High Dimensional Phenomenon in Convex Geometry and Spectral Theory of Random Graphs

by

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ABSTRACT

In this thesis, we study high dimensional phenomena arising in convexity and probablistic combinatorics. The main object of the first part is high dimensional convex bodies. We study random almost spherical sections of a convex body, which is related to Dvoretzky's theorem. We also investigate the mass distribution in a convex body with respect to its maximum volume ellipsoid. Furthemore, we study the approximation of convex bodies by polytopes with few facets. We also construct a special class of convex bodies which we use to define affine surface area.

The second part of the thesis is devoted to the study of nodal domains of Erdös Rényi graphs. An Erdös Rényi graph $G(n, p)$ is a random graph with n vertices where any two vertices are connected by an edge with probability p independently of other edges. Consider an eigenvector of the adjacency matrix of such random graph. A nodal domain corresponding to this eigenvector is a connected component of the set of vertices where the vector has a constant sign. It was proved by Dekel et. al. that with high probability, there are exactly two nodal domains for every non-leading eigenvector. We show that the sizes of these two nodal domains are almost exactly equal to each other.

CHAPTER 1

Introduction

1.1 Geometric Functional Analysis

A convex body $K \subseteq \mathbb{R}^n$ is a convex, compact subset with non-empty interior. The study of convex bodies can be traced back to Euclid. In the past century, functional analysis provided a new perspective on the study of convex bodies. The connection between the two subjects is simple: If we embed a finite-dimensional normed space X to $\mathbb{R}^n$, its unit ball becomes an origin-symmetric convex body in $\mathbb{R}^n$. Moreover, for any origin-symmetric convex body $K \subseteq \mathbb{R}^n$, the gauge function $\|x\|_K := \inf\{r > 0 \,:\, x \in rK\}$ defines a norm. The most natural examples of such convex bodies are the Euclidean ball B_2^n, the unit cube which is the l_∞-ball B_∞^n, and, correspondingly, the l_1-ball B_1^n. In functional analysis, Banach-Mazur distance for two n-dimensional normed spaces $X,\, Y$ is defined as

$$d_{BM}(X,\, Y) = \inf\left\{\|T\| \left\|T^{-1}\right\| : T\in GL(X,\, Y)\right\},$$

where $GL(X,\, Y)$ is the collection of linear isomorphisms from X to Y. Notice that d_{BM} is not a metric but its $\log$ is. However, it is more convenient to use d_{BM} in the study of linear operators. Suppose we embedded both X and Y to $\mathbb{R}^n$ (we can choose any isomorphic embedding) and let B_X and B_Y be the corresponding unit balls of X and Y respectively. Then, the Banach Mazur distance of X and Y can be expressed in terms of B_X and B_Y:

$$d_{BM}(B_X,\, B_Y) := \inf\{R \geq 1 \,:\, \exists T \in GL_n(\mathbb{R}) \text{ such that } TB_Y \subseteq B_X \subseteq RTB_Y\},$$

where $GL_n(\mathbb{R})$ is the general linear group of degree n. This definition is geometric: We try to rotate, stretch, and compress B_Y so that it lies in B_X. Meanwhile, we want to scale the transformed B_Y by the smallest possible factor R so that it contains B_X. For example, for Banach Mazur distance between the cube B_∞^n and B_2^n is $\sqrt{n}$. The ratio of the outer and inner radius of B_∞^n is exactly $\sqrt{n}$. Also, for the crosspolytope B_1^n, we also have $d_{BM}\left(B_1^n,\, B_2^n\right)=\sqrt{n}$. Influenced by this, we often consider a convex body K and its affine linear images as one object.

High dimensional measure concentration also plays an important role. To give a brief idea of what is this high dimensional phenomenon means, consider a unit sphere S^{n-1}. If we intersect S^{n-1} with a centered strip with width about $\frac{1}{\sqrt{n}}$, the remaining covers more than 99% of the volume of ball. In other words, a $\frac{C}{\sqrt{n}}$ neighbourhood of the equator basically covers almost everything volumetrically. This contradicts our intuition from small-dimensional world, but it demonstrates the measure concentration phenomenon. Relying on measure concentration on sphere, in 1971, Milman found a quantitative proof of Dvoretzky theorem. Roughly speaking, for any origin-symmetric convex body $K \subseteq \mathbb{R}^n$, there exists a large dimension k such that a random k-dimensional subspace F satisfies $d_{BM}\left(K\cap F,\, B_2^k\right)\leq 1+\varepsilon$ with high probability. Precisely, the theorem is stated as follows:

Theorem 1.1. *[24, Theorem 5.2.10] Let $K \subseteq \mathbb{R}^n$ be an origin-symmetric convex body. Let M be the average of $\|\theta\|_K$ over S^{n-1} (with respect to the normalized Haar measure on S^{n-1}) and $b=\max_{\theta\in S^{n-1}}\|\theta\|_K$. For any $\varepsilon\in(0,\,1)$, if $k\leq C_1\varepsilon^2\log^{-1}(2/\varepsilon)\,n\left(\frac{M}{b}\right)^2$, then*

$$\nu_{n,k}\left(\left\{F\in Gr_{n,k}\,:\,(1-\varepsilon)\,M<\|\cdot\|_{K\cap F}<(1+\varepsilon)\,M\right\}\right)\geq 1-\exp\left(C_2\varepsilon^2 k\right),$$

where $\nu_{n,k}$ is the normalized Haar measure on Grassmannian $Gr_{n,k}$. For some convex bodies, k can be of order n. For instance, an n-dimensional crosspolytope is one of such examples. Also, C, C_1, C_2 etc, shall denote absolute constants that may change from line to line.

Among all affine linear images of K, there are two classical choices: isotropic and John's position (named after Fritz John). Isotropic position refers to the case where K is volumetrically balanced. An isotropic convex body K is a convex body of volume 1 with the following property: Suppose X is a random vector uniformly distributed on K, then, $\mathbb{E}X=0$ and for any $\alpha,\,\beta\in S^{n-1}$, $\mathbb{E}\langle X,\,\alpha\rangle^2=\mathbb{E}\langle X,\,\beta\rangle^2$ (the second moments in any direction are the same).

John's position focuses on the geometric shapes of convex bodies. A convex body K is in John's position if K contains B_2^n and B_2^n has the maximum volume among all ellipsoids contained in K. In general, we will let $E_K\subseteq K$ be the maximum volume ellipsoid (John ellipsoid), so that it has the maximum volume among all ellipsoids that are contained in K. (For the existence and uniqueness of the maximum volume ellipsoid E_K, we refer to [24, Proposition 2.1.6]) A theorem of F. John shows that K is isotropic in a different sense if it is in John's position:

Theorem 1.2. *[24, Theorem 2.1.10] For a convex body $K \subseteq \mathbb{R}^n$ we have $E_K = B_2^n$ if and only if there exists a finite subset in the intersection of boundaries of K and B_2^n, $\{u_i\}_{i=1}^m \subseteq \partial K \cap S^{n-1}$, and positive numbers $\{c_i\}_{i=1}^m$ such that $\sum_{i=1}^m c_i u_i = 0$ and $\sum_{i=1}^m c_i u_i u_i^\top = I_n$.*

To see why it is isotropic, consider the random vector X defined on $\{u_i\}_{i=1}^m$ with $\mathbb{P}(X = u_i) = \frac{c_i}{\sum_{i=1}^m c_i}$. Then, X satisfies $\mathbb{E}X = 0$ and $\mathbb{E}\langle X, \alpha\rangle^2 = \mathbb{E}\langle X, \beta\rangle^2$ for any $\alpha, \beta \in S^{n-1}$. As an application of John's theorem, we are able to show that for any convex body $K \subseteq \mathbb{R}^n$, we have

$$d_{BM}(B_2^n, K) \leq \sqrt{n}$$

(see [24, Theorem 2.1.3]). In particular, this bound is sharp. For instance, if we pick K to be B_∞^n or B_1^n, then the equality holds.

1.1.1 Overview of Results in Asymptotic Geometric Analysis

Here we describe the results of this thesis in convex geometry.

The Upper Bound in Dvoretzky Theorem and Milman–Schechtman Theorem After the Milman's Dvoretzky theorem is proved, with $\varepsilon > 0$ fixed, one ask whether the cricitical dimension $k \simeq \left(\frac{M}{b}\right)^2 n$ is optimal. This is answered by Milman and Schechtman in 1997. For a fixed $\varepsilon > 0$, they define the Dvoretzky's dimension $k(K)$ for a convex body K to be the largest integer k such that

$$\nu_{n,k}\left(\{F \in Gr_{n,k} : (1-\varepsilon) M < \|\cdot\|_{K\cap F} < (1+\varepsilon) M\}\right) > p_{n,k} := \frac{n}{n+k}.$$

Theorem 1.3. *(Milman–Schechtman [24, Theorem 5.3.4]). Fix $\varepsilon > 0$. There exist constants $C_1, C_2 > 0$ such that for any an origin-symmetric convex body $K \subseteq \mathbb{R}^n$, we have*

$$C_1 n(M/b)^2 \leq k(K) \leq C_2 n(M/b)^2$$

whenever $\frac{M}{b} > c\sqrt{\frac{\log n}{n}}$.

We improve Milman-Schechtman's theorem by removing the condition $\frac{M}{b} > c\sqrt{\frac{\log n}{n}}$. In our case we define $p_{n,k} = \frac{1}{2}$ instead of $p_{n,k} = \frac{n}{n+k}$, which is more natural. Therefore, we have $k(K) \simeq n(M/b)^2$ for any origin-symmetric convex bodies with no restriction. Let

$$f(k) := \nu_{n,k}\left(\{F \in Gr_{n,k} : (1-\varepsilon) M < \|\cdot\|_{K\cap F} < (1+\varepsilon) M\}\right).$$

More precisely, we show:

Theorem. *(Theorem 2.5 in Chapter 2) There exists a constant $c_1 > 0$ such that the following holds: For any origin-symmetric convex body $K \subseteq \mathbb{R}^n$, if $k \geq c_1 \left(\frac{M}{b}\right)^2 n$, then*

$$f(k) \leq \exp\left(-\frac{c_0}{4}k\right) < \frac{1}{2}.$$

Together with Millman's result (Theorem 1.1), we realize that there is a threshold effect happenning at the level $k \simeq \left(\frac{m}{b}\right)^2 n$ for the function $f(k)$. This result does not put any assumption on the convex body, and thus proves that the lower bound of Milman is actually not a bound, but a formula which is precise up to a constant. Finding such formulas is rare in Geometric Functional Analysis.

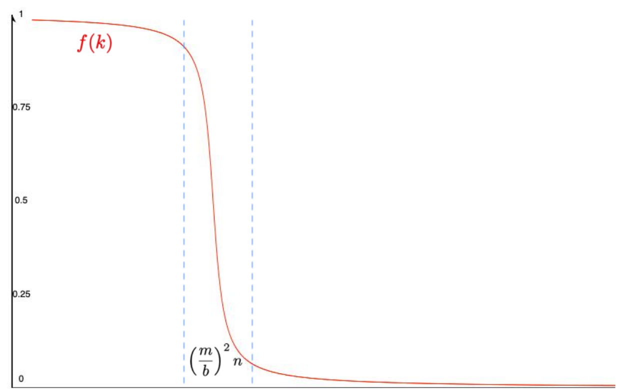


Figure 1.1.1: Threshold Phenomenon of $f(k)$

This result is a joint work with F. Wei. It is published in [35]. We present it in Chapter 2.

Does the maximum volume ellipsoid contain the barycenter? The title is a problem asked by S. Vempala. Does the maximum volume ellipsoid contain the barycenter of a convex body? If not, how much does one need to scale the maximum volume ellipsoid so that it contains the barycenter? Roughly speaking, we realize that the maximum volume ellipsoid does not contain significant volumetric information about the convex body. It is shown that one can find a convex body K with the following property: inflating the maximum volume ellipsoid by a factor $(1 - o(1))\, n$, it does not contain the barycenter of K. In order to interpret this, it is worth mentioning that the maximum volume ellipsoid contains K, if it is inflated by a factor n.

The importance of this question stems from its relation to the efficiency of algorithms for convex bodies. The efficiency of many such algorithms depends on the "roundness" of the body. This can be measured in two ways:

1. the traditional way, as the ratio of the radii of the circumscribed to the inscribed ball;
2. as the ratio of the radii of the smallest ball that contains the most points (say 1/2 of the volume) to the inscribed ball.

For instance, the complexity of sampling algorithms grows quadratically with the latter ratio. Thus, a common pre-processing step is to find a good rounding–in other words, find an ellipsoid for which this ratio is reasonably small and then map it to the unit ball using an affine transformation. This can be done in a randomized polynomial in n time algorithm by estimating the inertia ellipsoid (defined by the covariance matrix of the uniform random point from K), wherein the complexity depends logarithmically on the initial ratio of the radii, but as a large degree polynomial in the dimension. The other possible candidate is the maximum volume ellipsoid. This ellipsoid is difficult to construct in general, but for explicit polytopes, a simple iterative algorithm identifies the inscribed ellipsoid of the maximum volume quite efficiently. The polytopes here are formulated by linear constraints. For instance, here is a polytope defined by linear inequalities:

$$P := \{x \in \mathbb{R}^n \,:\, \forall i \in [m]\ \langle u_i,\, x\rangle \leq a_i\}$$

where $\{u_i\}_{i=1}^m \subseteq \mathbb{R}^n$ and $\{a_i\}_{i=1}^m \subseteq \mathbb{R}$. In particular, each facet of the polytope comes from a linear constraint. This algorithm was developed by L. G. Khachiyan [39]. Recently, Y. Lee and A. Sidford have provided a faster algorithm [42]. In contrast to the inertia ellipsoid, whose construction requires sampling, the John ellipsoid is constructed deterministically. The John ellipsoid can be used to reduce the ratio (1) but it can be as large as n, which is the dimension of the body. On the other hand, the inertia ellipsoid yields the bound $O(\sqrt{n})$ for the ratio (2). Indeed, the counterexample K above implies that the ratio (2) for maximum volume ellipsoid can still be of order n. So the next question will be: if we restrict convex bodies to polytopes with few facets, can we provide $O\left(\sqrt{n}\right)$ for ratio (2) for maximum volume ellipsoid? Indeed, if ratio (2) can be of order $O\left(\sqrt{n}\right)$ for the maximum volume ellpsoid, then the deterministic algorithm we mentioned before has a faster compute volume of polytope. It turns out we can construct a polytope P with $O\left(n^2\right)$ facets so that its inflated maximum volume ellipsoid does not contain the barycenter. In this case, the scaled factor is slightly weaker: it is of order $\frac{n}{\log n}$ instead of $\left(1 - o\left(1\right)\right) n$. Nevertheless, this example shows that the ratio (2) for the maximum volume ellipsoid can be of order $\frac{n}{\log n}$ even if we restrict ourselves to polytopes with $O(n^2)$ facets. This result was published in [32]. We present it in Chapter 3.

Approximation of convex bodies by polytopes and John's position Here we consider a classical question: how well can one approximates a convex body by polytopes with few facets or vertices? Certainly there are several ways to quantify the approximation. Here we consider the Banach-Mazur distance we described above. In the non-symmetric case, the Banach-Mazur distance $d_{BM}(K, L)$ is defined by

$$d_{BM}(K, L) := \inf \left\{ \begin{array}{c} r \geq 1 \,:\, \exists T \in GL_n(\mathbb{R}) \text{ and } x, y \in \mathbb{R}^n \\ \text{such that } K - x \subset T(L - y) \subset r(K - x) \end{array} \right\}. \tag{1.1}$$

The current results have a big gap between approximating origin-symmetric convex bodies and non-symmetric convex bodies. To see that, we consider approximating convex bodies in $\mathbb{R}^n$ by polytopes with $O(n)$ facets. It is known that one can approximate an origin-symmetric convex body with a Banach-Mazur distance $O(\sqrt{n})$. Non-symmetric convex bodies are only known to have distance at most $O(n)$. In particular, the approximating polytope P is constructed based on John's Theorem 1.13. Specifically, let $\{u_i\}_{i=1}^m$ be the contact points that appeared in John's theorem 1.13, the polytope P is defined as

$$P := \{x \in \mathbb{R}^n \,:\, \forall u_i \in J \, \langle u_i, x \rangle \leq 1\}$$

where J is a subset of $\{u_i\}_{i=1}^m$ with cardinality of order $O(n)$. (The choice of J is based on a result of [**?**].)

There are more results in this problem if more facets are allowed. However, the gap between the symmetric case and non-symmetric case remains. We will discuss the progress more in Chapter 4.

In the non-symmetric case, we are facing a new obstacle: it is not clear what is a good choice for the center of scaling. In other words, it is not clear which x we should pick in (1.1). Since John's theorem works very well in the symmetric case, one of the natrual candidate will be the center of maximum volume ellpisoid as suggested from the example P we discussed above. Thus, we study the following problem:

Problem. (Problem 4.1 Chapter (4)) Let $R = o(n)$, and $K \subset \mathbb{R}^n$ be a convex body is in John's position. Is there a polytope P with a polynomial number of facets in n, such that

$$K \subset P \subset RK \ ?$$

We have an unexpected negative answer:

Theorem. *(Theorem 4.2 in Chapter (4)) For a sufficiently large n and for any $c_0\sqrt{n} \leq R \leq c_1 n$, there exists a convex body $K \subset \mathbb{R}^n$ whose John's ellipsoid is centered at the origin, and such that any polytope P satisfying*

$$K \subset P \subset RK,$$

has at least $\exp\left(C\log\left(\frac{R^2}{n}\right)\frac{n}{R^2}n\right)$ *facets, where* $c_0,\ c_1,\ C > 0$ *are some universal constants.*

To interpret this theorem, if $R = o\left(n\right)$, then $\exp\left(C\log\left(\frac{R^2}{n}\right)\frac{n}{R^2}n\right)$ has a faster growth than polynomial of n. Furthermore, if $R = O\left(\sqrt{n}\right)$, then the number of facets can be exponential in n. Therefore, in the non-symmetric case, the maximum volume ellpsoid does not yield anything useful in contrast to the symmetric case. This result was published in [31]. We present it in Chapter 4.

From polytope to measure generated set, and floating bodies Suppose a polytope P has m vertices $x_1, \ldots, x_m \in \mathbb{R}^n$, then we can describe P as the following set

$$P = \left\{\sum_{i=1}^{m} \lambda_i x_i \ :\ \lambda_i \geq 0 \,\text{and}\, \sum_{i=1}^{m} \lambda_i = 1\right\}.$$

One may view $\sum_{i=1}^{m} \lambda_i$ and $\sum_{i=1}^{m} \lambda_i x_i$ as integrals. Suppose μ is a Borel measure on $\mathbb{R}^n$. B. Slomka defines the following set

$$\mathrm{M}\left(\mu\right) := \left\{\int_{\mathbb{R}^n} f\left(x\right) x \,\mathrm{d}\mu\left(x\right) \ :\ 0 \leq f\left(x\right) \leq 1 \,\text{and}\, \int_{\mathbb{R}^n} f\left(x\right)\,\mathrm{d}\mu\left(x\right) = 1\right\}.$$

The reason we introduced the extra condition $f\left(x\right) \leq 1$ is because otherwise $\mathrm{M}\left(\mu\right) = \mathrm{conv}\left(\mathrm{supp}\left(\mu\right)\right)$. If $\mu = \sum_{i=1}^{m} \delta_{x_i}$ where δ_x is the Dirac measure on x, then, $\mathrm{M}\left(\mu\right) = P$. Also, one can verify that $\mathrm{M}\left(\mu\right)$ is a convex body if μ is non-degenerate ($\mathrm{supp}\left(\mu\right) \not\subseteq H$ for any affine subspace H of $\mathbb{R}^n$) with bounded support and $\mu(\mathbb{R}^n) > 1$. In this way, we generalize polytopes to measure-generating sets. We explore various properties of this new type of convex sets and show that a special collection of them is very close to floating bodies.

Floating bodies are introduced independently in [**?**] and [67]. For a convex body K, if we cut off δ volume in every direction, the remaining is called a floating body, K_δ.

Precisely,

$$K_\delta = \cap_{|K \cap H^-| = \delta} H^+,$$

where H is a hyperplane and H^+, H^- are the corresponding closed half-spaces. One of the most stunning results about the floating body is its relation to affine surface area. Affine surface area is an analogue of surface area that is $SL_n(\mathbb{R})$ and translation invariant. For a smooth convex body K, the affine surface area is defined by

$$as(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \, \mathrm{d}\mu_K,$$

where μ_K is the surface area measure on ∂K and κ_K is the (generalized) Gaussian curvature.

In 1990, Schï¿œtt and Werner [67] showed that the affine surface area arises as a limit of the volume difference of the convex body and its floating body. In particular, this was the first way to define the affine surface area for convex bodies with a general (non-smooth) boundary.

In short, we have

$$\lim_{\delta \searrow 0} \frac{|K| - |K_\delta|}{c_n \delta^{2/(n+1)}} = as(K).$$

Consider μ uniform measure on convex body K. More specifically, let $\mathrm{d}\mu_\delta(x) = \frac{1}{\delta}\mathbf{1}_K \, \mathrm{d}m$ ($\mathrm{d}m$ is the Lebesgue measure) where $\mathbf{1}_K$ is the indicator function of K. Then, $\mathrm{M}(\mu_\delta)$ behaves similarly to the floating body K_δ. Also, the same type of result on affine surface area (and l_p version of affine surface area) can be deduced as well. Precisely, we have

Theorem 1.4. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $\phi : K \to (0, \infty)$ be a continuous function. We define*

$$\mathrm{M}_\delta(K, \phi) := \mathrm{M}\left(\frac{\phi(x)}{\delta}\mathbf{1}_K(x) \, \mathrm{d}x\right).$$

Then,

$$\lim_{\delta \searrow 0} \frac{|K| - |\mathrm{M}_\delta(K, \phi)|}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \, \mathrm{d}\mu_K(x), \tag{1.2}$$

where $c_n = 2\frac{n+1}{n+3}\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$. *In particular, if* $\phi = 1$, *then,* $\mathrm{M}_\delta(K, \phi) = \mathrm{M}_\delta(K)$ *and* $\int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \, \mathrm{d}\mu_K(x) = as(K)$.

We also show the smoothness of boundary of $\mathrm{M}_\delta(K)$ and its relation to K_δ. This is a joint work with B. Slomka and E. Werner. The contents are from [34] and partially from [33]. This part will be presented in Chapter 5.

1.2 Nodal Domains for graphs

Nodal domains of the eigenfunctions of the Laplacian on smooth manifolds have been studied for more than a century. For the detail, we refer the readers to the book [78]. If $f : M \to \mathbb{R}$ is such an eigenfunction on a manifold M, then the nodal domain is a connected component of the set where the function f has a constant sign. Precisely, for an eigenfunction f, a nodal domain is a maximal connected subset such that f is strictly positive (or strictly negative) on this subset.

Let's consider the simplest example: the Laplace operator on an open interval $[0, 1] \subseteq \mathbb{R}$ with Dirichlet boundary condition. If we list the eigenvalues in an increasing order, then the nth eigenfunction is $\sin(n\pi x)$ up to a scaling.

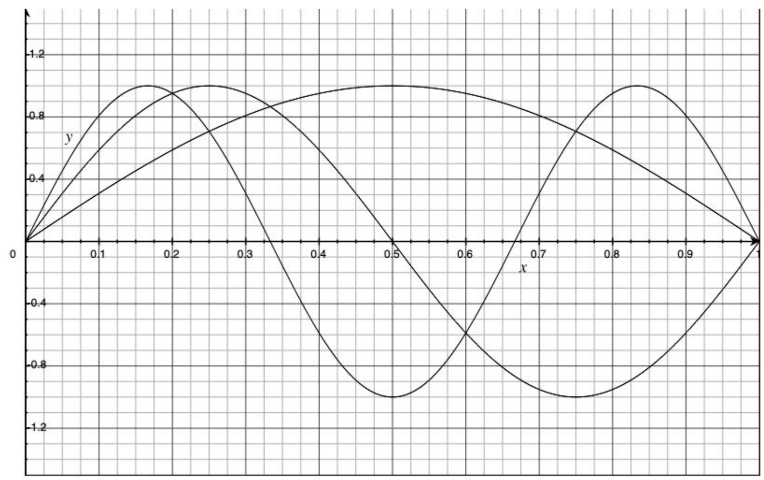


Figure 1.2.1: Nodal Domains

Thus, there are exactly n nodal domains for nth eigenfunction. Indeed, this is exactly the maximal number of possible nodal domains. The following celebrated result was discovred by Courant in 1923: there are at most $n + r$ nodal domains corresponding to nth eigenfunctions and r is the multiplicity of that eigenavlue. For the fruitful results in the study of nodal domains, we refer to the book of Zelditch[78].

Since Laplace operator also exists for graphs, there is a discrete version of nodal domains for graph. In the graph case, we consider the adjacency matrix A: the ijth entry equals 1 if there is an edge connecting ith and jth vertices. If there is no edge connecting them, then the corresponding entry equals 0. For an eigenvector u of A, a strong nodal domain is defined as the maxmal connected subset such that u is strictly positive (or strictly negative) on this subset. The definition of weak nodal domains will be the same as strong nodal domain but replacing the condition strictly positive(or strictly negative) by non-negative(or non-positive).

In 2008, Dekel, Lee, and Linial[**?**] discovered that nodal domains for an Erdös-Rï¿œnyi $G(n,\, p)$ graph with a fixed p behave different from the eigenfunctions of the Laplacian on a manifold. Namely, the number of nodal domains for any non-leading eigenvector of $G(n,\, p)$ is bounded by a constant depending only in p. Later, their result was improved by Arora and Bhaskara, who shows that there are exactly 2 nodal domains for non-leading eigenvector (non-leading means the corresponding eigenvalue is not the largest one). In [**?**] Dekel, Lee, and Linial pioneered the study of the nodal domains for graphs. This study was motivated by the usefulness of the eigenvectors of graphs in a number of partitioning and clustering algorithms, see [**?**] and the references therein.

Here, we establish another natural property of nodal domains. Namely, we will show that with high probability, the nodal domains are balanced, i.e. each one of them contains close to $n/2$ vertices with high probability. Unlike the previous ones, this property does not follow from the combination of the no-gaps and the ℓ_∞ delocalization. Indeed, the vector $u \in S^{n-1}$ with $n/3$ coordinates equal to $\sqrt{\frac{2}{n}}$ and the rest $2n/3$ coordinates equal to $-\frac{1}{\sqrt{2n}}$ satisfies both properties. Moreover, for such vector, $\sum_{j=1}^{n} u(j) = 0$, so it is orthogonal to the vector $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$ which is close to the leading eigenvector with high probability.

We prove that the nodal domains are roughly of the same size. Here we break the eigenvalues into two types: The bulk eigenvalues refer to intermediate eigenvalues and the edge eigenvalues are those close to the extreme of the spectrum. The precise definitions appear in formulations of the theorems as well as in Chapter 6. Here we state our theorem in bulk and edge cases:

Theorem 1.5. *(Bulk case) There is $c \in (0,\, 1)$ such that the following holds. Let $G(n,\, p)$ be an Erdös-Rï¿œnyi Graph with $p \in \left[n^{-c},\, \frac{1}{2}\right]$. Fix $\varepsilon,\, \kappa \in (0,\, 1)$ and $\rho > 1$. Suppose n is sufficiently large. Let u_α be an eigenvector of $G(n,\, p)$ for $\alpha \in [\kappa n, n - \kappa n]$. Then there exists $\eta = \eta(\varepsilon,\, \kappa) > 0$ such that, for a sufficiently large n,*

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + \varepsilon\right) n\right) \leq n^{-\eta}$$

where P and N are the two nodal domains corresponding to u_α.

Theorem 1.6. *(Edge case) Let $G(n,\, p)$ be an Erdös-Rï¿œnyi Graph with $p \in (0,\, 1)$. Fix a sufficiently large $\rho > 0$. Suppose n is sufficiently large. Let u_α be a non-leading eigenvector of $G(n,\, p)$ with $\min\{\alpha,\, n - \alpha\} \leq (\log n)^{\rho \log\log n}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$*

such that

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + n^{-\frac{1}{6}+\varepsilon}\right) n\right) \leq n^{-\delta}$$

where P and N are the two nodal domains corresponding to u_α.

This is a joint work with M. Rudelson. We present this result in Chapter 6.

1.3 Notations and Preliminaries

We denote $[m] := \{1, 2, 3, \ldots, m\}$ for $m \in \mathbb{N}$. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\{e_i\}_{i=1}^n$ denotes the standard orthonormal basis. For $x,\, y \in \mathbb{R}^n$,

- $\langle x,\, y\rangle$ denotes the standard inner product,
- $|x|$ denotes the Euclidean norm, and
- x_i denotes the ith component of x.

For a Borel measurable subset $K \subseteq \mathbb{R}^n$, we define $\mathbf{1}_K$ to be the indicator function of K. Furthermore, we set $|K|$ be the volume of K. (The Lebesgue integral of $\mathbf{1}_K$.)

By B_2^n we denote the unit Euclidean ball. Let $GL_n(\mathbb{R})$ denote the group of invertible linear transformations.

Convex Geometry

We begin with the definition of convexity and convex bodies.

Definition 1.7. A subset $A \subseteq \mathbb{R}^n$ is called convex if for any points $x,\, y \in A$ we have $\lambda x + (1-\lambda)\, y \in A$ for $\lambda \in [0,\, 1]$. A convex body $K \subseteq \mathbb{R}^n$ is a convex, compact subset with non-empty interior.

There are several functions corresponding to a convex body:

Definition 1.8. (Radial funciton, support function and gauge function) For a convex body $K \subseteq \mathbb{R}^n$ containing the origin, the radial function $\rho : \mathbb{R}^n \to \mathbb{R}_+$ is defined as

$$\rho(x) = \max\{t > 0\,,\, tx \in K\}.$$

The support function is

$$h_K(x) := \sup\{\langle x,\, y\rangle \,:\, y \in K\} \qquad \forall x \in \mathbb{R}^n$$

and the gauge function is

$$\|x\|_K := \inf\{r > 0 \,:\, x \in rK\}\,.$$

In the case K is origin-symmetric, then $\|x\|_K$ defiines a norm on $\mathbb{R}^n$. Notice that we have

$$\|x\|_K = \frac{1}{\rho(x)}.$$

Definition 1.9. For a convex body K, let x_K denote the barycenter of K. Specifically,

$$x_K := \frac{1}{|K|}\int_K x \, \mathrm{d}m.$$

If we embedded an n dimensional normed space X to $\mathbb{R}^n$, then its dual space X^* is also embedded to $\mathbb{R}^n$ automatically via the standard inner product in $\mathbb{R}^n$. Specifically, we can define the dual norm as

$$\forall x \in \mathbb{R}^n, \quad \|x\|_{X^*} := \max_{y \in B_X} \{\langle x, y\rangle\}.$$

This relation is naturally adapted to convex bodies.

Definition 1.10. For a convex body $K \subseteq \mathbb{R}^n$ containing 0, its dual K° is defined as

$$K^\circ := \{x \in \mathbb{R}^n \,:\, \forall y \in K, \, \langle x, y\rangle \leq 1\}.$$

Remark 1.11. There are several properties with respect to the $\circ$ operation:

1. K° is also a convex body containing the origin.
2. (Order reversing) $K \subseteq L$ implies $L^\circ \subseteq K^\circ$.
3. $(K^\circ)^\circ = K$.
4. $\left(\cap_{i=1}^d K_i\right)^\circ = \mathrm{conv}\left(\{K_1^\circ, \cdots, K_d^\circ\}\right)$ where $\mathrm{conv}\,(K, \, L)$ is the convex hull of K, L.

Definition 1.12. Suppose $K \subseteq \mathbb{R}^n$ is a convex body with smooth boundaries. For $x \in \partial K$, we define $N(x)$ to be the unique outer normal vector at $x \in \partial K$. The Gaussian curvature $\kappa(x)$ is defined as

$$\kappa(x) := \det\left(dN(x)\right).$$

John's Decomposition

Let $K \subset \mathbb{R}^n$ be a convex body in John's position. Recall a point $u \in \mathbb{R}^n$ is a contact point of K and B_2^n if $u \in \partial K \cap \partial B_2^n$. A classical theorem of F. John provides a decomposition of identity operator in terms of contact points.([24, Theorem 2.1.10])

Theorem 1.13. *Let K be a convex body in $\mathbb{R}^n$ that contains B_2^n. Then, K is in John's position if and only if there exist contact points $u_1, .., u_m$ and $c_1, .., c_m > 0$ such that*

1. $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$, and
2. $\sum_{i=1}^m c_i u_i = 0$.

Moreover, one may choose $m \leq \binom{n+1}{2} + 1$.

Measure Concentration

We will state a few standard results in measure concentration.

1.3.1 Concentration on the Sphere

Let σ_{n-1} denote the normalized Haar measure on S^{n-1}. We will introduce two concentration inequalities on sphere. The following inequality is the concentration inequality for Lipschitz functions on the sphere (see, e.g., [24, Theorem 3.2.2]):

Theorem 1.14. *Let $f : S^{n-1} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant b. Then, for every $t > 0$,*

$$\sigma_{n-1}(\{x \in S^{n-1} : |f(x) - \mathbb{E}(f)| \geq bt\}) \leq 4\exp(-C_4 t^2 n),$$

where $C_4 > 0$ is a universal constant.

The proposition below provides an upper bound for the measure of a spherical cap (see, e.g., [24, Remark 3.1.8]):

Proposition 1.15. *Let $A_t = \{\theta \in S^{n-1}, \theta_1 > t\}$, then $\sigma_{n-1}(A_t) \leq 2\exp(-C_3 t^2 n)$ where $C_3 > 0$ is a universal constant.*

1.3.2 Log-concave measures

Log-convave measures can be treated as a functional form of convex bodies. Before we move on to the detail, we begin with the definition.

Definition 1.16. A Borel probability measure μ on $\mathbb{R}^n$ is called log-concave if for all non-empty compact subsets A, B of $\mathbb{R}^n$ and all $0 < \lambda < 1$ we have

$$\mu\left((1-\lambda)A + \lambda B\right) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}.$$

A non-negative function f defined on $\mathbb{R}^n$ is log-concave if, for any $\lambda \in (0,1)$ and $x, y \in \mathbb{R}^n$, we have

$$f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}. \tag{1.3}$$

It is shown by Borell[**?**] that a non-degenerate ($\mathrm{supp}\,(\mu) \not\subset H$ for any $n-1$ dimensional affine subspace H of $\mathbb{R}^n$) Boral probability measure μ is log-concave if and only if there exists a log-concave function f such that $\mathrm{d}\mu = f\,\mathrm{d}m$. ($\mathrm{d}m$ is the usual Lebesgue measure on $\mathbb{R}^n$.) Thus, suppose that $K \subseteq \mathbb{R}^n$ is a convex body, then, $\mathbf{1_K} dx$ is a log-concave measure on $\mathbb{R}^n$. Many results in the theory of convex bodies can be extended to log-concave measures. In some cases, it is even more convenient to discuss in log-concave measure instead. For instance, the k dimeinsonal marginal of a convex body K is a log-concave measure.

We state Borell's theorem ([24, Theorem 1.5.7]) and its application on comparison of moments ([24, Theorem 3.5.11]):

Theorem 1.17. *Let $K \subset \mathbb{R}^n$ be a convex body with volume $|K| = 1$. Let U be a closed, convex and symmetric set such that $|K \cap U| = \delta > 1/2$. Then, for any $t > 1$, we have*

$$|K \cap (tU)^c| \leq \delta \left(\frac{1-\delta}{\delta}\right)^{\frac{t+1}{2}}.$$

Theorem 1.18. *Let μ be a non-degenerate log-concave probability measure on $\mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}$ is a seminorm, then, for any $q > p \geq 1$, we have*

$$(\mathbb{E}|f|^p)^{1/p} \leq (\mathbb{E}|f|^q)^{1/q} \leq C_5 \frac{q}{p} (\mathbb{E}|f|^p)^{1/p},$$

where $C_5 > 0$ is some universal constant.

In the end, we include one more theorem about log-concave probability measures (Corollary 1 in [41]):

Theorem 1.19. *For each $0 < b < 1$ there exists a constant C_b such that for every log-concave probability measure μ and every measurable convex symmetric set U with $\mu(U) = b$ we have*

$$\mu(tU) \leq C_b t \mu(U) \ \mathit{for} \ \ t \in [0, 1].$$

CHAPTER 2

The Upper Bound in Dvoretzky theorem and Milman–Schechtman Theorem

For an origin-symmetric convex body K, let $M = M(K) := \int_{S^{n-1}} \|x\|_K d\nu_n$ and $b = b(K) := \sup\{\|x\|_K \,,\, x \in S^{n-1}\}$ be the mean and the maximum of the norm over the unit sphere. In 1971, V. D. Milman proved the following Dvoretzky-type theorem [24, Theorem 5.2.10]:

Theorem 2.1. *Let K be a symmetric convex body in $\mathbb{R}^n$. Assume that $\|x\|_K \le b|x|$ for all $x \in \mathbb{R}^n$. For any $\epsilon \in (0,1)$, there is $k \ge C_\epsilon (M/b)^2 n$ such that*

$$\nu_{n,k}\{F \in G_{n,k} : (1-\varepsilon)M < \|\cdot\|_{K\cap F} < (1+\varepsilon)M\} > 1 - \exp(-\tilde{c}k)$$

where $\tilde{c} > 0$ is a universal constant, $C_\epsilon > 0$ is a constant depending only on ϵ.

The quantity C_ϵ was of the order $\epsilon^2 \log^{-1}(\frac{1}{\epsilon})$ in the original proof of V. D. Milman. It was improved to the order of ϵ^2 by Y. Gordon [26] and later, with a simpler argument, by G. Schechtman [62].

In 1997, V. D. Milman and G. Schechtman [56] found that the bound on k appearing in Theorem 2.1 is essentially optimal. More precisely, they proved the following theorem.

Theorem 2.2. *(Milman–Schechtman [24, Theorem 5.3.4]). Let K be a symmetric convex body in $\mathbb{R}^n$. For $\epsilon \in (0,1)$, define $k(K)$ to be the largest dimension k such that*

$$\nu_{n,k}\left(\{F \in G_{n,k} : \forall x \in S^{n-1} \cap F \,,\, (1-\varepsilon)M < \|x\|_K < (1+\varepsilon)M\}\right) > p_{n,k} = \frac{n}{n+k}.$$

Then,

$$\tilde{C}_\epsilon n(M/b)^2 \ge k(K) \ge \bar{C}_\epsilon n(M/b)^2$$

when $\frac{M}{b} > c(\frac{\log(n)}{n})^{\frac{1}{2}}$ for some universal constant $c > 0$, where $\|\cdot\|_F$ denotes the norm corresponding to the convex body $K \cap F$ in F, and $\tilde{C}_\epsilon, \bar{C}_\epsilon > 0$ are constants depending only on ϵ.

Because the Dvoretzky-Milman theorem cannot guarantee the lower bound with small $\frac{M}{b}$ for $p_{n,k} = \frac{n}{n+k}$, the original proof required an assumption that $\frac{M}{b} > c(\frac{\log(n)}{n})^{\frac{1}{2}}$ for some c. In [24, p. 197], S. Artstein-Avidan, A. A. Giannopoulos, and V. D. Milman addressed it as an open question whether one can prove the same result when we define $p_{n,k}$ to be a constant, instead of $\frac{n}{n+k}$. If we define $p_{n,k} = \frac{1}{2}$, the lower estimate on $k(K)$ is a direct result of Dvoretzky-Milman theorem 2.1, but the upper bound was unknown. Our main result is the following theorem:

Theorem 2.3. *Let K be a symmetric convex body in $\mathbb{R}^n$. Fix a constant $\epsilon \in (0,1)$, let $k = k(K)$ be the largest dimension such that*

$$\nu_{n,k}\{F \in G_{n,k} : (1-\varepsilon)M < \|\cdot\|_{K\cap F} < (1+\varepsilon)M\} > \frac{1}{2}.$$

Then,

$$Cn(M/b)^2 \geq k(X) \geq \bar{C}_\epsilon n(M/b)^2$$

where $C > 0$ is a universal constant and $\bar{C}_\epsilon > 0$ is a constant depending only on ϵ.

In the next section, we will provide a proof of Theorem 2.1with no restriction on $\frac{M}{b}$. In fact, from the proof, one can see that $\frac{1}{2}$ can be replaced by any $c \in (0,1)$ or $1 - \exp(-\tilde{c}k)$, which is the probability appearing in Milman-Dvoretzky theorem.

2.1 Proof of Theorem 2.3

Let $H \subseteq \mathbb{R}^n$ be an k-dimensional subsapce of $\mathbb{R}^n$. Consider the orthogonal projection P_k from $\mathbb{R}^n$ to H.

Lemma 2.4. *For $k > \frac{8}{c_0}$ we have*

$$\nu_n\left(\left\{x \in S^{n-1} \;:\; |P_k\left(x\right)| < \frac{1}{4}\sqrt{\frac{k}{n}}\right\}\right) \leq 4\exp\left(-\frac{c_0}{4}k\right).$$

Proof. First, we will show $\mathbb{E}|P_k(x)|$ about $\sqrt{\frac{k}{n}}$ where the expectation is taken over the normalized Haar measure on S^{n-1}. To see that, observe that

$$\mathbb{E}|P_k|^2 = \mathbb{E}\sum_{i=1}^{k}|x_i|^2 = \frac{k}{n}.$$

Using the fact that $|P_k|$ is a 1-Lipschitz function on S^{n-1}, by Theorem 1.14 we have

$$\nu_n(||P_k(x)| - \mathbb{E}|P_k(x)||^2 > t) \leq 4\exp(-c_0 tn).$$

Then,

$$\begin{aligned}
\mathbb{E}|P_k|^2 - (\mathbb{E}|P_k|)^2 &= \mathbb{E}(|P_k|(x) - \mathbb{E}|P_k|)^2 \\
&< \int_0^\infty \nu_n(||P_k(x)| - \mathbb{E}|P_k(x)||^2 > t)dt \\
&\leq \int_0^\infty 4\exp(-c_0 tn)dt = \frac{4}{c_0 n}.
\end{aligned}$$

If $k > \frac{8}{c_0}$, then we have

$$\mathbb{E}\left(|P_k|\right) \geq \sqrt{\frac{k}{n} - \frac{4}{c_0 n}} \geq \sqrt{\frac{k}{2n}}.$$

We apply Theorem 1.14 again and get

$$\begin{aligned}
\nu_n(|P_k| < \frac{1}{4}\sqrt{\frac{k}{n}}) &< \nu_n\left(||P_k| - \mathbb{E}|P_k|| > \mathbb{E}(|P_k|) - \frac{1}{4}\sqrt{\frac{k}{n}}\right) \\
&\leq 4\exp(-c_0(\mathbb{E}(|P_k|) - \frac{1}{4}\sqrt{\frac{k}{n}})^2 n) \\
&\leq 4\exp(-c_0(\frac{1}{2}\sqrt{\frac{k}{n}})^2 n) \leq 4\exp(-\frac{c_0}{4}k).
\end{aligned}$$

$\square$

Theorem 2.5. *There exists a concstant $c_1 > 0$ such that the following holds: For any origin-symmetric convex body $K \subseteq \mathbb{R}^n$. If $k \geq c_1\left(\frac{M}{b}\right)^2 n$, then*

$$\nu_{n,k}\left(\{F \in G_{n,k} : (1-\varepsilon)M < \|\cdot\|_{K\cap F} < (1+\varepsilon)M\}\right) \leq \exp\left(-\frac{c_0}{4}k\right) < \frac{1}{2}.$$

(The last inequality guarantees that $\exp\left(-\frac{c_0}{4}k\right)$ *won't be too big.)*

Proof. We may assume $\|e_1\|_K = b$, then $K \subset S = \{x \in \mathbb{R}^n : |x_1| < \frac{1}{b}\}$, thus

$\|x\|_K \geq \|x\|_S = b|\langle x, e_1\rangle|$. This implies

$$\begin{array}{ll} & \{V \in G_{n,k} \,:\, \forall x \in V \cap S^{n-1}\,,\, (1-\epsilon)M < \|x\|_K < (1+\epsilon)M\} \\ \subset & \{V \in G_{n,k} \,:\, \forall x \in V \cap S^{n-1}\,,\, \|x\|_S < (1+\epsilon)M\} \\ = & \{V \in G_{n,k} \,:\, \sup_{x\in V\cap S^{n-1}} |\langle x, e_1\rangle| < (1+\epsilon)\frac{M}{b}\} \\ = & \{V \in G_{n,k} \,:\, |P_V(e_1)| < (1+\epsilon)\frac{M}{b}\} \end{array} \tag{2.1}$$

where P_V is the orthogonal projection from $\mathbb{R}^n$ to V. If V is uniformly distributed on $G_{n,k}$ and x is uniformly distributed on S^{n-1}, then $|P_{V_0}(x)|$ and $|P_V(e_1)|$ are equi-distributed for any fixed k-dimensional subspace V_0. Therefore,

$$\begin{aligned} &\nu_{n,k}\left(\left\{V \in G_{n,k} \,:\, |P_V(e_1)| < (1+\epsilon)\frac{M}{b}\right\}\right) \\ =&\nu_n\left(\left\{x \in S^{n-1} \,:\, |P_{V_0}(x)| < (1+\epsilon)\frac{M}{b}\right\}\right). \end{aligned}$$

As shown in the Remark 5.2.2(iii) of [24, p. 164], the ratio $\frac{M}{b} \geq \frac{c'}{\sqrt{n}}$. Thus, there exists a constant c_1 such that if k is the smallest integer greater than $c_1\left(\frac{M}{b}\right)^2 n$, then,

$$\frac{1}{4}\sqrt{\frac{k}{n}} \geq (1+\varepsilon)\,\frac{M}{b} \qquad \text{and} \qquad k \geq \frac{16}{c_0}.$$

Now, by Lemma 2.4, we get

$$\nu_{n,k}\{F \in G_{n,k} : (1-\varepsilon)M < \|\cdot\|_{K\cap F} < (1+\varepsilon)M\} \leq 4\exp(-\frac{c_0}{4}k)$$

and the last term is smaller than $\frac{1}{2}$ since $k \geq \frac{16}{c_0}$. □

Now we can prove Theorem 2.3 as a corollary of Theorem 2.5 and Theorem 2.1:

Proof. [Proof of Theorem 2.3] Theorem 2.1 shows that if $C_\epsilon(M/b)^2 n > \frac{\log(2)}{\tilde{c}}$, then there is $k \geq C_\epsilon(M/b)^2 n$ such that

$$\nu_{n,k}\{F \in G_{n,k} : (1-\varepsilon)M < \|\cdot\|_F < (1+\varepsilon)M\} > 1 - \exp(-\tilde{c}k) > \frac{1}{2}.$$

Otherwise, $(M/b)^2 n < \frac{\log(2)}{\tilde{c}C_\epsilon}$. Therefore, $k(K) \geq \min\{\frac{\tilde{c}C_\epsilon}{\log(2)}, C_\epsilon\}(M/b)^2 n$. Combining it with Theorem 2.5

$$c_1\left(\frac{M}{b}\right)^2 n \geq k(K) \geq \min\left\{\frac{\tilde{c}C_\epsilon}{\log(2)}, C_\epsilon\right\}\left(\frac{M}{b}\right)^2 n.$$

$\square$

Remark. (1) It is worth noticing that the number $\frac{1}{2}$ plays no special role in our proof. Thus, if we define the Dvoretzky dimension to be the largest dimension such that

$$\nu_{n,k}\{F \in G_{n,k} : (1-\varepsilon)M < \|\cdot\|_{K\cap F} < (1+\varepsilon)M\} > c$$

for some $c \in (0,1)$, then exactly the same proof will work. We will still have $k(K) \sim (\frac{M}{b})^2 n$. Similarly, if we fix ϵ and replace $\frac{1}{2}$ by $1-\exp(-\tilde{c}k)$, then the lower bound of $k(K)$ is the one from Theorem 2.1. For k bigger than some absolute constant, we have $1-\exp(-\tilde{c}k) > \frac{1}{2}$. Thus, the upper bound is still of order $\left(\frac{M}{b}\right)^2 n$. Therefore, we can replace $\frac{1}{2}$ by $1-\exp(-\tilde{c}k)$ in Theorem A. With this choice, it also shows Theorem 2.1 provides an optimal k depending on M, b.

