. Moreover, there exists a universal vector bundle $\tilde{\mathcal{E}} \to U(n,d) \times C$ such that for any closed point $[E] \in U(n,d)$, the restriction $\tilde{\mathcal{E}}|_{C\times[E]}$ is a stable bundle of degree d, isomorphic to E. Note that the universal $\tilde{\mathcal{E}}$ is not unique since we can obtain other universal vector bundles by tensoring $\tilde{\mathcal{E}}$ with the pullback of any line bundle on $U(n,d)$. Let $\rho : U(n,d) \times C \to U(n,d)$ be the projection map. Using the same arguments as in the proof of Lemma V.5, one can show that $R^1\rho_*\tilde{\mathcal{E}} = 0$ if $d > 2n(g-1)$. In this case, $\rho_*\tilde{\mathcal{E}}$ is a vector bundle over $U(n,d)$. Let $\mathbb{P}((\rho_*\tilde{\mathcal{E}})^{\oplus N})$ be the projectivization of $(\rho_*\tilde{\mathcal{E}})^{\oplus N}$.

Proposition VI.41. [11, Theorem 3.26] *Suppose $(n,d) = 1$ and $d > 2n(g-1)$. Then we have an isomorphism*

$$\overline{\mathcal{M}}_C^{0+}(\mathrm{Gr}(n,N),d) \cong \mathbb{P}((\rho_*\tilde{\mathcal{E}})^{\oplus N}).$$

Moreover, the above identification gives an isomorphism between the universal N-pair $(\mathcal{E}, S)$ and $(q^*(\tilde{\mathcal{E}}) \otimes \mathcal{O}(1), S')$, where S' is induced by the tautological section of the anti-tautological line bundle $\mathcal{O}(1)$ on the projective bundle $\mathbb{P}((\rho_*\tilde{\mathcal{E}})^{\oplus N})$.

Similar results hold for moduli spaces of δ-stable parabolic N-pairs, when δ is sufficiently small. Let δ_+ be the smallest critical value. When $0 < \delta < \delta_+$, we have

- If $(E, \{f_p\}, s)$ is a δ-stable parabolic N-pair then $(E, \{f_p\})$ is a parabolic semistable bundle.

- Conversely, if $(E, \{f_p\})$ is stable, then $(E, \{f_p\}, s)$ is δ-stable for any non-zero choice of $s \in H^0(E \otimes \mathcal{O}_C^N)$.

Let $U(n, d, \underline{a}, \underline{m})$ be the moduli space of S-equivalence classes of semistable parabolic bundles of rank n, degree d and parabolic type $(\underline{a}, \underline{m})$. For $\delta \in (0, \delta_+)$, we denote the moduli space of δ-stable parabolic N-pairs by $\overline{\mathcal{M}}_C^{\mathrm{par},0+}(\mathrm{Gr}(n,N), d, \underline{a})$.

Theorem VI.42. *Suppose $\mu_{\mathrm{par}}(E) > 2g - 1 + |I|$ and there is no strictly semistable parabolic vector bundle in $U(n, d, \underline{a}, \underline{m})$. Then for $0 < \delta < \delta_+$, we have an isomorphism*

$$\overline{\mathcal{M}}_C^{\mathrm{par},0+}(\mathrm{Gr}(n,N), d, \underline{a}) \cong \mathbb{P}((\rho_* \mathcal{E})^{\oplus N}),$$

where $\rho : C \times U(n, d, \underline{a}, \underline{m}) \to U(n, d, \underline{a}, \underline{m})$ is the projection map. Moreover, the above identification gives an isomorphism between the N-pairs $(\mathcal{E}, S)$ and $(q^*(\tilde{\mathcal{E}}) \otimes \mathcal{O}(1), S')$, where $\mathcal{S}'$ is induced by the tautological section.

Proof. The proof is identical to the proof of Theorem VI.41. □

6.5.1 Verlinde invariants and parabolic GLSM invariants

We first recall the definition of theta line bundles over moduli spaces of S-equivalence classes of semistable parabolic bundles. Then we generalize it to the moduli space of δ-stable parabolic N-pairs.

Recall that $I = \{p_1, \ldots, p_k\}$ is the set of ordinary marked points. Let $\mathrm{P}_l' \subset \mathrm{P}_l$ be the subset of partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying $\lambda_1 < l$. Let $\underline{\lambda} = (\lambda_{p_1}, \ldots, \lambda_{p_k})$, where $\lambda_{p_i} = (\lambda_{1,p_i}, \ldots, \lambda_{n,p_i})$ is a partition in P_l', for $1 \le i \le k$. For each partition

λ_p, $p \in I$, let $r_p = (r_{1,p}, \dots, r_{l_p,p})$ be the sequence of jumping indices of λ_p (i.e. $l > \lambda_{1,p} = \cdots = \lambda_{r_{1,p},p} > \lambda_{r_{1,p}+1,p} = \cdots = \lambda_{r_{2,p},p} > \dots$). For $1 \leq i \leq l_p$, let $m_{i,p} = r_{i,p} - r_{i-1,p}$. We define the parabolic weights $a_p = (a_{1,p}, \dots, a_{l_p,p})$ by $a_{j,p} = l - 1 - \lambda_{r_{j,p},p}$ for $1 \leq j \leq l_p$. The assumption $\lambda_{p_i} \in \mathrm{P}'_l$ ensures that $a_{l_{p_i},p_i} < l$. Let $\underline{a} = (\underline{a}_p)_{p \in I}$ and $\underline{m} = (m_p)_{p \in I}$ be the parabolic type determined by $\underline{\lambda}$. In the following discussion, we will denote the parabolic type by $\underline{\lambda}$.

