Ideals of Subspace Arrangements

by

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To my mum,
your strength and compassion shape the person I strive to be.

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ABSTRACT

Given a collection of t subspaces in an n-dimensional $\mathbb{K}$-vector space W, we can associated to them t vanishing ideals in the symmetric algebra $\mathcal{S}(W^*) = \mathbb{K}[x_1, x_2, \ldots, x_n]$. As a subspace is defined by a set of linear equations, its vanishing ideal is generated by linear forms, so it is a linear ideal. Conca and Herzog showed that the Castelnuovo-Mumford regularity of the product of t linear ideals is equal to t. Derksen and Sidman showed that the Castelnuovo-Mumford regularity of the intersection of t linear ideals is at most t. We show that analogous results hold when we replace the symmetric algebra $\mathcal{S}(W^*)$ with the exterior algebra $\bigwedge(W^*)$ and work over a field of characteristic 0. To prove these results we rely on the functoriality of free resolutions and construct a functor Ω from the category of polynomial functors to itself. The functor Ω transforms resolutions of ideals in the symmetric algebra to resolutions of ideals in the exterior algebra. We use our regularity bound on the intersection of t linear ideals to prove Noether's degree bound on the minimal generating invariant polynomials of a finite group acting on $\bigwedge(W^*)$. We also provide a fast algorithm to compute the invariant monomials of a finite abelian group.

CHAPTER I

Introduction

Mathematicians and human beings in general have always been fascinated by objects exhibiting symmetry. In this thesis we investigate questions related to functions that satisfy symmetry conditions, *invariant polynomials*. This fits under the general umbrella of the field of algebra and more specifically invariant theory. We ask questions with a computational focus that we can investigate with tools from commutative algebra, representation theory, and combinatorics.

An example of an invariant polynomial is $p(x, y) = x + y$. Under the action that swaps the variable x with the variable y the polynomial remains unchanged. Another such polynomial is $q(x, y) = xy$. A central goal of invariant theory is to determine a minimal set of generating invariants (i.e., invariant polynomials) for the action of a group on the set of variables. In the case of the above action on the variables x and y, one can show that every other invariant is a polynomial in p and q. This means that if $r(x, y)$ is another invariant then there is some polynomial f such that $r(x, y) = f(p, q)$. We say that $\{p(x, y), q(x, y)\}$ is a generating set of invariants.

When explicitly computing a generating set is difficult, we investigate a related question: determining the smallest integer β such that a set of generating invariants of degree $\leq \beta$ exists. Emmy Noether proved that for a finite group G of order n

we have that $\beta(G) \leq n$, when the ground field has characteristic zero. This famous result is known as *Noether's degree bound*. In the example above, the group acting is $\mathbb{Z}/(2)$, a group of order two. We also noticed that we have a set of generating invariants of degree two. Applying Noether's result to this problem we can conclude that $\beta = 2$ (assuming that the ground field has characteristic zero).

In general, explicitly finding invariants is a fairly computationally expensive problem, but for finite abelian groups we provide a good algorithm to compute invariant polynomials which we present in Chapter III. In particular, we observed experimentally that highest-degree invariants exhibit an interesting combinatorial structure. In the case of finite abelian groups, invariant polynomials correspond to zero-sums modulo an integer, a topic of interest to number theorists [16]. In this context, computing the maximal degree of a generating invariant is equivalent to computing Davenport's constant, a constant in number theory measuring the length of longest non-shortenable zero-sums.

In characteristic zero, a result of Barbara Schmid [33] tells us that only cyclic groups ($G = \mathbb{Z}/(d)$) achieve Noether's degree bound. Thus, if G is not cyclic, then $\beta(G) < |G|$. It is widely unknown what the actual value of β is, even in the case of finite abelian groups. We discuss this topic in Chapter III and we provide a bound for $\beta(G)$ for $G = (\mathbb{Z}/(d))^r$.

Many mathematicians have worked on determining if analogs of Noether's degree bound hold in more general settings. In [5] Harm Derksen showed that there is a connection between upper bounds for minimal sets of invariants and generators of ideals of *subspace arrangements*. Derksen showed that generating invariant polynomials can be computed from a set of polynomials vanishing on a certain subspace arrangement. In Chapter II we present the connection between invariant theory and the study of subspace arrangement.

By a subspace arrangement we mean a finite collection of subspaces in Euclidean space. Questions about the complement of a real hyperplane arrangement date back to the mid-1800's, whilst the more recent trend of research investigates general subspace arrangements in combinatorics, topology, and complexity theory (see the survey [2]). In this study we investigate these objects from an algebraic perspective. There are two main types of algebraic structures associated to a subspace arrangement: the cohomology ring of the complement of a hyperplane arrangement and the vanishing ideal of a subspace arrangement. Both types are discussed in the survey [32]. Here we study topics related to the vanishing ideal of a subspace arrangement.

In 1999 Derksen conjectured that the vanishing ideal of a union of t subspaces is generated by polynomials of degree at most t. He used this conjecture on subspace arrangements to establish a bound on the degree of invariants of finite groups. Specifically, he proved that in the non-modular case (when the group order does not divide the characteristic of the base field) Noether's degree bound holds if the conjecture holds for $t = |G|$. Bernd Sturmfels made an even stronger conjecture: the vanishing ideal of a union of t subspaces has Castelnuovo-Mumford regularity t. Derksen's result on the connection between invariants and subspace arrangements sparked our interest in studying ideals associated to subspace arrangements to prove results in invariant theory. We study the complexity of these ideals by either computing an explicit set of generators or by describing a resolution for the ideals as modules over a ring.

Suppose that $W_1, W_2, \dots, W_t$ are subspaces of an n-dimensional $\mathbb{K}$-vector space $W \cong \mathbb{K}^n$ and let $I_1, I_2, \dots, I_t \subseteq \mathbb{K}[x_1, x_2, \dots, x_n]$ be the vanishing ideals of $W_1, W_2, \dots, W_t$. These vanishing ideals are *linear* ideals in the sense that they are generated by linear forms. Conca and Herzog showed that the Castelnuovo-Mumford regularity of the product ideal $I_1 I_2 \cdots I_t$ is equal to t (see [4]). Derksen and Sidman proved Sturmfels' conjecture, namely they showed that the Castelnuovo-Mumford regularity of the intersection ideal $I_1 \cap I_2 \cap \cdots \cap I_t$ is at most t (see [8]); similar results hold for more general ideals constructed from linear ideals (see [9]). Because it is possible to use the regularity of an ideal to bound the degree of its generators, then a regularity result yields a degree bound for the generators.

One contribution of our work is to use the subspace arrangement approach to study the product and the intersection of linear ideals over the exterior algebra, as described in Chapter V. Over the symmetric algebra $\mathcal{S}(W) = \mathbb{K}[x_1, x_2, \ldots, x_n]$, we have good bounds on the Castelnuovo-Mumford regularity (hereafter just referred to as regularity). We leverage these results for the symmetric algebra $\mathcal{S}(W)$ to prove similar regularity bounds over the exterior algebra $\bigwedge(W)$.

In the literature, monomial and square-free ideals over the exterior algebra have been studied in relation to their analogues in the symmetric algebra. In particular, monomial ideals in the exterior algebra have been studied in [1]. Using square-free modules in the exterior algebra, one can define a generalization of Alexander's duality (see [28]). In the context of hyperplane arrangements, the homology and the cohomology rings of the complement of the arrangement are modules over the exterior algebra and have been studied in [11]. These results rely on the idea of creating a connection between resolutions over the symmetric algebra and resolutions over the exterior algebra. Our approach also relies on a similar idea, even though it exploits a different method: a functor on polynomial functors.

Our methods allow us to study any finite wedge product of linear ideals in the exterior algebra. In particular, we have the following result.

Theorem I.1. *Assume that V is a finite-dimensional vector space over a field of characteristic zero. In the exterior algebra $\bigwedge(V)$, the wedge product of a finite number of linear ideals has a linear resolution.*

Specifically, this theorem is a direct consequence of our main result, Theorem V.5, which establishes that the wedge product of t linear ideals is t-regular. In general, we are interested in computing the regularity of a module because this numerical invariant gives us a measure of its complexity. Even for ideals that are simple to describe, it can be hard to explicitly compute their regularity. Moreover, even prime ideals can have very large regularity as shown in [22] by McCullough and Peeva's counterexample to the Eisenbud-Goto conjecture. Our result shows that working with ideals constructed from linear ideals, we have the best possible regularity bound irrespective of whether we work over the symmetric algebra or the exterior algebra.

To study ideals in the exterior algebra, we construct a way to transfer information between the symmetric algebra and the exterior algebra. To this goal, we consider ideals of subspace arrangements that are stable under the action of the general linear group and study them using the tools of representation theory. Specifically, in Chapter IV we describe a functor, Ω, on the category of graded polynomial functors. The functor Ω is the transpose functor used by Sam and Snowden [29–31] to study modules over twisted commutative algebras. The functor Ω will transfer homological properties from equivariant resolutions over the symmetric algebra to equivariant resolutions over the exterior algebra.

Ideals with the additional structure of a group representation exhibit interesting behavior even in simple examples. In fact, the Hilbert series of the vanishing ideal of a hyperplane arrangement of d hyperplanes in n-dimensional space is just $t^d/(1-t)^n$. However, the Hilbert series of an ideal which is stable under a group action is a much more interesting object. In fact, one can define the notion of equivariant Hilbert series of $\mathrm{GL}(V)$-equivariant ideals. In Chapter V, we compute equivariant Hilbert series of ideals of subspace arrangements and use these computations to write down equivariant resolutions of ideals associated to subspace arrangements. The resolutions considered will be $\mathrm{GL}(V)$-equivariant, meaning that all modules in the resolution will be $\mathrm{GL}(V)$-representations and all maps in the resolution will be maps of $\mathrm{GL}(V)$-representations.

Our regularity bound on the intersection of t linear ideals is also used in Chapter VI to derive a result in non-commutative invariant theory. In particular, we generalize Derksen's subspace approach to the exterior algebra and prove the following the result.

Theorem I.2. *In characteristic zero, for the action of a finite group G on the exterior algebra $\bigwedge(V)$, we have that Noether's degree bound holds that is, $\beta_V(G) \leq |G|$, for every finite dimensional vector space V.*

We establish the theorem by considering a subspace arrangement of cardinality $t = |G|$ associated to the group G. The regularity bound on the ideal of the subspace arrangement in Theorem I.1 gives us the degree bound on a minimal set of generating invariants. The idea of using polynomial functors to establish results in invariant theory goes back to the times of Weyl and it continues nowadays. One can find examples of this trend of research in our work and the work of Snowden [35].

Furthermore, in Chapter III we also study subspace arrangements that are invariant under the action of a finite abelian group and describe a set of generators of the associated ideal. When considering a finite abelian group, we show that its associated vanishing ideal is a binomial ideal. We completely characterize a set of binomial generators for these vanishing ideals and discuss the connection between these binomials and the invariant monomials of the group action.

CHAPTER II

Noether's bound and subspace arrangements

In this chapter we provide historical context for Noether's degree bound for invariants of finite groups and describe the connection between this bound and ideals of subspace arrangements. As we will prove later in Chapter VI, some of the arguments generalize from the polynomial ring to the exterior algebra.

2.1 Degree bounds on invariants

Let G be a finite group of order $|G|$ and let G act on a n-dimensional vector space V over a field $\mathbb{K}$. The action of G on V induces an action of G on the ring $\mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]$ of polynomial functions on V. Determining a set of invariants for a given action is a central question in invariant theory. A first step towards computing a set of generating invariants is to determine a bound on the degrees of generating invariants. To study degree bounds for rings of invariants, we will introduce the notion of a minimal set of generating invariants.

Definition II.1. Let G be a finite group acting on $S = \mathbb{K}[x_1, \ldots, x_n]$. Let S^G denote the subring of invariant polynomials i.e.,

$$S^G = \{f \in S \mid g \cdot f = f, \forall g \in G\}.$$

A set of polynomials $\{f_1, \ldots, f_r\} \subset S^G$ is a minimal set of generating invariants if

(i) the set $\{f_1, \ldots, f_r\}$ generates the ring of invariants as a $\mathbb{K}$-algebra, that is, $\mathbb{K}[f_1, \ldots, f_r] = S^G$;

(ii) no proper subset of $\{f_1, \ldots, f_r\}$ generates S^G.

An invariant f_i in a minimal set of generating invariants $\{f_1, \ldots, f_r\}$ is called a minimal generating invariant.

Given a minimal set of generating invariants, we can ask for a bound on their degrees.

Definition II.2. We let $\beta_V(G)$ be the maximal degree of a homogeneous minimal generating invariant for the representation V of G. We denote by $\beta(G)$ the maximum of $\beta_V(G)$ over all representations V of G.

A priori, it is not clear whether $\beta(G)$ will have a finite value. There are two main cases. When the characteristic of the field $\mathbb{K}$ divides the order of the group $|G|$, we say that we are in the *modular case*. In the modular case, we have representations with minimal invariants of arbitrarily large degree, so that $\beta(G)$ cannot be finite (see [27]).

When the characteristic of the field does not divide the order of the group or the characteristic is zero, we say that we are in the *non-modular case*. In this thesis we will only consider the non-modular case. Thus, we will discuss representations of finite groups in the non-modular case.

In this context, we have an extremely useful tool: the Reynolds operator.

Definition II.3. The Reynolds operator of G is the averaging operator $\mathcal{R}_G : S \to S^G$ given by

$$\mathcal{R}_G(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f,$$

for any $f \in S$.

2.2 Noether's degree bound

In 1915 Emmy Noether [23] proved a beautiful result in characteristic zero: $\beta(G) \leq |G|$. Since then, mathematicians have worked on filling the "Noether gap": proving that the same bound holds in the non-modular case. Working independently, Fleischmann [13] and Fogarty [14] provided the first complete proofs of Noether's degree bound in the non-modular case in 2000-2001. Fogarty's proof was simplified by Benson [5] and here we present this argument. First, consider the following definition.

Definition II.4. Let J be an ideal in S, a commutative ring with unity. Suppose that G is a group of automorphisms of S. Then, we define

$$J^G = \{f \in J \mid g \cdot f = f, \forall g \in G\}.$$

The core of Benson's proof is the following lemma ([5, Lemma 3.2.1]).

Lemma II.5 (Benson's Lemma)**.** *Let S be a commutative ring with unity, G a group of automorphisms of S, and J a G–stable ideal in S. If $|G|$ is invertible in S, then*

$$J^{|G|} \subseteq J^G S.$$

Proof. Let $\{f_\alpha\}$ be a collection of $|G|$ elements of J, indexed by $\alpha \in G$. Consider $p = \prod_\alpha f_\alpha$ and notice that any element in $J^{|G|}$ can be written as such a product. On the other hand, for any given τ in G we have that

$$\prod_{\alpha \in G} (\tau\alpha \cdot f_\alpha - f_\alpha) = 0,$$

because there exists an α such that $\alpha = \tau^{-1}$ in G. If we expand this product, for each choice of a subset $H \subseteq G$ we get a monomial of the type

$$\prod_{\alpha \in H} (\tau\alpha \cdot f_\alpha) \prod_{\alpha \in G-H} (-f_\alpha).$$

Thus, we get that

$$\sum_{H\subseteq G}\prod_{\alpha\in H}(\tau\alpha\cdot f_\alpha)\prod_{\alpha\in G-H}(-f_\alpha)=0.$$

For each $\tau\in G$ we get an equation in the f_α. Summing over all $\tau\in G$ we get

$$\sum_{\tau\in G}\sum_{H\subseteq G}\prod_{\alpha\in H}(\tau\alpha\cdot f_\alpha)\prod_{\alpha\in G-H}(-f_\alpha)=0$$

Changing the order of summation and collecting all the negative signs, we get

$$\sum_{H\subseteq G}(-1)^{|G-H|}\sum_{\tau\in G}\prod_{\alpha\in H}(\tau\alpha\cdot f_\alpha)\prod_{\alpha\in G-H}f_\alpha=0.$$

The summand for $H=\emptyset$ is $\pm|G|\prod_\alpha f_\alpha$, a constant multiple of $p=\prod_\alpha f_\alpha$. For any $H\neq\emptyset$, we have that the corresponding summand lies in J^GS, as acting on it with an element $\sigma\in G$ only has the effect of re-indexing the sum over all $\tau\in G$. Therefore,

$$p=\pm\frac{1}{|G|}\sum_{\emptyset\neq H\subseteq G}(-1)^{|G-H|}\sum_{\tau\in G}\prod_{\alpha\in H}(\tau\alpha\cdot f_\alpha)\prod_{\alpha\in G-H}f_\alpha\in J^GS,$$

so for every $p\in J^{|G|}$, we have that $p\in J^GS$, as required. □

The last ingredient required to prove Noether's degree bound in the non-modular case is the following lemma ([5, Lemma 2.2]).

Lemma II.6 (Derksen's Lemma)**.** *Let $\mathfrak{m}$ be the maximal homogeneous ideal of $S=\mathbb{K}[x_1,\ldots,x_n]$. Suppose that $\{f_1,\ldots,f_r\}$ generate $\mathfrak{m}^GS$ as an ideal. Then $S^G=\mathbb{K}[\mathcal{R}_G(f_1),\ldots,\mathcal{R}_G(f_r)]$.*

Using the previous results, we can now prove Noether's degree bound in the non-modular case.

Theorem II.7. *Let G be a finite group acting on a finite dimensional vector space V over $\mathbb{K}$. Suppose that $|G|$ is invertible in $\mathbb{K}$. Then for every V we have that $\beta_V(G)\leq|G|$. Thus, $\beta(G)\leq|G|$ in the non-modular case.*

Proof. Let $S = \mathbb{K}[V]$, the algebra of polynomial functions on V. Let $d = |G|$. Let $\mathfrak{m}$ be the ideal consisting of all polynomials with zero constant term. By Lemma II.6 we have that a set of ideal generators for $\mathfrak{m}^G S$ yields a set of generators for S^G. So a bound on the degrees of the generators of $\mathfrak{m}^G S$ will yield a bound on the degrees of the generators of S^G. We will show that there are no minimal generating invariants of degree $d+1$ or larger by showing that $\mathfrak{m}^G S$ has no minimal generators of degree $d+1$ or larger.

Notice that $\mathfrak{m}$ is G-stable as the action of G preserves degrees. Applying the Lemma II.5, we get

$$\mathfrak{m}^d \subseteq \mathfrak{m}^G S.$$

Choose a set X of homogeneous generators of the ideal $\mathfrak{m}^G S$. The ideal $\mathfrak{m}^d$ contains all polynomials of degree d and is contained in $\mathfrak{m}^G S$. So without loss of generality we may assume that X contains all monomials of degree d. We can remove all polynomials of degree $> d$ from X, because they can be written as S-linear combinations of monomials of degree d contained in X. Now Lemma II.6 yields a system of algebra generators of S^G of degree $\leq d$. $\square$

2.3 The subspace arrangement approach

In 1999 Derksen ([5]) showed that there is a connection between upper bounds for minimal sets of invariants and generators of ideals of *subspace arrangements*.

Definition II.8. A subspace arrangement $\mathcal{A} = \{W_1, \ldots, W_t\} \subset W$ is a finite collection of subspaces W_i in a fixed ambient vector space $W \cong \mathbb{K}^n$.

The union of the subspaces in a subspace arrangement is an algebraic set i.e., its vanishing ideal is defined by a set of polynomials.

Definition II.9. For a subspace arrangement $\mathcal{A} = \{W_1, \ldots, W_t\} \subset W$, let J_i be the vanishing ideal of W_i

$$J_i = \mathbb{I}(W_i) = \{f \in \mathbb{K}[x_1, \ldots, x_n] \mid f(x) = 0, \forall x \in W_i\}.$$

To the subspace arrangement $\mathcal{A}$ we associate the vanishing ideal $I_{\mathcal{A}}$

$$I_{\mathcal{A}} = \mathbb{I}(\mathcal{A}) = \mathbb{I}\big(\bigcup_i W_i\big) = \bigcap_i J_i.$$

Derksen showed that computing invariants for a finite group is equivalent to the finding ideal generators of the vanishing ideal of a certain subspace arrangement. In 2002, working with Sidman ([8]), they proved a bound for the generators of the vanishing ideal of a subspace arrangement and gave a different proof of Noether's degree bound in the non-modular case.

To describe his approach, we introduce a subspace arrangement associated to the action of the group G on the n dimensional vector space V.

Definition II.10. The graph of the action of g, denoted V_g, is a set of points in $V \oplus V$ given by

$$V_g = \{(v, g \cdot v) \mid v \in V\}.$$

Moreover, we define $\mathcal{A}_G$ to be the subspace arrangement associated to the action of G on V:

$$\mathcal{A}_G = \bigcup_{g \in G} V_g.$$

Notice that for the identity element $1_G \in G$, the graph of 1_G is Δ, the diagonal of $V \oplus V$. Derksen proved that a set of ideal generators for the vanishing ideal of $\mathcal{A}_G$ can be used to produce a set of generating invariants for $\mathbb{K}[V]^G$.

To present his result, we introduce the Hilbert ideal J_G.

Definition II.11. Let G be a finite group acting on a polynomial ring S. Let $\mathfrak{m}$ be the maximal homogeneous ideal of S. We define the Hilbert ideal of G as

$$J_G = \mathfrak{m}^G S.$$

Derksen proved the following result relating subspace arrangements and invariant theory.

Theorem II.12 (Theorem 3.1 in [5])**.** *Let J_G be the Hilbert ideal for the action of G on $S = \mathbb{K}[x_1, \ldots, x_n]$. Consider $I_G = \mathbb{I}(\mathcal{A}_G)$ in the ring $\mathbb{K}[\mathbf{x}, \mathbf{y}]$, where $\mathbb{K}[\mathbf{x}, \mathbf{y}] = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We have that:*

$$(I_G + (y_1, \ldots, y_n)) \cap \mathbb{K}[x_1, \ldots, x_n] = J_G.$$

We can now apply Theorem II.6 to get a set generating invariants from the generators of the Hilbert ideal.

Corollary II.13 (Corollary 3.2 in [5])**.** *Let $\{p_i(\mathbf{x}, \mathbf{y})\}$ be a set of generators for I_G in $\mathbb{K}[\mathbf{x}, \mathbf{y}]$. Then $\{\mathcal{R}_G(p_i(x, 0))\}$ is a set of generating invariants for $\mathbb{K}[\mathbf{x}]^G$.*

Proof. Let $\{p_i(\mathbf{x}, \mathbf{y})\}$ be a set of generators for I_G. Notice that for each i, we have that $p_i(\mathbf{x}, \mathbf{y}) = p_i(\mathbf{x}, \mathbf{0}) + q_i(\mathbf{x}, \mathbf{y})$, for $q_i(\mathbf{x}, \mathbf{y}) \in (y_1, \ldots, y_n)$. So

$$(p_i(\mathbf{x}, \mathbf{y}), y_1, \ldots, y_n) = (p_i(\mathbf{x}, \mathbf{0}), y_1, \ldots, y_n),$$

for all i. Thus, $(p_i(\mathbf{x}, \mathbf{y}), y_1, \ldots, y_n) \cap \mathbb{K}[x_1, \ldots, x_n] = (p_i(\mathbf{x}, \mathbf{0}))$, for each i. By Theorem II.12, we have that $\{p_i(\mathbf{x}, \mathbf{0})\}$ is a set of generators for J_G. Then by Theorem II.6, we have that $\{R_G(p_i(\mathbf{x}, \mathbf{0}))\}$ generate the ring of invariants, as claimed. $\square$

The above corollary gives us the following bound on the degree of minimal generating invariants.

Corollary II.14. *Let $\{p_i(\mathbf{x}, \mathbf{y})\}$ be a set of generators for I_G in $\mathbb{K}[\mathbf{x}, \mathbf{y}]$. Suppose that for all i we have that $\deg(p_i) \leq d$. Then $\beta_V(G) \leq d$.*

Proof. Notice that

$$\deg(\mathcal{R}_G(p_i(\mathbf{x}, \mathbf{0}))) \leq \deg(p_i(\mathbf{x}, \mathbf{0})) \leq \deg(p_i(\mathbf{x}, \mathbf{y})).$$

Thus, by the assumption and Corollary II.13, we have that the ring of invariants is generated in degree $\leq d$. □

Using Derksen's approach, we see that bounds on the degrees of the generators of I_G give us bounds on β. In particular, we will see that the bound we get on the generators of the vanishing ideal of the subspace arrangement $\mathcal{A}_G$ is Noether's degree bound: $|G|$.

2.4 The subspace theorem

For the vanishing ideal of a subspace arrangement of cardinality t, Derksen wanted to determine a bound on the maximal degree of its minimal generators. It turns out that the sleek approach to this problem is via a homological invariant: the Castelnuovo-Mumford regularity of the intersection of t linear ideals.

For $S = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring in n variables and M a graded module over R, a free resolution of M is an exact complex that approximates M by a sequence of free modules S^{a_i}:

$$\cdots S^{a_1} \to S^{a_0} \to M \to 0,$$

We can encode the multiplicity and the degree of S^{a_i} in a graded vector space E_i and write the complex as

$$\cdots S \otimes E_1 \to S \otimes E_0 \to M \to 0.$$

Then E_i will be naturally isomorphic to graded torsion module $\mathrm{Tor}_i(M, \mathbb{K})$.

The regularity of M is a numerical invariant depending on the E_i's that measures the complexity of M. Modules with low regularity have simple minimal resolutions.

Definition II.15. Let M be a graded module of S and let $E_i \cong \mathrm{Tor}_i(M, \mathbb{K})$ be the graded torsion module appearing at step i of a graded minimal free resolutions of M. Define

$$\deg(E_i) := \max\{d : (E_i)_d \neq 0\},$$

if $E = \{0\}$, then we define $\deg(E) = -\infty$. We say that M is s-regular if

$$\deg(E_i) \leq s + i,$$

for all i. Then the regularity of M, denoted by $\mathrm{reg}(M)$, is the smallest integer s such that M is s-regular.

Conca and Herzog in [4] showed that the product of t linear ideals has the best possible regularity: $\mathrm{reg}(J_1 J_2 \cdots J_t) = t$. On the other hand, Derksen and Sidman [8] proved the following result.