(2) Usually, we are only interested in $\epsilon \in (0,1)$. In the lower bound, $\bar{C}_\epsilon = o_\epsilon(1)$. It is a natural question to ask if we could improve the upper bound from a universal constant C to $o_\epsilon(1)$. Unfortunately, it is not possible due to the following observation. Let $K = \mathrm{conv}(B_2^n, Re_1)^\circ$. By passing from the intersection on K to the projection of K°, one can show that $k(K)$ does not exceed the maximum dimension k such that $\nu_n(P_k(Rx) < 1+\epsilon) > \frac{1}{2}$. Choosing $R = \sqrt{\frac{n}{l}}$, we get $n(\frac{M}{b})^2 \sim l$ and $k(X) \sim l$ by Theorem 1.14 and a similar argument to that of Lemma 2.4. This example shows that no matter what $\frac{M}{b}$ is, one cannot improve the upper bound in Theorem A from an absolute constant C to $o_\epsilon(1)$.

CHAPTER 3

Barycenter and maximum volume ellipsoid

In this chapter we focus on the question on the relation between the barycenter of a convex body K and its maximum volume ellipsoid E_K. A natural question asked by S. Vempala is whether the barycenter of K lies in a small dilation of its John ellipsoid. We formulate it as the following conjecture.

Conjecture 3.1. *For any convex body K in $\mathbb{R}^n$, the John ellipsoid of K scaled by a factor of $O(\sqrt{n})$ about the ellipsoid's center will contain at least half of the volume of K.*

This can be formulated in terms of the barycenter of K. We will show in Section 4 that Conjecture 3.1 is equivalent to the following conjecture:

Conjecture 3.2. *For any convex body K in $\mathbb{R}^n$, the John ellipsoid of K scaled by a factor of $O(\sqrt{n})$ about the ellipsoid's center will contain the barycenter of K.*

The main result is the following:

Theorem 3.3. *For a sufficiently large $n \in \mathbb{N}$,*

1. There exists a convex body $K \subset \mathbb{R}^n$ such that its barycenter does not lie in the John ellipsoid scaled by a factor of $(1 - C_0\sqrt{\frac{\log(n)}{n}})n$ about the ellipsoid's center, where $C_0 > 0$ is a universal constant.

2. There exists a polytope $P \subset \mathbb{R}^n$ with $O(n^2)$ facets such that its barycenter does not lie in the John ellipsoid scaled by a factor of $C_1\frac{n}{\log(n)}$ about the ellipsoid's center, where $C_1 > 0$ is a universal constant.

Remark: It is well known that for any convex body $K \subset \mathbb{R}^n$, the John ellipsoid of K scaled by a factor n about the ellipsoid's center contains the original body K. (see [24, Theorem 2.1.3 and Remark 2.1.4])

Thus, the example in Theorem 3.3(1) is the asymptotically optimal in the sense that $\lim_{n\to+\infty} \frac{(1-C_0\sqrt{\frac{\log(n)}{n}})n}{n} = 1$.

A consequence of this theorem is the following:

Corollary 3.4. *For a sufficiently large $n \in \mathbb{N}$,*

1. There exists a convex body $K \subset \mathbb{R}^n$ such that the center of its John ellipsoid B_K is 0 and
$$\mathrm{vol}\left(\left(1 - C_0'\sqrt{\frac{\log(n)}{n}}\right) nB_K \cap K\right) \leq \frac{1}{2}\mathrm{vol}(K),$$
where $C_0' > 0$ is a universal constant.

2. There exists a polytope $P \subset \mathbb{R}^n$ with $O(n^2)$ facets such that the center of its John ellipsoid B_P is 0 and
$$\mathrm{vol}\left(C_1' \frac{n}{\log(n)} B_P \cap P\right) \leq \frac{1}{2}\mathrm{vol}(P),$$
where $C_1' > 0$ is a universal constant.

Thus, Conjecture 3.1 and Conjecture 3.2 are not true due to Theorem 3.3 and Corollary 3.4. In particular, both conjectures will not hold even if one restricts the collection of convex bodies to polytopes with $O(n^2)$ facets.

This Chapter is structured as follows. The proof of Theorem 3.3 is presented in Section 3.1. Corollary 3.4 and the relation between Conjecture 3.1 and 3.2 are examined in Section 3.2.

3.1 Proof of Theorem 3.3

Since the result of Theorem 3.3 is not affected by applying an affine transformation on K or P, Theorem 3.3 can be rephrased in the following way: Recall that x_K denotes the barycenter for a convex body K and $|\cdot|$ denotes the Euclidean norm. We have

Theorem 3.5. *For a sufficiently large $n \in \mathbb{N}$,*

1. There exists a convex body $K \subset \mathbb{R}^n$ in John's position such that $|x_K| \geq \left(1 - C_0\sqrt{\frac{\log(n)}{n}}\right) n$ where $C_0 > 0$ is a universal constant.

2. There exists a convex polytope $P \subset \mathbb{R}^n$ in John's position with $O(n^2)$ facets such that $|x_P| \geq C_1 \frac{n}{\log(n)}$ where $C_1 > 0$ is a universal constant.

We write points in the form $x = (y, t)$ where $y \in \mathbb{R}^{n-1}$ corresponds to $\{e_i\}_{i=1}^{n-1}$ and $t \in \mathbb{R}$ corresponds to e_n. For a convex body K, we write $x_K := (y_K, t_K)$. Observe that, for $R > 0$,

$$\begin{array}{rcl} & t_K - R & \geq 0 \\ \Leftrightarrow & \frac{1}{\mathrm{vol}(K)} \int_K (t-R)dx & \geq 0 \\ \Leftrightarrow & \int_K (t-R)dx & \geq 0. \end{array}$$

Also, for a convex body $K \subset \mathbb{R}^n$, let $K_t := \{y \in \mathbb{R}^{n-1}\,,\,(y,t) \in K\}$, which is a slice of the convex body K. Let $[a_K, b_K]$ be the orthogonal projection of K to the span of e_n.

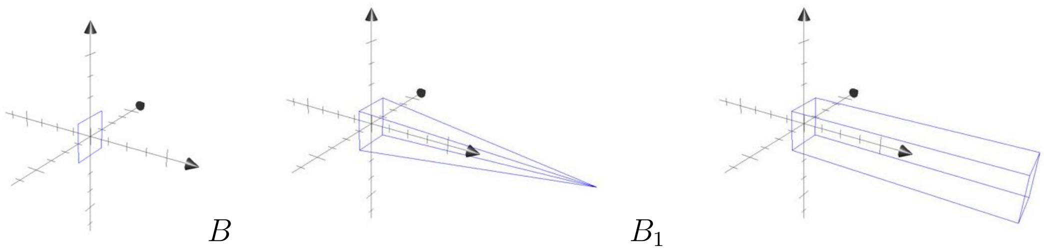
Assuming $0 \in K_t$ for all $t \in [a_K, b_K]$, let $\rho_K(\cdot, t)$ denote the radial function of K_t as a convex body in $\mathbb{R}^{n-1}$. Then,

$$\begin{aligned} & \int_K (t-R)dx \\ = & \int_{a_K}^{b_K} (t-R) \int_{K_t} dy dt \\ = & \int_{a_K}^{b_K} (t-R)(n-1)\kappa_{n-1} \int_0^{\rho_K(\theta,t)} \int_{S^{n-2}} r^{n-2} dr d\sigma_{n-2}(\theta) dt \\ = & \kappa_{n-1} \int_{S^{n-2}} \int_{a_K}^{b_K} \rho_K(\theta,t)^{n-1}(t-R) dt d\sigma_{n-2}(\theta), \end{aligned}$$

where κ_n denote the volume of B_2^n. With $|x_K| \geq |t_K|$, we conclude

$$\begin{aligned} & \int_{S^{n-2}} \int_{a_K}^{b_K} \rho_K(\theta,t)^{n-1}(t-R) dt d\sigma_{n-2}(\theta) \geq 0 \\ \Rightarrow & |x_K| \geq R. \end{aligned} \tag{3.1}$$

Before moving on to the proof of the main theorem, we examine two simple convex bodies in $\mathbb{R}^n$.



B $\qquad B_1$ $\qquad B_2$

Figure 3.1.1: B, B_1, and B_2

Let $0 \in B \subset \mathbb{R}^{n-1}$ be a $n-1$ dimensional convex body. We define $B_1, B_2 \subset \mathbb{R}^n$ as

$$
\begin{aligned}
B_1 &:= \{(y,t) \in \mathbb{R}^n \ , \ y \in B \text{ and } t \in [0, n+1]\}, \text{ and} \\
B_2 &:= \{(y,t) \in \mathbb{R}^n \ , \ y \in \frac{t}{n+1} B \text{ and } t \in [0, n+1]\}.
\end{aligned}
$$

In other words, B_1 is a cylinder and B_2 is a cone. (see figure 3.1.1) Both of them have the same base B and height $n+1$. We have $t_{B_1} = \frac{n+1}{2}$.

For t_{B_2}, using the fact that B_2 is a cone, we have

$$
\begin{aligned}
t_{B_2} &= \langle x_{B_2}, e_n \rangle \\
&= \frac{1}{\text{vol}(B_2)} \int_{B_2} t dx \\
&= \frac{n}{(n+1)\text{vol}(B)} \int_0^{n+1} \text{vol}(B) (\frac{t}{n+1})^{n-1} t dt \\
&= n.
\end{aligned}
$$

Comparing these two examples, we see that x_{B_2} is much closer to its base. For the same reason, the convex hull of B_2^n and ne_n, which is in John's position, has a barycenter that lies in B_2^n, because its shape is similar to that of a cone.

We will construct examples in the Theorem 3.3 as the intersection of two convex bodies, $Q \cap L$. Q and L will satisfy the following:

1. Q is in John's position. L contains B_2^n. Thus, $Q \cap L$ is also in John's position.

2. L will be a cone (or a cylinder) with the property that $Q \cap L$ and L have a similar shape. Therefore, $x_{Q \cap L}$ behaves like the barycenter of a cone (or a cylinder).

3.1.1 Construction of Q

The following proposition is related to the contact points decomposition of the identity:

Proposition 3.6. *Let $u_1, ..., u_m$ be unit vectors in $\mathbb{R}^{n-1} = span\{e_1, .., e_{n-1}\} \subset \mathbb{R}^n$, and $c_1, ..., c_m > 0$ be some positive numbers such that*

$$\sum_{i=1}^m c_i u_i \otimes u_i = I_{n-1} \quad \textit{and} \qquad \sum_{i=1}^m c_i u_i = \vec{0}.$$

Set $v_i = (\sqrt{1-\frac{1}{n^2}}u_i, \frac{1}{n}) \in \mathbb{R}^n$ for $i = 1, ..., m$ and $v_0 = (\vec{0}, -1)$. With $c_i' = \frac{c_i}{1-\frac{1}{n^2}}$ and $c_0' = \frac{n}{n+1}$, we obtain

$$\sum_{i=0}^m c_i' v_i \otimes v_i = I_n, \quad \textit{and} \qquad \sum_{i=0}^m c_i' v_i = \vec{0}.$$

Proof. From the definition of v_i, we have

$$v_i \otimes v_i = \frac{1}{n^2} e_n \otimes e_n + \frac{1}{n}\sqrt{1-\frac{1}{n^2}}(e_n \otimes u_i + u_i \otimes e_n) + (1 - \frac{1}{n^2}) u_i \otimes u_i.$$

We know that $n - 1 = \mathrm{Tr}(I_{n-1}) = \mathrm{Tr}(\sum_{i=1}^m c_i u_i \otimes u_i) = \sum_{i=1}^m c_i$. Thus, we have

$$\sum_{i=0}^m c_i v_i \otimes v_i = \frac{n-1}{n^2} e_n \otimes e_n + (1 - \frac{1}{n^2}) I_{n-1}.$$

where we use the fact that $\sum_{i=1}^m c_i = n - 1$ and $\sum_{i=1}^m c_i u_i = \vec{0}$. Now let $c_i' = \frac{c_i}{1-\frac{1}{n^2}} = \frac{n^2 c_i}{n^2-1}$ for $i = 1, ..., m$ and $c_0' = \frac{n}{n+1}$. We then have

$$\sum_{i=1}^m c_i' v_i \otimes v_i + c_0'(-e_n) \otimes (-e_n) = \frac{n-1}{n^2-1} e_n \otimes e_n + I_{n-1} + \frac{n}{n+1} e_n \otimes e_n = I_n.$$

Also,

$$\sum_{i=1}^m c_i' v_i - c_0' e_n = (\frac{n-1}{n}\frac{n^2}{n^2-1} - \frac{n}{n+1}) e_n = \vec{0}.$$

$\square$

The points $\{u_j\}_{j=1}^{2(n-1)} = \{\pm e_i\}_{i=1}^{n-1}$ with $c_j = \frac{1}{2}$ satisfy the assumption of Proposition 3.6. We set

$$A := \left\{ \left(\pm\sqrt{1-\frac{1}{n^2}}e_i, \frac{1}{n} \right) \right\}_{i=1}^{n-1} \cup \left\{ \left(\vec{0}, -1 \right) \right\}$$

and

$$Q := \{x \in \mathbb{R}^n \, , \, \forall u \in A \ \langle x, u \rangle \leq 1\}.$$

The set A is the collection of contact points of Q. By Proposition 3.6 and Theorem 1.13, Q is in John's position.

Let $B_\infty^{n-1} := \{y \in \mathbb{R}^{n-1} \, , \, \forall i = 1, 2, \cdots, n-1 \ |\langle y, e_i \rangle| \leq 1\}$ be the unit cube in $\mathbb{R}^{n-1}$.

$$\begin{aligned} Q &= \{x \in \mathbb{R}^n \, , \, \forall u \in A \ \langle x, u \rangle \leq 1\} \\ &= \{(y, t) \in \mathbb{R}^n \, , \, y \in \frac{n-t}{\sqrt{n^2-1}} B_\infty^{n-1} \text{ and } t \in [-1, n]\}. \end{aligned}$$

Q is in John's position and it is a cone with base $\frac{n+1}{\sqrt{n^2-1}} B_\infty^{n-1}$ and height $n+1$. Thus, Q_t is $\frac{n-t}{\sqrt{n^2-1}} B_\infty^{n-1}$ for $t \in [-1, n]$. Since the radial function of B_∞^{n-1} is $\rho_{B_\infty^{n-1}}(\theta) = \frac{1}{\max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}}$. We have

$$\rho_Q(\theta, t) = \frac{1}{\max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}} \frac{n-t}{\sqrt{n^2-1}}. \tag{3.2}$$

3.1.2 Proof of Theorem 3.3(1)

We define

$$L := \{(y, t) \in \mathbb{R}^n \, , \, y \in (2 + \frac{t}{n}) B_2^{n-1} \text{ and } t \in [-1, n]\}. \tag{3.3}$$

In particular, L_t is equal to $(2 + \frac{t}{n}) B_2^{n-1}$ and the radial function is $\rho_L(\theta, t) = 2 + \frac{t}{n}$.

Fix $R_0 = n - \frac{C_0}{2}\sqrt{\log(n)n}$ for some $C_0 > 0$ that we will determine later. Then, we have $\rho_L(\theta, R_0) = 3 - \frac{C_0}{2}\sqrt{\frac{\log(n)}{n}}$. By (3.2),

$$\rho_Q(\theta, R_0) = \frac{1}{\max\{|\langle \theta, e_i \rangle|\}_{i=1}^{n-1}} \frac{C_0}{2} \frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}}.$$

We split S^{n-2} into two components by defining

$$O_1 := \{\theta \in S^{n-2}, \rho_Q(\theta, R_0) \leq \rho_L(\theta, R_0)\}.$$

For a sufficiently large n, we have

$$\begin{aligned} O_1 = \quad & \{\theta \in S^{n-2}\,,\ \frac{1}{(3 - \frac{C_0}{2}\sqrt{\frac{\log(n)}{n}})}\frac{C_0}{2}\frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}} \leq \max\{|\langle\theta, e_i\rangle|\}_{i=1}^{n-1}\} \\ \subset \quad & \{\theta \in S^{n-2}\,,\ \frac{C_0}{6}\frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}} \leq \max\{|\langle\theta, e_i\rangle|\}_{i=1}^{n-1}\} \\ \subset \quad & \cup_{i=1}^{n-1}\{\theta \in S^{n-2}, \frac{C_0}{6}\frac{\sqrt{\log(n)n}}{\sqrt{n^2-1}} \leq |\langle\theta, e_i\rangle|\}. \end{aligned}$$

Due to Proposition 1.15, the measure of O_1 can be bounded:

$$\begin{aligned} \sigma_{n-2}(O_1) &\leq 4n\exp(-\frac{1}{36}C_3C_0^2\frac{n^2}{n^2-1}\log(n)) \\ &\leq 4\exp\left((1 - \frac{1}{36}C_3C_0^2)\log(n)\right). \end{aligned}$$

By setting $C_0 := \sqrt{\frac{72}{C_3}}$, for a sufficiently large n, we have

$$\sigma_{n-2}(O_1) \leq 4\exp(-\log(n)) \leq \frac{1}{2}. \tag{3.4}$$

Moreover, $\rho_L(\theta, t)$ is increasing with respect to $t \in [-1, n]$, while $\rho_Q(\theta, t)$ is decreasing with respect to $t \in [-1, n]$. We may conclude that,

$$\forall\theta \in O_1^c\,,\ \rho_Q(\theta, t) \geq \rho_L(\theta, t) \text{ for } t \in [-1, R_0]. \tag{3.5}$$

We define K to be the intersection of Q and L, $K = Q \cap L$. Then, we have $K_t = Q_t \cap L_t$ and thus $\rho_K(\theta, t) = \min\{\rho_Q(\theta, t), \rho_L(\theta, t)\}$.

By (3.1), it is sufficient to prove

$$\int_{S^{n-1}}\int_{-1}^{n}\rho_K(\theta, t)^{n-1}(t - R)dtd\sigma_{n-2}(\theta) \geq 0, \tag{3.6}$$

with $R = n - C_0\sqrt{\log(n)n}$. For the inner integral in (3.6):

$$\begin{aligned}
&\int_{-1}^{n} \rho_K(\theta,t)^{n-1}(t-R)dt \\
\leq &\int_{-1}^{R_0} \rho_K(\theta,t)^{n-1}(t-R)dt \\
= -&\int_{-1}^{R} \rho_K(\theta,t)^{n-1}(R-t)dt + \int_{R}^{R_0} \rho_K(\theta,t)^{n-1}(t-R)dt.
\end{aligned}$$

For the first component, with $\rho_K(\theta,t) \leq \rho_L(\theta,t) = 2 + \frac{t}{n}$, we have

$$\int_{-1}^{R} \rho_K(\theta,t)^{n-1}(R-t)dt \leq \int_{-1}^{R} (2+\frac{t}{n})^{n-1}(R-t)dt.$$

The integral on the right side is computable via integration by parts:

$$\begin{aligned}
&\int_{-1}^{R} (2+\frac{t}{n})^{n-1}(R-t)dt \\
= &(2+\frac{t}{n})^{n}(R-t)\Big]_{-1}^{R} + \int_{-1}^{R} (2+\frac{t}{n})^{n}dt \\
= -&(2-\frac{1}{n})^{n}(R+1) + \frac{n}{n+1}(2+\frac{R}{n})^{n+1} - \frac{n}{n+1}(2-\frac{1}{n})^{n+1} \\
\leq &\frac{n}{n+1}(2+\frac{R}{n})^{n+1}.
\end{aligned}$$

Thus,

$$\int_{-1}^{R} \rho_K(\theta,t)^{n-1}(R-t)dt \leq \frac{n}{n+1}(2+\frac{R}{n})^{n+1}. \tag{3.7}$$

For $\theta \in O_1^c$, due to (3.5) we have $\rho_K(\theta,t) = \rho_L(\theta,t) = (2+\frac{t}{n})$ for $t \in [-1,n]$. Thus, we have the equality when $\theta \in O_1^c$:

$$\int_{R}^{R_0} \rho_K(\theta,t)^{n-1}(t-R)dt = \int_{R}^{R_0} \rho_L(\theta,t)^{n-1}(t-R)dt.$$

Again, the integral on the right side is computable:

$$\begin{aligned}
&\int_{R}^{R_0} (2+\frac{t}{n})^{n-1}(t-R)dt \\
= &(2+\frac{t}{n})^{n}(t-R)\Big]_{R}^{R_0} - \int_{R}^{R_0} (2+\frac{t}{n})^{n}dt \\
= &(2+\frac{R_0}{n})^{n}(R_0-R) - \frac{n}{n+1}(2+\frac{R_0}{n})^{n+1} + \frac{n}{n+1}(2+\frac{R}{n})^{n+1}.
\end{aligned}$$

Observe that, for a sufficiently large n, we have

$$(R_0 - R) = \frac{C_1}{2}\sqrt{\log(n)n} > 4 > 2\frac{n}{n+1}(2 + \frac{R_0}{n}).$$

Hence, the previous equality can be bounded:

$$\begin{array}{rc} & (2+\frac{R_0}{n})^n(R_0-R) - \frac{n}{n+1}(2+\frac{R_0}{n})^{n+1} + \frac{n}{n+1}(2+\frac{R}{n})^{n+1} \\ \geq & \frac{1}{2}(2+\frac{R_0}{n})^n(R_0-R). \end{array}$$

We conclude that, for any $\theta \in O_1^c$,

$$\int_R^{R_0} \rho_K(\theta,t)^{n-1}(t-R)dt \geq \frac{1}{2}(2+\frac{R_0}{n})^n(R_0-R). \tag{3.8}$$

Now we can derive the main inequality (3.6). First, we split the integral:

$$\begin{array}{rl} & \int_{S^{n-1}}\int_{-1}^{n} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta) \\ = & \int_{S^{n-1}}\int_{-1}^{R} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta) \\ & +\int_{S^{n-1}}\int_{R}^{R_0} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta) \\ & +\int_{S^{n-1}}\int_{R_0}^{n} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta). \end{array}$$

By (3.7), the first summand satisfies

$$\int_{S^{n-1}}\int_{-1}^{R} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta) \geq -\frac{n}{n+1}(2+\frac{R}{n})^{n+1}.$$

According to (3.8) and (3.4), the second summand satisfies

$$\begin{array}{rl} & \displaystyle\int_{S^{n-1}}\int_{R}^{R_0} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta) \\ \geq & \displaystyle\int_{O_1^c}\int_{R}^{R_0} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta) \\ \geq & \displaystyle\frac{1}{4}(2+\frac{R_0}{n})^n(R_0-R). \end{array}$$

Noticing that the third summand is non-negative, we conclude that

$$\begin{array}{rl} & \int_{S^{n-1}} \int_{-1}^{n} \rho_K(\theta,t)^{n-1}(t-R)dtd\sigma_{n-2}(\theta) \\ \geq & -\frac{n}{n+1}(2+\frac{R}{n})^{n+1} + \frac{1}{4}(2+\frac{R_0}{n})^n(R_0-R). \end{array}$$

With $\frac{1}{4}(R_0-R) > 2 > \frac{n}{n+1}(2+\frac{R}{n})$ and $(2+\frac{R_0}{n})^n > (2+\frac{R}{n})^n$, we get

$$-\frac{n}{n+1}(2+\frac{R}{n})^{n+1} + \frac{1}{4}(2+\frac{R_0}{n})^n(R_0-R) > 0$$

for a sufficiently large n. Hence,

$$\int_{S^{n-1}} \int_{-1}^{n} \rho_K(\theta,t)(t-R)dtd\sigma_{n-2}(\theta) > 0.$$

We conclude from (3.1) that

$$|x_K| > R = n - C_0\sqrt{\log(n)n} = (1 - C_0\sqrt{\frac{\log(n)}{n}})n.$$

3.1.3 Proof of Theorem 3.3 (2)

To construct P in Theorem 3.3 (2) we define a cylinder L_2, which is the intersection of $O(n^2)$ number of halfspaces and set $P := Q \cap L_2$, where Q is the same as above.

Let $\{\epsilon_n\}$ be a decreasing sequence. Later we will specify ϵ_n, but for now we assume that

$$\frac{10}{n} < \epsilon_n < 1 \quad \text{, and} \tag{3.9}$$

$$\lim_{n\to+\infty} \epsilon_n = 0. \tag{3.10}$$

Let

$$A' := \{\pm(1-\epsilon_n)e_i \pm \sqrt{1-(1-\epsilon_n)^2}e_j\}_{i,j<n\, i\neq j},$$

and

$$L_2 := \{(y,t) \in \mathbb{R}^n, \langle y,u\rangle \leq 1 \ \forall u \in A' \text{ and } t \in [-1,n]\}.$$

We have $|A'| = 4n(n-1)$ and L_2 is a cylinder with

$$L_{2,t} = \{y \in \mathbb{R}^{n-1}, \langle y,u\rangle \leq 1 \ \forall u \in A'\}$$

for $t \in [-1, n]$. Let $P = Q \cap L_2$. Since $B_2^n \subset L_2$ and Q is in John's position, P is in John's position. Following the same approach from the proof of Theorem 3.5 (1), we want to show

$$\int_{S^{n-2}} \int_{-1}^{n} \rho_P(\theta, t)^{n-1} (t - \frac{1}{5}\epsilon_n n) dt d\sigma_{n-2}(\theta) > 0. \tag{3.11}$$

Then, we can conclude $|x_P| > \frac{1}{5}\epsilon_n n$.

For convenience, let $Q' := Q_{\epsilon_n n}$ and $L' := L_{2,\epsilon_n n}$. Also, let $\rho_{Q'}(\cdot) := \rho_Q(\cdot, \epsilon_n n)$ and $\rho_{L'}(\cdot) := \rho_{L_2}(\cdot, \epsilon_n n)$. We will show that for the majority of $\theta \in S^{n-2}$, $\rho_P(\theta, t) = \rho_{L_2}(\theta, t)$ for $t \in [-1, \epsilon_n n]$. In the case that $\rho_P(\theta, t) \neq \rho_{L_2}(\theta, t)$ for some t in $[-1, \epsilon_n n]$, $\rho_P(\theta, t)$ will be nicely bounded.

Proposition 3.7.

With the notation above, let

$$O_2 := \{\theta \in S^{n-2}\,,\ \rho_{Q'}(\theta) \leq \rho_{L'}(\theta)\}.$$

For a sufficiently large n, we have

$\forall \theta \in O_2$, $\rho_{Q'}(\theta) \leq 4\sqrt{\epsilon_n n}$.
$\sigma_{n-2}(O_2) \leq 4n \exp(-\frac{C_6}{\epsilon_n})$, *where $C_6 > 0$ is a universal constant.*

Proof. Let $y \in \partial Q' \cap L'$. Then, there exists i such that $|y_i| = (1-\epsilon_n)\frac{n}{\sqrt{n^2-1}} = \rho_{Q'}(\frac{y}{|y|})$. Following the conditions from the definition of L', we have, for $j \neq i$,

$$\begin{aligned} (1-\epsilon_n)|y_i| + \sqrt{1-(1-\epsilon_n)^2}|y_j| &\leq 1 \\ \Rightarrow \quad \sqrt{1-(1-\epsilon_n)^2}|y_j| &\leq 1-(1-\epsilon_n)^2 \\ \Rightarrow \quad |y_j| &\leq \sqrt{1-(1-\epsilon_n)^2}, \end{aligned}$$

where for the second inequality we use $\frac{n}{\sqrt{n^2-1}} \geq 1$.

From the previous argument, $y \in \partial Q' \cap L'$ implies that

$$|y| \leq \sqrt{(n-2)(1-(1-\epsilon_n)^2) + (1-\epsilon_n)^2 \frac{n^2}{n^2-1}}. \tag{3.12}$$

By (3.9) and (3.10), we have $0 < (1-(1-\epsilon_n)^2) = 2\epsilon_n - \epsilon_n^2 \leq 2\epsilon_n$ and $n\epsilon_n > 1$. Hence, (3.12) becomes

$$\begin{aligned} |y| \leq & \sqrt{(n-2)(1-(1-\epsilon_n)^2) + (1-\epsilon_n)^2 \frac{n^2}{n^2-1}} \\ \leq & \sqrt{2\epsilon_n n + 2} \\ \leq & 2\sqrt{\epsilon_n n}, \end{aligned}$$

which proves Claim (1) in Proposition 3.7.

For $\theta \in O_2$, we have $\rho_Q(\theta)\theta \in \partial Q' \cap L'$. There exists i such that $|(\rho_{Q'}(\theta)\theta)_i| = (1-\epsilon_n)\frac{n}{\sqrt{n^2-1}}$. By (3.10), we have $(1-\epsilon_n)\frac{n}{\sqrt{n^2-1}} > \frac{1}{2}$ for large n.

Together with $\rho_{Q'}(\theta) \leq 2\sqrt{\epsilon_n n}$,

$$|\theta_i| = \frac{(1-\epsilon_n)\frac{n}{\sqrt{n^2-1}}}{\rho_{Q'}(\theta)} \geq \frac{1}{2\rho_{Q'}(\theta)} \geq \frac{1}{4\sqrt{\epsilon_n n}}. \tag{3.13}$$

Thus, inequality (3.13) leads to the following inclusion:

$$O_2 \subset \cup_{i=1}^{n-1} \left\{ \theta \in S^{n-2} \, , \, |\theta_i| \geq \frac{1}{4\sqrt{\epsilon_n n}} \right\}.$$

By Proposition 1.15,

$$\sigma_{n-2}\left(\left\{ \theta \, , \, |\theta_i| \geq \frac{1}{4\sqrt{\epsilon_n n}} \right\} \right) \leq 4\exp\left(-\frac{C_3}{16\epsilon_n} \frac{n-1}{n} \right) \leq 4\exp\left(-\frac{C_6}{\epsilon_n} \right).$$

Therefore, using the union bound, we conclude that

$$\sigma_{n-2}\left(O_2\right) \leq 4n\exp\left(-\frac{C_6}{\epsilon_n} \right).$$

□

Proposition 3.8. *With the notation above, there exists a constant $C_7 > 0$ such that if the sequence $\{\epsilon_n\}$ satisfies $\frac{C_7}{\log(n)} > \epsilon_n$ for a sufficiently large n, then*

$$\sigma_{n-2}(\{\theta \, , \, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \leq 4\exp(-\frac{C_8}{\epsilon_n}),$$

where $C_8 > 0$ is a universal constant.

Proof. Let $\|\cdot\|$ be the norm on $\mathbb{R}^{n-1}$ such that L' is the unit ball that corresponds to the norm $\|\cdot\|$. More specifically, for $y \in \mathbb{R}^{n-1}$,

$$\|y\| = \max_{1\leq i,j<n\,,\, i\neq j}\{(1-\epsilon_n)|y_i| + \sqrt{1-(1-\epsilon_n)^2}|y_j|\}.$$

Let $g = (g_1, g_2, .., g_{n-1})$ be the standard Gaussian random vector in $\mathbb{R}^{n-1}$. Then,

$$\begin{aligned}
\mathbb{E}\|g\| &= \mathbb{E}\max_{1\leq i,j<n\,,\, i\neq j}\{(1-\epsilon_n)|g_i| + \sqrt{1-(1-\epsilon_n)^2}|g_j|\} \\
&\leq 2\mathbb{E}\max_{i=1,..,n-1}|g_i| \leq c'\sqrt{\log(n)},
\end{aligned}$$

where $c' > 0$ is a universal constant and the last inequality is a classical result for the extreme value of independent Gaussian random variables.

Using the standard polar integration, we obtain the following inequality,

$$\int_{S^{n-2}} \|\theta\| d\sigma_{n-2}(\theta) \leq \frac{c''}{\sqrt{n}}\mathbb{E}\|g\|,$$

where $c'' > 0$ is a universal constant. Thus, $\mathbb{E}_{\sigma_{n-2}}\|\theta\| \leq c'c''\sqrt{\frac{\log(n)}{n}}$. Moreover, $\sup_{\theta\in S^{n-2}}\|\theta\| \leq 1$ due to the fact that $B_2^{n-1} \subset L'$. Therefore, the function $\theta \to \|\theta\|$ is 1-Lipschitz on S^{n-2}. We set $C_7 > 0$ to be small enough so that $\frac{1}{2}\frac{1}{5\sqrt{\epsilon_n n}} > c'c''\sqrt{\frac{\log(n)}{n}}$. Since $\rho_{L'}(\theta) = \frac{1}{\|\theta\|}$, we have the equality

$$\{\theta \in S^{n-2}\,,\, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\} = \{\theta \in S^{n-2}\,,\, \|\theta\| \geq \frac{1}{5\sqrt{\epsilon_n n}}).$$

Furthermore, the inequality $\mathbb{E}_{\sigma_{n-2}}\|\theta\| \leq \frac{1}{2}\frac{1}{5\sqrt{\epsilon_n n}}$ implies

$$\left\{\theta \in S^{n-2}\,,\, \|\theta\| \geq \frac{1}{5\sqrt{\epsilon_n n}}\right\} \subset \left\{\theta \in S^{n-2}\,,\, |\|\theta\| - \mathbb{E}\|\theta\|| > \frac{1}{10\sqrt{\epsilon_n n}}\right\}.$$

Together with Theorem 1.14, we may conclude that

$$\begin{aligned}
& \sigma_{n-2}(\{\theta \in S^{n-2}\,,\, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \\
\leq\ & \sigma_{n-2}(\{\theta \in S^{n-2}\,,\, |\|\theta\| - \mathbb{E}\|\theta\|| > \frac{1}{10\sqrt{\epsilon_n n}})\} \\
\leq\ & 4\exp(-\frac{C_8}{\epsilon_n}),
\end{aligned}$$

where we use Theorem 1.14 in the last inequality.

$\square$

Now we are able to prove Theorem 3.3 (2).

Proof. [Proof of Theorem 3.3 (2)]

We want to choose ϵ_n so that

$$\sigma_{n-2}(O_2) < \frac{1}{4}, \tag{3.14}$$

and

$$\sigma_{n-2}(\{\theta \in S^{n-2}, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \leq \frac{1}{4}, \tag{3.15}$$

for a large n.

According to Proposition 3.7, the first condition can be achieved if $\epsilon_n < \frac{c}{\log(n)}$ for some $c > 0$ when n is large.

Moreover, we also want to choose $\epsilon_n < \frac{c'}{\log(n)}$ so that we can apply Proposition 3.8 to get $\sigma_{n-2}(\{\theta \in S^{n-2}, \rho_{L'}(\theta) \leq 5\sqrt{\epsilon_n n}\}) \leq \frac{1}{4}$. Therefore, we can set $\epsilon_n = \frac{c''}{\log(n)}$ for some $c'' > 0$ so that (3.14) and (3.15) hold.

Recall that from (3.11) our goal is to show that

$$\int_{S^{n-2}} \int_{-1}^{n} \rho_P(\theta, t)^{n-1}(t - \frac{1}{5}\epsilon_n n) dt d\sigma_{n-2}(\theta) > 0.$$

Since $P = Q \cap L_2$,

$$\rho(\theta, t) = \min\{\rho_Q(\theta, t), \rho_{L_2}(\theta, t)\} = \min\{\rho_Q(\theta, t), \rho_{L'}(\theta)\}.$$

We handle the inner integral differently for $\theta \in O_2$ and $\theta \notin O_2$.

In the case that $\theta \notin O_2$:
First, we have $\rho_Q(\theta, \epsilon_n n) \geq \rho_{L_2}(\theta, \epsilon_n n)$. Thus, $\rho_P(\theta, t) = \rho_{L'}(\theta)$ for $t \in [-1, \epsilon_n n]$. This is

because $\rho_{L'}(\theta)$ is a constant and $\rho_P(\theta,t)$ is decreasing with respect to t. Thus,

$$\begin{aligned} & \int_{-1}^{n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt \\ \geq & \int_{-1}^{\epsilon_n n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt \\ = & \int_{-1}^{\epsilon_n n} \rho_{L'}(\theta)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt. \end{aligned}$$

We split the integral to two parts:

$$\begin{aligned} & \int_{-1}^{\varepsilon_n n} \rho_{L'}(\theta)^{n-1}\left(t-\frac{1}{5}\varepsilon_n\right) \mathrm{d}t \\ = & \int_{-1}^{2\frac{1}{5}\epsilon_n n+1} \rho_{L'}(\theta)^{n-1}\left(t-\frac{1}{5}\epsilon_n n\right) dt \\ & + \int_{2\frac{1}{5}\epsilon_n n+1}^{\epsilon_n n} \rho_{L'}(\theta)^{n-1}\left(t-\frac{1}{5}\epsilon_n n\right) dt. \end{aligned}$$

Due to the symmetry of the integrand with respect to $t=\frac{1}{5}\epsilon_n n$, the first summand is 0. For the second summand, we have

$$\begin{aligned} & \int_{\frac{2}{5}\epsilon_n n+1}^{\epsilon_n n} \rho_{L'}(\theta)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt \\ \geq & (\epsilon_n n-\frac{2}{5}\epsilon_n n-1)\rho_{L'}(\theta)^{n-1}(\frac{1}{5}\epsilon_n n+1) \\ \geq & \frac{(\epsilon_n n)^2}{10}\rho_{L'}(\theta)^{n-1}, \end{aligned}$$

where in the second to last inequality we used that $\frac{2}{5}\epsilon_n n+1 \leq \frac{1}{2}\epsilon_n n$ by (3.9). We conclude that

$$\forall\theta\in O_2^c\,,\int_{-1}^{n}\rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt \geq \frac{(\epsilon_n n)^2}{10}\rho_{L'}(\theta)^{n-1}. \tag{3.16}$$

In the case that $\theta\in O_2$:

From Proposition 3.7, we know that $\rho_Q(\theta,\epsilon_n n)\leq 2\sqrt{\epsilon_n n}$. Therefore, since $\rho_Q(\theta,t)$ is linear on $[-1,n]$ and $\rho_Q(\theta,n)=0$, we see that for any $t\in[-1,n]$,

$$\rho_Q(\theta,t)\leq\frac{n+1}{n-\epsilon_n n}2\sqrt{\epsilon_n n}\leq 4\sqrt{\epsilon_n n},$$

for a sufficiently large n. We have

$$\int_{-1}^{n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt \geq \int_{-1}^{\frac{1}{5}\epsilon_n n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt,$$

because the integrand is positive for $t > \frac{1}{5}\epsilon_n n$. Then, using the estimate of $\rho_Q(\theta,t) \leq 4\sqrt{\epsilon_n n}$,

$$\begin{aligned}
&\int_{-1}^{\frac{1}{5}\epsilon_n n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt \\
\geq\ & -(\frac{1}{5}\epsilon_n n+1)(4\sqrt{\epsilon_n n})^{n-1}(1+\frac{1}{5}\epsilon_n n) \\
\geq\ & -\frac{4}{25}(\epsilon_n n)^2(4\sqrt{\epsilon_n n})^{n-1},
\end{aligned}$$

where in the last inequality we used $\frac{1}{5}\epsilon_n n + 1 \leq \frac{2}{5}\epsilon_n n$, which is valid for a large n. Therefore, we have

$$\forall \theta \in O_2\,,\ \int_{-1}^{n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dt \geq -\frac{4}{25}(\epsilon_n n)^2(4\sqrt{\epsilon_n n})^{n-1}. \tag{3.17}$$

Now we are able to derive the main inequality.

$$\begin{aligned}
&\int_{S^{n-2}}\int_{-1}^{n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dtd\sigma_{n-2}(\theta) \\
=\ & \int_{O_2}\int_{-1}^{n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dtd\sigma_{n-2}(\theta) \\
&+\int_{O_2^c}\int_{-1}^{n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dtd\sigma_{n-2}(\theta).
\end{aligned} \tag{3.18}$$

Applying (3.16), the second summand satisfies

$$\int_{O_2^c}\int_{-1}^{n} \rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dtd\sigma_{n-2}(\theta) \geq (\epsilon_n n)^2 \int_{O_2^c} \frac{1}{10}\rho_{L'}(\theta)^{n-1}d\sigma_{n-2}(\theta).$$

Let $U := \{\theta\,,\ \rho_{L'}(\theta) \geq 5\sqrt{\epsilon_n n}\}$. From (3.14) and (3.15) we know that $\sigma_{n-2}(U\cap O_2^c) \geq \frac{1}{2}$ for a large n. Since the integrand is positive,

$$\int_{O_2^c} \frac{1}{10}\rho_{L'}(\theta)^{n-1}d\sigma_{n-2}(\theta) \geq \int_{U\cap O_2^c} \frac{1}{10}\rho_{L'}(\theta)^{n-1}d\sigma_{n-2}(\theta).$$

Thus,

$$\int_{O_2^c}\int_{-1}^{n}\rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dtd\sigma_{n-2}(\theta)\geq(\epsilon_n n)^2\int_{U\cap O_2^c}\frac{1}{10}\rho_L(\theta)^{n-1}d\sigma_{n-2}(\theta).$$

For the first summand of (3.18), we apply (3.17) and (3.14) to get

$$\begin{aligned}&\int_{O_2}\int_{-1}^{n}\rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dtd\sigma_{n-2}(\theta)\\ \geq\ &-(\epsilon_n n)^2\sigma_{n-2}(O_2)\frac{4}{25}(4\sqrt{\epsilon_n n})^{n-1}.\end{aligned}$$

Combining the inequalities for the two summands together we have

$$\begin{aligned}&\int_{S^{n-2}}\int_{-1}^{n}\rho_P(\theta,t)^{n-1}(t-\frac{1}{5}\epsilon_n n)dtd\sigma_{n-2}(\theta)\\ \geq\ &(\epsilon_n n)^2[\frac{1}{20}(5\sqrt{\epsilon_n n})^{n-1}-\frac{1}{25}(4\sqrt{\epsilon_n n})^{n-1}]\\ \geq\ &0.\end{aligned}$$

Therefore, the barycenter is at least $C_1\frac{n}{\log(n)}$ away from 0, where $C_1 := \frac{c''}{5}$.

□

3.2 The relation between the conjectures

Let $K\subset\mathbb{R}^n$ be a convex body in John's position and X be a random vector uniformly distributed in K. Let M_K denote the median of $|X|$, which is the unique value satisfying

$$\mathbb{P}(|X|\leq M_K)=\frac{1}{2}.$$

Lemma 3.9. *Let $K\subset\mathbb{R}^n$ be a convex body. Let X be a random vector uniformly distributed in K. Let M_K denote the median of $|X|$. Then, we have*

$$\frac{M_K}{\sqrt{2}}\leq(\mathbb{E}|X|^2)^{1/2}\leq C_9 M_K,$$

where $C_9>0$ is a universal constant.

Proof. The first inequality is standard:

$$\mathbb{E}|X|^2 \ge \mathbb{E}(|X|^2 \mathbf{1}_{|X|\ge M_K}) \ge \frac{1}{2} M_K^2$$

Thus, the first inequality can be obtained by taking square root on both sides.

To prove the second one, let R be the number such that $\mathbb{P}(|X| \le R) = \frac{2}{3}$. We can apply Theorem 3.3 with $U = RB_2^n$ and $\delta = \frac{2}{3}$ to get

$$\mathbb{P}(|X| > tR) \le \frac{\sqrt{2}}{3} 2^{-t/2} \text{ for } t > 1.$$

A simple integration shows that

$$\mathbb{E}|X|^2 \le cR^2,$$

for a universal constant $c > 0$.

Now we apply Theorem 1.19 with $b = \frac{2}{3}$ and $U = RB_2^n$ to obtain

$$\mathbb{P}(|X| \le M_K B_2^N)) \le C_b \frac{M_K}{R} \mathbb{P}(|X| \le R),$$

which implies that $M_K \ge \frac{3}{C_{2/3}} R$.

□

We could also relate $\mathbb{E}|X|^2$ and the barycenter x_K of K when K is in John's position.