Let $U(n, d, \underline{\lambda})$ denote the moduli space of S-equivalence classes of semistable parabolic vector bundles of rank n, degree d and parabolic type $\underline{\lambda}$. We recall the construction of $U(n, d, \underline{\lambda})$ and we will use the same notations as in Section 6.3. The family of semistable parabolic vector bundles is bounded. Therefore there exists a sufficiently large $m \in \mathbb{N}$ such that for any semistable parabolic bundle $(E, \{f_p\})$, it can be realized as a quotient $q : H^0(E(m)) \otimes \mathcal{O}_C(-m) \twoheadrightarrow E$. Let V be a vector space of dimension $\chi_m := \chi(E(m))$. Define an open subset $Z' \subset \mathrm{Quot}_C^{n,d}(V \otimes \mathcal{O}_C(-m))$ which consists of points $[q]$ such that the quotient sheaf E is locally free and q induces an isomorphism $V \xrightarrow{\sim} H^0(E(m))$. For each marked point $p \in I$, we consider the restriction of the universal quotient sheaf $\widetilde{\mathcal{E}}_p := \widetilde{\mathcal{E}}|_{Z \times \{p\}}$. Let Fl_{m_p} denote the flag bundle of $\widetilde{\mathcal{E}}_p$ of type $m_p = (m_{i,p})$. Define T to be the fiber product

$$T := \mathrm{Fl}_{m_{p_1}} \times_Z \cdots \times_Z \mathrm{Fl}_{m_{p_k}}.$$

Given a parabolic type $\underline{\lambda}$, one can choose a $\mathrm{SL}(V)$-linearized ample line bundle L' such that the moduli space of semistable parabolic vector bundles of type $\underline{\lambda}$ is the

GIT quotient

$$U(n,d,\underline{\lambda}) = T^{ss} /\!\!/_{L'} \mathrm{SL}(V)$$

where T^{ss} denotes the open semistable locus in T.

We assume that

$$ld - |\underline{\lambda}| \equiv 0 \mod n.$$

Recall that $d_{i,p} = a_{i+1,p} - a_{i,p}$ for $1 \le i \le l_p$, where $a_{l_p+1,p} := l-1$. Let $\widetilde{\mathcal{Q}}_{i,p}$ be the universal quotient bundle of rank $r_{i,p} = \sum_{j=1}^{i} m_{j,p}$ over Fl_{m_i}. Set

$$e = \frac{ld - \sum_{p\in I}\sum_{i=1}^{l_p} d_{i,p} r_{i,p}}{n} + l(1-g).$$

Notice that $\sum_{i=1}^{l_p} d_{i,p} r_{i,p} = n(l-1) - |a_p| = |\lambda_p|$. Let $\pi : T \times C \to T$ be the projection to the first factor. Let $x_0 \in C$ be the distinguished marked point which is away from I. Following [62], we consider the following line bundle over T:

$$\Theta_{\widetilde{\mathcal{E}}} = \left(\det R\pi_*(\widetilde{\mathcal{E}})\right)^{-l} \otimes \bigotimes_{p\in I} \tilde{L}_{m_p} \otimes (\det \widetilde{\mathcal{E}}_{x_0})^e$$

where $\tilde{L}_{m_p}$ are the Borel-Weil-Bott line bundles defined by

$$\tilde{L}_{m_p} = \bigotimes_{i=1}^{l_p} \det \widetilde{\mathcal{Q}}_{i,p}^{d_{i,p}}.$$

The calculation in the proof of [62, Théorème 3.3] shows that $\Theta_{\widetilde{\mathcal{E}}}$ descends to a line bundle $\Theta_{\underline{\lambda}} \to U(n,d,\underline{\lambda})$. Global sections of $\Theta_{\underline{\lambda}}$ are called *generalized theta functions* and the space of global sections $H^0(\Theta_{\underline{\lambda}})$ is isomorphic to the dual of the space of *conformal blocks* (cf. [8] and [62]). The *GL Verlinde numbers* are defined to be Euler characteristics of the type $\chi(U(n,d,\underline{\lambda}), \Theta_{\underline{\lambda}})$ (see [58]).

The following lemma shows that we can define similar theta line bundles on the moduli spaces of parabolic N-pairs.

Lemma VI.43. *Let $\mathcal{E}$ be the universal bundle over $\overline{\mathcal{M}}_C^{\mathrm{par},0+}(\mathrm{Gr}(n,N),d,\underline{\lambda})\times C$ and let $q:\overline{\mathcal{M}}_C^{\mathrm{par},0+}(\mathrm{Gr}(n,N),d,\underline{\lambda})\to U(n,d,\underline{\lambda})$ be the forgetful morphism. Then we have the following identification*

$$q^*\Theta_{\underline{\lambda}}=\big(\det R\pi_*(\mathcal{E})\big)^{-l}\otimes\bigotimes_{p\in I}L_{m_p}\otimes(\det\mathcal{E}_{x_0})^e,$$

where L_{λ_p} are the Borel-Weil-Bott line bundles defined by

$$L_{m_p}=\bigotimes_{i=1}^{l_p}\det\mathcal{Q}_{i,p}^{d_{i,p}}.$$

Proof. By definition, $\Theta_{\underline{\lambda}}$ is the descent of $\Theta_{\widetilde{\mathcal{E}}}=\big(\det R\pi_*(\widetilde{\mathcal{E}})\big)^{-l}\otimes\bigotimes_{p\in I}\tilde{L}_{m_p}\otimes(\det\widetilde{\mathcal{E}}_{x_0})^e$. Let $\tilde{q}:R\to T$ be the flag bundle map, which is in particular flat. Then we have