Theorem II.16 (Theorem 2.1 in [8])**.** *If $I_{\mathcal{A}}$ is the vanishing ideal of the subspace arrangement $\mathcal{A} = \{W_1, \ldots, W_t\}$, then $I_{\mathcal{A}}$ is t-regular.*

In fact, the regularity of $I_{\mathcal{A}}$ may well be much smaller than t, so that the cardinality of the subspace arrangement is only an upper bound for the regularity of $I_{\mathcal{A}}$.

Notice that, as the first torsion modulo E_0 records the degrees of the minimal generators of a module M, we have that the regularity of M is an upper bound on the degrees of the generators of M. Specifically, if $\{f_i\}$ is a set of minimal generators for M, then

$$\deg(E_0) = \max_i\{\deg(f_i)\}.$$

Therefore, if M is t-regular, then $\deg(E_0) \leq t$ so that M is generated is degree $\leq t$.

Now for a finite group G acting of a vector space V, consider the subspace arrangement

$$\mathcal{A}_G = \bigcup_{g \in G} V_g = \bigcup_g \{(v, g \cdot v) | v \in V\} \subset V \oplus V.$$

Then the ideal $I_G = \mathbb{I}(\mathcal{A}_G)$ is the intersection of $|G|$ linear ideals. Thus, I_G is $|G|$-regular by Theorem II.16. As the regularity is an upper bound on the degrees of the generators, I_G is generated in degree at most $|G|$. However, recall that Corollary II.14 states that a bound on the degrees of the generators of I_G is also a bound on the degrees of the generators of S^G. Thus, we have that $\mathrm{reg}(I_G) \leq |G|$ implies that $\beta_V(G) \leq t$. Therefore, as this holds for every representation V, we have that Noether's degree bound $\beta(G) \leq |G|$ holds in the non-modular case, via the subspace arrangement approach.

In the next chapter we will discuss bounds on $\beta(G)$ in the non-modular case when G is a finite abelian group. We will see that there is often a large gap between Noether's degree bound and the actual value of β. However, the value of β is generally unknown, so we set up an algorithm to compute invariants. We also use the characters of G to find a better bound on $\beta(G)$ for $G = (\mathbb{Z}/(d))^r$.

CHAPTER III

Invariants of abelian groups

In this chapter, we study invariants of finite abelian groups. A general bound for invariants of finite abelian groups was conjectured by Schmid (see [33]). This conjecture is now known to be false in general, but Olson proved in [25] that it is true for abelian p-groups using the group algebra over $\mathbb{F}_p$. We show how a similar approach, using group algebras over $\mathbb{C}$ instead of $\mathbb{F}_p$, can give bounds for $\beta(G)$ where $G = (\mathbb{Z}/(m))^r$. In Section 3.3 we give an efficient algorithm for finding generating invariants for finite abelian groups by adapting an algorithm of Derksen and Kemper (see [7]) for torus invariants.

3.1 Invariants of abelian groups

Let G be an abelian group acting on a vector space V of dimension n. We assume that we are in the non-modular case, which means that $|G|$ is invertible in $\mathbb{K}$. We will also assume that $\mathbb{K}$ contains a $|G|$-th root of unity. A multiplicative character is a group homomorphism $G \to \mathbb{K}^\star$, where $\mathbb{K}^\star = \mathbb{K} - \{0\}$ is the multiplicative group of units. The product of two multiplicative characters is again a multiplicative character and the set of multiplicative characters form a group $\widehat{G}$.

Because the group G is abelian, the action of G on V is diagonalizable. In particular, we can choose a basis for V such that $\mathbb{K}[V] = \mathbb{K}[x_1, \dots, x_n]$ and for each

$g \in G$

$$g \cdot x_j = \chi_j(g) x_j,$$

where $\chi_j : G \to \mathbb{K}$ is a multiplicative character, i.e., it has the property $\chi_j(gh) = \chi_j(g)\chi_j(h)$.

Moreover, for any abelian group G, we can choose an isomorphism

$$G \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r),$$

where $d_i \mid d_{i+1}$ for all i. In particular, when we identify G with the group $\mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r)$ we will use additive notation for the group structure in G. In this context, a set of generators of G is $g_1 = (1, 0, \ldots, 0), g_2 = (0, 1, 0, \ldots, 0), \ldots, g_r = (0, 0, \ldots, 0, 1)$, where g_i has order d_i for $i = 1, 2, \ldots, r$.

Let ζ_i be a d_i-th root of unity. Notice that the diagonal action of g_i on x_j is given by

$$g_i \cdot x_j = \chi_j(g_i) x_j = \zeta_i^{w_{ij}} x_j,$$

for some $w_{ij} \in \mathbb{Z}/(d_i)$. We call w_{ij} the weight of the action of g_i on x_j. The integer vector

$$\mathbf{w}_j = (w_{1j}, \ldots, w_{rj}) \in \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r)$$

is the weight associated to χ_j, or the weight of the action of G on x_j. By associating multiplicative characters with their weight, we get an isomorphism between the group of characters $\widehat{G}$ and $\mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r)$. So G and $\widehat{G}$ are isomorphic. We can also consider

$$\mathbf{w}^i = (w_{i1}, \ldots, w_{in}) \in (\mathbb{Z}/(d_i))^n$$

as the weight of the action of g_i on V.

Let M denote the set of non-constant monomials in $\mathbb{K}[x_1, \ldots, x_n]$. As the action of the abelian group G is diagonal, we can choose a set of invariant monomials as a minimal set of generating invariants. The monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ will be invariant if $g_i \cdot \mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{a}}$ for all the generators g_i of G. In terms of the characters χ_j of the variables x_j, we have that $\mathbf{x}^{\mathbf{a}}$ will be an invariant if $\prod_{j=1}^{n} \chi_j^{a_j} = 1$. In terms of the weights of the generators g_i, we have that $\mathbf{x}^{\mathbf{a}}$ is an invariant monomial if $\sum_j a_j w_{ij}$ is zero in $\mathbb{Z}/(d_i)$ for all i, in other words, $\sum_j a_j \mathbf{w}_j = 0$ in $\widehat{G}$.

Definition III.1. Let $W = (w_{ij})$ be the matrix of the weights of the action of G on $\mathbb{K}[x_1, \ldots, x_n]$. The j-th column of W is $\mathbf{w}_j$, the weight of the action of G on x_j, whilst the i-th row of W is $\mathbf{w}^i$, the weight of the action of g_i on V. We have that $\mathbf{x}^{\mathbf{a}}$ is an invariant monomial if $W\mathbf{a} = \mathbf{0}$, where the i-th entry of the vector on the right hand side lies in $\mathbb{Z}/(d_i)$. We say that the vector $W\mathbf{a}$ is the weight of the monomial $\mathbf{x}^{\mathbf{a}}$.

Notice that with this set up, checking if a monomial $\mathbf{x}^{\mathbf{a}}$ is invariant is equivalent to a linear algebra check: compute $W\mathbf{a}$ and check for each i that the i-th entry is zero modulo d_i. If we have a bound t on $\beta_V(G)$, the maximum degree of a minimal invariant, there is a (very inefficient) algorithm to find the invariant monomials: check all monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ with $\sum a_j \leq t$ by computing $W\mathbf{a}$. There are better algorithms and we will describe one in 3.3. First we discus bounds on $\beta_V(G)$ for G an abelian group.

3.2 Bounds on invariants for abelian groups

We first introduce some notation.

Definition III.2. For any abelian group G, choose an isomorphism

$$G \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r),$$

where $d_i \mid d_{i+1}$ for $i = 1, 2, \ldots, r-1$. Let $+$ be the group operation on G. A sum

$\sum_{i\in I} b_i$ of elements in G is called non-shortenable if there is no nonempty proper subset $I' \subsetneq I$ such that $\sum_{i\in I'} b_i = 0$ in G.

The following example shows that Noether's degree bound is sharp when G is cyclic.

Example III.3. Suppose that $G = \mathbb{Z}/(d)$ is the cyclic group generated by $g = 1+(d)$. Let g act on the polynomial ring $\mathbb{K}[x]$ by $g \cdot x = \zeta_d x$. Then the invariant ring is $\mathbb{K}[x]^G = \mathbb{K}[x^d]$ and $\beta(G) = d = |G|$.

At the end of this section we will bound the length of non-shortenable zero-sums in $\bigoplus_{i=1}^r \mathbb{Z}/(d_i)$.

We start with a lemma that tells us that in the non-modular case ($\mathrm{char}(\mathbb{K}) \nmid |G|$) Noether's degree bound ($|G|$) is achieved precisely when G is a cyclic group (see Schmid [33]).

Lemma III.4. *Suppose that $\sigma = \sum_{i\in I} b_i$ is a non-shortenable zero-sum in the finite abelian group G. Then $|I| \le |G|$. If $|I| = |G|$, then G is cyclic and for all i we have that $b_i = b$, where b is some generator of G.*

Proof. Let $I = \{1, 2, \ldots, s\}$, so that that $\sigma = \sum_{i=1}^s b_i$. Let σ_k be the sub-sum of the first k terms of σ, namely $\sigma_k = \sum_{i=1}^k b_i$. By assumption, $\sigma = \sigma_s = 0$.

We claim that $\sigma_k \neq \sigma_j$ for all $k \neq j$. Otherwise, suppose without loss of generality that $k < j$ and $\sigma_k = \sigma_j$. Then

$$\sigma_j = \sum_{i=1}^{j} b_i = \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{j} b_i = \sigma_k,$$

implies that $\sum_{i=k+1}^j b_i = 0$. However, this contradicts the assumption that σ is non-shortenable.

Hence, all sub-sums σ_k are distinct elements in G. As there are $s = |I|$ sub-sums and $|G|$ elements in G, we conclude that $|I| \le |G|$.

Finally, suppose that $s = |G|$, we want to show that G is cyclic. As all σ_k are distinct and $|G| = |I|$, we have that each element of G appears exactly once as some sub-sum σ_k. Notice that by assumption $\sigma = \sigma_s = 0$. If $|G| = 1$, then the claim is trivially true. Assume that $|G| \geq 2$. As every element of G appears as a sub-sum, we have that, in particular, b_2 has to appear as some sub-sum.

Suppose that $b_2 = \sigma_k$ for some $k \geq 2$. Then,

$$b_2 = \sigma_k = \sum_{i=1}^{k} b_i = b_1 + b_2 + \ldots b_k.$$

The above equation implies that $b_1 + \sum_{i=3}^{k} b_i = 0$. However, this contradict the assumption that σ is non-shortenable. Hence, $b_2 = \sigma_1 = b_1$.

Because there is no particular assumption about b_2, we could relabel any other b_i as b_2. Thus, $b_i = b_1$ for all i and the set of sub-sums is $\{b_1, 2b_1, \ldots, sb_1\}$. Recall that the set of sub-sums is the whole group G. Inspecting the sub-sums we can conclude that $G = \langle b_1 \rangle$. Therefore, G is cyclic. $\square$

It is natural to ask how far is any specific group from achieving Noether's degree bound. We have the following general lower bound.

Lemma III.5. *If $G \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r)$, then $\beta(G) \geq 1 + \sum_{i=1}^{r}(d_i - 1)$*

Proof. Let $\widehat{G} \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r)$ be the group of multiplicative characters (with additive notation) and $e_1, \ldots, e_r$ be the generators of $\widehat{G}$, so that $e_1 = (1, 0, \ldots, 0), \ldots, e_r = (0, 0, \ldots, 0, 1)$. If we take the weights

$$\underbrace{g_1, \ldots, g_1}_{d_1 - 1}, \underbrace{g_2, \ldots, g_2}_{d_2 - 1}, \ldots, \underbrace{g_r, \ldots, g_r}_{d_r - 1}, g_1 + \cdots + g_r \in \widehat{G}$$

then the sum of all weights is 0, but no proper nonempty subsum is 0. If we use these weights as the weights in a representation of G, then there is a generating invariant in degree $\sum_{i=1}^{r}(d_i - 1) + 1$. $\square$

Given this lower bound for $\beta(G)$, Barbara Schmid [33] made the following conjecture:

Conjecture III.6. *For $G \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r)$ we have that $\beta(G) = 1 + \sum(d_i - 1)$.*

Her conjecture is equivalent to the following one.

Conjecture III.7. *Consider a sum of t elements $\{b_1, \ldots, b_t\}$ in $G \cong \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_r)$, where $t = \sum(d_i - 1)$. Suppose that $\sigma = \sum_{i=1}^{t} b_i$ is such that*

(i) $b_i \neq 0$ for all i,

(ii) the sum is non-shortenable,

(iii) the sum is non-zero.

Then every element of $G - \{0\}$ appears as a subsum of σ.

If Conjecture III.7 is true, then adding any element b to σ results either in a shortenable sum or, for $b = -\sigma$, in a non-shortenable zero-sum. Therefore, the longest non-shortenable zero-sum in G has size $\leq 1 + t = 1 + \sum(d_i - 1)$. This shows that Conjecture III.7 implies Schmid's conjecture (Conjecture III.6).

It has been shown that the conjecture fails for groups of rank five. In fact, in [18] the authors introduce the following infinite family of groups for which the conjecture fails:

$$G_n = \mathbb{Z}/(2) \oplus (\mathbb{Z}/(2n))^4,$$

where $n \geq 3$. In that context, the authors study the Davenport constant $D(G)$ of non-shortenable zero-sums in G: the largest positive integer k such that a non-shortenable zero-sum of k elements exists in G. Notice that $D(G)$ is equal to $\beta_V(G)$ for V the regular representation of G. Even though the conjecture has been disproved in ranks greater or equal to five, it is still open in smaller ranks. In fact, by Lemma

III.4 the conjecture holds for cyclic groups. We will discuss rank two in section 3.4, when we introduce an algorithm to compute a minimal set of generating invariant monomials. In the remaining of this section we discuss two cases: G is a p-group or $G \cong (\mathbb{Z}/(d))^r$.

Olson [25] proved that the conjecture holds for p-groups. Before we present his proof for arbitrary p-groups, we give a simpler argument for the special case where $G \cong (\mathbb{Z}/(2))^r$.

Proposition III.8. *Consider $G \cong \mathbb{Z}/(2)^r$. Then any sum of r elements $\{b_i\}$ such that*

(i) $b_i \neq 0$ for all i,

(ii) the sum is non-shortenable,

(iii) the sum is non-zero,

is such that every element of $G - \{0\}$ appears as a subsum.

Proof. We can view G as an r-dimensional vector space over the field $\mathbb{Z}/(2)$. Because the sum is non-shortenable, the vectors $b_1, \ldots, b_r$ are linearly independent. Therefore, they form a basis and span G as a vector space. Therefore, any element in $G - \{0\}$ appears as a subsum, as required. □

Notice that when a set $\{b_i\}$ of r elements in $G \cong (\mathbb{Z}/(2))^r$ satisfies the hypotheses of the proposition, then all possible subsums are non-zero. In fact, there are $2^r - 1$ elements in $G - \{0\}$ and

$$\sum_{i=1}^{r} \binom{r}{i} = 2^r - 1$$

possible subsums. To prove the conjecture for small ranks, one could think of counting the number of distinct sub-sums of $\sigma = \sum b_i$ and compare that count with number of elements in G. Such an argument turns out to be tricky, especially when one tries to pin down how many subsums are actually distinct. A case by case argument can be made for $(\mathbb{Z}/(3))^2$, but it is not very enlightening and it will not be included here.

Olson [25] proved that for any prime p the conjecture holds for p-groups i.e., for any $G \cong \bigoplus_{i=1}^{r} \mathbb{Z}/(p^{e_i})$. Notice that for this result we will think of G multiplicatively, not additively as in the previous paragraphs. We present Olson's original idea, slightly adapted using the language of ring theory.

Theorem III.9 (Olson's theorem on p-groups)**.** *Let $G = \prod_{i=1}^{r} \mathbb{Z}/(p^{e_i})$. Let k be the smallest positive integer such that for any elements $g_1, g_2, \ldots, g_k$ in G there exists a nonempty subset $J \subseteq \{1, 2, \ldots, s\}$ such that $\prod_{i \in J} g_i = 1$ in G. Then*

$$k = 1 + \sum_{i=1}^{r} (p^{e_i} - 1).$$

Proof. The inequality $k \geq 1 + \sum (p^{e_i} - 1)$ is given by Lemma III.5.

For the inequality in the other direction, consider

$$R = \frac{\mathbb{F}_p[x_1, \ldots, x_r]}{(x_1^{p^{e_1}} - 1, x_2^{p^{e_2}} - 1, \ldots, x_r^{p^{e_r}} - 1)},$$

the group algebra of G with coefficients in $\mathbb{F}_p$. Then any element of G is represented by some monomial g in R. For $g_1, \ldots, g_s \in G$, consider the product p in R

$$p = (1 - g_1)(1 - g_2) \cdots (1 - g_s).$$

We will show that when $s \geq 1 + \sum (p^{e_i} - 1)$ the product p will be zero in R. Because p is equal to zero, then that some term has to cancel the constant term 1. In particular, that means that for some nonempty subset $J \subseteq \{1, 2, \ldots, s\}$ the product $\prod_{j \in J} g_j$ of the g_j's will have to be equal to 1 in R. It follows that $k \leq 1 + \sum (p^{e_i} - 1)$, as required.

Left to show is that the product $p = \prod_{j=1}^{s}(1-g_j)$ is 0 in R whenever $s \geq 1+\sum(p^{e_i}-1)$. The following observation is the key idea of the argument: if for some $t \in J$ we have that $g_t = a \cdot b$ in R, then

$$1-g_t = (1-a)+a(1-b) \in (1-a, 1-b).$$

Because each g_j is a monomial in R, then g_j is a product of the variables x_i. Applying the observation above repeatedly to each monomial g_j, we get

$$1-g_j \in (1-x_1, 1-x_2, \ldots, 1-x_r),$$

for $j = 1, \ldots, s$. It follows that

$$\prod_{j=1}^{s}(1-g_j) \in (1-x_1, 1-x_2, \ldots, 1-x_r)^s.$$

The ideal on the right is generated by all $\prod_{i=1}^{r}(1-x_i)^{\sigma_i}$ with $\sigma_1+\sigma_2+\cdots+\sigma_r = s$. As $s > \sum(p^{e_i}-1)$, there has to be some i such that $\sigma_i > p^{e_i}-1$, so that $\sigma_i \geq p^{e_i}$. However, in R we have that

$$(1-x_i)^{p^{e_i}} = 1-x_i^{p^{e_i}} = 0.$$

Thus, we get $\prod_{i=1}^{r}(1-x_i)^{\sigma_i} = 0$. Therefore, we have $(1-x_1, \ldots, 1-x_r)^s = (0)$ and $\prod_{j=1}^{s}(1-g_j) = 0$, as claimed. $\square$

Next, we present a way to generalize Olson's approach to prove a bound on $\beta_V(G)$ for V the regular representation of $G = (\mathbb{Z}/(d))^r$. For G a finite abelian group, let $\chi : G \to \mathbb{C}^\star = \mathbb{C} - \{0\}$ be a multiplicative character. Consider $\mathbb{C}[G]$, the group algebra of G over $\mathbb{C}$. Then χ extends to a linear map $\mathbb{C}[G] \to \mathbb{C}$. The characters $\chi \in \widehat{G}$ are linearly independent, so for an element $u \in \mathbb{C}[G]$ we have $u = 0$ if and only if $\chi(u) = 0$ for all multiplicative characters $\chi \in \widehat{G}$.

Studying Olson's proof, we discovered that we can produce a novel characterization of when a set of elements in a group G contains a zero-sum subset. We generalize Olson's idea to produce the following criterion.

Corollary III.10. *Let G be a finite group and consider $g_1, \ldots, g_k \in G$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. Suppose that $\prod_i^k (g_i - \lambda_i)$ is such that*

$$\chi\big(\prod_{i=1}^{k}(g_i - \lambda_i)\big) = \prod_{i=1}^{k}(\chi(g_i) - \lambda_i) = 0$$

for all characters χ of G. Then we have $\prod_{i=1}^{k}(g_i - \lambda_i) = 0$ in $\mathbb{C}[G]$ and $\prod_{i \in J} g_i = 1$ for some nonempty subset $J \subseteq \{1, 2, \ldots, k\}$.

The criterion provided by Corollary III.10 above can be used to provide a bound on $\beta_V(G)$ for the regular representation $V = \mathbb{C}[G]$. In particular, we want to determine how large does k need to be such that for any choice of $g_1, g_2, \ldots, g_k \in G$ there exist $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$ such that

$$\prod(g_i - \lambda_i) = 0.$$

Recall that in the context of abelian groups we have that the character group $\widehat{G}$ is isomorphic to the actual group G. By switching the roles of the group elements and the characters, we can rephrase the above problem in the following way.

Question III.11. *Let G be an abelian group of size n and let $\widehat{G}$ be its character group. If we want that for any choice of k characters $\chi_1, \chi_2, \ldots, \chi_k$ there exists $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$ such that*

$$\prod_{i=1}^{k}(\chi_i(g) - \lambda_i) = 0,$$

for all $g \in G$, how large does k need to be?

We will establish a bound on k for $G \cong \big(\mathbb{Z}/(m)\big)^r$ by considering the fact that characters of the group G have a limited set of possible values. Similar bounds appear in the literature for more general cases (see [16, Theorem 3.6] and [12]), but we provide a new, short, and self-contained proof for the special case of $\big(\mathbb{Z}/(m)\big)^r$.

Theorem III.12. *Let $G = \big(\mathbb{Z}/(m)\big)^r$. Let k be the smallest positive integer such that for any multiplicative characters $\chi_1, \chi_2, \ldots, \chi_k$ there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ such that $\prod_{i=1}^k (\chi_i(g) - \lambda_i) = 0$ for all $g \in G$. Then*

$$k < \frac{r \log(m)}{\log\left(\frac{m}{m-1}\right)} \leq mr \log(m).$$

Proof. Let

$$S_c = \{g \in G \mid \chi_i(g) \neq \lambda_i, \text{ for } i = 1, \ldots, c\}.$$

We can let $S_0 = G$. We will show that there is a way to pick each λ_i so that the cardinality of S_c is decreasing. We will get an upper value on the value k by finding a c large enough so that $S_c = \emptyset$.

Start with χ_1 and notice that χ_1 has only m possible values. One of these values appears at least $\frac{n}{m}$ times in the character table of G. Let this value be λ_1. Then

$$|S_1| \leq m - \frac{n}{m} = \Big(1 - \frac{1}{m}\Big)n = \Big(1 - \frac{1}{m}\Big)|S_0|.$$

Next, consider χ_2. On the set S_1, we have that χ_2 can take at most m values. So there is some value λ_2 that appears at least $\frac{|S_1|}{m}$ times. Thus, we have that

$$|S_2| \leq |S_1| - \frac{|S_1|}{m} = \Big(1 - \frac{1}{m}\Big)|S_1| \leq \Big(1 - \frac{1}{m}\Big)^2 |S_0|.$$

Proceeding in this fashion, we get that

$$|S_c| \leq |S_{c-1}| - \frac{|S_{c-1}|}{m} = \Big(1 - \frac{1}{m}\Big)|S_{c-1}| \leq \Big(1 - \frac{1}{m}\Big)^c n.$$

Thus, we need to find a k such that

$$\Big(1 - \frac{1}{m}\Big)^k n < 1,$$

implying that $S_k = \emptyset$.

Recalling that $n = m^r$, we need to solve the equation

$$\left(1 - \frac{1}{m}\right)^{\alpha} m^r = 1$$

and then we have that for $k < \alpha$, we have the desired result. We have that

$$\alpha = \frac{r \log(m)}{\log\left(\frac{m}{m-1}\right)}.$$

Recall that $\log(x) \geq 1 - \frac{1}{x}$ for $x > 1$. Thus,

$$\log\left(\frac{m}{m-1}\right) \geq \frac{1}{m}.$$

Therefore,

$$k < \frac{r \log(m)}{\log\left(\frac{m}{m-1}\right)} \leq mr \log(m),$$

as claimed. □

In the next section we present an algorithm to compute a minimal generating set of invariant monomials given the weights of the action of the group. We have run several computations using this algorithm and the results brought us to a conjecture about the structure of highest degree invariants of finite abelian groups of rank two.

3.3 An algorithm to compute invariants for abelian groups

We use an original modification of an algorithm of Derksen and Kemper ([7]) to compute invariants of finite abelian groups over $\mathbb{K} = \mathbb{C}$. In their book [7], Derksen and Kemper briefly mention that their algorithm for invariants of algebraic tori could be adapted to the case of finite abelian group. We explicitly produce this adaptation and we introduce two new features that speed up the computations: the use of degree lexicographic ordering and the requirement that each monomial is checked at most once.

Let $G = \bigoplus_{i=1}^{r} \mathbb{Z}/(d_i)$ with generators $g_1 = (1, 0, \ldots, 0)$, $g_2 = (0, 1, \ldots, 0)$, $\ldots$, $g_r = (0, 0, \ldots, 1)$ of order $d_1, d_2, \ldots, d_r$ respectively. Suppose that G acts diagonally on $S = \mathbb{K}[x_1, \ldots, x_n]$ so that

$$g_i \cdot x_j = \zeta_i^{w_{ij}} x_j,$$

where ζ_i a primitive d_i-th root of unity. Recall that we call $\mathbf{w}_j = (w_{1j}, \ldots, w_{rj})$ the weight of x_j and we collect all weights in the matrix $W = (w_{ij})$. Then a monomial $\mathbf{x}^{\mathbf{a}}$ has weight $W\mathbf{a}$.

The group of multiplicative characters $\widehat{G}$ is isomorphic to the additive group of weights $U = \{(u_1, \ldots, u_r) \mid u_i \in \mathbb{Z}/(d_i)\}$. We can partition M, the set of non-constant monomials in $\mathbb{K}[x_1, \ldots, x_n]$, into sets $M_{\mathbf{w}}$ of non-constant monomials of weight $\mathbf{w}$. Notice that the invariant non-constant monomials will be precisely the ones lying in the set $M_{\mathbf{0}}$. The algorithm will create minimal generating sets $T_{\mathbf{w}}$ of monomials in $M_{\mathbf{w}}$. We start with the monomials x_j in the sets $T_{\mathbf{w}_j}$ and carefully build all the sets $T_{\mathbf{w}}$. In particular, $T_{\mathbf{0}}$ will be a minimal set of generating invariants. We will use Noether's degree bound $t = |G|$ to force termination of the algorithm. The algorithm would also end without this criterion, but the criterion may avoid unnecessary computations. Notice that one can adapt the algorithm to just produce all minimal invariant of degree up to any chosen t.