Lemma 3.10. *There exists $C_{10}, C_{11} > 0$ such that, for any convex body $K \subset \mathbb{R}^n$ in John's position, we have*

$$|x_K|^2 \le \mathbb{E}|X|^2 \le C_{10}|x_K|^2 + C_{11}n,$$

where X is a random vector uniformly distributed in K.

This result was proved by M. Fradelizi, G. Paouris and C. Schï¿œtt in [22].

Here we present a different proof.

Proof. Since K is in John's position, there exist $\{u_i\}_{i=1}^m \subset S^{n-1}$ and $\{c_i\}_{i=1}^m$ with $c_i > 0$ such that $\sum_{i=1}^m c_i u_i = 0$ and $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$.

In particular,

$$\mathbb{E}|X|^2 = \mathbb{E}\sum_{i=1}^{m} c_i(\langle X, u_i\rangle)^2.$$

Also, $|x_K|^2 = \sum_{i=1}^{m} c_i(\langle x_K, u_i\rangle)^2$.

Given that u_i is a contact point of K, we have $\langle x, u_i\rangle \le 1$ for all $x \in K$. As a consequence, with $\langle x_K, u_i\rangle = \mathbb{E}\langle X, u_i\rangle$ we have

$$0 \le \mathbb{E}|\langle X, u_i\rangle| - |\langle x_K, u_i\rangle| \le 2.$$

Here, the first inequality follows from Jensen's inequality while the second one relies on an elementary observation that for any random variable $Y \le 1$,
$\mathbb{E}|Y| = 2\mathbb{E}\max\{Y, 0\} - \mathbb{E}Y \le 2 + |\mathbb{E}Y|$. Thus,

$$|\langle x_K, u_i\rangle|^2 \le (\mathbb{E}|\langle X, u_i\rangle|)^2 \le 3(|\langle x_K, u_i\rangle|^2 + 2).$$

According to Theorem 1.18, we have

$$\mathbb{E}|\langle X, u_i\rangle| \le (\mathbb{E}|\langle X, u_i\rangle|^2)^{1/2} \le 2C_5\mathbb{E}|\langle X, u_i\rangle|.$$

Therefore, we can conclude that

$$|x_K|^2 \le \mathbb{E}|X|^2 \le 12C_5(|x_K|^2 + 2\sum c_i) \le C|x_K|^2 + C'n,$$

where the last inequality uses the fact that $\sum_{i=1}^{m} c_i = n$.

$\square$

Corollary 3.11. *Conjecture 3.2 and Conjecture 3.1 are equivalent.*

Proof. Let $K \subset \mathbb{R}^n$ be a convex body. Since the result is invariant under affine transformations, we may assume that K is in John's position. Let X be a random vector uniformly distributed in K and M_K be the median of the random variable $|X|$.

Suppose Conjecture 3.1 is true. There exists a universal constant $C > 0$ such that
$|x_K| \le C\sqrt{n}$. According to Lemma 3.9 and Lemma 3.10,

$$
\begin{aligned}
M_K &\leq \sqrt{2}(\mathbb{E}|X|^2)^{1/2} \\
&\leq \sqrt{2}\sqrt{C_{10}|x_K|^2 + C_{11}n} \\
&\leq \sqrt{2n}\sqrt{C_{10}C^2 + C_{11}}.
\end{aligned}
$$

This argument is valid for any convex body K; therefore, Conjecture 3.2 is true.

On the other hand, assuming Conjecture 3.2 is valid, there exists a universal constant $C > 0$ such that $M_K \leq C\sqrt{n}$. Again, according to Lemma 3.9 and Lemma 3.3,

$$
\begin{aligned}
|x_K| &\leq (\mathbb{E}|X|^2)^{1/2} \\
&\leq C_9 M_K \\
&\leq C_9 C\sqrt{n}.
\end{aligned}
$$

Therefore, Conjecture 3.1 is true.

□

The examples in Corollary 3.4 will be examples K, P, which are constructed in Theorem 3.3. For Corollary 3.4(2), the result will follow by $|x_P| \leq C_9 M_P$. Corollary 3.4(1) is a more delicate situation, and so the same argument does not apply. Observe that, for $R > 0$,

$$K \cap RB_2^n \subset K \cap \{x \in \mathbb{R}^n, \langle x, e_1\rangle \leq R\}.$$

It is sufficient to show a stronger statement:

$$\text{vol}(K \cap \{x \in \mathbb{R}^n, \langle x, e_n\rangle \leq R\}) \leq \text{vol}(K \cap \{x \in \mathbb{R}^n, \langle x, e_1\rangle > R\})$$

for $R = n - C_0'\sqrt{\log(n)n}$. Adapting the notations from the proof, this is equivalent to show

$$\int_{S^{n-1}} \int_{-1}^{n} \rho_K(\theta, t)^{n-1}\text{sign}(t - R)dtd\sigma_{n-2} > 0.$$

The proof of this statement is almost identical to the proof of Theorem 3.3(1).

CHAPTER 4

Approximation of Convex Bodies by Polytopes with few facets

One of the most natural questions in convex geometry is how well a convex body in $\mathbb{R}^n$ can be approximated by polytopes with as few facets (or vertices) as possible. How closely a polytope approximates a convex body can be measured in different ways. We refer to the papers [29, 15] for a more detailed discussion. In this chapter, we are interested in the Banach-Mazur distance which, for convex bodies $K, L \subseteq \mathbb{R}^n$, is defined by

$$d_{BM}(K,L) := \inf\{r \geq 1 \,:\, \exists T \in GL_n(\mathbb{R}) \text{ and } x, y \in \mathbb{R}^n \text{ such that } K \subset TL \subset rK\}.$$

Interestingly, John's theorem 1.13 provides a way to approximate convex bodies by polytopes. We put K in John's position and let contact points $\{u_i\}_{i=1}^m$ and positive constants $\{c_i\}_{i=1}^m$ be the pairs that are described in Theorem 1.13 with $m \leq \binom{n+1}{2} - 1$. Observe that for each $i \in [m]$, we have $\langle x, u_i \rangle \leq 1$ for every $x \in K$. Thus, K is contained in the polytope

$$P := \{x \in \mathbb{R}^n \,:\, \forall i \in [m]\,,\, \langle u_i, x\rangle \leq 1\}\,,$$

which has m facets. Furthermore, P is also in John's positions with the same pairs of contact points $\{u_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^m$. Suppose K is origin-symmetric, we may replace $\{u_i\}_{i=1}^m$ by $\{u_{i,\pm}\}_{i=1}^m$ where $u_{i,+} = u_i$ and $u_{i,-} = u_j$. (Correspondingly, we set $c_{i,\pm} = \frac{1}{2}c_i$) So we may assume P is origin-symmetric. Suppose $x \in P$, using the identity decomposition $I_n = \sum_{i=1}^m c_i u_i u_i^\top$, we have

$$\|x\|_2^2 = \sum_{i=1}^m c_i \langle u_i, x\rangle^2 \leq \sum_{i=1}^m c_i, \tag{4.1}$$

and the last term is equal to n due to $\mathrm{Tr} I_n = \mathrm{Tr} \sum_{i=1}^m c_i u_i u_i^\top$. Therefore, we have the relation $B_2^n \subseteq K \subseteq P \subseteq nB_2^n$. That means we can find a Polytope P with about n^2 facets such that

$$d_{BM}\left(K,\, P\right) \leq \sqrt{n}.$$

This bound is indeed sharp. To see that, we have the following lower bound for approximating the unit ball by polytope:

$$d_{BM}(B_2^n, P) \geq c\sqrt{\frac{n}{\log(\frac{m}{n})}}$$

where m is the number of facets or vertices of P. This was proven independently (and using different methods) in [7],[14], [18] and [25]. Developing from this construction of P and, along with tools such as tensor algebra and Chebyshev's polynomial, Barvinok [8], in 2012, showed that for any symmetric convex body $K \subset \mathbb{R}^n$, one can find a polytope P with a number of facets, m, which is at least polynomial in n, such that $d_{BM}(K, P) = O(\sqrt{\frac{n\log(n)}{\log(m)}})$. In the fine scale, which means that $d_{BM}\left(K,\, P\right) \leq 1+\varepsilon$, this result is improved by Naszï¿œdi, Nazarov, and Ryabogin [57]. We remark that the result of [57] also works in the non-symmetric setting that we describe below.

For the non-symmetric case, one has to modify the definition of the Banach-Mazur distance: For convex bodies $K, L \subseteq \mathbb{R}^n$, the Banach-Mazur distance $d_{BM}(K, L)$ is defined by

$$d_{BM}(K,L) := \inf \left\{ \begin{array}{c} r \geq 1 \;:\; \exists T \in GL_n(\mathbb{R}) \text{ and } x, y \in \mathbb{R}^n \\ \text{such that } K - x \subset T(L-y) \subset r(K-x) \end{array} \right\}.$$

The polytope P constructed from the contact points from John's theorem provides a similar but weaker result. In the case P is not origin-symmetric, we no longer have $\langle u_i,\, x\rangle^2 \leq 1$ for $x \in P$ and thus 4.1 does not hold. Instead, relying on $\sum_{i=1}^m c_i u_i = 0$, we have

$$x = \sum_{i=1}^m c_i \langle x,\, u_i\rangle u_i = \sum_{i=1}^m c_i \left(\langle x,\, u_i\rangle - \min_j \langle x,\, u_j\rangle\right) u_i.$$

Then, using $\langle x,\, u_i\rangle \leq 1$, we obtain

$$\langle x,\, x\rangle \leq \sum_{i=1}^m c_i \left(\langle x,\, u_i\rangle - \min_j \langle x,\, u_j\rangle\right) = -\sum_{i=1}^m c_i \min_j \langle x,\, u_j\rangle \leq n\,|x|$$

and conclude $|x| \leq n$. Therefore, the same argument leads to $d_{BM}\left(K,\, P\right) \leq n$.

The choice of the origin x of K in this definition is crucial. For a symmetric convex body, classical choices such as the barycenter, the center of the John ellipsoid, and the Santalï¿œ point, all coincide with the center of symmetry. However, this is not the case for a general convex body, a fact that makes the choice of origin an obstacle.

The following results make use of the barycenter as the origin. The first result, by Szarek [72], states that for any convex body in $\mathbb{R}^n$, there exists a polytope with either m facets or m vertices, such that $d_{BM}(K, P) \leq \frac{n}{\log(\frac{m}{n})}$. Using a random method, Brazitikos, Chasapis, and Hioni obtain an upper bound of the order of $\frac{n}{\sqrt{\log(\frac{m}{n})}}$, where m is the number of vertices.

Here, we examine the case by taking the origin of a convex body $K \subseteq \mathbb{R}^n$ as the center of its John ellipsoid. We investigate the following problem.

Problem 4.1. Let $R = o(n)$, and $K \subset \mathbb{R}^n$ be a convex body is in John's position. Is there a polytope P with a polynomial number of facets in n, such that

$$K \subset P \subset RK\ ?$$

We prove the following main theorem.

Theorem 4.2. *For a sufficiently large n and for any $c_0\sqrt{n} \leq R \leq c_1 n$, there exists a convex body $K \subset \mathbb{R}^n$ whose John's ellipsoid is centered at the origin, and such that any polytope P satisfying*

$$K \subset P \subset RK,$$

has at least $\exp(C \log(\frac{R^2}{n})\frac{n}{R^2}n)$ facets, where $c_0,\ c_1,\ C > 0$ are some universal constants.

Remark :

1. Notice that the inclusion relations are invariant under linear transformations. We may assume that the constructed body K is in John's position.

2. For each $R \in [c_0\sqrt{n},\ c_1 n]$, the constructed body K in the Theorem 4.2 is a polytope. Moreover, for any polytope P satisfying $K \subset P \subset RK$, we have:

$$\text{number of facets of } P \geq \frac{c}{n} \cdot (\text{number of facets of } K)\,.$$

A direct consequence of Theorem 4.2 solves Problem 4.1 in the negative:

Corollary 4.3. *Let $R_n \to +\infty$ be a positive increasing sequence that satisfies $\lim_{n\to\infty} \frac{R_n}{n} \to 0$. For any constant $k > 0$, there exists a convex body $K \subset \mathbb{R}^n$ in John's position such that for a sufficiently large n such that there is no polytope that has at most n^k number of facets that satisfy*

$$K \subset P \subset R_n K.$$

In the other extreme, we have the following corollary:

Corollary 4.4. *For a sufficiently large n, there exists a convex body $K \subset \mathbb{R}^n$ in John's position such that there is no polytope P that has less than $\exp(cn)$ number of facets, and satisfies*

$$K \subset P \subset \sqrt{n}K,$$

where $c > 0$ is a universal constant.

As we previously mentioned, for a symmetric convex body K, there exists a polytope P with $O(n^2)$ facets such that $K \subset P \subset \sqrt{n}K$. Corollary 4.4 shows that approximating a non-symmetric body, in the same scale of $\sqrt{n}$ could be much more expensive.

The fact that Theorem 4.2 cannot provide a better result when $R = o(\sqrt{n})$ is not surprising. Using a net argument, one can derive the following:

Proposition 4.5. *Suppose $B_2^n \subset K \subset RB_2^n$. For a sufficiently small $\delta > 0$, there exists a polytope P_δ with no more than $\exp(c\log(\frac{2R}{\delta})n)$ facets such that*

$$(1-\delta)P_\delta \subset K \subset P_\delta.$$

Applying the proposition to convex bodies in John's position, we conclude the following.

Corollary 4.6. *Let K be a convex body in $\mathbb{R}^n$ in John's position, where n is sufficiently large. Then, there exists a polytope P with at most $\exp(c\log(n)n)$ facets such that*

$$\frac{1}{2}P \subset K \subset P,$$

where $c > 0$ is a universal constant.

The proof of Theorem 4.2 is presented in Section 4.1. In Section 4.2 we prove Propostion 4.5.

4.1 Proof of the main result

In this section, we prove Theorem 4.2. The proof will be divided into three main propositions. The body K is obtained by intersecting a simplex in John's position with a large number of halfspaces. As long as each halfspace contains the John ellipsoid of the simplex, the new body will be in John's position as well. The construction of the body uses both certain structures and randomness.

Let Δ_n be the regular simplex in $\mathbb{R}^n$ that has an inner radius equal to 1. Using the symmetry of Δ_n and uniqueness of the John ellipsoid, it is not difficult to check that Δ_n is in John's position. Suppose $u_1, \cdots, u_{n+1}$ are contact points of Δ_n. Then, $\langle u_i, u_j \rangle = -\frac{1}{n}$ for $i \neq j$ and $\{-nu_i\}_{i=1}^{n+1}$ are the vertices of Δ_n. We can express Δ_n and Δ_n° in terms of $\{u_i\}_{i=1}^{n+1}$:

$$\Delta_n = \left\{x \in \mathbb{R}^n \,:\, \langle x,\, u\rangle \leq 1 \,\forall u \in \{u_i\}_{i=1}^{n+1}\right\}, \qquad \Delta_n^\circ = \mathrm{conv}\left(\{u_i\}_{i=1}^{n+1}\right).$$

4.1.1 Lower bound of facets

The first proposition below shows how to determine if a convex body cannot be approximated by polytopes that have few facets with a fixed origin.

Proposition 4.7. *Let $K := \{x \in \mathbb{R}^n \,:\, \langle x, y_i \rangle \leq 1 \quad \forall i \in [m]\} \cap L$, where $y_1, ..., y_m$ are vectors in $\mathbb{R}^n$ and L is a convex body in $\mathbb{R}^n$ that has 0 as an interior point. Suppose there are points $x_1, ..., x_m \in K$ such that for some $R > 1$, we have*

$$\langle x_i, y \rangle \begin{cases} = 1 & \text{if } y = y_i, \\ \leq \frac{1}{2R} & \text{if } y = y_j \text{ with } i \neq j. \\ \leq \frac{1}{2R} & \text{if } y \in L^\circ \end{cases}$$

Then, there is no polytope P that has fewer than $\frac{m}{2R}$ facets such that

$$K \subset P \subset RK.$$

Proof. Suppose there exist $\{w_i\}_{i=1}^{m_1} \subset \mathbb{R}^n$ such that $P := \{x \in \mathbb{R}^n \,:\, \langle x, w_l \rangle \leq 1 \,\forall l \in [m_1]\}$ satisfies

$$K \subset P \subset RK.$$

The first inclusion indicates that $\{w_i\}_{i=1}^{m_1} \subset K^\circ$. The second inclusion is equivalent to the following: $\forall x \in \partial K$, $R\langle x, w_l \rangle \geq 1$ for some $l \in [m_1]$. Due to $\langle x_i,\, y_i \rangle = 1$, we also have $x_i \in \partial K$ for $i \in [m]$.

For $l \in [m_1]$, let O_l be the sub-collection of $\{x_i\}_{i=1}^m$ such that $R\langle x_i, w_l \rangle \geq 1$. Observe that $K^\circ = \mathrm{conv}(\{y_i\}_{i=1}^m, L^\circ)$. Thus, w_l can be expressed as a convex combination:

$$w_l = \sum_{i=1}^m \lambda_i y_i + \lambda_{m+1} y,$$

where $y \in L^\circ$, $\lambda_i \geq 0$, and $\sum_i^{m+1} \lambda_i = 1$.

This expression is not necessarily unique, but we fix one such expression. Taking inner product with Rx_i we have

$$\begin{aligned} R\langle x_i, w_l\rangle &= \sum_{i\neq j}^{m} \lambda_j R\langle x_i,\, y_j\rangle + \lambda_{m+1}R\langle x_i,\, y\rangle + \lambda_i R \\ &\leq \frac{1}{2} + \lambda_i R. \end{aligned}$$

If $x_i \in O_l$, then $\lambda_i \geq \frac{1}{2R}$. Due to $\sum_{i=1}^{m+1} \lambda_i = 1$, we conclude that $|O_l| \leq 2R$.

Observe that $\cup_{l\in[m_1]} O_l = \{x_i\}_{i\in[m]}$; we conclude that $m_1 \geq \frac{m}{2R}$. Therefore, P has at least $\frac{m}{2R}$ facets. $\square$

The example in the main theorem will be in the form $K := \{x : \langle x, y_i\rangle \leq 1 \ \forall i \in [m]\} \cap \Delta_n$, where $\{y_i\}_{i=1}^m \subset S^{n-1}$ and Δ_n is a regular simplex in John's position. Then, we will find $\{x_i\}_{i=1}^m$, which satisfies the assumption of Proposition 4.7.

4.1.2 Structure

The following is a deterministic statement about points in S^{n-1}.

Proposition 4.8. *Let $S := S^{n-1} \cap \{x : \langle \beta, x\rangle = 0\}$ for some $\beta \in S^{n-1}$. For $\theta \in S$, let $\theta^{\downarrow} := -\frac{1}{8}\beta + \sqrt{1-\left(\frac{1}{8}\right)^2}\theta$ and $\theta^{\uparrow} := \sqrt{1-(\frac{1}{7})^2}\beta + \frac{1}{7}\theta$. Then,*

1. For $\alpha, \theta \in S$, $\langle \alpha^{\downarrow}, \theta^{\uparrow}\rangle > 0$ implies $\langle \alpha, \theta\rangle > \frac{3}{4}$, and
2. $\langle \theta^{\uparrow}, \theta^{\downarrow}\rangle = \frac{1}{C_0}$ for $\theta \in S$ where $C_0 := \frac{1}{\frac{1}{7}\sqrt{1-\left(\frac{1}{8}\right)^2}\left(1-\sqrt{\frac{48}{63}}\right)} > 1$.

In our construction of K, y_i will be $\theta_i^{\uparrow}$ for some $\theta_i \in S$ and x_i will be $C_0\theta_i^{\downarrow}$. In particular, the first statement of Proposition 4.8 implies that $\langle x_i, y_j\rangle < 0$ when $\langle \theta_i,\, \theta_j\rangle < \frac{3}{4}$.

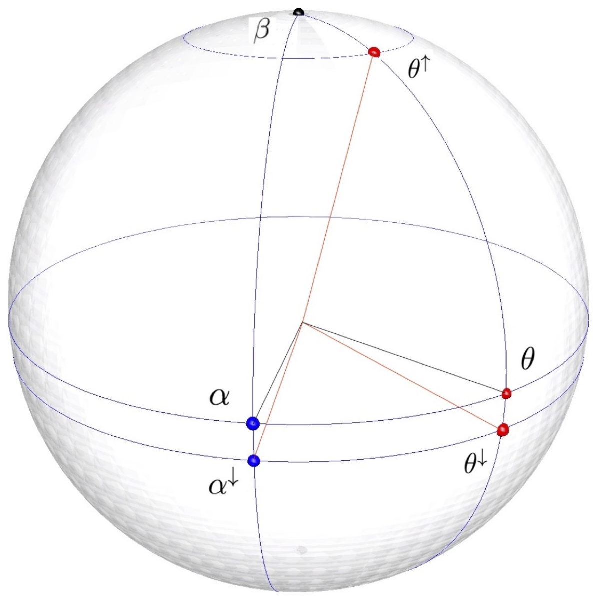


Figure 4.1.1: Inner Product for shifted unit vectors

Proof. We fix $\theta \in S$. For $\alpha \in S$, it can be expressed as

$$\alpha = s\theta + \sqrt{1-s^2}\alpha',$$

where $s = \langle \alpha, \theta \rangle$ and $\alpha' = \frac{\alpha - s\theta}{|\alpha - s\theta|}$. Notice that $\alpha' \perp \theta$ and $\alpha' \in S$. Thus,

$$\begin{aligned} \langle \alpha^{\downarrow}, \theta^{\uparrow} \rangle &= \langle -\frac{1}{8}\beta + \sqrt{1-\left(\frac{1}{8}\right)^2}(s\theta + \sqrt{1-s^2}\alpha'), \sqrt{1-(\frac{1}{7})^2}\beta + \frac{1}{7}\theta \rangle \\ &= -\frac{1}{8}\sqrt{1-(\frac{1}{7})^2} + s\frac{1}{7}\sqrt{1-\left(\frac{1}{8}\right)^2}. \end{aligned}$$

Suppose $\langle \alpha^{\downarrow}, \theta^{\uparrow} \rangle > 0$. Then, $s \geq \frac{7}{8}\sqrt{\frac{48}{49}\frac{64}{63}} = \sqrt{\frac{48}{63}} > \frac{3}{4}$.

Observe that $\langle \theta^{\downarrow}, \theta^{\uparrow} \rangle$ is the same for all $\theta \in S$,

$$\begin{aligned} \langle \theta^{\downarrow}, \theta^{\uparrow} \rangle &= \frac{1}{7}\sqrt{1-\left(\frac{1}{8}\right)^2} - \frac{1}{8}\sqrt{1-(\frac{1}{7})^2} \\ &= \frac{1}{7}\sqrt{1-\left(\frac{1}{8}\right)^2}\left(1-\sqrt{\frac{48}{63}}\right). \end{aligned}$$

Since $0 < \frac{1}{7}\sqrt{1-\left(\frac{1}{8}\right)^2}\left(1-\sqrt{\frac{48}{63}}\right) < 1$, we conclude that $C_0 = \frac{1}{\frac{1}{7}\sqrt{1-\left(\frac{1}{8}\right)^2}\left(1-\sqrt{\frac{48}{63}}\right)} > 1$. $\square$

4.1.3 Randomness

In the construction, we will choose $\{\theta_i\}$ independently and uniformly from a probability distribution. In order to apply Proposition 4.7, we need to choose θ_i so that $\langle x_i,\, y\rangle \le \frac{1}{2R}$ for all $y \in \Delta_n^\circ$. Equivalently, $\rho_{\Delta_n}(\theta_i^\downarrow)$ needs to be larger than $2RC_0$. The uniform randomness on S does not work in this case. Thus, a probability that is compatible with the structure of Δ_n is required.

The following proposition provides a tail bound for hypergeometric distribution.

Proposition 4.9. *For a sufficiently large $n \in \mathbb{N}_+$, let k be a positive integer satisfying $100 < k < \frac{1}{2e^8}n$. Suppose $I,\ J$ are chosen independently and uniformly from $\{W \subset [n] : |W| = k\}$. Then,*

$$\mathbb{P}\left(|I \cap J| \ge \frac{k}{2}\right) \le \left(\frac{2k}{n}\right)^{k/5}.$$

Proof. We may assume that J is fixed. For any positive integer $1 \le l \le k$,

$$\mathbb{P}(|I \cap J| = l) = \frac{\binom{k}{l}\binom{n-k}{k-l}}{\binom{n}{k}}. \tag{4.2}$$

For positive integers $a \ge b$, $\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b(b-1)\cdots 1}$. A standard estimate of $\binom{a}{b}$ is the following:

$$(\frac{a}{b})^b \le \binom{a}{b} \le (\frac{ea}{b})^b.$$

Applying these bounds to (4.2), we have

$$\begin{aligned} \mathbb{P}(|I \cap J| = l) \le & \left(\frac{ek}{l}\right)^l \left(\frac{e(n-k)}{k-l}\right)^{k-l} \left(\frac{k}{n}\right)^k \\ = & \ e^k \left(\frac{k^2}{ln}\right)^l \left(\frac{n-k}{n}\right)^{k-l} \left(\frac{k}{k-l}\right)^{k-l}. \end{aligned} \tag{4.3}$$

Assuming that $l \ge \frac{k}{2}$ and $2k < n$,

$$(\frac{k^2}{ln})^l \le (\frac{2k}{n})^l \le (\frac{2k}{n})^{\frac{k}{2}}.$$

Also, using $(1+x) \le e^x$ for $x \in \mathbb{R}$ we have

$$(\frac{k}{k-l})^{k-l} = (1+\frac{l}{k-l})^{k-l} \le e^l \le e^k.$$

Together with $(\frac{n-k}{n})^{k-l} \le 1$,

$$\begin{aligned} \mathbb{P}(|I \cap J| = l) &\le \exp\left(k - \log\left(\frac{n}{2k}\right)\frac{k}{2} + k\right) \\ &= \exp\left(2k - \log\left(\frac{n}{2k}\right)\frac{k}{2}\right). \end{aligned}$$

If $\frac{n}{2k} \ge e^8$, then $2k \le \log(\frac{n}{2k})\frac{k}{4}$. We have the following:

$$\mathbb{P}(|I \cap J| = l) \le \exp(-\log(\frac{n}{2k})\frac{k}{4}) \qquad \forall l \ge \frac{k}{2}.$$

Using an union bound,

$$\mathbb{P}(|I \cap J| \ge \frac{k}{2}) \le k\exp(-\log(\frac{n}{2k})\frac{k}{4}) \le \exp(-\log(\frac{n}{2k})\frac{k}{5}) = (\frac{2k}{n})^{k/5},$$

where the last inequality requires $k \ge 100$. □

4.1.4 The construction

Let Δ_n be the $n-$dimensional simplex in John's position and $u_1, \cdots, u_{n+1}$ be its contact points. We define $S := S^{n-1} \cap \{x \in \mathbb{R}^n : \langle x, u_1 \rangle = 0\}$. Notice that $\Delta'_n = \Delta_n \cap \{\langle x, u_1\rangle = 0\}$ is a $n-1$ dimensional regular simplex. Let $v_1, ..., v_n \in S$ such that $\{c_n n v_i\}_{i=1}^n$ are vertices of Δ'_n. It is not hard to verify that $c_n \to 1$ as $n \to \infty$. Let $1 \le k \le n$. For $I \subset [n]$ with $|I| = k$, we define

$$v_I = \frac{\sum_{i \in I} v_i}{|\sum_{i \in I} v_i|} \in S.$$

Proposition 4.10. *For a sufficiently large n, let $1 \le k \le \frac{n+3}{4}$. Suppose $I, J \subset \{W \subset [n] \ , |W| = k\}$ satisfy $|I \cap J| < \frac{k}{2}$. Then,*

$$\langle v_I, v_J \rangle \le \frac{3}{4}.$$

Proof. Because $\{v_i\}_{i=1}^n$ are in the vertex directions of a regular simplex, $\langle v_i, v_j \rangle = -\frac{1}{n-1}$

if $i \neq j$. We have

$$\begin{aligned}\langle \sum_{i\in I} v_i, \sum_{i\in I} v_i\rangle &= \sum_{i,j\in I}\Big(-\frac{1}{n-1}+\delta_{ij}(1+\frac{1}{n-1})\Big)\\ &= -\frac{k^2}{n-1}+k(1+\frac{1}{n-1})\\ &= k(1-\frac{k-1}{n-1}).\end{aligned}$$

Thus,

$$v_I = \frac{c_{n,k}}{\sqrt{k}}\sum_{i\in I} v_i,$$

where $c_{n,k} = \frac{1}{\sqrt{1-\frac{k-1}{n-1}}}$. Suppose $1 \leq k \leq \frac{n+3}{4}$, then $1 \leq c_{n,k} \leq \sqrt{\frac{4}{3}}$.

Let $J \subset [n]$ with $|J| = k$. Then,

$$\begin{aligned}\frac{k}{c_{n,k}^2}\langle v_I, v_J\rangle &= \sum_{i\in I}\sum_{j\in J}\langle v_i, v_j\rangle\\ &= \sum_{i\in I}\sum_{j\in J}\Big(-\frac{1}{n-1}+(1+\frac{1}{n-1})\delta_{ij}\Big)\\ &= -\frac{k^2}{n-1}+|I\cap J|(1+\frac{1}{n-1})\\ &\leq |I\cap J|(1+\frac{1}{n-1}).\end{aligned}$$

Suppose $|I \cap J| < \frac{k}{2}$ and n is large enough. Then,

$$\langle v_I, v_J\rangle \leq \frac{c_{n,k}^2}{2}(1+\frac{1}{n-1}) \leq \frac{2}{3}(1+\frac{1}{n-1}) < \frac{3}{4}.$$

$\square$

We are now ready to prove the main theorem.

Proof of Theorem 4.2 . For a sufficiently large n, we fix $100 \leq k \leq \frac{n}{2e^8}$. Then, n and k satisfy the assumptions in Proposition 4.9 and Proposition 4.10. Let $m \subset \mathbb{N}$ be an integer that we will specify later. Let $I_1, I_2, \ldots, I_m$ be chosen independently and uniformly from $\{W \subset [n] : |W| = k\}$. Let u_1 be the β described in Proposition 4.8. In particular, $S := S^{n-1} \cap \{x : \langle u_1, x\rangle = 0\}$. We adapt the definition of $\theta^{\uparrow}$ and $\theta^{\downarrow}$ for $\theta \in S$. Let

$$K := \Delta_n \cap (\cap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, u_{I_i}^{\downarrow}\rangle \leq 1\}).$$

By Propositoin 4.9, we have $\mathbb{P}(|I_i \cap I_j| \geq \frac{k}{2}) \leq (\frac{2k}{n})^{\frac{k}{5}}$. An union bound argument shows that

$$\mathbb{P}(\exists 1 \leq i < j \leq m \text{ such that } |I_i \cap I_j| \geq \frac{k}{2}) \leq \binom{m}{2}(\frac{2k}{n})^{\frac{k}{5}} < m^2(\frac{2k}{n})^{\frac{k}{5}}.$$

By setting $m = (\frac{n}{2k})^{k/20}$, we have

$$\mathbb{P}(\exists 1 \leq i < j \leq m \text{ such that } |I_i \cap I_j| \geq \frac{k}{2}) \leq (\frac{2k}{n})^{\frac{k}{10}}. \tag{4.4}$$

Since the probability is strictly smaller than 1, there exists a sample such that $|I_i \cap I_j| < \frac{k}{2}$ for all $1 \leq i < j \leq m$. From now on, we fix such a sample.

We want to apply Proposition 4.7 with $L = \Delta_n$, $y_i = v_{I_i}^{\uparrow}$, and $x_i = \frac{1}{\langle v_{I_i}^{\downarrow}, v_{I_i}^{\uparrow}\rangle} v_{I_i}^{\downarrow} = C_0 v_{I_i}^{\downarrow}$, where C_0 is the constant defined in Proposition 4.8. We start verifying the assumptions that are described in Proposition 4.7.

First, $\Delta_n^{\circ} = \text{conv}\{u_1, \cdots, u_{n+1}\}$. Because $\{c_n n v_i\}_{i=1}^m \subset \Delta_n^{'} \subset \Delta_n$, $\langle c_n n v_i, u_j\rangle \leq 1$ for $i \in [n]$ and $j \in [n+1]$. Thus, for any $I \subset [n]$ with $|I| = k$ and $j \in [n+1]$,

$$\langle v_I, u_j\rangle = \frac{c_{n,k}}{\sqrt{k}} \sum_{i \in I} \langle v_i, u_j\rangle \leq \frac{c_{n,k}}{\sqrt{k}} k \frac{1}{c_n n} \leq \frac{c_{n,k}\sqrt{k}}{c_n n}.$$

Since $\langle -u_i, u_j\rangle = -(1 + \frac{1}{n})\delta_{ij} + \frac{1}{n} \leq \frac{1}{n}$,

$$\begin{aligned}
\langle v_I^{\downarrow}, u_j\rangle &= \frac{1}{8}\langle -u_1, u_j\rangle + \sqrt{1 - \left(\frac{1}{8}\right)^2} \langle v_I, u_j\rangle \\
&\leq \frac{1}{8}\frac{1}{n} + \sqrt{1 - \left(\frac{1}{8}\right)^2} \frac{c_{n,k}\sqrt{k}}{c_n n}.
\end{aligned}$$

Since $\frac{n}{2e^8} \geq k \geq 1$, $1 \leq c_{n,k} \leq \sqrt{\frac{4}{3}}$ and $c_n \to 1$ as $n \to +\infty$,

$$\begin{aligned}
\frac{1}{8}\frac{1}{n} + \sqrt{1 - \left(\frac{1}{8}\right)^2} \frac{c_{n,k}\sqrt{k}}{c_n n} &\leq \frac{1}{n} + \frac{1}{c_n}\sqrt{\frac{4}{3}}\frac{\sqrt{k}}{n} \\
&\leq \left(\frac{1}{c_n}\sqrt{\frac{4}{3}} + 1\right)\frac{\sqrt{k}}{n} \\
&\leq 3\frac{\sqrt{k}}{n}.
\end{aligned}$$

We conclude that

$$\langle C_0 v_I^{\downarrow}, u_j \rangle \leq 3C_0 \frac{\sqrt{k}}{n}.$$

For $y \in \Delta_n^{\circ}$, it can be written as a convex combination of $\{u_i\}_{i=1}^{n+1}$. Thus, the same inequality holds:

$$\langle C_0 v_{I_i}^{\downarrow},\, y \rangle \leq 3C_0 \frac{\sqrt{k}}{n} \qquad \forall y \in \Delta_n^{\circ}. \tag{4.5}$$

Let $i, j \in [m+1]$ with $i \neq j$. Applying Proposition 4.10, we obtain $\langle v_{I_i}, v_{I_j} \rangle < \frac{3}{4}$ since $|I_i \cap I_j| < \frac{3}{4}$. According to Proposition 4.8, we obtain $\langle C_0 v_{I_i}^{\downarrow}, v_{I_j}^{\uparrow} \rangle < 0$. In the case $i = j$, by definition we have $\langle C_0 v_{I_i}^{\downarrow}, v_{I_j}^{\uparrow} \rangle = 1$. To summarize,

$$\langle c_1 v_{I_i}^{\downarrow}, y \rangle \begin{cases} = 1 & \text{if } y = v_{I_i}^{\uparrow}, \\ \leq 0 & \text{if } y = v_{I_j}^{\uparrow} \text{ with } j \neq i, \\ \leq 3C_0 \frac{\sqrt{k}}{n} & \text{if } y \in \Delta_n^{\circ}. \end{cases} \tag{4.6}$$

Now, we can apply Proposition 4.7 with $m = (\frac{n}{2k})^{k/10}$, $y_i = u_{I_i}^{\uparrow}$, $x_i = C_0 u_{I_i}^{\downarrow}$, $L = \Delta_n$ and $R = \frac{n}{6C_0\sqrt{k}}$ with the condition that $100 \leq k \leq \frac{n}{2e^8}$. Expresssing these relations in terms of R and n, we have

$$k = (\frac{n}{6C_0 R})^2, \qquad m = \left(\frac{18C_0^2 R^2}{n}\right)^{(\frac{n}{6C_0 R})^2/20}, \quad \text{and } \frac{\sqrt{2}e^4}{6C_0}\sqrt{n} \leq R \leq \frac{n}{60C_0}.$$

The lower bound of the facets of the polytope P in Proposition 4.7 is $\frac{m}{2R}$. To simplify m, we further restrict $R > \sqrt{en}$ so that $\frac{R^2}{n} > e$. Since $C_0 > 1$, we have $\log(18C_0^2) > 0$. Thus,

$$\log(\frac{18C_0^2 R^2}{n}) = \log(18C_0^2) + \log(\frac{R^2}{n}) \geq \log(\frac{R^2}{n}) > 1 > 0.$$

Then,

$$\begin{aligned} \frac{m}{2R} &= \exp(-\log(2R) + \log(\frac{18C_0^2 R^2}{n})\frac{n^2}{720C_0^2 R^2}) \\ &\geq \exp(-\log(2R) + \frac{1}{720C_0^2}\log(\frac{R^2}{n})\frac{n^2}{R^2}) \\ &\geq \exp(-\log(2n) + C' \log(\frac{R^2}{n})\frac{n^2}{R^2}), \end{aligned}$$

when $C' = \frac{1}{720C_0^2} > 0$. In order to take care of the $\log(2n)$ term we need to check the last term carefully. First,

$$\frac{d}{dR}\log(\frac{R^2}{n})\frac{n^2}{R^2} = -\frac{2n^2}{R^3}\left(\log(\frac{R^2}{n}) - 1\right) < 0$$

for $R > \sqrt{en}$. Let $c_1 = \min\left\{1, \sqrt{\frac{C'}{8}}, \frac{1}{60C_0}\right\}$. Suppose $R = c_1 n$, we have

$$C'\log(\frac{R^2}{n})\frac{n^2}{R^2} = \frac{C'}{c_1^2}\log(n) + \frac{C'}{c_1^2}\log({c_1}^2) > 4\log(n),$$

where the last inequality holds for large n. Together with $2\log(n) \geq \log(2n)$,

$$\frac{1}{2}C'\log(\frac{R^2}{n})\frac{n^2}{R^2} \geq \log(2n)$$

when $R = c_1 n$. Since $\log(\frac{R^2}{n})\frac{n}{R^2}$ is a decreasing function for $R > \sqrt{en}$, $\frac{1}{2}C'\log(\frac{R^2}{n})\frac{n}{R^2} \geq \log(2n)$ for $\sqrt{en} < R < c_1 n$. Therefore, we conclude that for $c_0\sqrt{n} < R < c_1 n$,

$$\frac{m}{2R} \geq \exp(C\log(\frac{R^2}{n})\frac{n^2}{R^2})$$

where $C > 0$ is an universal constants. Therefore, for $c_0\sqrt{n} \leq R \leq c_1 n$, there exists a convex body $K \subset \mathbb{R}^n$ in John's position such that no polytope P that has less than $\exp(C\log(\frac{R^2}{n})\frac{n^2}{R^2})$ facets satisfies

$$K \subset P \subset RK.$$

□

4.2 Upper bound for small R

Proposition 4.11. *Suppose $B_2^n \subset K \subset RB_2^n$. For $0 < \delta < 1$, there exists a polytope P_δ with no more than $\exp(c\log(\frac{2R}{\delta})n)$ facets such that $(1-\delta)P_\delta \subset K \subset P_\delta$. Here $c > 0$ is an universal constant.*

Proof. Let $B_2^n \subset K \subset RB_2^n$ be a convex body. Let $h : S^{n-1} \to [1, R]$ be the support function of K. Observe that h is also the gauge function of $\frac{1}{R}B_2^n \subset K^\circ \subset B_2^n$. Thus, h is a $R-$Lipschitz function.

Let $\mathscr{N}$ be a $\frac{\delta}{2R}$-net of S^{n-1}. We define

$$P_\delta := \{x \in \mathbb{R}^n \,:\, \forall \alpha \in \mathscr{N} \, \langle \alpha, \, x \rangle \leq h(\alpha)\} .$$

Thus, P_δ is a polytope with at most $|\mathscr{N}|$ number of facets. Recall that by a volumetric argument, the size of a $\varepsilon-$net on S^{n-1} is bounded by $\exp\left(c \log\left(\frac{1}{\varepsilon}\right) n\right)$ for an universal constant $c > 0$. Hence, P_δ has no more than $\exp\left(c \log\left(\frac{2R}{\delta}\right) n\right)$ number of facets.

Since

$$K = \left\{x \in \mathbb{R}^n \,:\, \forall \alpha \in S^{n-1} \, \langle \alpha, \, x \rangle \leq h(\alpha)\right\},$$

we have $K \subset P_\delta$. Observe that

$$(1-\delta) P_\delta = \{x \in \mathbb{R}^n \,:\, \forall \alpha \in \mathscr{N} \, \langle \alpha, \, x \rangle \leq (1-\delta) h(\alpha)\} .$$

For $x \in \partial K$, there exists θ such that $\langle x, \, \theta \rangle = h(\theta)$. We pick $\alpha \in \mathscr{N}$ such that $\|\alpha - \theta\| < \frac{\delta}{2R}$. Then,

$$\langle x, \, \alpha \rangle = \langle x, \, \theta \rangle + \langle x, \, \alpha - \theta \rangle \geq h(\theta) - |x| \, |\alpha - \theta| > h(\theta) - R\frac{\delta}{2R} = h(\theta) - \frac{\delta}{2}. \quad (4.7)$$

Since h is a $R-$Lipstichz continuous function, we have

$$h(\theta) \geq h(\alpha) - \frac{\delta}{2}.$$

Together with $h(\alpha) \geq 1$, the equation (4.7) becomes

$$\begin{aligned} \langle x, \, \alpha \rangle &> h(\alpha) - \delta \\ &\geq (1-\delta) h(\alpha) . \end{aligned}$$

Thus, $x \notin (1-\delta) P_\delta$. In paricular, we conclude the radial function of K is always greater than the radial funciton of $(1-\delta) P_\delta$. Therefore, we have $(1-\delta) P_\delta \subset K$. $\square$

CHAPTER 5

Ulam's floating body

5.1 Introduction

5.1.1 Metronoids

We study a new family of convex bodies $\mathrm{M}_\delta(K)$ associated to K, where $0 < \delta < |K|$ is a parameter. Given a Borel measure μ on $\mathbb{R}^n$, the *metronoid* associated to μ is the convex set defined by

$$\mathrm{M}(\mu) = \bigcup_{\substack{0 \le f \le 1, \\ \int_{\mathbb{R}^n} f \, \mathrm{d}\mu = 1}} \left\{ \int_{\mathbb{R}^n} y f(y) \, \mathrm{d}\mu(y) \right\},$$

where the union is taken over all functions $0 \le f \le 1$ for which $\int_{\mathbb{R}^n} f \, \mathrm{d}\mu = 1$ and $\int_{\mathbb{R}^n} y f(y) \, \mathrm{d}\mu(y)$ exists. Note that for a discrete measure of the form $\sum_{i=1}^N \delta_{x_i}$, the corresponding metronoid is the convex hull of $x_1, \ldots, x_N$. Hence $\mathrm{M}(\mu)$ can be thought of as a fractional extension of the convex hull.

Our main object $\mathrm{M}_\delta(K)$ is the metronoid generated by the uniform measure on K with total mass $\delta^{-1}|K|$. Namely, let μ be the measure whose density with respect to Lebesgue measure is $\delta^{-1}\mathbf{1}_K$. Then $\mathrm{M}_\delta(K) := \mathrm{M}(\mu)$. It turns out that $M_\delta(K)$ is intimately related to the following long-standing problem proposed by Ulam, see e.g., [**?**, **?**, **?**, 23]: Is a solid of uniform density which floats in water in every position a Euclidean ball? While counterexamples were found in $\mathbb{R}^2$ (convex and non-convex) and $\mathbb{R}^3$ (only non-convex), this problem remains open in arbitrary dimensions. For a full account of the progress made on this problem, see [**?**] and references therein.

As we show in Section 5.2.2 below, along with a precise description of Ulam's problem, one can restate Ulam's problem in terms of $\mathrm{M}_\delta(K)$ as follows: If $\mathrm{M}_\delta(K)$ is a Euclidean ball, must K be a Euclidean ball as well? For that reason, we call $\mathrm{M}_\delta(K)$ an *Ulam floating body*. As far as we know, this construction and its relation to Ulam's problem is not mentioned anywhere in the literature.

We also define weighted variations of $\mathrm{M}_\delta(K)$ where the weight is given by a positive continuous function $\phi : K \to \mathbb{R}$. Namely, we define

$$\mathrm{M}_\delta(K, \phi) := \mathrm{M}\left(\frac{\phi(x)}{\delta}\mathbf{1}_K(x)\,\mathrm{d}x\right).$$

To understand $\mathrm{M}_\delta(K)$ geometrically, for every direction $\theta \in \mathbb{S}^{n-1}$, let $H(\delta, \theta)$ be the hyperplane orthogonal to θ that cuts a set of volume δ from K. That is

$$C_\delta(\theta) = K \cap \{x : \langle x, \theta\rangle \geq \langle y_\theta, \theta\rangle\}$$

has volume δ for any $y_\theta \in H(\delta, \theta)$. Then, the barycenter of $C_\delta(\theta)$ is a point on the boundary of $\mathrm{M}_\delta(K)$. The proof of these properties will be shown in Section 5.2.

As illustrated in Figure 5.1.1, the body $\mathrm{M}_\delta(K)$ is closely related to the convex floating body K_δ, introduced independently in [**?**] and [67]. Using the above notation, we have that

$$K_\delta = \bigcap_{\theta \in \mathbb{S}^{n-1}} \{x : \langle x, \theta\rangle \leq \langle y_\theta, \theta\rangle\},$$

which is a non-empty convex set for a sufficiently small $0 < \delta$. In fact, $\mathrm{M}_\delta(K)$ is isomorphic to K_δ in the sense that $K_{\frac{e-1}{e}\delta} \subseteq \mathrm{M}_\delta(K) \subseteq K_{\frac{1}{e}\delta}$. We discuss this property in the more general case of weighted Ulam floating bodies in Section 5.2.3 below (also see Theorem 5.1).