$$\begin{aligned}\tilde{q}^*(\Theta_{\mathcal{E}})&=\big(\det R\pi_*(\widetilde{\mathcal{E}})\big)^{-l}\otimes\bigotimes_{p\in I}\tilde{L}_{m_p}\otimes(\det\widetilde{\mathcal{E}}_{x_0})^e\\&=\big(\det R\pi_*(\widetilde{\mathcal{E}}\otimes\mathcal{O}_{\mathbb{P}}(1))\big)^{-l}\otimes\Big\{\bigotimes_{p\in I}\tilde{L}_{m_p}\otimes\mathcal{O}_{\mathbb{P}}(1)\Big\}\otimes(\det\widetilde{\mathcal{E}}_{x_0}\otimes\mathcal{O}_{\mathbb{P}}(|\lambda_p|))^e,\end{aligned}$$

which descends to $\big(\det R\pi_*(\mathcal{E})\big)^{-l}\otimes\bigotimes_{p\in I}L_{m_p}\otimes(\det\mathcal{E}_{x_0})^e$, □

Parabolic N-pairs can be viewed as parabolic GLSM data. We give the following definition of parabolic GLSM invariants.

Definition VI.44. For a generic value of δ and partitions $\lambda_{p_1}, \dots \lambda_{p_k} \in \mathrm{P}'_l$, we define the δ-stable parabolic GLSM invariant with insertions $V_{\lambda_{p_1}}, \dots, V_{\lambda_{p_k}}$ by

$$\langle \det(E)^e | V_{\lambda_{p_1}}, \dots, V_{\lambda_{p_k}} \rangle_{C,d}^{l,\delta,\mathrm{Gr}(n,N)}$$
$$= \chi(\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N), d, \underline{\lambda}), \big(\det R\pi_*(\mathcal{E})\big)^{-l} \otimes \bigotimes_{p\in I} L_{\lambda_p} \otimes (\det \mathcal{E}_{x_0})^e).$$

In the ($\delta = 0+$)-chamber, to relate the parabolic GLSM invariants with GL Verlinde numbers, we recall the following result from [13].

Lemma VI.45. [13, Theorem 3.1] *Let $f : X \to Y$ be a surjective morphism of projective varieties with rational singularities. Assume that the general fiber of f is rational, i.e., $f^{-1}(y)$ is an irreducible rational variety for all closed points in a dense open subset of Y. Then $f_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in K_0(Y)$.*

Let $U_C^s(n,d,\underline{\lambda}) \subset U_C(n,d,\underline{\lambda})$ denote the (possibly empty) open subset which parametrizes isomorphism classes of stable vector bundles.

Corollary VI.46. *Suppose $d > n(g-1)$ and $U_C^s(n,d,\underline{\lambda})$ is non-empty. Then the ($\delta =$ $0+$)-stable parabolic GLSM invariants are equal to the corresponding GL Verlinde numbers, i.e.,*

$$\langle \det(E)^e | V_{\lambda_{p_1}}, \dots, V_{\lambda_{p_k}} \rangle_{C,d}^{l,\delta,\mathrm{Gr}(n,N)} = \chi(U(n,d,\underline{\lambda}), \Theta_{\underline{\lambda}}).$$

Proof. According to [71, Theorem 1.1], the moduli space $U(n,d,\underline{\lambda})$ is a normal projective variety with only rational singularities. Let $[E]$ be a closed point in $U(n,d,\underline{\lambda})$, where E is a stable parabolic bundle. Then the fibre of q over $[E]$ is $\mathbb{P}H^0(E)$. When

$d > n(g-1)$, any bundle E must have non-zero sections by Riemann-Roch. Therefore the image of the forgetful morphism q contains the non-empty open subset $U_C^s(n,d,\underline{\lambda}) \subset U_C(n,d,\underline{\lambda})$. Since $\overline{\mathcal{M}}_C^{0+}(\mathrm{Gr}(n,N),d)$ is proper and $U_C(n,d,\underline{\lambda})$ is irreducible, the morphism q is surjective. Then the corollary follows from Lemma VI.43, Lemma VI.45 and the projection formula. □

Remark VI.47. It follows from [72, Proposition 4.1] that $U_C^s(n,d,\underline{\lambda})$ is non-empty if

$$(n-1)(g-1) + \frac{|I|}{l} > 0, \tag{6.17}$$

where $|I|$ is the number of marked points. This condition is automatically satisfied when $g \geq 2$. When $g = 1$, we require $|I|$ to be non-empty. Therefore, inequality (6.17) is a primarily a condition for the genus 0 case.

We end this section by studying the relation of parabolic GLSM invariants with respect to the embedding (6.16).

Lemma VI.48. *Let $\mathcal{E}$ and $\mathcal{E}'$ be the universal bundle over $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N), d + nd_D, \underline{\lambda}) \times C$ and $\overline{\mathcal{M}}_C^{\mathrm{par},\delta}(\mathrm{Gr}(n,N), d, \underline{\lambda}) \times C$, respectively. Denote the corresponding Borel-Weil-Bott line bundles by L_{λ_p} and L'_{λ_p}, respectively. Let $\mathcal{D}_d = \big(\det R\pi_*(\mathcal{E}')\big)^{-l} \otimes \bigotimes_{p\in I} L'_{\lambda_p} \otimes (\det \mathcal{E}'_{x_0})^{e'}$ and $\mathcal{D}_{d+nk} = \big(\det R\pi_*(\mathcal{E})\big)^{-l} \otimes \bigotimes_{p\in I} L_{\lambda_p} \otimes (\det \mathcal{E}_{x_0})^{e}$ be the corresponding determinant line bundles. Then*

$$\iota_D^* \mathcal{D}_{d+nk} = \mathcal{D}_d \otimes \big((\det \mathcal{E}_{x_0})^{kl} \otimes (\det \mathcal{E}_D)^{-l}\big).$$