Algorithm III.13. *Fix a lexicographic ordering on the space of weights U and a degree lexicographic ordering on the space of monomials M. Then execute the following routine.*

1: INPUT weights $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of the variables $x_1, \ldots, x_n$

2: INPUT $t = |U|$

3: set $T_{\mathbf{w}_j} = \{x_j\}$ for $j = 1, 2, \ldots, n$ and $T_{\mathbf{w}} = \emptyset$ for all other $\mathbf{w}$

4: set $R = \{x_1, x_2, \ldots, x_n\}$

while $R \neq \emptyset$ ***do***

  find smallest monomial m *in* R

  let k *be maximal such that* $x_k \mid m$

  for $j = k, k+1, \ldots, n$ ***do***

    $m' = mx_j$; $w' =$ *weight of* m'

    if *no* $m_0 \in T_{\mathbf{w}'}$ *divides* m' ***then***

      $T_{\mathbf{w}'} = T_{\mathbf{w}'} \cup \{m'\}$

      if $\deg(m') < t$ ***then***

        $R = R \cup \{m'\}$

      end if

    end if

  end for

  $R = R - \{m\}$

end while

OUTPUT $\{(\mathbf{w}, T_{\mathbf{w}})\}$ *for* $\mathbf{w} \in U$

Notice that the algorithm ends as the degree of the monomials in R are weakly increasing but the list R will be emptied when the degree of all new monomials m' are $\geq t$. The algorithm is as efficient as possible as the ordering on the variables forces us to consider each monomial at most once. For example, for $j > i$, when we choose $x_i \in R$ we will multiply it with x_j to obtain the monomial $x_i x_j$, but we will not multiply x_j by x_i when we choose $x_j \in R$ as $x_i < x_j$. However, we will show that we do have a minimal list of generating monomials in each $T_{\mathbf{w}}$.

Lemma III.14. *Let* m *be a monomial of degree* k *and weight* $\mathbf{w}$ *that does never appear in* R*. Then at the end of Algorithm III.13, we have that* $m \in (T_{\mathbf{w}})$*.*

Proof. Towards contradiction, let m be the smallest monomial such that $m \notin R$ at any point and $m \notin (T_{\mathbf{w}})$ at the end of the algorithm. Let $m = ax_k$, for a some monomial of weight $\mathbf{u}$ and x_k the largest variable appearing in m.

Case 1: We have that $a \in R$ at some point. If $a \in R$, then we have to check $m = ax_k$ in the algorithm. If $m = ax_k$ is checked and m is not added to R, then there is $m_0 \in T_{\mathbf{w}}$ such that $m_0 \mid m$. Thus, $m \in (T_{\mathbf{w}})$, contradiction.

Case 2: We have that $a \notin R$. Because $a < m$, by the minimality of m we have that $a \in (T_{\mathbf{u}})$. As a is a monomial and a lies in the monomial ideal $(T_{\mathbf{u}})$, there is some a_1 in $T_{\mathbf{u}}$ such that $a = a_0 a_1$. Notice that this means that a_0 is a monomial of weight $\mathbf{0}$. Then $m = ax_k = a_0 a_1 x_k$. Notice that $a_1 x_k$ has weight $\mathbf{w}$, the same weight as m, because a_0 has weight $\mathbf{0}$. Moreover, as $a_1 \in T_{\mathbf{u}}$, we have that the monomial $a_1 x_k$ is checked by the algorithm. Then either $a_1 x_k \in T_{\mathbf{w}}$, implying that $m \in (T_{\mathbf{w}})$, or there is some $m_0 \in T_{\mathbf{w}}$ such that $m_0 \mid a_1 x_k \mid m$, yielding again that $m \in (T_{\mathbf{w}})$. Either way, $m \in (T_{\mathbf{w}})$, which is a contradiction.

Therefore, as both cases lead to a contradiction, we can conclude that no such m exists. $\square$

The Lemma above allows to show that the lists $T_{\mathbf{w}}$ do indeed generate the ideals of non-zero monomials of degree $\leq t$.

Proposition III.15. *Let $M_{\mathbf{w}}$ denote the set of all non-constant monomials of weight $\mathbf{w}$ in $\mathbb{K}[x_1, \ldots, x_n]$. For every degree k such that $1 \leq k \leq t$, when every monomial m of $\deg(m) < k$ has been removed from the list R in Algorithm III.13, we have that*

$$(T_{\mathbf{w}})_{\leq k} = (M_{\mathbf{w}})_{\leq k},$$

so the monomials of degree $\leq k$ in the list $T_{\mathbf{w}}$ generate the same ideal as the non-constant monomials of weight $\mathbf{w}$ and degree $\leq k$.

Proof. As $T_{\mathbf{w}} \subseteq M_{\mathbf{w}}$, it is clear that $(T_{\mathbf{w}})_{\leq k} \subseteq (M_{\mathbf{w}})_{\leq k}$.

We will prove the other inclusion by induction on k. For $k = 1$, by line 3 of the Algorithm III.13 we have that $(T_{\mathbf{w}_j})_{\leq 1} = (x_j) = (M_{\mathbf{w}_j})_{\leq 1}$ for $j = 1, \ldots, n$, whilst for all other $\mathbf{w}$ we have that $(T_{\mathbf{w}})_{\leq 1} = (0) = (M_{\mathbf{w}})_{\leq 1}$.

Suppose that the claim is true for all $d < k$. Let m be any non-constant monomial of weight $\mathbf{w}$ and degree k, so that $m \in (M_{\mathbf{w}})_{\leq k}$. If $m \in T_{\mathbf{w}}$, then the claim is true. If $m \notin T_{\mathbf{w}}$, then in the algorithm we either checked m or did not.

If $m \notin T_{\mathbf{w}}$ and we checked m, then we decided not to add m to $T_{\mathbf{w}}$. Thus, there is some $m_0 \in T_{\mathbf{w}}$ such that $m_0 \mid m$. Say $\deg(m_0) = d$. Then $m \in (T_{\mathbf{w}})_{\leq d} \subset (T_{\mathbf{w}})_{\leq k}$. So $m \in (T_{\mathbf{w}})_{\leq k}$, as desired.

If we did not check the monomial m, then $m \notin R$ at any point. Then by Lemma III.14, we have that $m \in (T_{\mathbf{w}})$. As the degree of m is k, we can conclude that $m \in (T_{\mathbf{w}})_{\leq k}$, as desired. $\square$

Next, we show that each $T_{\mathbf{w}}$ is a minimal set of generators for $(M_{\mathbf{w}})_{\leq t}$.

Proposition III.16. *Let $M_{\mathbf{w}}$ denote the set of all non-constant monomials of weight $\mathbf{w}$ in $\mathbb{K}[x_1, \ldots, x_n]$. The set $T_{\mathbf{w}}$ produced at the end of Algorithm III.13 is a minimal set of generators for $(T_{\mathbf{w}}) = (M_{\mathbf{w}})_{\leq t}$.*

Proof. Suppose that $T_{\mathbf{w}}$ is not minimal, so there is some $m \in T_{\mathbf{w}}$ that can be omitted and still generate $(T_{\mathbf{w}}) = (M_{\mathbf{w}})_{\leq k}$. As $m \in T_{\mathbf{w}}$, we checked m in the algorithm and decided to add it to $T_{\mathbf{w}}$. Thus, there is no $m_0 \in T_{\mathbf{w}}$ such that $m_0 \mid m$. Therefore, m is a minimal generator of $(T_{\mathbf{w}})$, contradiction. $\square$

We only need to establish that the set $T_{\mathbf{w}}$ generates the whole ideal $(M_{\mathbf{w}})$. By Proposition III.15, we only need to show that $(M_{\mathbf{w}})_{\leq t} = (M_{\mathbf{w}})$, for $t = |G|$.

Proposition III.17. *Let G be a finite abelian group acting on $S = \mathbb{K}[x_1, \ldots, x_n]$ and let $t = |G|$. Let $M_{\mathbf{w}}$ denote the set of non-constant monomials of weight $\mathbf{w}$ in S and let $(M_{\mathbf{w}})_{\leq t}$ denote the ideal generated by all non-constant monomials of weight $\mathbf{w}$ and degree $\leq t$. Then we have that $(M_{\mathbf{w}})_{\leq t} = (M_{\mathbf{w}})$.*

Proof. Consider the ring $R = \mathbb{K}[x_1, \ldots, x_n, z]$, where z is a new variable of weight $-\mathbf{w}$. Suppose that m is a monomial in $S = \mathbb{K}[x_1, \ldots, x_n]$ of weight $\mathbf{w}$ and degree $> t$. We will show that m is not a minimal generator of $(M_{\mathbf{w}})$. Notice that mz is a monomial of weight $\mathbf{0}$ and degree $> t$. By Noether's degree bound in the ring R, we have that $mz = ab$, where a, b are invariant monomials. Without loss of generality, assume that $z \nmid a$, then $a \mid m$, so m is not a minimal generator. $\square$

In conclusion, we have the following result.

Theorem III.18. *Let G be a finite abelian group acting on $S = \mathbb{K}[x_1, \ldots, x_n]$. Let $T_{\mathbf{w}}$ be the set of monomials of weight $\mathbf{w}$ produced by Algorithm III.13. Then $T_{\mathbf{0}}$ is a minimal set of generating invariants, so that $S^G = \mathbb{K}[T_{\mathbf{0}}]$. Moreover, $(M_{\mathbf{w}}) = S^G T_{\mathbf{w}}$, where $(M_{\mathbf{w}})$ is the ideal of all non-constant monomials of weight $\mathbf{w}$.*

Proof. Notice that $S^G = \mathbb{K}[M_{\mathbf{0}}]$ as all non-constant invariants are in $M_{\mathbf{0}}$. By Proposition III.17, $T_{\mathbf{0}}$ generates $M_{\mathbf{0}}$ and by Proposition III.16, $T_{\mathbf{0}}$ is a minimal generating set. Finally, if $m \in (M_{\mathbf{w}})$, then by Proposition III.17, there is some $b \in T_{\mathbf{w}}$, such that $m = ab$. Then m and b both have weight $\mathbf{w}$, so a has weight $\mathbf{0}$. Therefore, $a \in S^G$, as claimed. $\square$

We showcase Algorithm III.13 with the following example.

Example III.19. Consider $G \cong \mathbb{Z}/(3) \times \mathbb{Z}/(3)$ acting on $\mathbb{K}[x_1, x_2, x_3]$ with weight matrix

$$W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We have nine weights in U and the algorithm starts with $x_1 \in S_{(1,0)}$, $x_2 \in S_{(0,1)}$, and $x_3 \in S_{(1,1)}$. We provide in the table below the monomials produced by Algorithm III.13. Each column indexed by $\mathbf{w}$ contains the monomials in the space $S_{\mathbf{w}}$ at the end of the algorithm. The table is organized by the degree of the monomials (shown in the first column).

degree	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
1		x_2		x_1	x_3				
2			x_2^2		x_1x_2	x_2x_3	x_1^2	x_1x_3	x_3^2
3	x_1^3, x_2^3, x_3^3	$x_1^2x_3$	$x_1x_3^2$	$x_2^2x_3$		$x_1x_2^2$	$x_2x_3^2$	$x_1^2x_2$	$x_1x_2x_3$
4	$x_1x_2x_3^2$,		$x_1^2x_2x_3$			$x_1^2x_3^2$	$x_1x_2^2x_3$	$x_2^2x_3^2$	$x_1^2x_2^2$
5	$x_1^2x_2^2x_3$								

Notice that the algorithm terminates well before reaching degree $9 = |G|$. This is because for any possible new monomial m of weight $\mathbf{w}$, there is some monomial $m_0 \in S_{\mathbf{w}}$ such that $m_0 \mid m$.

We implemented the algorithm in MATHLAB, as each check and operation can be reduced to a linear algebra computation. After running several iterations of the program, we observed a pattern that brought us to the following conjecture about the highest degree minimal invariants in $G \cong \mathbb{Z}/(n) \oplus \mathbb{Z}/(n)$.

Conjecture III.20. *All minimal invariants of highest degree $2n-1$ are given by monomials of the type $x_i^{n-1}m$, for some variable x_i and some non square-free monomial m such that $\deg(m) = n$ and $x_i \nmid m$.*

Olson [24] showed that $\beta(\mathbb{Z}/(n)\oplus\mathbb{Z}/(n)) = 2n-1$. Even though the exact value of β for any abelian group is still unknown, questions about non-shortenable zero-sums continue to arise from different fields of mathematics: combinatorics, graph theory, and number theory. A recent survey of results on this topic from the point of view of number theory is [16].

In general the structure of longest non-shortenable zero-sums is not completely understood. On the other hand, for $G \cong (\mathbb{Z}/(n))^2$, Lettl and Schmid [20] have proved the existence of longest non-shortenable zero-sums with one term x of multiplicity $n-1$. These longest non-shortenable zero-sums correspond precisely to highest degree minimal invariants. Thus, their term x of multiplicity $n-1$ corresponds to the x_i^{n-1} factor in our conjecture about highest degree invariant monomials.

It was recently proved that in all finite abelian groups of rank two, all highest degree invariant monomials are of this type. Progress on proving the conjecture is described in [17]. In the number theory context, a group G satisfies our conjecture if G has "Property **B**". First, a large number of groups of rank two had been shown to satisfy "Property **B**". In fact, the "Property **B**" conjecture had been reduced to the case that $G \cong (\mathbb{Z}/(p))^2$ for every odd prime p. Finally, Reiher proved that $(\mathbb{Z}/(p))^2$ has "Property **B**" in [26] and the conjecture was proved for every finite abelian group of rank 2.

In our Conjecture III.20 we show that when $G \cong (\mathbb{Z}/(p))^2$, the requirement that m is not square-free is redundant as it is a consequence of the other conditions.

Proposition III.21. *Suppose that p is an odd prime and $S = \sum_{i=1}^{2p-1} b_i$ is a longest non-shortenable zero-sum in $G \cong \mathbb{Z}/(p) \oplus \mathbb{Z}(p)$. If $b_1 = \ldots = b_{p-1}$, then the other p elements b_i cannot all be distinct.*

Proof. Suppose, towards contradiction, that all the other elements are distinct. After a change of coordinates we can assume that $b_1 = (0,1) \in \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$. As

$$(p-1)b_1 + \sum_{i \geq p} b_i = 0,$$

we have that $\sum_{i \geq p} b_i = (0,1)$. Thus, we are looking for p distinct elements $\{b_p, \ldots, b_{2p-1}\}$ summing to $(0,1)$. Relabel these p elements $a_1, \ldots, a_p$.

Let $a_j = (a_{j1}, a_{j2})$. The condition above requires that $\sum a_{j1} \equiv 0 \mod p$, whilst $\sum a_{j2} \equiv 1 \mod p$. Recall that in $\mathbb{Z}/(p)$ a zero-sum of p element a_{j1} is such that either a proper subsum is zero or all elements a_{j1}'s are equal. We consider these two cases separately.

Case 1: All a_{j1}'s are equal. Say $a_{j1} = d$ for all i. Then all the a_{j2}'s must be distinct, as all the a_j's are distinct. Notice that the set of p distinct a_{j2}'s in $\mathbb{Z}/(p)$ is such that $\{a_{j2}\} = \mathbb{Z}/(p)$. Thus, the set of a_j's is a full coset of $\mathbb{Z}/(p)$ in G i.e., $\{a_j\} = \{(d,0), \ldots, (d,p-1)\}$ in G. However, this means that

$$\sum_j a_i = \sum_{j=0}^{p-1} (d,j) = 0$$

as $p \cdot d \equiv 0 \mod p$ and $\sum_{j=0}^{p-1} j \equiv 0 \mod p$. Thus, $S = \sum b_i$ contains the zero-sum $\sum a_j = \sum_{i \geq p} b_i$, contradicting the assumption that S is non-shortenable.

Case 2: A proper subsum of the a_{j1}'s is equal to zero. Relabel these terms so that $\sum_{j=1}^{t} a_{j1} = 0$ for $t < p$. Then

$$\sum_{j=1}^{t} a_j = (\sum_{j=1}^{t} a_{j1}, \sum_{j=1}^{t} a_{j2}) = (0,d),$$

for $\sum_{j=1}^{t} a_{j2} = d \in \mathbb{Z}/(p)$. If $d = 0$, then we have the shortenable subsum $\sum_{j=1}^{t} a_j = 0$ in S, which is a contradiction. If $d \neq 0$, then $\sum_{i=1}^{p-d} b_i = (p-d)b_1 = (0, p-d)$. Thus,

$$\sum_{i=1}^{p-d} b_i + \sum_{i=p}^{p+t-1} b_i = (0, p-d) + \sum_{j=1}^{t} a_j = (0, p-d) + (0,d) \equiv (0,0),$$

in $G \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$, As $d \neq 0$ and $t < p$, this contradicts the assumption that S is non-shortenable.

As both cases lead to a contradiction, we can conclude that the elements $\{b_p, \ldots, b_{2p-1}\}$ cannot be distinct. $\square$

We finish this chapter by connecting invariant monomials to the theory of subspace arrangements.

3.4 Subspace arrangements and abelian groups

Consider the subspace arrangement $\mathcal{A}_G$ associated to a finite abelian group G acting diagonally on a vector space V

$$\mathcal{A}_G = \bigcup_{g \in G} V_g = \bigcup_{g \in G} \{(v, g \cdot v) \mid v \in V\} \subseteq V \oplus V.$$

We have that the action of V gives an action on $\mathbb{K}[V]$, so that G acts on the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, where x_i is the i-th coordinate function on V. We let the ring of polynomials on $V \oplus V$ be $\mathbb{K}[\mathbf{x}, \mathbf{y}]$ and we notice that the action of G defining $\mathcal{A}_G$ corresponds to G acting trivially on the first set of n coordinates, whilst we have the original action on the second set of n coordinates. Consider

$$I_G = \mathbb{I}(\mathcal{A}_G) = \bigcap_{g \in G} \mathbb{I}(V_g).$$

Notice that each $I_g = \mathbb{I}(V_g)$ is a binomial linear ideal as it is defined by the equations

$$y_j = g \cdot x_j.$$

In general, the intersection of binomial ideals need not be binomial. However, because our ideal $I_G = \bigcap I_g$ arises from a group action, we can prove that our ideal I_G is a generated by binomials. Our argument is an adaptation of the proof that toric ideals are binomial ideals as in Lemma 4.1 in [36].

Recall that we collect the weights of the action of $G = \prod \mathbb{Z}/(d_i)$ into the matrix $W = (w_{ij})$, where w_{ij} is the weight of the i-th generator of G on the j-th variable x_j. We think of a weight vector $\mathbf{w}$ as living in the space $U = \bigoplus \mathbb{Z}/(d_i)$, where we think of $\mathbb{Z}/(d_i)$ as the set $\{0, 1, \ldots, d_i - 1\}$ and with addition modulo d_i. We start by defining a particular family of binomials.

Definition III.22. Let $I = (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ be a sequence of four weights $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in U$ such that

(i) $W\mathbf{b} = W\mathbf{d}$,

(ii) $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$.

Let $\mathcal{U}$ be the set of sequences satisfying conditions (i) and (ii). For $I \in \mathcal{U}$, we define the binomial f_I by

$$f_I = \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{c}}\mathbf{y}^{\mathbf{d}}.$$

The family of binomials $\{f_I\}$ is precisely what we need to obtain the ideal I_G.

Proposition III.23. *Let f_I for $I \in \mathcal{U}$ be defined as above. Then for $I_G = \mathbb{I}(\mathcal{A}_G)$, we have that*

$$I_G = \mathrm{span}_{\mathbb{K}}\{f_I \mid I \in \mathcal{U}\}.$$

Therefore, I_G is a binomial ideal.

Proof. The easy inclusion is $\{f_I\} \subseteq I_G$. Let $g \in G$ and consider any $(\mathbf{x}, g \cdot \mathbf{x})$ in V_g. Let $\{g_i\}$ be a set of generators for G and suppose that $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$. We let $\mathbf{w}^i$ be the i-th row of W, corresponding to the action of g_i on $\mathbb{K}[x_1, \ldots, x_n]$. We have that

$$f_I(\mathbf{x}, g \cdot \mathbf{x}) = (\prod_i \zeta_i^{\beta_i \mathbf{w}^i \cdot \mathbf{b}})\mathbf{x}^{\mathbf{a}+\mathbf{b}} - (\prod_i \zeta_i^{\beta_i \mathbf{w}^i \cdot \mathbf{d}})\mathbf{x}^{\mathbf{c}+\mathbf{d}} = (\prod_i \zeta_i^{\beta_i \mathbf{w}^i \cdot \mathbf{b}} - \zeta_i^{\beta_i \mathbf{w}^i \cdot \mathbf{d}})\mathbf{x}^{\mathbf{a}+\mathbf{b}} = 0,$$

because by condition (ii) $\mathbf{a}+\mathbf{b}=\mathbf{c}+\mathbf{d}$ and by condition (i) $W\mathbf{b}=W\mathbf{d}$ so that $\beta_i\mathbf{w}^i\cdot\mathbf{b}=\beta_i\mathbf{w}^i\cdot\mathbf{d}$ for all i. Since f_I vanishes on $(\mathbf{x}, g\cdot\mathbf{x})$ for all generators $g\in G$, we have that $f_I\in I_G$. Therefore, $f_I\in I_G$ for every $I\in\mathcal{U}$.

We now prove the harder direction, $I_G\subseteq \operatorname{span}_{\mathbb{K}}\{f_I\}$. The proof is by contradiction. Fix a term order $<$ on $S=\mathbb{K}[\mathbf{x},\mathbf{y}]$ and suppose that $f\in I_G$ cannot be written as an $\mathbb{K}$-linear combination of the binomials f_I. Find the f in I_G with this property and such that $\operatorname{in}(f)$ is minimal with respect to the chosen order, say $\operatorname{in}(f)=\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$. As $f\in I_G$, we have that $f(\mathbf{x}, g_i\cdot\mathbf{x})=0$ for every generator g_i of G. In particular, the term $\zeta_i^{\mathbf{w}^i\cdot\mathbf{b}}\mathbf{x}^{\mathbf{a}+\mathbf{b}}$ must cancel. Hence, there is some monomial $\mathbf{x}^{\mathbf{c}}\mathbf{y}^{\mathbf{d}}<\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$ such that $\zeta_i^{\mathbf{w}^i\cdot\mathbf{d}}\mathbf{x}^{\mathbf{c}+\mathbf{d}}$ cancels $\zeta_i^{\mathbf{w}^i\cdot\mathbf{b}}\mathbf{x}^{\mathbf{a}+\mathbf{b}}$. Thus, $\zeta_i^{\mathbf{w}^i\cdot\mathbf{b}}=\zeta_i^{\mathbf{w}^i\cdot\mathbf{d}}$ for all i, so that $W\mathbf{c}=W\mathbf{d}$. Moreover, we have that $\mathbf{a}+\mathbf{b}=\mathbf{c}+\mathbf{d}$. Thus, $I=(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ satisfies conditions (i) and (ii), so that $I\in\mathcal{U}$. For $f_I=\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}-\mathbf{x}^{\mathbf{c}}\mathbf{y}^{\mathbf{d}}$, consider

$$f'=f-f_I.$$

As f could not be written as a $\mathbb{K}$-linear combination of binomials $\{f_I|I\in\mathcal{U}\}$, f' also has this property. However, $\operatorname{in}(f')<\operatorname{in}(f)$, contradicting our initial assumption on f. Therefore, no such f exists and I_G is indeed a binomial ideal. □

Recall that Derksen [5] proved an algorithm for finding the generators of the Hilbert ideal J_G from the generators of the vanishing ideal I_G of the subspace arrangement $\mathcal{A}_G$ for the group G. In the context of abelian groups, the invariant monomials can be found by looking at initial ideals. Specifically, when we fix a term order $<$ such that $x_j<y_k$ for all j,k, we can consider the monomials in $\operatorname{in}(I_G)$ only in the x variables.

These monomials will precisely be the generators of I^G. By the Proposition III.23, these monomials arise precisely from the binomials

$$f_I = \mathbf{x}^{\mathbf{a}} - \mathbf{y}^{\mathbf{a}},$$

where $I = (\mathbf{a}, \mathbf{0}, \mathbf{0}, \mathbf{a})$ and $W\mathbf{a} = 0$.

Notice that in Algorithm III.13 we obtain lists $S_{\mathbf{w}}$ of monomials of the same weight $\mathbf{w}$. We can consider these $S_{\mathbf{w}}$ as lists of exponent vectors $\mathbf{b}, \mathbf{d}$ in U such that $W\mathbf{b} = W\mathbf{d}$. One can wonder whether the exponent vectors in $S_{\mathbf{w}}$ are enough to produce a set of binomial generators for I_G. We definitely have enough vectors in $S_{\mathbf{0}}$ to generate all the invariant monomials as $(S_{\mathbf{0}}) = \mathrm{in}(I_G) \cap \mathbb{K}[x_1, \ldots, x_n]$. However, the fact that we have enough monomials in $S_{\mathbf{0}}$ to generate the intersection of the initial ideal with $\mathbb{K}[x_1, \ldots, x_n]$ is not enough to establish that we have enough monomials in the set $\{S_{\mathbf{w}} \mid \mathbf{w} \in U\}$ to generate the whole initial ideal. On the other hand, if we were able to produce a set of monomials $\{\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}\}$ that generates the initial ideal $\mathrm{in}(I_G)$, then we would be able to produce a list of binomial generators f_I for I_G. Specifically, we would need to consider all f_I where $I = (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \in \mathcal{U}$ is such that $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} \in \mathrm{in}(I_G)$ and $\mathbf{x}^{\mathbf{c}} \leq \mathbf{x}^{\mathbf{a}}$.