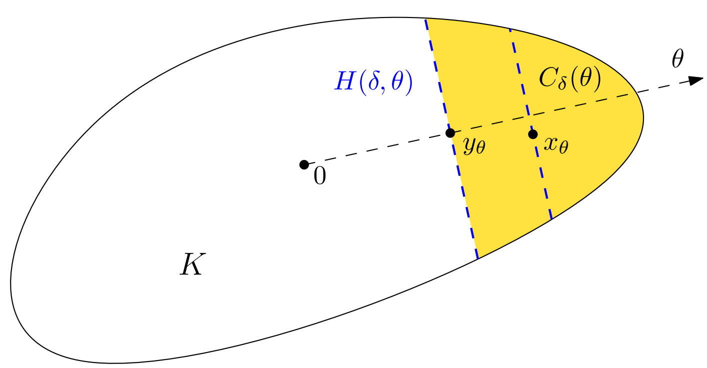


Figure 5.1.1: $H(\delta, \theta)$ is the hyperplane orthogonal to θ that cuts a set $C_\delta(\theta)$ of volume δ from a convex body K. The point x_θ is the barycenter of $C_\delta(\theta)$. Then, $K_\delta \subset \{x : \langle x, \theta\rangle \leq \langle y_\theta, \theta\rangle\}$ while ${}_\delta(K) \subset K \cap \{x : \langle x, \theta\rangle \leq \langle x_\theta, \theta\rangle\}$.

The convex floating body is a natural variation of Dupin's floating body [19] from 1822. Dupin's floating body $K_{[\delta]}$ is defined as the body whose boundary is the set of points that are the barycenters of all the sections of K of the form $K \cap H(\delta, \theta)$, where $H(\delta, \theta)$ are the aforementioned hyperplanes that cut a set of volume δ from K. However, while K_δ coincides with $K_{[\delta]}$ whenever $K_{[\delta]}$ is convex (e.g., for centrally-symmetric K, see [54]), in the non-centrally symmetric case, Dupin's floating body need not be convex, as in the case of some triangles in $\mathbb{R}^2$ (see e.g., [44]). Restating the above, every point on the boundary of K_δ is the barycenter of $K \cap H(\delta, \theta)$ for some θ, but the converse holds only if Dupin's floating body is convex.

Note that our construction $M_\delta(K)$ corresponds nicely to both definitions, that of the floating body and that of the convex floating body in the sense that it enjoys being convex as well as having the property that a point is on the boundary of $\mathrm{M}_\delta(K)$ if and only if it is the barycenter of a set of volume δ that is cut off by a hyperplane.

5.1.2 Main results

We present three main theorems concerning Ulam floating bodies. While the first result establishes an explicit relation between (weighted) floating bodies and (weighted) Ulam floating bodies, the other two results are the analogous counterparts to the classical floating bodies.

5.1.2.1 Relation to floating bodies

Our first theorem shows that (weighted) Ulam floating bodies are isomorphic, in a sense, to (weighted) floating bodies. Weighted floating bodies were introduced in [74] (also see [9, **?**] for recent applications) as follows. Let $K \subseteq \mathbb{R}^n$ be a convex body, $0 < \delta$, and $\phi : K \to \mathbb{R}$ be integrable and such that $\phi > 0$ almost everywhere with respect to Lebesgue measure. For a hyperplane H in $\mathbb{R}^n$, let $H^\pm$ be the half-spaces separated by H. Then the weighted floating body $F_\delta(K, \phi)$ is defined as

$$F_\delta(K,\phi) = \bigcap \left\{ H^- : \int_{H^+ \cap K} \phi(x)\, \mathrm{d}x \le \delta \right\}.$$

Note that for $\phi \equiv 1$, we have that $F_\delta(K, \phi) = K_\delta$.

We prove the following.

Theorem 5.1. *Let K be a convex body in $\mathbb{R}^n$, and let $\phi : K \to \mathbb{R}^+$ be an integrable log-concave function. Then for all $0 < \delta < |K|$, we have*

$$F_{\frac{e-1}{e}\delta}(K,\phi) \subseteq \mathrm{M}_\delta(K,\phi) \subseteq F_{\frac{\delta}{e}}(K,\phi).$$

In particular, for $\phi \equiv 1$ we have that

$$K_{\frac{e-1}{e}\delta} \subseteq \mathrm{M}_\delta\left(K, \phi\right) \subseteq K_{\frac{\delta}{e}}.$$

5.1.2.2 Smoothness of Ulam floating bodies

Our second main result states that the boundary $\partial \mathrm{M}_\delta\left(K\right)$ of an Ulam floating body $\mathrm{M}_\delta\left(K\right)$ is always smoother than the boundary of K.

Theorem 5.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body, Suppose that $\partial K \in C^k$ for some $k \geq 0$. Then for any $0 < \delta < |K|$, we have that $\partial \mathrm{M}_\delta\left(K\right) \in C^{k+1}$.*

We remark that in the case of the convex floating body, an analogous result to Theorem 5.2 is known only in the centrally-symmetric case [54]. The main reason for this is that the proof in [54] relies on the above mentioned fact that in the centrally-symmetric case the convex floating convex body and Dupin's floating body coincide.

5.1.2.3 Affine Surface Area

The affine surface area was introduced by W. Blaschke [10] in 1923 for smooth convex bodies in Euclidean space of dimensions 2 and 3, and extended to $\mathbb{R}^n$ by K. Leichtweiss [43]. Given a convex body $K \subseteq \mathbb{R}^n$ with a sufficiently smooth boundary, let $\kappa_K\left(x\right)$ be the generalized Gaussian curvature at $x \in \partial K$ (see Definition 1.12), and μ_K the surface area measure on ∂K. The affine surface area of K is defined by

$$as\left(K\right) = \int_{\partial K} \kappa_K\left(x\right)^{\frac{1}{n+1}}\, \mathrm{d}\mu_K.$$

Even though it proved to be much more difficult to extend the notion of affine surface area to general convex bodies than other notions, like surface area measures or curvature measures, successively such extensions were achieved, by e.g., Leichtweiss [43], Lutwak [47], who also proved the long conjectured upper semicontinuity of affine surface area [47] and by Schï¿œtt and Werner [67] who showed that the affine surface area arises as a limit of the volume difference of the convex body and its floating body. All these extensions coincide as was shown in [65, **?**].

Affine surface area is among the most powerful tools in equiaffine differential geometry (see Andrews [3, 4], A. Stancu [70, 71], Ivaki [37], Ivaki and Stancu [38] and Ludwig and Reitzner [46]). It appears naturally as the Riemannian volume of a smooth convex hypersurface with respect to the affine metric (or Berwald-Blaschke metric), see e.g., the thorough monograph of Leichtweiss [44] or the book by Nomizu and Sasaki [58]. In particular the upper semicontinuity proved to be critical in the solution of the affine Plateau problem by Trudinger and Wang [73].

Applications of affine surface areas have been manifold. For instance, affine surface area appears in best and random approximation of convex bodies by polytopes, see Bï¿œrï¿œczky Jr. [12, 11], Gruber [27, 28, 30], Ludwig [45], Reitzner [59], Schï¿œtt [64, 66] and Grote and Werner, [?] and Schï¿œtt and Werner [68]. Furthermore, recent contributions indicate astonishing developments which open up new connections of affine surface area to, e.g., concentration of volume (e.g. [21, 49]), spherical and hyperbolic spaces [?, ?], geometric inequalities [52, 76] and information theory (e.g. [6, 17, 50, 51, 77, ?]).

The L_p-affine surface area is a generalization of the classical affine surface area and a central part in the L_p-Brunn-Minkowski theory. It was introduced by Lutwak [48] for $p > 1$ (see also Hug [36] and Meyer and Werner [55]) and extended for all $p \in [-\infty, \infty]$ in [69]. For $-\infty < p < \infty$, the L_p-affine surface area of a convex body $K \subseteq \mathbb{R}^n$ is given by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x), \tag{5.1}$$

where $N_K(x)$ is the outer normal of K at x. For $p = \pm\infty$, it is given by

$$as_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} d\mu_K(x). \tag{5.2}$$

As in the case of the classical affine surface area, several geometric extensions for the L_p-affine surface area have been proven. We refer to [69, 75] and references therein. These extensions all involve a construction of a special family of convex bodies $\{K_t\}_{t>0}$ which is related to a given convex body K, where the L_p-affine surface area can be written as a limit involving their volume difference.

We prove the following theorem which shows that this can also be achieved using weighted Ulam floating bodies.

Theorem 5.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $\phi : K \to (0,\infty)$ be a continuous function. Then*

$$\lim_{\delta \searrow 0} \frac{|K| - |\mathrm{M}_\delta(K,\phi)|}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \, \mathrm{d}\mu_K(x), \tag{5.3}$$

where $c_n = 2\frac{n+1}{n+3}\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$.

For $-\infty \le p \le \infty$, $p \ne -n$, define the function $\phi_p : \partial K \to [0, \infty]$ by

$$\phi_p(x) = \frac{\langle x,\ N_K(x)\rangle^{\frac{n(n+1)(p-1)}{2(n+p)}}}{\kappa_K(x)^{\frac{n(p-1)}{2(n+p)}}}. \tag{5.4}$$

Note that $\phi_1(x) = 1$ for all $x \in \partial K$. If $\kappa_K(x) = 0$, which is the case, e.g., when $K = P$ is a polytope and x belongs to an $(n-1)$-dimensional facet of P, then

$$\phi_p(x) = \begin{cases} \infty & p > 1 \text{ or } p < -n \\ 0 & -n < p < 1. \end{cases}$$

If $\kappa_K(x) = \infty$, which is the case, e.g., when $K = P$ is a polytope and x is a vertex of P, then

$$\phi_p(x) = \begin{cases} 0 & p > 1 \text{ or } p < -n \\ \infty & -n < p < 1. \end{cases}$$

If K and p are such that ϕ_p is continuous on ∂K, we extend ϕ_p to a continuous function on K which we call again ϕ_p.

Applying Theorem 5.3 with ϕ_p yields the following extension of L_p-affine surface areas.

Corollary 5.4. *Let $K \subseteq \mathbb{R}^n$ be a convex body. If ϕ_p is continuous on K, then*

$$\lim_{\delta \searrow 0} \frac{|K| - |\mathrm{M}_\delta(K, \phi_p)|}{\delta^{\frac{2}{n+1}}} = c_n\, as_p(K).$$

In particular, for $p = 1$ we have

$$\lim_{\delta \searrow 0} \frac{|K| - |M_\delta(K)|}{\delta^{\frac{2}{n+1}}} = c_n\, as_1(K).$$

5.1.3 Some additional notation

We denote by $B_2^n(u, \rho)$ the Euclidean ball with radius $\rho > 0$ centered at u. For $u, v \in \mathbb{R}^n$, $[u, v]$ will denote the line segment between u and v. We denote the interior of a set $C \subseteq \mathbb{R}^n$ by $\mathrm{int}(C)$. In the sequel, we will always assume that our convex body K contains the origin in its interior. Let O_n denote the orthogonal group of dimension n.

In Section 5.2 we discuss some properties of Ulam floating bodies, and prove Theorems 5.1 and 5.2. Section 5.3 is devoted for the proof of Theorem 5.3.ester.

5.2 Properties of Ulam floating bodies

5.2.1 Basic properties

We begin with the basic property for $\mathrm{M}(\mu)$:

Proposition 5.5. *Consider a Borel measure μ with bounded support, absolutely continuous, and $1 < \mu(\mathbb{R}^n) < +\infty$. Let*

$$R(\theta, \lambda) := \sup\{R \in \mathbb{R},\ \mu(\{x \in \mathbb{R}^n : \langle x, \theta\rangle \geq R\} \geq \lambda)\}$$

and $f_\theta(x)$ be the indicator function of $\{x : \langle x, \theta\rangle \geq R(\theta, 1)\}$. Then,

$$x_\mu(\theta) := \int_{\mathbb{R}^n} x f_\theta(x)\ \mathrm{d}m \in \mathrm{M}(\mu)\,, \text{and} \quad h_{\mathrm{M}(\mu)}(\theta) = \langle x_\mu(\theta), \theta\rangle.$$

Furthermore, for any sufficiently small $\eta > 0$, there exists an $\varepsilon > 0$ such that if $y \in \mathrm{M}(\mu)$ satisfies $\langle y, \theta\rangle > h_{\mathrm{M}(\mu)}(\theta) - \varepsilon$, then $|y - x_\mu(\theta)| \leq \eta$. In particular, it means that $x_\mu(\theta)$ is the unique extreme point of $\mathrm{M}(\mu)$ in θ direction.

Proof. Fix $\theta \in S^{n-1}$. By definition, $x_\mu(\theta) \in \mathrm{M}(\mu)$. Now let $y \in \mathrm{M}(\mu)$ and $0 \leq f(x) \leq 1$ be a function such that $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\mu(x) = 1$ and $\int_{\mathbb{R}^n} x f(x)\,\mathrm{d}\mu(x) = y$. Notice that we have $\int_{\langle\theta, x\rangle \geq R(\theta, \kappa)} f_\theta(x)\ \mathrm{d}\mu(x) = \kappa$ for $\kappa \in (0, 1]$. Thus, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f_\theta(x)\langle x, \theta\rangle\,\mathrm{d}\mu(x) &= \int_{\langle\theta, x\rangle \geq R(\theta, \kappa)} f_\theta(x)\langle x, \theta\rangle d\mu(x) + \int_{\langle\theta, x\rangle < R(\theta, \kappa)} f_\theta(x)\langle x, \theta\rangle\,\mathrm{d}\mu(x) \\ &> \underbrace{\int_{\langle\theta, x\rangle \geq R(\theta, \kappa)} f_\theta(x)\langle x, \theta\rangle d\mu(x) + R(\theta, 1)(1-\kappa)}_{L_\kappa}.\end{aligned}$$

Notice that, due to absolutely continuity of μ, L_κ is a monotone increasing function and converge to $\int_{\mathbb{R}^n} f_\theta(x)\langle x, \theta\rangle\,\mathrm{d}\mu(x)$ as $\kappa \nearrow 1$. From now on, we fix $\kappa = \int_{\langle x, \theta\rangle \geq R(\theta, 1)} f(x)\ \mathrm{d}\mu(x)$, we have

$$\int_{\mathbb{R}^n} f(x)\langle x, \theta\rangle\,\mathrm{d}\mu(x) \leq (1-\kappa) R(\theta, 1) + \int_{\langle\theta, x\rangle \geq R(\theta, 1)} f(x)\langle x, \theta\rangle\,\mathrm{d}\mu(x)\,. \qquad (5.5)$$

For $\langle x,\, \theta\rangle \geq R\left(\theta,\, \kappa\right)$, $f\left(x\right) \leq 1 = f_\theta\left(x\right)$. Then,

$$
\begin{aligned}
&\int_{\langle x,\,\theta\rangle \geq R(\theta,\,\kappa)} f_\theta\left(x\right)\langle x,\, \theta\rangle \,\mathrm{d}\mu\left(x\right) - \int_{\langle \theta,\,x\rangle \geq R(\theta,\,1)} f\left(x\right)\langle x,\, \theta\rangle \,\mathrm{d}\mu\left(x\right)\\
=&\int_{\langle x,\,\theta\rangle \geq R(\theta,\,\kappa)} \left(f_\theta\left(x\right) - f\left(x\right)\right)\langle x,\, \theta\rangle \,\mathrm{d}\mu\left(x\right) - \int_{\langle x,\,\theta\rangle \in [R(\theta,\,1),\,R(\theta,\,\kappa)]} f\left(x\right)\langle x,\, \theta\rangle \,\mathrm{d}\mu\left(x\right)\\
\geq&\int_{\langle x,\,\theta\rangle \geq R(\theta,\,\kappa)} \left(f_\theta\left(x\right) - f\left(x\right)\right)R\left(\theta,\,\kappa\right)\,\mathrm{d}\mu\left(x\right) - R\left(\theta,\,\kappa\right)\int_{\langle x,\,\theta\rangle \in [R(\theta,\,1),\,R(\theta,\,\kappa)]} f\left(x\right)\,\mathrm{d}\mu\left(x\right)\\
=&R\left(\theta,\kappa\right)\left(\kappa - \kappa\right) = 0.
\end{aligned}
$$

Replacing the second summand in 5.5, we have

$$
\int_{\mathbb{R}^n} f\left(x\right)\langle x,\, \theta\rangle \,\mathrm{d}\mu\left(x\right) \leq L_\kappa < \langle x_\mu\left(\theta\right),\, \theta\rangle.
$$

In particular, this implies that $h_{\mathrm{M}(\mu)}\left(\theta\right) = \langle x_\mu\left(\theta\right),\, \theta\rangle$. Next, let's consider the distance between x and y. Let $M = \sup_{x\in\mathrm{supp}(\mu)}|x|$. Then, we have

$$
\begin{aligned}
|y - x| &\leq M\int_{\mathbb{R}^n} \left|f\left(x\right) - f_\theta\left(x\right)\right|\,\mathrm{d}\mu\left(x\right)\\
&= M\left[\int_{\langle \theta,\,x\rangle \geq R(\theta,\,1)} f_\theta\left(x\right) - f\left(x\right)\,\mathrm{d}\mu\left(x\right) + \int_{\langle \theta,\,x\rangle < R(\theta,\,1)} f_\theta\left(x\right)\,\mathrm{d}\mu\left(x\right)\right]\\
&= 2M\left[1 - \kappa\right].
\end{aligned}
$$

For any given $\eta > 0$, we set $\varepsilon = h_{\mathrm{M}(\mu)}\left(\theta\right) - L_{1-\frac{\eta}{2M}}$. If $\langle y,\, \theta\rangle \geq h_{\mathrm{M}(\mu)}\left(\theta\right) - \varepsilon$, then, we have $L_{1-\frac{\eta}{2M}} \leq L_\kappa$. By monotoniciy, we have $1 - \frac{\eta}{M} \leq \kappa$. Therefore, it leads to $|y - x| \leq \eta$. □

Corollary 5.6. *Consider a Borel measure μ with bounded support, absolutely continuous, and $1 < \mu\left(\mathbb{R}^n\right) < +\infty$. Then, $\mathrm{M}\left(\mu\right)$ is a convex compact set.*

Proof. Convexity is immediate from its definition. So we want to show $\mathrm{M}\left(\mu\right)$ is closed. If a sequence of points in $\mathrm{M}\left(\mu\right)$ converge to some y, then, y is a boundary point of $\mathrm{M}\left(\mu\right)$. No □

For $\theta \in \mathbb{S}^{n-1}$ and $d \in \mathbb{R}$, we define the hyperplane orthogonal to θ at distance d from the origin by $H\left(\theta,\, d\right) := \{x \in \mathbb{R}^n \,:\, \langle x,\, \theta\rangle = d\}$. We also define one of the closed half-spaces determined by $H(\theta,\, d)$ by $H^+(\theta,\, d) := \{x \in \mathbb{R}^n \,:\, \langle x,\, \theta\rangle \geq d\}$. The function

$$\begin{aligned} \mathbb{S}^{n-1} \times \mathbb{R} &\longrightarrow \left[0, \int_K \phi(z)dz\right] \\ (\theta, d) &\longrightarrow \delta(\theta,\, d) := \int_{K\cap H^+(,\theta)d} \phi(z)\ \mathrm{d}z \end{aligned}$$

is continuous in the product metric. Observe also that the function $(\theta, r) \to (\theta, \delta(\theta,\, r))$ is a bijection from

$$\left\{(\theta, r) \,:\, \theta \in \mathbb{S}^{n-1},\, -h_K(-\theta) \leq r \leq h_K(\theta)\right\}$$

to $\mathbb{S}^{n-1} \times \left[0,\, \int_K \phi(x)\ \mathrm{d}x\right]$. We denote

$$(\theta,\, \delta) \to (\theta,\, d(\theta, \delta)) \tag{5.6}$$

as the inverse function of $(\theta,\, d) \to (\theta,\, \delta(\theta,\, d))$, which is also a continuous function. Abusing the notation we denote

$$H^+(,\, \theta)\,\delta := H^+(,\, \theta)\, d(\theta,\, \delta), \tag{5.7}$$

Let $h_{\mathrm{M}_\delta(K,\phi)}(\theta)$ be the support function of $\mathrm{M}_\delta(K, \phi)$. By definition of $\mathrm{M}_\delta(K, \phi)$,

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \max_{x\in \mathrm{M}_\delta(K,\phi)} \langle \theta,\, x\rangle = \sup_{0\leq f\leq 1, \int_K \frac{f(y)\phi(y)}{\delta}\ \mathrm{d}y=1} \int_K \langle y,\, \theta\rangle \frac{f(y)}{\delta}\phi(y)\, dy. \tag{5.8}$$

It follows from 5.5 that the maximum in the above equation is attained for the function

$$f = \mathbf{1}_{K\cap H^+(\theta,\delta)}$$

and this maximal function is unique as $\phi(y)\, \overrightarrow{1}_K\, \mathrm{d}y$ is absolutely continuous with respect to Lebesgue measure.

Proposition 5.7. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $\phi : K \to (0, \infty)$ be a continuous function. Let $\theta \in \mathbb{S}^{n-1}$ and $\delta \in \left(0, \int_K \phi(y)\ \mathrm{d}y\right)$. Then, the barycenter of $K \cap H^+(,\, \theta)\,\delta$ with respect to the weight function ϕ,*

$$x_{K,\phi}(\theta,\, \delta) := \frac{\int_{K\cap H^+(,\theta)\delta} y\phi(y)\ \mathrm{d}y}{\delta}$$

is the unique point in $\partial \mathrm{M}_\delta(K, \phi)$ with normal θ. In particular, $\mathrm{M}_\delta(K, \phi)$ is strictly convex.

Moreover,

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \frac{\int_{K\cap H^+(,\theta)\delta}\langle\theta,\, y\rangle\phi(y)\,\mathrm{d}y}{\delta}.$$

Extending by limit, $h_{\mathrm{M}_\delta(K,\phi)}$ *is a continuous function on* $\mathbb{S}^{n-1}\times\left[0, \int_K \phi(y)\,\mathrm{d}y\right]$ *and* $h_{\mathrm{M}_0(K,\phi)}$ *is the support function* h_K *of* K*.*

We remark that we will use $x(\theta,\,\delta)$ in short for $x_{K,\,\phi}(\theta,\,\delta)$ whenever there is no ambiguity (which is actually everywhere, except for the proof of Theorem 5.2).

Proof. We only need to show that $h_{\mathrm{M}_\delta(K,\phi)}$ is continuous as a function of θ and δ. We put $g(\theta,\,d) = \int_{K\cap H^+(\theta,\,d)}\langle\theta,\,y\rangle\phi(y)\,\mathrm{d}y$. Then g is continuous in the product metric. By the above, the function $(\theta,\,\delta)\to(\theta,\,d(\theta,\delta))$ is continuous in the product metric. Now

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \frac{g(\theta,\,d(\theta,\,\delta))}{\delta},$$

and therefore it is continuous for $0<\delta\leq\int_K\phi(y)\,\mathrm{d}y$, $\theta\in\mathbb{S}^{n-1}$. Moreover, for all $\theta\in\mathbb{S}^{n-1}$ and for all $\delta\in(0,\int_K\phi(y)\,\mathrm{d}y]$,

$$d(\theta,\delta)\leq h_{\mathrm{M}_\delta(K,\phi)}(\theta)\leq h_K(\theta).$$

Note that for $\delta=0$, $d(\theta,0)=h_K(\theta)$. Let $\theta_0\in\mathbb{S}^{n-1}$ be fixed. For $\varepsilon>0$, there exists an open ball O containing $(\theta_0,\,0)\in\mathbb{S}^{n-1}\times\left[0,\int_K\phi(y)\,\mathrm{d}y\right]$ such that for $(\theta_1,\,\delta_1)\in O$ we have $|h_K(\theta_0)-d(\theta_1,\,\delta_1)|<\varepsilon$. Thus, we conclude that $\left|h_K(\theta_0)-h_{\mathrm{M}_{\delta_1}(K,\phi)}(\theta_1)\right|<\varepsilon$ and hence $h_{\mathrm{M}_\delta(K,\phi)}(\theta)$ is continuous at $(\theta_0,\,0)$ if we define $h_{\mathrm{M}_0(K,\phi)}(\theta_0):=h_K(\theta_0)$. $\square$

5.2.2 Ulam's floating body problem

Let $K\subseteq\mathbb{R}^n$ be a body with a uniform density $0<\rho<1$. Suppose we put K in a liquid of uniform density 1, such that the surface of the liquid is orthogonal to the direction u. Let g be the barycenter of K, and b its center of buoyancy, that is the barycenter of the portion of K which is submerged in the liquid. We say that K floats in equilibrium in direction u if the barycenter of K is directly above its buoyancy center, namely $g-b$ is parallel to u.

A well-known fact in hydrostatics which was pointed out to us by Ning Zhang (see e.g., [**?**, Theorem 2]) states that if a body floats in liquid, then its barycenter, its center of buoyancy, and the barycenter of the portion of the body that is above the surface of the liquid, are all collinear. In terms of $M_\delta(K)$, this property translates to the following proposition:

Proposition 5.8. *Let $K \subseteq \mathbb{R}^n$ be a convex body with $x_K = 0$ (x_K is the barycenter of K) and $|K| = 1$. Then,* $\mathrm{M}_{1-\delta}(K) = -\frac{\delta}{1-\delta}\mathrm{M}_{\delta}(K)$.

Remark 5.9. An immediate consequence of the above proposition is that for any convex body $K \subseteq \mathbb{R}^n$, $\mathrm{M}_{\frac{1}{2}}(K)$ is centrally-symmetric. Moreover, by Theorem 5.1 and Proposition 5.12, it follows that $\mathrm{M}_{\frac{1}{2}}(K)$ is isomorphic to B_2^n.

Proof. Recall that $h_{M_\delta(K)}(\theta) = \langle x(\theta, \delta), \theta\rangle$ where

$$x(\theta,\ \delta) := \frac{\int_{K\cap H^+(,\theta)\delta} y\,\mathrm{d}y}{\delta}$$

and $H^+(\theta,\ \delta)$ is the halfspace in direction θ such that $|K \cap H^+(\theta,\ \delta)| = \delta$. Observe that

$$0 = \mathrm{bar}(K) = \int_K x\,\mathrm{d}x = \int_{K\cap H^+(\theta,\,\delta)} x\,\mathrm{d}x + \int_{K\cap H^-(\theta,\,\delta)} x\,\mathrm{d}x,$$

which is equivalent to

$$0 = \delta x(\theta, \delta) + (1-\delta)\, x(-\theta, 1-\delta).$$

Therefore, $x(-\theta, 1-\delta) = -\frac{\delta}{1-\delta}x(\theta,\ \delta)$, which is equivalent to $\mathrm{M}_{1-\delta}(K) = -\frac{\delta}{1-\delta}\mathrm{M}_{\delta}(K)$. $\square$

As mentioned in the introduction, Ulam's long-standing floating problem asks whether the only body of uniform density that floats in equilibrium in every orientation must be a Euclidean ball. A direct consequence of Proposition 5.8 is that Ulam's floating problem can be restated in terms of $M_\delta(K)$:

Corollary 5.10. *Ulam's floating problem is equivalent to the following problem: Suppose $M_\delta(K)$ is a Euclidean ball. Must K be a Euclidean ball?*

We remark that this new form of Ulam's problem remains open if one replaces $M_\delta(K)$ with the convex floating body K_δ. Another related open problem asks whether a convex body K is centrally-symmetric if and only if K_δ is symmetric. When replaced with $M_\delta(K)$, this problem seems also interesting. Note that Auerbach's counterexample in [**?**] to Ulam's problem in the plane, provides an example for a non-centrally-symmetric convex body $K \subseteq \mathbb{R}^2$ for which $M_\delta(K)$ is a Euclidean ball, thus answer both of the above problems in this case.

5.2.3 Connection to floating bodies.

We begin with the proof of Theorem 5.1:

Proof of Theorem 5.1 . By Proposition 5.7 we have that

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \frac{1}{\delta}\int_{K\cap\{y\in\mathbb{R}^n\,:\,\langle y,\,\theta\rangle\geq d(\theta,\,\delta)\}} \langle x,\,\theta\rangle\phi(x)\;\mathrm{d}x \geq d(\theta,\,\delta) \geq h_{F_\delta(K,\,\phi)}(\theta).$$

Therefore, $F_\delta(K,\phi) \subseteq \mathrm{M}_\delta(K,\phi)$.

Fix $\delta > 0$ and $\theta \in \mathbb{S}^{n-1}$. For $\beta \in \mathbb{S}^{n-1}$, let $H_\beta^+ := \{y \in \mathbb{R}^n \,:\, \langle y,\,\beta\rangle \geq \langle x(\theta,\,\delta),\,\beta\rangle\}$. Consider the function $g_\beta(t) := \int_{\{y\,:\,\langle y,\,\beta\rangle = t\}} \mathbf{1}_{K\cap H^+(\theta,\,\delta)}(y)\,\phi(y)\;\mathrm{d}y$. Since ϕ is log-concave, it follows by Prï¿œkopa-Leindler's inequality that g_β is also log-concave. By [**?**, Lemma 5.4] (a generalization of Grï¿œnbaum's inequality), we have that

$$\frac{1}{e}\int g_\beta(t)\;\mathrm{d}t \leq \int_{t\geq\langle x(\theta,\,\delta),\,\beta\rangle} g_\beta(t)\;\mathrm{d}t \leq \left(1-\frac{1}{e}\right)\int g_\beta(t)\;\mathrm{d}t$$

or equivalently,

$$\frac{1}{e}\int_{K\cap H^+(\theta,\,\delta)} \phi(y)\;\mathrm{d}y \leq \int_{H_\beta^+\cap K\cap H^+(\theta,\,\delta)} \phi(y)\;\mathrm{d}y \leq \left(1-\frac{1}{e}\right)\int_{K\cap H^+(\theta,\,\delta)} \phi(y)\;\mathrm{d}y.$$

Taking $\beta = \theta$, we have $H_\theta^+\cap K\cap H^+(\theta,\,\delta) = H_\theta^+\cap K$. Since $\int_{H_\theta^+\cap K}\phi(y)\;\mathrm{d}y \leq \left(1-\frac{1}{e}\right)\delta$, we obtain

$$h_{F_{\left(1-\frac{1}{e}\right)\delta}(K,\phi)}(\theta) \leq d\left(\theta,\,\left(1-\frac{1}{e}\right)\delta\right) \leq \langle x(\theta,\,\delta),\,\theta\rangle = h_{\mathrm{M}_\delta(K,\phi)}(\theta),$$

and thus $F_{\left(1-\frac{1}{e}\right)\delta}(K,\,\phi) \subseteq \mathrm{M}_\delta(K,\phi)$. On the other hand (see Figure 5.2.1), for $\beta \in \mathbb{S}^{n-1}$ we have

$$\int_{H_\beta^+\cap K} \phi(y)\;\mathrm{d}y \geq \int_{H_\beta^+\cap K\cap H^+(\theta,\,\delta)} \phi(y)\;\mathrm{d}y \geq \frac{\delta}{e} = \int_{H^+\left(\beta,\,\frac{\delta}{e}\right)\cap K} \phi(y)\;\mathrm{d}y.$$

Hence, $d\left(\beta,\,\frac{\delta}{e}\right) \geq \langle x(\theta,\,\delta),\,\beta\rangle$. Therefore we have

$$x(\theta,\,\delta) \in \bigcap_{\beta\in\mathbb{S}^{n-1}} \left\{y \,:\, \langle y,\,\beta\rangle \leq d\left(\theta,\,\frac{\delta}{e}\right)\right\} = F_{\frac{\delta}{e}}(K,\,\phi).$$

Since $\mathrm{M}_\delta(K,\phi)$ and $F_{\frac{\delta}{e}}(K,\,\phi)$ are convex sets, we conclude that $\mathrm{M}_\delta(K,\phi) \subseteq F_{\frac{\delta}{e}}(K,\,\phi)$.

□

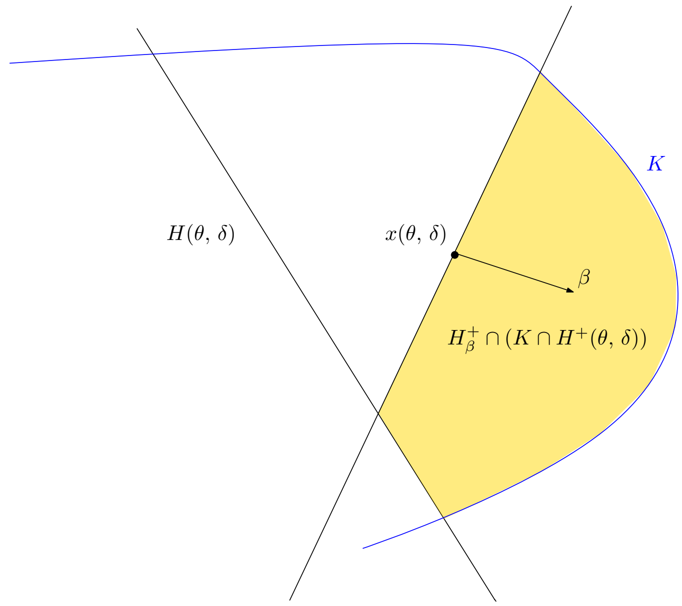


Figure 5.2.1: $H_\beta^+\cap(K\cap H^+(\theta,\,\delta))$

The L_p centroid bodies were introduced by Lutwak and Zhang [52] (using a different normalization) as follows: For a convex body K in $\mathbb{R}^n$ of volume 1 and $1\le p\le\infty$, the L_p centroid body $Z_p(K)$ is this convex body whose support function is given by:

$$h_{Z_p(K)}(\theta)=\left(\int_K|\langle x,\theta\rangle|^p dx\right)^{1/p}. \tag{5.9}$$

It is known that the floating body K_δ is close to some L_p *centroid body* of K. More precisely, one has:

Theorem 5.11. *([?, Theorem 2.2]) Let K be a symmetric convex body of volume 1. For $\delta\in\left(0,\frac{1}{2}\right)$, we have*

$$c_1 Z_{\log\left(\frac{e}{2\delta}\right)}(K)\subseteq K_\delta\subseteq c_2 Z_{\log\left(\frac{e}{2\delta}\right)}(K),$$

where $c_1,c_2>0$ are universal constants.

We obtain a similar result for Ulam floating bodies:

Proposition 5.12. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1 . Then there is an absolute constant $c_1>0$ such that for all $\delta<\frac{1}{e}$*

$$c_1 Z_{\log\left(\frac{e}{2\delta}\right)}(K)\subseteq K_\delta\subseteq \mathrm{M}_\delta(K)\subseteq e Z_{\log\left(\frac{1}{\delta}\right)}(K).$$

Proof. The first inclusion holds by Theorem 5.11. The second one, $K_\delta\subseteq\mathrm{M}_\delta(K)$, follows from Theorem 5.1. By Hï¿œlder's inequality, we have for $p \in [1, \infty]$,

$$\int_{K \cap H^+(\theta,\delta)} \langle y, \theta \rangle \,\mathrm{d}y \leq \left(\int_{K \cap H^+(\theta,\delta)} 1^q \,\mathrm{d}y \right)^{\frac{1}{q}} \left(\int_K |\langle \theta, y \rangle|^p \,\mathrm{d}y \right)^{\frac{1}{p}}$$
$$= \delta^{\frac{1}{q}} h_{Z_p(K)}(\theta),$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Dividing both sides by δ, we get

$$h_{\mathrm{M}_\delta(K)}(\theta, \delta) \leq \left(\frac{1}{\delta} \right)^{\frac{1}{p}} h_{Z_p(K)}(\theta).$$

Putting $p = \log\left(\frac{1}{\delta}\right)$ yields

$$h_{\mathrm{M}_\delta(K)}(\theta, \delta) \leq e\, h_{Z_{\log\left(\frac{1}{\delta}\right)}(K)}(\theta).$$

Therefore, we have that

$$\mathrm{M}_\delta(K) \subseteq e\, Z_{\log\left(\frac{1}{\delta}\right)}(K).$$

□

5.2.4 Smoothness of Ulam floating bodies

In this section we prove Theorem 5.2. To this end, let $\rho_v(\cdot)$ denote the radial function of K with center v. That is,

$$\rho_v(\theta) = \max\left\{ r \in \mathbb{R}^+ \,:\, v + r\theta \in K \right\}.$$

We will need the following fact, which can be found implicitly in e.g., [63].

Fact 5.13. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Then, the following are equivalent:*

1. K has C^k boundary;
2. The function $(v, \theta) \to \rho_v(\theta)$ is C^k for every $v \in \mathrm{int}(K)$ and $\theta \in \mathbb{S}^{n-1}$;
3. There exists $v \in \mathrm{int}(K)$ such that $\theta \to \rho_v(\theta)$ is C^k.

Proof of Theorem 5.2. For $a \in \mathbb{R}^n \backslash \{0\}$, let $H := \{x \,:\, \langle x, a \rangle = 1\}$, $\delta(a) = |K \cap \{\langle x, a \rangle \geq 1\}|$, and $U(a) := \int_{K\cap\{\langle x,\, a\rangle\geq 1\}} x\,\mathrm{d}x$. We would like to show that

$$\nabla\delta(a) = \frac{1}{\|a\|}\int_{K\cap H} x\,\mathrm{d}x \tag{5.10}$$

$$DU = \frac{1}{\|a\|}\left(\int_{K\cap\{\langle x,\, a\rangle=1\}} x_i x_j\,\mathrm{d}x\right)_{i,j\in[n]}. \tag{5.11}$$

Equation (5.10) was proved in [53, Lemma 5]. Using the same ideas, we prove (5.11) as follows. Pick a direction θ so that θ is not parallel to a, and let $H_\varepsilon := \{x\,:\,\langle x,\, a+\varepsilon\theta\rangle = 1\}$. As illustrated in Figure 5.2.2, we also define:

$$K_-(\varepsilon) = \mathrm{int}(K)\cap\{y\in\mathbb{R}^n\,:\,\langle y,\, a\rangle\geq 1,\,\langle y,\, a+\varepsilon\theta\rangle\leq 1\},$$
$$K_+(\varepsilon) = \mathrm{int}(K)\cap\{y\in\mathbb{R}^n\,:\,\langle y,\, a\rangle\leq 1,\,\langle y,\, a+\varepsilon\theta\rangle\geq 1\}.$$

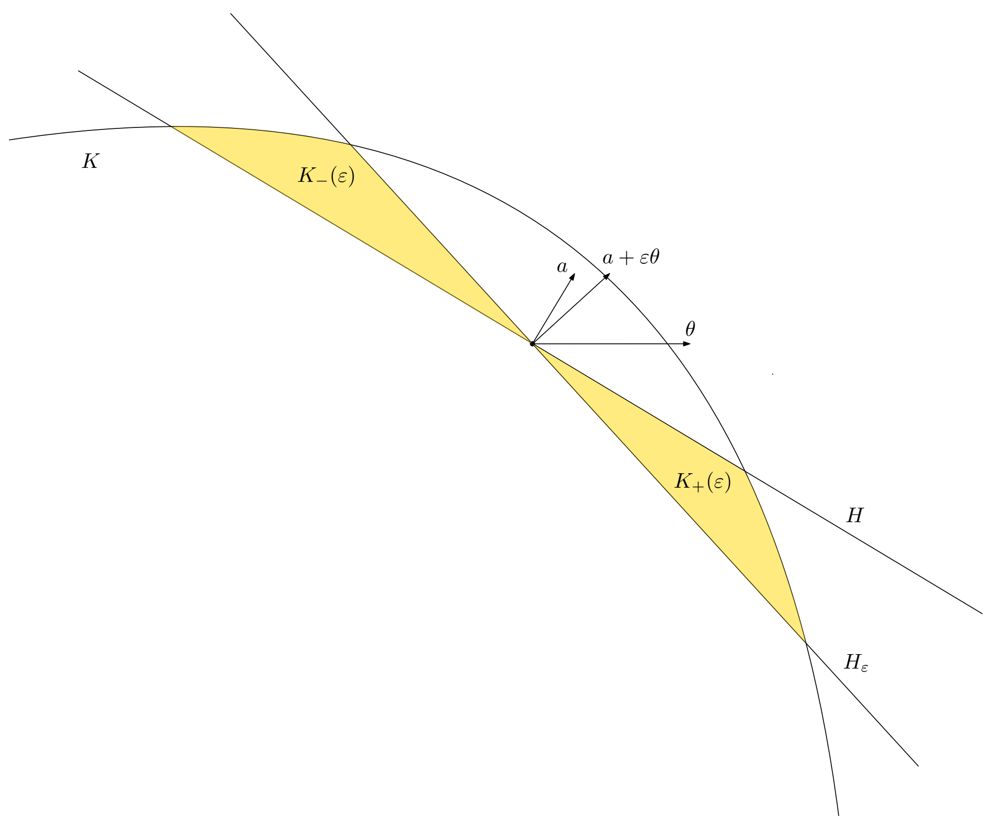


Figure 5.2.2: $K_\pm(\varepsilon)$

Let U_j denote the jth coordinate of U. We have

$$U_j\left(a+\varepsilon\theta\right)-U_j\left(a\right)=\int_{K_+(\varepsilon)}\langle x,\,e_j\rangle\,\mathrm{d}x-\int_{K_-(\varepsilon)}\langle x,\,e_j\rangle\,\mathrm{d}x.$$

From now on we choose $\varepsilon>0$ small enough so that $\langle a,\,a+\varepsilon\theta\rangle>0$. For $y\in\mathbb{R}^n$, we write y uniquely in the form $x+t\frac{a}{\|a\|}$, where $x=y+\frac{1-\langle y,\,a\rangle}{\langle a,\,a\rangle}a$ and $t=-\frac{1-\langle y,\,a\rangle}{\langle a,\,a\rangle}\|a\|$. Notice that $x\in H$. Then,

$$\begin{aligned}
&\{y\in\mathbb{R}^n\,:\,\langle y,\,a\rangle\geq 1,\,\langle y,\,a+\varepsilon\theta\rangle\leq 1\}=\\
&\left\{x+ta\,:x\in H,\,t\in\mathbb{R},\,\langle x+t\frac{a}{\|a\|},\,a\rangle\geq 1,\,\langle x+t\frac{a}{\|a\|},\,a+\varepsilon\theta\rangle\leq 1\right\}=\\
&\left\{x+ta\,:x\in H,\,0\leq t\leq\frac{-\varepsilon\langle x,\,\theta\rangle\,\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}\right\}=\\
&\left\{x+ta\,:x\in H,\,\langle x,\,\theta\rangle\leq 0,\,0\leq t\leq\frac{-\varepsilon\langle x,\,\theta\rangle\,\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}\right\}.
\end{aligned}$$

Thus,

$$K_-\left(\varepsilon\right)=\left\{x+ta\,:x\in H,\,\langle x,\,\theta\rangle\leq 0,\,0\leq t\leq\frac{-\varepsilon\langle x,\,\theta\rangle\,\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}\right\}\cap\operatorname{int}\left(K\right).$$

Let

$$O_-\left(\varepsilon\right):=\left\{x\in H\,:\,\langle x,\,\theta\rangle\leq 0,\,\left[x,\,x+\frac{-\varepsilon\langle x,\,\theta\rangle\,\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}a\right]\cap\operatorname{int}\left(K\right)\neq\emptyset\right\}.$$

For $x\in H$ such that $\langle x,\,\theta\rangle\leq 0$, we have that

$$\frac{-\varepsilon\langle x,\,\theta\rangle\,\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}=\frac{\varepsilon\left|\langle x,\,\theta\rangle\right|\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}=\frac{\left|\langle x,\,\theta\rangle\right|\|a\|}{\langle a,\,a\rangle\varepsilon^{-1}+\langle a,\,\theta\rangle}$$

decrease to 0 as $\varepsilon\searrow 0$. Thus, $O\left(\varepsilon\right)$ shrinks to

$$\begin{aligned}
O_-\left(0\right)&=\{x\in H\,:\,\langle x,\,\theta\rangle\leq 0,\,[x,\,x]\cap\operatorname{int}\left(K\right)\neq\emptyset\}\\
&=\{x\in H\cap\operatorname{int}\left(K\right)\,:\,\langle x,\,\theta\rangle\leq 0\}.
\end{aligned}$$

For $x\in O_-\left(\varepsilon\right)$, let $0\leq t_1\left(\varepsilon,\,x\right)\leq t_2\left(\varepsilon,\,x\right)\leq\frac{-\varepsilon\langle x,\,\theta\rangle}{\langle a,\,a+\varepsilon\theta\rangle}\|a\|$ be defined such that

$$\left\{x+ta\,:\,0\leq t\leq\frac{-\varepsilon\langle x,\,\theta\rangle\,\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}\right\}\cap\operatorname{int}\left(K\right)=\{x+ta\,:\,t_1\left(\varepsilon,\,x\right)<t<t_2\left(\varepsilon,\,x\right)\}.$$

Then, by Fubini's theorem, we have

$$\begin{aligned}\int_{K_-(\varepsilon)} \langle y,\, e_j\rangle \,\mathrm{d}y &= \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)} \langle x + t\frac{a}{\|a\|},\, e_j\rangle \,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)} \langle x,\, e_j\rangle \,\mathrm{d}t\,\mathrm{d}x + \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)} \langle t\frac{a}{\|a\|},\, e_j\rangle \,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

We analyze each of the above terms, separately, as follows.