Proof. Consider the short exact sequence

$$0 \to \mathcal{E}' \to \iota_D^* \mathcal{E} \to \mathcal{E}_D \to 0.$$

Then we have

$$\iota_D^* \det R\pi_*(\mathcal{E}) = \det R\pi_*(\mathcal{E}') \otimes \det \mathcal{E}_D$$

and

$$\iota_D^* L_{\lambda_p} = L'_{\lambda_p}, \quad \iota_D^* \mathcal{E}_{x_0} = \mathcal{E}'_{x_0}.$$

This concludes the proof. □

Corollary VI.49. *We have*

$$\chi(\overline{\mathcal{M}}_d, \mathcal{D}_d \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_d}) = \chi(\overline{\mathcal{M}}_{d+nd_D}, \mathcal{D}_{d+nd_D} \otimes \lambda_{-1}(\mathcal{E}_D^\vee)^N \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{d+nd_D}}).$$

Proof. Using the same argument as in the proof of [72, Theorem 3.1], one can show that $(\det \mathcal{E}_{x_0})^{kl} \otimes (\det \mathcal{E}_D)^{-l}$ and the trivial sheaf $\mathcal{O}$ are algebraically equivalent. The corollary follows from Proposition VI.40 and Lemma VI.48.

□

6.6 Parabolic δ-wall-crossing in rank two case

In this section, we prove Theorem I.10 in the rank two case. According to Remark VI.8, when $n = 1$, the moduli space of δ-stable parabolic N-pairs is independent of δ. In fact, by Theorem VI.42, the moduli space of δ-stable parabolic N-pairs of rank 1 is isomorphic to a projective bundle over $U(1, d, \underline{a}, \underline{m})$ for all δ. Therefore, the δ-wall-crossing is trivial in the rank one case.

Let us restate Theorem I.10 in the rank 2 case.

Theorem VI.50. *Assume $n = 2$. Suppose that $N \geq 2 + l$, $d > 2g - 2 + k$, and δ is generic. Then,*

$$\langle \det(E)^e | V_{\lambda_{p_1}}, \ldots, V_{\lambda_{p_k}} \rangle_{C,d}^{l,\delta,\mathrm{Gr}(2,N)}$$

is independent of δ.

The proof of the above theorem is very similar to the one given in Section 5.2. We fix the degree d and the parabolic type $\underline{a}$. For a critical value δ_c, the underlying vector bundle of a strictly δ_c-semistable parabolic N-pair (E, s) must split: $E = L \oplus M$ where L, M are line bundles of degrees d' and d'', respectively, and $s \in H^0(L \otimes \mathcal{O}_C^N)$. Let $\underline{a}'$ and $\underline{a}''$ be the induced parabolic structures on L and M, respectively. Then the following equalities hold:

$$d' + \delta_c + \frac{|\underline{a}'|}{l} = \frac{d + \delta_c}{2} + \frac{|\underline{a}|}{2l}, \text{ and} \tag{6.18}$$

$$d'' + \frac{|\underline{a}''|}{l} = \frac{d + \delta_c}{2} + \frac{|\underline{a}|}{2l}. \tag{6.19}$$

Since L has non-zero sections, we have $d' > 0$. The equality (6.18) implies that

$$\delta_c < d + \frac{|\underline{a}| - 2|\underline{a}'|}{l} \leq d + k,$$

where $k = |I|$ is the number of ordinary marked points.

Let $\nu > 0$ be a small real number such that δ_c is the only critical value in $[\delta_c - \nu, \delta_c + \nu]$. For simplicity, we denote by $\mathcal{M}_{\delta_c}^+$ (resp., $\mathcal{M}_{\delta_c}^-$) the moduli space $\overline{\mathcal{M}}_C^{\mathrm{par},\delta_c+\nu}(\mathrm{Gr}(n,N), d, \underline{a})$ (resp., $\overline{\mathcal{M}}_C^{\mathrm{par},\delta_c-\nu}(\mathrm{Gr}(n,N), d, \underline{a})$). Let $\mathcal{W}_{\delta_c}^+$ be the subscheme of $\mathcal{M}_{\delta_c}^+$ parametrizing $(\delta_c + \nu)$-pairs which are not $(\delta_c - \nu)$-stable. Similarly, we denote by $\mathcal{W}_{\delta_c}^-$ the subscheme of $\mathcal{M}_{\delta_c}^-$ which parametrizes $(\delta_c - \nu)$-pairs which are not $(\delta_c + \nu)$-stable.

Let (E, s) be an N-pair in $\mathcal{W}_{\delta_c}^-$. It follows from the definition that there exists a short exact sequence

$$0 \to L \to E \to M \to 0,$$

satisfying the following properties:

1. L, M are line bundles of degree d' and d'', respectively, such that $d' + d'' = d$.
2. $s \in H^0(L \otimes \mathcal{O}_C^N)$.
3. $d' + \delta_c + |\underline{a}'|/l = (d + \delta_c)/2 + |\underline{a}|/(2l)$, where $\underline{a}'$ is the induced parabolic structure on L. Equivalently, we have $d'' + |\underline{a}''|/l = (d + \delta_c)/2 + |\underline{a}|/(2l)$, where $\underline{a}''$ is the induced parabolic structure on M.

Notice that L and M are unique since L is the saturated subsheaf of E containing s. Similarly, for a parabolic pair (E, s) in $\mathcal{W}_{\delta_c}^+$. There exists a unique sub line bundle M of E of degree d'' which fits into a short exact sequence

$$0 \to M \to E \to L \to 0.$$

Here $s \notin H^0(M \otimes \mathcal{O}_C^N)$ and the degree d'' satisfies $d'' + |\underline{a}''|/l = (d + \delta_c)/2 + |\underline{a}|/(2l)$.