CHAPTER IV

The omega functor

In the second half of this thesis we will study ideals in the exterior algebra, a skew commutative ring. To aid our investigation, we use a tool that allows us to transfer information from the symmetric algebra to the exterior algebra. To be able to exploit this tool, we need to consider ideals of subspace arrangements that are stable under the action of the general linear group and study them using representation theory. Specifically, we construct resolutions for these ideals and we transfer homological properties from resolutions over the symmetric algebra to resolutions over the exterior algebra. To be able to do so, we need to define a functor on the category of polynomial functors which we call the functor Ω.

Our functor Ω is the transpose functor of Sam and Snowden [29–31]. They used the transpose functor in conjunction with Koszul duality to define their Fourier transform in [30, Section 6.6]. Moreover, in [31] they use this Fourier transform to establish that finitely generated modules over the twisted commutative algebra $\mathrm{Sym}(W \otimes \mathbb{C}^\infty)$ have finite regularity.

We will use the functor Ω to establish a regularity bound on ideals of subspace arrangements over the exterior algebra using known bounds over the symmetric algebra. For any finitely generated module M over the twisted commutative algebra

$\mathrm{Sym}(W \otimes \mathbb{C}^\infty)$, we exploit a connection between $\mathrm{Tor}_i(\Omega(M), \mathbb{K})$ and $\Omega(\mathrm{Tor}_i(M, \mathbb{K}))$ which is also a consequence of the properties of their Fourier transform. However, we only consider the case where M is a module functor associated to a subspace arrangement. In this case, using the transpose functor Ω is sufficient to establish our results.

4.1 Polynomial functors

In our discussion of polynomial functors we follow the classical treatment of Macdonald [21]. Let us fix a field $\mathbb{K}$ of characteristic 0. Let us denote by **Vec** the category of finite dimensional $\mathbb{K}$-vector spaces whose morphisms are the $\mathbb{K}$-linear maps. This abelian category also has a tensor product, which makes **Vec** into a symmetric monoidal category.

Definition IV.1. A functor $\mathcal{F}$ from **Vec** to **Vec** is a polynomial functor if the map

$$\mathcal{F} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a polynomial mapping for all finite dimensional $\mathbb{K}$-vector spaces X, Y. We say that $\mathcal{F}$ is homogeneous of degree d if $\mathcal{F}(\lambda h) = \lambda^d \mathcal{F}(h)$ for every linear map $h \in \mathrm{Hom}(X, Y)$ and every scalar $\lambda \in \mathbb{K}$.

Let $\mathcal{F}$ be a polynomial functor. We will consider the category of polynomial functors **Poly**. The morphisms in **Poly** are natural transformations of functors. If $\mathcal{F}$ and $\mathcal{G}$ are polynomial functors, we define the direct sum functor $\mathcal{F} \oplus \mathcal{G} : \mathbf{Vec} \to \mathbf{Vec}$ by $(\mathcal{F} \oplus \mathcal{G})(X) = \mathcal{F}(X) \oplus \mathcal{G}(X)$ for every finite dimensional vector space $X \in \mathrm{Obj}(\mathbf{Vec})$, and

$$(\mathcal{F} \oplus \mathcal{G})(h) = \begin{pmatrix} \mathcal{F}(h) & 0 \\ 0 & \mathcal{G}(h) \end{pmatrix} \in \mathrm{Hom}(\mathcal{F}(X) \oplus \mathcal{G}(X), \mathcal{F}(Y) \oplus \mathcal{G}(Y))$$

for every linear map $h : X \to Y$. We can also define the tensor product of two polynomial functors $\mathcal{F}$ and $\mathcal{G}$ by $(\mathcal{F} \otimes \mathcal{G})(X) = \mathcal{F}(X) \otimes \mathcal{G}(X)$ for every finite dimensional vector space and $(\mathcal{F} \otimes \mathcal{G})(h) = \mathcal{F}(h) \otimes \mathcal{G}(h) : \mathcal{F}(X) \otimes \mathcal{G}(X) \to \mathcal{F}(Y) \otimes \mathcal{G}(Y)$ for any linear map $h : X \to Y$. This makes **Poly** into an abelian symmetric monoidal category. If $\mathcal{F}$ and $\mathcal{G}$ are homogeneous polynomial functors of degree d and e respectively, then $\mathcal{F} \otimes \mathcal{G}$ is homogeneous of degree $d + e$.

For categories **A** and **B** we denote the category of all functors from **A** to **B** by $\mathbf{Fun}(\mathbf{A}, \mathbf{B})$. Morphisms in $\mathbf{Fun}(\mathbf{A}, \mathbf{B})$ are natural transformations. We can view **Poly** as a subcategory of $\mathbf{Fun}(\mathbf{Vec}, \mathbf{Vec})$.

For an n-dimensional vector space V, let $\mathrm{GL}(V) \subseteq \mathrm{Hom}(V, V)$ be the group of invertible linear maps from V to V. A polynomial functor $\mathcal{F}$ gives a polynomial map $\mathrm{Hom}(V, V) \to \mathrm{Hom}(\mathcal{F}(V), \mathcal{F}(V))$ that restricts to a group homomorphism $\rho : \mathrm{GL}(V) \to \mathrm{GL}(\mathcal{F}(V))$. This means that $\mathcal{F}(V)$ is a polynomial representation of $\mathrm{GL}(V)$.

A partition is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. For each partition λ one can define a polynomial functor $\mathcal{S}_\lambda : \mathbf{Vec} \to \mathbf{Vec}$ that is homogeneous of degree $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. For a finite dimensional vector space V, the representation $\mathcal{S}_\lambda(V)$ is an irreducible representation of $\mathrm{GL}(V)$. The space $\mathcal{S}_{(d)}(V) = \mathrm{Sym}^d(V)$ is the d-th symmetric power of V whilst the space $\mathcal{S}_{(1,1,\ldots,1)}(V) = \mathcal{S}_{(1^d)}(V) = \bigwedge^d(V)$ is the d-th exterior power of V. It follows from Schur's lemma that

$$\mathrm{Hom}(\mathcal{S}_\lambda, \mathcal{S}_\mu) = \begin{cases} \mathbb{K} & \text{if } \lambda = \mu; \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

Every polynomial functor is naturally equivalent to a finite direct sum of $\mathcal{S}_\lambda$'s. By grouping the $\mathcal{S}_\lambda$'s together we see that every polynomial functor $\mathcal{P} \in \mathbf{Poly}$ is naturally equivalent to a direct sum $\mathcal{P} = \bigoplus_d \mathcal{P}_d$, where $\mathcal{P}_d$ is a homogeneous polynomial functor of degree d. We will denote the full subcategory of homogeneous polynomial functors of degree d by $\mathbf{Poly}_d$. For more details, the interested reader can consult [21, p. 150].

Let $\mathbf{Rep}_V$ denote the category of finite dimensional rational representations of $\mathrm{GL}(V)$ where the morphism are $\mathrm{GL}(V)$-equivariant linear maps.

Lemma IV.2. *A polynomial functor $\mathcal{P}$ on the category of finite dimensional vector spaces* $\mathbf{Vec}$ *induces a functor $\mathcal{P}_V$ on the category of $\mathrm{GL}(V)$-representations $\mathbf{Rep}_V$.*

Proof. Let us consider a $\mathrm{GL}(V)$-representation $\rho_U : \mathrm{GL}(V) \to \mathrm{GL}(U)$. The polynomial functor $\mathcal{P}$ gives a polynomial map $\mathrm{Hom}(U,U) \to \mathrm{Hom}(\mathcal{P}(U), \mathcal{P}(U))$ which restricts to a representation $\mathrm{GL}(U) \to \mathrm{GL}(\mathcal{P}(U))$. The composition $\mathrm{GL}(V) \to \mathrm{GL}(U) \to \mathrm{GL}(\mathcal{P}(U))$ makes $\mathcal{P}(U)$ into a representation of $\mathrm{GL}(V)$.

Let ϕ be a $\mathrm{GL}(V)$-equivariant map from U to U', so that for all $g \in \mathrm{GL}(V)$ the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "\rho_U(g)"] & U' \arrow[d, "\rho_U(g)"] \\ U \arrow[r, "\phi"] & U' \end{tikzcd} .$$

Applying $\mathcal{P}$ to this diagram, we notice that the resulting diagram also commutes as

$$\mathcal{P}(\rho_U(g))\mathcal{P}(\phi) = \mathcal{P}(\rho_U(g)\phi) = \mathcal{P}(\phi\rho_U(g)) = \mathcal{P}(\phi)\mathcal{P}(\rho_U(g)),$$

by functoriality of $\mathcal{P}$ and our assumptions on ϕ. This shows that $\mathcal{P}(\phi) : \mathcal{P}(U) \to \mathcal{P}(U')$ is $\mathrm{GL}(V)$-equivariant.

We conclude that $\mathcal{P}$ induces a functor from $\mathbf{Rep}_V$ to itself. $\square$

We can consider the category $\mathbf{Poly}_V$ of polynomial functors from $\mathbf{Rep}_V$ to itself. Morphisms in the category $\mathbf{Poly}_V$ are $\mathrm{GL}(V)$-equivariant natural transformations.

An object $\mathcal{P}$ is the category **Poly** induces an object $\mathcal{P}_V$ in $\mathbf{Poly}_V$ by Lemma IV.2.

4.2 The category GPoly

We also consider the category **GVec** of graded vector spaces. The objects of **GVec** are graded vector spaces $V = \bigoplus_{d=0}^{\infty} V_d$ such that V_d is finite dimensional for all d. A morphism $\phi : V \to W$ in the category **GVec** is a linear map that respects the grading, i.e., $\phi(V_d) \subseteq W_d$ for all d. The tensor product of two graded vector spaces V, W in **GVec** is defined by $(V \otimes W)_d = \bigoplus_{e=0}^{d} V_e \otimes W_{d-e}$. This makes **GVec** into a symmetric monoidal category.

Next we describe the full subcategory **GPoly** in the functor category $\mathbf{Fun}(\mathbf{Vec}, \mathbf{GVec})$.

Definition IV.3. An object $\mathcal{F}$ in **GPoly** is a functor in $\mathbf{Fun}(\mathbf{Vec}, \mathbf{GVec})$ with the property that

$$V \mapsto \mathcal{F}(V)_d$$

is a homogeneous polynomial functor of degree d. Morphisms in **GPoly** are natural transformations.

An example of a functor in **GPoly** is the functor $\mathcal{S} = \mathrm{Sym}$, mapping a vector space V to the symmetric algebra $\mathcal{S}(V) = \mathrm{Sym}(V)$ on V. Similarly, another such functor in **GPoly** is the exterior functor $\bigwedge$ that maps a vector space V to its exterior algebra $\bigwedge(V)$.

The category **GPoly** is a symmetric monoidal category via the tensor structure inherited from **GVec**. In **GPoly** we have that

$$((\mathcal{F} \otimes \mathcal{G})(V))_d = (\mathcal{F}(V) \otimes \mathcal{G}(V))_d = \bigoplus_{e=0}^{d} \mathcal{F}(V)_e \otimes \mathcal{G}(V)_{d-e}.$$

We will also view $\mathbb{K}$ as an object in **GPoly** as the functor that sends every vector space to the graded vector space $\mathbb{K}$ concentrated in degree 0. The object $\mathbb{K}$

is the identity in the monoidal category **GPoly**. This means that we have a natural equivalence $\kappa : \mathbb{K} \otimes \mathcal{F} \to \mathcal{F}$ for every object $\mathcal{F}$ in **GPoly**.

4.2.1 Algebras and modules in GPoly

We will define algebra functors and module functors in **GPoly**. These are objects in **GPoly** that satisfy axioms analog to the axioms of algebras and modules, respectively.

Definition IV.4. An object $\mathcal{R}$ in **GPoly** is called an algebra functor if it comes equipped with a multiplication $\mu : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$ (i.e., a natural transformation of the functor $\mathcal{R} \otimes \mathcal{R}$ to the functor $\mathcal{R}$) and an identity $\mathbf{1} : \mathbb{K} \to \mathcal{R}$ that satisfy the following axioms.

Connected $\mathbf{1}_0 : \mathbb{K}_0 \to \mathcal{R}_0$ is a natural equivalence. Hence, we assume $\mathcal{R}_0(V) \cong \mathbb{K}$ for all vector spaces V;

Identity the following diagram commutes

$$\begin{tikzcd}
\mathbb{K} \otimes \mathcal{R} \arrow[r, "\kappa"] \arrow[d, "\mathbf{1} \otimes \mathrm{Id}_{\mathcal{R}}"'] & \mathcal{R} \arrow[d, "\mathrm{Id}_{\mathcal{R}}"] \\
\mathcal{R} \otimes \mathcal{R} \arrow[r, "\mu"'] & \mathcal{R}
\end{tikzcd} ;$$

Associative the following diagram commutes

$$\begin{tikzcd}
(\mathcal{R} \otimes \mathcal{R}) \otimes \mathcal{R} \arrow[rr, "\cong"] \arrow[d, "\mu \otimes \mathrm{Id}_{\mathcal{R}}"'] & & \mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{R}) \arrow[d, "\mathrm{Id}_{\mathcal{R}} \otimes \mu"] \\
\mathcal{R} \otimes \mathcal{R} \arrow[r, "\mu"'] & \mathcal{R} & \mathcal{R} \otimes \mathcal{R} \arrow[l, "\mu"]
\end{tikzcd} .$$

We will define a (left) module in a similar fashion.

Definition IV.5. Given an algebra functor $(\mathcal{R}, \mu, \mathbf{1})$, a left module functor $\mathcal{M}$ over $\mathcal{R}$ is an object $\mathcal{M}$ in **GPoly** equipped with a natural transformation $\nu : \mathcal{R} \otimes \mathcal{M} \to \mathcal{M}$ that satisfies the following axioms.

Identity the following diagram commutes

$$\begin{tikzcd}
\mathbb{K} \otimes \mathcal{M} \arrow[rr, "\mathbf{1}\otimes \mathrm{Id}_{\mathcal{M}}"] \arrow[dr, "\kappa"'] & & \mathcal{R} \otimes \mathcal{M} \arrow[dl, "\nu"] ; \\
& \mathcal{M} &
\end{tikzcd}$$

Associative the following diagram commutes

$$\begin{tikzcd}
(\mathcal{R} \otimes \mathcal{R}) \otimes \mathcal{M} \arrow[rr, "\cong"] \arrow[d, "\mu\otimes \mathrm{Id}_{\mathcal{M}}"'] & & \mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{M}) . \arrow[d, "\mathrm{Id}_{\mathcal{R}} \otimes \nu"] \\
\mathcal{R} \otimes \mathcal{M} \arrow[r, "\nu"'] & \mathcal{R} & \mathcal{R} \otimes \mathcal{M} \arrow[l, "\nu"]
\end{tikzcd}$$

Notice that for any vector space V and for every algebra functor $\mathcal{R}$, we have that $\mathcal{R}(V)$ is a $\mathbb{K}$-algebra. Similarly, $\mathcal{M}(V)$ is a left module over $\mathcal{S}(V)$. Moreover, the above axioms give us that for every $f \in \mathrm{Hom}(\mathbf{Vec})$, we have that $\mathcal{R}(f)$ is a homomorphism of $\mathbb{K}$-algebras. The symmetric algebra functor $\mathcal{S} = \mathrm{Sym}$ is an example of an algebra functor in **GPoly**. If $\mathcal{M}$ is a left module functor over $\mathcal{S}$, then for every $V \in \mathrm{Obj}(\mathbf{Vec})$ we have that $\mathcal{M}(V)$ is a left module over $\mathcal{S}(V)$.

Example IV.6. Consider the symmetric algebra functor $\mathcal{S}$. For any n–dimensional vector space V, we have the maximal homogeneous ideal $\mathcal{M}(V) = (x_1, \dots, x_n)$ in $\mathcal{S}(V) = \mathbb{K}[x_1, \dots, x_n]$. Thus, in **GPoly** we have a module functor $\mathcal{M}$ over $\mathcal{S}$ defined on $\mathrm{Obj}(\mathbf{Vec})$ by $V \mapsto \mathcal{M}(V)$. Notice that a minimal equivariant resolution for $\mathcal{M}$ is the Koszul resolution:

$$\cdots \to \mathcal{S} \otimes \mathcal{S}_{(1,1,1)} \to \mathcal{S} \otimes \mathcal{S}_{(1,1)} \to \mathcal{S} \otimes \mathcal{S}_{(1)} \to \mathcal{M} \to 0,$$

an infinite resolution. We will discuss resolutions of modules in **GPoly** at the end of this chapter.

4.2.2 Connections to twisted commutative algebras

The constructions in this sections are closely related to the notion of twisted commutative algebras studied by Sam and Snowden ([29–31]). In particular, our definition of the category **GPoly** is closely related to one interpretation of the category $\mathcal{V}$ in [29]. The difference is that we prefer to work with a graded category of polynomial functors and allow for infinite direct sums, rather than considering a category whose objects are representations of GL_∞.

Algebras in **GPoly** satisfy the same axioms as twisted commutative algebras in the category $\mathcal{V}$. For this reason, our algebra functors are twisted commutative algebras if one prefers to consider them as objects in the category $\mathcal{V}$ instead of the category **GPoly**.

Next, we will consider the functor Ω on the category **GPoly**. The functor Ω is the translation of the transpose functor on the category of representations of the symmetric group to the context of representations of the general linear group. Concretely, Ω maps the Schur functor $\mathcal{S}_\lambda$ to the Schur functor $\mathcal{S}_{\lambda'}$.

Towards the end of the chapter we will use Ω to establish a connection between modules over Sym and modules over $\bigwedge$. In [31] the authors had already established this connection and used it to prove regularity results. For example, for a fixed d Sam and Snowden prove that a finitely generated module over the twisted commutative algebra $V \mapsto \mathrm{Sym}(V^d)$ has finite regularity ([31, Corollary 7.8]). Moreover, using their results, one can establish that a finitely generated module over the twisted commutative algebra $V \mapsto \bigwedge(V^d)$ also has finite regularity. Furthermore, Snowden used twisted commutative algebras to give bounds to the minimal resolution of invariant rings of finite groups in [35].

We include in the next sections a self-contained treatment of the subject, for the benefit of the reader. We start by constructing the functor Ω from **GPoly** to itself. One important feature of Ω is that $\Omega(\mathcal{S}) = \bigwedge$. In general, for any homogeneous polynomial functor $\mathcal{F}_d$ of degree d, $\Omega(\mathcal{F}_d)$ will be another homogeneous polynomial functor of degree d. In fact, we will first construct Ω_d, the dth graded piece of Ω, a functor from the category of homogeneous polynomial functors of degree d to itself. The functor Ω can be found in the literature in the context of GL_∞-representations [30, p. 1102]. In that context Ω is called the transpose functor and it is defined for representations of the infinite symmetric group. The transpose functor is then transferred to GL_∞-representations via Schur-Weyl duality. For the convenience of the reader, we present a construction which does not require previous knowledge of the structure theory of GL_∞ representations.

4.3 Definition of Ω_d on the category $\mathbf{Poly}_d$

Let $\mathbf{Poly}_d$ be the full subcategory of **Poly** consisting of homogeneous polynomial functors of degree d. To be able to define Ω_d we will need to go through a multi-step process. The first sections will aim to define a functor $\Omega_{V,d}$ on $\mathbf{Poly}_d$ for any fixed vector space V. Then we will define Ω_d as a direct limit of functors $\Omega_{V,d}$.

We start with a construction from category theory. For a functor $\mathcal{F} : \mathbf{A} \to \mathbf{B}$, we can define the functor $\mathcal{F}^* : \mathbf{Fun}(\mathbf{B}, \mathbf{C}) \to \mathbf{Fun}(\mathbf{A}, \mathbf{C})$ by $\mathcal{F}^*(\mathcal{G}) = \mathcal{G} \circ \mathcal{F}$, for any functor $\mathcal{G} : \mathbf{B} \to \mathbf{C}$. Similarly, for any fixed $\mathcal{G} \in \mathbf{Fun}(\mathbf{B}, \mathbf{C})$, we can define $\mathcal{G}_* : \mathbf{Fun}(\mathbf{A}, \mathbf{B}) \to \mathbf{Fun}(\mathbf{A}, \mathbf{C})$ by $\mathcal{G}_*(\mathcal{F}) = \mathcal{G} \circ \mathcal{F}$, for any functor $\mathcal{F} : \mathbf{A} \to \mathbf{B}$.

We fix a vector space V of dimension n. Let us consider the category $\mathbf{Fun}(\mathbf{Vec}, \mathbf{Rep}_V)$. We define the functor $\mathcal{T}_V : \mathbf{Vec} \to \mathbf{Rep}_V$ as the functor $-\otimes V$ which acts by mapping $W \in \mathrm{Obj}(\mathbf{Vec})$ to $W \otimes V \in \mathrm{Obj}(\mathbf{Rep}_V)$. Notice that $\mathrm{GL}(V)$ acts on $W \otimes V$ by trivial action on W and left multiplication on V. Let us fix a degree d such that $n \geq d$, where we recall that n is the dimension of the fixed vector space V. In the category $\mathbf{Fun}(\mathbf{Rep}_V, \mathbf{Rep}_V)$ we consider the full subcategory $\mathbf{Poly}_{V,d}$ of homogeneous polynomial functors of degree d.

Recall that an object $\mathcal{P}_d$ in $\mathbf{Poly}_d$ induces an object $\mathcal{P}_{V,d}$ in $\mathbf{Poly}_{V,d}$ by Lemma IV.2. Finally, let us consider the functor $\mathcal{H}_{V,d} : \mathbf{Rep}_V \to \mathbf{Vec}$, defined as $\mathrm{Hom}_{\mathrm{GL}(V)}(\bigwedge^d(V), —)$. On objects, we have that $\mathcal{H}_{V,d}$ maps a $\mathrm{GL}(V)$-representation U to its $\bigwedge^d(V)$-isotopic component.

Definition IV.7. For a polynomial functor $\mathcal{P}_d$ of degree c, the functor $\Omega_{V,d}(\mathcal{P}_d)$: $\mathbf{Vec} \to \mathbf{Vec}$ is defined by

$$\Omega_{V,d}(\mathcal{P}_d) = (\mathcal{H}_{V,d})_* \mathcal{T}_V^*(\mathcal{P}_{V,d}) = \mathcal{H}_{V,d} \circ \mathcal{P}_{V,d} \circ \mathcal{T}_V$$

The following commuting diagram illustrates the effect of $\Omega_{V,d}(\mathcal{P}_d)$ on objects in the category $\mathbf{Vec}$.

$$\begin{tikzcd}
W \arrow[r, "\mathcal{T}_V"] \arrow[d, "\Omega_{V,d}(\mathcal{P}_d)"'] & W \otimes V \arrow[d, "\mathcal{P}_{V,d}"] \\
\Omega_{V,d}(\mathcal{P}_d)(W) & \mathcal{P}_{V,d}(W \otimes V) \arrow[l, "\mathcal{H}_{V,d}"]
\end{tikzcd}$$

From our definition of $\Omega_{V,d}(\mathcal{P}_d)$, it is clear that this functor depends on the choice of the polynomial functor $\mathcal{P}_d$ and the choice of a vector space V. Our goal is to be able to define a new functor, $\Omega_d : \mathbf{Poly}_d \to \mathbf{Poly}_d$. To be able to do so, we consider the following lemma.

Lemma IV.8. *The functor $\Omega_{V,d}(\mathcal{P}_d)$ on* $\mathbf{Vec}$ *is a homogeneous polynomial functor of degree d.*

Proof. Recall the assumption that P_d was itself a homogeneous polynomial functor of degree d. We have defined $\Omega_{V,d}(\mathcal{P}_d) = \mathcal{H}_{V,d} \circ \mathcal{P}_{V,d} \circ \mathcal{T}_V$ so that to establish the claim we need to analyze the three functors used here. First, notice that $\mathcal{T}_V$ is a polynomial functor, being in particular a homogeneous linear functor. Moreover, we are given that $\mathcal{P}_d$ is a homogeneous polynomial functor of degree d and the induced functor $\mathcal{P}_{V,d}$ still retains this property. Finally, $\mathcal{H}_{V,d}$ is a homogeneous linear functor being the restriction to the $\mathrm{GL}(V)$-invariant component of the homogeneous linear functor $\mathcal{H}_{V,d} = \mathrm{Hom}(\bigwedge^d V, -)$. Thus, the composition of these three functor is a homogeneous polynomial functor of overall degree d.

$\square$

The above lemma allows us to define for every $\mathcal{P}_d \in \mathbf{Poly}_p$ a new object in $\mathbf{Poly}_d$, namely $\Omega_{V,d}(\mathcal{P}_d)$. Notice that since we defined $\Omega_{V,d}$ as a composition of functors, its effect on morphisms in $\mathbf{Poly}_d$ (which are natural transformations between polynomial functors) is just the composition of the functors $(\mathcal{H}_{V,d})_\star$ and $\mathcal{T}_V^\star$.

4.3.1 The functor $\Omega_{V,d}$ on Schur functors

To understand the effect of the functor $\Omega_{V,d}$ on $\mathbf{Poly}$, we will first study the polynomial functor $\Omega_{V,d}(\mathcal{S}_\lambda)$ in $\mathbf{Poly}$, for $\mathcal{S}_\lambda$ the Schur functor associated to λ, a partition of d.

Lemma IV.9. *Let λ be a partition of n and $d \geq \dim V$. The polynomial functor $\Omega_{V,d}(\mathcal{S}_\lambda)$ is naturally equivalent to $\mathcal{S}_{\lambda'}$.*

Proof. We have already showed that $\Omega_{V,d}(\mathcal{S}_\lambda)$ is a homogeneous polynomial functor of degree d. Notice that the functor $\mathcal{S}_\lambda \mathcal{T}_V = \mathcal{S}_\lambda(- \otimes V)$ can be decomposed using the following formula

$$\mathcal{S}_\lambda(- \otimes V) = \bigoplus (\mathcal{S}_\mu(-) \otimes \mathcal{S}_\nu(V))^{a_{\lambda,\mu,\nu}} = \ldots \oplus \mathcal{S}_{\lambda'}(-) \otimes \bigwedge^d(V) \oplus \ldots,$$

where $a_{\lambda,\mu,\nu}$ is the Kronecker coefficient (the tensor product multiplicity for the corresponding representations of the symmetric group). We notice that in this decomposition the isotypic component of $\bigwedge^d(V)$ is given by $\mathcal{S}_{\lambda'}(-) \otimes \bigwedge^d(V)$ corresponding to the Kronecker coefficient $a_{\lambda,\lambda',(1^d)} = 1$. Consider now the effect of the functor $\mathcal{H}_{V,d}$.