Firstly, we have that

$$\begin{aligned}\left|\int_{O_-(\varepsilon)} \int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)} \langle t\frac{a}{\|a\|},\, e_j\rangle \,\mathrm{d}t\,\mathrm{d}x\right| &\leq \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)} t\,\mathrm{d}t\,\mathrm{d}x \\ &\leq \int_{O_-(\varepsilon)} \int_{0}^{\frac{-\varepsilon\langle x,\,\theta\rangle\|a\|}{\langle a,\,a+\varepsilon\theta\rangle}} t\,\mathrm{d}t\,\mathrm{d}x \\ &\leq \frac{1}{2}\frac{\varepsilon^2\,\|a\|^2}{\langle a,\,a+\varepsilon\theta\rangle^2}\int_{O_-(\varepsilon)} \langle x,\,\theta\rangle^2\,\mathrm{d}x.\end{aligned}$$

Since $O_-(\varepsilon)$ is bounded and shrinks as ε decreases, we conclude that

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_{O_-(\varepsilon)} \int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)} \langle t\frac{a}{\|a\|},\, e_j\rangle \,\mathrm{d}t\,\mathrm{d}x = 0.$$

Secondly, we have that

$$\frac{\int_{O_-(\varepsilon)} \int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)} \langle x,\, e_j\rangle \,\mathrm{d}t\,\mathrm{d}x}{\varepsilon} = \int_H \frac{\left(t_2(x,\,\varepsilon) - t_1(x,\,\varepsilon)\right)\langle x,\, e_j\rangle \mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon}\,\mathrm{d}x.$$

Fix $\varepsilon_0 > 0$. For $\varepsilon_0 > \varepsilon > 0$, we have that

$$\left|\frac{\left(t_2(x,\,\varepsilon) - t_1(x,\,\varepsilon)\right)\langle x,\, e_j\rangle \mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon}\right| \leq \frac{|\langle x,\,\theta\rangle|\,\|a\|}{\langle a,\,a\rangle - \varepsilon_0\,|\langle a,\,\theta\rangle|}\,|\langle x,\,e_j\rangle|\,\mathbf{1}_{O_-(\varepsilon_0)},$$

where the function on the right hand side is integrable.

Suppose $x \notin O_-(0)$. Then, $\frac{(t_2(x,\varepsilon)-t_1(x,\varepsilon))\langle x,e_j\rangle\mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon} \to 0$ as $\varepsilon \searrow 0$ since $\mathbf{1}_{O_-(\varepsilon)}(x) = 0$ for small $\varepsilon > 0$. For $x \in O_-(0)$, we have $t_1(x) = 0$ and $t_2(x) = \frac{-\varepsilon\langle x,\theta\rangle\|a\|}{\langle a,a+\varepsilon\theta\rangle}$ for sufficiently small ε. We conclude that, as $\varepsilon \searrow 0$,

$$\frac{\left(t_2(x,\,\varepsilon) - t_1(x,\,\varepsilon)\right)\langle x,\, e_j\rangle \mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon} \to \frac{-\langle x,\,\theta\rangle\langle x,\,e_j\rangle}{\|a\|}\mathbf{1}_{O_-(0)}(x).$$

By Lebesgue's dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0} -\frac{\int_{K_-(\varepsilon)}\langle x,\, e_j\rangle\,\mathrm{d}x}{\varepsilon}\\
=&\lim_{\varepsilon\searrow 0} -\frac{\int_{O_-(\varepsilon)}\int_{t_1(\varepsilon,x)}^{t_2(\varepsilon,x)}\langle x,\, e_j\rangle\,\mathrm{d}t\,\mathrm{d}x}{\varepsilon}\\
=&\frac{1}{\|a\|}\int_{K\cap H\cap\{\langle x,\,\theta\rangle\le 0\}}\langle x,\,\theta\rangle\langle x,\, e_j\rangle\,\mathrm{d}x.
\end{aligned}
$$

Via the same argument, one also shows that

$$
\lim_{\varepsilon\searrow 0}\frac{\int_{K_+(\varepsilon)}\langle x,\, e_j\rangle\,\mathrm{d}x}{\varepsilon}=\frac{1}{\|a\|}\int_{K\cap H\cap\{\langle x,\,\theta\rangle\ge 0\}}\langle x,\,\theta\rangle\langle x,\, e_j\rangle\,\mathrm{d}x.
$$

Thus we conclude that

$$
\lim_{\varepsilon\searrow 0}\frac{U_j\left(a+\varepsilon\theta\right)-U_j\left(a\right)}{\varepsilon}=\frac{1}{\|a\|}\int_{K\cap H}\langle x,\,\theta\rangle\langle x,\, e_j\rangle\,\mathrm{d}x.
$$

This completes the proof of (5.11).

Next, we show that $DU\left(a\right)$ and $\nabla\delta\left(a\right)$ are C^k functions.

Pick $v\in\operatorname{int}\left(K\right)\cap H$. Let σ_a be the normalized Haar measure on $S\left(a\right)=\mathbb{S}^{n-1}\cap a^{\perp}$. Then

$$
\begin{aligned}
\int_{K\cap H}x\,\mathrm{d}x&=\left(n-1\right)\left|B_2^{n-1}\right|\int_{S(a)}\int_0^{\rho_v(\theta)}r^{n-2}\left(v+r\theta\right)\,\mathrm{d}r\,\mathrm{d}\sigma_a\left(\theta\right)\\
&=\left|B_2^{n-1}\right|\int_{S(a)}\left(\rho_v^{n-1}\left(\theta\right)v+\frac{n-1}{n}\rho_v^{n}\left(\theta\right)\theta\right)\,\mathrm{d}\sigma_a\left(\theta\right). \qquad (5.12)
\end{aligned}
$$

Fix $a_0\in\mathbb{R}^n$ so that $\operatorname{int}(K)\cap\{\langle x,\, a_0\rangle=1\}\neq\emptyset$ and let $v_0\in\operatorname{int}\left(K\right)\cap\{\langle x,\, a_0\rangle=1\}$. By Fact 5.13, $(v,\,\theta)\to\rho_v\left(\theta\right)$ is C^k, and hence the function $F_{a_0}:\mathbb{R}^n\times O_n\to\mathbb{R}^n$ defined by

$$
\left(v,\,T\right)\mapsto\left|B_2^{n-1}\right|\int_{S(a_0)}\left(\rho_v^{n-1}\left(T\theta\right)v+\frac{n-1}{n}\rho_v^{n}\left(T\theta\right)T\theta\right)\,\mathrm{d}\sigma_{a_0}\left(\theta\right)
$$

is also C^k. We can find a smooth function $a\mapsto\left(v\left(a\right),\,T\left(a\right)\right)$ in a neighborhood of a_0 so that $v\left(a\right)\in\operatorname{int}\left(K\right)\cap\{\langle x,\, a\rangle=1\}$ and $T\left(a\right)S\left(a_0\right)=\mathbb{S}^{n-1}\cap a^{\perp}$. Indeed, for a close to a_0, we define the unique two-dimensional rotation $T\left(a\right)$ satisfying $T\left(a\right)\frac{a_0}{\|a_0\|}=\frac{a}{\|a\|}$ and $T\left(a\right)v=v$ for all $v\in\operatorname{span}\left(a,\,a_0\right)^{\perp}$. In particular, $a\mapsto T\left(a\right)$ is a smooth function around a_0. Also, $T\left(a\right)\left(S\left(a_0\right)\right)=S\left(a\right)$. Let $v\left(a\right)$ be the projection of v_0 onto $\{\langle x,\, a\rangle=1\}$. In other words,

$$v(a) := v_0 - \langle v_0, \frac{a}{\|a\|}\rangle \frac{a}{\|a\|} + \frac{a}{\|a\|^2},$$

which is again smooth when $a \neq 0$. Also, $v(a_0) = v_0$, and $v(a) \in \operatorname{int}(K)$ if a is close to a_0.

Next, we express $\nabla\delta$ in terms of $v(a)$ and $T(a)$: By (5.12) we have

$$\begin{aligned}
\nabla\delta(a) &= \int_{K\cap\{\langle x,\, a\rangle=1\}} x\,\mathrm{d}x \\
&= \frac{1}{\|a\|}\left|B_2^{n-1}\right| \int_{S(a)} \left(\rho_{v(a)}^{n-1}(\theta)\, v(a) + \frac{n-1}{n}\rho_{v(a)}^{n}(\theta)\,\theta\right) \mathrm{d}\sigma_a(\theta) \\
&= \frac{1}{\|a\|}\left|B_2^{n-1}\right| \int_{S(a_0)} \left(\rho_{v(a)}^{n-1}(T(a)\,\theta)\, v(a) + \frac{n-1}{n}\rho_{v(a)}^{n}(T(a)\,\theta)\, T(a)\,\theta\right) \mathrm{d}\sigma_{a_0}(\theta) \\
&= \frac{1}{\|a\|} F_{a_0}(v(a),\, T(a)).
\end{aligned}$$

We conclude that $\nabla\delta(a)$ is C^k and thus $\delta(a)$ is C^{k+1}.

Recall that $\delta(\theta,\, d) = |K \cap \{\langle x,\, \theta\rangle \geq d\}|$. Consider the function from $\mathbb{S}^{n-1} \times \mathbb{R}$ to $\mathbb{S}^{n-1} \times \mathbb{R}$ defined by

$$(\theta,\, d) \mapsto \left(\theta,\, \delta\left(\frac{1}{d}\theta\right)\right) = (\theta,\, \delta(\theta,\, d)).$$

By the above, it is C^{k+1} whenever $\operatorname{int}(K) \cap \{\langle x,\, \theta\rangle = d\} \neq \emptyset$. Thus, its inverse function $(\theta,\, d(\theta,\, \delta))$ is also C^{k+1} for $(\theta,\, \delta) \in \mathbb{S}^{n-1} \times [0,\, |K|]$. Repeating the same argument as for $\nabla\delta(a)$ implies that $U(a)$ is also C^{k+1}.

Recall that if $d(\theta,\, \delta) > 0$,

$$x_K(\theta,\, \delta) = \frac{1}{\delta}\int_{K\cap\{\langle x,\, \theta\rangle \geq d(\theta,\, \delta)\}} x\,\mathrm{d}x = \frac{1}{\delta} U\left(\frac{\theta}{d(\theta,\, \delta)}\right).$$

Therefore, for a fixed $0 < \delta < |K|$, and θ such that $d(\theta,\, \delta) > 0$, the function $\theta \mapsto \frac{x_K(\theta,\, \delta)}{\|x_K(\theta,\, \delta)\|}$ is C^{k+1}. Moreover, it is invertible since $\mathrm{M}_\delta(K)$ is strictly convex. Thus its inverse, denoted by $G_\delta : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is also C^{k+1}. Therefore, the radial function of $\mathrm{M}_\delta(K)$, which is given by $\rho(\theta) = \|x(G_\delta(\theta),\, \delta)\|$ is also C^{k+1}.

Finally, we need to show that $\theta \to x_K(\theta,\, \delta)$ is C^{k+1} whenever $d(\theta,\, \delta) \leq 0$. Indeed, we may choose some vector $v \in \mathbb{R}^n$ and consider $M_\delta(v + K)$. Then, $x_K(\theta,\, \delta) = x_{v+K}(\theta,\, \delta) - v$. Clearly, we can always choose v such that, for $v + K$, $d(\theta,\, \delta) > 0$. Thus, following the same argument, we can show $x_{v+K}(\theta,\, \delta)$ is C^{k+1}. As a consequence,

$x_K(\theta,\delta)$ is C^{k+1}. Therefore, we conclude that $\rho(\theta)$ is C^{k+1} on $\mathbb{S}^{n-1}$. By Fact (5.13), the boundary of $\mathrm{M}_\delta(K)$ is C^{k+1}. $\square$

5.3 Relation to p-affine surface area

This section is devoted to the proof of Theorem 5.3.

5.3.1 Preliminary results

For the proof of Theorem 5.3, we will need a few preliminary results.

First, we focus on $\mathrm{M}_\delta(\rho B_2^n, \phi)$, where ρB_2^n is the Euclidean ball centered at 0 and with radius, and $\phi(x)$ is a constant function. By symmetry, we know that $\mathrm{M}_\delta(\rho B_2^n, \phi)$ is again a Euclidean ball with the same center. Let $\Delta(\rho,\delta)$ be the difference of the radius of ρB_2^n and $\mathrm{M}_\delta(\rho B_2^n, \phi)$. If $\phi : \rho B_2^n \to (0,\infty)$, is a constant function, $\phi(x) = s$, for all $x \in \rho B_2^n$, then, we define $\Delta(\rho,\delta,s)$ to be the difference of radius of ρB_2^n and $\mathrm{M}_\delta(\rho B_2^n, s)$. One easily verifies that

$$\Delta(\rho,\delta,s) = \Delta\left(\rho, \frac{\delta}{s}\right). \tag{5.13}$$

Proposition 5.14. *$\lim_{\delta\searrow 0} \Delta(\rho,\delta)/\delta^{\frac{2}{n+1}}\rho^{\frac{n+1}{n-1}} = c_n$, where $c_n = \frac{1}{2}\frac{n+1}{n+3}\left(\frac{n+1}{|B_2^{n-1}|}\right)^{\frac{2}{n+1}}$.*

Proof. We denote $h(\rho,\delta)$ to be height of the cap of ρB_2^n which has volume δ. To be specific, $h(\rho,\delta)$ satisfies the equality

$$\delta = \left|B_2^{n-1}\right| \int_0^{h(\rho,\delta)} g^{n-1}(t)\, \mathrm{d}t,$$

where $g(t) = \left(\rho^2 - (\rho - t)^2\right)^{1/2}$. Moreover,

$$g(t) = \left(\rho^2 - (\rho - t)^2\right)^{1/2} = \rho\left(1 - (1 - t/\rho)^2\right)^{1/2} = \rho\,(2 - t/\rho)^{1/2}(t/\rho)^{1/2}.$$

We have

$$\delta = \left|B_2^{n-1}\right| \rho^{n-1} \int_0^{h(\rho,\delta)} (2 - t/\rho)^{\frac{n-1}{2}} (t/\rho)^{\frac{n-1}{2}}\, \mathrm{d}t.$$

Thus, we have the inequality

$$\left|B_2^{n-1}\right| \rho^{n-1} \left(2-h\left(\rho,\,\delta\right)/\rho\right)^{\frac{n-1}{2}} \int_0^{h(\rho,\,\delta)} \left(t/\rho\right)^{\frac{n-1}{2}}\ \mathrm{d}t \le\ \delta$$
$$\le\ \left|B_2^{n-1}\right| \rho^{n-1} 2^{\frac{n-1}{2}} \int_0^{h(\rho,\,\delta)} \left(t/\rho\right)^{\frac{n-1}{2}}\ \mathrm{d}t.$$

Since

$$\int_0^{h(\rho,\,\delta)} \left(t/\rho\right)^{\frac{n-1}{2}}\ \mathrm{d}t = \frac{2}{n+1} h\left(\rho,\,\delta\right)^{\frac{n+1}{2}} \rho^{-\frac{n-1}{2}},$$

we obtain

$$\frac{1}{2}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}} \le \frac{h\left(\rho,\,\delta\right)}{\delta^{\frac{2}{n+1}}} \le \frac{1}{2-h\left(\rho,\,\delta\right)/\rho}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}}.$$

We conclude that

$$\lim_{\delta\searrow 0} \frac{h\left(\rho,\,\delta\right)}{\delta^{\frac{2}{n+1}}} = \frac{1}{2}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}}.$$

Recall that

$$\Delta\left(\rho,\,\delta\right) = \frac{\left|B_2^{n-1}\right| \int_0^{h(\rho,\,\delta)} tg\left(t\right)^{n-1}\ \mathrm{d}t}{\left|B_2^{n-1}\right| \int_0^{h(\rho,\,\delta)} g\left(t\right)^{n-1}\ \mathrm{d}t}.$$

To compute the next limit, we apply twice L'Hospital's Rule,

$$\lim_{h\to 0}\frac{h}{\Delta} = \lim \frac{h\int_0^h h^{n-1}\,\mathrm{d}t}{\int_0^h tg^{n-1}\,\mathrm{d}t} \overset{L}{=} \lim \frac{\int_0^h g^{n-1}\,\mathrm{d}t + hg\left(h\right)^{n-1}}{hg\left(h\right)^{n-1}} = 1 + \lim \frac{\int_0^h g^{n-1}\,\mathrm{d}t}{hg\left(h\right)^{n-1}}$$
$$\overset{L}{=} 1 + \lim \frac{\rho^{n-1}\left(2-\frac{r}{\rho}\right)^{\frac{n-1}{2}}\left(\frac{r}{\rho}\right)^{\frac{n-1}{2}}}{\rho^n\left(\frac{1}{\rho}\frac{n+1}{2}\left(\frac{r}{\rho}\right)^{\frac{n-1}{2}}\left(2-\frac{r}{\rho}\right)^{\frac{n-1}{2}} - \frac{1}{\rho}\frac{n-1}{2}\left(\frac{r}{\rho}\right)^{\frac{n+1}{2}}\left(2-\frac{r}{\rho}\right)^{\frac{n-3}{2}}\right)}$$
$$= 1 + \lim \frac{\left(2-\frac{r}{\rho}\right)}{\frac{n+1}{2}\left(2-\frac{r}{\rho}\right) - \frac{n-1}{2}\left(\frac{r}{\rho}\right)} = 1 + \frac{2}{n+1} = \frac{n+3}{n+1}.$$

So,

$$\lim_{\delta\searrow 0}\frac{\Delta\left(\rho,\,\delta\right)}{\delta^{\frac{2}{n+1}}} = \lim_{\delta\searrow 0}\frac{h\left(\rho,\,\delta\right)}{\delta^{\frac{2}{n+1}}}\cdot\frac{\Delta\left(\rho,\,\delta\right)}{h\left(\rho,\,\delta\right)} = \frac{1}{2}\,\frac{n+1}{n+3}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}}.$$

□

We will also need the next lemma from [67]:

Lemma 5.15. *Let K and L be convex bodies in $\mathbb{R}^n$ such that $0 \in$ int(L) and such that $L \subseteq K$. Then*

$$|K| - |L| = \frac{1}{n} \int_{\partial K} \langle x, N(x) \rangle \left(1 - \left| \frac{\|x_L\|}{\|x\|} \right|^n \right) \mathrm{d}\mu_K(x),$$

where x_L is the unique point in the intersection $\partial L \cap [0, x]$.

For the next lemma we need a notion that was introduced in [67]. Let K be a convex body in $\mathbb{R}^n$ and let $x \in \partial K$ be such that $N_K(x)$ is unique. We put $r(x)$ to be the radius of the biggest Euclidean ball contained in K that touches K in x,

$$r(x) = \max\{\rho : B_2^n(x - \rho N_K(x), \rho) \subseteq K\}.$$

If $N_K(x)$ is not unique, $r(x) = 0$. It was shown in [67, Lemma 5] that for any convex body K in $\mathbb{R}^n$ and any $0 \le \alpha < 1$,

$$\int_{\partial K} r(x)^{-\alpha} d\mu(x) < \infty. \tag{5.14}$$

Lemma 5.16. *Let K be a convex body in $\mathbb{R}^n$. Let $x \in \partial K$ and let $x_{M,\delta} = \partial \left(\mathrm{M}_\delta \left(K, \phi \right) \right) \cap [0, x]$. Then*

$$\frac{\langle x,\, N_K\left(x\right) \rangle}{\delta^{\frac{2}{n+1}}} \left(1 - \left| \frac{\|x_{M,\delta}\|}{\|x\|} \right|^n \right) \le c\, n\, r(x)^{-\frac{n-1}{n+1}},$$

where c is a constant independent of x and δ.

Proof. Let $x_{F,\delta} = \partial \left(F_\delta \left(K, \phi \right) \right) \cap [0, x]$. By Theorem 5.1, we have that $F_\delta \left(K, \phi \right) \subseteq M_\delta \left(K, \phi \right)$ and hence $\|x_{\mathbb{F},\delta}\| \le \|x_{M,\delta}\|$. Therefore

$$\frac{\langle x,\, N_K\left(x\right) \rangle}{\delta^{\frac{2}{n+1}}} \left(1 - \left| \frac{\|x_{M,\delta}\|}{\|x\|} \right|^n \right) \le \frac{\langle x,\, N_K\left(x\right) \rangle}{\delta^{\frac{2}{n+1}}} \left(1 - \left| \frac{\|x_{F,\delta}\|}{\|x\|} \right|^n \right)$$

and it was shown in [67], Lemma 8, that the latter is smaller than or equal $c\, n\, r(x)^{-\frac{n-1}{n+1}}$. □

The next lemma was proved in [67]. There, and in the proof of the main theorem, we need the indicatrix of Dupin (see, e.g., [68]). A theorem of Alexandrov [2] and Busemann and Feller [16] shows that the indicatrix of Dupin exists almost everywhere on ∂K and is an ellipsoid or an elliptic cylinder. We also use the notation $C(r, h)$ for the cap of a Euclidean ball with radius r and height h.

Lemma 5.17. *[67] Let K be a convex body in $\mathbb{R}^n$ with $0 \in \partial K$ and $N_K(0) = -e_n = (0, \cdots, 0, -1)$. Suppose the indicatrix of Dupin at 0 exists and is an $(n-1)$-dimensional* *sphere with radius $\sqrt{\rho}$. Let ξ be an interior point of K.*

(i) Let H be the hyperplane orthogonal to $N_K(0)$ and passing through z in $[0,\xi]$. We put $z_n = \langle z, e_n\rangle$. Then we have for $0 \le z_n \le \rho$,

$$\left|K \cap H^+\right| \le f(z_n)^{n-1} \left|C(\rho, z_n)\right|.$$

(ii) Let $d = dist\left(z, B_2^n(\rho\, e_n, \rho)^C\right)$. There is $\varepsilon_0 > 0$ such that we have for all $z \in [0,\xi]$ with $\|z\| \le \varepsilon_0$

$$d \le z_n \le d + \frac{2\, d^2}{\rho\langle \frac{\xi}{\|\xi\|}, N_K(0)\rangle^2}.$$

(iii) There is $\varepsilon_0 > 0$ and an absolute constant $c > 0$ such that for all $z \in [0,\xi]$ with $\|z\| \le \varepsilon_0$ and all hyperplanes H passing through z

$$\left|K \cap H^+\right| \ge f(\gamma)^{-n+1} \left|C(\rho, d(1 - c(f(\gamma) - 1))\right|.$$

Here, $\gamma = 2\sqrt{2\,\rho\, d}$ and f is a monotone function on $\mathbb{R}^+$ such that $\lim_{t\to} f(t) = 1$.

Lemma 5.18. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Moreover, we assume that $0 \in \partial K$ and that $N_K(0) = -e_n$ is the unique outer normal to ∂K at 0. Let $\phi : K \to (0, \infty)$ be a continuous function. We set $H_t^+ = H^+(-e_n, -t) = \{y : \langle y, e_n\rangle < t\}$. Then, for each $t > 0$, there exists $r > 0$ such that for any $\delta > 0$,*

$$\mathrm{M}_\delta\left(K, \phi\right) \cap B_2^n\left(0,\, r\right) = \mathrm{M}_\delta\left(K \cap H_t^+, \phi\right) \cap B_2^n\left(0,\, r\right).$$

Proof. It is obvious that

$$\mathrm{M}_\delta\left(K \cap H_t^+, \phi\right) \cap B_2^n\left(0,\, r\right) \subseteq \mathrm{M}_\delta\left(K, \phi\right) \cap B_2^n\left(0,\, r\right).$$

Therefore, it is sufficient to show the other inclusion. Let $d \ge 0$. Observe that if $(\theta,\, d)$ is sufficiently close to $(-e_n,\, 0)$, then $H^+(\theta, -d) \cap K \subseteq H_t^+$, where $H^+(\theta, -d) = \{y : \langle y, -\theta\rangle < d\}$. As noted in (5.6), the function $d\left(\theta, \delta\right)$ is continuous in $(\theta,\, \delta)$. Therefore, there exists $\delta_0 > 0$ and $\varepsilon > 0$ such that

$$K \cap H^+\left(\theta,\, d\left(\theta,\, \delta\right)\right) \subseteq H_t^+, \tag{5.15}$$

for $\|\theta - (-e_n)\| < \varepsilon$ and $0 \le \delta < \delta_0$. For each x in the interior of K, let $\delta\left(x\right)$ be the value such that $x \in \partial \mathrm{M}_{\delta(x)}\left(K, \phi\right)$ and $\theta\left(x\right)$ denote the unique outer normal at x of $\mathrm{M}_{\delta(x)}\left(K, \phi\right)$.
Claim : For any $\delta_0 > 0$ and $\varepsilon > 0$, there exists $r > 0$ such that $\delta\left(x\right) < \delta_0$ and $\|\theta\left(x\right) - (-e_n)\| < \varepsilon$, for $x \in \mathrm{int}\left(K\right) \cap B_2^n\left(0,\, r\right)$.

Indeed, note that $\mathrm{M}_{\delta_0}(K,\phi)$ is strictly contained in K. Thus, $0 \notin \mathrm{M}_{\delta_0}(K,\phi)$. Since $\mathrm{M}_{\delta_0}(K,\phi)$ is convex, there exists $r>0$ so that $B_2^n(0,\, r) \cap \mathrm{M}_{\delta_0}(K,\phi) = \emptyset$. Then, $\delta(x) < \delta_0$ for $x \in \operatorname{int}(K) \cap B_2^n(0,\, r)$.
It remains to show that there exists $r>0$ such that $\|\theta(x) - (-e_n)\| < \varepsilon$ for $\operatorname{int}(K) \cap B_2^n(0,\, r)$. Suppose it is false. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\operatorname{int}(K)$ such that $x_k \to 0$ and such that $\|\theta(x_k) - (-e_n)\| > \varepsilon$. By the compactness of $\mathbb{S}^{n-1}$, we may replace $(x_k)_{k\in\mathbb{N}}$ by a subsequence, again denoted by $(x_k)_{k\in\mathbb{N}}$, so that $\theta(x_k)$ converges to some $\theta_1 \neq -e_n$. Moreover, $\delta(x_k) \to 0$ since the first claim is true. Continuity of $h_{\mathrm{M}_{\delta(K,\phi)}}(\theta)$ implies that $h_{\mathrm{M}_{\delta(x_k)}(K,\phi)}(\theta(x_k)) \to h_K(\theta_1)$. As $-e_n$ is the unique outer normal to ∂K in 0, $h_K(\theta_1) > \langle 0,\, \theta_1\rangle = 0$. Therefore, we obtain a contradiction, as $h_{\mathrm{M}_{\delta(x_k)}(K,\phi)}(\theta(x_k)) = \langle x_k,\, \theta(x_k)\rangle$, which converges to 0 as $x_k \to 0$. This completes the proof of the claim.
Hence, with the assumptions on δ_0 and ε, we conclude that there exists $r>0$ such that for $x \in \operatorname{int}(K) \cap B_2^n(0,\, r)$,

$$K \cap H^+(\theta(x)\,,\, d(\theta(x)\,,\, \delta(x))) \subseteq H_t^+.$$

Let $x \in M_\delta(K,\, \phi) \cap B_2^n(0,\, r)$. Since $x \in \operatorname{int}(K) \cap B_2^n(0,\, r)$,

$$K \cap H^+(\theta(x),\, d(\theta(x),\, \delta(x))) \subseteq H_t^+,$$

and thus $x \in \mathrm{M}_{\delta(x)}\left(K \cap H_t^+, \phi\right)$. Moreover, notice that $\delta(x) \geq \delta$ and hence we have

$$\mathrm{M}_{\delta(x)}\left(K \cap H_t^+, \phi\right) \subseteq \mathrm{M}_\delta\left(K \cap H_t^+, \phi\right).$$

Hence, $x \in \mathrm{M}_\delta\left(K \cap H_t^+, \phi\right)$. Therefore, $\mathrm{M}_\delta(K,\phi) \cap B(0,\, r) \subseteq \mathrm{M}_\delta\left(K \cap H_t^+, \phi\right) \cap B(0,\, r)$. $\square$

5.3.2 Proof of Theorem 5.3

Recall that x_M is the unique point in $\partial(\mathrm{M}_\delta(K,\phi)) \cap [0,x]$. We will sometimes write in short x_M for $x_{M,\delta}$. By Lemmas 5.15 and 5.16, we have that

$$\lim_{\delta\to 0} \frac{|K| - |\mathrm{M}_\delta(K,\phi)|}{\delta^{\frac{2}{n+1}}} = \frac{1}{n}\int_{\partial K} \lim_{\delta\to 0} \delta^{-\frac{2}{n+1}} \langle x,\, N_K(x)\rangle \left(1 - \left|\frac{\|x_M\|}{\|x\|}\right|^n\right) \mathrm{d}\mu_K(x).$$

For $x \in \partial K$ fixed, the goal is to understand

$$\lim_{\delta\searrow 0} \frac{1}{n}\int_{\partial K} \delta^{-\frac{2}{n+1}} \langle x,\, N_K(x)\rangle \left(1 - \left|\frac{\|x_M\|}{\|x\|}\right|^n\right) \mathrm{d}\mu_K(x).$$

As x and x_M are collinear and as for all $0 \leq a \leq 1$,

$$1 - na \leq (1-a)^n \leq 1 - na + \frac{n(n-1)}{2} a^2,$$

we get for δ sufficiently small that

$$\begin{aligned} \frac{\|x - x_M\|\|}{\|x\|} \left(1 - \frac{n-1}{2} \frac{\|x - x_M\|\|}{\|x\|}\right) \leq \frac{1}{n} \left(1 - \left|\frac{\|x_M\|}{\|x\|}\right|^n\right) = \\ \frac{1}{n} \left[1 - \left(1 - \frac{\|x - x_M\|}{\|x\|}\right)^n\right] \leq \frac{\|x - x_M\|\|}{\|x\|}. \end{aligned} \tag{5.16}$$

(i) We assume first that the indicatrix of Dupin at $x \in \partial K$ is an ellipsoid. In fact, by a change of the coordinate system, we may also assume that $x = 0$ and $N_K(0) = -e_n$. Let $\zeta \in \mathbb{R}^n$ be the origin in the previous coordinate system. Let $y_{M,\delta} := \partial(\mathrm{M}_\delta(K, \phi)) \cap [0, \zeta]$. Notice that $\|y_{M,\delta}\| = \|x - x_{M,\delta}\|$ and that $y_{M,\delta} \to 0$ as $\delta \searrow 0$. Thus

$$\lim_{\delta \searrow 0} \langle x,\, N_K(x) \rangle \frac{\|x - x_{M,\delta}\|}{\|x\|} = \lim_{\delta \searrow 0} \langle \zeta,\, e_n \rangle \frac{\|y_{M,\delta}\|}{\|\zeta\|} = \lim_{\delta \searrow 0} \langle y_{M,\,\delta},\, e_n \rangle. \tag{5.17}$$

There exists a volume preserving positive definite linear transform T such that $N_{TK}(0) = -e_n$ and such that the indicatrix of Dupin at 0 becomes a Euclidean ball with radius $\sqrt{\rho}$ (see, e.g., equation (5) in [68]). Moreover, ρ satisfies

$$\kappa_K(0) = \frac{1}{\rho^{n-1}}.$$

Let H^+ be the halfspace such that

$$\delta = \int_{K \cap H^+} \phi(y)\, \mathrm{d}y \quad \text{ and } \quad y_{M,\,\delta} = \frac{\int_{K \cap H^+} y \phi(y)\, \mathrm{d}y}{\delta}.$$

As T is volume preserving, $\int_{TK \cap TH^+} \phi(T^{-1}y)\, \mathrm{d}y = \delta$, and thus

$$\begin{aligned} T y_{M,\,\delta} &= \int_{K \cap H^+} T y \phi(y)\, \mathrm{d}y/\delta = \int_{TK \cap TH^+} y \phi\left(T^{-1}y\right)\, \mathrm{d}y/\delta \\ &\in \partial \mathrm{M}_\delta\left(TK, \phi \circ T^{-1}\right). \end{aligned}$$

As a consequence we have

$$[0,\, T\zeta] \cap \partial \mathrm{M}_\delta\left(TK, \phi \circ T^{-1}\right) = T y_{M,\,\delta},$$

$$\phi\left(T^{-1}0\right) = \phi\left(0\right),$$

and

$$\langle Ty_{M,\delta},\, e_n\rangle = \langle y_{M,\delta},\, Te_n\rangle = \langle y_{M,\delta},\, e_n\rangle.$$

Hence we have reduced the problem to the case when the indicatrix of Dupin at $0 \in \partial K$ is a Euclidean sphere with radius $\sqrt{\rho}$ and $\kappa_K\left(0\right) = \frac{1}{\rho^{n-1}}$.
Moreover, ∂K can be approximated in 0 by a Euclidean ball $B_2^n(\rho e_n, \rho)$ of radius ρ and center ρe_n in the following sense (see, e.g., [69, Proof of Lemma 23]):
Let $\varepsilon > 0$ be given. Let $B_2^n\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right)$ be the Euclidean ball centered at $(1-\varepsilon)\rho e_n$ whose radius is $(1-\varepsilon)\,\rho$. Similarly, let $B_2^n\left((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho\right)$ be the Euclidean ball centered at $(1+\varepsilon)\rho$ with radius $(1+\varepsilon)\rho$. Then,

$$\begin{aligned} 0 \in \partial\left[B_2^n(\rho e_n, \rho)\right],\;\; & 0 \in \partial\left[B_2^n\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right)\right], \\ & 0 \in \partial\left[B_2^n\left((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho\right)\right], \end{aligned}$$

and

$$N_{B_2^n(\rho e_n,\rho)} = N_{B_2^n((1-\varepsilon)\rho e_n,(1-\varepsilon)\rho)} = N_{B_2^n((1+\varepsilon)\rho e_n,(1+\varepsilon)\rho)} = -e_n$$

and (see, e.g., [69, Proof of Lemma 23]) there exists Δ_ε^0 such that for $0 < t < \Delta_\varepsilon^0$, the half-space $H_t^+ = \{y : \langle y, e_n\rangle \le t\}$ determined by the hyperplane orthogonal to e_n through the point te_n is such that

$$\begin{aligned} H_t^+ \cap B_2^n\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right) &\subseteq H_t^+ \;\cap\; K \\ &\subseteq H_t^+ \;\cap B_2^n\left((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho\right). \end{aligned} \tag{5.18}$$

By continuity of ϕ there exists $s > 0$ such that for all $y \in \text{int}(B_2^n(0, s))$,

$$(1-\varepsilon)\,\phi(0) \le \phi(y) \le (1+\varepsilon)\,\phi(0). \tag{5.19}$$

We will apply Lemma 5.18 with $t = \Delta_\varepsilon^0$ simultaneously to K, $B_2^n\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right)$ and $B_2^n\left((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho\right)$ with weights ϕ, $(1-\varepsilon)\,\phi\left(0\right)$, and $(1+\varepsilon)\,\phi\left(0\right)$ respectively. Let $H_{\Delta_\varepsilon}^+ = \{y : \langle y, e_n\rangle \le \Delta_\varepsilon\}$. We choose $\Delta_\varepsilon \le \Delta_\varepsilon^0$ so small that

$$H_{\Delta_\varepsilon}^+ \;\cap\; B_2^n\left((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho\right) \;\subseteq B_2^n(0, \min\{s,\, r\}),$$

where r is given by Lemma 5.18. We denote

$$d_{M,\,\delta}^- = \text{dist}\left(y_{M,\,\delta}, B_2^n\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right)^c\right)$$

and

$$d^{+}_{M,\,\delta} = \text{dist}\left(y_{M,\,\delta}, B^n_2\left((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho\right)^c\right).$$

Boundedness of ϕ on $B^n_2\,(0, s)$ and (5.18) imply that for $\delta \geq 0$,

$$\begin{aligned} \mathrm{M}_\delta\left(B^n_2\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right) \cap H^+_{\Delta_\varepsilon}, (1-\varepsilon)\,\phi\,(0)\right) &\subseteq \mathrm{M}_\delta\left(K \cap H^+_{\Delta_\varepsilon}, \phi\right) \\ &\subseteq \mathrm{M}_\delta\left(B^n_2\left((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho\right) \cap H^+_{\Delta_\varepsilon}, (1+\varepsilon)\,\phi\,(0)\right). \end{aligned}$$

By Lemma 5.18 and the choice of Δ_ε we have

$$\begin{aligned} \mathrm{M}_\delta\left(B^n_2\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right), (1-\varepsilon)\,\phi\,(0)\right) \cap H^+_{\Delta_\varepsilon} &\subseteq\ \mathrm{M}_\delta\left(K, \phi\right) \cap H^+_{\Delta_\varepsilon} \\ \subseteq\ \mathrm{M}_\delta\left(B^n_2\left((1+\varepsilon)\rho e_n, (1-\varepsilon)\rho\right), (1+\varepsilon)\,\phi\,(0)\right) \cap H^+_{\Delta_\varepsilon}. \end{aligned}$$

Choose δ so small that $y_{M,\delta} \in H^+_{\Delta_\varepsilon}$. Then

$$y_{M,\,\delta}\ \notin \text{int}\ \left(\mathrm{M}_\delta\left(B^n_2\left((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho\right), (1-\varepsilon)\,\phi\,(0)\right)\right)$$

and

$$y_{M,\,\delta}\ \in\ \text{int}\ \left(\mathrm{M}_\delta\left(B^n_2\left((1-\varepsilon)\rho e_n, (1+\varepsilon)\rho\right), (1+\varepsilon)\,\phi\,(0)\right)\right).$$

Thus, we conclude that

$$d^{-}_{M,\,\delta} \leq \Delta\left((1-\varepsilon)\,\rho,\ (1-\varepsilon)\,\delta\phi\,(0)\right) \quad \text{and} \quad d^{+}_{M,\,\delta} \geq \Delta\left((1+\varepsilon)\,\rho,\ (1+\varepsilon)\,\delta\phi\,(0)\right),$$

where $\Delta\left((1+\varepsilon)\,\rho,\ (1+\varepsilon)\,\delta\phi\,(0)\right)$ and $\Delta\left((1-\varepsilon)\,\rho,\ (1-\varepsilon)\,\delta\phi\,(0)\right)$ are the differences of the radii of $(1+\varepsilon)\,\rho B^n_2$ and $\mathrm{M}_\delta\left(\rho B^n_2, (1+\varepsilon)\phi(0)\right)$, and of $(1-\varepsilon)\,\rho B^n_2$ and $\mathrm{M}_\delta\left(\rho B^n_2, (1-\varepsilon)\phi(0)\right)$, respectively. Applying Lemma 5.17(ii) with $z = y_{M,\,\delta}$ and Proposition 5.14 for sufficiently small δ, yields

$$(1-\varepsilon)^{\frac{n+1}{n-1}+\frac{2}{n+1}} \leq \frac{\langle y_{M,\,\delta},\, e_n\rangle}{c_n \delta^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}} \phi(0)^{\frac{2}{n+1}}} \leq (1+\varepsilon)^{\frac{n+1}{n-1}+\frac{2}{n+1}}.$$

Since $\varepsilon > 0$ can be chosen arbitrary, we obtain, also using (5.17),

$$\lim_{\delta\to 0} \phi(x)^{\frac{2}{n+1}}\ \langle x,\ N_K\,(x)\rangle \frac{\|x - x_{M,\delta}\|}{\|x\|\,\delta^{\frac{2}{n+1}}} = c_n\ \rho(x)^{-\frac{n-1}{n+1}} = c_n\ \kappa_K(x)^{\frac{1}{n+1}}.$$

(ii) Now we assume that x is such that the indicatrix of Dupin at x is an elliptic cylinder.

We will show that then

$$\lim_{\delta\to 0}\langle x,\ N_K\left(x\right)\rangle\frac{\|x-x_{M,\delta}\|}{\|x\|\,\delta^{\frac{2}{n+1}}}=0.$$

We only need to show that $\lim_{\delta\to 0}\langle x,\ N_K\left(x\right)\rangle\frac{\|x-x_{M,\delta}\|}{\|x\|\delta^{\frac{2}{n+1}}}\leq 0$.

We may assume that the first k axes of the elliptic cylinder have infinite lengths and the others not. Then, as above (see, e.g., [69, Proof of Lemma 23]) for all $\varepsilon>0$ there is an approximating ellipsoid $\mathcal{E}$ and Δ_ε such that the hyperplane $H\left(N_K(x),x-\Delta_\varepsilon)N_K(x)\right)$ orthogonal to $N_K(x)$ through the point $x-\Delta_\varepsilon N_K(x)$ is such that

$$H^+\left(N_K(x),x-\Delta_\varepsilon)N_K(x)\right)\ \cap\ \mathcal{E}\subseteq H^+\left(N_K(x),x-\Delta_\varepsilon)N_K(x)\right)\ \cap\ K$$

and such that the lengths of the k first principal axes of $\mathcal{E}$ are larger than $\frac{1}{\varepsilon}$. As noted above, there is a support hyperplane H_δ to $F_\delta\left(K,\phi\right)$ such that $x_{F,\delta}\in H_\delta$ and such that $\delta=\int_{K\cap H_\delta^+}\phi(y)dy$ [74]. Then

$$\delta\geq\min_{y\in K}\phi(y)|K\cap H_\delta^+|\geq\min_{y\in K}\phi(y)|\mathcal{E}\cap H_\delta^+|.$$

As above, we may assume that the approximating ellipsoid $\mathcal{E}$ is a Euclidean ball with radius $\rho=\rho(x)$ where $\rho\geq\frac{1}{\varepsilon}$. Then

$$\begin{aligned}
\langle x,\ N_K\left(x\right)\rangle\frac{\|x-x_{M,\delta}\|}{\|x\|\,\delta^{\frac{2}{n+1}}} &\leq \langle x,\ N_K\left(x\right)\rangle\frac{\|x-x_{F,\delta}\|}{\|x\|\,\delta^{\frac{2}{n+1}}}\\
&\leq \frac{\langle\frac{x}{\|x\|},N_K(x)\rangle\ \|x-x_{F,\delta}\|}{\left(\min_{y\in K}\phi(y)\right)^{\frac{2}{n+1}}\left(|B_2^n(x-\rho N_K(x),\rho)\cap H_\delta^+|\right)^{\frac{2}{n+1}}}\\
&\leq \frac{\rho^{-\frac{n-1}{n+1}}}{c_n\left(\min_{y\in K}\phi(y)\right)^{\frac{2}{n+1}}}.
\end{aligned}$$

The last inequality can be shown using similar methods as in the case (i). Or, one notices that we are precisely in the situation of Lemmas 7 and 10 of [67] where exactly this estimate is proved. As ρ is arbitrarily small, the proof is completed.

CHAPTER 6

Size of nodal domains for $G(n, p)$ graph

6.1 Introduction

Let A_p be the adjacency matrix of $G(n, p)$. We denote eigenvalues of A_p by $\lambda_1 \geq \cdots \geq \lambda_n$ and the corresponding unit eigenvectors by $u_1, \ldots, u_n$. Recall the theorem of nodal domain theorem of Dekel, Lee and Linial [**?**] which improved by Arora and Bhaskara [5]:

Theorem 6.1. *For $p \geq n^{-1/19+\varepsilon}$, $D > 0$, and v an eigenvector of A_p for non-first eigenvalue λ. Then, v has precisely 2 nodal domains with probability greater than $1-n^{-D}$.*

Before we state our theorems, we seperate the eigenvalues of A_p in the following 2 types: Let $\kappa \in (0, 1)$, we say λ_i is an bulk eigenvalue if $i \in [\kappa n, n-\kappa n]$. For $\rho > 0$, we say λ_i is an edge eigenvalue if $\min\{i, n-i\} \leq (\log n)^{\rho \log\log n}$. To explain why we call them bulk and edge eigenvalues, let us introduce the semicircle law

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$$

where $(4-x^2)_+ = \max\{4-x^2, 0\}$, which arises as the limiting ditribution of eigenvalues of many random matrix models. Consider a $n \times n$ wigner matrix $W = \{w_{ij}\}_{i,j=1}^n$. Precisely, the entries of W are independent, mean 0 and $\mathbb{E}|w_{ij}|^k$ exists for every $k \in \mathbb{N}$. Furthermore, the variance of the off-diagonal and diagonal terms are 1 and C respectively. Let $\mu_1, \cdots, \mu_n$ denote the eigenvalues of W in a non-increasing order. We define the empirical measure of its eigenvalues as

$$\mathrm{Em}(W) = \sum_{i=1}^{n} \frac{1}{n}\delta_{\mu_i}.$$

Then, it is known that asymptotically, we have $\mathrm{EM}\left(\frac{1}{\sqrt{n}}W\right)$ converge weakly to $\rho_{sc}(x)\,\mathrm{d}x$. Our matrix A_p is not far away from this model. Let $\overrightarrow{1} \in S^{n-1}$ be the vector such that every component equals $\frac{1}{\sqrt{n}}$. The entries of

$$\frac{1}{\sqrt{p\,(1-p)}}A_p - \frac{pn}{\sqrt{p\,(1-p)}}\overrightarrow{1}\overrightarrow{1}^\top$$

have mean 0, variance 1 and bounded kth moments for every k. In other words, it is also a wigner matrix. Thus, we expect the eigenvalues of A_p are distributed according to the semi-circle law (with a proper scaling) with one exception: The leading eigenvalue is much larger due to the rank 1 shift $\frac{pn}{\sqrt{p(1-p)}}\overrightarrow{1}\overrightarrow{1}^\top$. Bulk eigenvalues refer to eigenvalues appeared in the bulk region of the semi-circle and edge eigenvalues correspond to eigenvalues close to the edge. (i.e. eigenvalues of $\frac{1}{\sqrt{p(1-p)n}}A_p$ close to $\{-2,\,2\}$) We prove that the nodal domains are roughly of the same size both for the bulk and for the edge eigenvectors. Yet the methods of proof in these cases are entirely different. Let us consider the bulk case first as the proof in this case is shorter.

Theorem 6.2. *(Bulk case) There is $c \in (0,\,1)$ such that the following holds. Let $G\,(n,\,p)$ be an Erdös-Rï¿œnyi Graph with $p \in \left[n^{-c},\,\frac{1}{2}\right]$. Fix $\varepsilon,\,\kappa \in (0,\,1)$. Suppose n is sufficiently large. Let u_α be an eigenvector of $G\,(n,\,p)$ with $\alpha \in [\kappa n,\,n-\kappa n]$. Then there exists $\eta = \eta\,(\varepsilon,\,\kappa) > 0$ such that, for a sufficiently large n,*

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2}+\varepsilon\right)n\right) \leq n^{-\eta},$$

where P is the collection of indexes of u_α with positive components and N is the collection of indexes of u_α with negative components

Remark 6.3. By Theorem 6.1, we know that P and N are exactly the two nodal domains with probability greater than $1-n^{-D}$.