Let $\tilde{\mathcal{L}}_{d'}$ be a Poincaré bundle over $\mathrm{Pic}^{d'} C \times C$ and let $p : \mathrm{Pic}^{d'} C \times C \to \mathrm{Pic}^{d'} C$ be the projection. If $d' > 2g - 1$, the higher derived image $R^1 p_* \tilde{\mathcal{L}}_{d'} = 0$. Let $U = (R^0 p_* \tilde{\mathcal{L}}_{d'})^N$. We define $Z_{d'} := \mathbb{P}U \times \mathrm{Pic}^{d''} C$. Let $\mathcal{M}_{d''}$ be a Poincaré bundle over $\mathrm{Pic}^{d''} C \times C$. Note that $H^0(\mathrm{Pic}^{d'} C, \mathrm{End}\, U) = H^0(\mathrm{Pic}^{d'} C \times C, U^\vee \otimes \tilde{\mathcal{L}}_{d'} \otimes \mathcal{O}^N) =$

$H^0(\mathbb{P}U \times C, \mathcal{O}_{\mathbb{P}U}(1) \otimes \tilde{\mathcal{L}}_{d'} \otimes \mathcal{O}^N)$. The identity automoprhism of U gives rise to a tautological section of $\mathcal{L}_{d'} \otimes \mathcal{O}^N$, where $\mathcal{L}_{d'} := \mathcal{O}_{\mathbb{P}U}(1) \otimes \tilde{\mathcal{L}}_{d'}$. This tautological section induces an injective morphism $g : \mathcal{M}_{d''}\mathcal{L}_{d'}^{-1} \to \mathcal{M}_{d''} \otimes \mathcal{O}^N$. Let a' and a'' be parabolic weights such that (6.18) and (6.19) hold. Let $\mathcal{L}_{d',a'}$ and $\mathcal{M}_{d'',a''}$ be the unique parabolic line bundles corresponding to $\mathcal{L}_{d'}$ and $\mathcal{M}_{d''}$, respectively. Note that we have an injection

$$i : \mathcal{P}ar\mathcal{H}om(\mathcal{L}_{d',a'}, \mathcal{M}_{d'',a''}) \hookrightarrow \mathcal{H}om(\mathcal{L}_{d,,a,}, \mathcal{M}_{d'',a''}).$$

Let f be the composition $g \circ i$. Denote by $\mathcal{F}_{d',a'}$ the cokernel of f. Let $\pi : Z_{d'} \times C \to Z_{d'}$ be the projection. By abuse of notation, we use the same notations $\mathcal{M}_{d'',a''}$ and $\mathcal{L}_{d',a'}$ to denote the pullback of the corresponding universal line bundles to $Z_{d'} \times C$.

The flip loci $\mathcal{W}^{\pm}_{\delta_c}$ are characterized by the following proposition.

Proposition VI.51. *Assume* $(d-\delta)/2 - k > 2g - 1$ *for* $\delta \in [\delta_c - \nu, \delta_c + \nu]$. *Let* $\mathcal{V}^+_{d',a'} = R^0\pi_*(\mathcal{F}_{d',a'})$ *and* $\mathcal{V}^-_{d',a'} = R^1\pi_*(\mathcal{P}ar\mathcal{H}om(\mathcal{M}_{d'',a''}, \mathcal{L}_{d',a'}))$. *Then the flip loci* $\mathcal{W}^-_{\delta_c}$ *is a disjoint union* $\sqcup \mathcal{W}^-_{d',a'}$, *where* (d', a') *satisfies (6.18) and* $\mathcal{W}^-_{d',a'}$ *is isomorphic to*

$$\mathcal{W}^-_{d',a'} \cong \mathbb{P}(\mathcal{V}^-_{d',a'}).$$

Similarly, the flip loci $\mathcal{W}^+_{\delta_c}$ *is a disjoint union* $\sqcup \mathcal{W}^+_{d',a'}$, *where* $\mathcal{W}^+_{d',a'}$ *is isomorphic to*

$$\mathcal{W}^+_{d',a'} \cong \mathbb{P}(\mathcal{V}^+_{d',a'}).$$

Let $q_\pm : \mathcal{W}^{\pm}_{d',a'} \to Z_{d'}$ *be the projective bundle maps. Then the maps* $\mathcal{W}^{\pm}_{d',a'} \to \mathcal{M}^{\pm}_{\delta_c}$ *are regular embeddings with normal bundles* $q_\pm^*\mathcal{V}^{\mp}_{d',a'}(-1)$. *Moreover we have the* *following two short exact sequences of universal bundles*

$$0 \to \tilde{q}_-^* \mathcal{L}_{d',a'} \to \mathcal{E}_{\delta_c}^-|_{\mathcal{W}_{d',a'}^- \times C} \to \tilde{q}_-^* \mathcal{M}_{d'',a''} \otimes \mathcal{O}_{\mathcal{W}_{d',a'}^-}(-1) \to 0, \tag{6.20}$$

$$0 \to \tilde{q}_+^* \mathcal{M}_{d'',a''} \otimes \mathcal{O}_{\mathcal{W}_{d',a'}^+}(1) \to \mathcal{E}_{\delta_c}^+|_{\mathcal{W}_{d',a'}^+ \times C} \to \tilde{q}_+^* \mathcal{L}_{d',a'} \to 0, \tag{6.21}$$

where $\mathcal{E}_{\delta_c}^\pm$ *are the universal bundles over* $\mathcal{M}_{\delta_c}^\pm$ *and* $\tilde{q}_\pm : \mathcal{W}_{d',a'}^\pm \times C \to Z_{d'} \times C$ *are the projective bundle maps.*