Only the image of the isotypic component of $\bigwedge^d(V)$ will be non-zero. In particular,

$$\mathcal{H}_{V,d}(\mathcal{S}_{\lambda'}(-) \otimes \bigwedge^n(V)) \cong (\bigwedge^n(V)^* \otimes \mathcal{S}_{\lambda'}(-) \otimes \bigwedge^n(V))^{\mathrm{GL}(V)} \cong \mathcal{S}_{\lambda'}(-),$$

where all the isomorphisms are natural equivalences. □

Notice that in the above proof, we studied the image of $\Omega_{V,d}(\mathcal{S}_\lambda)$ by examining the isotypic component of $\bigwedge^d(V)$ in $\mathcal{S}_\lambda(- \otimes V)$. When we apply the functor $\Omega_{V,d}$, we will often use this computational approach to understand its effect on polynomial functors. In particular, one can use this approach to show that $\Omega_{V,d}$ behaves well with respect to direct sums. The proof of the following lemma is left to the reader.

Lemma IV.10. *For polynomial functors $\mathcal{P}_d$ and $\mathcal{P}_d$ the functors $\Omega_{V,d}(\mathcal{P}_d \oplus \mathcal{P}'_d)$ and $\Omega_{V,d}(\mathcal{P}_d) \oplus \Omega_{V,d}(\mathcal{P}'_d)$ are naturally equivalent.*

Moreover, we have that $\Omega_{V,d}$ behaves well with respect to tensor products.

Lemma IV.11. *Let λ and μ be partitions of d and e respectively. We have that $\Omega_{V,d+e}(\mathcal{S}_\mu \otimes \mathcal{S}_\nu)$ is naturally equivalent to $\Omega_{V,d}(\mathcal{S}_\mu) \otimes \Omega_{V,e}(\mathcal{S}_\nu)$ if $\dim V \geq d + e$.*

Proof. Recall that an application of the Littlewood Richardson rule gives that:

$$\begin{aligned} \mathcal{S}_\mu \otimes \mathcal{S}_\nu &= \bigoplus_\lambda \mathcal{S}_\lambda^{c^\lambda_{\mu\nu}} \\ &= \bigoplus_\lambda \mathcal{S}_\lambda^{c^{\lambda'}_{\mu'\nu'}}, \end{aligned}$$

by the properties of the Littlewood Richardson coefficients. Applying $\Omega_{V,d+e}$ to this equations, we get that

$$\Omega_{V,d+e}(\mathcal{S}_\mu \otimes \mathcal{S}_\nu) \cong \bigoplus_\lambda \Omega_{V,d+e}(\mathcal{S}_\lambda)^{c^{\lambda'}_{\mu'\nu'}} \cong \bigoplus_\lambda \mathcal{S}_{\lambda'}^{c^{\lambda'}_{\mu'\nu'}}.$$

On the other hand, we have that

$$\Omega_{V,d}(\mathcal{S}_\mu) \otimes \Omega_{V,e}(\mathcal{S}_\nu) \cong \mathcal{S}_{\mu'} \otimes \mathcal{S}_{\nu'} = \bigoplus_\lambda \mathcal{S}_{\lambda'}^{c^{\lambda'}_{\mu'\nu'}}.$$

As $\Omega_{d+e}(\mathcal{S}_\mu \otimes \mathcal{S}_\nu)$ and $\Omega_d(\mathcal{S}_\mu) \otimes \Omega_e(\mathcal{S}_\nu)$ have the same direct sum decomposition in terms of Schur functors, they are naturally equivalent polynomial functors.

□

4.4 The functor Ω_V on Poly

So far we have seen the effect of $\Omega_{V,d}$ on Schur functors, on direct sums, and on tensor products. Using the graded structure of **Poly**, we can define Ω_V.

Definition IV.12. Let $\mathcal{P}$ be an object in the category **Poly**. We can decompose $\mathcal{P}$ in its graded pieces i.e., $\mathcal{P} = \bigoplus \mathcal{P}_d$, where $\mathcal{P}_d$ is a homogeneous polynomial functor of degree d. We define

$$\Omega_V(\mathcal{P}) = \bigoplus_d \Omega_{V,d}(\mathcal{P}_d)$$

However, for any homogeneous polynomial functor of degree d, we can choose a natural equivalence so that

$$\mathcal{P}_d \cong \bigoplus_{\lambda \dashv d} \mathcal{S}_\lambda^{m_\lambda},$$

for some integers m_λ. Then, using the previous results, we obtain that

$$\Omega_{V,d}(\mathcal{P}_d) \cong \Omega_{V,d}(\bigoplus \mathcal{S}_\lambda^{m_\lambda}) \cong \bigoplus \mathcal{S}_{\lambda'}^{m_\lambda}.$$

In fact, we will actually often just think of the functor $\Omega_{V,d}$ on $\mathbf{Poly}_d$ in terms of its effect on Schur functors. Moreover, we use this point of view to compute the effect of Ω_V on $\mathcal{P} \in \mathrm{Obj}(\mathbf{Poly})$:

$$\Omega_V(\mathcal{P}) = \bigoplus_d \Omega_{V,d}(\mathcal{P}_d) \cong \bigoplus_d \bigoplus_{\lambda \dashv d} \mathcal{S}_{\lambda'}^{m_\lambda}.$$

Finally, we want to show that our definition is independent of the choice of V i.e., if V' is another vector space of dimension $m \geq d$, then for any $\mathcal{P}_d \in \mathbf{Poly}_d$ the functors $\Omega_{V,d}(\mathcal{P}_d)$ and $\Omega_{V',d}(\mathcal{P}_d)$ are naturally equivalent.

Lemma IV.13. *Let V, V' be two vector spaces of dimensions n, m, respectively, such that $n, m \geq d$. For every $\mathcal{P}_d \in \mathbf{Poly}_d$ we have that $\Omega_{V,d}(\mathcal{P}_d)$ and $\Omega_{V',d}(\mathcal{P}_d)$ are naturally equivalent functors.*

Proof. Let $\mathcal{P}_d \in \mathbf{Poly}_d$. Then there exists a natural equivalence $\psi : \mathcal{P}_d \to \bigoplus \mathcal{S}_\lambda^{m_\lambda}$. By our definition of $\Omega_{V,d}, \Omega_{V',d}$ on $\mathrm{Hom}(\mathbf{Poly}_d)$, we have that $\Omega_{V,d}(\psi), \Omega_{V',d}(\psi)$ are natural equivalences in $\mathrm{Hom}(\mathbf{Poly}_d)$. Moreover, recall that $\Omega_{V,d}(\bigoplus \mathcal{S}_\lambda^{m_\lambda}) \cong \bigoplus \mathcal{S}_{\lambda'}^{m_\lambda}$, by our results on the effect of $\Omega_{V,d}$ on Schur functors. Let us call this natural equivalence ϕ. Similarly, there exist a natural equivalence $\phi' : \Omega_{V',d}(\bigoplus \mathcal{S}_\lambda^{m_\lambda}) \to \bigoplus \mathcal{S}_{\lambda'}^{m_\lambda}$. We will define $\eta_{\mathcal{P}_d} : \Omega_{V,d}(\mathcal{P}_d) \to \Omega_{V',d}(\mathcal{P}_d)$ to be

$$\eta_{\mathcal{P}_d} = \Omega_{V',d}(\psi)^{-1} \circ \phi'^{-1} \circ \mathrm{Id} \circ \phi \circ \Omega_{V,d}(\psi),$$

or the top horizontal map in the following commuting diagram:

$$\begin{tikzcd}
\Omega_{V,d}(\mathcal{P}_d) \arrow[rr, "\eta_{\mathcal{P}_d}"] \arrow[d, "\Omega_{V,d}(\psi)"] & & \Omega_{V',d}(\mathcal{P}_d) \\
\Omega_{V,d}(\bigoplus \mathcal{S}_\lambda^{m_\lambda}) \arrow[d, "\phi"] & & \Omega_{V',d}(\bigoplus \mathcal{S}_\lambda^{m_\lambda}) \arrow[u, "\Omega_{V',d}(\psi)^{-1}"'] \\
\bigoplus \mathcal{S}_{\lambda'}^{m_\lambda} \arrow[rr, "\mathrm{Id}"] & & \bigoplus \mathcal{S}_{\lambda'}^{m_\lambda} \arrow[u, "\phi'^{-1}"']
\end{tikzcd} .$$

In conclusion, notice that since $\eta_{\mathcal{P}_d}$ is a composition of natural equivalences in $\mathrm{Hom}(\mathbf{Poly}_d)$, it is itself a natural equivalence in $\mathrm{Hom}(\mathbf{Poly}_d)$. $\square$

As two vector spaces of dimension greater than d yield naturally equivalent functors $\Omega_{V,d}(\mathcal{P}_d)$ and $\Omega_{V',d}(\mathcal{P}_d)$ in $\mathbf{Poly}_d$, for any functor $\mathcal{P}$ in $\mathbf{Poly}$ of degree d (not necessarily homogeneous), we have that choosing vector spaces V, V' of dimension greater than d will yield the naturally equivalent functors $\Omega_V(\mathcal{P})$ and $\Omega_{V'}(\mathcal{P})$.

4.4.1 The functor Ω_V and the tensor structure of Poly

Consider two Schur functors $\mathcal{S}_\mu, \mathcal{S}_\nu$, where μ, ν are partitions of d and e, respectively. Using the results from the previous section, we get that:

$$\Omega_V(\mathcal{S}_\mu \otimes \mathcal{S}_\nu) = \Omega_{V,d+e}(\mathcal{S}_\mu \otimes \mathcal{S}_\nu) \cong \Omega_{V,d}(\mathcal{S}_\mu) \otimes \Omega_{V,e}(\mathcal{S}_\nu) \cong \Omega_V(\mathcal{S}_\mu) \otimes \Omega_V(\mathcal{S}_\nu)$$

where V is a vector space of dimension greater or equal to $d+e$.

However, we have not explicitly produced a natural equivalence between these functor. In particular, we have not studied how the functor Ω_V interacts with the symmetric tensor structure of **Poly**. Recall that for every $\mathcal{P}, \mathcal{P}' \in$ **Poly**, there are natural equivalence $s_{\mathcal{P},\mathcal{P}'}, s_{\mathcal{P}',\mathcal{P}}$, where

$$\mathcal{P} \otimes \mathcal{P}' \xrightarrow{s_{\mathcal{P},\mathcal{P}'}} \mathcal{P}' \otimes \mathcal{P} \xrightarrow{s_{\mathcal{P}',\mathcal{P}}} \mathcal{P} \otimes \mathcal{P}' \quad ,$$

such that $s_{\mathcal{P}',\mathcal{P}} \circ s_{\mathcal{P},\mathcal{P}'} = \mathrm{Id}_{\mathcal{P}\otimes\mathcal{P}'}$. Similarly, there are natural equivalences $s_{\Omega_V(\mathcal{P}),\Omega_V(\mathcal{P}')}$, $s_{\Omega_V(\mathcal{P}'),\Omega_V(\mathcal{P})}$ such that

$$s_{\Omega_V(\mathcal{P}'),\Omega_V(\mathcal{P})} \circ s_{\Omega_V(\mathcal{P}),\Omega_V(\mathcal{P}')} = \mathrm{Id}_{\Omega_V(\mathcal{P})\otimes\Omega_V(\mathcal{P}')} \, .$$

The question arising from this set up is whether we can produce a natural equivalence $\psi_{\mathcal{P},\mathcal{P}'} : \Omega_V(\mathcal{P}) \otimes \Omega_V(\mathcal{P}') \to \Omega_V(\mathcal{P} \otimes \mathcal{P}')$ compatible with the tensor structure. In practice, we want to determine if the following diagram commutes:

$$\begin{tikzcd}
\Omega_V(\mathcal{P} \otimes \mathcal{P}') \arrow[r, "\Omega_V(s_{\mathcal{P},\mathcal{P}'})"] & \Omega_V(\mathcal{P}' \otimes \mathcal{P}) \\
\Omega_V(\mathcal{P}) \otimes \Omega_V(\mathcal{P}') \arrow[u, "\psi_{\mathcal{P},\mathcal{P}'}"] \arrow[r, "s_{\Omega_V(\mathcal{P}),\Omega_V(\mathcal{P}')}"] & \Omega_V(\mathcal{P}') \otimes \Omega_V(\mathcal{P}) \arrow[u, "\psi_{\mathcal{P}',\mathcal{P}}"']
\end{tikzcd} .$$

As every polynomial functor is a direct sum of Schur functors, it will be enough to study this diagram for $\mathcal{P} = \mathcal{S}_\lambda$ and $\mathcal{P}' = \mathcal{S}_\mu$, for λ a partition of d and μ a partition of e.

Proposition IV.14. *We can define $\psi_{\mathcal{S}_\lambda,\mathcal{S}_\mu}$ so that the above diagram commutes up to sign $(-1)^{de}$.*

Proof. First we will define $\psi_{\mathcal{S}_\lambda,\mathcal{S}_\mu}$ and from our definition we will conclude that the diagram only commutes up to a sign.

As remarked before, $\Omega_{V,d}(\mathcal{S}_\lambda)(W)$ is the multiplicity space of the isotypic component of the irreducible $\mathrm{GL}(V)$-representation $\bigwedge^d(V)$ inside $\mathcal{S}_\lambda(W \otimes V)$. Thus, to find a natural equivalence from $\Omega_{V,d}(\mathcal{S}_\lambda) \otimes \Omega_{V,e}(\mathcal{S}_\mu)$ to $\Omega_{V,d+e}(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu)$ we need to exhibit for any finite dimensional vector space W an isomorphism from the tensor product of the isotypic component of $\bigwedge^d(V)$ in $\Omega_{V,d}(\mathcal{S}_\lambda)(W)$ and the isotypic component of $\bigwedge^e(V)$ in $\Omega_{V,e}(\mathcal{S}_\mu)(W)$ to the isotypic component of $\bigwedge^{d+e}(V)$ in $\Omega_V(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu)$.

Going back to the definition of $\Omega_{V,d}(\mathcal{P}_d)(W)$, we notice that this is equivalent to producing an isomorphism $\phi = \psi_{\mathcal{S}_\lambda,\mathcal{S}_\mu}(W)$:

$$\begin{array}{c}
\mathrm{Hom}_{\mathrm{GL}(V)}(\bigwedge^d(V), \mathcal{S}_\lambda(W \otimes V)) \otimes \mathrm{Hom}_{\mathrm{GL}(V)}(\bigwedge^e(V), \mathcal{S}_\mu(W \otimes V)) \\
\Big\downarrow{\scriptstyle \phi} \\
\mathrm{Hom}_{\mathrm{GL}(V)}(\bigwedge^{d+e}(V), \mathcal{S}_\lambda(W \otimes V) \otimes \mathcal{S}_\mu(W \otimes V))
\end{array}$$

for every W.

Note that $\bigwedge^d(V) \otimes \bigwedge^e(V)$ has a unique sub-representation isomorphic to $\bigwedge^{d+e}(V)$. Now we define ϕ as follows. If $f : \bigwedge^d(V) \to \mathcal{S}_\lambda(W \otimes V)$ and $g : \bigwedge^e(V) \to \mathcal{S}_\mu(W \otimes V)$ are $\mathrm{GL}(V)$-equivariant linear maps, then we define $\phi(f \otimes g)$ as the restriction of $f \otimes g : \bigwedge^d(V) \otimes \bigwedge^e(V) \to \mathcal{S}_\lambda(W \otimes V) \otimes \mathcal{S}_\mu(W \otimes V)$ to the sub-representation $\bigwedge^{d+e}(V) \subseteq \bigwedge^d V \otimes \bigwedge^e V$. We can extend ϕ to a linear map.

By Schur's lemma any $\mathrm{GL}(V)$-equivariant map in $f : \bigwedge^d(V) \to \mathcal{S}_\lambda(W \otimes V)$ can be written as

$$f = \mathrm{Id} \otimes a : \bigwedge^d(V) \to \bigwedge^d(V) \otimes \mathcal{S}_{\lambda'}(W) \subseteq \mathcal{S}_\lambda(W \otimes V),$$

for a some constant map from $\bigwedge^d(V)$ to $\mathcal{S}_{\lambda'}(W)$, as $\mathcal{S}_{\lambda'}(W)$ is the multiplicity space of the isotypic component of $\bigwedge^d(V)$ in $\mathcal{S}_\lambda(W \otimes V)$. Similarly, letting b be a constant map to $\mathcal{S}_{\mu'}(W)$, we will have that $g \in \mathrm{Hom}_{\mathrm{GL}(V)}(\bigwedge^e(V), \mathcal{S}_\mu(W \otimes V))$ can be written as $g = \mathrm{Id} \otimes b$. As the multiplicity space of $\bigwedge^{d+e}(V)$ in $\mathcal{S}_\lambda(W \otimes V) \otimes \mathcal{S}_\mu(W \otimes V)$ is precisely $\mathcal{S}_{\lambda'}(W) \otimes \mathcal{S}_{\mu'}(W)$, we have that ϕ sends $\Sigma_i \mathrm{Id} \otimes a_i \otimes \mathrm{Id} \otimes b_i$ to $\Sigma_i \mathrm{Id} \otimes a_i \otimes b_i$. As this map is an injective map between isomorphic spaces, it is an isomorphism.

In the diagram below:

$$\begin{tikzcd}
\bigwedge^d(V) \otimes \bigwedge^e(V) \arrow[d, "s_{\wedge^d(V), \wedge^e(V)}"] & & \bigwedge^{d+e}(V) \arrow[ll, hook'] \arrow[dll, hook'] \\
\bigwedge^e(V) \otimes \bigwedge^d(V) & &
\end{tikzcd}.$$

the map $s_{\wedge^d(V), \wedge^e(V)}$ takes the pure tensor $a \otimes b$ to $(-1)^{de} b \otimes a$. Thus, the diagram only commutes up to sign $(-1)^{de}$.

So applying $\mathrm{Hom}(—, \mathcal{S}_\lambda \otimes \mathcal{S}_\mu(W \otimes V))$ to the diagram above, we obtain a new diagram that only commutes up to sign $(-1)^{de}$.

$$\begin{tikzcd}
\mathrm{Hom}(\bigwedge^d(V) \otimes \bigwedge^e(V), \mathcal{S}_\lambda \otimes \mathcal{S}_\mu(W \otimes V)) \arrow[r] & \mathrm{Hom}(\bigwedge^{d+e}(V), \mathcal{S}_\lambda \otimes \mathcal{S}_\mu)(W) \\
\mathrm{Hom}(\bigwedge^e(V) \otimes \bigwedge^d(V), \mathcal{S}_\lambda \otimes \mathcal{S}_\mu(W \otimes V)) \arrow[u] \arrow[ur] &
\end{tikzcd}.$$

Notice that restricting the spaces above to the relevant invariant subspaces does not affect the sign in the above diagram.

The map

$$\Omega_V(s_{\mathcal{S}_\mu, \mathcal{S}_\lambda})(W) : \Omega_V(\mathcal{S}_\mu \otimes \mathcal{S}_\lambda)(W) \to \Omega_V(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu)(W)$$

is just given by $\Sigma_i \mathrm{Id} \otimes b_i \otimes a_i \mapsto \Sigma_i \mathrm{Id} \otimes a_i \otimes b_i$, so that on pure tensors we have that

$\mathrm{Id} \otimes b \otimes a \mapsto \mathrm{Id} \otimes a \otimes b$. Finally, consider the diagram below

$$\begin{tikzcd} \Omega_V(\mathcal{S}_\lambda)(W) \otimes \Omega_V(\mathcal{S}_\mu)(W) \arrow[r, "\phi"] & \Omega_V(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu)(W) \\ \Omega_V(\mathcal{S}_\mu)(W) \otimes \Omega_V(\mathcal{S}_\lambda)(W) \arrow[r, "\phi"] \arrow[u, "s_{\Omega_V(\mathcal{S}_\mu)(W),\Omega_V(\mathcal{S}_\lambda)(W)}"] & \Omega_V(\mathcal{S}_\mu \otimes \mathcal{S}_\lambda)(W) \arrow[u, "\Omega_V(s_{\mathcal{S}_\lambda,\mathcal{S}_\mu})(W)"'] \end{tikzcd} .$$

We can conclude that the diagram only commutes up to sign $(-1)^{de}$ as first going right and then up sends $\mathrm{Id} \otimes b \otimes \mathrm{Id} \otimes a \mapsto \mathrm{Id} \otimes a \otimes b$, whilst first going up then right send $\mathrm{Id} \otimes b \otimes \mathrm{Id} \otimes a \mapsto (-1)^{de} \mathrm{Id} \otimes a \otimes b$

□

4.5 Definition of $\Omega(\mathcal{P})$

After examining the effect of $\Omega_{V,d}$ on the tensor structure of **Poly**, we want to be able to work with a functor defined independently from the choice of the vector space V. For every $i \geq 0$, define $V_i \cong \mathbb{K}^i$ as the vector space of all sequences $(a_1, a_2, a_3, \dots) \in \mathbb{K}^\infty$ with $a_j = 0$ for all $j > i$. Let ρ_{ji} be the inclusion $\rho_{ji} : V_i \to V_j$.

Suppose that $i \leq j$. Consider $\Omega_{d,V_j}(\mathcal{P}_d)$, by definition we have that $\Omega_{d,V_j}(\mathcal{P}_d) \in \mathrm{Hom}_{\mathrm{GL}(V_j)}(\bigwedge^d V_j, \mathcal{P}_d(V_j \otimes -))$. The restriction of $\Omega_{d,V_j}(\mathcal{P}_d)$ to its subspace $\bigwedge^d V_i$ is $\mathrm{GL}(V_i)$-equivariant and the image is contained in $\mathcal{P}_d(V_i \otimes -)$. So we have a natural transformation $g_{i,j} : \Omega_{d,V_j}(\mathcal{P}_d) \to \Omega_{d,V_i}(\mathcal{P}_d)$. The kernel of this natural transformation consists exactly of all isotypic components $\mathcal{S}_{\lambda'}$ where λ has more than i parts. There is a unique splitting $f_{j,i} : \Omega_{d,V_i}(\mathcal{P}_d) \to \Omega_{d,V_j}(\mathcal{P}_d)$ such that $g_{i,j} \circ f_{j,i}$ is the identity. We have a direct system

$$\Omega_{d,V_0} \xrightarrow{f_0} \Omega_{d,V_1} \xrightarrow{f_1} \Omega_{d,V_2} \xrightarrow{f_2} \cdots$$

where $f_i = f_{i+1,i}$.

Definition IV.15. We define $\Omega_d(\mathcal{P}_d)$ to be the following direct limit of the maps

$f_{j,i}$:

$$\Omega_d(\mathcal{P}_d) = \varinjlim \Omega_{d,V_i}(\mathcal{P}_d).$$

Notice that the abelian category of polynomial functor is cocomplete so this functor is well-defined as direct limits exist. Finally, recall our definition of Ω_V for $\mathcal{P} \in$ **Poly** decomposed as $\mathcal{P} = \bigoplus_d \mathcal{P}_d$:

$$\Omega_V(\mathcal{P}) = \bigoplus_d \Omega_{V,d}(\mathcal{P}_d).$$

Similarly, we have the following definition for Ω.

Definition IV.16. Let $\mathcal{P} \in$ **Poly** be decomposed as $\mathcal{P} = \bigoplus_d \mathcal{P}_d$. We define $\Omega(\mathcal{P})$ to be:

$$\Omega(\mathcal{P}) = \bigoplus_d \Omega_d(\mathcal{P}_d).$$

We can notice that our definition of Ω_V is compatible with our definition of Ω:

$$\Omega(\mathcal{P}) = \bigoplus_d \Omega_d(\mathcal{P}_d) = \bigoplus_d \varinjlim \Omega_{d,V_i}(\mathcal{P}_d) = \varinjlim \bigoplus_d \Omega_{d,V_i}(\mathcal{P}_d) = \varinjlim \Omega_{V_i}(\mathcal{P}),$$

as direct sums and direct limits commute.

4.6 The functor Ω on GPoly

Recall that the category **GPoly** is a functor category where each functor $\mathcal{F} \in$ **GPoly** can be decomposed as

$$\mathcal{F} = \bigoplus \mathcal{F}_d$$

with $\mathcal{F}_d$ a homogeneous polynomial functor of degree d. Moreover, recall that each polynomial functor is naturally equivalent to a direct sum of $\mathcal{S}_\lambda$'s. Thus we can also notice that

$$\mathcal{F} \cong \bigoplus \mathcal{S}_\lambda^{m_\lambda},$$

for λ's of any length, but with the stipulation that for each polynomial functor $\mathcal{F}_d$ and any $W \in \mathrm{Obj}(\mathbf{Vec})$, we have that $\mathcal{F}_d(W) \cong \bigoplus_{\lambda \dashv d} \mathcal{S}_\lambda^{m_\lambda}(W)$ is a finite dimensional vector space.

In the previous section we have seen how to define Ω on any polynomial functor. We can extend this definition to any direct sum of polynomial functors, even though the direct sum itself may not be a polynomial functor. Thus, we can define the functor Ω on the category **GPoly**. In particular, we have that $\mathcal{S}$, the symmetric algebra functor, is not a polynomial functor because for any $W \in \mathrm{Obj}(\mathbf{Vec})$ we have that $\mathcal{S}(W)$, the symmetric algebra on W, is an infinite dimensional vector space. However, $\mathcal{S}$ is a direct sum of polynomial functors as each graded piece $\mathcal{S}^d(W)$ is a finite dimensional vector space. Thus $\mathcal{S}$ is an object in **GPoly**.

Definition IV.17. For any functor $\mathcal{F} \in \mathrm{Obj}(\mathbf{GPoly})$, where $\mathcal{F} = \bigoplus \mathcal{F}_d$, we let

$$\Omega(\mathcal{F}) = \bigoplus \Omega_d(\mathcal{F}_d).$$

In particular, for $\mathcal{S} = \bigoplus \mathcal{S}_(d)$, we have that

$$\Omega(\mathcal{S}) = \bigoplus \Omega_d(\mathcal{S}_(d)) \cong \bigoplus \bigwedge^d = \bigwedge$$

from our results on the effect of Ω on Schur functors.