The proof relies on Theorem 1.1 from [13]. (see Theorem 6.5 in section 6.2 for the statement of the theorem)

For the edge case, the bound similar to Theorem 6.5 has not been established yet. On the other hand, the gaps between the eigenvalues near the edges of the spectrum are much larger. The eigenvalue gap is at least $n^{-2/3-o(1)}$ for edge eigenvalues while it is of order $n^{-1-o(1)}$ for bulk eigenvalues. Also, the edge eigenvalues enjoy stronger rigidity properties than the bulk ones.

Theorem 6.4. *(Edge case) Let $G\,(n,\,p)$ be an Erdös-Rï¿œnyi Graph with $p \in (0,\,1)$. Fix a sufficiently large $\rho > 0$. Suppose n is sufficiently large. Let u_α be a non-leading eigenvector of $G\,(n,\,p)$ with $\min\{\alpha,\,n-\alpha\} \leq (\log n)^{\rho \log\log n}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$*

such that

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + n^{-\frac{1}{6}+\varepsilon}\right) n\right) \leq n^{-\delta}$$

where P is the collection of indexes of u_α with positive components and N is the collection of indexes of u_α with negative components.

Our goal here is to show that with high probability we have

$$\sum_{i=1}^{n} \mathrm{sign} u\,(i) = o\,(n)$$

for an eigenvector u of A_p. This can be derived by Markov inequality if $\mathbb{E}\left(\sum_{i=1}^{n} \mathrm{sign}\,(u\,(i))\right)^2 = o\,(n^2)$. The later equation can be derived if for $i \neq j$,

$$\mathbb{E}\mathrm{sign}\,(u\,(i)\,u\,(j)) = o\,(1)\,. \tag{6.1}$$

The proof on both bulk and edge case are aiming to show (6.1). Yet, the approaches are completely different.

6.2 Tools

Theorem 6.5. *[13, Theorem 1.1] Fix arbitrary constants $\mathrm{d}, \kappa > 0$ Let A be an $n \times n$ be the adjacency matrix of a $G(n, p)$ graph with $n^{-1+\mathrm{d}} \leq p \leq 1/2$. Let $v_1, \ldots, v_n$ be its eigenvectors corresponding to the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. For any polynomial $f : \mathbb{R} \to \mathbb{R}$ for any $n \geq n(f)$, $j \in [\kappa n : n - \kappa n]$ and any $q \in S^{n-1}, q \perp \overrightarrow{1}$, there exists an $\nu > 0$ such that*

$$|\mathbb{E}f(n\langle q, v_j\rangle^2) - \mathbb{E}f(g^2)| \leq n^{-\nu}.$$

We will also use a partial case of the no-gaps delocalization theorem [61].

Theorem 6.6. *Let $p \in (0, 1)$ be an arbitrary constant. Let A be the adjacency matrix of a $G\,(n, p)$ graph. Then, for any $\varepsilon > Cn^{-1/7}$with probability at least $1 - \exp\,(-\varepsilon n)$ any non-leading eigenvector $v \in S^{n-1}$ satisfies*

$$\left|\left\{j \,:\, |v\,(j)| \leq c'\varepsilon^{11/2} n^{-1/2}\right\}\right| \leq \varepsilon n.$$

Next, we will introduce a generalized Wigner matrix model:

Condition 6.7. Let $H = (h_{ij})$ be a Hermitian $n \times n$ random matrix with $\mathbb{E}h_{ij} = 0$ and variances $\sigma_{ij}^2 := \mathbb{E}h_{ij}^2$. Denote by $\Sigma := \left(\sigma_{ij}^2\right)$ the matrix of variances. H satisfies this condition if

1. $\sum_{i=1}^{n} \sigma_{ij}^2 = 1$.

2. There exists $\delta_W > 0$ such that 1 is a simple eigenvalue of Σ and Spec $(B) \subseteq [-1+\delta_W,\, 1-\delta_W] \cup \{1\}$.

3. There is a constant C_W, independent of n, such that $\max_{ij} \left\{\sigma_{ij}^2\right\} \leq \frac{C_W}{n}$.

4. h_{ij} have a uniformly subsexponential decay: There exists a constant $\nu > 0$, independent of n,such that for any $x \geq 1$ and $1 \leq i, j \leq n$ we have

$$\mathbb{P}\left(|h_{ij}| > x\sigma_{ij}\right) \leq \nu^{-1} \exp\left(-x^{\nu}\right).$$

Here is a summarized partial results of strong local statistic results and its consequences, rigidity of eigenvalues and l_∞-norm of eigenvectors, by Erdös, Yau and Yin:

Theorem 6.8. *[20, Theorem 2.1, 2.2] Let $H = (h_{ij})$ be Hermitian $n \times n$ random matrix. Suppose H satisfies Condition 6.7. Then, there exist positive constants $A_{sls} > 1$, C_{sls}, c_{sls} and $\phi_{sls} < 1$ depending only on ν and δ_W and C_W from Condition 6.7 such that the following estimates hold for any sufficiently large $n \geq n_0(\nu,\, \delta_W,\, C_W)$. Let $\varphi_n := (\log n)^{\log\log n}$, $G(z)$ be the Green funtion of H, $m(z)$ be the Stieltjes transform of H and $m_{sc}(z)$ be the Stieltjes transform of the semicircle law. We have*

1. $$\mathbb{P}\left(\sup_{z \in S_{A_0}} |m(E+\mathrm{i}\eta) - m_{sc}(E+\mathrm{i}\eta)| \leq \frac{\varphi_n^{4A_{sls}}}{N\eta}\right) \geq 1 - \exp\left(-\varphi_n^{\phi_{sls}A_{sls}}\right) \quad (6.2)$$

   where $S_{A_0} = \left\{z = E + \mathrm{i}\eta \,:\, |E| \leq 5,\, n^{-1}\varphi_n^{10A_0} < \eta \leq 10\right\}$.

2. $$\mathbb{P}\left(\forall\alpha \,:\, |\lambda_\alpha - \gamma_\alpha| < \varphi_n^{A_{sls}} \left[\min(\alpha,\, n-\alpha+1)\right]^{-1/3} n^{-2/3}\right) \geq 1 - \exp\left(-\varphi_n^{\phi_{sls}A_{sls}}\right), \quad (6.3)$$

   where γ_α is the expected location of αth eigenvalue for random matrix satisfying semicircle law. In other words, γ_α satisfies $\int_{2-\gamma_\alpha}^{2} \frac{1}{2\pi}\sqrt{4-x^2}\,\mathrm{d}x = \frac{\alpha}{n}$.

3. $$\mathbb{P}\left(\forall\alpha,\, \|u_\alpha\|_\infty \leq \frac{\varphi_n^C}{\sqrt{n}}\right) \quad (6.4)$$

The following level repulsion condition was introduced in [40, Definition 1.3], which are satisfied for many generalized wigner matrices.

Condition 6.9. (Level Repulsion on Edge) A random Hermitian matrix H is said to satisfy level repulsion at the edge, if for any $C_{LR} > 0$, and $\varepsilon_{LR} > 0$, there exists $\delta_{LR} > 0$, with probability at least $1 - n^{-\delta_{LR}}$

$$\max_{E \subseteq \left[2 - n^{-2/3} \varphi_n^{C_{LR}},\, 2 + n^{-2/3} \varphi_n^{C_{LR}}\right]} \mathcal{N}\left(E - n^{-2/3-\varepsilon_{LR}},\, E + n^{-2/3-\varepsilon_{LR}}\right) < 2. \tag{6.5}$$

Theorem 6.10. *[1, Theorem 2.12](Isotropic local semicircle law) H is a generalized Wigner matrix and for all $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that $\mathbb{E}\left|\sqrt{n} h_{ij}\right|^k \le C_k$ for all n, i and j. Then, for every s, $D > 0$, $0 < \varepsilon < 1/3$, and deterministic unit vectors $v,\, w \in \mathbb{C}^n$ we have*

$$\sup_{z \in S_W} \mathbb{P}\left(\left|\langle v,\, G(E + \mathrm{i}\eta)\, w\rangle - \langle v,\, w\rangle m(E + \mathrm{i}\eta)\right| > n^s \left(\sqrt{\frac{\operatorname{Im} m_{sc}(E + \mathrm{i}\eta)}{n\eta}} + \frac{1}{n\eta}\right)\right) \le n^{-D}$$

where $\eta = n^{-2/3-\varepsilon}$ and $n \ge n(s,\, D)$.

Remark 6.11. To understand what this inequalities means, let's consider $z = E + \mathrm{i}\eta$ with $|E - 2| \le n^{-2/3+\varepsilon}$. We have

$$\operatorname{Im} m_{sc}(z) \le \sqrt{|E - 2| + \eta} \le 2n^{-1/3+\varepsilon/2}.$$

Then,

$$\sqrt{\frac{\operatorname{Im} m_{sc}(E + \mathrm{i}\eta)}{n\eta}} + \frac{1}{n\eta} \le \frac{2}{n\eta} \le 2n^{-\frac{1}{3}+\varepsilon}.$$

We obtain

$$\sup_{|E-2| \le n^{-2/3+\varepsilon}} \mathbb{P}\left(\left|\langle v,\, G(E + \mathrm{i}\eta)\, w\rangle - m_{sc}(E + \mathrm{i}\eta)\right| > 3n^{-\frac{1}{3}+s+\varepsilon}\right) \le n^{-D}$$

for $n \ge n(s,\, D)$. Suppose $E,\, E' \in \mathbb{R}$ satisfies $|E - E'| < n^{-13}$. Then,

$$\begin{aligned}
&\left|\langle v,\, G(E + \mathrm{i}\eta)\, w\rangle - \langle v,\, G(E' + \mathrm{i}\eta)\, w\rangle\right| \\
\le &\left|\sum_\alpha \langle u_\alpha,\, v\rangle \langle u_\alpha,\, w\rangle \frac{E - E'}{(\lambda_\alpha - E - \mathrm{i}\eta)(\lambda_\alpha - E' - \mathrm{i}\eta)}\right| \\
\le & n n^{-13} \eta^{-2} \le n^{-10} < n^{-\frac{1}{3}+s+\varepsilon}.
\end{aligned}$$

Now taking a union bound we obtain the following Corollary:

Corollary 6.12. *(Isotropic local semicircle law) H is a generalized Wigner matrix and for all $k \in \mathbb{N}$ there exists a constant C_k such that $\mathbb{E}\left|\sqrt{n}h_{ij}\right|^k \leq C_k$ for all n, i and j. Then, for every $s,\, D > 0$ and $0 < \varepsilon < 1/3$, we have*

$$\mathbb{P}\left(\sup_{|E-2|\leq n^{-2/3+\varepsilon}} \left|\langle v,\, G\left(E+\mathrm{i}\eta\right)w\rangle - m_{sc}\left(E+\mathrm{i}\eta\right)\right| < 4n^{-\frac{1}{3}+s+\varepsilon}\right) \geq 1-n^{-D} \tag{6.6}$$

where $\eta = n^{-2/3-\varepsilon}$ and $n \geq n\left(s,\, D\right)$.

In other words, $G\left(E+\mathrm{i}\eta\right)$ behaves like $m_{sc}\left(E+\mathrm{i}\eta\right)I_n$. Next, we have the isotropic delocalization theorem:

Theorem 6.13. *[1, Theorem 2.16](Isotropic delocalization) H is a generalized Wigner matrix and for all $k \in \mathbb{N}$ there exists a constant C_k such that $\mathbb{E}\left|\sqrt{n}h_{ij}\right|^k \leq C_k$ for all n, i and j. Then, for every $s,\, D > 0$ and deterministic unit vector $v, \in \mathbb{C}^n$ we have*

$$\mathbb{P}\left(\max_{\alpha\in[n]} \left|\langle u_\alpha,\, v\rangle\right|^2 < n^{s-1}\right) \geq 1-n^{-D}$$

when n is sufficiently large.

We also needs a Hanson-Wright Inequality for i.i.d subgaussian vectors:

Theorem 6.14. *[60]Let $X = \left(X_1, \ldots, X_n\right) \in \mathbb{R}^n$ be a random vector with independent components X_i with satisfy $\mathbb{E}X_i = 0$, and $\left\|X_i\right\|_{\psi_2} \leq K$. Let A be an $n\times n$ matrix. Then, for every $t \geq 0$,*

$$\mathbb{P}\left(\left|X^\top AX - \mathbb{E}X^\top AX\right| > t\right) \leq 2\exp\left(-c\min\left(\frac{t^2}{K^4\left\|A\right\|_{HS}^2},\, \frac{t}{K^2\left\|A\right\|}\right)\right) \tag{6.7}$$

Theorem 6.15. *(Berry-Esseen) Suppose $X = \left(X_1, \ldots X_n\right)$ are i.i.d random varaibles with mean 0 and variance σ_i^2. Let $\rho_i = \mathbb{E}\left|X_i\right|^3$. Consider $S_n = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n \sigma_i^2}}$. Let F_n and Φ be the cumulative distribution function of S_n and standard norml distribution respectively. Then,*

$$\sup_{x\in\mathbb{R}} \left|F_n\left(x\right) - \Phi\left(x\right)\right| \leq \left(\sum_{i=1}^n \sigma_i^2\right)^{-1/2} \max_i \frac{\rho_i}{\sigma_i^2}.$$

6.3 Bulk eigenvector

Consider a graph G with the adjacency matrix A, and let $v \in S^{n-1}$ be its eigenvector. In order to show that $\sum_{j=1}^n \operatorname{sign}\bigl(v(j)\bigr) = o(n)$, consider a random pair of distinct indices $(k, l) \subset [n]$, which is uniformly chosen among all such pairs. We will check below that if $\mathbb{E}\mathrm{sign}(v_j(k) \cdot v_j(l)) = o(1)$, then the nodal domains are of the size close to $n/2$. We are going to establish this bound for the adjacency matrix of a typical $G(n, p)$ graph. Since sign is not a continuous function, it is hard to approach this task directly. Instead, we will approximate the function sign by a suitable polynomial f and show that $\mathbb{E}\bigl[f(v_j(k) \cdot v_j(l)) \mid A\bigr] = o(1)$ where the expectation is taken with respect to the random pair (k, l) and A is the adjacency matrix of a typical $G(n, p)$ graph, i.e., it is chosen from some set of adjacency matrices whose probability is $1 - o(1)$. After that, we will have to estimate the error of this approximation. To implement the first step, we will use Theorem 6.5 to derive a similar bound for the expectation of an even polynomial of four random coordinates of the eigenvector. This will lead to a stronger bound for an even polynomial of two random coordinates. Finally, applying the latter bound to a one-variable polynomial of the product of two coordinates, we will get the desired estimate.

Let us formulate this statement precisely. Let $v_j \in S^{n-1}$ be a bulk eigenvector of the $G(n, p)$ graph, and let $g_1, \ldots, g_n \sim N(0, 1)$ be independent standard normal random variables. Denote by $\mathbb{E}_{(k,l)}$ the expectation with respect to the random pair of coordinates (k, l), where the matrix A is regarded as fixed.

Lemma 6.16. *Let A, v_j be as in Theorem 6.5. Let (k, l) be a uniformly chosen random pair of elements of $[n]$. For any* even *polynomial $F : \mathbb{R}^2 \to \mathbb{R}$, there exists a $\nu > 0$ and a set $\mathcal{A}_F \in \mathbf{Mat}_{sym}(n)$ such that for all sufficiently large n,*

$$\mathbb{P}(A \in \mathcal{A}_F) \geq 1 - n^{-\nu},$$

and for any $A \in \mathcal{A}_F$,

$$|\mathbb{E}_{(k,l)} F(n^{1/2} v_j(k), n^{1/2} v_j(l)) - \mathbb{E}F(g_1, g_2)| \leq n^{-\nu}.$$

Proof. The proof breaks in two parts. First, we will show that the statement of Theorem 6.5 holds for any $q \in S^{n-1}$ such that $|\text{supp}(q)| \leq 4$. It is enough to prove the statement for $f(x) = x^d$. Without loss of generality, assume that $q = \sum_{j=1}^4 \alpha e_j$ with $\sum_{j=1}^4 \alpha_j^2 = 1$. Set

$\beta := \langle \overrightarrow{1}, q \rangle = n^{-1/2} \sum_{j=1}^{4} \alpha_j$. Then

$$|\beta| \leq \frac{4}{\sqrt{n}}, \quad q_0 := q - \beta \overrightarrow{1} \perp \overrightarrow{1} \quad \text{and} \quad \|q_0\|_2 = 1 + O(n^{-1/2}). \tag{6.8}$$

Recall that $w := \overrightarrow{1} - v_1$ satisfies

$$\|w\|_2 \leq 2\frac{\log n}{\sqrt{n}}, \tag{6.9}$$

see [61, Theorem 3].

Let us check that for any $d \in \mathbb{N}$,

$$\mathbb{E}(n\langle q, v_j\rangle^2)^d \leq C(d)$$

for some function $C(d) > 0$. Indeed, since $\left\langle \overrightarrow{1}, v_j \right\rangle = \langle w, v_j \rangle$,

$$\begin{aligned}
\mathbb{E}(n\langle q, v_j\rangle^2)^d = \mathbb{E}(n\langle q_0 + \beta\sqrt{n}w, v_j\rangle^2)^d &\leq 2^{2d}\left(\mathbb{E}(n\langle q_0, v_j\rangle^2)^d + \beta^{2d} n^d \|w\|_2^{2d}\right) \\
&\leq 2^{2d}\left(\mathbb{E}(2g_1^2)^d + \left(16\frac{\log^2 n}{n}\right)^d\right) \leq C(d).
\end{aligned}$$

where we used (6.8), (6.9) and Theorem 6.5 in the second inequality. By Cauchy-Schwarz inequality, this means that for any $k \in \mathbb{N}$,

$$\mathbb{E}|\sqrt{n}\,\langle q, v_j\rangle\,|^k \leq C'(k). \tag{6.10}$$

Therefore, for any $d \in \mathbb{N}$,

$$\begin{aligned}
\left|\mathbb{E}(n\langle q, v_j\rangle^2)^d - \mathbb{E}g^{2d}\right| &\leq \left|\mathbb{E}(n\langle q, v_j\rangle^2)^d - \mathbb{E}(n\langle \frac{q_0}{\|q_0\|_2}, v_j\rangle^2)^d\right| + \left|\mathbb{E}(n\langle \frac{q_0}{\|q_0\|_2}, v_j\rangle^2)^d - \mathbb{E}g^{2d}\right| \\
&\leq \left|\mathbb{E}(n\langle q, v_j\rangle^2)^d - \frac{1}{\|q_0\|_2^{2d}}\mathbb{E}(n\langle q - \beta\overrightarrow{1}, v_j\rangle^2)^d\right| + n^{-\nu} \\
&\leq \left|\mathbb{E}(n\langle q, v_j\rangle^2)^d - \mathbb{E}(n\langle q - \beta w, v_j\rangle^2)^d\right| + 2n^{-\nu} \\
&\leq \sum_{j=1}^{n}\binom{2d}{j}\mathbb{E}|\sqrt{n}\langle q, v_j\rangle|^{2d-j} \cdot \left(8\frac{\log n}{\sqrt{n}}\right)^j + 2n^{-\nu} \leq n^{-\nu'}
\end{aligned}$$

for large n. Here, the third inequality follows from Theorem 6.5, the fourth one from (6.8) and (6.9), and the last one from (6.10). This shows that the conclusion of Theorem 6.5 holds for any $q \in S^{n-1}$ supported on four coordinates. The same argument can be used to prove this statement for any fixed number of coordinates, but we would not need it here.

Let us extend the conclusion of Theorem 6.5 to even polynomials of four variables. Consider an even monomial $G(x_1, \ldots, x_4) := x_1^{d_1} \cdot x_2^{d_2} \cdot x_3^{d_3} \cdot x_4^{d_4}$ with $d = d_1 + d_2 + d_3 + d_4 \in 2\mathbb{N}$. Note that for this monomial, $G(\sqrt{n}v_j(k_1), \ldots, \sqrt{n}v_j(k_4))$ can be represented as a finite linear combination of $(\sqrt{n}\,\langle q, v_j\rangle)^d$ for different values of $q \in S^{n-1}$, $\text{supp}(q) \subset \{k_1, \ldots, k_4\}$. Hence,

$$\left|\mathbb{E}G(\sqrt{n}v_j(k_1), \ldots, \sqrt{n}v_j(k_4)) - \mathbb{E}G(g_1, \ldots, g_4)\right| \leq n^{-\nu} \tag{6.11}$$

and this ine quality can be extended to all even polynomials of four variables.

Now, let $F : \mathbb{R}^2 \to \mathbb{R}$ be an even polynomial. Let $s \in [\kappa n : n - \kappa n]$. For a pair $(i,j) \in \binom{[n]}{2}$, define a random variable

$$Y_{(i,j)} = F(\sqrt{n}v_s(i), \sqrt{n}v_s(j)) - \mathbb{E}F(g_i, g_j),$$

where $g_1, \ldots, g_n$ are independent $N(0,1)$ random variables. Then for any distinct $i, j, k, l, \in [n]$,

$$\begin{aligned}
|\mathbb{E}Y_{(i,j)}Y_{(k,l)}| &= |\mathbb{E}F(\sqrt{n}v_s(i), \sqrt{n}v_s(j))F(\sqrt{n}v_s(k), \sqrt{n}v_s(l)) \\
&\quad - \mathbb{E}F(\sqrt{n}v_s(i), \sqrt{n}v_s(j))\mathbb{E}F(g_k, g_l) \\
&\quad - \mathbb{E}F(g_i, g_j)\mathbb{E}F(\sqrt{n}v_s(k), \sqrt{n}v_s(l)) \\
&\quad + \mathbb{E}F(g_i, g_j)\mathbb{E}F(g_k, g_l)| \\
&\leq |\mathbb{E}F(g_i, g_j)F(g_k, g_l) - 2\mathbb{E}F(g_i, g_j) \cdot \mathbb{E}F(g_k, g_l) \\
&\quad + \mathbb{E}F(g_i, g_j)F(g_k, g_l)| + n^{-\nu} \\
&= n^{-\nu},
\end{aligned}$$

where we used (6.11) with $G_1(x_1, x_2, x_3, x_4) = F(x_1, x_2)F(x_3, x_4),\ G_2(x_1, x_2, x_3, x_4) = F(x_1, x_2)$, and
$G_3(x_1, x_2, x_3, x_4) = F(x_3, x_4)$ to derive the inequality. A similar calculation shows that $|\mathbb{E}Y_{(i,j)}Y_{(k,l)}| = O(1)$ when i, j, k, l are not necessarily distinct. Hence,

$$\mathbb{E}\left(\frac{1}{\binom{n}{2}} \sum_{(i,j) \in \binom{[n]}{2}} Y_{(i,j)}\right)^2 \leq \frac{1}{\binom{n}{2}^2} \sum_{(i,j,k,l) \in \binom{[n]}{4}} \mathbb{E}Y_{(i,j)}Y_{(k,l)} + O(n^{-1}) \leq n^{-\nu}.$$

The Markov inequality implies that there exists a set $\mathcal{A}'_F \in \mathbf{Mat}_{sym}(n)$ such that for all sufficiently large n,

$$\mathbb{P}(A \in \mathcal{A}'_F) \geq 1 - n^{-\nu/2},$$

and for any $A \in \mathcal{A}'_F$,

$$\left| \frac{1}{\binom{n}{2}} \sum_{(i,j) \in \binom{[n]}{2}} F(\sqrt{n} v_s(i), \sqrt{n} v_s(j)) - \mathbb{E} F(g_1, g_2) \right| = \left| \frac{1}{\binom{n}{2}} \sum_{(i,j) \in \binom{[n]}{2}} Y_{(i,j)} \right| \leq n^{-\nu/4}.$$

The lemma is proved. □

Applying the previous lemma to a polynomial $F(x, y) = f(x \cdot y)$ for a one-variable polynomial f, we derive the following corollary.

Corollary 6.17. *Let A, v_j be as in Theorem 6.5. Let (k, l) be a uniformly chosen random pair of elements of $[n]$. For any polynomial $f : \mathbb{R} \to \mathbb{R}$, there exists a $\nu > 0$ and the set $\mathcal{A}_f \in \mathbf{Mat}_{sym}(n)$ such that for all sufficiently large n,*

$$\mathbb{P}(A \in \mathcal{A}_f) \geq 1 - n^{-\nu},$$

and for any $A \in \mathcal{A}_f$,

$$|\mathbb{E}_{(k,l)} f(n v_j(k) \cdot v_j(l)) - \mathbb{E} f(g_1 g_2)| \leq n^{-\nu}.$$

To prove that the nodal domains are balanced, we will use Corollary 6.17 with f being an *odd* polynomial approximating sign(x) on some interval $[r, R]$. Since f is odd, $\mathbb{E} f(g_1 g_2) = 0$. Hence, assuming that the nodal domains are unbalanced, it would be enough to show that $|\mathbb{E}_{(k,l)} f(n v_j(k) \cdot v_j(l))|$ is non-negligible to get a contradiction. The values of r and R will be chosen so that the absolute values of most of the coordinates will fall into this interval. A simple combinatorial calculation will show that if the nodal domains are unbalanced, then $\mathbb{E}_{(k,l)}$sign$(v_j(k) \cdot v_j(l)) = \Omega(1)$. Indeed, assume that for a given matrix A and vector v_j,

$$|P| \vee |N| \geq \left(\frac{1}{2} + \varepsilon\right).$$

Then

$$\mathbb{E}_{(k,l)} \text{sign}(v_j(k) \cdot v_j(l)) = \binom{n}{2}^{-1} \cdot \left[\binom{|P|}{2} + \binom{|N|}{2} - |P| \cdot |N| \right] \geq 4\varepsilon^2 + O(n^{-1}).$$

This reduces our task to the comparison between this quantity and $|\mathbb{E}_{(k,l)} f(n v_j(k) \cdot v_j(l))|$. To achieve it, we construct f approximating $\text{sign}(x)$ pointwise on the set $[-R, -r] \cup [r, R]$

and show that the contribution of the coordinates falling outside of this set is negligible. For the interval $(-r, r)$, this will be done using the no-gaps delocalization. Handling the set $(-\infty, -R) \cup (R, \infty)$ is more delicate. Since the polynomial is unbounded on this set, we will control the L_2 norm of f and use the Markov inequality. This argument requires constructing the polynomial f which approximates $\operatorname{sign}(x)$ in two metrics simultaneously: uniformly on the set $[-R, -r] \cup [r, R]$ and in $L_2(\mu)$ norm on $\mathbb{R}$. The measure μ here will be the probability measure on $\mathbb{R}$ defined by

$$\mu(B) = \mathbb{P}(g_1 g_2 \in B).$$

Instead of controlling two metrics at the same time, we will introduce one Sobolev norm which will be stronger than both metrics. Such norm can be chosen in many different ways. We will chose a particular way which makes the argument shorter.

Let $\eta : \mathbb{R} \setminus \{0\} \to (0, \infty)$ and $\psi : \mathbb{R} \to (0, \infty)$ be even functions such that

- $\eta \in C^1((0, \infty))$, $\psi \in C^1(\mathbb{R})$;
- $\eta(x), \psi(x) = \frac{1}{\pi} \exp(-x/2)$ for all $x \geq 2$;
- $\eta(x) \geq \phi(x)$ for all $x > 0$, and $\eta \in L_1(\mathbb{R})$.

Consider a weighted Sobolev space H defined as the completion of the space of $C^1(\mathbb{R})$ functions for which the norm

$$\|f\|_H^2 := \int_{\mathbb{R}} f^2(x)\eta(x)\, dx + \int_{\mathbb{R}} (f'(x))^2 \psi(x)\, dx$$

is finite. Note that $H \subset C(\mathbb{R})$. Indeed, for any $M > 0$, $a < b$, $a, b \in [-M, M]$ and any $f \in C^1(\mathbb{R})$,

$$\begin{aligned} |f(b) - f(a)| = \left| \int_a^b f'(a)\, dx \right| &\leq \left(\min_{x \in [-M,M]} \psi(x) \right)^{-1} \cdot \int_a^b |f'(x)|\psi(x)\, dx \\ &\leq \left(\min_{x \in [-M,M]} \psi(x) \right)^{-1} \cdot \left(\int_a^b (f'(x))^2 \psi(x)\, dx \right)^{1/2} \left(\int_a^b \psi(x)\, dx \right)^{1/2} \\ &\leq \left(\min_{x \in [-M,M]} \psi(x) \right)^{-1} \cdot \|f\|_H \cdot \left(\max_{x \in [-M,M]} \psi(x) \right)^{1/2} \cdot (b-a)^{1/2}, \end{aligned} \tag{6.12}$$

and the same inequality holds for the completion.

We will need the following lemma.

Lemma 6.18. *Let $h \in C^1(\mathbb{R})$ be an odd function such that $\|h\|_\infty + \|h'\|_\infty < \infty$. Then for any $\delta > 0$, there exists an odd polynomial Q satisfying $\|Q - h\|_H < \delta$.*

Proof. Denote by $\mathbb{P}$ the set of all polynomials. Let E_{odd} be the set of all odd functions $h \in C^1(\mathbb{R})$ such that $\|h\|_\infty + \|h'\|_\infty < \infty$. It is enough to prove that $E_{odd} \subset \mathrm{Cl}_H\mathbb{P}$. Indeed, if this is proved, then for any $\delta > 0$ there exists $q \in \mathbb{P}$ such that $\|h - q\|_H < \delta$. Setting $Q(x) = \frac{1}{2}(q(x) - q(-x))$ to make the polynomial odd would finish the proof.

Assume to the contrary that $E_{odd} \not\subset \mathrm{Cl}_H\mathbb{P}$. Then there exists $h \in \mathrm{Cl}_H(E_{odd}) \setminus \{0\}$ such that $\langle h, x^n \rangle_H = 0$ for any $n \in \{0\} \cup \mathbb{N}$. We will prove that this assumption leads to a contradiction. To this end, set

$$F(z) = \int_{\mathbb{R}} h(x)e^{zx}\eta(x)\,dx + \int_{\mathbb{R}} h'(x)ze^{zx}\psi(x)\,dx.$$

Using the Cauchy-Schwarz inequality, one can check that the function F is analytic in $\{z : |\mathrm{Re}(z)| < 1/2\}$ and

$$F^{(n)}(0) = \int_{\mathbb{R}} h(x)x^n\eta(x)\,dx + \int_{\mathbb{R}} h'(x)nx^{n-1}\psi(x)\,dx = \langle h, x^n \rangle_H = 0.$$

Hence, $F(z) = 0$, and applying this conclusion to $z = it$, $t \in \mathbb{R}$, we see that h satisfies the equality

$$\big(h\eta - (h'\psi)'\big)^\wedge = 0 \text{ and thus } h\eta - (h'\psi)' = 0$$

in the sense of distributions. Since the function $h\eta$ is continuous on $(0, \infty)$, h satisfies the differential equation

$$h(x)\eta(x) - (h'(x)\psi(x))' = 0 \tag{6.13}$$

pointwise for all $x \in (0, \infty)$. This in turn means that h'' is well-defined on $(0, \infty)$. Actually, with a little effort, one can prove that this differential equation is satisfied for all $x \in \mathbb{R}$, but we would not need it for our proof.

Since $h \in \mathrm{Cl}_H(E_{odd})$, h is an odd continuous function. For $x \geq 2$, (6.13) reads

$$h(x) + \frac{1}{2}h'(x) - h''(x) = 0,$$

and so $h(x) = C_1\exp(\lambda_1 x) + C_2\exp(\lambda_2 x)$ with

$$\lambda_1 = \frac{1 - \sqrt{17}}{4}, \quad \lambda_2 = \frac{1 + \sqrt{17}}{4}$$

for all $x \geq 2$. Since $\lambda_2 > 1/2$ and $h \in H$, $C_2 = 0$. Without loss of generality, assume that $h(2) > 0$, i.e., $C_1 > 0$. Then $h'(2) < 0$ and since $h(0) = 0$, $h(2) > 0$, there exists $x \in (0, 2)$ such that $h'(x) > 0$. Denote

$$x_0 = \sup\{x \in (0,2) :\ h'(x) > 0\}.$$

Then $h'(x_0) = 0$ and since $h'(x) \leq 0$ for $x > x_0$, we have $h(x_0) > 0$. Hence, (6.13) implies that $h''(x_0) > 0$. Therefore $h'(x) > 0$ for some $x > x_0$, which contradicts the definition of x_0. This contradiction finishes the proof of the lemma. $\square$

We are now ready to prove the main result of this section.

Proof. [Proof of Theorem 6.2] Fix an $\varepsilon > 0$, and let Ω be the event that $|P| \vee |N| \geq (1/2 + \varepsilon)n$. Let (k, l) be a uniformly chosen random pair of distinct elements of $[n]$. Assume that Ω occurs. Then

$$\mathbb{P}(v(k)v(l) > 0 \mid A) \geq \frac{\binom{(1/2+\varepsilon)n}{2} + \binom{(1/2-\varepsilon)n}{2}}{\binom{n}{2}} = \frac{1}{2} + 2\varepsilon^2 + O(n^{-1}) \tag{6.14}$$

and

$$\mathbb{P}(v(k)v(l) < 0 \mid A) \leq \frac{\left(\frac{1}{4} - \varepsilon^2\right) n^2}{\binom{n}{2}} = \frac{1}{2} - 2\varepsilon^2 + O(n^{-1}). \tag{6.15}$$

By Theorem 6.6, for $r = c\varepsilon^{22}$,

$$\mathbb{P}\left(|\{j \in [n] :\ |v(j)| \leq r^{1/2}n^{-1/2}\}| \geq (\varepsilon^2/8)n\right) \leq \exp(-c\varepsilon n).$$

Let Ω_{large} be the event that $|\{j \in [n] :\ |v(j)| \leq r^{1/2}n^{-1/2}\}| \leq (\varepsilon^2/8)n$, and assume that $\Omega \cap \Omega_{large}$ occurs. Then

$$\mathbb{P}(n|v(k)| \cdot |v(l)| \leq r \mid A) \leq \mathbb{P}(|v(k)| \wedge |v(l)| < r^{1/2}n^{-1/2} \mid A) \leq 1 - \frac{\binom{(1-(\varepsilon^2/8))n}{2}}{\binom{n}{2}} \leq \frac{\varepsilon^2}{4}. \tag{6.16}$$

Let $R \geq (c_0\varepsilon)^{-4}$, where the constant $c_0 > 0$ will be chosen later. Since $\|v\|_2 = 1$,

$$|\{j \in [n] :\ |v(j)| \geq R^{1/2}n^{-1/2}\} \leq \frac{n}{R} \leq (c_0\varepsilon)^4 n,$$

so

$$\mathbb{P}(n|v(k)|\cdot|v(l)| \geq R \mid A) \leq \mathbb{P}(|v(k)| \geq R^{1/2}n^{-1/2} \text{ or } |v(l)| \geq R^{1/2}n^{-1/2} \mid A) \leq 2(c_0\varepsilon)^4. \tag{6.17}$$

Summarizing (6.14), (6.15), (6.16), and (6.17), and choosing c_0 small enough, we conclude that on the event $\Omega \cap \Omega_{large}$,

$$\mathbb{P}(nv(k)v(l) \in [r, R] \mid A) \geq \frac{1}{2} + \frac{3}{2}\varepsilon^2 + O(n^{-1})$$

and

$$\mathbb{P}(nv(k)v(l) \in [-r, -R] \mid A) \leq \frac{1}{2} - \frac{3}{2}\varepsilon^2 + O(n^{-1}).$$

Let $h \in C^\infty(\mathbb{R})$ be an odd function such that $h(x) = \operatorname{sign}(x)$ for any $x \notin (-r, r)$. Lemma 6.18 and inequality (6.12) imply that there exists an odd polynomial Q such that $\|h - Q\|_{L_2(\phi\, dx)} < \varepsilon$ and

$$\max_{x \in [-R,R]} |h(x) - Q(x)| \leq \frac{\varepsilon^2}{2}.$$

By Corollary 6.17, there exists $\mathcal{A}_Q$ with $\mathbb{P}(A \in \mathcal{A}_Q) \geq 1 - n^{-\nu}$ such that for any $A \in \mathcal{A}_Q$,

$$\mathbb{E}_{(k,l)} Q(nv(k)v(l)) \leq \mathbb{E} Q(g_1 g_2) + n^{-\nu} = n^{-\nu},$$

for sufficiently large n, since the polynomial Q is odd. We will provide a lower estimate of this expectation in terms of $\mathbb{P}(\Omega)$. We have

$$\mathbb{E}_{(k,l)} Q(v(k)v(l)) = \mathbb{E}_{(k,l)} Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)| \leq R} + \mathbb{E}_{(k,l)} Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)| > R}.$$

Let us estimate these terms separately. On the event $\Omega \cap \Omega_{large}$,

$$\begin{aligned} \mathbb{E}[Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)| \leq R} \mid A] &\geq \left(1 - \frac{\varepsilon^2}{2}\right) \mathbb{P}(nv(k)v(l) \in [r, R] \mid A) \\ &\quad - \left(1 + \frac{\varepsilon^2}{2}\right) \mathbb{P}(nv(k)v(l) \in [-R, -r] \mid A) \\ &\quad - \left(1 + \frac{\varepsilon^2}{2}\right) \mathbb{P}(nv(k)v(l) \in [-r, r] \mid A) \\ &\geq 2\varepsilon^2 + O(n^{-1}). \end{aligned}$$

If $A \in \mathcal{A}_{Q^2}$, then

$$\mathbb{E}[Q^2(nv(k)v(l)) \mid A] \leq \mathbb{E} Q^2(g_1 g_2) + n^{-\nu} \leq \left(\|f\|_{L_2(\phi\, dx)} + \varepsilon\right)^2 + n^{-\nu} \leq C.$$

Hence, by (6.17) and Cauchy-Schwarz inequality, for any $A \in \mathcal{A}_{Q^2}$,

$$\mathbb{E}\left[Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)|>R} \mid A\right] \leq (\mathbb{P}[n|v(k)v(l)| \geq R \mid A])^{1/2} \cdot \left(E[Q^2(nv(k)v(l)) \mid A]\right)^{1/2}$$
$$\leq C(c_0\varepsilon)^2 \leq \frac{\varepsilon^2}{2}$$

if c_0 is chosen sufficiently small. Thus, if $A \in \mathcal{A}_{Q^2}$ and the event $\Omega \cap \Omega_{large}$ occurs and n is sufficiently large to absorb the $O(n^{-1})$ term, then

$$\mathbb{E}\left[Q(nv(k)v(l)) \mid A\right] \geq \frac{\varepsilon^2}{4},$$

and so, $A \notin \mathcal{A}_Q$. This means that $\Omega \cap \Omega_{large} \cap \{A \in \mathcal{A}_{Q^2} \cap \mathcal{A}_Q\} = \varnothing$, and so

$$\mathbb{P}(\Omega) \leq \mathbb{P}(\Omega_{large}^c) + \mathbb{P}(A \in \mathcal{A}_{Q^2}^c) + \mathbb{P}(A \in \mathcal{A}_Q^c) \leq n^{-\nu}.$$

The theorem is proved. □

6.4 Edge Eigenvector

In this section we will prove Theorem 6.4:

Theorem. *(Edge case) Let $G(n,\, p)$ be an Erdös-Rï¿œnyi Graph with $p \in (0,\, 1)$. Fix a sufficiently large $\rho > 0$. Suppose n is sufficiently large. Let u_α be a non-leading eigenvector of $G(n,\, p)$ with $\min\{\alpha,\, n-\alpha\} \leq \varphi_n^\rho = (\log n)^{\rho \log\log n}$. Denote by P and N the nodal domains of this eigenvector. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + n^{-\frac{1}{6}+\varepsilon}\right) n\right) \leq n^{-\delta}.$$

Let A_p be the adjacency matrix of the $G(n,\, p)$ graph with a fixed $p \in (0,\, 1)$. Suppose u is a non-leading edge eigenvector. We are aiming to show that

$$\mathbb{E}\left(\operatorname{sign}\left(u\left(1\right) u\left(2\right)\right)\right) \leq n^{-1/3+\varepsilon} \tag{6.18}$$

for a sufficiently small $\varepsilon > 0$. If proved, it leads to

$$\mathbb{E}\left(\sum_i \operatorname{sign} u\left(i\right)\right)^2 = n + \sum_{i \neq j} \mathbb{E}\operatorname{sign}\left(u\left(i\right) u\left(j\right)\right) \leq n + \binom{n}{2} n^{-1/3+\varepsilon} \leq n^{5/3+\varepsilon}.$$

By Markov's inequality, we can derive a bound for $\mathbb{P}\left(|\sum_i \operatorname{sign} u\left(i\right)| \geq n^{5/6+\varepsilon}\right)$ and thus prove Theorem 6.4. Due to technical difficulties, we would not derive (6.18) directly. Instead, we find an event $\mathscr{A}$ so that

$$\mathbb{E}\left(\operatorname{sign}\left(u\left(1\right)u\left(2\right)\right)\mid\mathscr{A}\right)\leq n^{-1/3+\varepsilon}.$$

For the event $\mathscr{A}$, $\mathbb{P}\left(\mathscr{A}^{c}\right)\leq n^{-\delta}$ where $\delta>0$ is ε dependent. Suppose we are able to find such event, then, we have

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{i}\operatorname{sign}u\left(i\right)\right|\geq n^{5/6+\varepsilon/2}\right)&\leq\mathbb{P}\left(\mathscr{A}^{c}\right)+\mathbb{P}\left(\left|\sum_{i}\operatorname{sign}u\left(i\right)\right|\geq n^{5/6+\varepsilon}\mid\mathscr{A}\right)\\&\leq n^{-\delta}+n^{-\varepsilon}\leq n^{-\delta'},\end{aligned}$$

which finishes the proof of Theorem (6.4).

The adjacency matrix A_p can be represented as a rank one shift of a scaled Wigner matrix:

$$\tilde{A}:=H+\sqrt{\frac{pn}{1-p}}\overrightarrow{1}\overrightarrow{1}^{\top}\tag{6.19}$$

where $H_{ij}=\left(h_{ij}\right)$ is a Wigner matrix with

$$h_{ij}=\begin{cases}\sqrt{\frac{1-p}{p}}\frac{1}{\sqrt{n}} & \text{with probability } p,\\ -\sqrt{\frac{p}{1-p}}\frac{1}{\sqrt{n}} & \text{with probability } 1-p.\end{cases}\tag{6.20}$$

and $\overrightarrow{1}\in S^{n-1}$ is the vector such that every component equals $\frac{1}{\sqrt{n}}$.

The matrix H satisfies both Condition 6.7 and Condition 6.9.

Lemma 6.19. *The matrix H satisfies Condition 6.9.*

The proof of this lemma is based on Green function comparison theorem. The proof is almost identical to that of [40, Proposition 2.4] and relies on comparison with the GOE matrix. We omit the details.

In this section, we will fix a sufficiently large $\rho>0$ appearing in the Theorem 6.4. In particular, we require that $\rho>4A_{sls}$ where A_{sls} is the constant from the strong local statistic Theorem 6.8.

6.4.1 Outline of the proof

To lighten the notation, assume that A_p is $n+2$ by $n+2$. It is convenient to break the matrix $\tilde{A}$ into the blocks:

$$\tilde{A} = \begin{bmatrix} D & W^\top \\ & \\ W & B \\ & \end{bmatrix}, \tag{6.21}$$

where B is of size $n \times n$ and D is of size 2×2. Let $G(z) := \frac{1}{B-z}$ be the Green function of B. We will write the eigenvalues of $\tilde{A}$ in terms of B, W and D:

Proposition 6.20. *Any $\lambda \in R$ satisfying*

$$\det\left(W^\top G(\lambda) W - D + \lambda I_2\right) = 0 \tag{6.22}$$

is an eigenvalue λ of $\tilde{A}$. Furthermore, let $q \in \mathbb{R}^2$ be a non-trivial null vector of $W^\top G(\lambda) W - D + \lambda I_2$. Then, $\begin{bmatrix} q \\ -G(\lambda) W q \end{bmatrix}$ is an eigenvector corresponding to λ.

Proof. Assume that

$$\det\left(W^\top G(\lambda) W - D + \lambda I_2\right) = 0.$$

Let $q \in \mathbb{R}^2$ be a non-trivial null vector of $W^\top G(\lambda) W - D + \lambda I_2$. Then, we have

$$\begin{bmatrix} D-\lambda & W^\top \\ & \\ W & B-\lambda \\ & \end{bmatrix} \begin{bmatrix} q \\ -G(\lambda) W q \end{bmatrix} = \vec{0}.$$

Therefore, we have λ is an eigenvalue of $\tilde{A}$ and $u = \begin{pmatrix} q \\ -G(\lambda) W q \end{pmatrix}$ is the corresponding eigenvector. ∎

Up to a scaling, we have $q = \begin{bmatrix} 1 \\ -\frac{w_1^\top G(\lambda) w_1 - d_{11} + \lambda}{w_1^\top G(\lambda) w_2 - d_{12}} \end{bmatrix}$ where w_1, w_2 are the column vectors of W and $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}$. Therefore,

$$\operatorname{sign}(u(1)\,u(2)) = \operatorname{sign}\left(-\frac{w_1^\top G(\lambda) w_1 - d_{11} + \lambda}{w_1^\top G(\lambda) w_2 - d_{12}}\right). \tag{6.23}$$

Our goal is to estimate $\mathbb{E}\text{sign}\left(-\frac{w_1^\top G(\lambda)w_1-d_{11}+\lambda}{w_1^\top G(\lambda)w_2-d_{12}}\right)$. Notice that $G(z)$ depends only on B, but λ depends on W, B and D. Instead on dealing with λ directly, we will study

$$s(E) := \text{sign}\left(-\frac{w_1^\top G(E)\,w_1 - d_{11} + E}{w_1^\top G(E)\,w_2 - d_{12}}\right)$$

for a constant E. To achieve this, it is necessary to know how the matrix B looks like.