Proof. The proposition is a straightforward generalization of [73, (3.7)-(3.12)]. We sketch the proof here. By definition, we have tautological extensions of parabolic vector bundles

$$0 \to \tilde{q}_-^* \mathcal{L}_{d',a'} \to \mathcal{E}_{d',a'}^- \to \tilde{q}_-^* \mathcal{M}_{d'',a''} \otimes \mathcal{O}_{\mathcal{W}_{d',a'}^-}(-1) \to 0$$
$$0 \to \tilde{q}_+^* \mathcal{M}_{d'',a''} \otimes \mathcal{O}_{\mathcal{W}_{d',a'}^+}(1) \to \mathcal{E}_{d',a'}^+ \to \tilde{q}_+^* \mathcal{L}_{d',a'} \to 0$$

over $\mathbb{P}(\mathcal{V}_{d',a'}^-)$ and $\mathbb{P}(\mathcal{V}_{d',a'}^+)$, respectively. By the universal properties of $\mathcal{W}_{d',a'}^\pm$, the tautological extensions induce injections $\mathcal{W}_{d',a'}^\pm \to \mathcal{M}_{\delta_c}^\pm$. Next, we show that these injections induce the following exact sequences:

$$0 \to T\mathbb{P}(\mathcal{V}_{d',a'}^-) \to T\mathcal{M}_{\delta_c}^-|_{\mathbb{P}(\mathcal{V}_{d',a'}^-)} \to \mathcal{V}_{d',a'}^+(-1) \to 0, \tag{6.22}$$

$$0 \to T\mathbb{P}(\mathcal{V}_{d',a'}^+) \to T\mathcal{M}_{\delta_c}^+|_{\mathbb{P}(\mathcal{V}_{d',a'}^+)} \to \mathcal{V}_{d',a'}^-(-1) \to 0. \tag{6.23}$$

Let (E,s) be a point in the image of $\mathcal{W}_{d',a'}^-$. Let (L,s') be the destabilizing sub-pair and let M be the corresponding quotient line bundle. By Corollary VI.37, the moduli spaces $\mathcal{M}_{\delta_c}^\pm$ are smooth. The tangent space of $\mathcal{M}_{\delta_c}^-$ at (E,s) can be described by the hypercohomology

$$\mathbb{H}^1\big(\mathcal{P}ar\mathcal{E}nd(E) \to E \otimes \mathcal{O}_C^N\big).$$

By using the standard deformation argument, one can show that the tangent space $T_{(E,s)}\mathbb{P}\big(\mathcal{V}^-_{d',a'}\big)$ is given by the hypercohomology

$$\mathbb{H}^1 = \mathbb{H}^1\Big(\mathcal{P}ar\mathcal{H}om(M,E) \oplus \mathcal{O}_C \to L \otimes \mathcal{O}_C^N\Big).$$

Here the first component of the morphism is the composition $\mathcal{P}ar\mathcal{H}om(M,E) \hookrightarrow \mathcal{H}om(M,E) \to \mathcal{O}_C \xrightarrow{s'} L \otimes \mathcal{O}_C^N$ and the second component of the morphism is given by s'. The vanishing of the hypercohomology groups $\mathbb{H}^0$ and $\mathbb{H}^2$ of the complex $\mathcal{P}ar\mathcal{H}om(M,E) \to L \otimes \mathcal{O}_C^N$ can be obtained by studying the long exact sequence of hypercohomology groups

$$0 \to \mathbb{H}^0 \to H^0(\mathcal{P}ar\mathcal{H}om(M,E)) \oplus \mathbb{C} \to (H^0(L))^N \to$$
$$\to \mathbb{H}^1 \to H^1(\mathcal{P}ar\mathcal{H}om(M,E) \oplus \mathcal{O}_C) \to (H^1(L))^N \to \mathbb{H}^2 \to 0.$$

Here $H^0(\mathcal{P}ar\mathcal{H}om(M,E)) = 0$ because E is a nonsplit extension of M by L and $H^0(\mathcal{P}ar\mathcal{H}om(M,L)) = 0$. The morphism from $\mathbb{C}$ to $(H^0(L))^N$ is injective since it is multiplication by ϕ. Therefore $\mathbb{H}^0 = 0$. It follows from the assumption $(d-\delta)/2-k > 2g-1$ that $d' > 2g-1$ and hence $H^1(L) = 0$. Therefore $\mathbb{H}^2 = 0$.

The short exact sequence (6.22) follows from the hypercohomology long exact sequence of the following short exact sequence of two-term complexes.

$$\begin{array}{ccccccccc}
0 \to & \mathcal{P}ar\mathcal{H}om(\mathcal{M}_{d'',a''}, \mathcal{E}^-_{d',a'}(-1)) \oplus \mathcal{O} & \to & \mathcal{P}ar\mathcal{E}nd(\mathcal{E}^-_{d',a'}, \mathcal{E}^-_{d',a'}) & \to & \mathcal{P}ar\mathcal{H}om(\mathcal{L}_{d',a'}, \mathcal{M}_{d'',a''}(-1)) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{L}_{d',a'} \otimes \mathcal{O}^N & \longrightarrow & \mathcal{E}^-_{d',a'} \otimes \mathcal{O}^N & \longrightarrow & \mathcal{M}_{d'',a''}(-1) \otimes \mathcal{O}^N & \longrightarrow 0
\end{array}$$