4.7 Equivariant resolutions

Let us fix a vector space U of dimension n. The ring of polynomial functions on U can be identified with the symmetric algebra $S(U^*)$. Let $R = S(U^*)$ and let $\mathfrak{m}$ be the homogeneous maximal ideal in R. Given a module M over R we can construct a minimal resolution by defining $D_0 := M$ and $E_0 := D_0/\mathfrak{m}D_0$. We can then extend in a unique way the homogeneous section $\phi_0 : E_0 \to D_0$ of the homogeneous quotient map $\pi_0 : D_0 \to E_0$ to a R-module homomorphism $\phi_0 : R \otimes E_0 \to D_0$. The tensor product $R \otimes E_0$ is naturally graded as a tensor product of graded vector spaces and ϕ_0 is homogeneous with respect to this grading. Letting D_1 be the kernel of ϕ_0, we see how to proceed inductively to construct a free resolution of M. The resolution is finite by Hilbert's syzygy theorem which states that $D_i = 0$ for $i > n$. In the resulting minimal free resolution:

$$0 \to R \otimes E_t \to \cdots \to R \otimes E_0 \to M \to 0,$$

we can naturally identify E_i with $\mathrm{Tor}_i(M, \mathbb{K})$.

Moreover, suppose that U is a representation of a linearly reductive algebraic group G. If G also acts on the module M and the multiplication map $m : R \times M \to M$ is G-equivariant, then each graded piece M_d of $M = \bigoplus M_d$ is a G-module. By linear reductivity, we can choose the maps ϕ_i to be G-equivariant giving each E_i the structure of a graded G-module. In particular, we can choose a decomposition of each E_i into irreducible G-representations. In the case of $G = \mathrm{GL}(V)$ and M a polynomial representation of G, we will have a decomposition of each E_i in the equivariant resolution of M in terms of the Schur functors $\mathcal{S}_\lambda$'s. As a result we have the following equivariant resolution of the G-module M:

$$0 \to R \otimes \bigoplus \mathcal{S}_\lambda(V)^{m_\lambda^n} \to \cdots \to R \otimes \bigoplus \mathcal{S}_\lambda(V)^{m_\lambda^0} \to M \to 0.$$

In particular, we have that $M/\mathfrak{m}M = \mathrm{Tor}_0(M, \mathbb{K})$ can be decomposed in irreducible $\mathrm{GL}(V)$-representations as $\bigoplus \mathcal{S}_\lambda(V)^{m_\lambda^0}$.

4.7.1 Resolutions in GPoly and $\boldsymbol{\Omega}$

Consider a module functor $\mathcal{M} \in \mathrm{Obj}(\mathbf{GPoly})$ over the algebra functor $\mathcal{R} \in \mathrm{Obj}(\mathbf{GPoly})$. A resolution for $\mathcal{M}$ is constructed analogously to a resolution for a $\mathrm{GL}(V)$-module. In particular, the construction relies on the fact that **GPoly** is a semisimple category as each object is naturally equivalent to a direct sum of simple objects: the irreducible polynomial functors $\mathcal{S}_\lambda$'s.

The object $\mathcal{M}$ in **GPoly** is equipped with a natural equivalence

$$\mathcal{M} \cong \bigoplus \mathcal{S}_\lambda \otimes A_\lambda,$$

where A_λ is the multiplicity space of $\mathcal{S}_\lambda$, a vector space recording the multiplicity of the polynomial functor $\mathcal{S}_\lambda$.

Let $\mathcal{N}$ be another $\mathcal{R}$-module functor, where $\mathcal{N} \cong \bigoplus \mathcal{S}_\lambda \otimes B_\lambda$. We have that the map $\psi : \mathcal{M} \to \mathcal{N}$ can be viewed as a map

$$\bigoplus \mathcal{S}_\lambda \otimes A_\lambda \to \bigoplus \mathcal{S}_\lambda \otimes B_\lambda$$

and being a homomorphism in **GPoly**, we have that $\psi = \bigoplus Id \otimes \psi_\lambda$, where each $\psi_\lambda : A_\lambda \to B_\lambda$ is just a linear map. Thus each ψ_λ has a section $\phi_\lambda : B_\lambda \to A_\lambda$ allowing us to construct $\phi := \bigoplus Id \otimes \phi_\lambda : \mathcal{N} \to \mathcal{M}$, a section of ψ.

As **GPoly** is an abelian category for each $\phi \in \mathrm{Hom}(\mathbf{GPoly})$, there exists an object $\mathcal{K} \in \mathrm{Obj}(\mathbf{GPoly})$ which is the kernel of $\phi : \mathcal{N} \to \mathcal{M}$.

To construct the minimal resolution of $\mathcal{M}$, we will consider the map $\psi_0 : (\mathcal{M}/\mathfrak{m}\mathcal{M}) \otimes \mathcal{R} \to \mathcal{M}$, where $\mathfrak{m}$ is the positively graded part of $\mathcal{R} \in \mathbf{GPoly}$, the maximal homogeneous module functor of $\mathcal{R}$. We will define $\mathcal{D}_0 := \mathcal{M}$ and $\mathcal{E}_0 := \mathcal{M}/\mathfrak{m}\mathcal{M}$. Then $\mathcal{D}_1 := \ker(\phi_0)$, where ϕ_0 is the $\mathcal{R}$-module functor morphism arising from a section of ψ_0 as discussed above. Inductively, we define $\mathcal{D}_i = \ker \phi_{i-1}$ and $\mathcal{E}_i = \mathcal{D}_i/\mathfrak{m}\mathcal{D}_i$. Then we will let $\phi_i : \mathcal{E}_i \to \mathcal{D}_i$ be the section of the quotient map $\psi_i : \mathcal{D}_i \to \mathcal{E}_i$ and we will extend ϕ_i to a $\mathcal{R}$-module functors map $\phi_i : \mathcal{R} \otimes \mathcal{E}_i \to \mathcal{D}_i$. As a result we obtain the following minimal resolution for $\mathcal{M}$:

$$\cdots \to \mathcal{R} \otimes \bigoplus \mathcal{S}_\lambda^{m_\lambda^i} \to \cdots \to \mathcal{R} \otimes \bigoplus \mathcal{S}_\lambda^{m_\lambda^0} \to \mathcal{M} \to 0,$$

where $\mathcal{E}_i \cong \bigoplus \mathcal{S}_\lambda^{m^i_\lambda}$.

Notice that for $W \in \mathrm{Obj}(\mathbf{Vec})$ and $\mathcal{R} = \mathcal{S}(W)$, a minimal resolution for $\mathcal{M}(W)$ will terminate as by Hilbert's syzygy theorem $\mathcal{D}_i = 0$ when $i > \dim(W)$. However, the categorical construction of the resolution of $\mathcal{M}$ may be infinite.

Given a minimal resolution of $\mathcal{M}$ constructed as above, we can apply Ω to the resolution to obtain

$$\cdots \to \Omega(\mathcal{R}) \otimes \bigoplus \mathcal{S}_{\lambda'}^{m^i_\lambda} \to \cdots \to \Omega(\mathcal{R}) \otimes \bigoplus \mathcal{S}_{\lambda'}^{m^0_\lambda} \to \Omega(\mathcal{M}) \to 0.$$

In particular, notice that if $\mathcal{R} = \mathcal{S}$, the symmetric algebra functor, then applying Ω to a resolution of the $\mathcal{S}$-module functor $\mathcal{M}$ will result in a resolution of $\Omega(\mathcal{M})$, a module over the algebra functor $\Omega(\mathcal{R}) \cong \bigwedge$.

4.8 Castelnuovo-Mumford regularity of modules in GPoly

For a finite dimensional graded $\mathbb{K}$-vector space $E = \bigoplus E_d$, we define

$$\deg(E) := \max\{d : E_d \neq 0\}.$$

If $E = \{0\}$, then we define $\deg(E) = -\infty$. For $\mathcal{M}, \mathcal{R}$ in $\mathrm{Obj}(\mathbf{GPoly})$, let $\mathcal{M}$ be a module functor over the algebra functor $\mathcal{R}$. For every V in $\mathrm{Obj}(\mathbf{Vec})$ we have that $\mathcal{M}(V)$ is a module, in the usual sense, over the $\mathbb{K}$-algebra $\mathcal{R}(V)$. We have that $\mathcal{M}(V)$ is s-regular if $\deg(\mathrm{Tor}_i(M(V), \mathbb{K})) \leq s + i$, for all i. In particular, notice that using the minimal resolution constructed above, we have that $\mathrm{Tor}_i(\mathcal{M}(V), \mathbb{K})) = \mathcal{E}_i(V)$. Thus, we have that $\mathcal{M}(V)$ is s-regular if

$$\deg(\mathcal{E}_i(V)) \leq s + i,$$

for all i.

The *Catelnuovo-Mumford regularity* $\operatorname{reg}(\mathcal{M}(V))$ of $\mathcal{M}(V)$ is the smallest integer s such that $\mathcal{M}(V)$ is s-regular. We define the *regularity* of $\mathcal{M} \in \operatorname{Obj}(\mathbf{GPoly})$ to be

$$\lim_{\dim(V)\to\infty} \operatorname{reg}(\mathcal{M}(V))$$

if the limit exists.

Proposition IV.18. *Let $\mathcal{M}$ be a module over $\mathcal{R}$ in* **GPoly** *with regularity d. Then $\Omega(\mathcal{M})$ is a module over $\Omega(\mathcal{R})$ with regularity d.*

Proof. As the module $\mathcal{M}$ in **GPoly** comes equipped with a multiplication map $\nu : \mathcal{R} \otimes \mathcal{M} \to \mathcal{M}$, we have that $\Omega(\nu) : \Omega(\mathcal{R}) \otimes \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$ equips $\Omega(\mathcal{M})$ with the structure of a $\Omega(\mathcal{R})$-module. Consider a minimal resolution for $\mathcal{M}$ as constructed above:

$$\cdots \to \mathcal{R} \otimes \mathcal{E}_i \to \cdots \to \mathcal{R} \otimes \mathcal{E}_0 \to \mathcal{M} \to 0.$$

We can apply Ω to the resolution to obtain

$$\cdots \to \Omega(\mathcal{R}) \otimes \Omega(\mathcal{E}_i) \to \cdots \to \Omega(\mathcal{R}) \otimes \Omega(\mathcal{E}_0) \to \Omega(\mathcal{M}) \to 0.$$

Notice that for every vector space V, we have that $\deg(\Omega(\mathcal{E}_i)(V)) \leq \deg(\mathcal{E}_i(V))$. In fact, it is possible that $\mathcal{E}_i(V) \neq 0$ but $\Omega(E_i)(V) = 0$ for $\dim V \leq i$. However, for $\dim(V)$ large enough, we have that $\deg(\Omega(\mathcal{E}_i)(V)) = \deg \mathcal{E}_i(V)$. Hence for $\dim(V)$ large enough, we also have that $\Omega(\mathcal{E}_i)(V) = \operatorname{Tor}_i(\Omega(\mathcal{M})(V), \mathbb{K})$. We have that $\mathcal{M}(V)$ is s-regular if $\max_i\{\deg(\mathcal{E}_i(V)) - i\} \leq s$. Thus, for $\dim(V)$ large enough, we have that

$$\max_i\{\deg(\Omega(\mathcal{E}_i)(V)) - i\} = \max_i\{\deg(\mathcal{E}_i(V)) - i\} \leq s,$$

so $\Omega(\mathcal{M})(V)$ is s-regular whenever $\mathcal{M}(V)$ is s-regular.

Therefore,

$$d = \operatorname{reg}(\mathcal{M}) = \lim_{\dim(V)\to\infty} \operatorname{reg}(\mathcal{M}(V)) = \lim_{\dim(V)\to\infty} \operatorname{reg}(\Omega(\mathcal{M})(V)) = \operatorname{reg}\Omega(\mathcal{M}).$$

□

CHAPTER V

Resolutions of ideals associated to subspace arrangements

5.1 The module functors of a subspace arrangement

For Y a $\mathbb{K}$-vector space, a subspace arrangement $\mathcal{A} = \{Y_1, \ldots, Y_t\}$ is a collection of linear subspaces in Y. The ideal associated to $\mathcal{A}$ is the vanishing ideal of the subspace arrangement: $I_{\mathcal{A}} = \mathbb{I}(\mathcal{A}) = \mathbb{I}(Y_1 \cup \cdots \cup Y_t)$. Moreover, we can define $J_{\mathcal{A}} = \prod_i \mathbb{I}(Y_i) = \mathbb{I}(Y_1)\mathbb{I}(Y_2)\cdots\mathbb{I}(Y_t)$

Let $W = Y^*$. Then we have that $I_{\mathcal{A}}, J_{\mathcal{A}}$ are ideal in $\mathrm{Sym}(Y^*) = \mathcal{S}(W)$.

Definition V.1. Let V be any object in **Vec** and let $Z = V^*$. In the polynomial ring $\mathcal{S}(W \otimes V)$ we define $\mathcal{I}_{\mathcal{A}}(V)$ to be the vanishing ideal of the subspace arrangement $\mathcal{A} \otimes Z$, i.e.,

$$\mathcal{I}_{\mathcal{A}}(V) = \mathbb{I}(Y_1 \otimes Z \cup \cdots \cup Y_t \otimes Z).$$

Moreover, we define $J_{\mathcal{A}}$ to be the product ideal

$$\mathcal{J}_{\mathcal{A}}(V) = \mathbb{I}(Y_1 \otimes Z)\mathbb{I}(Y_2 \otimes Z)\cdots\mathbb{I}(Y_t \otimes Z).$$

For any subspace arrangement $\mathcal{A}$, we can now construct the module functors $\mathcal{I}_{\mathcal{A}}, \mathcal{J}_{\mathcal{A}}$ in **GPoly** for the algebra functor $\mathcal{S}(W \otimes -)$. For any object V in **Vec**, we have already defined $\mathcal{I}_{\mathcal{A}}(V)$ and $\mathcal{J}_{\mathcal{A}}(V)$. Notice that for every vector space V, we have that $\mathcal{I}_{\mathcal{A}}(V), \mathcal{J}_{\mathcal{A}}(V)$ are homogeneous ideals in $\mathcal{S}(W \otimes V)$ so that can define monomorphisms $\mathcal{I}_{\mathcal{A}}, \mathcal{J}_{\mathcal{A}} \hookrightarrow \mathcal{S}(W \otimes -)$. Thus for every d, we have that $(\mathcal{I}_{\mathcal{A}})_d, (\mathcal{J}_{\mathcal{A}})_d$ are polynomial functors of degree d giving $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{A}}$ the structure of objects in **GPoly**.

To show that $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{A}}$ are module functors in **GPoly**, as they inherit a multiplication map from $\mathcal{S}(W \otimes -)$, we only need to define $\mathcal{I}_{\mathcal{A}}, \mathcal{J}_{\mathcal{A}}$ on $\mathrm{Hom}(\mathbf{Vec})$. For ease of notation, will proceed with the definition for $\mathcal{I}_{\mathcal{A}}$, but the same construction works for $\mathcal{J}_{\mathcal{A}}$. In fact, the reader may substitute $\mathcal{J}_{\mathcal{A}}$ for $\mathcal{I}_{\mathcal{A}}$ in the following paragraphs without affecting the results. Let $f : V_1 \to V_2$ be in $\mathrm{Hom}(\mathbf{Vec})$. Then also $Id \otimes f : W \otimes V_1 \to W \otimes V_2$ is in $\mathrm{Hom}(\mathbf{Vec})$, As $\mathcal{S}(W \otimes -)$ is an object in **GPoly**, we have $\mathcal{S}(\mathrm{Id} \otimes f) : \mathcal{S}(W \otimes V1) \to \mathcal{S}(W \otimes V_2)$ in $\mathrm{Hom}(\mathbf{GVec})$.

Proposition V.2. *For any $f : V_1 \to V_2$ in $\mathrm{Hom}(\mathbf{Vec})$, we have that $\mathcal{I}_{\mathcal{A}}(f)$ defined as $\mathcal{S}(\mathrm{Id} \otimes f)|_{\mathcal{I}_{\mathcal{A}}(V_1)}$ is an element in $\mathrm{Hom}(\mathbf{GVec})$ such that*

$$\mathcal{S}(\mathrm{Id} \otimes f)|_{\mathcal{I}_{\mathcal{A}}(V_1)} : \mathcal{I}_{\mathcal{A}}(V_1) \to \mathcal{I}_{\mathcal{A}}(V_2).$$

Moreover, this shows that $\mathcal{I}_{\mathcal{A}}$ is a module functor in **GPoly**.

Proof. Notice that since $\mathcal{I}_{\mathcal{A}}(V_1)$ is an ideal in $\mathcal{S}(W \otimes V_1)$, we can restrict $\mathcal{S}(\mathrm{Id} \otimes f)$ to $\mathcal{I}_{\mathcal{A}}(V_1)$. Moreover, we have that in degree one $\mathcal{S}(\mathrm{Id} \otimes f)_1 = 1d \otimes f$ and that $\mathcal{I}_{\mathcal{A}}(V)$ is a linear ideal for any vector space V meaning that $\mathcal{I}_{\mathcal{A}}(V)$ is generated in degree one. Thus, we only need to show that the linear generators of $\mathcal{I}_{\mathcal{A}}(V_1)$ get mapped by $\mathcal{S}(\mathrm{Id} \otimes f)_1 = \mathrm{Id} \otimes f$ to the linear generators of $\mathcal{I}_{\mathcal{A}}(V_2)$ and that $\mathcal{S}(\mathrm{Id} \otimes f)$ is an algebra homomorphism, so that it maps ideals to ideals.

In general, for any $g \in \mathrm{Hom}(\mathbf{Vec})$ by our definition of an algebra functor $\mathcal{R}$ in **GPoly** we have that $\mathcal{R}(g)$ is an algebra homomorphism. In particular, $\mathcal{S}(\mathrm{Id} \otimes f)$ is algebra homomorphism.

With regards to the generators of $\mathcal{I}_{\mathcal{A}}(V_1)$, let $w \in I_{\mathcal{A}}$, so that $w = 0$ on $\mathcal{A}$. Then for any $v \in V$ we have that $w \otimes v$ is a linear generator of $\mathcal{I}_{\mathcal{A}}(V)$. Moreover, if $w \otimes v_1 \in \mathcal{I}_{\mathcal{A}}(V_1)$ then $\mathcal{I}_{\mathcal{A}}(f)(w \otimes v_1) = w \otimes f(v_1)$ is a linear generator of $\mathcal{I}_{\mathcal{A}}(V_2)$ as $v_2 = f(v_1) \in V_2$ and $w \in I_{\mathcal{A}}$.

Finally, as $\mathcal{I}_{\mathcal{A}}(f)$ is the restriction of the functorial map $\mathcal{S}(\mathrm{Id} \otimes f)$, we have that $\mathcal{I}_{\mathcal{A}}(f)$ is itself functorial.

□

Using the construction above, we can establish our main result.

Theorem V.3. *For any subspace arrangement $\mathcal{A}$ of size t consider the module functor $\mathcal{I}_{\mathcal{A}}$ in* **GPoly**. *For any vector space V, we have that $\Omega(\mathcal{I}_{\mathcal{A}})(V)$ is a t-regular* $\mathrm{GL}(V)$*-equivariant ideal in $\Omega(\mathcal{S}(W \otimes V)) = \bigwedge(W \otimes V)$.*

Proof. Derksen and Sidman proved in [8] that the intersection of t linear ideals is t-regular and this implies that $\mathcal{I}_{\mathcal{A}}(V)$ is t-regular. We have previously seen that $\Omega(\mathcal{S}(W \otimes V)) = \bigwedge(W \otimes V)$ and that the image under Ω of a module functor is a module functor. Furthermore, Ω is an exact functor on **GPoly** so that monomorphisms are sent under Ω to monomorphisms. Thus $\Omega(\mathcal{I}_{\mathcal{A}})(V)$ is an ideal in $\bigwedge(W \otimes V)$. Finally, we have already established that Ω preserves the regularity of a module functor. Therefore, for every vector space V, we have that $\Omega(\mathcal{I}_{\mathcal{A}})(V)$ is t-regular. □

Conca and Herzog showed in [4] that the product of t linear ideals is t-regular and this implies that $\mathcal{J}_{\mathcal{A}}(V)$ is also t-regular for al V. Therefore, the same result holds for $\Omega(\mathcal{J}_{\mathcal{A}})(V)$. Moreover, Dersken and Sidman in [9] produce regularity bounds for a more general class of ideals constructed from linear ideals in the symmetric algebra. One can adapt Theorem V.3 to establish that the same class of ideals in the exterior algebra has the same regularity bounds.

Consider the module functor $\mathcal{J}_{\mathcal{A}}$ for the product ideal $J_{\mathcal{A}}$. We can characterize the ideal $\Omega(\mathcal{J}_{\mathcal{A}})(V)$ in the exterior algebra $\bigwedge(W \otimes V)$.

Proposition V.4. *Let $\mathcal{A}$ be the a subspace arrangement of cardinality t and let J_i be the vanishing ideal of the i-th subspace in $\mathcal{A}$. We have that for every finite dimensional vector space V*

$$\Omega(\mathcal{J}_{\mathcal{A}})(V) = J_1(V) \wedge J_2(V) \wedge \cdots \wedge J_t(V).$$

Proof. Notice that $\mathcal{J}_{\mathcal{A}}(V) = J_1(V)J_2(V)\cdots J_t(V)$, using the multiplication structure of the symmetric algebra $\mathcal{S}(W \otimes V)$. The functor Ω maps the multiplication map of $\mathcal{S}$ to a multiplication map in $\Omega(\mathcal{S})$. Up to scalars, there is a unique $\mathrm{GL}(V)$-equivariant multiplication in $\bigwedge$, namely the multiplication given by $\wedge$. $\square$

Using the proposition above, we can reformulate Theorem V.3 as general statement in commutative algebra.

Theorem V.5. *The wedge product of t linear ideals in the exterior algebra is t-regular.*

Proof. Every linear ideal J_i is a vanishing ideal of a subspace W_i. Consider the subspace arrangement $\mathcal{A}$ given by a set of linear ideals. Applying Proposition IV.18 we conclude that the associated module functor $\Omega(\mathcal{J}_{\mathcal{A}})(V)$ in $\bigwedge(W \otimes V)$ is t-regular. Using Proposition V.4 for V a 1-dimensional vector space, we conclude that the wedge product of the linear ideals is t-regular. $\square$

5.2 Equivariant Hilbert series and examples

Let V be an n-dimensional vector space. We denote by $s_\lambda(x_1, \ldots, x_n)$ the symmetric function which is the character of the irreducible representation $\mathcal{S}_\lambda(V)$. To the polynomial functor $\mathcal{S}_\lambda$ we associate the symmetric function $s_\lambda = s_\lambda(x_1, x_2, \dots)$ in infinitely many variables. The character of the symmetric algebra functor $\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{S}_d$ is $\sigma = 1 + s_1 + s_2 + \cdots$. We call this series of symmetric function the equivariant Hilbert series of $\mathcal{S}(V)$. The equivariant Hilbert series of the functor $V \mapsto \mathcal{S}(V \oplus V) = \mathcal{S}(V) \otimes \mathcal{S}(V)$ is $(1 + s_1 + s_2 + s_3 + \cdots)^2 = 1 + (2s_1) + (3s_2 + s_{1,1}) + \cdots$. As a result of the properties of Ω, we have that the character of $\Omega(\mathcal{S}_\lambda)(V)$ is $s_{\lambda'}(x_1, \dots, x_n)$. For a polynomial functor $\mathcal{F}$, we consider the symmetric function $H^e(\mathcal{F})$ such that $H^e(\mathcal{F})(V)$ is the character of $\mathcal{F}(V)$ as a representation of $\mathrm{GL}(V)$. We refer to $H^e(\mathcal{F})$ as the equivariant Hilbert series of the polynomial functor $\mathcal{F}$. We state the following result for $\mathcal{J}_\mathcal{A}$, the product module functor of a subspace arrangement $\mathcal{A}$, but the same result holds for $\mathcal{I}_\mathcal{A}$, the intersection module functor.

Theorem V.6. *Let $\mathcal{A}$ be a subspace arrangement. Consider $\mathcal{J}_\mathcal{A}(V)$, its associated* $\mathrm{GL}(V)$*-equivariant product ideal in the symmetric algebra, and $\Omega(\mathcal{J}_\mathcal{A})(V)$, the ideal in the exterior algebra obtained by applying Ω to $\mathcal{J}_\mathcal{A}$. We have that*

$$H^e(\Omega(\mathcal{J}_\mathcal{A})(V)) = \omega(H^e(\mathcal{J}_\mathcal{A}(V))),$$

where ω is the involution on the ring of symmetric functions sending s_λ to $s_{\lambda'}$.

Proof. Consider an equivariant resolution of $\mathcal{J}_\mathcal{A}$. As $\mathcal{J}_\mathcal{A}$ has a linear resolution, we can read off $H^e(\mathcal{J}_\mathcal{A})$ from the resolution (as discussed by Derksen in [6]). Apply Ω to the resolution. Each Schur functor $\mathcal{S}_\lambda$ is mapped by Ω to $\mathcal{S}_{\lambda'}$. The effect on $H^e(\mathcal{J}_\mathcal{A})$ is to change each s_λ to $s_{\lambda'}$. Thus the new equivariant Hilbert series is $\omega(H^e(\mathcal{J}_\mathcal{A}))$. However we now have a resolution of $\Omega(\mathcal{J}_\mathcal{A})$, so $\omega(H^e(\mathcal{J}_\mathcal{A}))$ is its equivariant Hilbert series. □

Consequently, an equivariant Hilbert series of $\Omega(\mathcal{J}_\mathcal{A})$ can immediately be obtained from an equivariant Hilbert series of $J_\mathcal{A}$. As we have a recursive combinatorial for

$H^e(\mathcal{J}_\mathcal{A})$ from [6], we can find write down a resolution for $\mathcal{J}_\mathcal{A}$ and obtain a resolution for $\Omega(\mathcal{J}_\mathcal{A})$ from its equivariant Hilbert series $H^e(\Omega(\mathcal{J}_\mathcal{A}) = \omega(H^e(\mathcal{J}_\mathcal{A}))$.