Let $\{\mu_\alpha\}_{\alpha=1}^n$ be the eigenvalues of B arranged in a non-increasing order and $\{u_\alpha\}_{\alpha=1}^n$ be the corresponding unit eigenvectors. Observe that, up to a scaling factor $\sqrt{\frac{n+2}{n}}$, B is a Wigner matrix with a rank 1 shift:

$$B = M + \sqrt{\frac{p(n+2)}{(1-p)}}ll^\top,$$

where M is the lower right n by n minor of H, and $l \in \mathbb{R}^n$ is the vector with all its components equal to $\frac{1}{\sqrt{n+2}}$. $\sqrt{\frac{n+2}{n}}M$ is a generalized Wigner matrix satisfying both Conditions 6.7 and 6.9.

The proof will be break into 4 steps:

1. From Typical Sample of H To Typical Sample of M

Here we are facing one obstacle. We want to fix a typical sample M to compute $s(\lambda)$. In particular, we want the level repulsion event described in Condition 6.9 to hold for M. Unfortunately, if we choose $\mathscr{A}$ to be the event that every $n \times n$ principal minor of $\tilde{M}$ satisfies Condition 6.9, then we are not able to bound the probability nicely.

Indeed, we pick $\mathscr{A}$ to be what a typical H would looks like. $\mathscr{A}$ includes the event that H satisfies the level repulsion Condition with $\varepsilon_{LR} > 0$ and $C_{LR} = \rho$. In particular, $\mathbb{P}(\mathscr{A}^c) < n^{\delta_{LR}}$ for some δ_{LR} depending on ε_{LR} and ρ. However, we cannot condition on $\mathscr{A}$ directly , in this way we will lose the independence of B, W and D when we try to estimate $s(E)$. Therefore, in the first step we will define $\mathscr{A}$ and show the following:

$$\mathbb{E}\left(|\mathbf{1}_{\mathbf{H}\text{ is typical}} - \mathbf{1}_{\mathbf{M}\text{ is typical}}|\right) \text{ is small enough.}$$

2. A Typical Sample of B

In the second step, we will show that if we fix a typical M, then B behaves like a Wigner matrix. We expect B to behave like a Wigner matrix with an exceptional eigenvector almost parallel to l with eigenvalue close to $\sqrt{\frac{p(n+2)}{1-p}}$. We will quantify these properties in Definition (6.28) in section 6.4.3.

3. $\textbf{Concentration of}\ w_i^\top G(E)\, w_j - d_{ij} + E$

Thirdly, we will derive concentration of $w_i^\top G(E)\, w_j$ for $i,\, j \in \{1,\, 2\}$. By definition,

$$w_i^\top G(E)\, w_j = \sum_{\alpha \in [n]} \frac{1}{\mu_\alpha - E} \langle w_i,\, u_\alpha \rangle \langle w_j,\, u_\alpha \rangle.$$

If E is much closer to an eigenvalue μ_{α_E} than any other eigenvalues, then, we expect $w_i^\top G(E)\, w_j$ to be dominated by the term $\frac{1}{\mu_{\alpha_E} - E} \langle w_i,\, u_{\alpha_E} \rangle \langle w_j,\, u_{\alpha_E} \rangle$. We will show that by freezing a typical B, with high probability in W and D we have

$$\forall i,\, j \in \{1,\, 2\} \;\; w_i^\top G(E)\, w_j \simeq -\delta_{ij} + \frac{\langle w_i,\, u_{\alpha_E} \rangle \langle w_j,\, u_{\alpha_E} \rangle}{\mu_{\alpha_E} - E} \tag{6.24}$$

4. $\textbf{Estimate of}\ \mathbf{s}(\lambda)$

Finally, we will derive the theorem by showing that

$$\mathbb{E}\left(s(\lambda_\alpha)\, \mathbf{1}_{\text{H is typical}}\right) = n^{-1/3 + C\varepsilon_{LR}}.$$

Once all these lemmas are proved, the main theorem follows immediately.

6.4.2 A typical sample of M

Let M be an $n \times n$ principal submatrix of H. Let $\{\nu_\alpha\}_{\alpha=1}^n$ be the eigenvalues of M arranged in a non-increasing order and let $\{v_\alpha\}_{\alpha=1}^n$ be the corresponding unit eigenvectors. Let $G_M(z) := (M - z)^{-1}$ be the Green function of M and $m_M(z) := \frac{1}{n} \sum_{\alpha=1}^n \frac{1}{\nu_\alpha - z}$ be the Stieltjes transform of M.

A special role in the proof will be played by the level repulsion property, and the strength of the level repulsion has to be carefully chosen for matrices of different sizes. Let $t > 0$. We will say that an $m \times m$ symmetric matrix B satisfies the level repulsion property with parameter t if for any two distinct eigenvalues $\nu,\, \nu'$ of A in $\left[2 - n^{-2/3}\varphi_n^{3\rho},\, 2 + n^{-2/3}\varphi_n^{3\rho}\right]$, we have

$$|\nu - \nu'| > t.$$

Denote the set of such matrices by $\mathscr{L}\mathscr{R}(n, t)$. Lemma 6.21 asserts that

$$\mathbb{P}(M \in \mathscr{L}\mathscr{R}(n, n^{-2/3-\varepsilon_{LR}})) \geq 1 - n^{-\delta_{LR}}$$

for some $\delta_{LR} > 0$. We start with a lemma showing that the parameter t in the definition of level repulsion can be adjusted without significantly changing this probability.

Lemma 6.21. *Let $C > 0$. Let M be an $n \times n$ symmetric random matrix. There exists $\theta \in (1/2, 1)$ which depends on the distribution of M such that*

$$\mathbb{P}\left(M \in \mathscr{L}\mathscr{R}\left(n, \theta n^{-2/3-\varepsilon_{LR}} - 4\frac{\varphi_n^C}{n}\right)\right) - \mathbb{P}\left(M \in \mathscr{L}\mathscr{R}(n, \theta n^{-2/3-\varepsilon_{LR}})\right) \leq n^{-1/3+2\varepsilon_{LR}}.$$

Proof. For $k \geq 0$, denote

$$P_k := \mathbb{P}\left(M \in \mathscr{L}\mathscr{R}\left(n, n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}\right)\right).$$

Then $P_k \in (0, 1)$ form an increasing sequence. Hence, there exists $k \leq 4n^{1/3-2\varepsilon_{LR}}$ such that

$$P_{k+4} - P_k \leq n^{-1/3+2\varepsilon_{LR}}.$$

This implies the lemma if we choose θ so that $\theta n^{-2/3-\varepsilon_{LR}} = n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}$ and note that $\theta > 1/2$. $\square$

We will fix this value of θ throughout the rest of the proof, but the distribution of M will be specified later in Theorem 6.24.

Definition 6.22. Let us collect the properties of the $n \times n$ submatrices of H which we will use throughout the proof. Fix a sufficiently small $\varepsilon_{LR} > 0$ and set

$$\eta = n^{-2/3-2\varepsilon_{LR}}. \tag{6.25}$$

Denote by $\mathscr{A}_{(n,k)}$ the set of symmetric $n \times n$ matrices M having the following properties:
Isotropic local semicircular law:

$$\sup_{|E-2| \leq n^{-2/3+3\varepsilon_{LR}}} \sup_{x,y \in \{e_i\}_{i=1}^n \cup \{l\}} |\langle x,\, G_M(E + \mathrm{i}\eta)\, y\rangle - \langle x,\, y\rangle m_{sc}(E + \mathrm{i}\eta)| < 3n^{-\frac{1}{3}+3\varepsilon_{LR}}, \tag{6.26}$$

Rigidity of eigenvalues:

$$|\nu_\alpha - \gamma_\alpha| \leq \varphi_n^{A_{sls}} \left[\min(\alpha,\, n - \alpha + 1)\right]^{-1/3} n^{-2/3}, \tag{6.27}$$

where γ_α is defined so that $\int_{2-\gamma_\alpha}^2 \frac{2}{\pi}\sqrt{4 - x^2}dx = \frac{\alpha}{n}$.
l_∞-delocalization of eigenvectors:

$$\forall \alpha,\ \|v_\alpha\|_\infty \leq \frac{\varphi_n^C}{\sqrt{n}}, \tag{6.28}$$

Isotropic delocalization of eigenvectors:

$$\max_{\alpha\in[n]} |\langle v_\alpha,\, l\rangle|^2 < n^{\varepsilon_{LR}-1}, \tag{6.29}$$

Level repulsion at the edge: $M \in \mathscr{LR}\left(n, \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}\right)$, i.e.,
for any two distinct eigenvalues ν, ν' of M in $\left[2 - n^{-2/3}\varphi_n^{3\rho},\, 2 + n^{-2/3}\varphi_n^{3\rho}\right]$, we have

$$|\nu - \nu'| > \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}. \tag{6.30}$$

A typical Wigner matrix belongs to the set $\mathscr{A}_{(n,0)}$. However, we need this fact not for a single matrix M, but for all $n \times n$ principal submatrices of the $(n+2)\times(n+2)$ matrix H. Denote by $H^{(k)}$ the $(n+1)\times(n+1)$ principal submatrix of H with row and column k removed. Similarly, denote by $H^{(i,j)}$ the $n \times n)$ principal submatrix of H with rows and columns i, j removed. The properties (6.26) – (6.27) hold with an overwhelming probability, which allows to use a union bound while establishing them. In contrast to it, condition (6.5) holds only with probability $1 - n^{-\delta_{LR}}$ for some $\delta_{LR} > 0$, which is too weak to be combined with the union bound. To guarantee that the level repulsion holds with high probability for all principal submatrices, we show that the eigenvalues of these submatrices are located closely to the eigenvalues of the original matrix. To this end, we need the following lemma.

Lemma 6.23. *Let J be an $m\times m$ symmetric matrix satisfying conditions (6.27) and (6.28), with $n = m$. Let $k \in [m]$, and let $J^{(k)}$ be the $(m-1)\times(m-1)$ principal submatrix of J with row and column k removed. Let $\mu \in \left[2 - n^{-2/3}\varphi_n^{3\rho},\, 2 + n^{-2/3}\varphi_n^{3\rho}\right]$ be an eigenvalue of $J^{(k)}$. If J or $J^{(k)}$ satisfies (6.9), then there exists an eigenvalue λ of J such that*

$$0 \le \lambda - \mu \le \frac{\varphi_n^C}{n}.equationequation \tag{6.31}$$

Consequently, if one of the matrices J or $J^{(k)}$ satisfies (6.5), then the other one satisfies the same condition with a extra lose of $\frac{\varphi_n^C}{n}$.

Proof. Note that μ is an eigenvalue of the matrix $J - e_k e_k^\top J$ as well since the k-th row of this matrix is 0. We will start with showing that there exists an eigenvalue λ of J satisfying

(6.31). Let G_J be the Green function of J. By Sylvester's determinant identity, we have

$$\begin{aligned}
0 &= \det\left(J - \mu - e_k e_k^\top J\right) \\
&= \det\left(J - \mu\right) \det\left(I_n - e_k e_k^\top J G_J\left(\mu\right)\right) \\
&= \det\left(J - \mu\right)\left(1 - e_k^\top J G_J\left(\mu\right) e_k\right).
\end{aligned}$$

If $\det\left(J - \mu\right) = 0$, then we are done. Otherwise, $1 - e_k^\top J G_J\left(\lambda\right) e_k = 0$, which can be rewritten as

$$\sum_\alpha \frac{\lambda_\alpha}{\lambda_\alpha - \mu} \langle e_k,\, u_\alpha \rangle^2 = 1,$$

where $\lambda_1 \geq \cdots \geq \lambda_m$ are the eigenvalues of J, and $u_1, \ldots, u_m$ are the corresponding unit eigenvectors.

By (6.27), for $\lambda_\alpha < 0$ we have $0 < \frac{\lambda_\alpha}{\lambda_\alpha - \mu} < \frac{2}{3}$. Then,

$$\sum_{\alpha, \lambda_\alpha < 0} \frac{\lambda_\alpha}{\lambda_\alpha - \mu} \langle e_k,\, u_\alpha \rangle^2 \leq \sum_{\alpha, \lambda_\alpha < 0} \frac{2}{3} \langle e_k,\, u_\alpha \rangle^2 \leq \frac{2}{3}.$$

Hence,

$$\sum_{\alpha, \lambda_\alpha > \mu} \frac{\lambda_\alpha}{\lambda_\alpha - \mu} \langle e_k,\, u_\alpha \rangle^2 \geq \frac{1}{3}.$$

Let β be the largest positive integer so that $\lambda_\beta > \mu$. Assume that $\beta > 1$, and let $\alpha < \beta$. If J satisfies (6.5), then

$$\lambda_\alpha - \mu \geq \lambda_{\beta - 1} - \lambda_\beta \geq n^{-2/3 - \varepsilon_{LR}}.$$

On the other hand, if $J^{(k)}$ satisfies (6.5), and μ' is the smallest eigenvalue of $J^{(k)}$ which is greater than μ. Due to the Cauchy interlacing theorem, we know that

$$\mu < \lambda_\beta < \mu' < \lambda_\alpha.$$

Then,

$$\lambda_\alpha - \mu \geq \mu' - \mu \geq n^{-2/3 - \varepsilon_{LR}}.$$

In both cases, (6.28) and (6.27) applied with $\alpha = 1$ imply

$$\sum_{\alpha < \beta} \frac{\lambda_\alpha}{\lambda_\alpha - \mu} \langle e_k,\, u_\alpha \rangle^2 \leq \beta \frac{\lambda_1}{n^{-2/3 - \varepsilon_{LR}}} \max_\alpha \|u_\alpha\|_\infty = O\left(n^{-1/3 + C\varepsilon}\right).$$

If $\beta = 1$, the inequality above is vacuous. Thus, in both cases,

$$\frac{\lambda_\beta}{\lambda_\beta - \mu} \langle e_k,\, u_\alpha \rangle^2 \geq \frac{1}{3} + O\left(n^{-1/3 + C\varepsilon}\right)$$

which in combination with (6.27), (6.28) leads to

$$\frac{\varphi_n^C}{n} \geq \lambda_\beta - \mu > 0$$

establishing (6.31). Since (6.31) holds for all $\mu \in \left[2 - n^{-2/3}\varphi_n^{3\rho},\, 2 + n^{-2/3}\varphi_n^{3\rho}\right]$, the second part of the lemma follows from (6.5) for one of the matrices J or $J^{(k)}$ and interlacing of their eigenvalues. □

Equipped with Lemma 6.23, we derive the desired result about the typical behavior of the principal submatrices. We remind the reader that for convenience, we consider graphs with $n+2$ vertices.

Theorem 6.24. *Let A_p be the adjacency matrix of a $G(n+2, p)$ graph, and let*

$$H = \frac{1}{\sqrt{p\,(1-p)\,(n+2)}} A_p - \sqrt{\frac{p(n+2)}{1-p}} \overrightarrow{1}\,\overrightarrow{1}^\top,$$

where $\overrightarrow{1} \in S^{n+1}$ is the vector such that every component equals $\frac{1}{\sqrt{n}}$. Let $\mathscr{A}$ be the set of $(n+2) \times (n+2)$ symmetric matrices H such that the matrix itself belongs to $\mathscr{A}_{(n+2,2)}$, all its principal $(n+1) \times (n+1)$ submatrices belong to $\mathscr{A}_{(n+1,3)}$, and all its principal $n \times n$ submatrices belong to $\mathscr{A}_{(n,4)}$.

Then

$$\mathbb{P}(H \in \mathscr{A}) \geq 1 - n^{-\delta}$$

for some $\delta = \delta(p,\, \rho,\, \varepsilon_{LR}) > 0$. Moreover, for any $i, j \in [n]$,

$$\mathbb{E}\left|\mathbf{1}_{\mathscr{A}_{(n,0)}}(H^{(i,j)}) - \mathbf{1}_{\mathscr{A}}(H)\right| \leq n^{-1/3 + 2\varepsilon_{LR}}.$$

Proof. Combining Theorem 6.8, Corollary 6.12, and Theorem 6.13 with the union bound shows that conditions (6.26) – (6.29) hold for the matrix H itself, as well as for all its $(n+1) \times (n+1)$ and $n \times n$ principal submatrices with probability at least $1 - n^{-1}$. In addition to it, (6.5) holds for H with $k = 2$ with probability at least $1 - n^{-\delta}$ (See 6.19 and Condition 6.9). Then Lemma 6.23, together with the properties (6.26) – (6.29) allow us to extend (6.5) with $k = 3$ to all its $(n+1) \times (n+1)$ principal minors. As these minors possess the same properties, (6.5) further extends with $k = 4$ to all $n \times n$ principal minors.

Let us prove the second inequality. Denote by $\mathscr{B}$ the set of all $(n+2)\times(n+2)$ symmetric matrices satisfying conditions (6.26) – (6.29). Then

$$\begin{aligned}
&\mathbb{P}\left(H^{(i,j)}\in\mathscr{A}_{(n,0)}\text{ and }H\notin\mathscr{A}\right)\\
\leq&\mathbb{P}\left(H^{(i,j)}\in\mathscr{A}_{(n,0)}\text{ and }H\notin\mathscr{A}\text{ and }H\in\mathscr{B}\right)+\mathbb{P}(H\notin\mathscr{B})\\
\leq&n^{-1}
\end{aligned}$$

since by Lemma 6.23, $\mathscr{A}_{(n,0)}\cap\mathscr{A}^c\cap\mathscr{B}\subset\mathscr{A}_{(n,0)}\cap\mathscr{A}^c_{(n+2,2)}\cap\mathscr{B}=\varnothing$. Also, notice that all the minors $H^{(i,j)}$ have the same distribution. Now we specify that the θ from Lemma 6.21 is chosen according to the distribution of $H^{(i,j)}$. Hence,

$$\mathbb{P}\left(H^{(i,j)}\notin\mathscr{A}_{(n,0)}\text{ and }H\in\mathscr{A}\right)\leq\mathbb{P}\left(H^{(i,j)}\notin\mathscr{A}_{(n,0)}\text{ and }H^{(i,j)}\in\mathscr{A}_{(n,4)}\right)\leq n^{-1/3+2\varepsilon_{LR}}$$

by Lemma 6.21. The result follows. $\square$

6.4.3 Introduction of the shift

In this section, we will derive the typical properties of all $n\times n$ principal submatrices of a normalized adjacency matrix of a $G(n+2,p)$ graph. Recall that we denoted such submatrix by B, and

$$B=M+\sqrt{\frac{p\,(n+2)}{(1-p)}}ll^\top \tag{6.32}$$

where M is an $n\times n$ principal submatrix of H, and $l=\left(\frac{1}{\sqrt{n+2}},\dots,\frac{1}{\sqrt{n+2}}\right)$ is almost a unit vector. We expect B to behave close to M in a sense that its non-leading eigenvalues and eigenvectors possess similar properties. The argument at this stage is deterministic. We fix the matrix $M\in\mathscr{A}_{(n)}$ and treat B as its rank one perturbation.

We start with showing that the non-leading edge eigenvalues of B are very close to that of M.

Lemma 6.25. *Let $M\in\mathscr{A}_{(n,0)}$ be an $n\times n$ symmetric matrix, and let B be as in (6.32). Let $\mu_1\geq\cdots\geq\mu_n$ be the eigenvalues of B. If β is such that $|\mu_{\beta+1}-2|\leq n^{-2/3}\varphi_n^{2\rho}$, then*

$$|\nu_\beta-\mu_{\beta+1}|\leq n^{-1+C\varepsilon_{LR}} \tag{6.33}$$

for some universal constant $C>0$. Futhermore, $\mu_{\beta+1}$ is an eigenvalue of M if and only if

$\langle l,\, v_\beta \rangle = 0$. *In the case $\mu_{\beta+1}$ is not an eigenvalue of M, we have*

$$\frac{\langle l,\, v_{\beta+1}\rangle^2}{\nu_\beta - \mu_{\beta+1}} \geq 1 - o\left(1\right) \tag{6.34}$$

Proof. Suppose that μ is an eigenvalue of B. By Sylvester's determinant identity we have

$$\begin{aligned} 0 &= \det\left(M - \mu I_n + \sqrt{\frac{p\left(n+2\right)}{1-p}}ll^\top\right) \\ &= \det\left(M - \mu I_n\right)\det\left(I_n + G_M\left(\mu\right)\sqrt{\frac{p\left(n+2\right)}{1-p}}ll^\top\right) \\ &= \det\left(M - \mu I_n\right)\left(1 + l^\top G_M\left(\mu\right)\sqrt{\frac{p\left(n+2\right)}{1-p}}l\right). \end{aligned}$$

$\left(1 + l^\top G_M\left(\mu\right)\sqrt{\frac{p(n+2)}{1-p}}l\right)$ equals to 0 if

$$\sum_{\alpha\in[n]} \frac{\langle l,\, v_\alpha\rangle^2}{v_\alpha - \mu} = -\frac{1}{\sqrt{\frac{p(n+2)}{1-p}}}. \tag{6.35}$$

Now we assume that $\langle l,\, v_\alpha\rangle \neq 0$ for $\alpha \in [n]$. Observe that the function $x \mapsto \sum_{\alpha\in[n]} \frac{\langle l,\, v_\alpha\rangle^2}{\nu_\alpha - x}$ have vertical asymptotes when $x = \nu_\alpha$ for $\alpha \in [n]$. Away from the asymptotes, it is a monotone increasing function. For $x \in (\nu_1, \infty)$, it increases from $-\infty$ at ν_1 to 0 when x approaches $+\infty$. For $x < \nu_n$, the function is positive. Hence, (6.35) hold at exactly n points. We conclude that the eigenvalues of M and B are interlacing:

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \nu_n. \tag{6.36}$$

If $\langle l,\, v_\alpha\rangle = 0$ for some $\alpha \in [n]$, then, ν_α is an eigenvalue of B with the same eigenvector v_α and the function we described above lose the corresponding vertical intercept at ν_α. One can easily verify that the eigenvalues are still interlacing.

For the leading eigenvalue, $\mu_1 \geq \frac{1}{2}\sqrt{\frac{p(n+2)}{1-p}}$ due to the fact that $|M| = O(1)$ by (6.27).

Suppose we have

$$\nu_\alpha - n^{-1+C\varepsilon_{LR}} \leq \mu_{\alpha+1} \leq \nu_\alpha$$

for $\alpha = 1, \ldots, \beta - 1$ and $\nu_\beta > 2 - n^{-2/3}\varphi_n^{2\rho}$. Now we check the case for ν_β. Assume that $\langle l,\, v_\beta\rangle \neq 0$.

Suppose that

$$\sum_{\alpha \neq \beta} \frac{\langle l,\, v_\alpha \rangle^2}{v_\alpha - E} \leq -1 + o\left(1\right) \tag{6.37}$$

for all $E \in (\nu_{\beta+1},\, \nu_\beta)$. Then, by (6.35),

$$\begin{aligned} \frac{\langle l,\, v_{\beta+1} \rangle^2}{\nu_\beta - \mu_{\beta+1}} &= -\frac{1}{\sqrt{\frac{p(n+2)}{1-p}}} - \sum_{\alpha \neq \beta} \frac{\langle l,\, v_\alpha \rangle^2}{v_\alpha - \mu_{\beta+1}} \\ &\geq 1 - o\left(1\right) \end{aligned}$$

By (6.29), we have $\langle l,\, v_{\beta+1} \rangle^2 < n^{\varepsilon_{LR}-1}$, which allows to conclude that

$$0 < \nu_\beta - \mu_{\beta+1} \leq n^{2\varepsilon_{LR}-1}$$

as required. In the case $\langle l,\, v_\beta \rangle = 0$, we have $\mu_{\beta+1} = \nu_b$. Hence, by an induction, the proof of our statement is complete. In particular, we also conclude that $\mu_{\beta+1} \in \{\nu_\alpha\}_{i=1}^n$ if and only if $\langle l,\, v_\beta \rangle = 0$.

It remains to verify (6.37). This will be done by comparing the right hand side of (6.37) with

$$\mathrm{Re}\,\langle l,\, G_M\left(E + \mathrm{i}\eta\right) l \rangle = \sum_{\alpha \in [n]} \frac{\nu_\alpha - E}{\left(\nu_\alpha - E\right)^2 + \eta^2} \langle l,\, v_\alpha \rangle^2.$$

Assume first that $\frac{1}{2}\nu_\beta + \frac{1}{2}\nu_{\beta+1} \leq E \leq \nu_\beta$. In view of (**??**),

$$\nu_{\beta+1} + \frac{1}{2} n^{-2/3-\varepsilon_{LR}} < \frac{\nu_{\beta+1} + \nu_\beta}{2} < E < \nu_\beta < \nu_{\beta-1} - n^{-2/3-\varepsilon_{LR}}.$$

(we omit the last inequality if $\beta = 1$.) Hence, for $\alpha \neq \beta$, we have

$$\left|E - \nu_\alpha\right| > \frac{1}{2} n^{-2/3-\varepsilon_{LR}} = \frac{1}{2} \eta n^{\varepsilon_{LR}} \tag{6.38}$$

(recall that $\eta = n^{-2/3-2\varepsilon_{LR}}$) and so

$$\frac{1}{\nu_\alpha - E} = \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right) \frac{\nu_\alpha - E}{\left(\nu_\alpha - E\right)^2 + \eta^2}.$$

Therefore,

$$\begin{aligned}
&\sum_{\alpha>\beta} \frac{1}{\nu_\alpha - E} \langle l,\, v_\alpha \rangle^2 \\
&= \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right) \sum_{\alpha>\beta} \frac{\nu_\alpha - E}{\left(\nu_\alpha - E\right)^2 + \eta^2} \langle l,\, v_\alpha \rangle^2 \\
&= \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right) \left(\operatorname{Re} \langle l,\, G_M\left(E + \mathrm{i}\eta\right) l \rangle - \sum_{\alpha\le\beta} \frac{\nu_\alpha - E}{\left(\nu_\alpha - E\right)^2 + \eta^2} \langle l,\, v_\alpha \rangle^2 \right).
\end{aligned}$$

since all the summands have the same sign. Now we will evaluate the two terms in the brackets. The first one can be approximated using the local semicircular law, and the second one is negligible, because the sum consists of a few terms, and each term is small. Indeed, by rigidity of eigenvalues (6.27), we have $|2 - \gamma_\beta| < n^{-2/3}\varphi_n^{4\rho}$ and thus $\beta < \varphi_n^{C\rho}$ for some constant $C > 0$. With the trivial bound $|\nu_\alpha - E| < 2n^{-2/3}\varphi_n^{3\rho}$, we get

$$\left| \sum_{\alpha\le\beta} \frac{\nu_\alpha - E}{\left(\nu_\alpha - E\right)^2 + \eta^2} \langle l,\, v_\alpha \rangle^2 \right| \le \beta \frac{n^{-2/3}\varphi_n^{3\rho}}{\eta^2} n^{-1+\varepsilon_{LR}} \le n^{-1/3+6\varepsilon_{LR}}$$

if n is sufficiently large. The isotropic local semicircular law (6.26) yields

$$\operatorname{Re} \langle l,\, G_M\left(E + \mathrm{i}\eta\right) l \rangle = \langle l,\, l \rangle m_{sc}\left(E + \mathrm{i}\eta\right) + O\left(n^{-1/3+3\varepsilon_{LR}}\right).$$

Using the fact that $m_{sc}\left(z\right) = \frac{-z+\sqrt{z^2-4}}{2}$ with the branch cut at $[-2,\, 2]$, we have $m_{sc}\left(E + \mathrm{i}\eta\right) = -1 + O\left(n^{-1/3}\varphi_n^{3\rho}\right)$. Thus,

$$\operatorname{Re} \langle l,\, G_M\left(E + \mathrm{i}\eta\right) l \rangle = -1 + O\left(n^{-1/3+3\varepsilon_{LR}}\right)$$

and we conclude that

$$\sum_{\alpha>\beta} \frac{1}{\nu_\alpha - E} \langle l,\, v_\alpha \rangle^2 \le -1 + o\left(1\right) \tag{6.39}$$

for all $E \in \left(\frac{1}{2}\nu_\beta + \frac{1}{2}\nu_{\beta+1},\, \nu_\beta\right)$. Since $E \mapsto \sum_{\alpha>\beta} \frac{1}{\nu_\alpha - E} \langle l,\, v_\alpha \rangle^2$ is increasing for $E > \nu_{\beta+1}$, the inequality above extends to all $E \in \left(\nu_{\beta+1},\, \nu_\beta\right)$. Together with

$$\sum_{\alpha<\beta} \frac{1}{\nu_\alpha - E} \langle l,\, v_\alpha \rangle^2 \le \beta \frac{1}{n^{-2/3-\varepsilon_{LR}}} n^{-1+\varepsilon_{LR}} = o\left(1\right)$$

for $E \in (\nu_{\beta+1},\, \nu_\beta)$, we conclude that all $E \in (\nu_{\beta+1},\, \nu_\beta)$ satisfy

$$\sum_{\alpha\neq\beta} \frac{1}{\nu_\alpha - E} \langle l,\, v_\alpha\rangle^2 \leq -1 + o\left(1\right),$$

completing the proof of the lemma. $\square$

Our next aim is comparing the Stieltjes transform of B to that of the semicircular law. This will be done via the comparison of the former to the Stieltjes transform of M.

Lemma 6.26. *Let $M \in \mathscr{A}_{(n)}$ be an $n \times n$ symmetric matrix, and let B be as in (6.32). Then*

$$\sup_{E\colon |E-2|\leq \varphi_n^{2\rho}} \left| m_B\left(E + \mathrm{i}\eta\right) - m_{sc}\left(E + \mathrm{i}\eta\right)\right| \leq n^{-1/3 + C\varepsilon_{LR}},$$

where

$$m_B\left(z\right) := \frac{1}{n} \sum_{\alpha=1}^{n} \frac{1}{u_\alpha - z}$$

is the Stieltjes transform of B and $\eta = n^{-2/3-2\varepsilon_{LR}}$.

Proof. Fix E such that $|E-2| \leq \varphi_n^{2A_{sls}}$. We estimate the real part and imaginary of the Stieltjes transform part separately. Let us start with the real part.

$$\operatorname{Re} m_B\left(E + \mathrm{i}\eta\right) = \frac{1}{n} \sum_\alpha \frac{\mu_\alpha - E}{\left(\mu_\alpha - E\right)^2 + \eta^2}.$$

The function $x \to \frac{x}{x^2+\eta^2}$ is decreasing when $|x| > \eta$. Based on this fact, let β be the smallest integer such that $\nu_\beta < E - \eta$. Recall that we have the interlacing property:

$$E - \eta > \nu_\beta \geq \mu_{\beta+1} \geq \nu_{\beta+1} \geq \mu_{\beta+2} \cdots \geq \mu_n \geq \nu_n.$$

Then, we have

$$\sum_{\alpha=\beta}^{n-1} \frac{\nu_\alpha - E}{\left(\nu_\alpha - E\right)^2 + \eta^2} \leq \sum_{\alpha=\beta+1}^{n} \frac{\mu_\alpha - E}{\left(\mu_\alpha - E\right)^2 + \eta^2} \leq \sum_{\alpha=\beta+1}^{n} \frac{\nu_\alpha - E}{\left(\nu_\alpha - E\right)^2 + \eta^2}.$$

Furthermore, as $\frac{x}{x^2+\eta^2}$ lies in $\left[-\frac{1}{2\eta}, \frac{1}{2\eta}\right]$ for all $x \in \mathbb{R}$, we have

$$\operatorname{Re} m_M\left(E + \mathrm{i}\eta\right) - \frac{\beta}{n\eta} \leq \operatorname{Re} m_B\left(E + \mathrm{i}\eta\right) \leq \operatorname{Re} m_M\left(E + \mathrm{i}\eta\right) + \frac{\beta}{n\eta},$$

and the bound for the real part follows.

For the imaginary part we have

$$\operatorname{Im} m_B(E+\mathrm{i}\eta) = \frac{1}{n}\sum_{\alpha} \frac{\eta}{(\lambda_B - E)^2 + \eta^2}.$$

The function $x \to \frac{\eta}{x^2+\eta^2}$ is increasing if $x < 0$. Let β be the smallest constant such that $\lambda_\beta < E$. We have

$$\sum_{\alpha=\beta+1}^{n-1} \frac{\eta}{(\nu_\alpha - E)^2 + \eta^2} \le \sum_{\alpha=\beta+1}^{n} \frac{\eta}{(\mu_\alpha - E)^2 + \eta^2} \le \sum_{\alpha=\beta}^{n} \frac{\eta}{(\nu_\alpha - E)^2 + \eta^2}.$$

Since $\frac{\eta}{x^2+\eta^2} \in \left[0,\, \frac{1}{\eta}\right]$ for all x, we conclude that

$$\operatorname{Im} m_M(E+\mathrm{i}\eta) - \frac{2\beta}{n\eta} \le \operatorname{Im} m_B(E+\mathrm{i}\eta) \ge \operatorname{Im} m_M(E+\mathrm{i}\eta) + \frac{2\beta}{n\eta}.$$

Due to (6.27) and $\int_{2-\gamma_\alpha}^{2} \frac{1}{2\pi}\sqrt{4-x^2}\,\mathrm{d}x = \frac{\alpha}{n}$ we have $\beta \le \varphi_n^{C\rho}$. We conclude that

$$|m_M(E+\mathrm{i}\eta) - m_B(E+\mathrm{i}\eta)| \le \varphi_n^{C\rho} n^{-1/3+2\varepsilon_{LR}}.$$

In view of (6.26),

$$|m_M(E+\mathrm{i}\eta) - m_{sc}(E+\mathrm{i}\eta)| = \left|\frac{1}{n}\sum_i \langle e_i,\, G(E+\mathrm{i}\eta)\, e_i\rangle - m_{sc}(E+\mathrm{i}\eta)\right| \le 3n^{-\frac{1}{3}+3\varepsilon_{LR}}$$

which in combination with the previous inequality finishes the proof. □

Next, we will derive the delocalization properties of edge eigenvectors of B.

Lemma 6.27. *Let $M \in \mathscr{A}_{(n,0)}$ be an $n \times n$ symmetric matrix, and let B be as in (6.32). Let $\mu_1 \ge \cdots \ge \mu_n$ be the eigenvalues of B, and let $u_1, \ldots, u_n$ be the corresponding unit eigenvectors. If β is such that $|\mu_{\beta+1} - 2| \le n^{-2/3}\varphi_n^{2\rho}$, then*

$$|\langle u_\beta,\, l\rangle| \le n^{-1+2\varepsilon_{LR}}. \tag{6.40}$$

and

$$\|u_\beta\|_\infty \le \frac{n^{1/6+6\varepsilon_{LR}}}{\sqrt{n}}. \tag{6.41}$$

Proof. As pointed out in Lemma 6.25, $\mu_{\beta+1}$ is an eigenvalue of M if and only if $\langle l,\, v_\beta\rangle = 0$.

In this case, we have $v_\beta = u_{\beta+1}$ so we are done.

Now we assume $\mu_{\beta+1}$ is not an eigenvalue of M. In particular, it satisfies (6.35). Using this equality, one can directly check that

$$u = \sum_{\alpha\in[n]} \frac{\langle l,\, v_\alpha\rangle}{\nu_\alpha - \mu_{\beta+1}} v_\alpha$$

is an eigenvector corresponding to eigenvalue $\mu_{\beta+1}$.

First, we provide a lower bound for $\|u\|_2$. By Lemma 6.25, we have $\frac{\langle l,\, v_{\beta+1}\rangle^2}{|\nu_\beta - \mu_{\beta+1}|} \ge \frac{1}{2}$. Using $\langle l,\, v_{\beta+1}\rangle^2 \le n^{-1+\varepsilon_{LR}}$ from (6.29), we bound the norm by one of the coefficients:

$$\|u\|_2^2 \ge \frac{\langle l,\, v_{\beta+1}\rangle^2}{|\nu_\beta - \mu_{\beta+1}|^2} = \frac{1}{\langle l,\, v_{\beta+1}\rangle^2}\left(\frac{\langle l,\, v_{\beta+1}\rangle^2}{|\nu_\beta - \mu_{\beta+1}|}\right)^2 \ge \frac{1}{4}n^{1-\varepsilon_{LR}}. \tag{6.42}$$

Recall from (6.35) that $\sum_{\alpha\in[n]} \frac{\langle l,\, v_\alpha\rangle^2}{\nu_\alpha - \mu_{\beta+1}} = -\frac{1}{\sqrt{\frac{p(n+2)}{1-p}}}$ and the left hand side is exactly $\langle u,\, l\rangle$. This yields

$$|\langle u_\beta,\, l\rangle| = \frac{\langle u,\, l\rangle}{\|u\|_2} \le n^{-1+2\varepsilon_{LR}}$$

if n is sufficiently large.

Now we will estimate $\|u\|_\infty = \max_{i\in[n]} \left|\sum_{\alpha\in[n]} \frac{\langle l,\, v_\alpha\rangle\langle e_i,\, v_\alpha\rangle}{\nu_\alpha - \mu_{\beta+1}}\right|$. We break the sum to isolating the main term:

$$\begin{aligned}
|\langle u,\, e_i\rangle| &\le \left|\frac{\langle l,\, v_\beta\rangle}{\nu_\beta - \mu_{\beta+1}}\right| \|v_\beta\|_\infty + \left|\sum_{\alpha\neq\beta} \frac{\langle l,\, v_\alpha\rangle\langle e_i,\, v_\alpha\rangle}{\nu_\alpha - \mu_{\beta+1}}\right| \\
&\le \left|\frac{\langle l,\, v_\beta\rangle}{\nu_\beta - \mu_{\beta+1}}\right| \|v_\beta\|_\infty + \sqrt{\sum_{\alpha\neq\beta} \frac{\langle l,\, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2}} \sqrt{\sum_{\alpha\neq\beta} \langle e_i,\, v_\alpha\rangle^2} \\
&\le \left|\frac{\langle l,\, v_\beta\rangle}{\nu_\beta - \mu_{\beta+1}}\right| \|v_\beta\|_\infty + \sqrt{\sum_{\alpha\neq\beta} \frac{\langle l,\, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2}}.
\end{aligned}$$

We will show below that

$$\sqrt{\sum_{\alpha\neq\beta} \frac{\langle l,\, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2}} \le n^{1/6+4\varepsilon_{LR}}. \tag{6.43}$$

If this inequality holds, (6.42) implies

$$\|u_{\beta+1}\|_\infty = \frac{\|u\|_\infty}{\|u\|_2} \le \frac{\left|\frac{\langle l, v_\beta\rangle}{\nu_\beta-\mu_{\beta+1}}\right| \|v_\beta\|_\infty}{\|u\|_2} + \frac{n^{1/6+4\varepsilon_{LR}}}{\|u\|_2}$$
$$\le \frac{\left|\frac{\langle l, v_\beta\rangle}{\nu_\beta-\mu_{\beta+1}}\right| \|v_\beta\|_\infty}{\left|\frac{\langle l, v_\beta\rangle}{\nu_\beta-\mu_{\beta+1}}\right|} + 4n^{1/6-1/2+5\varepsilon_{LR}} \le n^{-1/3+6\varepsilon_{LR}},$$

where we used $\|v_\beta\|_\infty \le \frac{\varphi_n^C}{\sqrt{n}}$ from (6.28) in the last inequality. This completes the proof of the lemma modulus (6.43).

Now we will focus on establishing (6.43) by comparing $\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle^2}{(\nu_\alpha-E)^2}$ with

$$\frac{1}{\eta}\mathrm{Im}\,\langle l,\, G_M\,(E+\mathrm{i}\eta)\,l\rangle = \frac{1}{\eta}\mathrm{Im}\sum_{\alpha\in[n]}\frac{\langle l,\, v_\alpha\rangle^2}{\nu_\alpha - E - \mathrm{i}\eta} = \sum_{\alpha\in[n]}\frac{\langle l,\, v_\alpha\rangle^2}{(\nu_\alpha - E)^2+\eta^2}$$

for any $E \in \left(\frac{\nu_\beta+\nu_{\beta+1}}{2},\, \nu_\beta\right)$ which includes $\mu_{\beta+1}$. The approach is basically the same as in approximation of $\sum_{\alpha\neq\beta}\frac{\langle l, v_\alpha\rangle^2}{v_\alpha - E}$ by $\mathrm{Re}\,\langle l,\, G\,(E+\mathrm{i}\eta)\,l\rangle$ in Lemma 6.25. As in this lemma, we use $|\nu_\alpha - E| > \frac{1}{2}\eta n^{\varepsilon_{LR}}$ for $\alpha\neq\beta$ to derive

$$\frac{\eta}{(\nu_\alpha - E)^2} = \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\frac{\eta}{(\nu_\alpha - E)^2+\eta^2}.$$

Thus,

$$\sum_{\alpha\neq\beta}\frac{\langle l,\, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2} = \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\left[\frac{1}{\eta}\mathrm{Im}\,\langle l,\, G_M\,(E+\mathrm{i}\eta)\,l\rangle - \frac{\langle l,\, v_\beta\rangle^2}{(\nu_\beta - \mu_{\beta+1})^2+\eta^2}\right]$$
$$\le \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\frac{1}{\eta}\mathrm{Im}\,\langle l,\, G_M\,(E+\mathrm{i}\eta)\,l\rangle.$$

By (6.26) we have

$$\mathrm{Im}\,\langle l,\, G\,(E+\mathrm{i}\eta)\,l\rangle = \mathrm{Im}\,m_{sc}\,(E+\mathrm{i}\eta) + O\left(n^{-1/3+3\varepsilon_{LR}}\right).$$

As $|E-2| < n^{-2/3}\varphi_n^{3\rho}$ and $\eta = n^{-2/3-2\varepsilon_{LR}}$, a direct estimate yields $\mathrm{Im}\,m_{sc}\,(E+\mathrm{i}\eta) = O\left(n^{-1/3}\varphi_n^{3\rho}\right)$. Therefore,

$$\sum_{\alpha\neq\beta}\frac{\langle l,\, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2} \le n^{1/3+4\varepsilon_{LR}}$$

proving (6.43) and finishing the proof of the lemma. $\square$

We have shown that if $M \in \mathscr{A}_{(n,0)}$, then the matrix B shares the spectral properties of M. Let us summarize these properties.

Definition 6.28. Denote by $\mathscr{T}_{(n,k)}$ the set of $n \times n$ symmetric matrix B with eigenvalues $\mu_1 \geq \cdots \geq \mu_n$ and unit eigenvectors $u_1, \ldots, u_n$ possessing the following properties.

Eigenvalue properties:

Definition 6.29. Isotropic local semicircular law:

$$\sup_{E:\,|E-2|\leq\varphi_n^{2\rho}} |m_B\left(E+\mathrm{i}\eta\right) - m_{sc}\left(E+\mathrm{i}\eta\right)| \leq n^{-1/3+C\varepsilon_{LR}}, \tag{6.44}$$

where $m_B\left(z\right) := \frac{1}{n}\sum_{\alpha=1}^{n}\frac{1}{u_\alpha - z}$ is the Stieltjes transform of B and $\eta = n^{-2/3-2\varepsilon_{LR}}$.

Rigidity of the eigenvalues:

$$\forall\alpha = 1, \ldots, n-1\, |\mu_{\alpha+1} - \gamma_\alpha| \leq 2\varphi_n^{A_{sls}}\left[\min\left(\alpha,\, n-\alpha+1\right)\right]^{-1/3} n^{-2/3}. \tag{6.45}$$

Leading eigenvalue:

$$\mu_1 \geq \frac{1}{2}\sqrt{\frac{p}{1-p}n}. \tag{6.46}$$

Edge eigenvector properties:

Definition 6.30. Isotropic delocalization:
for β such that $|\mu_\beta - 2| \leq n^{-2/3}\varphi_n^{2\rho}$, we have

$$\langle u_\beta,\, l\rangle = O\left(n^{-1+c\varepsilon_{LR}}\right). \tag{6.47}$$

ℓ_∞ delocalization:
for β such that $|\mu_\beta - 2| \leq n^{-2/3}\varphi_n^{2\rho}$,

$$\|u_\beta\|_\infty \leq \frac{n^{-1/6+4\varepsilon_{LR}}}{\sqrt{n}}. \tag{6.48}$$

Level repulsion at the edge: $B \in \mathscr{LR}\left(n, \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}\right)$, i.e.,
for any two distinct eigenvalues ν, ν' of B in $\left[2-n^{-2/3}\varphi_n^{3\rho},\, 2+n^{-2/3}\varphi_n^{3\rho}\right]$, we have

$$|\nu - \nu'| > \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}. \tag{6.49}$$

The matrices $B \in \mathscr{T}_{(n,1)}$ will be called typical below. In particular, we've shown that $M \in \mathscr{A}_{(n,0)}$ implies $B \in \mathscr{T}_{(n,1)}$.

Theorem 6.24 implies that probability close to 1, the normalized adjacency matrix of a $G(n,p)$ graph is typical along with its principal submatrices. We will formulate it as a corollary.

Corollary 6.31. *Let A_p be the adjacency matrix of a $G(n+2,p)$ graph, and let*

$$\tilde{A}_p = \frac{1}{\sqrt{p\,(1-p)\,(n+2)}} A_p.$$

In particular, we have

$$\tilde{A}_p = H + \sqrt{\frac{p(n+2)}{1-p}} \overrightarrow{1}\,\overrightarrow{1}^\top.$$

We can view $\mathscr{A}$ as an event of A_p. For convenience, we denote it as $\mathscr{T}$. Precisely, $\mathbf{1}_{\mathscr{T}}(A_p) = \mathbf{1}_{\mathscr{A}}(H)$. Then

$$\mathbb{P}(A_p \in \mathscr{T}) \geq 1 - n^{-\delta}$$

for some $\delta = \delta(p, \rho, \varepsilon_{LR}) > 0$. Moreover, for any $i, j \in [n]$,

$$\mathbb{E}\left|\mathbf{1}_{\mathscr{T}_{(n,1)}}(\tilde{A}_p^{(i,j)}) - \mathbf{1}_{\mathscr{T}}(A_p)\right| \leq n^{-1/3+2\varepsilon_{LR}}.$$

Proof. Except for (6.45) and (6.46), these are conditions have been derived from the corresponding conditions on H above. Condition (6.45) follows from the interlacing of the eigenvalues of $\tilde{A}_p$ and its principal submatrices. Finally, (6.46), follows from (6.27) for $\alpha = 1$ since

$$\mu_1 \geq \langle l,\, Bl \rangle \geq \sqrt{\frac{p\,(n+2)}{1-p}}\, \|l\|_2^4 - \lambda_1(M)\, \|l\|_2^2 \geq \frac{1}{2}\sqrt{\frac{p}{1-p}} n.$$

Both probability estimates follow now from Theorem 6.24. □

6.4.4 Concentration of $w_i^\top G\,(E)\, w_j - d_{ij} + E$

In this section, we fix an $n \times n$ matrix $B \in \mathscr{T}_{(n,1)}$. Let E be a constant such that $|E - 2| \leq n^{-2/3}\varphi_n^{2\rho}$. Let $\{\mu_\alpha\}_{\alpha=1}^n$ be eigenvalues of B arranged in the non-increasing order and let $\{u_\alpha\}_{\alpha=1}^n$ be the corresponding unit eigenvectors. Let $G\,(E) = \sum_\alpha \frac{1}{\mu_\alpha - E} u_\alpha u_\alpha^\top$ be the Green function of B.