One can prove the short exact sequence (6.23) similarly. By using the standard deformation argument, one can show that the tangent space $T_{(E,s)}\mathbb{P}\big(\mathcal{V}^{+}_{d',a'}\big)$ is given by the hypercohomology

$$\mathbb{H}^1 = \mathbb{H}^1\big(\mathcal{P}ar\mathcal{H}om(L,E) \oplus \mathcal{O}_C \to E \otimes \mathcal{O}_C^N\big).$$

Here the first component of the morphism is defined by sending the n sections of L to n sections of E and the second component of the morphism is defined by s. Then (6.23) follows from the hypercohomology long exact sequence of the following short exact sequence of complexes.

$$\begin{array}{ccccccccc}
0 \to & \mathcal{P}ar\mathcal{H}om(\mathcal{L}_{d',a'}, \mathcal{E}^{+}_{d',a'}) \oplus \mathcal{O} & \to & \mathcal{P}ar\mathcal{E}nd(\mathcal{E}^{+}_{d',a'}, \mathcal{E}^{+}_{d',a'}) & \to & \mathcal{P}ar\mathcal{H}om(\mathcal{M}_{d'',a''}(1), \mathcal{L}_{d',a'}) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{E}^{+}_{d',a'} \otimes \mathcal{O}^N & \longrightarrow & \mathcal{E}^{+}_{d',a'} \otimes \mathcal{O}^N & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

□

To prove Theorem VI.50, we need the following lemma.

Lemma VI.52. *Let* $\mathcal{D}_{\delta_c,\pm} = \big(\det R\pi_*(\mathcal{E}^{\pm}_{\delta_c})\big)^{-l} \otimes \bigotimes_{p\in I} L_{\lambda_p} \otimes (\det\,(\mathcal{E}^{\pm}_{\delta_c})_{x_0})^e$. *Then*

1. *the restriction of* $\mathcal{D}_{\delta_c,-}$ *to a fiber of* $\mathbb{P}(\mathcal{V}^{-}_{d',a'})$ *is* $\mathcal{O}(l\delta_c/2)$*, and*

2. *the restriction of* $\mathcal{D}_{\delta_c,+}$ *to a fiber of* $\mathbb{P}(\mathcal{V}^{+}_{d',a'})$ *is* $\mathcal{O}(-l\delta_c/2)$.

Proof. By (6.20), the restriction of $(\mathcal{E}^{-}_{\delta_c})_{x_0}$ to a fiber of $\mathbb{P}(\mathcal{V}^{-}_{d',a'})$ is $\mathcal{O}(-1)$ and the restriction of $\det R\pi_*(\mathcal{E}^{-}_{\delta_c})$ is $\mathcal{O}(\chi(M))$, where $\chi(M) = d'' + 1 - g$ is the Euler characteristic of M. The restriction of $L_{\lambda_p} = \bigotimes_{i=1}^{l_p} \det \mathcal{Q}_{i,p}^{d_{i,p}}$ is $\mathcal{O}(-(l-1) + |a''_p|)$. So $\mathcal{D}^{-}_{d',a'}$

restricts

$$
\begin{aligned}
&\mathcal{O}(-e + l\chi(M) - \sum_{p\in I}(l-1-|a''_p|)) \\
={}&\mathcal{O}\Big(- \frac{dl - 2(l-1)k + |a|}{2} + ld'' - k(l-1) + |a''|\Big) \\
={}&\mathcal{O}\Big(\frac{l\delta_c}{2}\Big).
\end{aligned}
$$

Assertion (2) can be proved similarly. □

Proof of Theorem VI.50. The proof is similar to the proof of Theorem V.11. We only sketch it here.

Case 1. We assume that $(d-\delta)/2 - k > 2g-1$ when δ is sufficiently close to δ_c. Then $\mathcal{M}^{\pm}_{\delta_c}$ are smooth. By using similar arguments as in the proof of [73, (3.18)], one can show that there exists the following diagram.

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{M}}_{\delta_c} & \\
 {}^{p_-}\swarrow & & \searrow^{p_+} \\
\mathcal{M}^-_{\delta_c} & & \mathcal{M}^+_{\delta_c}
\end{array}
$$

Here $p_\pm$ are blow-down maps onto the smooth subvarieties $\mathcal{W}^{\pm}_{d',a'} \cong \mathbb{P}\big(\mathcal{V}^{\pm}_{d',a'}\big)$, and the exceptional divisors $A_{d',a'} \subset \widetilde{\mathcal{M}}_{d',a'}$ are isomorphic to the fiber product $A_{d',a'} \cong \mathbb{P}\big(\mathcal{V}^-_{d',a'}\big) \times_{Z_{d'}} \times \mathbb{P}\big(\mathcal{V}^+_{d',a'}\big)$.

Since $p_\pm$ are blow-ups in smooth centers, we have $(p_\pm)_*([\mathcal{O}_{\widetilde{\mathcal{M}}_{\delta_c}}]) = [\mathcal{O}_{\mathcal{M}^{\pm}_{\delta_c}}]$. By the projection formula, we have

$$
\chi(\mathcal{M}^{\pm}_{\delta_c}, \mathcal{D}_{\delta_c,\pm}) = \chi(\widetilde{\mathcal{M}}_{\delta_c}, p^*_\pm(\mathcal{D}_{\delta_c,\pm})). \tag{6.24}
$$