5.2.1 Computing equivariant Hilbert series via polymatroids

Consider a subspace arrangement $\mathcal{A} = \{W_1, \dots, W_t\} \subset W$. Let $A = \{1, 2, \dots, t\}$ be the indexing set of the subspaces in $\mathcal{A}$. Each choice of subset $B \subset A$ gives us the subarrangement $\mathcal{B} = \{W_i | i \in B\}$ so that any subset of indexes B gives us a product ideal

$$J_B = \prod_{i \in B} J_i,$$

where $J_i = \mathbb{I}(W_i)$. Additionally, we can define a map $\phi_{B,C}$ between a subset B of size s and a subset C of size $s-1$. We have that $\phi_{B,C} \neq 0$ only if $C \subset B$. If $B = \{a_1, \dots, a_s\}$ and $C = \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_s\}$, then $\phi_{B,C} = (-1)^i \operatorname{Id}$.

These maps allow us to construct the following complex:

$$\mathcal{C} : 0 \to J_A \to \bigoplus_{|B|=t-1} J_B \to \cdots \bigoplus_{|B|=2} J_B \to J_1 \oplus J_2 \oplus \cdots \oplus J_t \to R \to 0$$

where, for example, the first map is the direct sum of the maps

$$\phi_{A,\{1,2,\dots,i-1,i+1,\dots,t\}} = (-1)^i \operatorname{Id}.$$

One can check that this is indeed a complex: the composition of two consecutive maps is zero. One can consider the homology of the complex, given by $\frac{\ker \phi}{\operatorname{im} \phi}$. Because our ideals are linear, a result of Conca and Herzog [4] tells us that the homology of the product complex is well-behaved. If the intersection of the subspaces is 0, then the homogeneous maximal ideal $\mathfrak{m}$ kills the homology of this complex and they show that the kth homology is concentrated in degree k. In fact, this fact is key in showing that the product of t linear ideals has regularity t.

For any $V \in \mathbf{Vec}$, when we apply the tensor trick to $\mathcal{A}$ to obtain $\mathcal{A} \otimes V$. As a result, we just tensor every term in the complex $\mathcal{C}$ with V to obtain the complex $\mathcal{C} \otimes V$. Even though the complex is not exact, we have that up to low degree terms

$$0 \approx \sum_{B \subseteq A} (-1)^{|B|} H^e(J_{\mathcal{B}}(V)),$$

so that for any V we have that

$$(-1)^{n+1} H^e(J_{\mathcal{A}}(V)) \approx \Sigma_{B \subset A} (-1)^{|B|} H^e(J_{\mathcal{B}}(V)).$$

In particular, we can use this approach to compute some equivariant Hilbert series inductively. Specifically, we have the following result.

Proposition V.7. *Let $\mathcal{A}$ be the subspace arrangement given by the union of t distinct lines $\mathbb{K}^m$. We have that*

$$H^e(J_{\mathcal{A}}) = \sigma^m - t\sigma + \texttt{lower degree terms}.$$

Proof. Consider the complex

$$0 \to \prod_{i \in A} J_i(V) \to \bigoplus_{|B|=t-1} J_B(V) \to \cdots \to J_1(V) \oplus J_2(V) \oplus \cdots \oplus J_t(V) \to R \to 0.$$

As the k-th homology of the complex is concentrated in degree k, even though the complex is not exact, we have that up to low degree terms

$$0 \approx \sum (-1)^{|B|} H^e(J_{\mathcal{B}})$$

so that

$$(-1)^{t+1} H^e(J_{\mathcal{A}}) \approx \sum_{B \subsetneq A} (-1)^{|B|} H^e(J_{\mathcal{B}})$$

We will prove the claim by induction on t. For $t = 0$, we have that $H^e(J_{\emptyset}) = H^e(R) = \sigma^m$, so the claim holds.

Assuming the result for all $\mathcal{B}$ such that $|B| < t$, we have that

$$(-1)^{t+1} H^e(J_{\mathcal{A}}) \approx \sum_{B \subsetneq A} (-1)^{|B|}(\sigma^m - |B|\sigma).$$

Let k be a non-negative number. The number of k-subsets in A is $\binom{t}{k}$. Thus, we have to prove that

$$\sum_{k<t} (-1)^k \binom{t}{k} = (-1)^{t+1}$$

and

$$\sum_{k<t} (-1)^k \binom{t}{k} (-k) = (-1)^{t+1}(-t).$$

Consider the generating function $f(x) = (1-x)^t = \sum_{k \leq t} (-1)^k \binom{t}{k} x^k$. Firstly, for $x = 1$ we have that

$$f(1) = 0 = \sum_{k \leq t} (-1)^k \binom{t}{k},$$

so $(-1)^{t+1} = -(-1)^t = \sum_{k<t} (-1)^k \binom{t}{k}$, as required. Secondly, consider $f'(1)$:

$$f'(1) = 0 = \sum_{k \leq t} (-1)^k \binom{t}{k} k.$$

Rearranging, we get that $(-1)^{t+1} t = -(-1)^t t = \sum_{k<t} (-1)^k \binom{t}{k} k$. Multiplying both sides by -1, we get the required equality.

Therefore, by induction, the original result holds. $\square$

We mentioned that the equivariant Hilbert series of the product ideal can be computed purely from the combinatorial structure of $\mathcal{A}$, meaning that the information provided by the polymatroid of $\mathcal{A}$ is sufficient to determine $H^e(J(V))$.

Definition V.8. Let $\mathcal{A} = \{W_1, \ldots, W_t\} \subset W$ and consider the index set $A = \{1, \ldots, t\}$. We associate to a subset $B = \{a_1, \ldots, a_s\} \subset \{1, \ldots, t\}$ the subarrangement $\mathcal{B} = \{W_{a_1}, \ldots, W_{a_s}\}$. The polymatroid (A, rk) associated to $\mathcal{A}$ is the power set $\mathfrak{P}(A)$ with rank function $rk : \mathfrak{P}(A) \to \mathbb{N}$ defined on $B \subseteq A$ by

$$\mathrm{rk}(B) = \dim(W) - \dim(\cap_{i \in B} W_i).$$

Notice that a polymatroid where all one element subsets have rank one is a matroid. To each subspace arrangement $\mathcal{A}$ we can associate a symmetric polynomial $P(A)$ associated to the polymatroid (A, rk). We have that $P(A)$ measures how far is the complex $\mathcal{C}$ from being exact. In particular, if $P(A) = 0$, we will have that the complex $\mathcal{C}$ is exact.

Definition V.9. Define the symmetric polynomial $P(A)$ is of degree $\leq t-1$ recursively as follows. Set $P(\emptyset) = 1$. Then

$$P(A) = u_0 + \cdots + u_{|A|-1}$$

when

$$\sum u_i = -\sum_{B \subsetneq A} (-1)^{|A|-|B|} \sigma^{rk(A)-rk(B)} P(B).$$

A result of Derksen gives us a way to compute the equivariant Hilbert series of $J_{\mathcal{A}}$ from the polymatroid of $\mathcal{A}$ via the symmetric polynomial $P(A)$.

Theorem V.10 (Derksen, Theorem 5.2 in [6])**.** *Let $\sigma = \sum s_i$. We have that*

$$\sigma^{(n-rk(A))} P(A) = \sum_{B \subseteq A} (-1)^{|B|} H^e(J_B)$$

Example V.11. Consider the two coordinate axes in a two-dimensional vector space W. As linear subspaces we can characterize the x-axis W_1 as the subspace $\{(x, y) \mid y = 0\}$ and similarly, the y-axis W_2 as the subspace $\{(x, y) \mid y = 0\}$. Then $\mathcal{A} = \{W_1, W_2\}$ is a subspace arrangement. The associated linear ideals are $J_1 = (y)$, $J_2 = (x)$, whilst the product ideal is $J_A = (x)(y) = (xy)$.

The polymatroid data (A, rk) associated to $\mathcal{A}$ is given below together with the resulting symmetric polynomials $P(B)$ for $B \subseteq A$

$$\mathrm{rk}(\emptyset) = 0, \quad P(\emptyset) = 1$$

$$\mathrm{rk}(\{i\}) = 1, \quad P(\{i\}) = 1$$

$$rk(\{1,2\}) = 2, \quad P(\{1,2\}) = P(A) = 1$$

Using this data, we get that $H^e(\mathcal{J}_{\mathcal{A}}) = \sigma^2 - 2\sigma + 1$.

Let $\mathcal{A}$ be a subspace arranement of cardinality t. Notice that the product of t linear ideals is generated in degree t and it is t-regular by Conca and Herzog's result. Thus, we notice that the minimal free resolution of its associate polynomial functor $\mathcal{J}_{\mathcal{A}}$ is a linear resolution. Then, we will have that the torsion module $E_i = \mathrm{Tor}_i(\mathcal{J}_{\mathcal{A}}, \mathbb{K})$ is the only torsion module in the resolution of degree $i + t$. The following result gives us a way to find the signed sum of the characters of the torsion modules from the equivariant Hilbert series H^e.

Corollary V.12 (Derksen, Corollary 5.3 in [6])**.** *Let $\sigma = \sum s_i$, let $\mathcal{A}$ be a subspace arrangement in $W \cong \mathbb{K}^m$, and let $H^e(\mathcal{J}_{\mathcal{A}})$ be the equivariant Hilbert series of the product ideal functor $\mathcal{J}_{\mathcal{A}}$. Let*

$$\sigma^{-m} H^e(\mathcal{J}_{\mathcal{A}}) = \sum_{\lambda, |\lambda| \geq t} (-1)^{|\lambda| - t} a_\lambda s_\lambda,$$

where $a_\lambda \in \mathbb{Z}_{\geq 0}$ is the multiplicity of s_λ. Then, for $E_d = \mathrm{Tor}_d(\mathcal{J}_{\mathcal{A}}, \mathbb{K})$, we have that

$$H^e(E_d) = \sum_{|\lambda| = d} a_\lambda s_\lambda.$$

Corollary V.12 holds because there is only one torsion module E_d of each degree d. Therefore, we will have that each s_λ in $\sigma^{-m} H^e$ with $|\lambda| = d$ will give us precisely a Schur functors $\mathcal{S}_\lambda$ appearing in E_d.

5.2.2 Example: powers of the maximal ideal

Consider the subspace arrangement $\mathcal{A} = \{Y_1, \ldots, Y_t\} \subset \mathbb{K}$ with all $Y_i = \{0\}$. This subspace arrangement is t copies of the zero dimensional subspace in a vector space Y of dimension one. The equivariant product ideal of this subspace arrangement is the t-th power of maximal ideal $\mathcal{M}(V) = (x_1, \ldots, x_n)$ for V an n-dimensional vector space. So we have $\mathcal{M}^t(V)$ in $\mathcal{S}(W \otimes V) \cong \mathcal{S}(V)$. From [8] we know that $\mathcal{M}^t(V)$ is t-regular and from Theorem V.3 we can establish that $\Omega(\mathcal{M}^t)(V)$ is also t-regular.

Applying Theorem V.10 to $H^e(\mathcal{M}^t)$ we get a minimal resolution for $\mathcal{M}^t$. Let $\sigma = \sum_i s_i = 1 + s_1 + s_2 + \ldots$ and notice that $H^e(\mathcal{S}) = \sigma$. For $t = 1$, we get that $H^e(\mathcal{M}) = \sigma - 1$, yielding the Koszul resolution for $\mathcal{M}$:

$$\cdots \to \mathcal{S} \otimes \mathcal{S}_{(1,1,1)} \to \mathcal{S} \otimes \mathcal{S}_{(1,1)} \to \mathcal{S} \otimes \mathcal{S}_{(1)} \to \mathcal{M} \to 0,$$

and the following Koszul resolution for $\Omega(\mathcal{M})(V)$:

$$\cdots \to \bigwedge \otimes \mathcal{S}_{(3)} \to \bigwedge \otimes \mathcal{S}_{(2)} \to \bigwedge \otimes \mathcal{S}_{(1)} \to \Omega(\mathcal{M}) \to 0.$$

For $t = 2$, we get that $H^e(\mathcal{M}^2) = \sigma - (1 + s_1)$. This gives the following resolution for $\mathcal{M}^2$:

$$\cdots \to \mathcal{S} \otimes \mathcal{S}_{(2,1,1)} \to \mathcal{S} \otimes \mathcal{S}_{(2,1)} \to \mathcal{S} \otimes \mathcal{S}_{(2)} \to \mathcal{M}^2 \to 0,$$

yielding the following resolution for $\Omega(\mathcal{M}^2)$:

$$\cdots \to \bigwedge \otimes \mathcal{S}_{(3,1)} \to \bigwedge \otimes \mathcal{S}_{(2,1)} \to \bigwedge \otimes \mathcal{S}_{(1,1)} \to \Omega(\mathcal{M}^2) \to 0.$$

5.2.3 Example: distinct lines in a plane

Consider the subspace arrangement $\mathcal{A}$ given by t distinct lines in a vector space Y of dimension two. We have that $\mathcal{A} = \{Y_1, \ldots, Y_t\} \subset \mathbb{K}^2$ and where for each i the subspace Y_i is a line in $\mathbb{K}^2$, such that all lines Y_i are distinct. By Proposition V.7, we have that

$$H^e(\mathcal{J}_{\mathcal{A}}) = \sigma^2 - t\sigma - Q(A),$$

where $Q(A)$ is a symmetric polynomial of degree less than t depending on the symmetric polynomial $P(A)$ of the polymatroid (A, rk).

In particular, for $t = 2$ we have that $\mathcal{A}$ consists of two lines in $\mathbb{K}^2$. We can assume these two lines to be the y-axis the x-axis, so that $J_1J_2 = (x, y)$ in $\mathbb{K}[x, y]$. For each vector space V of dimension n, we get the product ideals

$$\mathcal{J}_{\mathcal{A}}(V) = J_1(V)J_2(V) = (x_1, \dots, x_n)(y_1, \dots, y_n) = (x_iy_j)$$

and

$$\Omega(\mathcal{J}_{\mathcal{A}})(V) = J_1(V) \wedge J_2(V) = (x_1, \dots, x_n) \wedge (y_1, \dots, y_n) = (x_i \wedge y_j)$$

where $1 \leq i, j \leq n$.

Using the polymatroid of $\mathcal{A}$, we get that $H^e(\mathcal{J}_{\mathcal{A}}) = (\sigma - 1)^2$. Using this formula for the equivariant Hilbert series of $\mathcal{J}_{\mathcal{A}}$ we get the following resolution:

$$\cdots \to \mathcal{S}\otimes(\mathcal{S}_{(2,2)}\oplus\mathcal{S}^3_{(2,1,1)}\oplus\mathcal{S}^3_{(1,1,1,1)}) \to \mathcal{S}\otimes(\mathcal{S}^2_{(2,1)}\oplus\mathcal{S}^2_{(1,1,1)}) \to \mathcal{S}\otimes(\mathcal{S}_{(2)}\oplus\mathcal{S}_{(1,1)}) \to \mathcal{J}_{\mathcal{A}} \to 0,$$

yielding the following resolution for $\Omega(\mathcal{J}_{\mathcal{A}})$:

$$\cdots \to \bigwedge\otimes(\mathcal{S}_{(2,2)}\oplus\mathcal{S}^3_{(3,1)}\oplus\mathcal{S}^3_{(4)}) \to \bigwedge\otimes(\mathcal{S}^2_{(2,1)}\oplus\mathcal{S}^2_{(3)})) \to \bigwedge\otimes(\mathcal{S}_{(1,1)}\oplus\mathcal{S}_{(2)}) \to \Omega(\mathcal{J}_{\mathcal{A}}) \to 0.$$

5.2.4 Example: a line and a plane

Consider the subspace arrangement $\mathcal{A}$ given by a plane and a line normal to it in an ambient space of dimension three. We have that $\mathcal{A} = \{Y_1, Y_2\} \subset \mathbb{K}^3$, where we can assume that Y_1 is the $(x, y)-$ plane and that Y_2 is the z-axis. Then in $\mathbb{K}[x, y, z]$ we have that $J_1 = (z)$ and $J_2 = (x, y)$, so that $J = J_1J_2 = (zx, zy)$. We get that

$$H^e(\mathcal{J}_{\mathcal{A}}) = \sigma^3 - \sigma^2 - \sigma + 1.$$

Using the formula above for the equivariant Hilbert series of $\mathcal{J}_{\mathcal{A}}$ we get the following resolution:

$$\cdots \to \mathcal{S} \otimes (\mathcal{S}^3_{(3,1)} \oplus \mathcal{S}^5_{(2,2)} \oplus \mathcal{S}^{12}_{(2,1,1)} \oplus \mathcal{S}^9_{(1,1,1,1)}) \to$$
$$\to \mathcal{S} \otimes (\mathcal{S}_{(3)} \oplus \mathcal{S}^6_{(2,1)} \oplus \mathcal{S}^5_{(1,1,1)}) \to \mathcal{S} \otimes (\mathcal{S}^2_{(2)} \oplus \mathcal{S}^2_{(1,1)}) \to \mathcal{J}_{\mathcal{A}} \to 0,$$

yielding the following resolution for $\Omega(\mathcal{J}_{\mathcal{A}})$:

$$\cdots \to \bigwedge \otimes (\mathcal{S}^3_{(2,1,1)} \oplus \mathcal{S}^5_{(2,2)} \oplus \mathcal{S}^{12}_{(3,1,1)} \oplus \mathcal{S}^9_{(4)}) \to$$
$$\to \bigwedge \otimes (\mathcal{S}_{(1,1,1)} \oplus \mathcal{S}^6_{(2,1)} \oplus \mathcal{S}^5_{(3)}) \to \bigwedge \otimes (\mathcal{S}^2_{(1,1)} \oplus \mathcal{S}^2_{(2)}) \to \Omega(\mathcal{J}_{\mathcal{A}}) \to 0.$$

5.2.5 Example: three coordinate axes

Consider the subspace arrangement $\mathcal{A}$ given by the three coordinate axes in $\mathbb{K}^3$. We have that $\mathcal{A} = \{Y_1, Y_2, Y_3\}$, where $J_1 = (y, z)$, $J_2 = (x, z)$, and $J_3 = (x, z)$ in $\mathbb{K}[x, y, z]$. Consider in this case the *intersection* ideal $I_{\mathcal{A}} = J_1 \cap J_2 \cap J_3$. One can check that $I_{\mathcal{A}} = (xy, xz, yz)$. Being generated in degree two, we know that the regularity of $I_{\mathcal{A}}$ is at least two. One can check that $I_{\mathcal{A}}$ is the sum of three products of linear ideals,

$$I_{\mathcal{A}} = (xy) + (xz) + (yz) = (xy, xz, yz),$$

so that we can use a result of Derksen and Sidman [9] to conclude that $I_{\mathcal{A}}$ is 4-regular.

We can show that $I_{\mathcal{A}}$ has regularity two. In fact, we will show that $\mathcal{I}_{\mathcal{A}}(V)$ has regularity two for every vector space V, so that the functor $\mathcal{I}_{\mathcal{A}}$ has regularity two. Let us use the notation $\mathbf{x} = x_1, \ldots, x_n$, and similarly for $\mathbf{y}, \mathbf{z}$.

Proposition V.13. *Let V be a vector space of dimension n. Let $J_1(V) = (\mathbf{y}, \mathbf{z})$, $J_2(v) = (\mathbf{x}, \mathbf{z})$, $J_3(V) = (\mathbf{x}, \mathbf{y})$. Then the ideal*

$$\mathcal{I}_{\mathcal{A}}(V) = J_1(V) \cap J_2(V) \cap J_3(V) = (\mathbf{y}, \mathbf{z}) \cap (\mathbf{x}, \mathbf{z}) \cap (\mathbf{x}, \mathbf{y})$$

has regularity two.

Proof. We have that $\mathcal{I}_{\mathcal{A}}(V)$ is generated in degree at least two. In fact, one can check that

$$\mathcal{I}_{\mathcal{A}}(V) = (x_1y_1, \ldots, x_iy_j, \ldots, x_ny_n) + (x_1z_1, \ldots, x_iz_k, \ldots, x_nz_n) + (y_1z_1, , \ldots, y_jz_k, \ldots, y_nz_n),$$

where $1 \leq i, j, k \leq n$. As $\mathcal{I}_{\mathcal{A}}(V)$ is generated in degree two, it has regularity at least two. So to prove the proposition it is enough to show that the regularity is at most two.

Let $\mathbb{K}[X]$ be the coordinate ring of $X = Y_1 \otimes V$. We have that X is the subspace spanned by the x coordinate axes

$$\mathbb{K}[X] = \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]/J_1(V) = \frac{\mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]}{(\mathbf{y}, \mathbf{z})}.$$

Similarly, $\mathbb{K}[Y] = \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]/J_2(V)$ and $\mathbb{K}[Z] = \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]/J_3(V)$. Let π_i be the canonical projection $\pi_i : \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}] \to \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]/J_i(V)$. We then have a map $\pi = (\pi_1, \pi_2, \pi_3)$ such that

$$\pi : \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}] \to \mathbb{K}[X] \oplus \mathbb{K}[Y] \oplus \mathbb{K}[Z].$$

We have that π factors through $\mathbb{K}[\mathcal{A}] = \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]/\mathcal{I}_{\mathcal{A}}(V)$ giving us a map ϕ:

$$\phi : \mathbb{K}[\mathcal{A}] \to \mathbb{K}[X] \oplus \mathbb{K}[Y] \oplus \mathbb{K}[Z].$$

Notice that an element of $r \in \mathbb{K}[\mathcal{A}]$ can be represented as $r = f(\mathbf{x}) + g(\mathbf{y}) + h(\mathbf{z})$. Hence ϕ is given by $f(\mathbf{x}) + g(\mathbf{y}) + h(\mathbf{z}) \mapsto (f(\mathbf{x}), g(\mathbf{y}), h(\mathbf{z}))$. The image of ϕ is

$$U = \{(f, g, h) : f(0) = g(0) = h(0)\}$$

as the intersection of any two of the three subspaces is the origin and so $X \cap Y \cap Z$ is also 0. Thus, (f, g, h) determines a function on $\mathcal{A}$, the union $X \cup Y \cup Z$, precisely when the functions f, g, h agree at 0.

Consider the short exact sequence of modules

$$0 \to U \to \mathbb{K}[X] \oplus \mathbb{K}[Y] \oplus \mathbb{K}[Z] \to \mathbb{K}^2 \to 0,$$

where the last map is the projection $(f, g, h) \mapsto (f(0) - g(0), g(0) - h(0))$. Consider the following result on regularity (part of Corollary 20.19 in [10]).

Lemma V.14. *If A, B, C are finitely generated graded modules, and*

$$0 \to A \to B \to C \to 0$$

is exact, then $\operatorname{reg}(A) \leq \max\{\operatorname{reg}(B), \operatorname{reg}(C) + 1\}$.

Thus, we can bound the regularity of the first module A in a short exact sequence if we know bound on the regularity of the other two modules. In particular, we have that

$$\operatorname{reg}(U) \leq \max\{\operatorname{reg}(\mathbb{K}[X] \oplus \mathbb{K}[Y] \oplus \mathbb{K}[Z]), \operatorname{reg}(\mathbb{K}^2) + 1\} = 1.$$

Finally, consider the short exact sequence of modules

$$0 \to \mathcal{I}_{\mathcal{A}}(V) \to \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}] \to U \to 0,$$

where the last map is the restriction of the projection π. Using Lemma V.14 again, we conclude that

$$\operatorname{reg}(\mathcal{I}_{\mathcal{A}}(V)) \leq \max\{\operatorname{reg}(\mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]), \operatorname{reg}(U) + 1\} \leq 2.$$

As $\operatorname{reg}(\mathcal{I}_{\mathcal{A}}(V)) \geq 2$, we can conclude that $\operatorname{reg}(\mathcal{I}_{\mathcal{A}}(V)) = 2$. $\square$

As $\mathcal{I}_{\mathcal{A}}(V)$ is generated in degree two and has regularity two, we have that $\mathcal{I}_{\mathcal{A}}$ has a linear resolution. Thus, we can use its equivariant Hilbert series to write down its resolution. We have that

$$H^e(\mathcal{I}_{\mathcal{A}}) = \sigma^3 - (3\sigma - 2),$$

yielding the following resolution for $\mathcal{I}_{\mathcal{A}}$:

$$\cdots \to \mathcal{S} \otimes (\mathcal{S}^{6}_{(3,1)} \oplus \mathcal{S}^{9}_{(2,2)} \oplus \mathcal{S}^{21}_{(2,1,1)} \oplus \mathcal{S}^{15}_{(1,1,1,1)}) \to$$
$$\to \mathcal{S} \otimes (\mathcal{S}^{2}_{(3)} \oplus \mathcal{S}^{10}_{(2,1)} \oplus \mathcal{S}^{8}_{(1,1,1)}) \to \mathcal{S} \otimes (\mathcal{S}^{3}_{(2)} \oplus \mathcal{S}^{3}_{(1,1)}) \to \mathcal{I}_{\mathcal{A}} \to 0,$$

and for $\Omega(\mathcal{I}_{\mathcal{A}})$:

$$\cdots \to \bigwedge \otimes (\mathcal{S}^{6}_{(2,1,1)} \oplus \mathcal{S}^{9}_{(2,2)} \oplus \mathcal{S}^{21}_{(3,1)} \oplus \mathcal{S}^{15}_{(4)}) \to$$
$$\to \bigwedge \otimes (\mathcal{S}^{2}_{(1,1,1)} \oplus \mathcal{S}^{10}_{(2,1)} \oplus \mathcal{S}^{8}_{(3)}) \to \bigwedge \otimes (\mathcal{S}^{3}_{(1,1)} \oplus \mathcal{S}^{3}_{(2)}) \to \Omega(\mathcal{I}_{\mathcal{A}}) \to 0.$$

CHAPTER VI

Noether's bound over the exterior algebra

6.1 The subspace theorem for the exterior algebra

In the previous chapters we defined a functor on polynomial functors, Ω, that preserves regularity. In our discussion we remarked that for a subspace arrangement of cardinality t this implies that $\Omega(\mathcal{I}_{\mathcal{A}})$ is t-regular. We will reformulate this result in terms of the intersection ideal in the exterior algebra.