Denote by α_E the integer such that

$$|\mu_{\alpha_E} - E| = \min_\alpha |\mu_\alpha - E| .$$

In this section we will prove the following lemma:

Lemma 6.32. *Fix a sample of* $B \in \mathscr{T}_{(n,1)}$. *With probability greater than* $1-\exp\left(-c\left(p\right)\varphi_n\right)$ *(*$\varphi_n := \left(\log n\right)^{\log\log n}$*) in* w_1 *and* w_2*, we have*

$$\forall i,\, j \in \{1,\, 2\} \;\; w_i^\top G\left(E\right) w_j = -\left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\delta_{ij} + \frac{\langle w_i,\, u_{\alpha_E}\rangle\langle w_j,\, u_{\alpha_E}\rangle}{\mu_{\alpha_E} - E} + O\left(n^{-1/3+C\varepsilon}\right) \tag{6.50}$$

for all $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$ *and* $\alpha_E \in [n]$ *is the integer so that* $|\mu_{\alpha_E} - E| \leq \min_{\alpha\in[n]} |\mu_\alpha - E|$.

By level repulsion (6.49), we have

$$|\mu_\alpha - E| > \frac{1}{8} n^{-2/3-\varepsilon_{LR}} \tag{6.51}$$

for $\alpha \neq \alpha_E$. Now, we decompose G to separate the main term:

$$\begin{aligned} G\left(E\right) &= \sum_\alpha \frac{1}{\mu_\alpha - E} u_\alpha u_\alpha^\top = \sum_{\alpha\neq\alpha_E} \frac{1}{\mu_\alpha - E} u_\alpha u_\alpha^\top + \frac{1}{\mu_{\alpha_E} - E} u_{\alpha_E} u_{\alpha_E}^\top \\ &:= L\left(E\right) + \frac{1}{\mu_{\alpha_E} - E} u_{\alpha_E} u_{\alpha_E}^\top . \end{aligned}$$

For $i = 1,\, 2$, we express w_i as

$$w_i = \tilde{w}_i + \sqrt{\frac{p}{1-p}}\, l,$$

where $\tilde{w}_i$ has i.i.d components which has the distribution as (6.20). In particular, one can treat $\sqrt{n+2}\,\tilde{w}_i$ as an isotropic subgaussian vector whose entries have ψ_2-norms bounded by $K\left(p\right)$.

Our goal is to show that $w_i^\top L\left(E\right) w_j$ is concentrated about $-\delta_{i,j}$. To achieve that, we break it into the form

$$w_i^\top L\left(E\right) w_j = \tilde{w}_i^\top L\left(E\right)\tilde{w}_j + \sqrt{\frac{p}{1-p}}\, l^\top L\left(E\right)\tilde{w}_j + \sqrt{\frac{p}{1-p}}\,\tilde{w}_i^\top L\left(E\right) l + \frac{p}{1-p}\, l^\top L\left(E\right) l \tag{6.52}$$

and estimate each summand separately. We start with the bilinear term.

Lemma 6.33. *Fix an $n \times n$ matrix $B \in \mathscr{T}_{(n,0)}$. With probability greater than $1-\exp\left(-c\left(p\right)\varphi_n\right)$ ($\varphi_n := \left(\log n\right)^{\log\log n}$) in w_1 and w_2, we have*

$$\tilde{w}_i^{\top} L\left(E\right)\tilde{w}_j = -\left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\delta_{ij} + O\left(n^{-1/3+C\varepsilon_{LR}}\right) \tag{6.53}$$

for $E \in \left[2-n^{-2/3}\varphi_n^{2\rho},\, 2+n^{-2/3}\varphi_n^{2\rho}\right]$. Here, $O\left(n^{-2\varepsilon_{LR}}\right)$ and $O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ mean some deterministic functions of n with the prescribed asymptotic, and $c\left(p\right)$ is a constant that depends only on p.

Proof of Lemma (6.33). Fix $E \in \left[2-n^{-2/3}\varphi_n^{2A_{sls}},\, 2+n^{-2/3}\varphi_n^{2A_{sls}}\right]$. We will first estimate the expectation of $\tilde{w}_1^{\top} L\left(E\right)\tilde{w}_1$ and then use the Hanson-Wright inequality to derive the concentration.

Expectation

Since $\mathbb{E}_{\tilde{w}_1,\tilde{w}_2}\tilde{w}_1^{\top} L\left(E\right)\tilde{w}_2 = 0$ by independence of $\tilde{w}_1$ and $\tilde{w}_2$, and since $\mathbb{E}_{\tilde{w}_2}\tilde{w}_2^{\top} L\left(E\right)\tilde{w}_2 = \mathbb{E}_{\tilde{w}_1}\tilde{w}_1^{\top} L\left(E\right)\tilde{w}_1$, we have to evaluate only the last quantity. Using the fact that $\tilde{w}_1$ has independent entries with mean 0 and variance $\frac{1}{n+2}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\tilde{w}_1}\tilde{w}_1^{\top} L\left(E\right)\tilde{w}_1 &= \mathbb{E}_{\tilde{w}_1}\sum_{\alpha\neq\alpha_E}\frac{1}{\mu_\alpha - E}\langle u_\alpha,\, \tilde{w}_1\rangle^2 \\
&= \sum_{\alpha\neq\alpha_E}\frac{1}{\mu_\alpha - E}\frac{\sum_{i\in[n]} u_\alpha^2\left(i\right)}{n+2} \\
&= \frac{1}{n+2}\sum_{\alpha\neq\alpha_E}\frac{1}{\mu_\alpha - E}.
\end{aligned}$$

Recall that for all $\alpha \in [n-1]$, we have rigidity of eigenvalues (6.45):

$$\left|\mu_{\alpha+1}-\gamma_\alpha\right| \le 2\varphi_n^{A_{sls}}\left[\min\left(\alpha,\, n-\alpha+1\right)\right]^{-1/3} n^{-2/3}.$$

Hence, $\left|\left\{\alpha\,:\,\mu_\alpha > E,\,\&\,\alpha\neq\alpha_E\right\}\right| \le \varphi_n^{C\rho}$, and

$$\sum_{\alpha:\,\mu_\alpha>E\,\&\alpha\neq\alpha_E}\frac{1}{\mu_\alpha - E} \le \left|\left\{\alpha\,:\,\mu_\alpha > E,\,\&\,\alpha\neq\alpha_E\right\}\right|\cdot\frac{1}{4}n^{2/3+\varepsilon_{LR}} \le n^{2/3+2\varepsilon_{LR}} \tag{6.54}$$

We write

$$\frac{1}{\mu_\alpha - E} = \left(1+\frac{\eta^2}{\left(\mu_\alpha - E\right)^2}\right)\frac{\mu_\alpha - E}{\left(\mu_\alpha - E\right)^2+\eta^2},$$

and set $\eta := n^{-2/3-2\varepsilon_{LR}}$. With this choice of η, we have $\left|\mu_\alpha - E\right| > \frac{1}{4}n^{\varepsilon_{LR}}\eta$ from (6.51), and so $\left(1+\frac{\eta^2}{(\mu_\alpha-E)^2}\right)=1+O\left(n^{-2\varepsilon_{LR}}\right)$. Therefore,

$$\begin{aligned}
&\frac{1}{n}\sum_{\alpha:\,\mu_\alpha<E\,\&\,\alpha\neq\alpha_E}\frac{1}{\mu_\alpha-E} \\
&=\left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\sum_{\alpha:\,\lambda_\alpha<E\,\&\,\alpha\neq\alpha_E}\frac{1}{n}\frac{\mu_\alpha-E}{\left(\mu_\alpha-E\right)^2+\eta^2} && (6.55)\\
&=\left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\left[\operatorname{Re}m_B\left(E+i\eta\right)-\frac{1}{n}\sum_{\alpha:\,\mu_\alpha>E\,\text{or}\,\alpha=\alpha_E}\frac{\mu_\alpha-E}{\left(\mu_\alpha-E\right)^2+\eta^2}\right] \\
&=\left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\operatorname{Re}m_B\left(E+i\eta\right)+O\left(n^{-1/3+3\varepsilon_{LR}}\right), && (6.56)
\end{aligned}$$

where the last equality relies on (6.54). Combining (6.54) and (6.56), we get

$$\frac{1}{n}\sum_{\alpha\neq\alpha_E}\frac{1}{\mu_\alpha-E}=\left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\operatorname{Re}m_B\left(E+i\eta\right)+O\left(n^{-1/3+3\varepsilon_{LR}}\right).$$

We have $\operatorname{Re}m_B\left(E+i\eta\right)=\operatorname{Re}m_{sc}\left(E+\mathrm{i}\eta\right)+O\left(n^{-1/3+C\varepsilon_{LR}}\right)=-1+O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ by (6.44). Thus, if ε_{LR} is small enough, then

$$\frac{1}{n}\sum_{\alpha\neq\alpha_E}\frac{1}{\mu_\alpha-E}=-1+O\left(n^{-2\varepsilon_{LR}}\right).$$

We conclude that

$$\mathbb{E}_{\tilde{w}_1}\tilde{w}_1^\top L\left(E\right)\tilde{w}_1=-1+O\left(n^{-2\varepsilon_{LR}}\right).$$

Concentration

Next, we would like to apply Hanson-Wright inequality from Theorem 6.14. Tho this end, we need to estimate the operator norm and Hilbert Schmidt norm of $L\left(E\right)$. The operator norm can be estimated directly:

$$\|L\left(E\right)\|\leq\max_{\alpha\neq\alpha_E}\frac{1}{\mu_\alpha-E}\leq\frac{1}{4}n^{2/3+\varepsilon_{LR}}.$$

For the Hilbert Schmidt norm, a derivation similar to (6.56) yields

$$
\begin{aligned}
\|L(E)\|_{HS}^{2} &= \sum_{\alpha\neq\alpha_E} \frac{1}{(\mu_\alpha - E)^2} \\
&= (1+o(1)) \sum_{\alpha\neq\alpha_E} \frac{1}{(\mu_\alpha - E)^2 + \eta^2} = (1+o(1)) \frac{n}{\eta} \sum_{\alpha\neq\alpha_E} \frac{\eta}{n} \frac{1}{(\mu_\alpha - E)^2 + \eta^2} \\
&= (1+o(1)) \frac{n}{\eta} \left[\operatorname{Im} m_B(E+\mathrm{i}\eta) - \frac{\eta}{n} \frac{1}{(\mu_{\alpha_E} - E)^2 + \eta^2} \right] \\
&= (1+o(1)) \frac{n}{\eta} \left(\operatorname{Im} m_{sc}(E+\mathrm{i}\eta) + O\left(n^{-1/3+C\varepsilon_{LR}}\right) - \frac{\eta}{n} \frac{1}{(\mu_{\alpha_E} - E)^2 + \eta^2} \right),
\end{aligned}
\tag{6.57}
$$

where we used $|m_{sc}(E+\mathrm{i}\eta) - m(E+\mathrm{i}\eta)| \leq O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ from (6.44). Now, a direct computation shows that $\operatorname{Im}\,(m_{sc}(E+\mathrm{i}\eta)) = O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ and $\frac{\eta}{n}\frac{1}{\left(\mu_{\alpha_E}-E\right)^2+\eta^2} = O\left(\frac{1}{n\eta}\right) = O\left(n^{-1/3+2\varepsilon_{LR}}\right)$. Hence,

$$
\|L(E)\|_{HS}^{2} = \sum_{\alpha\neq\alpha_E} \frac{1}{(\mu_\alpha - E)^2} = (1+o(1)) \frac{n}{\eta} O\left(n^{-1/3+C\varepsilon_{LR}}\right) = O\left(n^{4/3+C\varepsilon_{LR}}\right).
$$

One can easily show that $\left\|\sqrt{n+2}\,\tilde{w}_1(i)\right\|_{\psi_2} \leq C\sqrt{\frac{1-p}{p}}$. An application of (6.7) with $X = \sqrt{n+2}\,\tilde{w}_1$ and $A = L(E)$ yields

$$
\mathbb{P}\left(\left| \tilde{w}_1^\top L(E)\tilde{w}_1 - \mathbb{E}_{w_1}\tilde{w}_1^\top L(E)\tilde{w}_1 \right| \geq \frac{t}{n+2} \right) \leq 2\exp\left(-c(p) \frac{t}{n^{2/3+C\varepsilon_{LR}}} \right).
$$

Taking $t = n^{2/3+C'\varepsilon_{LR}}$, we get $\tilde{w}_1^\top L(E)\tilde{w}_1 = \underbrace{-1 + O\left(n^{-2\varepsilon_{LR}}\right)}_{\mathbb{E}\tilde{w}_1^\top L(E)\tilde{w}_1} + O\left(n^{-1/3+C'\varepsilon_{LR}}\right)$ with probability at least $1 - \exp(-c(p)\varphi_n)$. (Recall that $\varphi_n = \log n^{\log\log n}$.)

Notice that, the same estimate works for $\tilde{w}_2$ and $\tilde{w}_1 + \tilde{w}_2$ as well: with probability at least $1 - \exp(-c(p)\varphi_n)$,

$$
(\tilde{w}_1 + \tilde{w}_2)^\top L(E)(\tilde{w}_1 + \tilde{w}_2) = \underbrace{\mathbb{E}\tilde{w}_1^\top L(E)\tilde{w}_1 + \mathbb{E}\tilde{w}_2^\top L(E)\tilde{w}_2}_{\mathbb{E}(\tilde{w}_1+\tilde{w}_2)^\top L(E)(\tilde{w}_1+\tilde{w}_2)} + O\left(n^{-1/3+C\varepsilon_{LR}}\right).
$$

Therefore, by the linearity, with probability at least $1 - \exp(-c(p)\varphi_n)$ we have

$$
\tilde{w}_1^\top L(E)\tilde{w}_2 = O\left(n^{-1/3+C\varepsilon_{LR}}\right),
$$

obtaining (6.53) for a fixed E.

To extend this to all $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$, we will use a net argument. Let $\mathscr{N}$ be a κ-net on $\left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$ with $\kappa = n^{-100}$ and assume that (6.53) holds for all $E \in \mathscr{N}$. Since $|\mathscr{N}|$ is polynomial in n, this event has probability bounded by $\exp\left(-c\left(p\right)\varphi_n\right)$.

Recall that the coordinates of $\sqrt{n+2}\tilde{w}_i$ are independent, centered, subgaussian random variables. One can easily deduce that $\left\|\sqrt{n+2}\tilde{w}_1\left(k\right)\right\|_{\psi_2} \leq C\sqrt{\frac{1-p}{p}}$. By Hoeffding's inequality,

$$\sqrt{n+2}\langle \tilde{w}_i,\, u_\alpha\rangle = \sum_{k=1}^{n} \sqrt{n+2}\tilde{w}_i\left(k\right) u_\alpha\left(k\right)$$

is also subgaussian since $\left\|u_\alpha\right\|_2 = 1$. Similarly, $(n{+}2)\left\|\tilde{w}_i\right\|_2^2$, being a sum of subexponential random variables, satisfies Bernstein's inequality. Together with a union bound, these two fact imply

$$\mathbb{P}\left(\exists \alpha \in [n]\,, i \in \{1,\, 2\}\ \left|\langle \tilde{w}_i,\, u_\alpha\rangle\right| \geq \frac{\varphi_n}{\sqrt{n+2}}\ \&\ \left\|w_i\right\|_2 \leq \varphi_n\right) \leq \exp\left(-c\left(p\right)n\right).$$

Assume that these two events occur in addition to the assumption that (6.53) holds for all $E \in \mathscr{N}$ which we already made. Let $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$, and choose $E' \in \mathscr{N}$ such that $|E - E'| < \kappa$. Suppose $\alpha_E \neq \alpha_{E'}$, then

$$\begin{aligned}
&\left|\tilde{w}_i^\top L\left(E\right)\tilde{w}_j - \tilde{w}_i^\top L\left(E'\right)\tilde{w}_j\right| \\
&\leq \left\|\tilde{w}_i\right\|_2 \left\|\tilde{w}_j\right\|_2 \sum_{\alpha \neq \alpha_E,\, \alpha_{E'}} \left|\frac{1}{\mu_\alpha - E} - \frac{1}{\mu_\alpha - E'}\right| + \left|\frac{\langle \tilde{w}_i,\, u_{\alpha_{E'}}\rangle\langle \tilde{w}_j,\, u_{\alpha_{E'}}\rangle}{\mu_{\alpha_{E'}} - E}\right| + \left|\frac{\langle \tilde{w}_1,\, u_{\alpha_E}\rangle\langle \tilde{w}_j,\, u_{\alpha_E}\rangle}{\mu_{\alpha_E} - E'}\right| \\
&\leq \left\|\tilde{w}_1\right\|_2 \left\|\tilde{w}_2\right\|_2 \sum_{\alpha \neq \alpha_E,\, \alpha_{E'}} \frac{4\kappa}{\eta^2} + \left|\frac{\langle \tilde{w}_i,\, u_{\alpha_{E'}}\rangle\langle \tilde{w}_2,\, u_{\alpha_{E'}}\rangle}{\mu_{\alpha_{E'}} - E'}\right| + \left|\frac{\langle \tilde{w}_i,\, u_{\alpha_E}\rangle\langle \tilde{w}_j,\, u_{\alpha_E}\rangle}{\mu_{\alpha_E} - E'}\right| \\
&\leq \left\|\tilde{w}_i\right\|_2 \left\|\tilde{w}_j\right\|_2 \frac{4n}{\eta^2}\kappa + \left|\frac{\langle \tilde{w}_i,\, u_{\alpha_{E'}}\rangle\langle \tilde{w}_j,\, u_{\alpha_{E'}}\rangle}{\mu_{\alpha_{E'}} - E}\right| + \left|\frac{\langle \tilde{w}_i,\, u_{\alpha_E}\rangle\langle \tilde{w}_j,\, u_{\alpha_E}\rangle}{\mu_{\alpha_E} - E'}\right|
\end{aligned}$$

Since $\alpha_E \neq \alpha_{E'}$, we have $\min\left\{\left|\mu_{\alpha_{E'}} - E\right|,\, \left|\mu_{\alpha_E} - E'\right|\right\} \geq \frac{1}{8}n^{2/3-\varepsilon_{LR}}$. Together with $\left|\langle \tilde{w}_i,\, u_\alpha\rangle\right| \leq \frac{\varphi_n}{\sqrt{n+2}}$, this yields

$$\left|\frac{\langle \tilde{w}_i,\, u_{\alpha_{E'}}\rangle\langle \tilde{w}_j,\, u_{\alpha_{E'}}\rangle}{\mu_{\alpha_{E'}} - E}\right| + \left|\frac{\langle \tilde{w}_i,\, u_{\alpha_E}\rangle\langle \tilde{w}_j,\, u_{\alpha_E}\rangle}{\mu_{\alpha_E} - E'}\right| = O\left(n^{-1/3+2\varepsilon_{LR}}\right).$$

Thus,

$$\left|\tilde{w}_i^\top L(E)\,\tilde{w}_j - \tilde{w}_i^\top L(E')\,\tilde{w}_j\right| \le \|\tilde{w}_i\|_2\,\|\tilde{w}_j\|_2\,\frac{4n}{\eta^2}\kappa + O\left(n^{-1/3+2\varepsilon_{LR}}\right)$$

As $\kappa = n^{-100}$, the difference is bounded by $O\left(n^{-1/3+2\varepsilon_{LR}}\right)$. The same bound holds for the case $\alpha_E = \alpha_{E'}$, and the proof is simpler, since the last two terms do not appear. Therefore, (6.53) holds for E as well. if constant C is appropriately adjusted. □

Next, we bound the linear and constant terms in (6.52).

Lemma 6.34. *Fix an $n\times n$ matrix $B \in \mathscr{T}_{(n,1)}$. With probability greater than $1-\exp\left(c(p)\,\varphi_n^C n\right)$, for any E such that $|E-2| \le n^{-2/3}\varphi_n^{2\rho}$,*

$$l^\top L(E)\,l = O\left(n^{-1/3+C\varepsilon_{LR}}\right), \text{ and } \tilde{w}_1^\top L(E)\,l = O\left(n^{-1/3+C\varepsilon_{LR}}\right). \tag{6.58}$$

Here, $c(p)$ is a constant that depends only on p.

Proof. Applcation of Hoeffding's inequality to $\langle \tilde{w}_i,\, u_\alpha\rangle$ yields

$$\mathbb{P}\left(\langle \tilde{w}_i,\, u_\alpha\rangle^2 \ge \frac{\varphi_n}{n+2}\right) \le \exp\left(-c(p)\,\varphi_n\right),$$

and so

$$\max_{\alpha,\, i}\langle \tilde{w}_i,\, u_\alpha\rangle^2 \le \frac{\varphi_n}{n}$$

with probability greater than $1-\exp\left(-c(p)\,\varphi_n\right)$. In view of this inequality and the fact that $\left(\sum_{\alpha\neq 1}\langle l,\, u_\alpha\rangle^2\right)^{\frac{1}{2}} = \left|P_{u_1^\perp} l\right| = O\left(\frac{\log^C n}{\sqrt{n}}\right)$,

$$\begin{aligned}\left|\sum_{\alpha\neq 1,\, \alpha_E}\frac{\langle \tilde{w}_i,\, u_\alpha\rangle\langle l,\, u_\alpha\rangle}{\mu_\alpha - E}\right| &\le \left(\sum_{\alpha\neq 1,\, \alpha_E}\langle l,\, u_\alpha\rangle^2\right)^{\frac{1}{2}}\left(\sum_{\alpha\neq 1,\, \alpha_E}\frac{\langle \tilde{w}_i,\, u_\alpha\rangle^2}{(\mu_\alpha - E)^2}\right)^{\frac{1}{2}} \\ &= O\left(\frac{\varphi_n^C}{n}\right)\sqrt{\sum_{\alpha\neq 1,\, \alpha_E}\frac{1}{(\mu_\alpha - E)^2}}.\end{aligned}$$

Again, one can approximate $\sum_{\alpha\neq 1,\, \alpha_E}\frac{1}{(\mu_\alpha - E)^2}$ by $\frac{n}{\eta}\operatorname{Im} m_{sc}(E+\mathrm{i}\eta)$ as before and obtain

$$\sum_{\alpha\neq 1,\, \alpha_E}\frac{1}{(\mu_\alpha - E)^2} = O\left(n^{4/3+C\varepsilon_{LR}}\right).$$

This shows that
$$\left|\sum_{\alpha\neq 1,\,\alpha_E}\frac{\langle\tilde{w}_i,\,u_\alpha\rangle\langle l,\,u_\alpha\rangle}{\mu_\alpha-E}\right|=O\left(n^{-1/3+C\varepsilon_{LR}}\right).$$
with probability greater than $1-\exp\left(-c\left(p\right)\varphi_n\right)$.

Furthermore, recall from (6.46) that $\mu_1\geq\frac{1}{2}\sqrt{\frac{p(n+2)}{1-p}}$. Thus $\left|\frac{\langle\tilde{w}_i,u_1\rangle\langle l,u_1\rangle}{\mu_1-E}\right|=o\left(\frac{1}{\sqrt{pn}}\right)$, and

$$\left|l^\top L\left(E\right)l\right|=\left|\sum_{\alpha\neq\alpha_E}\frac{\langle l,\,u_\alpha\rangle^2}{\mu_\alpha-E}\right|\leq\left(\frac{1}{4}n^{2/3+\varepsilon_{LR}}\sum_{\alpha\neq 1,\,\alpha_E}\langle l,\,\tilde{u}_\alpha\rangle^2\right)+\frac{1}{\mu_1-E}\leq n^{-1/3+C\varepsilon_{LR}}.$$

Again, this result can extend easily for all $E\in\left[2-n^{-2/3}\varphi_n^{2\rho},\,2+n^{-2/3}\varphi_n^{2\rho}\right]$ by a net argument. We omit the proof here since it is the same as the net argument in Lemma 6.33. $\square$

Combining Lemmas 6.33 and 6.34, we obtain Lemma 6.32.

6.4.5 Estimate of $s\left(\lambda\right)$

Recall that in Corollary 6.31, we denoted by $\mathscr{T}$ be the set of $(n+2)\times(n+2)$ symmetric matrices all whose $n\times n$ principal submatrices are typical in a sense that they satisfy the conditions in $\mathscr{T}_{(n,0)}$. Suppose that λ_α is an eigenvalue of $\tilde{A}_p$ and $v_\alpha\in\mathbb{R}^{n+2}$ is the corresponding unit corresponding eigenvector. As in (6.23),
$$\operatorname{sign}\left(v_\alpha\left(1\right)v_\alpha\left(2\right)\right)=s\left(\lambda_\alpha\right)=\operatorname{sign}\left(-\frac{w_1^\top G\left(\lambda_\alpha\right)w_1-d_{11}+\lambda_\alpha}{w_1^\top G\left(\lambda_\alpha\right)w_2-d_{12}}\right).$$

In this section, we will prove the following:

Lemma 6.35. *Let A_p be the adjacency matrix of a $G(n,p)$ graph, and let $\lambda_1\geq\cdots\geq\lambda_n$ be the eigenvalues of the matrix*
$$\tilde{A}_p=\frac{1}{\sqrt{p\left(1-p\right)n}}A_p.$$
Fix $2\leq\alpha\leq\varphi_n^\rho$. Then
$$\mathbb{E}\left(s\left(\lambda_\alpha\right)\cdot\mathbf{1}_{\mathscr{A}}(H)\right)=O\left(n^{-1/3+C\varepsilon_{LR}}\right).$$

As $\mathscr{T}$ pertains to all $n \times n$ principal submatrices, the same bound holds for

$$\mathbb{E}\left(\operatorname{sign}\left(v_\alpha\left(i\right)v_\alpha\left(j\right)\right)\cdot\mathbf{1}_{\mathscr{T}}(\tilde{A}_p)\right)$$

for any $i \neq j$.

Once this lemma is proved, Theorem 6.4 follows easily:

Proof. For $2 \leq \alpha \leq \varphi_n^\rho$, we have

$$\mathbb{E}\left((\sum_{i=1}^{n+2}\operatorname{sign}(u_\alpha(i)))^2|\mathscr{A}\right) = O(n^{5/3+C\varepsilon_{LR}}).$$

Applying Markov's inequlaity we get

$$\mathbb{P}\left(|\sum_{i=1}^{n+2}\operatorname{sign}(u_\alpha(i))| > n^{5/3+C'\varepsilon}\right) < n^{-\delta_{LR}} + n^{-\varepsilon_{LR}}.$$

□

The proof of this lemma will be based on the concentration we get from Lemma 6.32. Let B be the $n\times n$ principal submatrix containing the last n rows and columns. If $\tilde{A}_p \in \mathscr{T}$, then $B \in \mathscr{T}_{(n,0)}$.

Consider $\alpha = 2$ first. Let $\mu_1' \geq \mu_{n+1}'$ be the eigenvalues of the $(n+1)\times(n+1)$ matrix containing the last $(n+1)$ rows and columns of $\tilde{A}_p$. Per (6.45) for $\tilde{A}_p$, $\lambda_2 \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$, so interlacing and Lemma 6.23 imply that

$$\mu_2' \leq \lambda_2 \leq \mu_2' + \frac{\varphi_n^C}{n} < \mu_1'$$

where μ_1' satisfies (6.46). Repeating this argument for B, in view of (6.49) and (6.46), we conclude that $\lambda_2 \in [\mu_2, \mu_1]$. For $2 < \alpha \leq \varphi_n^\rho$, (6.49) similarly yields $\lambda_\alpha \in [\mu_\alpha, \mu_{\alpha-1}]$.

Condition on a submatrix B. Since $\alpha \leq \varphi_n^\rho$, by the estimate that $\int_{2-t}^{2}\frac{1}{2\pi}\sqrt{4-x^2}\,\mathrm{d}x \geq \frac{1}{2\pi}t^{3/2}$, we have $2-\gamma_\alpha \leq n^{-2/3}\varphi_n^\rho$ and thus $2-\mu_\alpha \leq n^{-2/3}\varphi_n^{2\rho}$ due to rigidity of eigenvalues (6.45).

Let $\mathscr{A}_{wGw}$ be the set of $n\times 2$ matrices W such that (**??**) in Lemma 6.32 holds. Specifically, $\mathscr{A}_{wGw}$ is defined by the condition

$$\forall i,\, j \in \{1,\, 2\}\;\; w_i^\top G\left(E\right)w_j = -\left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\delta_{ij} + \frac{\langle w_i,\, u_{\alpha_E}\rangle\langle w_j,\, u_{\alpha_E}\rangle}{\mu_{\alpha_E} - E} + O\left(n^{-1/3+C_1\varepsilon_{LR}}\right) \tag{6.59}$$

for all $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$ and a universal constant $C_1 > 0$. Here, $\alpha_E \in [n]$ is the integer so that $|\mu_{\alpha_E} - E| \leq \min_{\alpha\in[n]} |\mu_\alpha - E|$.

Before we move on to the proof directly, let us introduce another set. Let $\mathscr{A}_W$ be an set of W such that for $i \in \{1,\, 2\}$

$$n^{-1/3+\kappa\varepsilon_{LR}} \leq \sqrt{n}\,|\langle \tilde{w}_i,\, u_\alpha\rangle| \leq \log^2 n \tag{6.60}$$

where $\kappa \geq \max\{2C_1,\, 8\}$ and and

$$\tilde{w}_i = w_i - \sqrt{\frac{p}{1-p}}l.$$

Lemma 6.36. *Let the W be the $n \times 2$ block W of $\tilde{A}_p$ defined in (6.21). With the notation above, we have*

$$\mathbb{P}\left(W \in \mathscr{A}_W\right) \geq 1 - n^{-1/3+2\kappa\varepsilon_{LR}},$$

and

$$\mathbb{P}\left(\langle \tilde{w}_i,\, u_\alpha\rangle > 0\right) = \frac{1}{2} + O\left(n^{-1/3+5\varepsilon_{LR}}\right) \quad \textit{for } i = 1, 2. \tag{6.61}$$

Proof. The upper bound in (6.60) holds with the desired probability due to Hoeffding's inequality. We will estimate the probability that the lower bound holds and prove (6.61) at the same time. Let $X_k := \sqrt{n+2}\tilde{w}_1(k)\, u_\alpha(k)$. Since $\tilde{w}_1(k)$ has mean 0 and variance $\frac{1}{n+2}$, we set

$$S_n = \frac{\sum_{k\in[n]} X_k}{\sum_{k\in[n]} \mathbb{E}X_k^2} = \frac{\langle \tilde{w}_1,\, u_\alpha\rangle}{\sqrt{n+2}}.$$

Observe that $\mathbb{E}X_k^2 = u_\alpha(k)^2$ and $\mathbb{E}X_k^3 \leq c(p)\,|u_\alpha(k)|^3$ where $c(p) > 0$ is a constant depends on p. Let F_n and Φ be the cumulative distributions of S_n and the normal random variable respectively. By the Berry-Esseen Theorem 6.15, we have

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi(x)| \leq C\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{-1/2} \max_i \frac{\mathbb{E}\,|X_i|^3}{\mathbb{E}X_i^2} \leq c(p)\,\frac{\|u_\alpha\|_\infty}{\|u_\alpha\|_2}.$$

Recall from (6.41) in the defintion of $\mathscr{T}_{(n,0)}$, we have the l_∞-norm bound: $\sqrt{n}\,\|u_\alpha\|_\infty \leq n^{-1/3+4\varepsilon_{LR}}$. Together with $\|u_\alpha\|_2 = 1$ we get

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi(x)| \leq n^{-1/3+5\varepsilon_{LR}}$$

if n is large enough. Thus,

$$\mathbb{P}\left(\sqrt{n}\,|\langle\tilde{w}_1,\,u_\alpha\rangle| \le n^{-1/3+\kappa\varepsilon_{LR}}\right) \le \mathbb{P}\left(\sqrt{n}\,|g| \le n^{-1/3+\kappa\varepsilon_{LR}}\right) + 2n^{-1/3+5\varepsilon_{LR}} \le n^{-1/3+1.5\kappa\varepsilon_{LR}},$$

where $g \sim N(0,\,1)$ is a normal random variable. Furthermore, we also obtain (6.61) by comparing Φ and F_n. □

Proof of Lemma 6.35. By (6.22), if $\lambda \in \mathbb{R}$ is an eigenvalue of $\tilde{A}_p$, then

$$\det\left(W^\top G(\lambda)\,W - D + \lambda I_2\right) = 0.$$

Let

$$f(E) := \frac{\left(w_1^\top G(E)\,w_1 - d_{11} + E\right)\left(w_2^\top G(E)\,w_2 - d_{22} + E\right)}{\left(w_1^\top G(E)\,w_2 - d_{12}\right)^2}.$$

Thus, λ is an eigenvalue whenever $f(\lambda) = 1$. We will use the function $f(E)$ to determine the location of the eigenvalues.

Let $\mathscr{A}_D$ be the set of all 2×2 symmetric matrices D such that $\max_{i,j\in\{1,2\}}|d_{ij}| = O\left(c(p)\,n^{-1/2}\right)$. Assume that $W \in A_{wGw} \cap A_W$ and $D \in \mathscr{A}_D$. We will see below that this is a likely event.

Under these conditions, the argument becomes deterministic. By (6.41) from the definition of $\mathscr{T}_{(n,0)}$, we have $|\langle u_\alpha,\,l\rangle| \le n^{-1+2\varepsilon_{LR}}$. Hence,

$$\langle w_i,\,u_\alpha\rangle = (1+o(1))\,\langle\tilde{w}_i,\,u_\alpha\rangle$$

and in particular $\langle w_i,\,u_\alpha\rangle$ and $\langle\tilde{w}_i,\,u_\alpha\rangle$ have the same sign.

Observe that $E \mapsto w_1^\top G(E)\,w_1 - d_{11} + E$ is a strictly increasing function on $(\mu_\alpha,\,\mu_{\alpha-1})$. It tends to $-\infty$ as $E \to \mu_\alpha^+$ and $+\infty$ as $E \to \mu_{\alpha-1}^-$. Thus, it crosses 0 only once. Let E_0 be maximum of the roots of $w_1^\top G(E)\,w_1 - d_{11} + E$ and $w_2^\top G(E)\,w_2 - d_{22} + E$ on $(\mu_\alpha,\,\mu_{\alpha-1})$. Then by (6.59) and $|d_{ij}| = O\left(c(p)\,n^{-1/2}\right)$,

$$-\left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right) + \frac{\langle w_i,\,u_{\alpha_{E_0}}\rangle^2}{\mu_{\alpha_{E_0}} - E_0} + E_0 = 0$$

for some $i \in \{1,2\}$. As $\mu_{\alpha-1} > E_0 > \mu_\alpha \ge 2 - n^{-2/3}\varphi_n^{2\rho}$, this implies that $E_0 > \mu_{\alpha_{E_0}}$, and thus $\alpha_{E_0} = \alpha$. Moreover, $E_0 - 1 = 1 + O\left(n^{-2\varepsilon_{LR}}\right)$, and so

$$E_0 = \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\max\left\{\langle w_1,\,u_\alpha\rangle^2,\,\langle w_2,\,u_\alpha\rangle^2\right\} + \mu_\alpha.$$

For $E > E_0$, both $w_1^\top G(E)\,w_1 - d_{11} + E$ and $w_2^\top G(E)\,w_2 - d_{22} + E$ are positive.

Setting

$$E_1 = 2\max\left\{\langle w_1,\, u_\alpha\rangle^2,\, \langle w_2,\, u_\alpha\rangle^2\right\} + \mu_\alpha,$$

for $E \in [\mu_\alpha,\, E_1]$, we also have $\alpha_E = \alpha$, and

$$\left|\frac{\langle w_1,\, u_\alpha\rangle\langle w_2,\, u_\alpha\rangle}{\mu_\alpha - E}\right| \geq \left|\frac{\langle w_1,\, u_\alpha\rangle\langle w_2,\, u_\alpha\rangle}{\mu_\alpha - E_1}\right| \tag{6.62}$$

$$= \frac{1}{2}\min\left\{\left|\frac{\langle w_1,\, u_{\alpha_E}\rangle}{\langle w_2,\, u_\alpha\rangle}\right|,\, \left|\frac{\langle w_2,\, u_\alpha\rangle}{\langle w_1,\, u_\alpha\rangle}\right|\right\} > \log^{-2} n \cdot n^{-1/3+\kappa\varepsilon_{LR}} \tag{6.63}$$

by (6.60). So $w_1^\top G\left(E\right) w_2 - d_{12}$ has no zeros in the interval $[\lambda_\alpha,\, E_1]$. Furthermore, because

$$\min\left\{\left|\frac{\langle w_1,\, u_\alpha\rangle}{\langle w_2,\, u_\alpha\rangle}\right|,\, \left|\frac{\langle w_2,\, u_\alpha\rangle}{\langle w_1,\, u_\alpha\rangle}\right|\right\} \leq 1,$$

using (6.59) and $|d_{ij}| = O\left(c\left(p\right) n^{-1/2}\right)$ again, we get

$$\begin{aligned}\left(w_1^\top G\left(E_1\right) w_2 - d_{12}\right)^2 &= \left(\frac{\langle w_1,\, u_\alpha\rangle\langle w_2,\, u_\alpha\rangle}{\mu_\alpha - E_1} + \mathrm{O}\left(n^{-1/3+C_1\varepsilon}\right)\right)^2 \\ &= \left(\frac{1}{2}\min\left\{\left|\frac{\langle w_1,\, u_\alpha\rangle}{\langle w_2,\, u_\alpha\rangle}\right|,\, \left|\frac{\langle w_2,\, u_\alpha\rangle}{\langle w_1,\, u_\alpha\rangle}\right|\right\} + O\left(n^{-1/3+C_1\varepsilon_{LR}}\right)\right)^2 \\ &\leq \frac{1}{4} + o\left(1\right) \leq \frac{1}{2}.\end{aligned}$$

Together with

$$\left(w_1^\top G\left(E_1\right) w_1 - d_{11} + E_1\right)\left(w_2^\top G\left(E_1\right) w_2 - d_{22} + E_1\right) = 1 + o\left(1\right)$$

this yields $f\left(E_1\right) \geq 1$. Since $f\left(E_0\right) = 0$, there exists $\lambda \in \left(E_0,\, E_1\right)$ such that $f\left(\lambda\right) = 1$, which shows that $\lambda_\alpha \in \left(E_0,\, E_1\right)$.

Now we will focus on $s\left(\lambda_\alpha\right)$. Since $\lambda_\alpha > E_0$, the $w_1^\top G\left(\lambda_\alpha\right) w_1 - d_{11} + \lambda_\alpha$ is positive. Also,

$$w_1^\top G\left(\lambda_\alpha\right) w_2 - d_{12} = \frac{\langle w_1,\, u_\alpha\rangle\langle w_2,\, u_\alpha\rangle}{\mu_\alpha - \lambda_\alpha} + O\left(n^{-1/3+C\varepsilon_{LR}}\right),$$

and the magnitude of the leading term is significantly greater than $O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ by (6.62). Since $\mu_\alpha - \lambda_\alpha < 0$, the expression above has the same sign as $-\langle w_1,\, u_\alpha\rangle\langle w_2,\, u_\alpha\rangle$. Therefore, we conclude that

$$s\left(\lambda_\alpha\right) = \left(-\frac{w_1^\top G\left(\lambda_\alpha\right) w_1 - d_{11} + \lambda}{w_1^\top G\left(\lambda_\alpha\right) w_2 - d_{12}}\right) = \mathrm{sign}\left(s\left(\langle w_1,\, u_\alpha\rangle\langle w_2,\, u_\alpha\rangle\right)\right) = \mathrm{sign}\left(s\left(\langle \tilde{w}_1,\, u_\alpha\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right)\right)$$

for any $\tilde{A}_p \in \mathscr{T},\ W \in \mathscr{A}_{wGw} \cap \mathscr{A}_W$, and $D \in \mathscr{A}_D$.

It remains to estimate the expectation of $s(\lambda_\alpha)$. Recall that we conditioned on the block B, and W and D are independent of B. Denote this conditional expectation and probability by $\mathbb{E}_{W,D}$ and $\mathbb{P}_{W,D}$. We have

$$\begin{aligned}
|\mathbb{E}_{W,D}\left(s(\lambda_\alpha)\,\mathbf{1}_{\mathscr{T}}(A_p)\right)| \leq & |\mathbb{E}_{W,D}\left(s(\lambda_\alpha)\,\mathbf{1}_{\mathscr{T}}(A_p)\mathbf{1}_{\mathscr{A}_W}(W)\mathbf{1}_{\mathscr{A}_{wGw}}(W)\mathbf{1}_{\mathscr{A}_D}(D)\right)| \\
& + \mathbb{P}_{W,D}\left(W \notin \mathscr{A}_{wGw} \cup \mathscr{A}_W\right) + \mathbb{P}_{W,D}\left(D \notin \mathscr{A}_D\right) \\
= & |\mathbb{E}_{W,D}\left(\operatorname{sign}\left(s\left(\langle \tilde{w}_1,\, u_\alpha\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right)\right)\mathbf{1}_{\mathscr{T}}(A_p)\mathbf{1}_{\mathscr{A}_W}(W)\mathbf{1}_{\mathscr{A}_{wGw}}(W)\mathbf{1}_{\mathscr{A}_D}(D)\right)| \\
& + O\left(n^{-1/3+C'\varepsilon_{LR}}\right).
\end{aligned}$$

We can get rid of the indicators in the leading term in a similar way:

$$\begin{aligned}
& |\mathbb{E}_{W,D}\left(\operatorname{sign}\left(s\left(\langle \tilde{w}_1,\, u_\alpha\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right)\right)\mathbf{1}_{\mathscr{T}}(A_p)\mathbf{1}_{\mathscr{A}_W}(W)\mathbf{1}_{\mathscr{A}_{wGw}}(W)\mathbf{1}_{\mathscr{A}_D}(D)\right)| \\
\leq & |\mathbb{E}_{W,D}\left(\operatorname{sign}\left(s\left(\langle \tilde{w}_1,\, u_\alpha\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right)\right)\mathbf{1}_{\mathscr{T}}(A_p)\right)| + \mathbb{P}_{W,D}\left(W \notin \mathscr{A}_{wGw} \cup \mathscr{A}_W\right) + \mathbb{P}_{W,D}\left(D \notin \mathscr{A}_D\right) \\
\leq & |\mathbb{E}_{W,D}\left(\operatorname{sign}\left(s\left(\langle \tilde{w}_1,\, u_\alpha\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right)\right)\mathbf{1}_{\mathscr{T}}(A_p)\right)| + O\left(n^{-1/3+C'\varepsilon_{LR}}\right).
\end{aligned}$$

Removing the conditioning over B, we get

$$\begin{aligned}
& \left|\mathbb{E}\left(s(\lambda_\alpha)\,\mathbf{1}_{\mathscr{T}}(\tilde{A}_p)\right)\right| \\
\leq & |\mathbb{E}\left(\operatorname{sign}\left(s\left(\langle \tilde{w}_1,\, u_\alpha\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right)\right)\mathbf{1}_{\mathscr{T}}(A_p)\right)| + O\left(n^{-1/3+C'\varepsilon_{LR}}\right) \\
\leq & \left|\mathbb{E}\left(\operatorname{sign}\left(s\left(\langle \tilde{w}_1,\, u_\alpha\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right)\right)\mathbf{1}_{\mathscr{T}_{(n,1)}}(A_p^{(1,2)})\right)\right| \\
& + \mathbb{E}\left|\mathbf{1}_{\mathscr{T}}(A_p) - \mathbf{1}_{\mathscr{T}_{(n,1)}}(\tilde{A}_p^{(1,2)})\right| + O\left(n^{-1/3+C'\varepsilon_{LR}}\right)
\end{aligned}$$

In view of Corollary 6.31, the second term does not exceed $n^{-1/3+2\varepsilon_{LR}}$. To bound the first term, we condition again on the block $B = \tilde{A}_p^{(1,2)}$ such that $\tilde{A}_p^{(1,2)} \in \mathscr{T}_{(n,1)}$ and apply (6.61). By this inequality,

$$P_i := \mathbb{P}\left[\langle \tilde{w}_1,\, u_a\rangle \geq 0 \mid \tilde{A}_p^{(1,2)}\right] := \frac{1}{2} + p_i$$

where $p_i = O\left(n^{-1/3+5\varepsilon_{LR}}\right)$. Using the independence of $\tilde{w}_1$ and $\tilde{w}_2$, we get

$$\begin{aligned}
\mathbb{E}\left[\operatorname{sign}\left(\langle \tilde{w}_1,\, u_a\rangle\langle \tilde{w}_2,\, u_\alpha\rangle\right) \mid \tilde{A}_p^{(1,2)}\right] &= P_1P_2 + (1-P_1)(1-P_2) - P_1(1-P_2) - (1-P_1)P_2 \\
&= 4p_1p_2 = O\left(n^{-2/3+10\varepsilon_{LR}}\right).
\end{aligned}$$

Removing the conditioning completes the proof of Lemma 6.35. $\square$

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