We only need to compare $p_\pm^*(\mathcal{D}_{\delta_c,\pm})$ over $\widetilde{\mathcal{M}}_{\delta_c}$. Note that the restriction of $\mathcal{O}_{A_{d',a'}}(A_{d',a'})$ to $A_{d',a'}$ is $\mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{d',a'})}(-1)\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V}^-_{d',a'})}(-1)$. Therefore by Lemma VI.52, we have

$$p_-^*(\mathcal{D}_{\delta_c,-}) = p_+^*(\mathcal{D}_{\delta_c,+})\Big(-\frac{l\delta_c}{2}A_{\delta_c}\Big),$$

where $A_{\delta_c} = \sum_{(d',a')} A_{d',a'}$. For $1 \le j \le l\delta_c/2$, we have the short exact sequence

$$\begin{aligned} (6.25)\qquad & 0 \to p_+^*(\mathcal{D}_{\delta_c,+})(-jA_{d',a'}) \to p_+^*(\mathcal{D}_{\delta_c,+})(-(j-1)A_{d',a'}) \\ & \to p_+^*(\mathcal{D}_{\delta_c,+})\otimes\mathcal{O}_{A_{d',a'}}(-(j-1)A_{d',a'}) \to 0. \end{aligned}$$

Define

$$\begin{aligned} &\tilde{\mathcal{D}}_{d',a'} \\ =&\big(\det R\pi_*(\mathcal{L}_{d',a'})\otimes\det R\pi_*(\mathcal{M}_{d',a'})\big)^{-l}\otimes\bigotimes_{p\in I}\tilde{L}_{\lambda_p}\otimes(\det(\mathcal{L}_{d',a'})_{x_0}\otimes\det(\mathcal{M}_{d',a'})_{x_0})^e. \end{aligned}$$

Then by Lemma VI.52, the restriction of $\mathcal{D}_{i,+}$ to $A_{d',a'}$ is $\tilde{\mathcal{D}}_{d',a'}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{d',a'})}(-l\delta_c/2)$. By taking the Euler characteristic of (5.4), we obtain

$$\begin{aligned} &\chi(\widetilde{\mathcal{M}}_{\delta_c}, p_+^*(\mathcal{D}_{\delta_c,+})(-(j-1)A_{d',a'})) - \chi(\widetilde{\mathcal{M}}_{\delta_c}, p_+^*(\mathcal{D}_{\delta_c,+})(-jA_{d',a'})) \\ =&\chi\bigg(A_{d',a'}, \tilde{\mathcal{D}}_{d',a'}\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V}^+_{d',a'})}\Big(-\frac{l\delta_c}{2}+j-1\Big)\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V}^-_{d',a'})}(j-1)\bigg) \quad \text{for } 1\le j\le\frac{l\delta_c}{2}. \end{aligned}$$

Let $n^+_{d',a'}$ be the rank of $\mathcal{V}^+_{d',a'}$. By using the Riemann-Roch formula and (6.1), one can easily show that

$$n^+_{d',a'} = N(d''+1-g) - (d''-d'+1-g) + m_{a',a''},$$

where $m_{a',a''}$ is the number of marked points p such that $a'_p > a''_p$. A simple calculations shows that $n_+ > l\delta_c/2$ when $l \le N-2$. Hence every term in the Leray spectral sequence of the fibration $\mathbb{P}^{n^+_{d',a'}-1} \to A_{d',a'} \to \mathbb{P}(\mathcal{V}^-_{d',a'})$ vanishes. It implies that $\chi(\mathcal{M}^\pm_{\delta_c}, \mathcal{D}_{\delta_c,\pm}) = \chi(\widetilde{\mathcal{M}}_{\delta_c}, p^*_\pm(\mathcal{D}_{\delta_c,\pm}))$ when $(d-\delta)/2 - k > 2g-1$.

Case 2. When $(d-\delta_c)/2 - k \leq 2g-1$, the moduli spaces $\mathcal{M}^\pm_{\delta_c}$ are singular. As before, we choose a sufficiently large integer t such that $(d-\delta)/2-k+t > 2g-1$ when δ is sufficiently close to δ_c. Let $D = x_1+\cdots+x_t$ be a divisor, where x_i are distinct points on C away from $I \cup \{x_0\}$. We denote the moduli spaces $\overline{\mathcal{M}}_C^{\mathrm{par},\delta_c\pm\nu}(\mathrm{Gr}(n,N), d, \underline{a})$ and $\overline{\mathcal{M}}_C^{\mathrm{par},\delta_c\pm\nu}(\mathrm{Gr}(n,N), d+2t, \underline{a})$ by $\mathcal{M}^\pm_{\delta_c,d}$ and $\mathcal{M}^\pm_{\delta_c,d+2t}$, respectively. Let $\mathcal{E}_\pm$ and $\mathcal{E}'_\pm$ be the universal vector bundles on $\mathcal{M}^\pm_{\delta_c,d} \times C$ and $\mathcal{M}^\pm_{\delta_c,d+2t} \times C$, respectively. By Lemma VI.39, there are embeddings $\iota_D : \mathcal{M}^\pm_{\delta_c,d} \hookrightarrow \mathcal{M}^\pm_{\delta_c,d+2t}$ such that $\iota_*(\mathcal{O}^{\mathrm{vir}}_{\mathcal{M}^\pm_{\delta_c,d}}) = \lambda_{-1}((\mathcal{E}'^\vee_\pm)_D) \otimes \mathcal{O}_{\mathcal{M}^\pm_{\delta_c,d+2t}}$. By Corollary VI.49, it suffices to show that

$$\chi\big(\mathcal{M}^-_{\delta_c,d+2k}, \mathcal{D}'_{\delta_c,-} \otimes \lambda_{-1}(((\mathcal{E}'^\vee_-)_D)^N)\big) = \chi\big(\mathcal{M}^+_{\delta_c,d+2k}, \mathcal{D}'_{\delta_c,+} \otimes \lambda_{-1}(((\mathcal{E}'^\vee_+)_D)^N)\big).$$

The above equality can be proved using the same argument as in the proof of Case 2 in Theorem V.11. We omit the details. □

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