Definition VI.1. Let $\mathcal{A} = \{W_1, \ldots, W_t\} \subset W \cong \mathbb{K}^m$ be a subspace arrangement. Suppose that S_i be the set of linear forms vanishing on W_i and let J_i be the ideal generate by S_i in $E = \bigwedge(W^*)$. We define $\mathbb{I}'(\mathcal{A})$, the intersection ideal of $\mathcal{A}$ in E to be

$$\mathbb{I}'(\mathcal{A}) = \bigcap_i J_i.$$

A word of caution: we cannot consider the non-linear skew polynomials in $\bigwedge(W^*)$ as $\mathbb{K}$-valued functions on W. The reader may think of $\mathbb{I}'(\mathcal{A})$ as the vanishing ideal of $\mathcal{A}$ in E, but one must be careful not to abuse this heuristic.

Theorem VI.2 (Subspace theorem for the exterior algebra)**.** *If $\mathcal{A}$ is an arrangement of t subspaces in $\mathbb{K}^n$, then the ideal $\mathbb{I}'(\mathcal{A})$ in the exterior algebra $\bigwedge(x_1, \ldots, x_n)$ is t-regular. In particular, this ideal is generated in degree at most t.*

Proof. Consider the polynomial functor $\mathcal{I}'_{\mathcal{A}}$ associated to the intersection ideal $\mathbb{I}'(\mathcal{A})$ in the exterior algebra. We have that $\mathcal{I}'_{\mathcal{A}} = \Omega(\mathcal{I}_{\mathcal{A}})$ for $\mathcal{I}_{\mathcal{A}}$ the polynomial functor associated to the intersection ideal $\mathbb{I}(\mathcal{A})$ in the symmetric algebra. By a result of Derksen and Sidman ([8]), we know that $\mathbb{I}(\mathcal{A})$ is t-regular, so it is generated in degree at most t. Moreover, the same result also gives us that $\mathcal{I}_{\mathcal{A}}(V)$ is t-regular, for any finite dimensional vector space $V \in \mathrm{Obj}(\mathbf{Vec})$. Thus, $\mathcal{I}_{\mathcal{A}}$ is t-regular. Applying Ω, we can conclude that $\mathcal{I}'_{\mathcal{A}} = \Omega(\mathcal{I}_{\mathcal{A}})$ is also t regular, meaning that $\mathcal{I}'_{\mathcal{A}}(V)$ is t-regular for any $V \in \mathrm{Obj}(\mathbf{Vec})$. In particular, $\mathbb{I}'(\mathcal{A})$ is t-regular, so it is generated in degree at most t. $\square$

In Chapter 2 we introduced an algorithm of Derksen II.12 to obtain invariant polynomials for the action of a finite group G acting on V from the ideal generators of the vanishing ideal in the polynomial ring of the group subspace arrangement

$$\mathcal{A}_G = \bigcup_{g \in G} \{(v, g \cdot v) | v \in V\} \subset V \oplus V.$$

We can find explicit equations for $\mathcal{A}$ by picking a basis for $V \oplus V$ with coordinate functions $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Let $A(g) = (A(g)_{i,j})$ be the matrix representing the action of g on V. Then the linear subspace V_g is cut out by the linear equations $y_i = \sum_{j=1}^{n} A(g)_{i,j} x_j$, $i = 1, 2, \ldots, n$.

We can consider the ideal generated by the set

$$S_g = \left\{ y_1 - \sum_{j=1}^{n} A(g)_{1,j} x_j, \ldots, \sum_{j=1}^{n} A(g) x_{n,j} x_j \right\}$$

in the polynomial ring or in the skew polynomial ring. For $J_g = \mathbb{I}(V_g)$ in the polynomial ring $\mathbb{K}[\mathbf{x}, \mathbf{y}]$, we say that J_g is the ideal of functions vanishing on V_g. For $f \in J'_g$, the ideal generated by S_g in the exterior algebra $\bigwedge(\mathbf{x}, \mathbf{y})$, we have that the ring homomorphism ϕ_g given by the substitution $y_i \mapsto \sum_{j=1}^{n} A(g)_{i,j} x_j$ sends f to zero i.e.,

$\phi_g(f) = 0$. In this sense we mean that f "vanishes" on V_g. Moreover, the vanishing ideal of $\mathcal{A}_G$ in the polynomial ring is the intersection of the ideals $J_g = \mathbb{I}(V_g)$. The ideal $\mathbb{I}'(\mathcal{A})$ is the intersection of the ideals $J'_g = (S_g)$ generated by the sets S_g in $E = \bigwedge(\mathbf{x}, \mathbf{y})$. Notice that for $f \in \mathbb{I}'(\mathcal{A})$ we have that the substitution ϕ_g is such that $\phi_g(f) = 0$, for all g. In this sense we say that $\mathbb{I}'(\mathcal{A})$ is the "vanishing" ideal of $\mathcal{A}$ in $E = \bigwedge(\mathbf{x}, \mathbf{y})$.

In the next sections we will prove an analog of Derksen's result II.12 in the new context of invariant skew polynomials in the exterior algebra. We will first prove a classical result of invariant theory in the general setting of (possibly non-commutative) graded algebras.

6.2 Hilbert invariant theorem for graded algebras

Let A be a graded $\mathbb{K}$ algebra such that $A = \bigoplus_{d \geq 0} A_d$ and $A_0 \cong \mathbb{K}$. Let G be a linearly reductive group acting regularly on A by degree-preserving automorphisms. In particular, notice that A_d is a representation of G for every d. We denote the subspace of fixed points of G in A_d by

$$A_d^G = \{f \in A_d | \forall g \in G \; g \cdot f = f\}.$$

We denote by A_+ the ideal $\bigoplus_{d>0} A_d$ whilst A_+^G denotes $\bigoplus_{d>0} A_d^G$. Notice that for $A^G = \bigoplus_{d \geq 0} A_d^G$, we have that $A^G \cong \mathbb{K} \oplus A_+^G$ as $g \cdot a_0 = a_0$, for all $a_0 \in A_0 \cong \mathbb{K}$. Moreover, we let the Hilbert left ideal I_G be

$$I_G = AA_+^G = \{\sum a_i f_i \mid a_i \in A, f_i \in A_d^G, \text{ for some } d > 0\}.$$

We introduce the notion of a Reynolds operator in this general setting. For a linearly reductive group, the space V^G has a unique G-stable complement in V. This means that there is a unique linear G-invariant projection $V \to V^G$.

Definition VI.3. Let G be a linearly reductive group and V be a representation of G. We define $\mathcal{R}_G$, the Reynolds operator of G, to be the unique G-equivariant linear projection $\mathcal{R}_G : V \to V^G$.

Lemma VI.4. *For any G-representations V and W and G-equivariant map $\phi : V \to W$ we have that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\mathcal{R}_G"'] & W \arrow[d, "\mathcal{R}_G"] \\ V^G \arrow[r, "\phi|_{V^G}"] & W^G \end{tikzcd} \tag{6.1}$$

Corollary VI.5. *Notice that the operator $\mathcal{R}_G$ will satisfy the following properties:*

(i) For a G-stable subspace U, we have that $\mathcal{R}_G(U) = U^G$;

(ii) We have that $\mathcal{R}_G$ is a A^G-bimodule homomorphism, meaning that if $a, b \in A^G$, then $\mathcal{R}_G(afb) = a\mathcal{R}_G(f)b$, for any $f \in A$.

Proof.

(i) For the inclusion map $\phi : U \to V$ we have a commuting diagram (6.1), so $\mathcal{R} : V \to V^G$ restricts to the Reynolds operator $\mathcal{R} : U \to U^G$.

(ii) Define $\phi : A \to A$ by $\phi(f) = afb$. Then ϕ is a G-equivariant linear map, hence we have

$$\mathcal{R}_G(afb) = \mathcal{R}_G(\phi(f)) = \phi(\mathcal{R}_G(f)) = a\mathcal{R}_G(f)b.$$

□

We will use this general notion of a Reynolds operator applied to the graded algebra A to prove a version of Hilbert invariant theorem for A. We build up to this result with the following lemmas.

Lemma VI.6 (Ideal generators are algebra generators)**.** *Suppose that* $\{f_1, \ldots, f_r\}$ *generate* A_+ *as a left ideal, then* $A = \mathbb{K}\langle f_1, \ldots, f_r\rangle$*, the subalgebra generated by* $f_1, \ldots, f_r$*.*

Proof. As A is a graded algebra, it is enough to prove that for any homogeneous $g \in A$, we have that $g \in \mathbb{K}\langle f_1, \ldots, f_r\rangle$. We will prove the claim by induction on $d = \deg(g)$. If $\deg(g) = 0$, then the claim is obvious. Assume that $d = \deg(g) > 0$, so that $g \in A_+$. Then we can write $g = \sum a_i f_i$ and after cancellation we can assume that $\deg(a_i) + \deg(f_i) = d$ for all i. Thus, $\deg(a_i) < d$ and by induction $a_i \in \mathbb{K}\langle f_1, \ldots, f_r\rangle$. Therefore, $g \in \mathbb{K}\langle f_1, \ldots, f_r\rangle$, as required. □

Lemma VI.7 (Invariant ideal generators are invariant algebra generators)**.** *Suppose that homogeneous* $f_1, \ldots, f_r \in A_+^G \subset I_G$ *generate* I_G *as a left ideal. Then* $A^G = \mathbb{K}\langle f_1, \ldots, f_r\rangle$*, the free algebra of words in the letters* f_i*.*

Proof. As A^G is a graded algebra, it is enough to prove that for any homogeneous $g \in A^G$, we have that $g \in \mathbb{K}\langle f_1, \ldots, f_r\rangle$. If $\deg(g) = 0$, then the claim is obvious. Assume that $d = \deg(g) > 0$, so that $g \in A_+^G \subset I_G$. Then $g = \sum a_i f_i$. Apply the Reynolds operator $\mathcal{R}$ to g. We have that

$$g = \mathcal{R}(g) = \sum \mathcal{R}(a_i f_i) = \sum \mathcal{R}(a_i) f_i,$$

by the assumption that $f_i \in A_+^G$ and property (ii) of $\mathcal{R}$. Thus, g lies in the left ideal generated by $\{f_1, \ldots, f_r\}$ in A^G. This means that $\{f_1, \ldots, f_r\}$ is a set of left ideal generators for A_+^G in A^G. Therefore, by Lemma VI.6, we have that $A^G = \mathbb{K}\langle f_1, \ldots, f_r\rangle$, as required. □

Before stating our final lemma, we need the graded version of Nakayama's lemma.

Lemma VI.8 (Graded Nakayama's lemma)**.** *Suppose that* M *is a finitely generated graded left* A*-module such that* $A_+ M = M$*. Then we have* $M = 0$*.*

Proof. Let i be the smallest positive degree such that $M_i \neq 0$. Then $M_i \cap A_+M = 0$, but by assumption $A_+M = M$, so $M_i \cap A_+M = M_i$. Therefore, $M = 0$. □

Lemma VI.9 (Replacing ideal generators with invariant ideal generators)**.** *Suppose that $\{f_1, \ldots, f_r\}$ generate I_G as a left ideal. Then $\{\mathcal{R}(f_1), \ldots, \mathcal{R}(f_r)\}$ generate I_G as a left ideal.*

Proof. Notice that since I_G is G-stable, we have that $\mathcal{R}(f_i) \in I_G$, for all i, by property (i) of $\mathcal{R}$. Thus, the left ideal generated by $\{\mathcal{R}(f_1), \ldots, \mathcal{R}(f_r)\}$ is contained in I_G. Let J_G be the left ideal generated by $\{\mathcal{R}(f_1), \ldots, \mathcal{R}(f_r)\}$. We have that $J_G \subseteq I_G$. We want to show that this containment is actually an equality.

Consider the space I_G/A_+I_G. We have that G acts trivially on I_G/A_+I_G, so that the Reynolds operator is the identity on this space. In particular, $f_i + A_+I_G = \mathcal{R}(f_i) + A_+I_G$, for all i. Thus, $I_G = J_G + A_+I_G$. Consider the module I_G/J_G. We have that

$$A_+(I_G/J_G) = (J_G + A_+I_G)/J_G = I_G/J_G.$$

By Nakayama's lemma VI.8, we have that $I_G/J_G = 0$. Therefore $J_G = I_G$, as required. □

Theorem VI.10 (Hilbert invariant theorem)**.** *Suppose that $\{f_1, \ldots, f_r\}$ generate I_G as a left ideal. Then $A^G = \mathbb{K}\langle f_1, \ldots, f_r\rangle$, the free algebra of words in the letters f_i.*

Proof. Notice that by Lemma VI.9 we can assume without loss of generality that $f_i \in A_+^G$, for all i. Then the result is an immediate consequence of Lemma VI.7. □

6.3 Computing invariants over the exterior algebra

We begin we the definition of the usual terms from invariant theory in the context of the exterior algebra $E = \bigwedge(x_1, \ldots, x_n)$. Notice that we can think of the exterior algebra as

$$E = \frac{\mathbb{K}\langle x_1, \ldots, x_n \rangle}{(x_i x_j + x_j x_i, 1 \leq i \leq j \leq n)}.$$

In particular, E is a finite-dimensional graded algebra where $E_d = 0$ for $d > n$. We denote the multiplication in E by $\wedge$. As a $\mathbb{K}$-vector space, we have that

$$E_d = \operatorname{span}_{\mathbb{K}}\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}.$$

Moreover, we notice that every homogeneous ideal in E is a two-sided ideal: for homogeneous a, f we can rewrite any product $a \wedge f$ as $\pm f \wedge a$ using the skew commutative relation $x_i x_j = -x_j x_i$, where the sign $\pm$ is determined by the degrees of f and a. In fact, $a \wedge f = (-1)^{\deg(f)\deg(a)} f \wedge a$.

Definition VI.11. Let V be a an n-dimensional representation of G and consider the invariant skew polynomials for this action in $E = \bigwedge(V) = \bigwedge(x_1, \ldots, x_n)$. The *Hilbert ideal* J_G of G in E is the ideal generated by all the invariants of positive degree i.e., $J_G = E E_+^G$, where

$$E_+^G = \{p \in E_+ \mid g \cdot p = p, \forall g \in G\}.$$

Notice that $E_+^G = \bigoplus_{d>0} E_d^G$, where E_d^G is the space of invariant skew polynomials of degree d.

The key theorem in our method to compute invariant skew polynomials is an adaptation of Derksen's result (here Theorem II.12) to the exterior algebra context.

Theorem VI.12. *Let J_G be the Hilbert ideal for the action of G on $E = \bigwedge(x_1, \ldots, x_n)$. Consider the ideal $I_G = \mathbb{I}'(\mathcal{A}_G)$ in the ring $\bigwedge(x_1, \ldots, x_n, y_1, \ldots, y_n)$. We have that*

$$(I_G + (y_1, \ldots, y_n)) \cap \bigwedge(x_1, \ldots, x_n) = J_G.$$

Proof. The easy containment is $(I_G + (y_1, \ldots, y_n)) \cap \bigwedge(x_1, \ldots, x_n) \supseteq I_G$. Let $f(\mathbf{x}) \in I_G$ and rewrite $f(\mathbf{x})$ as $f(\mathbf{x}) = (f(\mathbf{x}) - f(\mathbf{y})) + f(\mathbf{y})$. As $f(\mathbf{y})$ is a skew polynomial of positive degree in the y variables, we have that $f(\mathbf{y}) \in (y_1, \ldots, y_n)$. On the other hand, we have that $(f(\mathbf{x}) - f(\mathbf{y})) \in J_G$ as for any $g \in G$, we have that

$$\phi_g(f(\mathbf{x}) - f(\mathbf{y})) = f(\mathbf{x}) - f(A(g)\mathbf{x}) = f(\mathbf{x}) - g \cdot f(\mathbf{x}) = 0,$$

because we assumed that f in an invariant in J_G. This establishes the first containment.

Next, we will prove the second containment $(I_G + (y_1, \ldots, y_n)) \cap \bigwedge(x_1, \ldots, x_n) \subseteq J_G$. Any $f(\mathbf{x}) \in \bigwedge(x_1, \ldots, x_n)$ that lies in $I_G + (y_1, \ldots, y_n)$ can be written as

$$f(\mathbf{x}) = p(\mathbf{x}, \mathbf{y}) + \sum c_i(\mathbf{x}) g_i(\mathbf{y}),$$

for $p(\mathbf{x}, \mathbf{y}) \in I_G$ and $\sum c_i(\mathbf{x}) g_i(\mathbf{y}) \in (y_1, \ldots, y_n)$. In particular, for each i we require that $c_i(\mathbf{x}) \in \bigwedge(x_1, \ldots, x_n)$ and $g_i(\mathbf{y})$ is a skew polynomial in the y variables alone with no constant term.

Let G act on $V \times V$ by the trivial action on the first copy of V, and the given action on the second copy of V. The Reynolds operator for this action is such that

$$\mathcal{R} : \bigwedge(\mathbf{x}) \otimes \bigwedge(\mathbf{y}) \to (\bigwedge(\mathbf{x}) \otimes \bigwedge(\mathbf{y}))^G = \bigwedge(\mathbf{x}) \otimes \bigwedge(\mathbf{y})^G.$$

In particular, $\mathcal{R}$ is a $\bigwedge(\mathbf{x})$-module homomorphism by property (ii) of the Reynolds operator. Notice that $\mathcal{R}|_{\bigwedge(\mathbf{x})} = \mathrm{Id}_{\bigwedge(\mathbf{x})}$, whilst $\mathcal{R}|_{\bigwedge(\mathbf{y})} : \bigwedge(\mathbf{y}) \to \bigwedge(\mathbf{y})^G$ is the usual Reynolds operator on $\bigwedge(\mathbf{y})$. Apply $\mathcal{R}$ to $f(\mathbf{x})$. We get that

$$\mathcal{R}(f(\mathbf{x})) = f(\mathbf{x}) = \mathcal{R}(p(\mathbf{x}, \mathbf{y})) + \sum \mathcal{R}(c_i(\mathbf{x})) \mathcal{R}(g_i(\mathbf{y})) = \mathcal{R}(p(\mathbf{x}, \mathbf{y})) + \sum c_i(\mathbf{x}) \mathcal{R}(g_i(\mathbf{y})).$$

Notice that I_G is G-stable, so we have that $\mathcal{R}(p(\mathbf{x}, \mathbf{y})) \in I_G$, by the property (i) of the Reynolds operator.

Consider the ring map

$$\delta : \bigwedge(\mathbf{x}, \mathbf{y}) \to \bigwedge(\mathbf{x}),$$

given by $h(\mathbf{x},\mathbf{y}) \mapsto h(\mathbf{x},\mathbf{x})$. Notice that this is the just the substitution $\phi_{\mathbf{1}}$ given by $\mathbf{y} = A(\mathbf{1}) = \mathbf{1x}$. In particular, δ acts as the identity on skew polynomials in the subring $\bigwedge(x_1,\ldots,x_n)$. When we apply δ to $f(\mathbf{x})$ we get

$$\delta(f(\mathbf{x})) = f(\mathbf{x}) = \delta\mathcal{R}(p(\mathbf{x},\mathbf{y})) + \sum \delta(c_i(\mathbf{x}))\delta\mathcal{R}(g_i(\mathbf{y})) = \delta\mathcal{R}(p(\mathbf{x},\mathbf{y})) + \sum c_i(\mathbf{x})\mathcal{R}(g_i(\mathbf{x})).$$

However, notice that any $h \in I_G$ must "vanish" on the subspace $V_{\mathbf{1}}$ associated to the identity of G, meaning that $\phi_{\mathbf{1}}(h) = 0$. Thus $h(\mathbf{x},\mathbf{x}) = 0$ for any $h \in I_G$. Hence, I_G is in the kernel of δ. In particular, $\delta\mathcal{R}(p(\mathbf{x},\mathbf{y})) = 0$. Applying this observation to our expression for $f(\mathbf{x})$, we conclude that:

$$f(\mathbf{x}) = \sum c_i(\mathbf{x})\mathcal{R}(g_i(\mathbf{x})).$$

As for any g we have that $\mathcal{R}(g)$ lies in the Hilbert ideal J_G, the above expression establishes that $f(\mathbf{x}) \in J_G$, as required. $\square$

6.4 Noether's degree bound over the exterior algebra

We conclude with our theorem providing a bound on the degree of the invariant skew polynomials in the exterior algebra. This is one of the main results of this dissertation.

Theorem VI.13 (Noether's bound for the exterior algebra)**.** *Let $\mathbb{K}$ be a field of characteristic zero and G a finite group acting on the finite dimensional vector space V. Then $\bigwedge(V^*)^G$ is generated in degree at most $|G|$.*

Proof. By subspace theorem for the exterior algebra Theorem VI.2, we have that the ideal $J_G = \mathbb{I}'(\mathcal{A}_G)$ is generated in degree at most $|G|$. Using Theorem VI.12, we get that the Hilbert ideal I_G is given by

$$(J_G + (y_1,\ldots,y_n)) \cap \bigwedge(x_1,\ldots,x_n).$$

This means that generators of I_G are obtained from generators of J_G by setting the y-variables equal to 0. Since J_G is generated in degree $\leq |G|$, so is I_G. Let $\{f_1, \ldots, f_r\}$ be a set of generators for J_G, then by Hilbert invariant theorem (Theorem VI.10) we have that $\bigwedge(\mathbf{x})^G = \langle \mathcal{R}(f_1), \ldots, \mathcal{R}(f_r) \rangle$. Notice that the Reynolds operator does not increase the degree of a skew-polynomial. Therefore, $\bigwedge(\mathbf{x})^G$ is generated in degree at most $|G|$, as required. $\square$

We can restate the above theorem by saying that in characteristic zero for G a finite group acting on $\bigwedge(V^*)$, we have that $\beta_V(G) \leq |G|$ for all V and G, so we have the Noether's degree bound $\beta(G) \leq |G|$ holds in this setting.

In general, Noether's degree bound does not hold in the non-commutative setting. Consider for example the ring

$$F = \frac{\mathbb{K}\langle x_1, \ldots, x_n \rangle}{(x_i x_j + x_j x_i, 1 \leq i < j \leq n)},$$

the skew polynomial ring in $x_1, \ldots, x_n$. Notice that the exterior algebra can be obtained as a quotient of F, namely $E = \frac{F}{(x_i^2, 1 \leq i \leq n)}$. Recent work of Kirkman, Kuzmanovich, and Zhang [19] shows that Noether's degree bound does not hold for a finite group G acting on F. In particular, they show that the group $\mathbb{Z}/(2)$ has a minimal invariant of degree 3.

Example VI.14 (Example 3.1 in [19])**.** Consider the permutation representation of $G = \mathbb{Z}/(2)$ on $\mathbb{K}\langle x, y \rangle/(xy + yx)$. This means that the generator g of G acts by swapping the variables x and y. We have a linear invariant $f_1 = x + y$. However, the quadratic invariant $f_2 = x^2 + y^2$ is not a minimal invariant as

$$f_1^2 = (x + y)^2 = x^2 + xy + yx + y^2 = x^2 + y^2 = f_2,$$

by the defining equations of the (-1)-skew polynomial ring F. The next minimal invariant is the cubic invariant $f_3 = x^3 + y^3$. One can show that the set $\{f_1, f_3\}$ is a set of minimal generating invariants. Thus, for this representation V of G we have that $\beta_V(G) = 3 > |G| = 2$.

Therefore, the exterior algebra case is special among non-commutative algebras. On the other hand, it also has features different from the ones of the symmetric algebra. In characteristic zero, given a finite group G acting on an n-dimensional vector space V, the associated action on the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ is such that $\beta_V(G) \leq \beta_{V_{\mathrm{reg}}}(G)$, where V_{reg} is the regular representation of G [33]. One says that the degree bound is achieved by the regular representation. However, this behavior does not carry to $\bigwedge(x_1, \ldots, x_n)$.

Example VI.15. Consider the group $G = \mathbb{Z}/(2)$ and let g generate G. Consider the action of g on $\mathbb{K}[x, y]$ given by $g \cdot x = x$ and $g \cdot y = -y$. The polynomial ring $\mathbb{K}[x, y]$ with this action is equivalent to the regular representation of G. In fact, we have a degree two invariant, y^2, which achieves Noether's degree bound so that $\beta(G) = 2$.

Consider now the same action on the variables x, y, but in the exterior algebra $\bigwedge(x, y)$. We only have linear invariants, specifically the non-zero constant multiples of x. If the regular representation did achieve the degree bound, we would have that $\beta(G) \leq 1$. However, consider now two copies of the same representation. The action of g on the variables x_1, y_1, x_2, y_2 is given by $g \cdot x_i = x_i$ and $g \cdot y_i = -y_i$, for $i = 1, 2$. In $\bigwedge(x_1, x_2, y_1, y_2)$ we now have a quadratic invariant: $y_1 \wedge y_2$. Thus, Noether's degree bound is achieved and we can conclude that $\beta(G) = 2$.

Moreover, Weyl's polarization theorem does not hold over $\bigwedge(x_1, \ldots, x_n)$ as the following example show. Specifically, if $\dim(V) = n$, it is not true that the highest degree invariants do appear in V^n, the representation given by the direct sum of n copies of the representation V.

Example VI.16. Let V be the one-dimensional representation of $G = \mathbb{Z}/(2) = \langle g \rangle$ given by $g \cdot x = -x$. Then the only invariants in $\bigwedge(x)$ are scalars. On the other hand, consider the representation V^2, where $g \cdot x_i = x_i$, for $i = 1, 2$. We do have maximal degree invariants now: we have the quadratic invariant $x_1 \wedge x_2$.

Given that Noether's degree bound does hold in the exterior algebra $\bigwedge(x_1, \ldots, x_n)$, we were brought to the following conjecture.

Conjecture VI.17. *Assume the base field has characteristic zero and let G be a finite group acting on $\bigwedge(V)$. We have that the highest degree invariants appear in $\bigwedge(V^{|G|})$.*

Therefore, we conjecture that we only need as many copies of V as there are elements in the group G to be guaranteed to observe highest degree invariants. Generalizing the example above, one can see that the conjecture holds for cyclic groups.

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